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Motives and homotopy theory

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1 Motives and homotopical algebra – an introduction

This chapter outlines the fundamental questions motivating the work assembled in this thesis. It is written with the explicit goal of being as accessible to non-mathematicians as possible. Thus, it is some kind of a response to many people's question "What do you do at work?" The first section finishes with a rough description of the papers accumulated in this thesis.

Section 2 contains a brief introduction to Beilinson's conjecture, which fueled, to a higher or lesser extent, the papers in this first part of the thesis. Section 3 begins with a gentle invitation to homotopical algebra and concludes with a survey of the three papers in this second part of the thesis.

1.1 Algebraic and arithmetic geometry

Many areas of mathematics deal with the problem of solving equations

$$f(x) = 0,$$

where f is some function. In many situations it is moreover necessary to solve not a single equation, as above, but instead simultaneously solve equations involving several functions $f_1, \ldots f_m$, each of which depends on several variables $x_1, \ldots, x_n$:

$$
\begin{aligned}
f_1(x_1, \ldots, x_n) &= 0, \\
f_2(x_1, \ldots, x_n) &= 0, \\
&\vdots \\
f_m(x_1, \ldots, x_n) &= 0.
\end{aligned}
\tag{1.1}
$$

Algebraic geometry is concerned with the case when the above functions f_1, ..., f_m are polynomial functions in the variables $x_1, \ldots, x_n$ such as $x_1^2 + 4x_2^2 - x_1 x_3$. Since polynomials are built only using addition and multiplication, they are simpler than functions such as $\sin(x)$, $\log(x)$, e^x or $|x|$, which are not primarily studied by algebraic geometers. The word *algebraic* in algebraic geometry refers to restricting one's attention to polynomial equations. The word *geometry* in algebraic geometry refers to the nature of this domain of mathematics: it applies every-day geometric intuition to solve algebraic problems. For example, solving a system of equations can be graphically illustrated by intersecting the solution sets of the individual equations.

Algebraic geometry is driven by the following questions:

Question 1.1. *1. Is there a solution $(x_1, \ldots, x_n)$ to a system of polynomial equations as in (1.1) above?*
2. If yes, what can we say more about the solutions? How many solutions are there? Can we, instead of merely counting the solutions, give a more meaningful description of such a solution set?

To describe what we know about this, we have to clarify *what* qualifies as a solution. This can be dramatically illustrated with the innocuous-looking equation

$$x^n + y^n = z^n. \tag{1.2}$$

Here n is a positive integer. For $n = 2$ this equation is the one from Pythagoras' theorem. The main interest in this equation lies with $n \geq 3$, which we assume now. We can ask the solution (x, y, z) to consist of three positive *real* numbers, or three positive *rational* numbers, for example. The answer depends dramatically on our choice: any triple

$$(x, y, z = \sqrt[n]{x^n + y^n})$$

is a real solution for arbitrary positive x and y. However, if we require the solutions to consist of positive *rational* numbers, the answer is entirely different: the only solutions are of the form $(x, y = 0, z = x)$ where x is arbitrary, and $(x = 0, y, z = y)$ where y is arbitrary. This result was suggested by Fermat in 1637, but it required the efforts of generations of mathematicians and an arsenal of mathematical techniques until Wiles proved this result in 1994.

This example is not a coincidence, but part of a more general phenomenon: even though the rational numbers are much more elementary than, say, real or complex numbers, solving equations is easier if we

enlarge our range of solutions. To illustrate this phenomenon, we consider a *single* polynomial equation in a *single* variable:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \tag{1.3}$$

where $a_n, \ldots, a_0$ are some constants. Again, depending on what we count as a solution, we may end up having no solution. For example,

$$x^2 + 1 = 0$$

has no real (or rational) solution, since $x^2 \geq 0$ for all real numbers x. The number x we are looking for would be a square root of -1, which does not exist in the reals. There is one way out, namely by enlarging our number system. For example, starting with the real numbers $\mathbf{R}$, we can enlarge our number system to complex numbers, which are obtained from $\mathbf{R}$ by adding a new number $\sqrt{-1}$ to it. This forces us to also include numbers such as $b\sqrt{-1}$ and finally $a + b\sqrt{-1}$ to be able to do addition and multiplication as in the reals. Thus, we end up considering the complex numbers:

$$\mathbf{C} = \{a + b\sqrt{-1} \text{ with } a, b \in \mathbf{R}\}.$$

It is a remarkable fact, the *fundamental theorem of algebra*, that even though we added only one "new" number, namely $\sqrt{-1}$, now *all* polynomial equations as in (1.3) become solvable. We refer to this fact by saying that $\mathbf{C}$ is an *algebraically closed field*.

This process of enlarging a number system to include solutions of all polynomial equations can always be done. If we apply this procedure to $\mathbf{Q}$, the rationals (instead of $\mathbf{R}$), it is not enough to just add $\sqrt{-1}$. Instead the process of formally adding solutions to polynomial equations is an infinite procedure in this case. In both cases, the passage from our original set of numbers to the one where we have added solutions of all polynomial equations is denoted by an overline, such as $\overline{\mathbf{Q}}$ or $\overline{\mathbf{R}}$. (The latter, as we have seen, is just $\mathbf{C}$).

The passage to an algebraically closed field (i.e., a large enough system of numbers) solves – by design – the problem of solving single polynomial equations. What about multiple equations? From manipulations with linear equations, we only expect solutions to exist if we have more variables than equations in (1.1), i.e., $n \leq m$. Let us inspect the case $n = m = 2$ more closely, which means that we are intersecting the solution sets of two equations in the plane.

We will focus on two "stupid" systems of equations. The first one is this, where we take two variables x and y (there is no typo, y does not appear in the equations):

$$
\begin{aligned}
x &= 0, \\
x &= 1.
\end{aligned}
$$

Clearly, no pair (x, y) will satisfy these two equations (no matter whether we consider rational, real, or complex solutions). Geometrically, these two equations correspond to attempting to intersect two parallel lines, which is impossible. On the other hand, any two non-parallel lines do intersect in exactly one point. In this sense, our notion of intersecting lines is not 100% predictable: starting with a pair of non-parallel lines we might over time turn one of the lines so that it becomes parallel to the other: all of a sudden, the intersection points of the two lines disappears. More precisely, the intersection points (which did exist as long as the lines were not parallel) exist went off to infinity. To match our expectation that we get one intersection point, we have to include this point at infinity. This is what *projective geometry* is about. It can be described in completely elementary terms, but for brevity's sake, we will move on.

Next, we turn to another "stupid" system of equations, again in the variables x and y:

$$
\begin{aligned}
x &= 0, \\
x &= 0.
\end{aligned}
\tag{1.4}
$$

Before, we had less solutions than expected (namely none), which we circumvented by considering solutions at infinity. Now, we have more solutions than expected (namely infinitely many): we are intersecting a line (given by $x = 0$) with itself. There is a 1%-, a 99%-, and a 100%-solution to this fundamental issue: the 1%-solution is to omit one equation: in this case we have two variables, but only one equation, so we "rightfully" have infinitely many solutions. This is only a 1%-solution since the above phenomenon also arises in more complicated situations, where it is not necessarily true that one of the equations can be

obtained from the others. A 99%-solution is a policy "we don't look at systems of equations which have more solutions than expected". Commutative algebra, which is the engine under the hood of algebraic geometry, tells us what causes this phenomenon. At least in theory, we can attempt to avoid systems of equations displaying such a pathology. A comprehensive 100%-solution to this has become possible in recent years thanks to the development of an enhanced version of algebraic geometry known as *derived algebraic geometry*. It is fundamentally enhancing the way we form intersections. A tiny glimpse of this changed meaning of intersection is mentioned in the introduction to homotopical algebra below (Section 3). The work presented in this thesis is partly motivated by the recent advent of derived algebraic geometry.

Answer (**to Question 1.1.1.**). *1. Polynomial equations need not have any rational solution. However, (singly) polynomial equations always have complex solutions (or, more generally, solutions in an algebraically closed field).*

2. *Systems of equations always have complex solutions if we have more variables than equations, provided that we are working in projective geometry. That is, we count intersection points at infinity (should they arise) as solutions, as well.*

3. *Solutions of systems of polynomial equations as in (1.1) have the expected dimension $n - m$ if we avoid the phenomenon alluded to after (1.4). Here, "dimension" refers roughly to the number of independent directions of our solution set.*

It is worth noting that in answering the questions, we actually changed the question. (This is something mathematicians often do: if you cannot answer the question, change it until you can.) Let's be honest about how severe these deviations (or restrictions) are. The restrictions imposed in the above three points are of fundamentally different nature: doing projective geometry, as required by Answer 1.1.2. turns out to be extremely natural. Avoiding the systems of equations with unexpectedly many solutions, as in Answer 1.1.3., is also doable: in a precise mathematical sense, it is not only a 99%- solution, but a 99.9999...%-solution. Better yet, derived algebraic geometry, which is being rapidly developed, allows to circumvent such restrictions much more comprehensively. It is the restriction in Answer 1.1.1. which is, by a large margin, the most severe one. Indeed, our knowledge is *much* more partial if we are seeking rational (as opposed to complex) solutions of polynomial equations. For example, describing rational solutions (x, y) of equations of the form

$$y^2 = x^3 + ax + b \quad (\text{with } a, b \in \mathbf{Q}), \tag{1.5}$$

is a matter of ongoing research. To emphasize that one is interested in more restricted solutions, such as rationals or even integers, one refers to the area as *arithmetic geometry*, as opposed to algebraic geometry whose classical focus is on solving equations in algebraically closed fields such as $\mathbf{C}$.

1.2 Symmetries of solution sets

To work towards a better understanding of Question 1.1.1. in the context of arithmetic geometry, it is helpful to turn to Question 1.1.2.: how can we describe the structure of algebraic solution sets beyond stating that there are / there are no solutions? This question admits a comprehensive answer in two separate cases: for systems of linear equations (which are studied in linear algebra), and for a (single) polynomial equation in one variable (which is the topic of Galois theory).

