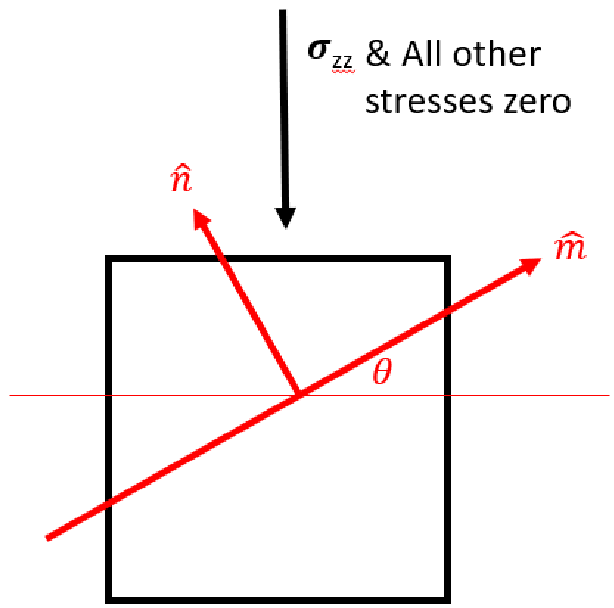
Stress & Strain exercises

Stress on an arbitrary surface /rotating the stress tensor

The force on an arbitrary surface element is calculated as $d\bar{F} = \boldsymbol{\sigma} \cdot d\bar{S} = \boldsymbol{\sigma} \cdot \hat{n} dS$. The meaning of the individual components of the stress tensor ($\boldsymbol{\sigma}$) is tied to the coordinate system. Often we want the stress on a surface with an arbitrary normal-vector $\hat{n}$. This can be calculated by projecting the tensor onto the normal with this operation: $\bar{\sigma}_{\hat{n}} = \boldsymbol{\sigma}\hat{n}$.



Now, consider a stress tensor where the only non-zero component is $\boldsymbol{\sigma}_{zz}$. We want to know the stresses in a coordinate system which is rotated θ degrees around the y-axis.

Tasks:

- Express $\hat{n}$ in terms of θ.
    - Let $c = \cos(\theta)$ and $s = \sin(\theta)$
- Write an expression for $\bar{\sigma}_{\hat{n}}$
- Write an expression for the normal stress with respect to $\hat{n}$.
- Write an expression for the shear stress along $\hat{m}$.

Looking at your results does it then make sense that in order to rotate a tensor to a new orthonormal basis then you apply the operation $\boldsymbol{\sigma}' = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^{\boldsymbol{T}}$, where **R** is a rotation matrix.

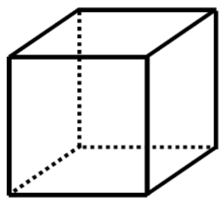
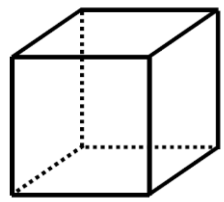
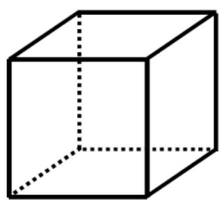
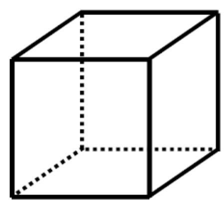
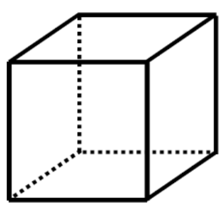
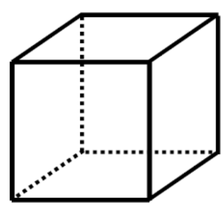
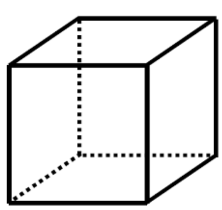
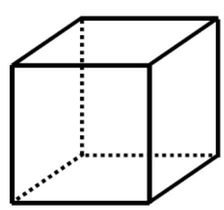
Book problems

6.3 A strong man pulls a jumbo airplane slowly but steadily, exerting a force of $\mathcal{F} = 2000$ N on a rope. The plane has $N = 32$ wheels, each touching the ground in a square area $A = 40 \times 40\ \text{cm}^2$. **(a)** Estimate the shear stress between the rubber and the tarmac. **(b)** Estimate the shear stress between the tarmac and his feet, each with area $A = 5 \times 25\ \text{cm}^2$.

6.4 **(a)** Estimate the maximal height h of a mountain made from rock with density about 3,000 kg m^{-3} when the maximal stress the material can tolerate before it deforms permanently is 300 MPa. **(b)** How high could it be on Mars where the surface gravity is 3.7 m s^{-2}? **(c)** Are the results reasonable?

