1. a) Probe

i) We need to balance the fluxes from the probe and the thermal shield

Without loss of generality, we assume that the area of the probe and thermal shield is the same (rods are of neglilible length)

Then we have the following situation (assuming that both the probe and the shield emit as black bodies)

$$F_{probe} = \sigma(T_{probe}^4 - T_{shield}^4)$$

$$F_{shield} = \sigma(T_{shield}^4 - T_{space}^4)$$

since the heat flow should be equal for black bodies in equilibrium

$F_{probe} = F_{shild} = F$

Adding the two we get

$$2F = \sigma \left(T_{probe}^4 - T_{space}^4 \right)$$

which would be the heat flow in the absence of the shield, F_0

So $F_1 = \frac{F_0}{2}$

If we approximate $T_{space}^4 << T_{probe}^4$ (it is far away from the sun after all), then balancing the internal luminosity with the flux gives $F = \frac{L}{A} = \sigma T_{probe}^4$, which must always hold true, so T_{probe} must go up with a factor of $(2)^{1/4}$ such that when the heat flow is diminished by a factor of 2

The temperature of the black-body makes up for it.

So $\underline{\underline{T_{probe} = (2)^{1/4} T_{probe,0}}}$

If we do not make that apuroximation we get

$$\sigma(T_{probe,1}^4 - T_{space}^4) = 2\sigma(T_{probe,0}^4 - T_{space}^4)$$

$$\underline{\underline{T_{probe,1} = (2T_{probe,0}^4 - T_{space}^4)^{1/4}}}$$

ii) This can be extended straightforwardly to the situation where we have N thermal shields by adding equation

So we get

$$\sigma(T_{probe,N}^4 - T_{space}^4) = (N+1)\sigma(T_0^4 - T_{space}^4)$$

$$T_{probe,N} = \left((N+1)\,T_{probe,0}^{4} - N\,T_{space}^{4}\right)^{1/4}$$

Assuming $\frac{T_{space}^{4}}{T_{probe,0}^{4}} \approx 0$

$$T_{probe,N} \approx \left((N+1)\,T_{probe,0}\right)^{1/4}$$

b) To determine T we must balance the total amount of energy coming in and out of the probe, i.e, thermal radiation from the probe, with internal flux + contributions from the sun at the probes position. The area over which it receives radiation is equal to $A_{sphere}/4$ (circle/sphere area)

$$A_{probe} \cdot \sigma \cdot T^4 = F_{probe} \cdot A_{probe} + F_{\odot} \cdot \frac{A_{\odot}}{4\pi D_{probe}^2} \cdot \frac{A_{probe}}{4}$$

Taking out Aprobe and inserting

$$F_{probe} = 5 \cdot 10^{-6} F_{\odot}$$

$$\sigma T^4 = 5 \cdot 10^{-6} F_{\odot} + F_{\odot} \cdot \frac{A_{\odot}}{16\pi D_{probe}^2}$$

$$= F_{\odot} \left(\frac{A_{\odot}}{16\pi D_{probe}^2} + 5 \cdot 10^{-6} \right)$$

$$F_{\odot} = \sigma \cdot T_{\odot}^4$$

$$T^4 = T_{\odot}^4 \left(\frac{A_{\odot}}{16\pi D_{probe}^2} + 5 \cdot 10^{-6} \right)$$

Remembering that $R_{\odot} \approx 7 \cdot 10^{8}$ m

and 1 AU $\approx 1.5 \cdot 10^{11}$ m

We get $T^4 = T_{\odot}^4 (5.5 \cdot 10^{-11} + 5 \cdot 10^{-6})$

$$T \approx T_{\odot} (5 \cdot 10^{-6})^{1/4} \approx 272 \text{ K}$$

10) Black hole Heat Bath

5) Freezing

a) The rate at which latent heat is released is

$$\dot{L} = L \cdot \rho \cdot \frac{dX(t)}{dt} \cdot A$$

where A is the area of the lake

b) $q = -k \nabla T$

not sure which T to put first

T is linear so $\nabla T(t) = \frac{T_0 - T_m}{X(t)}$

since we are looking for the heat flux, not heat flux density, we multiply by the area A_{lake}

$$q_{tot} = -k(T_0 - T_m) / X(t) \cdot A_{lake}$$

c) Setting these two equal

$$q_{tot} = \dot{L}$$

$$L \cdot \rho \cdot \cancel{A_{lake}} \cdot \frac{dX(t)}{dt} = -k \frac{T_0 - T_m}{X(t)} \cancel{A_{lake}}$$