Systems of linear equations are easy to solve, for example by eliminating one variable after another by means of adding and multiplying equations. What is more, the solutions have themselves a very linear shape. This means the following: a linear function f_1 is of the form $f_1(x_1, \ldots, x_n) = a_1 x_1 + \ldots a_n x_n$ for some numbers $a_1, \ldots, a_n$. If a tuple $(x_1, \ldots, x_n)$ satisfies

$$f_1(x_1, \ldots, x_n) = 0$$

and another tuple $(y_1, \ldots, y_n)$ satisfies the same equation, then a short computation involving the explicit form of f_1 shows that the sum $(x_1 + y_1, \ldots, x_n + y_n)$ also satisfies the equation:

$$f_1(x_1 + y_1, \ldots, x_n + y_n) = 0.$$

In other words: the sum of two solutions is again a solution. The same works for the remaining $f_2, \ldots, f_m$ (provided they are all linear). Mathematical objects having a linear shape, such as the solution set of a

system of linear equations, are called vector spaces. Vector spaces are easy to handle, thanks to the fact that we can always choose a coordinate system, similarly to the way we think about three-dimensional space, for example.

Polynomial equations in a single variable also have some additional structure. For example, the quadratic equation

$$x^2 + 1 = 0$$

has two solutions:

$$x = \pm\sqrt{-1}.$$

The $\pm$ sign is not only a shorthand indicating there are two solutions, but more importantly it shows how the two solutions are related: one is obtained from the other by changing $+$ to $-$ and vice versa. The symmetry displayed by the solutions of this equation is a very important asset. We denote this exchange of $+\sqrt{-1}$ and $-\sqrt{-1}$ by σ, i.e., $\sigma(a + b\sqrt{-1})$ is defined as $a - b\sqrt{-1}$. The function σ is called complex conjugation. It allows us to relate the complex numbers $\mathbf{C}$ to the real numbers $\mathbf{R}$ in the following way: a complex number $z = a + b\sqrt{-1}$ is a real number if and only if

$$\sigma(z) = z, \tag{1.6}$$

since this forces b to be zero. Geometrically, we can picture $\mathbf{C}$ as the plane, with $\mathbf{R} \subset \mathbf{C}$ being the horizontal axis and the replacement $z = a + b\sqrt{-1} \mapsto a - b\sqrt{-1}$ corresponding to mirroring at the horizontal axis.

As was mentioned above, systems of polynomial equations can be solved in the complex numbers. If we are instead tasked to find a real solution to a system of polynomial equations (whose coefficients are real), we can first look for complex solutions $(x_1, \ldots, x_n)$. These will be solutions in $\mathbf{R}$ precisely if they are unaffected by complex conjugation, i.e., if $\sigma(x_i) = x_i$ for all i.

This strategy can also be applied when we seek rational solutions: we pass from $\mathbf{Q}$ to the (infinitely bigger) algebraically closed field $\overline{\mathbf{Q}}$. Since we know we will find solutions there, we then have to determine what solutions are unaffected by the symmetry group (replacing the $\pm$ replacement above) of the passage from $\mathbf{Q}$ to its algebraic closure. Since we may well have no (or, as in the case of (1.2) above, very few) solutions, so we can (and should) instead try to specify the solutions in the bigger field $\overline{\mathbf{Q}}$ and also describe what the symmetry does to them.

This sounds fair enough, but suffers from two serious problem: first, describing the action of the symmetry group on non-linear objects (such as the non-linear solution set of polynomial equations) is difficult. Second, the symmetry group relating $\overline{\mathbf{Q}}$ and $\mathbf{Q}$ is infinitely more complicated than the one relating $\mathbf{C} = \overline{\mathbf{R}}$ to $\mathbf{R}$.

1.3   Linearization

Since we *can* fully control linear equations, the line of attack will now be this: first, instead of solving equations in a field such as $\mathbf{Q}$ or $\mathbf{R}$, say, we solve these equations in its algebraic closure $\overline{\mathbf{Q}}$ or $\overline{\mathbf{R}} = \mathbf{C}$. As was outlined in Section 1.1, this is possible and yields the expected answers. Second, we *linearize* the answer, motivated by the fact that we understand linear objects much better than non-linear ones. Third, we keep track of the symmetries arising from the passage from $\mathbf{Q}$ to $\overline{\mathbf{Q}}$, or from $\mathbf{R}$ to $\overline{\mathbf{R}}$. This should help us in finding solutions in $\mathbf{Q}$ or in $\mathbf{R}$. (There is no general, proud "then we are done", but at the end of the section, we will include a positive statement about solutions of polynomial equations.)

We have described the first and the third step to some extent above. Let us turn to the second. The strategy of linearization goes back to Leibniz and Newton, the founders of calculus. Their invention, the derivative of a function gives the best linear approximation to that function at a given point. The concept of a derivative also has a prominent place in algebraic and arithmetic geometry. We will also encounter the idea of linearization in the introduction of Section 3. In this section, however, the meaning of the term linearization is different and often goes by the name of *homology*. This notion originates in the early 20th century when Poincaré founded an area nowadays known as algebraic topology. Homology allows us to formalize the slogan

"Can we solve all problems that we expect to be able to solve?" (1.7)

We will explain this by means of a basic example of homology called *singular homology*. Consider a triangle Δ with three edges x, y, and z and vertices a, b, and c. For reasons that will become clear in a moment, we consider the *alternating* sum of its three edges

$$\partial(\Delta) = x - y + z. \tag{1.8}$$

(The $+$ and $-$ signs here are just book-keeping, i.e., a way of saying we count x and z once, and y once, but with a minus sign. We don't actually add the edges in the sense of moving or concatenating them.) In a similar vein, we can take a line segment (such as x, y or z) and consider its boundary, which in the case of x gives

$$\partial(x) = a - b.$$

We can now compute

$$\partial(\partial\Delta) = \partial(x - y + z) = (a - b) + (b - c) + (c - a) = 0.$$

(Here the alternating sums come in handy.) These computations tell us that we can only expect to find a triangle Δ satisfying (1.8) if $\partial(x) - \partial(y) + \partial(z) = 0$. The latter is a precondition for being able to solve the problem of finding Δ. Let us now suppose this precondition. Can we, then, always find Δ? Singular homology tells us whether it is possible and, if not, how badly it fails.

For example, we consider triangles and line segments in $X = \mathbf{R}^2 \backslash \{(0,0)\}$, the plane with the origin removed. In there, we have three line segments x, y, z as above, but there is no triangle such that $\partial(\Delta) = x - y + z$. The reason is the missing point in X. By contrast, for $X = \mathbf{R}^2$, this phenomenon does not happen.

Elaborating further on this idea, one assigns to any space X its so-called first singular homology of X, denoted by $H_1(X)$. Essentially, it measures, in the parlance of (1.7), how many triangles in X that we should be able to fill in, we actually can fill in. We have seen the examples $X = \mathbf{R}^2$ and $\mathbf{R}^2 \backslash \{(0,0)\}$. We may just as well apply it to X being the set of complex solutions of a system of polynomial equations. The important benefit is this: a space in general, and the solution set of polynomial equations in particular need not have any nice structure. However, homology does have a nice structure: we can add elements in it (by means of the book-keeping mentioned above), much the same way as we can add two elements of a line or two points in a plane. This linearity property of $H_1(X)$ makes it very useful. It is also very well computable. For example, we have

$$H_1(\mathbf{R}^2 \backslash \{(0,0)\}) = \mathbf{Z},$$

but

$$H_1(\mathbf{R}^2) = 0.$$

Let us summarize by saying that we have assigned to a system of polynomial equations the homology of its solution set, i.e., have transformed a non-linear object into a linear one. This transformation is a big simplification. Of course, it comes at the price that we may have suppressed essential features of our non-linear problem. The following formula, known as the Lefschetz trace formula, tells us that our simplification is not hopelessly naive: consider a map $f : X \to X$, the number of fixed points x, i.e., those points satisfying $f(x) = x$ is expressible in terms of the homology of X:

$$\#\{x \in X, f(x) = x\} = \sum_i (-1)^i \mathrm{tr}\left(H_i(X) \xrightarrow{f} H_i(X)\right). \tag{1.9}$$

It would require some more digression to completely specify the assumptions on X and on f and to completely explain the right hand side. The punchline of the above equation is nonetheless understandable: the non-linear question of solving the equation

$$f(x) = x$$

has been expressed by the linear (i.e., feasible) problem of computing the homology of X.

In addition to singular homology, which is closely linked to classical geometric intuition, there is a whole zoo of other homology theories which raise the question (1.7) for different kinds of problems. None of them is picturesque enough to be included in this introduction, so we just point out one important other such theory named étale cohomology. Étale cohomology achieves the seemingly impossible: on the one hand, it yields results compatible with our geometric intuition (along the lines of simplicial homology explained above) whenever our intuition is meaningful. On the other hand, it is applicable in arithmetic situations

where our geometric intuition breaks down. For example, it applies when we are interested in solutions of polynomial equations in the finite field $\mathbf{F}_p$. Computing in this field means that we only have the numbers $\{0, 1, \ldots, p - 1\}$, for a fixed prime number p, and moreover, whenever we get out of this set (by adding or multiplying two sufficiently large numbers), we divide by p and only remember the remainder of this division, which is again in $\{0, 1, \ldots, p - 1\}$. For example, in $\mathbf{F}_3$ we have $x^3 = x$ for any $x \in \{0, 1, 2\}$: dividing $2^3 = 8$ (in $\mathbf{Z}$) by 3 leaves a remainder of 2, so that $2^3 = 2$ in $\mathbf{F}_3$. An important feature of these fields is the existence of a map called *Frobenius map*: it maps any $x \in \mathbf{F}_p$ to $\mathrm{Fr}(x) := x^p$. Since $p = 0$ in the field $\mathbf{F}_p$, the binomial formula

$$(x + y)^p = x^p + px^{p-1}y + \frac{p(p-1)}{2}x^{p-2}y^2 + \cdots + \frac{p(p-1)}{2}x^2y^{p-2} + pxy^{p-1} + y^p$$

ensures that all summands but the outer two are zero. This means $(x+y)^p = x^p + y^p$ in these fields! (Unlike in high-school, where $(x + y)^2 = x^2 + y^2$ was always wrong.) Unlike $\overline{\mathbf{Q}}$, which exists, but is pretty implicit, the algebraic closure $\overline{\mathbf{F}_p}$ of $\mathbf{F}_p$ (obtained, once again, by adding solutions of all polynomial equations) is not too far from $\mathbf{F}_p$: an element $x \in \overline{\mathbf{F}_p}$ lies in $\mathbf{F}_p$ precisely if

$$\mathrm{Fr}(x) = x. \tag{1.10}$$

In a sense, the Frobenius map is a vague analogue of the complex conjugate, which similarity is most conspicuous when comparing (1.10) to (1.6).

