Compressible flow

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- So far we have worked predominantly with incompressible flows.

- That is convenient because:
  - ρ is constant and can be freely moved in and out of differential operators.
  - and the mass conservation eqn simplifies to $\nabla \cdot \boldsymbol{v} = \mathbf{0}$

- We have even applied it to air.
  - - everyday experience tells us it is pretty compressible.
    - - balloons
    - - sound waves (pressure/density waves)

Sound propagation

- I am sure you know that sound is pressure waves.
  - Changes in density
  - Changes in pressure
  - Small velocities

- We have previously talked about
  - Mechanical balances
  - Mass balance
  - Relationship between pressure and density? (Compressibility/Hooke/… Ch1)

- Do we have what we need to let us understand sound waves?

Lets revisit

- We have
    - $\rho \frac{D\boldsymbol{v}}{Dt} = f + \nabla \cdot \boldsymbol{\sigma}$ *momentum conservation*
    - $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$ *mass conservation*
- Assume ideal (viscosity free) fluid
    - $\boldsymbol{\sigma} = -\mathrm{p}\mathbf{I}$ *and* $\nabla \cdot \boldsymbol{\sigma} = -\nabla \mathrm{p}$

- So we have:
- $\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{g} - \frac{1}{\rho}\nabla p$
- Euler equations for ideal compressible flow

- $\rho \frac{D\boldsymbol{v}}{Dt} = f + \nabla \cdot \boldsymbol{\sigma}$
- $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$
- $\nabla \cdot \boldsymbol{\sigma} = -\nabla \mathrm{p}$

Now consider a small perturbation (SOUND)

- Start from hydrostatic equilibrium.
  - $\mathrm{p} = \mathrm{p}_0 \quad | \quad \rho = \rho_0 \quad | \quad v = 0$
- Consider small perturbation
  - $\mathrm{p} = \mathrm{p}_0 + \Delta \mathrm{p}$
  - $\rho = \rho_0 + \Delta\rho$
  - $v = \Delta v$
- Step 1: Insert in (1)
  - $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}} + \cancel{(\boldsymbol{v} \cdot \boldsymbol{\nabla})}\boldsymbol{v} = -\frac{1}{\rho_0}\nabla\Delta p$

Eqn14.12

1. $\frac{\partial v}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \cancel{\boldsymbol{g}} - \frac{1}{\rho}\nabla p$

2. $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$

   - $\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{v}) = 0$

3. $(\boldsymbol{v} \cdot \nabla)\Delta\rho$

Step 2: insert in the mass balance eqn…

- $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$

- $\nabla \cdot (\rho \boldsymbol{v}) =$
  - $(\boldsymbol{v} \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{v})$
  - $= (\boldsymbol{v} \cdot \nabla)\Delta\rho + (\rho_0 + \Delta\rho)(\nabla \cdot \boldsymbol{v})$
  - $= \cancel{(\boldsymbol{v} \cdot \nabla)\Delta\rho} + \rho_0(\nabla \cdot \boldsymbol{v}) + \cancel{\Delta\rho(\nabla \cdot \boldsymbol{v})}$
  - = small*small + big*small + small*small

- HENCE: $\frac{\partial \Delta\rho}{\partial t} = -\rho_0(\nabla \cdot \boldsymbol{v})$

- small perturb
  - $p = p_0 + \Delta p$
  - $\rho = \rho_0 + \Delta\rho$
  - $v = \Delta v$

- Apply $\frac{\partial}{\partial t}$ to (B)
  - $\frac{\partial^2 \Delta\rho}{\partial t^2} = -\rho_0 \left(\nabla \cdot \frac{\partial}{\partial t}\boldsymbol{v}\right)$
- Insert A on RHS
  - $\frac{\partial^2 \Delta\rho}{\partial t^2} = \nabla^2 \Delta p$
- Really close to a wave eqn. now
  - Needs relationship tween p and rho

$$\boldsymbol{A.}\quad \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}} = -\frac{1}{\rho_0}\nabla\Delta p$$

$$B.\quad \frac{\partial \Delta\rho}{\partial t} = -\rho_0(\nabla \cdot \boldsymbol{v})$$

- barotropic equation of state
  - $\mathrm{p} = \mathrm{p}(\rho)$
- Linearize:
  - $\Delta \mathrm{p} \approx \left.\frac{dp}{d\rho}\right|_0 \Delta\rho$
  - $\Delta \mathrm{p} \approx \frac{K_0}{\rho_0}\Delta\rho$
  - *(this is how we defined equilib. bulk modulus)*

- Inserting gives:
  - $\frac{\partial^2 \Delta\rho}{\partial t^2} = \frac{K_0}{\rho_0}\nabla^2 \Delta\rho$

  - Speed of sound is $c_0 = \sqrt{\frac{K_0}{\rho_0}}$

- $\frac{\partial^2 \Delta\rho}{\partial t^2} = \nabla^2 \Delta p$

Fluid	T	c_0
Glycerol	25	1920
Sea water	20	1521
Fresh water	20	1482
Lube oil	25	1461
Mercury	25	1449
Ethanol	25	1145
Hydrogen	27	1310
Helium	0	973
Water vapor	100	478
Neon	30	461
Humid air	20	345
Dry air	20	343
Oxygen	30	332
Argon	0	308
Nitrogen	27	363
Unit	°C	m s^{-1}

Empirical sound speeds in various liquids (above) and gases (below). The temperature of the measurement is also listed.