Chapter 6
Weak Formulations in Three Dimensions

6.1 Introduction

Albeit a bit repetitive, we follow similar constructions to those used in the one-dimensional analysis of the preceding chapters. This allows readers a chance to contrast and compare the differences between one-dimensional and three-dimensional formulations. To derive a direct weak form for a body, we take the balance of linear momentum $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}$ (denoting the strong form) and form a scalar product with an arbitrary smooth vector valued function $\boldsymbol{\nu}$, and integrate over the body (Fig. 6.1),

$$\int_{\Omega} (\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}) \cdot \boldsymbol{\nu} \, d\Omega = \int_{\Omega} \boldsymbol{r} \cdot \boldsymbol{\nu} \, d\Omega = 0, \tag{6.1}$$

where $\boldsymbol{r}$ is the residual and $\boldsymbol{\nu}$ is a test function. If we were to add a condition that we do this for all possible test functions ($\forall \boldsymbol{\nu}$), Eq. 6.1 implies $\boldsymbol{r} = \mathbf{0}$. Therefore, if every possible test function was considered, then

$$\boldsymbol{r} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \tag{6.2}$$

on any finite region in Ω. Consequently, the weak and strong statements would be equivalent provided the true solution is smooth enough to have a strong solution. Clearly, $\boldsymbol{r}$ can never be nonzero over any finite region in the body, because the test function will locate them. Using the product rule of differentiation,

$$\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) = (\nabla \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{\nu} + \nabla \boldsymbol{\nu} : \boldsymbol{\sigma} \tag{6.3}$$

leads to, $\forall \boldsymbol{\nu}$

$$\int_{\Omega} (\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) - \nabla \boldsymbol{\nu} : \boldsymbol{\sigma}) \, d\Omega + \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{\nu} \, d\Omega = 0, \tag{6.4}$$

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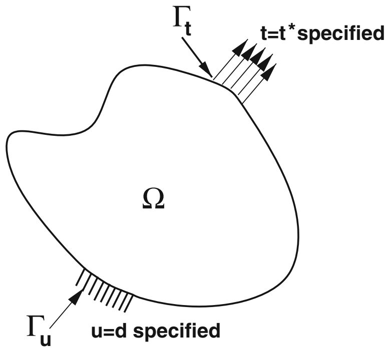

Fig. 6.1 A three-dimensional body

where we choose the $\boldsymbol{v}$ from an admissible set, to be discussed momentarily. Using the divergence theorem leads to, $\forall \boldsymbol{v}$,

$$\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{\sigma}\, d\Omega = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{n} \cdot \boldsymbol{v}\, dA, \tag{6.5}$$

which leads to

$$\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{\sigma}\, d\Omega = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \int_{\Gamma_t} \boldsymbol{t} \cdot \boldsymbol{v}\, dA. \tag{6.6}$$

If we decide to restrict our choices of $\boldsymbol{v}$'s to those such that $\boldsymbol{v}|_{\Gamma_u} = \mathbf{0}$, we have, where $\mathbf{d}$ is the applied boundary displacement on Γ_u, for infinitesimal strain linear elasticity

$$\text{Find } \boldsymbol{u}, \boldsymbol{u}|_{\Gamma_u} = d, \quad \text{such that } \forall \boldsymbol{v}, \boldsymbol{v}|_{\mathrm{u}} = \mathbf{0}$$

$$\underbrace{\int_{\Omega} \nabla \boldsymbol{v} : \boldsymbol{I\!E} : \nabla \boldsymbol{u}\, d\Omega}_{\stackrel{\text{def}}{=}\mathcal{B}(\boldsymbol{u},\boldsymbol{v})} = \underbrace{\int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, d\Omega + \int_{\Gamma_t} \boldsymbol{t} \cdot \boldsymbol{v}\, dA.}_{\stackrel{\text{def}}{=}\mathcal{F}(\boldsymbol{v})} \tag{6.7}$$

As in one-dimensional formulations, this is called a "weak" form because it does not require the differentiability of the stress $\boldsymbol{\sigma}$. In other words, the differentiability requirements have been *weakened*. It is clear that we are able to consider problems with quite irregular solutions. We observe that if we test the solution with all possible test functions of sufficient smoothness, then the weak solution is equivalent to the strong solution. *We emphasize that provided the true solution is smooth enough, the weak and strong forms are equivalent, which can be seen by the above constructive derivation.*