We can finally give justify the above-mentioned claim that linearization (i.e., passage to homology), combined with keeping track of symmetries, gives an answer to Question 1.1 (for solutions in $\mathbf{F}_p$). Let us write X for the set of solutions in $\mathbf{F}_p$ of the system (1.1). We also write $\overline{X}$ for the set of solutions in $\overline{\mathbf{F}_p}$. (We are thus redoing what we did at the end of Section 1.2, except for $\mathbf{F}_p$ instead of $\mathbf{Q}$ or $\mathbf{R}$. By (1.10), we know that fixed points of the Frobenius map are exactly solutions in $\mathbf{F}_p$.) Strikingly, the number of solutions in $\mathbf{F}_p$ can be expressed using étale cohomology, very much in the same spirit as in (1.9) above. The formula is known as Grothendieck's trace formula. It reads

$$\#X = \sum_i (-1)^i \mathrm{tr}\left(\mathrm{H}_{\mathrm{c}}^i(\overline{X}) \xrightarrow{\mathrm{Fr}} \mathrm{H}_{\mathrm{c}}^i(\overline{X})\right). \tag{1.11}$$

It is not the point of explaining all the notation in the right hand side, but only the following two aspects: X is a non-linear object, whereas the étale cohomology group $\mathrm{H}_{\mathrm{c}}^i(\overline{X})$ is a vector space, i.e., a linear object. The map Fr takes care of the symmetry inherent in the passage from $\mathbf{F}_p$ to $\overline{\mathbf{F}_p}$.

The upshot is this: we can count solutions of polynomial equations by linearizing the problem (by means of a suitable cohomology theory) and employing its finer structure, which expresses the symmetry of the passage the field $\mathbf{F}_p$ to its algebraic closure $\overline{\mathbf{F}_p}$.

Are we done? Far from it: we were initially looking for solutions in $\mathbf{Q}$, but we ended up talking about solutions in $\mathbf{F}_p$. The so-called local-to-global principle allows us, in restricted situations, including in particular the case of a (single) polynomial *quadratic* equation, to deduce the existence of rational solutions from solutions in $\{0, 1, \ldots, p^n - 1\}$ (for $n = 1$ this is $\mathbf{F}_p$ as above), for all p and all n, and $\mathbf{R}$. For most higher degree equations, such as (1.5), this method fails though.

1.4 Towards motives

In describing the strategy mentioned at the beginning of Section 1.3 we have not made explicit *how* we linearize the non-linear solution set. We have met singular homology and have noted the existence of étale cohomology. In addition to these two, there are a few others, such as de Rham cohomology, which is related to the solvability of differential equations. These different ways of linearizing algebraic-geometric objects all have their individual merits: they "see" different aspects: for example singular homology only works well for solutions in $\mathbf{C}$, but utterly fails to address solutions in $\mathbf{F}_p$, which is only seen by étale cohomology. On the other hand, the finer structures on complex solution sets offered by complex analysis are invisible to étale cohomology.

These different cohomology theories possess very similar formal features. They even yield identical results when we look look at the bare bones. (Recall that a key point in our strategy in Section 1.3 was the extra symmetry afforded by the transition from our initial field to its algebraic closure. The very existence of the

Frobenius map Fr appearing in (1.11) testifies the importance of this approach. By "bare bones" we mean striping off such extra structure.)

Motivic cohomology is the common principle behind these different ways of linearizing algebraic varieties. It is beyond the scope of this introduction to properly introduce these notions. We can only roughly state that motivic cohomology is given by the so-called *Chow groups*. For an algebro-geometric object X, the Chow group $\mathrm{CH}(X)$ consists of points, lines, surfaces, etc. in X. Here point refers to a solution of the polynomial equations defining X, and lines, surfaces etc. refer to letting points move (in 1, 2, etc. directions) in a polynomial way. Since we were initially interested in finding polynomial solutions, i.e., points in X, Chow groups are closely related to our object of interest. Moreover, in a sense that can be made precise, Chow groups (and motivic cohomology in general) are the most faithful way of linearizing algebraic varieties.

Because Chow groups are so closely related to the non-linear algebraic structure, computing Chow groups is very hard. We do have a few tools at our disposal, but our knowledge is much more limited even about supposedly basic questions. For example, the construction of singular homology makes it evident that there is no $\mathrm{H}_{-1}(X)$: after all we don't have any (-1)-dimensional analogue of points (which are 0-dimensional) and line segments (which are 1-dimensional). The corresponding assertion for motivic cohomology is, however, entirely non-trivial and only known in a few special cases, which will be mentioned again in Section 2.3.

The description of Chow groups should be compared to the construction of simplicial homology groups, whose elements arose similarly, namely by points, line segments, triangles etc., where now line segments etc. are not subject to the condition that they are of polynomial nature. Since the definition of Chow groups and singular homology is so similar (just that the condition on being polynomial is dropped in the latter), there is a map

$$\mathrm{CH}(X) \to \mathrm{H}(X), \tag{1.12}$$

for example, between the Chow group and the singular homology of X. The Hodge conjecture, one of the major open questions in algebraic geometry roughly says that given any element in $\mathrm{H}(X)$, subject to some natural restrictions, it is possible to find an element in $\mathrm{CH}(X)$ which maps to a multiple of the given one in $\mathrm{H}(X)$. This conjecture, and likewise its siblings including the Tate conjecture, are very interesting since they would allow to infer solutions of polynomial equations (broadly construed) from much more easily accessible objects, such as elements in the homology of X.

In relation to this sketch of arithmetic geometry and motives, the research presented in this thesis can be summarized (in a highly approximative way) as follows.

- Two papers *Arakelov motivic cohomology I, II* (surveyed in Section 2.1) develop a new cohomology theory which blends motivic cohomology and, roughly speaking, a variant of singular homology. In a way, this cohomology theory measures how much the right and left hand term in (1.12) differ.

- Using this notion of Arakelov motivic cohomology, the paper *Special L-values of geometric motives* (Section 2.2) develops a conjecture which unifies three important conjectures, including the Beilinson conjecture which relates rational solutions of polynomial equations with ones over $\mathbf{F}_p$ (more precisely with L-functions which are constructed out of counting points over $\mathbf{F}_p$). Beilinson's conjecture is almost the best kind of an answer we can get to Question 1.1 for rational solutions. (It is still a conjecture, though.)

- The proof that this unified conjecture is a valid reformulation of the classical ones is based on the papers *f-cohomology and motives over number rings* (Section 2.4) and *Artin-Tate motives over number rings* (Section 2.3).

- A paper *Algebraic K-theory at the infinite place* (Section 2.5) computes, again roughly speaking, the analogue of motivic cohomology for an unusual kind of number system.

- A group of three papers develops a theory which serves to do algebra in a situation where the sets of numbers we compute with are subject to deformations. The theory can be used as a computational tool to understand the enhanced meaning of intersections in derived algebraic geometry (see the discussion of Answer 1.1.3. above).

  The general statements of this theory are developed in the paper *Admissibility and rectification of colored symmetric operads* (Section 3.1). The paper *Homotopy theory of symmetric powers* (Section 3.2)

shows how to handle the requirements of this theory in practice. *Symmetric operads in abstract symmetric spectra* (Section 3.3) shows how to apply the theory to spectra, which bundle all the information given by a cohomology theory. This last part is also related to my earlier work discussed in Section 2.1.

2 Arithmetic geometry

Let us return to Question 1.1.2. above: how many rational solutions does a system of polynomial equations have? More generally, how many rational points does an algebraic variety $X/\mathbf{Q}$ (i.e., locally defined by rational polynomials) have?

This question can be made more precise by counting rational points $(x_1, \ldots, x_n) \in X(\mathbf{Q})$ whose denominators are bounded by some N and describe how it grows as N grows. This leads to so-called height ζ-functions.

Below, we will instead focus on the Chow groups $\mathrm{CH}(X)$ mentioned in Section 1.4. They are defined to be the free abelian group of all irreducible subvarieties in X, modulo rational equivalence, i.e., deforming cycles along a family parametrized by $\mathbf{P}^1$. More generally, one considers Bloch's higher Chow groups $\mathrm{CH}(X, n)$, which are built out of cycles on $X \times \mathbf{A}^n$ instead of cycles on X. Higher Chow groups are, up to torsion, isomorphic to Adams eigenspaces in higher algebraic K-theory by means of the Chern class map

$$K_n(X)^{(p)} \otimes \mathbf{Q} \xrightarrow{\cong} \mathrm{CH}^p(X, n) \otimes \mathbf{Q}.$$

This works for varieties X over a field. Since we will also be interested in finite type schemes over $\mathbf{Z}$, we use the left hand side in general as the definition of *motivic cohomology* (with rational coefficients), denoted by $\mathrm{H}_{\mathrm{M}}^{2p-n}(X, \mathbf{Q}(p))$.

Beilinson's conjecture relates the rank of these motivic cohomology groups, for $X/\mathbf{Q}$ with the vanishing or pole orders of an L-functions associated to X. Moreover, it expresses the value of the L-function at integers, up to a non-zero rational factor, in terms of motivic cohomology and another cohomology known as Deligne cohomology. (This non-zero rational factor is eliminated by the Bloch-Kato conjecture, which will not be discussed below.) In the remainder of this introduction, we will roughly outline the formulation of Beilinson's conjecture. For further details, the reader can consult [Kin03; Sch88], for example.

As a motivation, and also since they are needed later, we first discuss ζ-functions, which are closely related to the L-functions appearing in Beilinson's conjecture. Given an algebraic variety $X/\mathbf{F}_p$, Weil had the idea to assemble the number $\sharp X(\mathbf{F}_{p^n})$ of $\mathbf{F}_{p^n}$-valued points (i.e., solutions of the equations over all the finite extensions $\mathbf{F}_{p^n}$), into a function defined by

$$Z(X, t) := \exp\left(\sum_{n=1}^{\infty} \sharp X(\mathbf{F}_{p^n}) t^n / n\right).$$

The function $\zeta(X, s) := Z(X, p^{-s})$ can also be computed as an infinite product

$$\zeta(X, s) = \prod_x (1 - N(x)^{-s})^{-1}, \tag{2.1}$$

where the product ranges over closed points of X, i.e., all solutions of the equations with values in a finite field, whose cardinality is denoted by $N(x)$. As was outlined in Section 1.3, étale cohomology, more specifically the Grothendieck trace formula (1.11) serves the purpose of linearizing the problem of counting solutions. This trace formula implies

$$Z(X, t) = \prod_{i=0}^{2 \dim X} \det\left(\mathrm{id} - t\mathrm{Fr}^{-1} | \mathrm{H}_{\mathrm{c}}^i(X \times_{\mathbf{F}_p} \overline{\mathbf{F}_p}, \mathbf{Q}_\ell)\right)^{(-1)^{i+1}}. \tag{2.2}$$

Here Fr is the Frobenius map mentioned in Section 1.3, H_{c} denotes étale cohomology with compact support, and $\ell \neq p$ is a prime.

