Valley glacier

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Introduction

Valley glaciers initiate in high-altitude mountainous regions where average temperatures are low and accumulation rates (snowfall) are high (Figure 1). As time passes, consecutive years of snowfall accumulate, and the snow pack densifies and eventually transforms into glacier ice. As ice flows (creeps due to gravity) down-valley into warmer conditions, ablation (melting) becomes prevalent, thus removing mass. In a steady state, the (integrated) accumulation and ablation rates therefore balance over time.

In this exercise, we shall model the creep of glacier ice using a 2D vertical

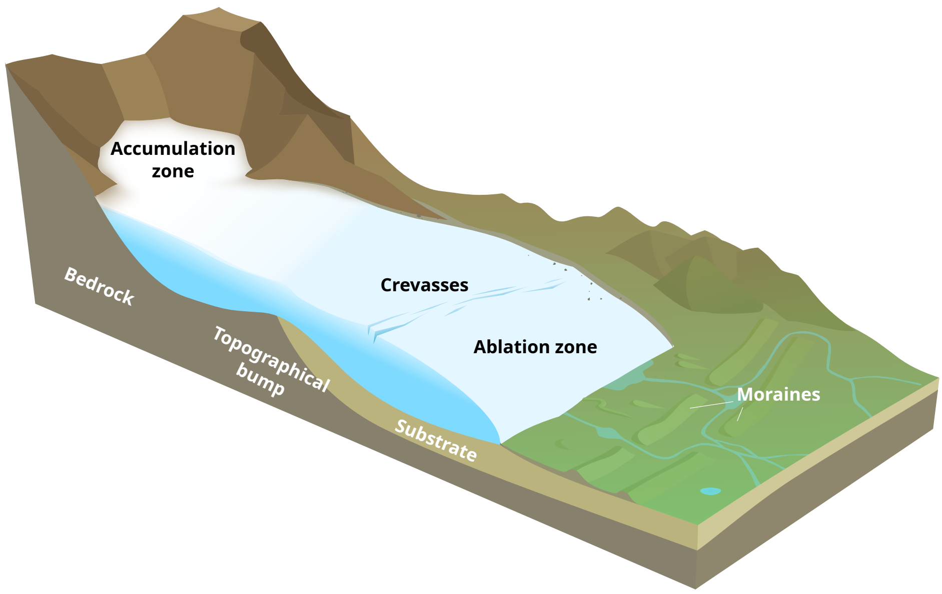

Figure 1: A valley glacier.

slab model (Figure 2). Specifically, we shall investigate how the flow and internal stresses of an idealized valley glacier respond to changes in boundary conditions.

Boundary conditions

The model assumes the glacier surface Γ_S (dashed black line in Figure 2) is stress free, i.e.

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_S, \tag{1}$$

and that the ice is frozen onto the bedrock on the *upper* part of the subglacial boundary $\Gamma_{B,1}$ (purple solid line in Figure 2), i.e.

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{B,1}. \tag{2}$$

How conditions at the left-hand boundary (Γ_L) and *lower* part of the subglacial boundary $\Gamma_{B,2}$ (green solid line in Figure 2) can influence the velocity and stress-field solutions is the main focus of this exercise.

Ice as a fluid

Although ice is a nonlinear viscoplastic material (so-called power-law fluid), we shall, for simplicity, treat it as a linear viscous fluid. Moreover, the viscosity of ice, η, depends strongly on temperature, T, which increases several orders in magnitude as T increases from $-40\,^\circ$C towards $0\,^\circ$C. In the following, we shall

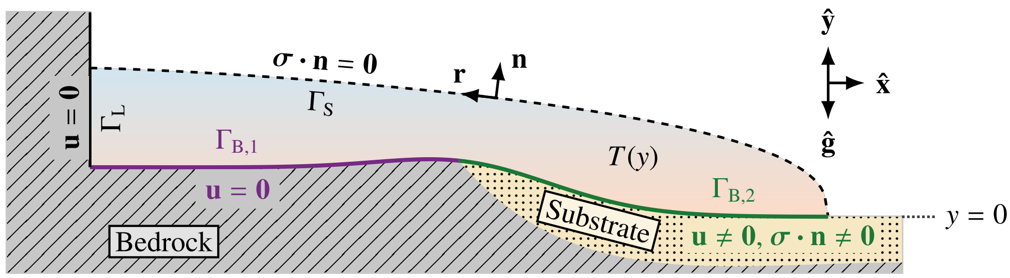

Figure 2: Valley glacier model: A vertical slab of ice with a non-uniform temperature profile, $T(y)$, rests partly on an un-deformable bedrock (hatched gray area) and partly on a deformable substrate (dotted yellow area). The two subglacial boundaries, $\Gamma_{B,1}$ (purple) and $\Gamma_{B,2}$ (green), admit spatially varying subglacial boundary conditions.