Assuming $X(t) = \sqrt{2Kt}$, $\frac{dX}{dt} = \frac{K}{\sqrt{2Kt}}$

Plugging this in we see

$$L \cdot \rho \cdot \frac{K}{\cancel{\sqrt{2Kt}}} = -k \frac{T_0 - T_m}{\cancel{\sqrt{2Kt}}}$$

$$K = \frac{k(T_m - T_0)}{L \cdot \rho}$$

So $X = \sqrt{2Kt}$ solves the system,

if $K = \frac{k(T_m - T_0)}{L \cdot \rho}$ $\square$

d) Taylor expand erf(λ) in λ

$$\mathrm{erf}(\lambda) \simeq \frac{2}{\sqrt{\pi}} \lambda$$

$\lambda \ll 1$ so $e^{\lambda^2} \simeq 1$

$$\lambda^2 \frac{2}{\cancel{\sqrt{\pi}}} = \frac{c(T_m - T_0)}{L \cancel{\sqrt{\pi}}}$$

$$\lambda = \sqrt{\frac{c(T_m - T_0)}{2L}}$$

Inserting we get

$$z = \sqrt{\frac{c \cdot (T_m - T_0)}{2L}} \cdot \sqrt{\frac{k}{\rho c} \cdot t}$$

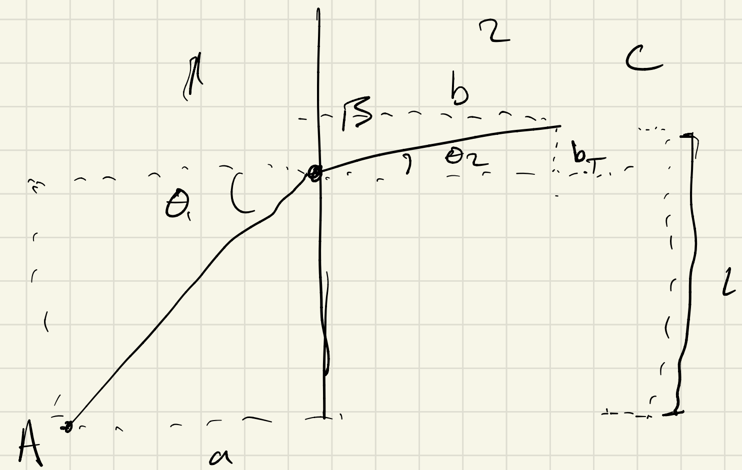
$$= \sqrt{2 \cdot \frac{(T_m - T_0) \cdot k}{L\rho} \cdot t} = \sqrt{2Kt} \quad \square$$

$$K = \frac{k \cdot (T_m - T_0)}{L\rho}$$ as before!!

e) The treatment should still be appropriate. With steel you will have more total latent heat to remove per meter the Freezing front moves but the higher k means that one can get rid of energy more efficiently. The two effects should therefore balance, making similar treatments appropriate!

2 Variational Physics

a) From the following diagram



And considering the total travel time T for a light ray to get from A to C through B we get

$$T = \frac{\sqrt{a^2 + a_T^2}}{V_1} + \frac{\sqrt{b^2 + b_T^2}}{V_2}$$

Where V_1, and V_2 are the speeds in the respective media

Note that refractive indices

$$n_1 = \frac{c}{v_1}, \quad n_2 = \frac{c}{v_2}$$

Since $b_T = l - a_T$ we can insert this and differentiate w.r.t. a_T

$$\frac{dT}{da_T} = \frac{a_T}{v_1\sqrt{a^2 + a_T^2}} + \frac{a_T - l}{v_2\sqrt{(l-a_T)^2 + b^2}}$$

$$\frac{a_T}{\sqrt{a^2 + a_T^2}} = \sin\theta_1, \quad \frac{a_T - l}{\sqrt{(l-a_T)^2 + b^2}} = -\sin\theta_2$$

This has a minimum when $\frac{dT}{da_T} = 0$

so

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

$$\Leftrightarrow \quad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

$\Leftarrow$ $\quad n_1 \sin\theta_1 = \sin\theta_2 \, n_2$ (using $n = \frac{c}{v}$)

$\Box$

c) The rope has no kinetic energy so this isn't really Euler-Lagrange

We do still need to minimize the potential energy of the rope under the assumption that the total length of the rope does not stretch.

leave out since uniform

$$U = \int_{-a}^{a} y\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx \cdot (\rho g)$$

$$L - l = 0, \quad L = \int_{-a}^{a} \left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$

We can think of this as a Lagrange multiplier

So in the style of calculus of variations we can write

$$\delta(U - \lambda L) = 0 = \delta\left(\int_{-a}^{a} (y-\lambda)\left(1+\left[\frac{dy}{dx}\right]^2\right)^{1/2} dx\right)$$

Since δ is arbitrary, the integral must be zero.

(Aha! this is where the Euler-Lagrange part comes in!)

Using the first integral of the integrand from Euler-Lagrange, we get that this is equal to

$$\frac{y-\lambda}{\left(1+\left[\frac{dy}{dx}\right]^2\right)^{1/2}} = C$$

(don't remember the constant, will determine later)

Separating variables

$$\frac{dy}{dx} = \sqrt{\left(\frac{y-\lambda}{c}\right)^2 - 1}$$

Remembering that the solution is

$$y = \lambda + c\cosh\left(\frac{x-c_2}{c}\right)$$

We determine λ, c, c_2 from boundary conditions.

From symmetry, $c_2 = \frac{1}{2}c$

$y(-a) = y(a) = 0$ so