This formula relates an entity of non-linear origin, namely the number of solutions of a polynomial system of equations, to something linear, namely the $\mathbf{Q}_\ell$-vector space of étale cohomology, endowed with its action of the Frobenius. (Saying that the right hand side is of linear nature refers to the cohomology groups being $\mathbf{Q}_\ell$-vector spaces. It is not to say, that the polynomial is actually linear in t. Instead, it is a rational function which is described precisely by the Weil conjectures proven by Dwork and Deligne.)

The formula (2.1) also makes sense if X is a finite type scheme over $\mathbf{Z}$. In this case we take the product over all closed points (i.e., $\mathbf{F}_{p^n}$-valued for some prime p and $n \geq 1$). These functions are vast extensions

of Riemann's ζ-function which is the special case $X = \operatorname{Spec}\mathbf{Z}$. Unlike for $X/\mathbf{F}_p$, handling such functions requires a lot more care: defined as above, they only converge for $Re(s) > \dim X$. It is expected that that they admit an analytic continuation to the entire complex plane, and that they satisfy a functional equation relating $\zeta(X,s)$ to $\zeta(X, d-s)$.

We now switch back to X being a projective smooth variety defined over $\mathbf{Q}$. A classical example is the elliptic curve E defined by (1.5) (or rather, its projective closure). The L-function of the motive $h^i(X)$ is defined by

$$L(h^i(X), s) := \prod_p \det(\mathrm{id} - \mathrm{Fr}^{-1}p^{-s}|\mathrm{H}^i(X, \mathbf{Q}_\ell)^{I_p})^{-1}.$$

This formula is closely related to (2.2), except for the presence of the inertia group I_p, which appears. To explain it, we choose a projective model $\mathcal{X}/\operatorname{Spec}\mathbf{Z}$ (by clearing all denominators). If we think of $\operatorname{Spec}\mathbf{Z}$ as being analogous to a curve C, and and $\mathcal{X}$ as a family of manifolds parametrized by C, the family will be smooth except at finitely many points. The invariants of the inertia group corresponds, in this analogy, to the invariants of the monodromy action, around these points, on the cohomology of a smooth fiber of the family $\mathcal{X}$. As above, (highly nontrivial) caveats concerning the independence of the choice of ℓ in each factor, the independence of a choice of embedding $\mathbf{Q}_\ell \subset \mathbf{C}$, the convergence, analytic continuation and functional equation apply to this definition of $L(h^i(X), s)$. We will neglect these here and in Section 2.2.

Beilinson's conjecture states that the pole order of $L(h^i(X), s)$ at an integer $s = m$ is expressible in terms of motivic cohomology. The complete statement would require introducing a number of further notions, so we just mention one special case, which is

$$\operatorname{ord}_{s=m} L(h^{i-1}(X), s) = \dim \mathrm{H}^i_{\mathrm{M}}(X, i-m)_{\mathbf{Z}} \tag{2.3}$$

for $i - 2m \geq 1$. If X has a projective regular model $\mathcal{X}$, the subscript $\mathbf{Z}$ at the right denotes the image of $\mathrm{H}^i_{\mathrm{M}}(\mathcal{X}, i-m)$ in the corresponding motivic cohomology group of X.

The special L-value is, according to Beilinson's conjecture, also closely related to motivic cohomology. Again, there are three different cases of the conjecture. For $i - 2m < 1$, it says that there is an isomorphism

$$\mathrm{H}^i_{\mathrm{M}}(X, \mathbf{Q}(i-m))_{\mathbf{Z}} \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{D}}(X, \mathbf{R}(i-m)). \tag{2.4}$$

The right hand vector space is Deligne cohomology, a cohomology mixing Betti cohomology with real coefficients, and de Rham cohomology (truncated by means of the Hodge filtration). Relative to natural $\mathbf{Q}$-lattices in these $\mathbf{R}$-vector spaces, Beilinson's conjecture asserts that the special L-value at $s = m$ is given by the determinant of this isomorphism.

2.1 Arakelov motivic cohomology

Beilinson's conjecture (in the special case (2.4) above, but even more prominently in the cases we have omitted above) suggests considering a cohomology theory

$$\widehat{\mathrm{H}}^*_{\mathrm{M}}(X, \mathbf{R}(*))$$

which measures the difference between $\mathrm{H}^*_{\mathrm{M}}(X, \mathbf{R}(*))$ and $\mathrm{H}^*_{\mathrm{D}}(X, \mathbf{R}(*))$, i.e., motivic cohomology (with real coefficients) and Deligne cohomology. Here X is a scheme over $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$. More precisely, we want a cohomology fitting into a long exact sequence

$$\ldots \to \widehat{\mathrm{H}}^i_{\mathrm{M}}(X, n) \to \mathrm{H}^i_{\mathrm{M}}(X, m) \to \mathrm{H}^i_{\mathrm{D}}(X, m) \to \widehat{\mathrm{H}}^{i+1}_{\mathrm{M}}(X, n) \to \ldots \tag{2.5}$$

The map in the middle is the Beilinson regulator, i.e., the Chern class for Deligne cohomology. More generally, to make the concept available for the Beilinson conjecture, which is about $h^i(X)$, which is only a piece of the motive of X, we want such a theory to be available for motives. A convenient such category is given by the category $\mathbf{DM}(\mathbf{Z})$ of Beilinson motives over $\operatorname{Spec}\mathbf{Z}$ introduced by Cisinski and Déglise.

The papers [HS15] (joint with Andreas Holmstrom) and [Sch15] develop such a formalism. While the desideratum in (2.5) is simple, it is nontrivial to actually construct such a theory. The difficulty is to refine the Beilinson regulator to a map between more structured objects: to construct $\widehat{\mathrm{H}}_{\mathrm{M}}$, it is not enough to

know the map on the level of cohomology groups, but one needs a refined presentation on the level of chain complexes. Such presentations were known, but are not compatible with pushforwards. The key idea which overcomes these difficulties and also makes the construction of $\widehat{H}_M$ highly canonical is a zig-zag

$$H_B \xrightarrow{\mathrm{id}\otimes 1} H_B \otimes H_D \xleftarrow{1\otimes\mathrm{id}} H_D.$$

Here H_B and H_D are spectra representing motivic cohomology and Deligne cohomology, respectively. Étale descent for Deligne cohomology implies that the right hand map is a weak equivalence. Thus, the homotopy fiber of the left hand map $\mathrm{id} \otimes 1$, which is well-defined, represents the sought-for $\widehat{H}_M$.

This very canonical way of constructing a cohomology theory can also be applied to a K-theoretic (as opposed to motivic cohomology) variant. It also quickly yields a higher arithmetic Riemann-Roch theorem.

In [Sch15], these abstract constructions are shown to extend the classical notions of arithmetic K-theory and arithmetic Chow groups. This is a subtle task since these classical constructions depend on explicit complexes computing, say, arithmetic K-theory, whereas the above construction is more conceptual, but inexplicit. In a nutshell, these comparison results are possible by upgrading the classical construction to the more structured framework of motivic spectra, and then using a strong unicity property for the homotopy fiber $\widehat{H}_B$: the only isomorphism of $\widehat{H}_B$ in the triangulated category $\mathbf{DM}(\mathbf{Z})$ which is compatible with the identity on H_B and the identity on H_D is the identity.

The idea of using motivic ring spectra to handle cohomology theories has also been used by various authors. In particular, Bunke, Nikolaus, and Tamme later refined the Beilinson regulator to a map of motivic E_∞-ring spectra, which is the most structured statement possible about this map [BNT15].

2.2 Special L-values

The paper [Sch16] is about a reformulation of Beilinson's above-mentioned conjecture.

For any motive M over $\mathbf{Z}$, the composition of morphisms in $\mathbf{DM}(\mathbf{Z})$ yields a natural pairing between (ordinary) motivic homology $H_*(M) = \mathrm{Hom}_{\mathbf{DM}(\mathbf{Z})}(H_B, M)$ and Arakelov motivic cohomology $\widehat{H}^*(M, d) := \mathrm{Hom}_{\mathbf{DM}(\mathbf{Z})}(M, \widehat{H}_B(d))$ introduced above:

$$H_i(M, \mathbf{R}) \otimes_{\mathbf{R}} \widehat{H}^{2-i}(M, \mathbf{R}(1)) \to \widehat{H}^2(\mathbf{Z}, \mathbf{R}(1)) = \widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathbf{Z}) = \mathbf{R}. \qquad (2.6)$$

Conjecture 2.7. This pairing is a perfect pairing for any constructible motive M in $\mathbf{DM}(\mathbf{Z})$.

This duality is of course in the same spirit as Poincaré duality for sheaves on an open manifold and also as Artin-Verdier duality for étale sheaves on $\mathrm{Spec}\,\mathbf{Z}$. Yet, this conjecture is much deeper. If M is of the form $M = i_*N$ for a geometric motive N over $\mathbf{F}_p$, this conjecture is equivalent to the conjunction of Beilinson's conjecture on agreement of rational and numerical equivalence and Parshin's conjecture. It also implies the independence of L-functions of the choice of ℓ. For $X/\mathbf{Z}$ being projective and regular, the conjecture is equivalent to the Beilinson-Soulé vanishing conjecture.

The vector spaces in pairing (2.6) (or more precisely, the alternating tensor products of their determinants, as i varies), carry a natural rational structure. For $H_i(M, \mathbf{R}) = H_i(M, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{R}$ this is the trivial one. The $\mathbf{Q}$-structure on $\widehat{H}^*(M, \mathbf{R})$ is obtained from the trivial one on $H^*(M, \mathbf{R})$ and the $\mathbf{Q}$-structure on $H_D^*(M, \mathbf{R})$ obtained by glueing the rational structure on Betti cohomology, and the $\mathbf{Q}$-structure on algebraic de Rham cohomology stemming from the isomorphism

$$H_{\mathrm{dR}}^*(X_{\mathbf{R}}) = H_{\mathrm{dR}}^*(X) \otimes_{\mathbf{Q}} \mathbf{R} \quad (\text{for } X/\mathbf{Q}).$$

In more concrete terms, the $\mathbf{Q}$-structure on $H_D^*(M)$ encodes periods, i.e., matrices of the form

$$\left(\int_{\gamma_i} \omega_j \right)$$

for bases $\gamma_i \in H_{B*}(X, \mathbf{Q})$ and $\omega_j \in H_{\mathrm{dR}}^*(X)$.

