Vibrations (ch24)

Aslak Grinsted Cont Mech 2022

- Motivation
  - Plate
  - PianoTeq
  - Earthquakes - Inge Lehmann
  - Bruel og Kjær

Redefine…

- Position of material particle at time t. (originally located at **x**)
  - $\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t)$
  - *Note: this is different from strain chapter where x was new pos.*
- So velocity field and acceleration field.
  - $\boldsymbol{v} = \partial \boldsymbol{u} / \partial t$
  - $\boldsymbol{w} = \partial^2 \boldsymbol{u} / \partial t^2$

TODO: consider D/Dt ..
more formally correct
Is this because these are
formulated as functions of orig pos.

- We aim to model small vibrations around an equilibrium state.
  - The initial state may already be stressed and deformed.
  - (E.g. a piano frame is already under tremendous stress)
  - (So we can ignore body forces.)

- Probably have a good idea where we are going with this.

  - $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{f}^* = \cancel{\boldsymbol{f} +} \nabla \cdot \boldsymbol{\sigma}$

- Hooke:
  - $\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda \boldsymbol{I} tr(\boldsymbol{\epsilon})$

- Definition of cauchy strain tensor
  - $\boldsymbol{\epsilon} = \boldsymbol{u} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^{\mathbf{T}})$

- It would be simple to update our elastostatics cheese
  - Finite Diff code to have a velocity field.

Combine

- Assume isotropic medium
  - (so that we can treat material properties as constants)

- And insert (not shown – but see ch9 in index notation)

- $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}$

- "Navier's equation of motion"

- You can already see it looks like a wave equation.
  - Second order time deriv. proportional to second order spatial deriv.
  - But now **u** is a vector.

Splitting u to transverse/longitudinal

- $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}$

- We can split u into two components
  - $\boldsymbol{u} = \boldsymbol{u_L} + \boldsymbol{u_T}$

  - $\boldsymbol{u}_{\mathrm{T}}$ is divergence free. $\nabla \cdot \boldsymbol{u}_{\mathrm{T}} = 0$
  - $\boldsymbol{u}_{\mathrm{L}}$ is curl free (purely divergent). $\nabla \times \boldsymbol{u}_{\mathrm{L}} = 0$

- Motivation for this split.
  - Naviers eqn has two RHS terms – one deals purely with divergence.

- Divergence in displacement is associated with density and thus pressure change.

- For this reason the longitudinal component is also called the pressure or p-component.

- The divergence free transverse component (no pressure change) is also called the shear component or S-component.

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}$$

Inserting…

- $\rho \frac{\partial^2 \boldsymbol{u_T}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u_T}$
- Using $\nabla \times (\nabla \times \boldsymbol{u}_L) = \nabla(\nabla \cdot \boldsymbol{u}_L) - \nabla^2 \boldsymbol{u}_L$
- $\rho \frac{\partial^2 \boldsymbol{u_L}}{\partial t^2} = (2\mu + \lambda) \nabla^2 \boldsymbol{u_L}$
- Completely separate (uncoupled equations.)

- Two wave equations, different speeds

$$\rho \frac{\partial^2 \boldsymbol{u_T}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u_T}$$

$$\rho \frac{\partial^2 \boldsymbol{u_L}}{\partial t^2} = (2\mu + \lambda) \nabla^2 \boldsymbol{u_L}$$

- $c_T = \sqrt{\frac{\mu}{\rho}}$
- $c_L = \sqrt{\frac{2\mu+\lambda}{\rho}}$

- Longitudinal faster than transverse.
  - **<u>P</u>**ressure faster than **<u>S</u>**hear
  - P & S conveniently also maps to Primary/secondary waves.

- Speed ratio can be rewritten in terms of poisson ratio.
- For normal materials (where $\nu = \frac{1}{3}$). $c_L = 2c_T$

- That's a wave-eqn…
- Natural to write the displacement as a harmonic series
  - Fourier – possible for any time dependent field
- So, if we figure something out for an oscillation of the form :

$$\boldsymbol{u}(\boldsymbol{x}) \cdot e^{-i\omega t}$$

Real part is the displacement. $\boldsymbol{u}(\boldsymbol{x}, t)$

Imag part: $\partial_t \boldsymbol{u}(\boldsymbol{x}, t)/\omega$

- Then we can superpose those to get to a more general soln.

Insert harmonic oscillation in Navier.