consider a simple linear temperature dependence to study the effect of a non-uniform viscosity. The strong form of the problem is therefore

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = -\boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \boldsymbol{\nabla}p &= \rho\mathbf{g} && \text{(Momentum balance)}\\
\boldsymbol{\nabla}\cdot\mathbf{u} &= 0 && \text{(Incompresibility)}\\
\boldsymbol{\tau} = 2\eta(T)\dot{\boldsymbol{\epsilon}} \quad \text{where} \quad \dot{\boldsymbol{\epsilon}} &= (\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^{\mathsf{T}})/2, && \text{(Linear rheology)}
\end{aligned}
$$

where $\eta(T)$ is yet to be specified.

Weak form

The weak form of the Stokes-flow problem becomes (see e.g. lecture slides), together with the above boundary conditions, is

$$
\begin{aligned}
\int_{\Omega} (\boldsymbol{\tau} : \boldsymbol{\nabla}\mathbf{v} - p\boldsymbol{\nabla}\cdot\mathbf{v} + q\boldsymbol{\nabla}\cdot\mathbf{u})\,\mathrm{d}A &= \int_{\Omega} \rho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}A + \int_{\Gamma} \mathbf{v}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}l \\
&= \int_{\Omega} \rho\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}A + \int_{\Gamma_{\mathrm{L}}} \mathbf{v}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}l + \int_{\Gamma_{\mathrm{B},2}} \mathbf{v}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\,\mathrm{d}l
\end{aligned}
\tag{3}
$$

where the boundary partitioning $\Gamma = \Gamma_{\mathrm{L}} \cup \Gamma_{\mathrm{S}} \cup \Gamma_{\mathrm{B},1} \cup \Gamma_{\mathrm{B},2}$ was inserted, $\mathbf{v}, q$ are the velocity and pressure weight-functions, respectively, and (1) and (2) were used. Recall that $\mathbf{v} = \mathbf{0}$ by construction wherever $\mathbf{u}$ is known (specified).

The exercise

0. In order to get started, let us first define the relevant physical constants, and load the glacier mesh "initial.xml" provided for the problem.
   We shall initially assume
   $$\eta(T) = \eta_0,$$
   and set the constants and load the mesh as follows:

Listing 1: Set physical constants and load mesh

```
from dolfin import * # load fenics functions
import matplotlib.pyplot as plt # for plotting

ky2ms,y2s = 3.17098e-5, 3.154e+7 # km/yr to m/s, yr to s
ex,ey = Constant((1,0)),Constant((0,1)) # unit vectors

g0, rho = 9.81, 917
g = (0, -g0) # gravity vector
eta0 = 4.5454e+14 # viscosity

mesh = Mesh("initial.xml");
boundaries = MeshFunction("size_t", mesh, \
                          "initial_facet_region.xml")
```

If you plot the mesh, you should see the mesh in Figure 3.

Because the subglacial boundary is curved and partitioned ($\Gamma_{B,1} \cup \Gamma_{B,2}$), it is useful to load additional information about the boundary partitioning in order to easily specify where given boundary conditions should apply. While you *can* use e.g. `near()` (following previous exercises), this quickly becomes inelegant for odd-shaped geometries like the present. The line

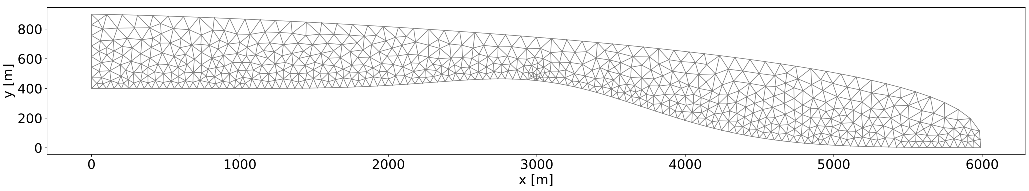

Figure 3: Mesh "initial.xml"

containing `MeshFunction()` loads the boundary partition information for later use.

1. As a first step, solve the simplified problem of

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{L}}, \Gamma_{\mathrm{B},2} \tag{4}$$

together with (1) and (2).