We can now state the second part of the L-values conjecture:

Conjecture 2.8. For a constructible motive M in $\mathbf{DM}(\mathbf{Z})$, the order of the L-function is given by

$$\mathrm{ord}_{s=0}\, L(M,s) = -\widehat{\chi}(M),$$

the negative Euler characteristic of Arakelov motivic cohomology of M. The special L-value is given, up to a non-zero rational factor, by

$$L^*(M,0) \equiv 1/\Pi_M \mod \mathbf{Q}^\times,$$

where Π_M denotes the determinant of the pairings (2.6) (more precisely, the alternating determinant for all i, with respect to the $\mathbf{Q}$-structures just mentioned).

There are three notable special cases of these conjectures: one is $M = i_*N$, as above. In this case Conjecture 2.8 is closely related to the Tate conjecture. For $M = \mathrm{M_c}(X)$, the motive with compact support of a scheme $X/\mathbf{Z}$, the pole order prediction is equivalent to a conjecture of Soulé. For M being a certain intermediate extension of the motive of a smooth projective variety $X/\mathbf{Q}$, the conjecture is closely related to Beilinson's conjecture. More formally, we have:

Theorem 2.9. *Assuming a motivic t-structure for motives over $\mathbf{Z}$ satisfying the usual expected properties, the above pair of conjectures is equivalent to the conjunction of Beilinson's conjecture, Tate's conjecture and Soulé's conjecture.*

The above conjectures are compatible with distinguished triangles of motives. In particular, thanks to the work of Borel, they hold for all Tate motives. They are also compatible with the functional equation.

It is a natural open question how to refine the above conjecture to an integral statement, along the lines of the Tamagawa number conjecture by Bloch-Kato.

A different approach to L-values, which gives an integral prediction, but only applies to L-functions of the form $L(M(X), s)$, where $X/\mathbf{Z}$ is projective and regular, has been initiated by Lichtenbaum and was pursued by Flach and Morin. We refer to the introduction of [Sch16] for references and further discussion, and also the recent work [FM16] which uses the above construction of Arakelov motivic cohomology and the above reformulation of Beilinson's conjecture.

2.3 Artin-Tate motives over number rings

An inspiring, but challenging feature of the world of motives is the fact that many foundational "facts" are still conjectures. For example, the Beilinson-Soulé vanishing, i.e., the vanishing of

$$K_{2p-i}(X)_{\mathbf{Q}}^{(p)} = \mathrm{H}_{\mathrm{M}}^i(X, \mathbf{Q}(p)) \overset{?}{=} 0 \quad \text{for } i < 0,$$

which is a triviality for Betti cohomology, is not at all clear.

There is one exception to this state of affairs, namely for the subcategory $\mathbf{DATM}(F) \subset \mathbf{DM}(F)$ of mixed Artin-Tate motives which is generated by motives of the form $M(E)(n)$, where E/F is a finite extension of the ground field F. For us, the ground field F is a number field or a finite field. For these fields, the Beilinson-Soulé vanishing is known. Levine and later Wildeshaus used this to establish a motivic t-structure on (Artin-)Tate motives. In [Sch11], these observations were extended to a triangulated category $\mathbf{DATM}(\mathcal{O}_F)$ of Artin-Tate motives over number rings. The category consists, by definition, of motives of the form $M(\mathcal{O}_E)$ and $M(\mathbf{F}_q)$, where E is a finite extension of F and $\mathbf{F}_q$ is a finite extension of some residue field of $\mathcal{O}_F$.

The main statements of this paper are summarized by the following theorem:

Theorem 2.10. *The category $\mathbf{DATM}(\mathcal{O}_F)$ carries a motivic t-structure, which on the one hand extends the one established by Levine-Wildeshaus, and on the other hand parallels the perverse t-structure on a curve. Moreover, there is a weight filtration formalism.*

For the subcategory $\mathbf{DATM}(\mathcal{O}_F) \subset \mathbf{DM}(\mathcal{O}_F)$, this result establishes what has been referred to as the "usual expected properties" in Theorem 2.9 above. In particular, for Artin-Tate motives the comparison of the above-mentioned L-values conjecture is unconditionally equivalent to the classical conjectures of Beilinson, Tate, and Soulé.

2.4 f-cohomology and motives over number rings

The paper [Sch12] develops the necessary theory to make sense of the intermediate extension functor $\eta_{!*}$ which shows up in relating Beilinson's conjecture to a conjecture for motives over $\mathbf{Z}$.

For special motives, the idea of this functor is simple: for $X/\mathbf{Q}$ is smooth and projective, there is an open subscheme $U \subset \operatorname{Spec} \mathbf{Z}$ and a smooth projective extension $\mathcal{X}_U$ of X. Then $\eta_{!*}(h^i(X))$ is defined as $j_{!*}h^{i+1}(\mathcal{X}_U)$, where $j_{!*}$ is the intermediate extension functor. To define this intermediate extension, one has to use an abelian category of mixed motives, which only exists conjecturally. (An unconditional subcategory is studied in Section 2.3.) The paper specifies the precise axioms on mixed motives we need to make this construction work. Moreover, based on these axioms, it shows that this functor $j_{!*}$ on motives is compatible with the usual one on ℓ-adic sheaves over $\operatorname{Spec} \mathbf{Z}$ via the ℓ-adic realization functor. Finally, the motivic cohomology of $\eta_{!*}h^i(X)$ is related to classical variants of motivic cohomology, such as the integral motivic cohomology groups $\mathrm{H}^i_{\mathrm{M}}(X, n)_{\mathbf{Z}}$ and the homologically trivial part of the Chow group.

2.5 K-theory at infinity

The paper [Sch14] is devoted to the K-theory of a new class of rings introduced by Durov [Dur07]. Durov's work allows to rigorously discuss the compactification $\overline{\operatorname{Spec} \mathbf{Z}}$ beyond the philosophy of Arakelov theory

$$\text{scheme over } \mathbf{Z} \text{ vs. complex analytic space over } \mathbf{C}.$$

This is made possible by using a relaxed notion of rings. These rings, called generalized rings are defined by their modules, extending the observation that the ring structure on a (usual) ring can be encoded via the free R-modules R^n, together with the map

$$R^2 \times R \times R \to R, ((x_1, x_2), y, z) \mapsto x_1 y + x_2 z.$$

The ring $\mathbf{Z}_\infty$ which serves as a replacement of the (usual) rings of p-adic integers $\mathbf{Z}_p$ is defined by declaring its free module of rank n to be

$$\mathbf{Z}_\infty(n) := \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n, \sum |x_i| \leq 1 \right\}.$$

The main result of this paper shows that the K-theory of $\mathbf{Z}_\infty$ and more general rings $\mathcal{O}$ (occurring at the infinite place of number rings) is governed by a smaller group, namely

$$E := \{ x \in \mathcal{O}, |x| = 1 \}.$$

For example, for $\mathcal{O} = \mathbf{Z}_\infty$, $E = \{\pm 1\}$. The algebraic K-theory of the Waldhausen category of free $\mathcal{O}$-modules can be computed as

$$K_i(\mathcal{O}) = \pi_i^s(BE_+, *),$$

the stable homotopy groups of the classifying space of E, equipped with a disjoint base point. The reason that the K-theory of these generalized rings is comparatively simple is the presence of the corners in the space $\mathbf{Z}_\infty(n)$ which implies strict constraints on automorphisms of $\mathbf{Z}_\infty(n)$: these are, it turns out, simply given by permuting the corners.

More recently, Haran [Har15] has proposed another type of generalized ring, which has the property that the free module of rank n is instead given by

$$\left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n, \sum |x_i|^2 \leq 1 \right\}.$$

It seems interesting to study the K-theory of these types of rings and to see if there is a closer relationship to Deligne cohomology than for Durov's rings.

3 Homotopical algebra

In Section 1.4, we encountered motives as being some kind of universal linearization of algebraic varieties. We will begin this section by an introduction to homological algebra, which we will initially frame as a tool to do linearization in a different sense.

The mathematical objects we are going to linearize are not just functions, which assigning a number $f(x)$ to a number x, but *functors*: they assign an object $F(X)$ to an object X. Functors are richer than functions since they are operating with objects X which have a richer internal structure than just a number. For example, F might be defined on all abelian groups, in which case the structure of abelian groups, as encoded by maps between them, must be respected by F.

To specify which functors are "linear", it is convenient to use the notion of an exact sequence

$$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0,$$

which is a shorthand for saying that b is surjective and its kernel is isomorphic to A via a. Here A, B, and C are abelian groups, for example. A functor F is called *exact* if it preserves short exact sequences. We view exact functors as being analogous to linear functions. Indeed, taking our cue from the dimension formula in linear algebra, we could view B as being some sort of "sum" of A and C

$$B = A \text{ "} + \text{" } C.$$

The quotation marks are huge here: it is not usually true, and indeed the whole point of homological algebra, that B is actually the direct sum of A and C! If we are willing, however, to indulge in a big-quotation-marks-attitude, then the condition that F be exact just means F is "linear", meaning

$$F(A \text{ "} + \text{" } C) = F(A) \text{ "} + \text{" } F(C).$$

Like in calculus, though, many interesting functors usually fail to be exact. For example, the functor

$$F : M \mapsto F(M) := M \otimes_{\mathbf{Z}} \mathbf{Z}/2 = M/2M$$

is not exact, since it maps the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{\text{pr}} \mathbf{Z}/2 \to 0 \tag{3.1}$$

to

$$0 \longrightarrow \mathbf{Z}/2 \xrightarrow{0} \mathbf{Z}/2 \xrightarrow{\text{id}} \mathbf{Z}/2 \longrightarrow 0 \tag{3.2}$$

which is no longer exact: the kernel of id is not isomorphic to $\mathbf{Z}/2$. *Deriving a functor* is a way to remedy its non-exactness (or non-"linearity"). More precisely, the derived functor of a functor F is the best exact approximation to F, comparable to the derivative being the best linear approximation of a function. In the above example (3.2), the easiest way (and, in a precise sense, the universal way) of reinstating "linearity" (i.e., exactness) is to extend the above sequence to

$$0 \longrightarrow \underline{\mathbf{Z}/2} \xrightarrow{\text{id}} \mathbf{Z}/2 \xrightarrow{0} \mathbf{Z}/2 \xrightarrow{\text{id}} \underline{\mathbf{Z}/2} \longrightarrow 0. \tag{3.3}$$

The right hand underlined copy of $\mathbf{Z}/2$ is the original $F(\mathbf{Z}/2)$, whereas the left one is the modification we have to insert to restore exactness. Therefore, we are led to stipulating that the derived tensor product should consist, in a sense that remains to be made precise, of the two underlined copies of $\mathbf{Z}/2$. On the other hand, if we regard (3.3) as a "linear equation"

$$\text{"the two } \underline{\mathbf{Z}/2}\text{"} = \text{"the remaining two } \mathbf{Z}/2\text{"}$$

we should also expect the derived tensor product, usually denoted by

$$\mathbf{Z}/2 \otimes_{\mathbf{Z}}^{\mathrm{L}} \mathbf{Z}/2,$$

to consist of the two non-underlined copies of $\mathbf{Z}/2$.