Cube exercise

1. A cube of some material is under stress as described by the stress tensor.
2. Draw the stresses on the cube.
3. Many materials are near incompressible (e.g. ice). For these pure pressure does not lead to deformation. Therefore, calculate the 'stress deviator' as $\tau_{ij} \equiv \sigma_{ij} - \delta_{ij}\frac{1}{3}\sum_k \sigma_{kk}$.
4. How do you think the cube will deform?

$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}$	Deformation
$\boldsymbol{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{bmatrix}$ $\sigma < 0$		
$\boldsymbol{\sigma_2} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$ $\sigma < 0$		
$\boldsymbol{\sigma_3} = \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{bmatrix}$ $\sigma < 0$		
$\boldsymbol{\sigma_4} = \begin{bmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix}$ $\tau > 0$		

Book problem

7.3 A displacement field is given by

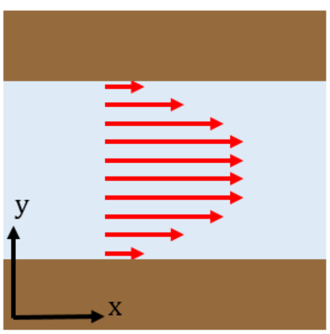
$$u_x = \alpha(x + 2y) + \beta x^2,$$
$$u_y = \alpha(y + 2z) + \beta y^2,$$
$$u_z = \alpha(z + 2x) + \beta z^2,$$

where α and βL are small (with L being the size of the body).
(a) Calculate the divergence and curl of this field. **(b)** Calculate Cauchy's strain tensor.

Crevasse exercise 1

The figure below illustrates the map plane view of a glacier constrained by valley sides. Assume velocities don't change along the flow direction ($\nabla_x v_x = 0$).

- Estimate the components of the 2D strain rate tensor.
- Calculate the principal strain rates.
- Which orientation would newly formed crevasses tend to open?



Width: 2000 m
Center Speed: 100 m/yr

Assume border wall speed = 0

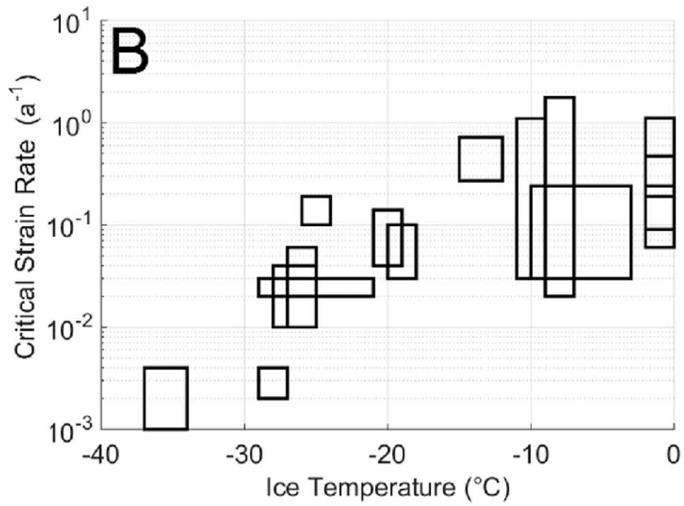
Crevasse exercise 2:

In this exercise you work on Greenland ice velocities derived from remote sensing data. Load the data from a random outlet glacier of your choice. Calculate and visualize as maps:

- The divergence of the velocity field
- Principal strain rates and principal directions

See if you can guess the principal directions try to guess the orientation of crevasse opening. Compare the real crevasse orientations seen in google maps.

Compare the principal extensional strain rate from your maps to the graph on the right - focus on an area where crevasses are just opening as that must be the 'critical strain rate'.



Is there a reasonable agreement?

There is a semi-empirical relation between deviatoric stress (τ) and strain rate ($\dot{\epsilon}$) for ice called Glen's flow law. A simplified scalar version can be written

$\dot{\epsilon} = A(T)\tau^3$ with $A(T) = A_0 e^{-Q/RT}$,

where $A_0 = 3.985 \cdot 10^{-13}\ \mathrm{s}^{-1}\mathrm{Pa}^{-3}$, and $Q = 60\ \mathrm{kJ\ mol}^{-1}$.

Final question. Take this relationship as given and use your own estimate of the 'critical strain rate' to estimate the tensile strength of ice. Compare your estimate to lab results from the literature (use google to find that).

I will put some code on Absalon to help you get started.

NOTE: The code depends on xarray, rasterio, and gdal. If you don't want to struggle with these dependencies then you can run it in the "geo" jupyter notebook on erda.

Source of graph. [LINK]