Setting up and solving this Stokes problem is similar to what we did in the pipe-flow exercise, but with different boundary conditions. That is, the weak form of the problem is the same (see the exercise solution if in doubt, but remember the physical constants are different).

Applying Dirichlet boundary conditions

With the boundary partitioning stored in "`boundaries`", we do not need to use `near()` for specifying Dirichlet boundary conditions, but may instead refer to the sub-boundaries directly using their integer identifiers[1] $1, 2, 3, 4$ for $\Gamma_{\mathrm{B},1}$,$\Gamma_{\mathrm{B},2}$,Γ_{L},Γ_{S}, repsectively:

Listing 2: Boundary conditions

```
bndB1,bndB2,bndL,bndS = 1,2,3,4 # boundary IDs
bc = [DirichletBC(W.sub(0),(0,0),boundaries,bndB1),\
DirichletBC(W.sub(0),(0,0),boundaries,bndB2),\
DirichletBC(W.sub(0),(0,0),boundaries,bndL)]
```

Recall that `W` (in the above) is a mixed velocity–pressure function-space, where the subcomponents are the separate velocity (`W.sub(0)`) and pressure (`W.sub(1)`) function-spaces, respectively.

Q: Solve the problem — which regions experience compressional and extensional deviatoric stresses?

2. Boundaries on which $\mathbf{u} = \mathbf{0}$ are called *no-slip* boundaries. Another useful boundary condition is the *slip* condition $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$. Hence, on slip boundaries the velocity need not be zero, but is required to be parallel to the boundary. A slip boundary does, meanwhile, not imply that the frictional drag exerted by the boundary on the fluid is zero. For this to be the case, the shear stress on the boundary must vanish too ($\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$), in which case the boundary condition is known as a *free-slip* condition.

[1]Note that the identifiers are freely chosen by the creator of the mesh.

Q: Is the solution, before or after the bump, sensitive to changing the no-slip condition on Γ_L to a slip condition? To answer this, impose that *only* u_x must vanish on Γ_L (u_y being unconstrained). Recall that `W.sub(0).sub(0)` and `W.sub(0).sub(1)` are the x- and y-components of the velocity function space.

Q: Why does specifying $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$ on Γ_L in our model actually constitute a free-slip condition rather than a regular slip condition, as it would seem to imply?
Hint: Did you include any boundary integrals up until now?

3. The viscosity of ice depends on the local ice temperature. Let us model the effect of temperature in terms of the simple linear viscosity–temperature relation

$$\eta(T) = \eta_0 \left(1 - \frac{9}{10} \frac{T - T_c}{T_w - T_c}\right), \tag{5}$$

where T_w is the warmer ice temperature in the valley, and T_c is the colder ice temperature at the glacier initiation. This produces a viscosity ratio between cold-to-warm ice of $\eta(T_c)/\eta(T_w) = 10$ (colder ice is harder). Furthermore, let us assume a linear ice-temperature increase with height

$$T(y) = T_w - \frac{T_w - T_c}{H} y, \tag{6}$$

such that $T(0\,\mathrm{m}) = T_w$ and $T(H) = T_c$. Combining the two expressions, yields

$$\eta(T) = \frac{\eta_0}{10} \left(1 + 9\frac{y}{H}\right). \tag{7}$$

In order to replace the constant viscosity $\eta = \eta_0$ with the function (7), replace the `eta = eta0` line in your code with an `Expression`, assuming a characteristic height of $H = 1000\,\mathrm{m}$:

Listing 3: Viscosity function

```
eta = Expression("eta0/10*(1+9*x[1]/H)", \
                  eta0=eta0, H=1000, degree=1)
```

(recall `x[1]` is the y-coordinate).

Q: How do the longitudinal- (τ_{xx}) and shear-stress (τ_{xy}) components change in response to including the temperature-dependant viscosity? Put differently, is there a change in which internal resistive stresses balance gravity?

4. Large accumulation (snowfall) anomalies can locally change the glacier surface profile, causing resistive viscous stresses ($\boldsymbol{\tau}$) to change so as to balance the change in driving force.

   Suppose that an unusually large surface profile anomaly develops as represented by the "accumulation.xml" mesh. Keeping the viscosity function $\eta(T)$ from above, load the new mesh and the corresponding "accumulation_facet_region.xml" boundary partitioning.

   Q: How do the compressional/extensional regimes change in response to the new geometry?

5. Crevasses can form in glaciers where extensional stresses exceed a critical material strength: $\tau_{xx} > \tau_{\mathrm{crit}} > 0$. In the present case, large extensional stresses occur on the down-stream side of the topographical bump. Suppose that a large crevasse develops there as represented by the "crevasse.xml" mesh.