- $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}$
- $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}) \cdot e^{-i\omega t}$

- $-\rho \omega^2 \boldsymbol{u} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}$

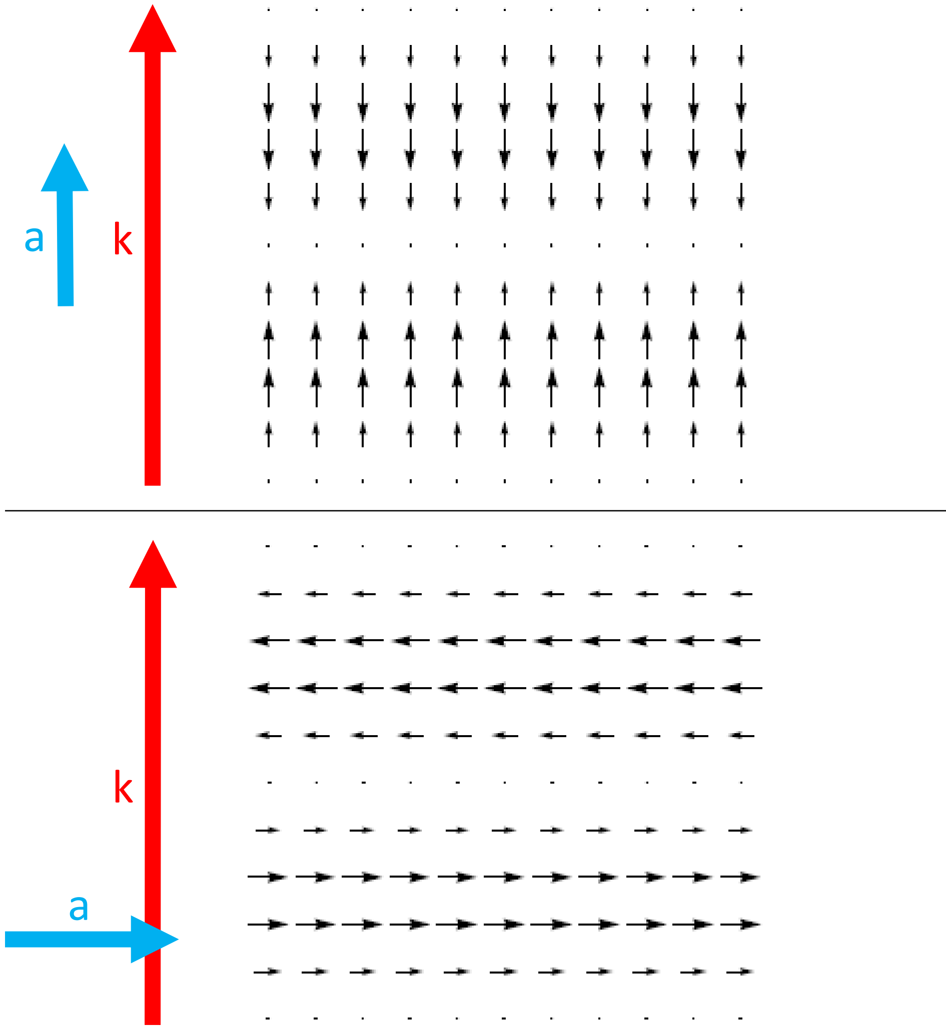
- Now we have a time independent equation for $\boldsymbol{u}(\boldsymbol{x}, \omega)$

$$-\rho\omega^2\boldsymbol{u} = \mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}$$

Consider a plane wave

- $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{a}\cdot e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$

- **k: propagation direction & wavelength**
- **a: “polarization vector” - complex**



$$-\rho\omega^2\boldsymbol{u} = \mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}$$
$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{a}\cdot e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

- If we insert t=0 then we can see how the wavelength $\frac{2\pi}{|\boldsymbol{k}|}$ is related to a, omega

- $-\rho\omega^2\boldsymbol{a} = \mu\boldsymbol{k}^2\boldsymbol{a} + (\lambda+\mu)\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{a}$

- Eigen-value equation

for the 3×3 matrix $\mu k^2\mathbf{1} + (\lambda+\mu)\boldsymbol{k}\boldsymbol{k} = \{\mu k^2\delta_{ij} + (\lambda+\mu)k_ik_j\}$.

Eigen-value equation

what can we use that for?

- For a given propagation direction ***k***

- You can find the 3 eigenvectors **a**
- **-**with corresponding eigenvalues $\rho\omega^2$

- I.e. 3 distinct ‘modes’ of polarization which are characterized by their own wavelength.

- Eigenvector 1: ***a** aligned with **k***
  - Eigenvalue 1: $\rho\omega^2 = (\lambda + 2\mu)\boldsymbol{k}^2$
  - Longitudinal – P-wave

- Eigenvectors 2&3: **a** perpendicular to **k**
  - Eigenvalue: $\rho\omega^2 = 2\mu\boldsymbol{k}^2$
  - Transverse – S-wave