Q5 - Nearly Ideal Flows & Sverdrups Balance

# Nearly ideal flow

* Fluids with no viscosity: Ideal fluids.
  + This essentially means high Reynolds number.

## Euler equation for incompressible ideal flow

* Forces on ideal homogenous fluid: Pressure and gravity:
* We insert into the Cauchy’s equation for dynamics and get:

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* + These two are the **Euler equations**. They govern the dynamics of incompressible ideal fluid.
  + They constitute a closed set of four equations: 1 for each velocity component and one for pressure.
  + The equation of motion for the pressure is indirectly defined by the divergence condition. We take the divergence of the first Euler equation:



* + - This is the Poisson equation! ( 🡪 With suitable boundary conditions this will give us the pressure field at a given time from the velocity field.
* **Boundary conditions:**
  + Newton’s Third Law demands pressure (in absence of surface tension) must be continuous across interface.
  + The fluid cannot accumulate in the material interface 🡪 Velocity must tangential to interface:
  + All in all:



* **Why useful to discuss ideal flows?**
  + Air and water can often be considered nearly ideal flows and the study of these flows have practical value.

## Application: Collapse of spherical cavity (DO NOT USE)

* **Ideal incompressible radial flow:**
  + Time-dependent purely radial flow:
  + Mass conservation 🡪 Mass flux is same for all
  + Euler equations of this:



* + - Where: does not depend on .
    - must decrease with otherwise we do not have mass conservation.
  + We can insert into the first equation and get and obtain as a direct function of , which then integrates to:

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* + - Here is the pressure at
    - We now let be the instantaneous radius of the cavity, the surface velocity and the surface pressure in the liquid. Setting and and using we obtain the second-order differential equation with respect to time:

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## Steady incompressible ideal flow

* Steady flow (no dependence in time) in a static gravitational field obeys the time-independent Euler equations:

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* True steady flow is only an approximation.
* A fluid in steady flow is effectively incompressible if it has speed much below the speed of sound.
* **Inlet-outlet asymmetry:**
  + The steady flow Euler equations can’t model inlet-outlet asymmetry by themselves. They need to be supplemented by the rule:
    - Inflow toward a channel from a wider region converges smoothly, whereas outflow from a channel into a wider channel continues as a well-defined jet.
* **Bernoulli’s theorem:**
  + Negative pressure gradient in steady-flow equation + absence of gravity for a flow accelerating in a direction 🡪 pressure must drop in that direction as well.
  + **Bernoulli’s theorem:**
    - is constant along streamlines.
  + **Using Bernoulli’s:**
    - **Venturi effect**: The pressure falls when fluid travels into a pipe because the velocity increases.
      * Duct with varying cross-section with a constant flow rate of incompressible fluid. We disregard gravity. Bernoulli gives us:
      * Average velocity: flowrate/cross-section: . Inserting and isolating for pressure:
      * This shows us: the pressure rises when the cross-section increases.

## Vorticity (Curl)

* The value of the Bernoulli field varies from streamline to streamline, but these can be related.
* **Asymptotically uniform flow**
  + A body moving in fluid is a common thing to run into.
  + A body is moving up against a fluid initially at rest with constant speed . Around the body the fluid is disturbed, but it is still far enough away from the motion.
  + Relativity of motion 🡪 This is the same as the flow moving towards the body with speed far upstream. 🡪 This means we can use Bernoulli’s law to calculate stuff:



* + - If we can be sure that all streamlines started infinitely far upstream 🡪 We could use these equations. That is not always the case due to vorticity (curl), as there might be circulating streamlines unconnected with the flow at infinity.
* **Vorticity field:**
  + Gradient of the Bernoulli field using Eulers equations for steady incompressible non-viscous flow:
    - Where the second last equality comes from the first Euler equation for steady incompressible flow.
    - We define the vorticity field as the curl of the velocity field:
    - For steady flow we thus get:



* + - The vorticity-field is a local measure of the circulation in a fluid. If there is none we see that so the Bernoulli field is constant along field lines.
* **Vortex lines:**
  + Field lines of vorticity field. Defined as curves that are always tangent to the vorticity field.
  + Solutions to the following ordinary differential equation:

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* + - Where is a running parameter for the curve. This running parameter has dimension of time length.
    - The vorticity field is divergence free 🡪 Vortex lines cannot emerge or terminate anywhere in the fluid, but in nice geometries they might be closed curves
    - Bernoulli field is constant along vortex lines.
* **Equation of motion for vorticity**
  + Including time-dependence in the derivation of the gradient of the Bernoulli field yields:



* + We take the curl on both sides and use that the curl of a gradient is 0 to arrive at the equation of motion for vorticity:



* + - **Lesson from this**: If the vorticity vanishes for some time , the gradient also vanishes, which means it can’t be generated again ‘naturally’🡪
      * Vorticity cannot be generated by the flow of an ideal fluid but must be present from the outset.

## Circulation

* **Circulation**: The global measure of rotation in a fluid given by:



* **Stokes theorem:** The circulation of a closed curve is equal the flux of its curl:

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* + We can hereby move between the local and the global description of rotation in a fluid.

## Potential flow

* If we have no viscosity and the flow is irrotational we can write the velocity field as a gradient:



* + Where flow potential. These flows follow a much simpler formalism than flows with rotation.
  + In incompressible fluid the divergence vanishes so the flow potential obeys Laplace’s equation:
  + We insert and into



* + We get . This means that cannot depend on spatial coordinates, but only on time. Inserting the Bernoulli function in place of we get and solve for pressure we get:

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* + - Where only depends on time.

# Sverdrups Balance

* Overall aim: Understanding large scale ocean circulation
* Assumptions:
  + Rotating Earth 🡪 Coriolis bodyforce
  + Steady and incompressible flow:
  + Large horizontal scales 🡪 Small horizontal gradients in 🡪 This + no gradients in and gives us that the advective acceleration is 0:
  + We in 2d:
  + Flow wind-driven (by some average windstress on the surface)
  + By these assumptions we know that the velocity-field obeys the following:

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* + - Where is pressure in ocean and is the stress tensor. The last term is a body force from the Coriolis force, where is the rotational velocity of the earth. is the windstress, which decreases only as we venture deeper into the ocean.
    - Which is Cauchy’s equation for dynamics in continuous matter set to zero as we have no acceleration.

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* + - Also incompressible:
* We want the and components of the velocity field. From the Cauchy’s equation above we get:
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  + = angle between the line from the given point to equator and the rotational axis.
* We define a level of zero motion (probably near the bottom of the ocean) and integrate vertically:
  + Where stays the same as its just the difference between top and bottom of ocean, i.e. only the top contributes exactly
* We likewise get for the y-equation:
* We now differentiate the -eq witch respect to and vice versa:
* We add them and get:
  + The first term is zero as it is the divergence and that is 0.
  + as does not depend on latitude.
  + We end up with:
  + We isolate and rewrite:
    - Where
    - **This is the motherfucking Sverdrups balance.**
* **What do we get from this?**
  + is northward volume transport pr. unit distance in the x-direction
  + If curl is 0 🡪 No transport northward.
* **SPØRG SCHAUSER OM KOORDINATSYSTEM. SPASSER.**