In pointing towards derived algebraic geometry around (1.4), we have discussed the self-intersection of the line $x = 0$ in the plane. Algebraically, this corresponds to the tensor product $k[y] \otimes_{k[x,y]} k[y]$, which is $k[y]$, i.e., corresponds to the line $x = 0$. An obvious modification of the discussion of $\mathbf{Z}/2 \otimes_{\mathbf{Z}}^{\mathrm{L}} \mathbf{Z}/2$ computes the derived tensor product $k[y] \otimes_{k[x,y]}^{\mathrm{L}} k[y] = [k[y] \xrightarrow{0} k[y]]$, which is the derived intersection of the line with itself.

We are now facing two questions:

Question 3.1.   *1. How can we rigorously define derived functors?*
2. How do we compute derived functors?

To comprehensively answer these questions, we will gradually consider more general situations of non-exactness. The first step requires the notion of a *quasi-isomorphism*. These are maps of chain complexes which induce an isomorphism on all homology groups (defined as the kernel of the differential modulo the image of the preceding differential). For example, the map of chain complexes (the map goes in the vertical direction, the horizontal maps are the differentials of the chain complex)

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \ldots \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array} \qquad (3.4)$$

is a quasi-isomorphism: the homology groups of both complexes are all 0, except at the spot involving the map pr, where homology is $\mathbf{Z}/2$. Note that saying this map is a quasi-isomorphism is simply restating the exact sequence (3.1). It is not hard to show that a functor F is exact in the sense above if and only if it preserves quasi-isomorphisms. It is helpful to think of quasi-isomorphisms as those maps which preserve the true content of a chain complex: at the end of the day we will not be interested so much in the complex itself, but rather only in its cohomology groups. For example, de Rham cohomology of a manifold is computed both by the complex of differential forms, and also by the complex of currents. Depending on the situation, one of the complexes may be better suited to computations than the other, but the core content (i.e., the cohomology) remains unchanged.

We have arrived at a point where we view exact functors as those preserving the core content of a mathematical object. This idea of core content is also important in non-abelian settings, most prominently in homotopy theory. Homotopy theorists regard two topological spaces as similar enough (by means of a fixed map $f : X \to Y$) whenever the induced map of homotopy groups $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ are isomorphisms for all $n \geq 0$. Such a map f is called a *weak equivalence*. The simplest example of a weak equivalence is the inclusion of a point inside an interval, $\mathrm{pt} \to I$.

We define a functor to be exact if it preserves weak equivalences. Once again, non-exact functors are ubiquitous. The following glueing functor is a typical example: it assigns to some diagram consisting of three spaces X, X' and X'' and two continuous maps x' and x'',

$$\begin{array}{ccc}
X & \xrightarrow{x'} & X' \\
{\scriptstyle x''}\downarrow & & \vdots \\
X'' & \cdots\cdots\rightarrow & X' \sqcup_X X''
\end{array} \qquad (3.5)$$

its pushout $X' \sqcup_X X''$, i.e., the space obtained by glueing X' and X'' along X, via the given maps. For example, the pushout of

$$\begin{array}{ccc}
\mathrm{pt} \sqcup \mathrm{pt} & \longrightarrow & \mathrm{pt} \\
\downarrow & & \\
\mathrm{pt} & &
\end{array} \qquad (3.6)$$

is a single point. However, if we replace the single copies of pt by intervals I (which are weakly equivalent,

and the maps are also weakly equivalent to the original ones):

$$\text{pt} \sqcup \text{pt} \xrightarrow{i_0 \sqcup i_1} I \tag{3.7}$$

$$\downarrow{\scriptstyle i_0 \sqcup i_1}$$

$$I,$$

the pushout is S^1, the circle. It is genuinely different from (i.e., not weakly equivalent to) the previous pushout: the winding number yields an isomorphism $\pi_1(S^1) = \mathbf{Z}$, but $\pi_1(\text{pt}) = 0$. Therefore, the pushout functor is not exact: it does not preserve weak equivalences between diagrams (which are by definition those maps of diagrams whose individual components are weak equivalences in the sense above).

Model categories, a far-reaching concept due to Quillen [Qui67] conveniently explain the above phenomena. They consist of the following data:

1. A category $\mathcal{C}$. In the above examples, we would take chain complexes of abelian groups or roof diagrams of topological spaces as in (3.5).

2. A class of maps in $\mathcal{C}$ called *weak equivalences*. These are the maps we consider to preserve the core content of an object. Above, we would take quasi-isomorphisms, i.e., maps inducing isomorphisms on homology, resp. weak equivalences of diagrams (i.e., maps inducing isomorphisms of homotopy groups for the three spaces involved.)

3. Two classes of maps called cofibrations and fibrations. Once the weak equivalences are specified, these two classes determine each other. In many model categories, there is a rather explicit set of generating cofibrations which formalize the intuition that any CW complex can be constructed by repeatedly (possibly infinitely) attaching cells.

These data are required to satisfy certain conditions and compatibilities. The most important condition is that for any object X, we must be able to find a weak equivalence, called a *cofibrant replacement*:

$$X' \xrightarrow{\sim} X$$

where X' is cofibrant, i.e., obtained from $\emptyset$ by repeatedly glueing "cells" (possibly infinitely many). Projective resolutions of modules, which are a key technique of homological algebra, are precisely the cofibrant replacements in a model category on chain complexes known as the projective model structure. Thus, the top line in (3.4) is a cofibrant replacement of the bottom line (i.e., the object $\mathbf{Z}/2$). Similarly, the diagram (3.7) is a cofibrant replacement of (3.6).

Model categories provide an answer for Question 3.1.2. above: suppose a functor $F : \mathcal{C} \to \mathcal{D}$ between two model categories preserves cofibrations and acyclic cofibrations (i.e., maps which are both cofibrations and weak equivalences). Such a functor is a *left Quillen functor*.[1] For example, the tensor product functor $- \otimes_{\mathbf{Z}} \mathbf{Z}/2$ is a left Quillen functor provided we understand cofibrations to be maps that glue in projective modules, as alluded to above. The functor which assigns to a diagram (3.5) its pushout is also a left Quillen functor provided that we understand cofibrant objects to be diagrams in which both maps x' and x'' are obtained by glueing in cells.

The derived functor of a left Quillen functor F, denoted $\mathrm{L}F$, is defined as

$$\mathrm{L}F(X) = F(X').$$

A little lemma known as Brown's lemma shows that (unlike F!), $\mathrm{L}F$ preserves weak equivalences. Moreover, $\mathrm{L}F$ is homotopically terminal among its peers, i.e., functors mapping to F and preserving all weak equivalences. In this precise sense, it is the optimal approximation of F by an exact functor. Being homotopically terminal only depends on the weak equivalences, not on the choice of (co)fibrations. In particular, if we

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[1]An entirely symmetric story can be told for deriving exact functors which are exact at the left (but fail to preserve exactness at the right). A dual notion of right Quillen functors serves to compute these. This subsumes (and greatly extends) the usual computation of sheaf cohomology by injective resolutions, for example. In the remainder of this introduction, we will focus on left Quillen functors.

had computed the left derived functor using a different class of cofibrations (which option is what makes model categories useful), the resulting derived functors would be weakly equivalent. For details the reader is referred to [Dwy+04].

Revisiting our examples, we see that

$$\mathbf{Z}/2 \otimes^{\mathrm{L}}_{\mathbf{Z}} \mathbf{Z}/2 = (\mathbf{Z} \xrightarrow{2} \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}/2 = (\mathbf{Z}/2 \xrightarrow{0} \mathbf{Z}/2),$$

i.e., just the terms in (3.3) which are not underlined. Similarly, the above derived pushout, better known as the homotopy pushout can be computed as

$$\mathrm{pt} \sqcup^{\mathrm{L}}_{\mathrm{pt} \sqcup \mathrm{pt}} \mathrm{pt} = I \sqcup_{\mathrm{pt} \sqcup \mathrm{pt}} I = S^1.$$

Nicely, the chain rule of calculus has the following analogue: given two composable left Quillen functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$, there is a weak equivalence

$$\mathrm{L}G \circ \mathrm{L}F \xrightarrow{\sim} \mathrm{L}(G \circ F).$$

We end this short introduction by putting the notion of a model category into the context of other homotopical-algebraic notions, by means of the following analogy between linear and homotopical algebra [FG12]:
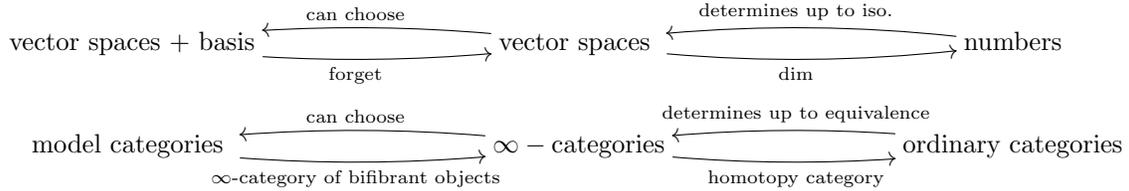
Two vector spaces are isomorphic if and only if their dimensions agree. Analogously, two ∞-categories are equivalent (by means of some given functor) if and only if their homotopy categories are equivalent. The classical example of a homotopy category is the derived category of a Grothendieck abelian category. The analogy also illustrates the shortcomings of the homotopy category (of a model or an ∞-category): for a linear map $f : V \to W$, we are unable to define $\dim \ker f$ if we only remember $\dim V$ and $\dim W$. Similarly, we are unable to do the majority of algebraic manipulations in the homotopy category of an ∞- (or model) category. This much about the right hand column. The left colum serves, so to speak, for doing concrete computations. Presentable ∞-categories arise from combinatorial model categories. Such a choice of a model structure is not unique, but allows for a convenient choice, similarly to choosing bases in vector spaces. Forgetting a basis of a vector space corresponds, in homotopy land, to constructing an ∞-category out of bifibrant (i.e., both cofibrant and fibrant) objects of a model category. It is inspiring to also view this analogy from a historical perspective: while matrices (i.e., vector spaces made concrete) appear in Gauss' 1801 *Disquisitiones Arithmeticae*, the axiomatic concept of a vector space emerged only in Peano's 1888 *Calcolo Geometrico* [Kle07]. Similarly, model categories (i.e., ∞-categories made concrete) were introduced by Quillen in 1967. It took some decades, until Lurie's 2012 *Higher topos theory* (based on earlier work of Joyal) gave us the notion of ∞-categories.

