Sverdrup balance and stream function

Lecture Notes

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AIM: understand large scale ocean circulation.

- Assume:
  - Rotating earth (coriolis bodyforce)
  - Steady and Incompressible
  - Large horizontal scales... small horizontal gradients in v.
  - "2d" $v_z = 0$
  - Wind-driven (by some average windstress)
- Q: what circulation would that give in the ocean?

- $\rho \frac{D\boldsymbol{v}}{Dt} = \rho \boldsymbol{g} - \nabla P + \frac{\partial \boldsymbol{\tau}}{\partial z} - 2\rho \boldsymbol{\Omega} \times \boldsymbol{v}$
- $\nabla \cdot \boldsymbol{v} = 0$

- $0 = \rho \boldsymbol{g} - \nabla P + \frac{\partial \boldsymbol{\tau}}{\partial z} - 2\rho \boldsymbol{\Omega} \times \boldsymbol{v}$

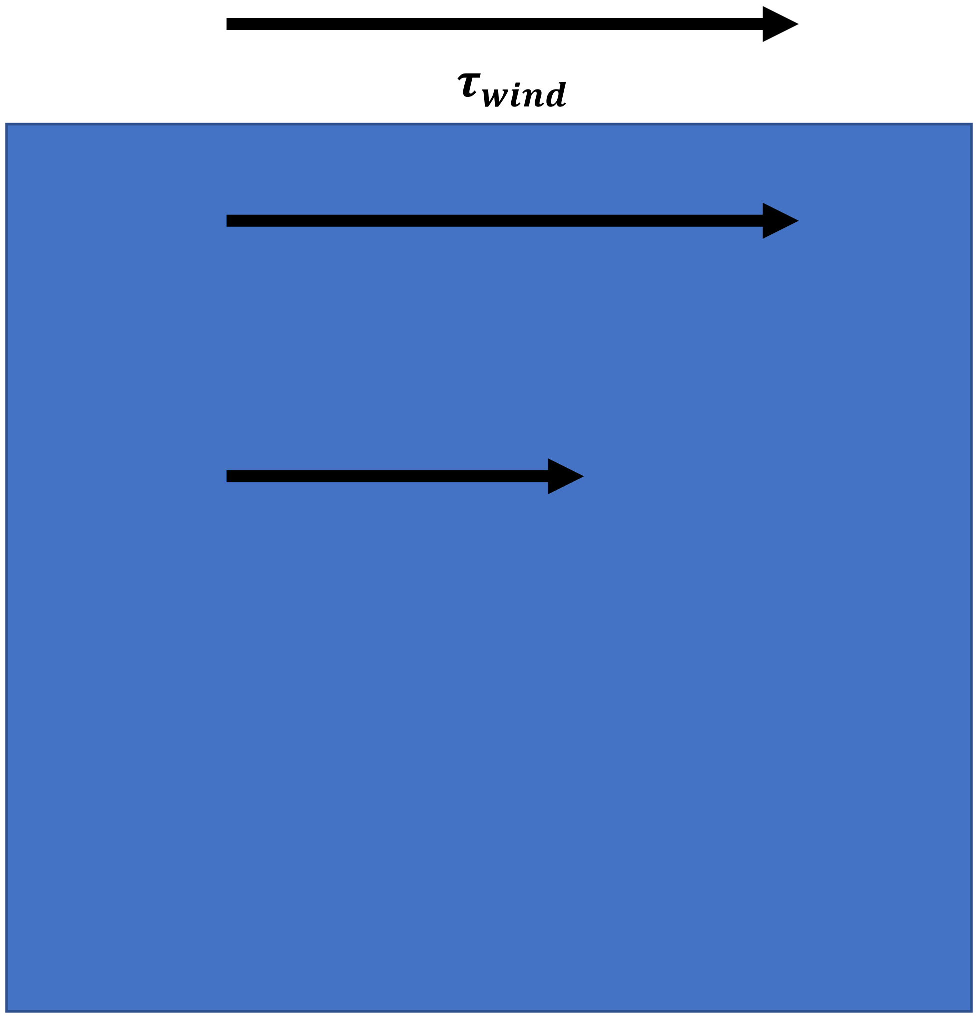
- $\boldsymbol{\tau} = \boldsymbol{\sigma_{iz}}$

Why dtau/dz

- We have not talked about internal friction.

- we have assumed
  - Wind driven
  - horizontal flow.

