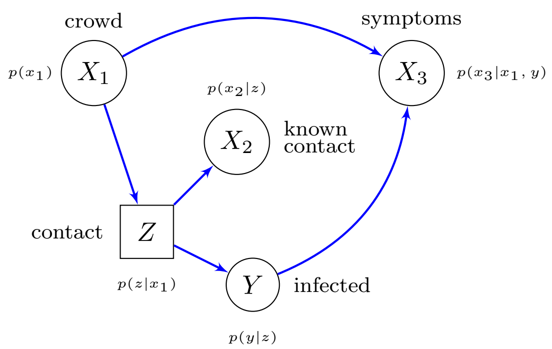
Extra Exercises for Models of Complex Systems

2022-02-09

Exercise 1. Consider the following Bayesian network over binary variables X_1, X_2, X_3, Y, Z.



$$p(y, x_1, x_2, x_3, z) = p(y|z)p(z|x_1)p(x_2|z)p(x_3|x_1, y)p(x_1)$$
$$p(y, x_1, x_2, x_3) = p(y|x_2, x_1)p(x_3|x_1, y)p(x_2|x_1)p(x_1)$$

Consider the following data, where in all tables $\hat{p} = \hat{p}(1|\ldots)$, the estimated conditional probability of a 1:

	y		
z	0	1	$\hat{p}$
0	9990	10	0.001
1	800	200	0.200

$p(y|z)$

	x_2		
z	0	1	$\hat{p}$
0	NN	0	0.0
1	50	50	0.5

$p(x_2|z)$

	x_3		
(x_1, y)	0	1	$\hat{p}$
(0,0)	980	20	0.02
(1,0)	950	50	0.05
(0,1)	20	80	0.80
(1,1)	15	85	0.85

$p(x_3|x_1, y)$

	z		
x_1	0	1	$\hat{p}$
0	990	10	0.01
1	190	10	0.05

$p(z|x_1)$

x_1		
0	1	$\hat{p}$
950	50	0.05

$p(x_1)$

Compute..

- ..the conditional probability $p(x_2|x_1)$ for $x_1 = 0, x_2 = 1$.
- ..the conditional probability $p(y|x_2, x_1)$ for $y = 1, x_1 = 1, x_2 = 0$.
- ..the probability $p(y, x_1, x_2, x_3)$ for $y = 0, x_1 = 0, x_2 = 1, x_3 = 1$.
- ..the conditional probability

$$p(y|x_1, x_2, x_3) = P(Y = y \mid X_1 = x_1, X_2 = x_2, X_3 = x_3)$$

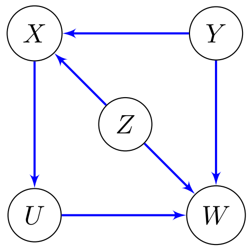
  for $y = 0, x_1 = 1, x_2 = 1, x_3 = 0$.

Exercise 2. How many directed graphs are I-equivalent to the directed chain $X_1 \to X_2 \to \ldots \to X_d$?

Exercise 3. Watch the first 4 minutes of the *Reachable* video available on the course page. On slide 11, Niels mentions that "it's probably a good idea to think in terms of the topological ordering of the nodes" and that the topological ordering may be useful for implementing the Reachable algorithm.

Discuss why the topological ordering is helpful in phase 1 of the Reachable algorithm. For example, provide an example of a Bayesian structure $\mathcal{G}$, nodes $\mathbf{Z}$ in $\mathcal{G}$, and a node $X \notin \mathbf{Z}$ and illustrate how, for your chosen setting, phase 1 benefits from traversing the nodes in the reverse topological ordering by contrasting it to another less suitable ordering.

Exercise 4. Determine which of the conditional independencies below hold for all distributions that factorize according to the Bayesian structure:



- $Z \perp U \mid X$
- $Z \perp Y \mid U$
- $X \perp W \mid U, Z, Y$
- $Z \perp U \mid X, W$
- $Y \perp U \mid X, Z$

Exercise 6. Consider the following two specifications of the joint distribution of three random variables X, Y, Z with $\mathrm{Val}(X) = \mathrm{Val}(Y) = \mathbb{N}_0$ and $\mathrm{Val}(Z) = \{0, 1\}$.

- $P(Z = 1) = 0.5$, $X \perp Y \mid Z$, $P(X \mid Z = 0) = P(Y \mid Z = 0)$ is a Poisson distribution with mean 1 and $P(X \mid Z = 1) = P(Y \mid Z = 1)$ is a Poisson distribution with mean 5.
- $X \perp Y$, $P(X) = P(Y)$ is a Poisson distribution with mean 5 and

$$Z = \begin{cases} 1 & \text{if } |X - Y| \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Draw the graphs of the Bayesian networks that correspond to the distributions. Write a program that simulates from either distribution and simulate 1000 triples (x_i, y_i, z_i). Make three plots for both distributions: one plot of y_i against x_i, one plot of y_i against x_i for those i with $z_i = 0$ and one plot of y_i against x_i for those i with $z_i = 1$. Comment on the results.
Hint for plotting: When plotting many values for discrete variables there is a lot of overplotting, which prevents you from seeing patterns. Using transparency *(`alpha = 0.3` in matplotlib's `scatter` function), and* jittering *(use e.g. `regplot` from the seaborn library) can help resolve this issue.*