3.1 Operads and their algebras in model categories

Above, we have outlined homological and homotopical algebra as a tool to apply homotopical methods to algebraic problems. Recently, however, the term homological algebra acquires a second flavor, namely by doing algebra in a context where the usual notions of rings and modules are understood up to a notion of weak equivalence. Such a development is fueled by the work of Toën and Vezzosi on derived algebraic geometry, and again Lurie's work on spectral algebraic geometry. My joint work with Dmitri Pavlov in [PS14a] adresses the question of doing homological algebra (in this latter sense) in a model-categorical context.

To do algebra, we need a multiplication. This is codified by endowing a model category $\mathcal{C}$ with an additional functor

$$- \otimes - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

playing the role of the tensor product (of chain complexes of abelian groups) or the cartesian product (of topological spaces). Such a structure is known as a *symmetric monoidal model category*. It allows us to talk about the standard notions of commutative monoids and their modules, namely objects $R \in \mathcal{C}$ with a multiplication map

$$\mu : R \otimes R \to R,$$

respectively, objects with a map $R \otimes M \to M$, satisfying the usual rules.

There are now the following immediate questions about a symmetric monoidal model category. The first asks for establishing the basis of homotopical linear algebra (broadly construed), whereas the second is needed to get homotopical commutative algebra (which forms the basis of homotopical algebraic geometry) off the ground.

Question 3.2. 1. *Given a monoid $R \in \mathcal{C}$, when does the category $\mathbf{Mod}_R(\mathcal{C})$ of R-modules inherit a model structure?*
2. *When does the category $\mathrm{Comm}(\mathcal{C})$ of commutative monoids in $\mathcal{C}$ inherit a model structure?*

In both cases the word "inherit" means that weak equivalences and fibrations in $\mathbf{Mod}_R(\mathcal{C})$ (respectively, $\mathrm{Comm}(\mathcal{C})$) are precisely those maps whose underlying map in $\mathcal{C}$ is in the corresponding class. We will use the term "inherit" in a similar way below.

Answer (to Question 3.2.1.). *The model structure on $\mathbf{Mod}_R(\mathcal{C})$ exists, for any R, whenever $\mathcal{C}$ satisfies the* monoid axiom *introduced by Schwede and Shipley [SS00]. The key point of this axiom is that for any $Y \in \mathcal{C}$ and any acyclic cofibration s, the map*

$$Y \otimes s$$

is a couniversal weak equivalence, i.e., a weak equivalence which remains a weak equivalence under any pushout.

In practice, the monoid axiom is a mild condition satisfied for all basic model categories. In Section 3.2, we discuss how to promote it to more involved model categories.

To describe our answer to Question 3.2.2., it is convenient to generalize the question. Originating in topology, but also relevant in algebraic geometry (in the guise of the multiplication in Deligne cohomology) is a situation where the multiplicative structure on some object is not just given by a map

$$\mu : R \otimes R \to R$$

as above, but instead the multiplication maps are parametrized by a space $O(2)$, so we consider a map

$$\mu : O(2) \otimes R \otimes R \to R.$$

A typical example in topology is the loop space $\Omega(X) = \mathrm{Hom}(S^1, X)$. The process of concatenating two loops in X gives a map $\Omega(X) \times \Omega(X) \to \Omega(X)$. But the way we choose our speed in traversing the first and then the second loop gives us a space parametrizing such binary operations. More generally, the space $\mathrm{A}_\infty(n)$ of n disjoint, linearly embedded, open intervals in $\mathbf{R}$ can be used to define a natural parametrized n-ary multiplication map

$$\mathrm{A}_\infty(n) \times \Omega(X)^{\times n} \to \Omega(X).$$

For different n, the spaces $\mathrm{A}_\infty(n)$ are naturally related by means of multiplication maps

$$\mathrm{A}_\infty(n) \times \prod_{i=1}^{n} \mathrm{A}_\infty(k_i) \to \mathrm{A}_\infty \left(\sum_{i=1}^{n} k_i \right),$$

and the above action on $\Omega(X)$ is compatible with this multiplication. Moreover, there is a Σ_n-action on $\mathrm{A}_\infty(n)$ which is naturally compatible with both multiplication maps. According to May, the collection of the $\mathrm{A}_\infty(n)$'s is called an *operad* known as the little intervals operad, and $\Omega(X)$ is an algebra over this operad. Contracting the intervals to their centers yields a map $\mathrm{A}_\infty(n) \to \mathrm{As}$, the associative operad, which is defined by $\mathrm{As}(n) = \Sigma_n$. This map of operads yields a restriction functor

$$\mathbf{Alg}_{\mathrm{As}}(\mathbf{Top}) \to \mathbf{Alg}_{\mathrm{A}_\infty}(\mathbf{Top})$$

between topological spaces with a strictly associative multiplication and those with a multiplication which is only associative up to homotopies specified by the A_∞-action. It is classically known that this functor is part of a Quillen equivalence, i.e., the homotopy categories are equivalent.

To also address the commutativity aspects of multiplication, one considers the operad E_∞ defined as the union over higher-dimensional little disks operads (for increasingly high-dimensional disks). Moreover, the commutative operad Comm is defined by $\mathrm{Comm}(n) = \mathrm{pt}$. Now, even though we have a natural weak equivalence $E_\infty \to \mathrm{Comm}$ of operads, it is *not* true (and classically known) that E_∞-algebras in **Top** are Quillen equivalent to strictly commutative algebras. The difference is that the Σ_n-action on the n-th level of Comm is not free, whereas it is free on $A_\infty(n)$ and $\mathrm{As}(n)$. These facts can be paraphrased by saying that a topological algebra, which is associative up to (coherent higher) homotopies can be strictified to (i.e., is weakly equivalent to) a strictly associative algebra. The corresponding statement for commutative algebras vs. E_∞-algebras is false.

With this motivation in hand, we come back to general questions about operadic algebras. The first question asks when we can do homotopical algebra over operads. (The case $O = \mathrm{Comm}$ is Question 3.2.2. above.) The second question asks whether doing homotopical algebra is sensitive to the choice of operad we use to model our algebras. In the above classical examples in $\mathcal{C} = \mathbf{Top}$, the answer is yes for A_∞ vs. As, but no for E_∞ vs. Comm. The third question asks whether our computations will be sensitive to our choice of model category. For example, there is a natural Quillen equivalence

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathrm{Sing} \tag{3.8}$$

between simplicial sets and topological spaces. It would be disturbing if homotopical algebra in simplicial sets would be genuinely different than in topological spaces. (It is not, it turns out.)

Question 3.3. *1. Given a symmetric operad O, when does the category $\mathbf{Alg}_O(\mathcal{C})$ of O-algebras inherit a model structure?*

2. When does a weak equivalence of operads $O \to P$ yield a Quillen equivalence

$$\mathbf{Alg}_P(\mathcal{C}) \to \mathbf{Alg}_O(\mathcal{C})?$$

I.e., when does the forgetful functor induce an equivalence of the homotopy categories

$$\mathbf{Ho}(\mathbf{Alg}_P(\mathcal{C})) \to \mathbf{Ho}(\mathbf{Alg}_O(\mathcal{C}))?$$

3. Given a Quillen equivalence $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, and an operad O in $\mathcal{C}$ (respectively P in $\mathcal{D}$), when are the adjunctions

$$\mathbf{Alg}_O(\mathcal{C}) \rightleftarrows \mathbf{Alg}_{F(O)}(\mathcal{D})$$
$$\mathbf{Alg}_{G(P)}(\mathcal{C}) \rightleftarrows \mathbf{Alg}_P(\mathcal{D})$$

Quillen equivalences?

These three questions form the basis for doing homotopical algebra in the afore-mentioned sense. They were adressed by Muro [Mur11; Mur14] for non-symmetric operads. This excludes operads such as Comm (yielding commutative monoids) or Lie (yielding Lie algebras). In [PS14a], we answer these questions for symmetric operads. Compared to non-symmetric operads, this is considerably more delicate, since the homotopical properties of Σ_n-actions, most importantly the homotopical properties of expressions such as

$$O(n) \otimes_{\Sigma_n} R^{\otimes n},$$

have to be taken into account.

In the answers below, we will omit mentioning certain mild technical conditions. These are usually of the following form: $\mathcal{C}$ is combinatorial (this is a mild set-theoretic size condition, and the condition that there is a set of generating cofibrations), and $\mathcal{C}$ satisfies a certain finiteness condition (which ensures that filtered colimits are exact). These two conditions are of technical nature, and can be relaxed further.

Below, we need the notion of an h-cofibration introduced by Batanin and Berger [BB13]. In practice (whenever $\mathcal{C}$ is left proper) being an acyclic h-cofibration is equivalent to being a couniversal weak equivalence. This notion already appeared in the monoid axiom above. In particular, this condition is decidedly weaker

than being an acyclic cofibration. The symbol $\square$ denotes the pushout product of maps. If s is a map $s : \emptyset \to X$ (from the initial object $\emptyset$), then $s^{\square n}$ is just the map $\emptyset \to X^{\otimes n}$.

To the best of our knowledge, the following existence criterion extends all other similar such criteria in the literature.

Answer (to Question 3.3.1.). *Suppose that for any Σ_n-equivariant object $Y \in \mathcal{C}$, and any acyclic cofibration s in $\mathcal{C}$, the map*

$$Z := Y \otimes_{\Sigma_n} s^{\square n} \tag{3.9}$$

is an acyclic h-cofibration.[2] Then $\mathbf{Alg}_O(\mathcal{C})$ inherits a model structure for any symmetric operad O.