   Keeping the viscosity function $\eta(T)$ from above, load the mesh and the corresponding "crevasse_facet_region.xml" boundary partitioning.

   Q: Since crevasses represent a break in the continuum across which stresses can not be transmitted, how does the formation of a crevasse locally influence the stress components τ_{ij}? How far upstream can the crevasse be "felt" (exert influence) in the components τ_{ij}?

6. On the lower subglacial boundary, $\Gamma_{\mathrm{B},2}$, the ice rests on a deformable substrate; that is, a sedimentary material that can withstand only a finite shear-stress before itself deforming. Hence, the substrate can provide *some* shear-stress resistance (frictional drag), but not enough to prevent the ice from sliding.

   Q: Considering once again the mesh "accumulation.xml" mesh, and let us assume for simplicity that $\mathbf{n} = \hat{\mathbf{y}}$ over $\Gamma_{\mathrm{B},2}$. Above what uniform sliding speed $u_x = u_{\mathrm{slide}}$ on $\Gamma_{\mathrm{B},2}$ does sliding affect τ_{xx}? (hint: u_{slide} is on the order of 1×10^{-3} km/yr to 1×10^{-2} km/yr).

   Q: Where do the largest gradients in τ_{xx} and τ_{xy} occur for large sliding speeds? Can you qualitatively explain why?

7. **Optional but encouraged question:**
   Instead of specifying the sliding speed u_{slide} as a Dirichlet boundary condition, we would, ideally, like to apply knowledge about the maximal possible traction the substrate can provide at the ice–substrate boundary. Consider therefore *not* specifying u_x on $\Gamma_{\mathrm{B},2}$—i.e. requiring only $u_y = 0$ as a Dirichlet

condition, leaving u_x free— but instead prescribing the shear-stress that the substrate provides at the ice–substrate boundary:

$$\mathbf{t} \equiv \boldsymbol{\sigma} \cdot \mathbf{n} = t_{\max}\mathbf{r} \quad \text{on} \quad \Gamma_{\mathrm{B},2}, \tag{8}$$

where $\mathbf{r}$ is the boundary parallel direction (see Figure 2), and $t_{\max}$ is the traction magnitude.

In `FEniCS`, $\mathbf{r}$ is given by rotating the boundary normal direction as follows:

Listing 4: Boundary normal and transverse directions

```
n = FacetNormal(mesh)
r = as_vector([n[1], -n[0]])
```

The stress boundary condition (8) is applied using the corresponding boundary integral in the weak form (3). To do so, we must first update the boundary integral measure, dl, with information about the boundary partitioning

Listing 5: Updating boundary integral measure

```
ds = Measure('ds', domain=mesh, subdomain_data=boundaries)
```

which allows splitting the boundary integral over the entire boundary into its contributions over the subdomains $\Gamma_\mathrm{L}, \Gamma_\mathrm{S}, \Gamma_{\mathrm{B},1}, \Gamma_{\mathrm{B},2}$. In addition to the complete boundary integral $\int_\Gamma \mathbf{v} \cdot \mathbf{t}\,\mathrm{d}l$, represented as usual by

`dot(v,t)*ds`,

we can now also prescribe the partial integrals by writing

`dot(v,t)*ds(1)` ...i.e. $\int_{\Gamma_{\mathrm{B},1}} \mathbf{v} \cdot \mathbf{t}\,\mathrm{d}l$,

`dot(v,t)*ds(2)` ...i.e. $\int_{\Gamma_{\mathrm{B},2}} \mathbf{v} \cdot \mathbf{t}\,\mathrm{d}l$,

`dot(v,t)*ds(3)` ...i.e. $\int_{\Gamma_{\mathrm{L}}} \mathbf{v} \cdot \mathbf{t}\,\mathrm{d}l$,

`dot(v,t)*ds(4)` ...i.e. $\int_{\Gamma_{\mathrm{S}}} \mathbf{v} \cdot \mathbf{t}\,\mathrm{d}l$,

for some boundary traction $\mathbf{t}$.

Q: Intuitively, if the frictional drag is small then the sliding speed should become large, and vice–versa. Determine $t_{\max}$ by trial-and-error such that the substrate fails to resist sliding, defined as the point where the resulting sliding speed departs from zero (hint: it is on the order of 100 kilopascal).

Q: How do the resulting down-stream stress components compare to those when directly specifying a sliding speed (previous question)?