Exercise: gravity wave

Credit: Marcin Grabski / CC-BY-2.0

You are standing at the shore of a very calm lake of length L=100m, and depth D = 10m. Your friend does a cannonball dive into the lake. Estimate how much time it would take for the fastest wave component to travel to the opposite shore and back. I.e., after which amount of time do you expect to be able to measure the first reflected signal?

Consider:

- Can we use "shallow water waves", "intermediate waves" or "deep water waves" (see chapter 25.3)?
- Is it phase or group velocity that we are interested in?

Exercise 2: Tsunami model

Credit: This exercise is built around an exercise made by Dion Häfner for the 2020 course.

The goal of this exercise is to build a simple 1.5-dimensional shallow water model of the ocean, and to validate it by simulating a Tsunami wave. To motivate and inspire you, you can take a look at what you can do with a more complete model here: https://www.youtube.com/watch?v=y2IxUvF7ip4

A shallow water model

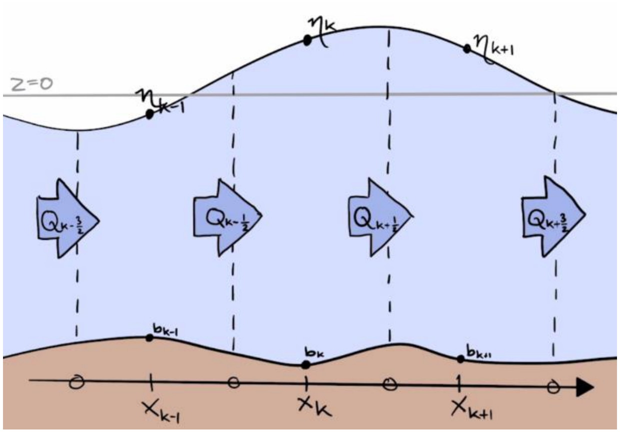
The dynamics we want to model are governed by the following set of partial differential equations:

$$\frac{\partial \eta}{\partial t} = -\nabla_x (u \cdot h)$$

$$\frac{\partial u}{\partial t} = -g \nabla_x \eta$$

where η is a sea surface height anomaly, u is the horizontal velocity, and $h = \eta - b$ is water depth. The two equations say that 1) the sea surface must change if there is a divergence in horizontal fluxes ($Q = u \cdot h$), and 2) slopes in the sea surface must give rise to horizontal flow.

Numerical implementation



In order to solve this equation numerically we have to decide on a discretization scheme. For this particular problem it is 'best' to use a so-called staggered grid. This means some variables (η_k, h_k, b_k) are defined on the grid points with the indices k, where as other variables ($u_{k+½}$, $Q_{k+½}$) are defined on the intermediate grid points. The advantage of the staggered grid is that the spatial derivative ∇_xin both equations can be approximated by first differences, and that we ensure mass conservation when we calculate the divergence of the fluxes in this manner.

In the first equation you will have to approximate $h_{k+½} = 0.5(h_k + h_{k+1})$.

Use a forward Euler approach for the timestepping. It is however critical for the numerical stability that you update u, before you calculate $\frac{\partial \eta}{\partial t}$. If you do this then the scheme is stable when

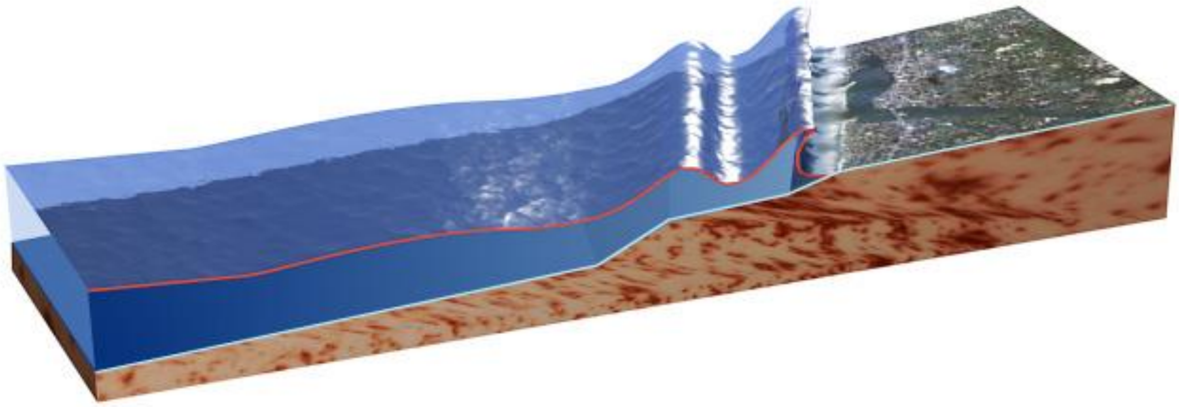
$$\Delta t \leq \frac{\Delta x}{\sqrt{g \cdot h_{max}}}$$

If you implemented your model correctly, total volume will be conserved to machine precision. You can verify that by monitoring $\int \eta dx$.

I've uploaded a python notebook to Absalon for you to use as a starting point here.

Verification

To verify our shallow water model, we will examine how we can excite a Tsunami (wave caused by displacement of water, such as an earthquake or landslide), and how the Tsunami behaves when approaching a coast.



Simulate the situation indicated in the above figure. In the middle of the ocean, an earthquake displaces water by 1m in a large area. The resulting wave then travels along the basin with a sloping bottom, until it (almost) reaches the coast (that means you will have to choose a suitable bottom topography D(x)).

Now, answer the following questions:

1. How does the wavelength of the Tsunami change when approaching the coast?
2. How fast does the Tsunami travel in the open ocean? Do you think this is a realistic value?
3. How fast do individual water particles travel in the crest of the Tsunami in the open ocean?
4. Should you use group or phase speed to characterize the velocity of the approaching Tsunami?
5. Why can we model a Tsunami that is generated in the open ocean (where the water column is, say, 4000m deep) with a "shallow water" model?

(This exercise is based on an exercise by Dion Häfner for the 2020 course)