# Hint for Problem (h)
# (Week 2)

September 14, 2021

Similar to how we fitted $\alpha(\omega)$ to a polynomial $P(\omega)$ in Problem (h), we now wish to fit it to a rational function, for which it is possible to include the singularity:

$$\alpha(\omega) \approx Q(\omega) = \frac{\sum_{j=0}^{n} a_j \omega^j}{1 + \sum_{j=1}^{n} b_j \omega^j} \tag{i}$$

We rearrange the equations:

$$\begin{aligned} Q(\omega) &= \frac{\sum_{j=0}^{n} a_j \omega^j}{1 + \sum_{j=1}^{n} b_j \omega^j} \\ \iff Q(\omega)\left(1 + \sum_{j=1}^{n} b_j \omega^j\right) &= \sum_{j=0}^{n} a_j \omega^j \qquad \text{(ii)} \\ \iff Q(\omega) &= \sum_{j=0}^{n} a_j \omega^j - \sum_{j=1}^{n} b_j (Q(\omega)\omega^j) \end{aligned}$$

This is a nonlinear equation (notice $Q(\omega)$ on both sides), but we can make a linear approximation by setting $Q(\omega) = \alpha(\omega)$ on the right hand side:

$$Q(\omega) \approx \sum_{j=0}^{n} a_j \omega^j - \sum_{j=1}^{n} b_j (\alpha(\omega)\omega^j) \tag{iii}$$

(a good approximation if $Q(\omega) \approx \alpha(\omega)$). By plugging in the $N = 1000$ calculated points of $\alpha(\omega)$, we then get a $N \times (2n+1)$ matrix $\mathbf{Q}$, which acts on a coefficient vector $\mathbf{c} = [a_0, \ldots, a_n, b_1, \ldots, b_n]^T$ as

$$(\mathbf{Qc})_i = \sum_{j=0}^{n} a_j \omega_i^j + \sum_{j=1}^{n} b_j (-\alpha(\omega_i)\omega_i^j) \tag{iv}$$

Together with the vector of $\alpha$-values, this defines a linear least-squares equation

$$\mathbf{Qc} \simeq \boldsymbol{\alpha} \tag{v}$$

where $\boldsymbol{\alpha} = [\alpha(\omega_1), \ldots, \alpha(\omega_N)]^T$, and $\mathbf{Q}$ is the length-$(2n+1)$ vector of basis functions

$$[1, \omega, \omega^2, \ldots, \omega^n, -\alpha(\omega)\omega, \ldots, -\alpha(\omega)\omega^n]$$

evaluated at each of the $N$ values of $\omega$.

Once you have calculated the $\mathbf{a}$ and $\mathbf{b}$ coefficients, you can plug them back into Eq. (5) in the assignment text to obtain the rational function $Q(\omega)$.