- Seems reasonable to expect
  - Differential velocities between layers.
  - Viscous Friction
  - i.e. we have some $\tau = \sigma_{iz}$

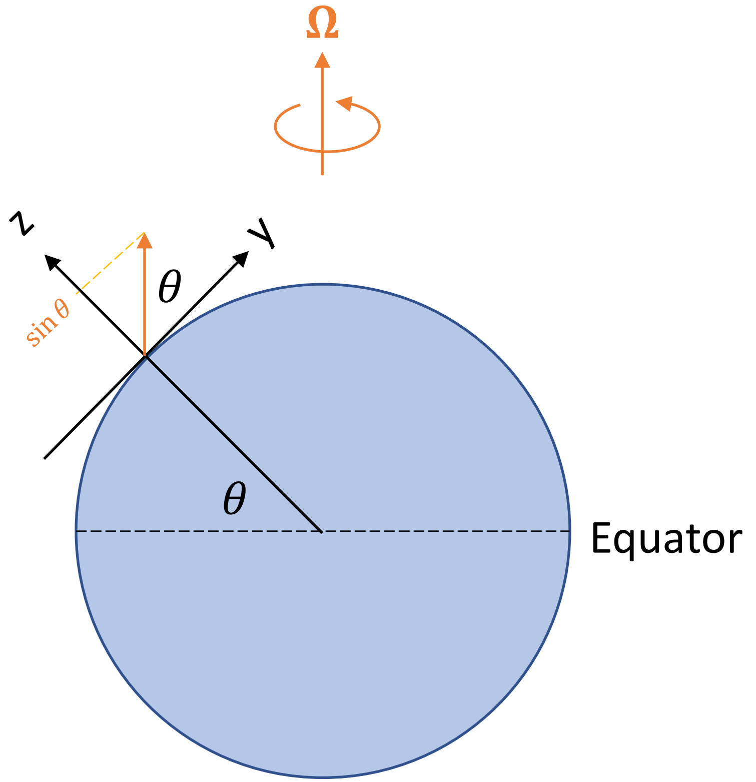


V=0 at Ocean bottom?

- $0 = \rho \boldsymbol{g} - \nabla P + \frac{\partial \boldsymbol{\tau}}{\partial z} - 2\rho \boldsymbol{\Omega} \times \boldsymbol{v}$

- We want x-y directions.
  - **X-eqn:** $-\rho f v_y = -\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial z}$
  - **Y-eqn:** $\rho f v_x = -\frac{\partial p}{\partial y} + \frac{\partial \tau_y}{\partial z}$

  - Where $f = 2\Omega \sin\theta$
  - $\Omega = \frac{2\pi}{86400\ \mathrm{s}} = 7.27 \cdot 10^{-5} s^{-1}$



Integrate vertically…

- Define level of no motion z_0. Here pressure gradients and shear stresses are zero.

- $-\rho f v_y = -\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial z}$
- $\rho f v_x = -\frac{\partial p}{\partial y} + \frac{\partial \tau_y}{\partial z}$

- $-f \int_{z_0}^{0} \rho v_y \, dz = -\int_{z_0}^{0} \frac{\partial p}{\partial x} + \tau_{wind}^{x}$
- $-f\rho U_y = -\int_{z_0}^{0} \frac{\partial p}{\partial x} dz + \tau_{wind}^{x}$

the integral is just the difference between top and bottom

- And similarly:
- $f\rho U_x = -\int_{z_0}^{0} \frac{\partial p}{\partial y} dz + \tau_{wind}^{y}$

- We can get rid of the pressure terms if we 'cross'-differentiate and subtract the two equations

- $\frac{\partial f \rho U_x}{\partial x} + \frac{\partial f \rho U_y}{\partial y} = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}$

- $f \cancel{\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right)} + U_y \frac{\partial f}{\partial y} + \cancel{U_x \frac{\partial f}{\partial x}} = \frac{1}{\rho} \left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right)$

Mass balance / div=0

f is a function of latitude only
(df/dx=0)

- From previous slides

- $-f \rho U_y = -\int_{z_0}^{0} \frac{\partial p}{\partial x} dz + \tau^x_{wind}$
- $f \rho U_x = -\int_{z_0}^{0} \frac{\partial p}{\partial y} dz + \tau^y_{wind}$

- $f = 2\Omega \sin \theta$

- $U_y = \frac{1}{\beta\rho} \nabla \times \boldsymbol{\tau_{wind}}$
- U_y: Northward volume transport per unit distance in the x direction.
- If curl is zero, then there is no northward transport.

- $U_y\beta = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}$
- Where $\beta = \frac{\partial f}{\partial y}$

SKIP

- Given the winds stress you can calculate U_y
- And from that obtain U_x
- $U_x = -\int \frac{\partial U_y}{\partial y} dx + k(y)$
- k(y): U_x should be zero at eastern boundary.

- $U_y = \frac{1}{\beta\rho} \nabla \times \boldsymbol{\tau_{wind}}$
- And mass conservation:
- $\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = 0$

Now link to stream function…

- $\psi = \int U_y dx$
- $U_x = -\frac{\partial \psi}{\partial y}$ and $U_y = \frac{\partial \psi}{\partial x}$
- $\psi = \frac{1}{\beta \rho} \int \nabla \times \tau_{wind} \; dx$
- $U_y = \frac{1}{\beta \rho} \nabla \times \tau_{wind}$
- $U_x = \int \frac{\partial U_y}{\partial y} dx + k(y)$