We define $\mathcal{C}$ to be *symmetric h-monoidal* if it satisfies the condition above, and also its non-acyclic counterpart (obtained by omitting the word acyclic above). The reason to include the non-acyclic part in this definition is explained in Section 3.2, as is the question of checking symmetric h-monoidality for a given model category $\mathcal{C}$.

Despite the condition of symmetric h-monoidality looking similar to the monoid axiom of Schwede and Shipley, the proof in the symmetric case is more involved. Its key point is to control homotopical properties of pushouts (in the category of operadic algebras) along a map of free operads. The seed crystal in the case of commutative monoids (i.e., the case $O = \mathrm{Comm}$) is the following: the coproduct of two free commutative monoids $\mathrm{Sym}(R)$ and $\mathrm{Sym}(S)$ is given by

$$\mathrm{Sym}(R) \sqcup \mathrm{Sym}(S) = \mathrm{Sym}(R \sqcup S),$$

and it can be computed as

$$\coprod_{a,b \geq 0} \Sigma_{a+b} \times_{\Sigma_a \times \Sigma_b} R^{\otimes a} \otimes S^{\otimes b},$$

a fancy way of writing the binomial formula from high-school. In general, the computation of arbitrary pushouts of operadic algebras is due to Harper [Har10].

Question 3.3.2. admits the following answer, where $\mathcal{C}$ is supposed to be symmetric h-monoidal (and satisfies some mild technical assumptions, as above).

Answer (to Question 3.3.2.). *A map $f : O \to P$ of operads induces a Quillen equivalence*

$$\mathbf{Alg}_O(\mathcal{C}) \rightleftarrows \mathbf{Alg}_P(\mathcal{C})$$

if and only if f is symmetric flat, i.e., its n-th level $f(n)$ is such that

$$f(n) \, \square_{\Sigma_n} \, s^{\square n} \tag{3.10}$$

is a weak equivalence for any cofibration s. (Again, the key case to keep in mind is $s : \emptyset \to X$ for X cofibrant, in which case the condition requires that $f(n) \otimes_{\Sigma_n} X^{\otimes n}$ is a weak equivalence.)

We have mentioned above that $\mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Top}) \rightleftarrows \mathbf{Alg}_{\mathrm{E}_\infty}(\mathbf{Top})$ is not a Quillen equivalence. This is explained by the fact that $\mathbf{Top}$ is not symmetric flat: taking coinvariants by a Σ_n-action does not preserve weak equivalences of topological spaces: for example $\mathbf{RP}^\infty$, which is a model for $B\mathbf{Z}/2 = (E\mathbf{Z}/2)/(\mathbf{Z}/2)$, is not weakly equivalent to $pt = pt/(\mathbf{Z}/2)$. This example suggests that symmetric flatness is a rarely satisfied condition. (It does hold, though, in any situation with rational coefficients, by Maschke's theorem.) This leads to the following

Question 3.4. *What to do if $\mathcal{C}$ is not symmetric h-monoidal or symmetric flat?*

A symmetric flatness condition also arises when we compare operadic algebras in model categories with operadic algebras in ∞-categories, as introduced by Lurie. Briefly, we show that the ∞-category underlying operadic O-algebras in $\mathcal{C}$ is equivalent to the ∞-category of algebras over the operadic nerve provided that $\mathcal{C}$ is symmetric flat with respect to the levelwise projective replacement $O' \to O$. In the parlance of the above

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[2]More generally, we need to require a similar property for a finite family $s_1, \ldots, s_e$ of acyclic cofibrations. A similar notational abuse will be done with the symmetric flatness condition below.

analogy between linear and homotopical algebra, the model category of operadic algebras is a "basis" of the analogous ∞-category, as it should.

Question 3.3.3. is also answered in [PS14a]. Instead of discussing the statement in detail, we just point out that the statements we obtain extend the ones of Schwede and Shipley (for modules over monoids) quite faithfully. For example, since the left adjoint in (3.8) is strong monoidal, we obtain a Quillen equivalence

$$\mathbf{Alg}_O(\mathbf{sSet}) \rightleftarrows \mathbf{Alg}_{|O|}(\mathbf{Top})$$

for any symmetric operad O in simplicial sets.

The proof of this statement uses a description of pushouts of operads due to Spitzweck [Spi01] and Berger–Moerdijk [BM09], which is by far the most involved algebraic input in this paper. This pushout description can also be used to give a new proof of Harper's above-mentioned formula for pushouts of operadic algebras.

3.2 Homotopy theory of symmetric powers

In a nutshell, even though the proofs of the above statements are involved, the final results take a very natural shape. A working mathematician, especially one working in motivic homotopy theory, will also ask:

Question 3.5. *How to verify the conditions of symmetric h-monoidality and symmetric flatness for a given model category $\mathcal{C}$?*

This question is adressed in the paper [PS15]. Our course of action is in the spirit of, say, the condition of a ring being Noetherian: it is easily or even trivially verified for basic rings (such as fields), and it is robust under various ring-theoretic constructions (localization, polynomial rings, completions), making it available for a large class of rings.

We check these conditions for basic model categories, such as chain complexes, simplicial abelian groups, or simplicial sets. For example, chain complexes are symmetric h-monoidal if and only if we work over a ground ring of characteristic 0. This recovers the well-known non-existence of a model structure on commutative dg-$\mathbf{F}_p$-algebras. Interestingly though, *simplicial* abelian groups are symmetric h-monoidal. This is derived from the fact that simplicial sets are symmetric h-monoidal. These facts and also the ones mentioned below are based on the following observation, which is partly due to Gorchinsky–Guletskii [GG16]: the conditions in (3.9) and (3.10) only have to be checked for *generating* (acyclic) cofibrations. This simplifies our task tremendously: for the basic model categories mentioned above it reduces to checking it for a few maps. For example, the generating acyclic cofibrations of simplicial sets are just $\partial \Delta^n \subset \Delta^n$.

We study the stability of the symmetricity properties under the two most common methods to construct of model categories: transfers and Bousfield localizations. *Transfer* refers to the situation that a model structure on a category $\mathcal{C}$ is turned into one on a category $\mathcal{D}$ by means of an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

such that weak equivalences and fibrations in $\mathcal{D}$ are the preimages (under G) of the ones in $\mathcal{C}$. This situation is ubiquitous, with a large class of examples arising from forgetful functors, including the functor forgetting some module structure, as alluded to in Question 3.2.1. above. The Quillen adjunction (3.8) is also an example of transfer.

Left Bousfield localizations are an equally important construction. The term localization derives from the localization of rings and modules. More generally, it refers to forcing a class S of maps to become weak equivalences while keeping the objects the same. In model categories, left Bousfield localizations are of the form

$$\mathcal{C} \rightleftarrows \mathcal{C}[S^{-1}],$$

where the right hand side is the same category, carrying the same cofibrations, but a larger class of weak equivalences. A prototypical example is the Bousfield localization of the category of presheaves with respect to some topology.

We show that for transfers or localizations which are sufficiently compatible with the monoidal structures, the symmetric h-monoidality and symmetric flatness of $\mathcal{C}$ passes to $\mathcal{D}$ (resp. to $\mathcal{C}[S^{-1}]$.) At this point, bundling the condition for acyclic and non-acyclic cofibrations in the definition of symmetric h-monoidality

becomes crucial: it would be impossible to carry the acyclic part of symmetric h-monoidality through a Bousfield localization, since we have in practice little control over the generating acyclic cofibrations of $\mathcal{C}[S^{-1}]$.

All of the above of course also works for the non-symmetric variants; in particular it shows how to propagate the monoid axiom along transfers and localizations.

3.3 Symmetric operads in symmetric spectra

In a third paper [PS14b], we apply the results of the preceding two papers to model categories of symmetric spectra. Motivated by the construction of motivic commutative ring spectra, which offer certain complications not present in the classical case of spectra of simplicial sets, we strive to make our results as general as possible: we consider spectra with values in a symmetric monoidal model category $\mathcal{C}$. In such a generality, the category $\mathbf{Sp}_R(\mathcal{C})$ of symmetric spectra is defined as modules over an arbitrary commutative monoid R in symmetric sequences. The model structure is obtained by first transferring it from a model structure on symmetric sequences (where one has a certain freedom), and then performing a Bousfield localization which corresponds to the stabilization in the classical case of spectra of simplicial sets. An important twist is the consideration of the positive stable model structure denoted $\mathbf{Sp}_R^+$ below. This concept is well-known in topology. It arises by forcing cofibrant objects X to be trivial in spectral degree 0 which causes expressions such as $X^{\otimes n}$ to have a free Σ_n-action. On the other hand, since the stabilization process allows us, roughly speaking, to disregard low spectral levels, the resulting model category will be equivalent to the usual (non-positive) stable model structure. This is, in a nutshell, the basis of the following implication, where we drop certain minor technical conditions on $\mathcal{C}$:

$$\mathcal{C} \text{ is h-monoidal and flat} \implies \mathbf{Sp}_R^+(\mathcal{C}) \text{ is symmetric h-monoidal and symmetric flat.}$$

Here, h-monoidality and flatness are the non-symmetric counterparts of the above notions, i.e., are obtained by omitting the coinvariants by the Σ_n-actions. These non-symmetric conditions are *much* weaker than the symmetric ones. They are satisfied for the following model categories $\mathcal{C}$, and many more:

- For $\mathcal{C} = \mathbf{Top}$ (or $\mathcal{C} = \mathbf{sSet}$) and R being freely generated by the circle S^1 is the classical one, the category $\mathbf{Sp}_{S^1}(\mathcal{C})$ is the classical category of symmetric spectra.

- Motivic spectra arise from $\mathcal{C} = \mathbf{sPSh}(\mathbf{Sm}/S)$, simplicial presheaves on the site of smooth schemes over some base scheme S, R being freely generated by $\mathbf{P}^1$, the projective line.

- For $\mathcal{C} = \mathbf{Ab}$ (or, to have a more meaningful model category $\mathcal{C} = \mathbf{Ch}(\mathbf{Ab})$, chain complexes of abelian groups) and R being freely generated by $\mathbf{Z}$, the category $\mathbf{Sp}_{\mathbf{Z}}(\mathbf{Ab})$ is the category of FI-modules appearing throughout the work of Church on representation stability and homological stability.

The main theorem in [Chu+14] states that FI-modules over a Noetherian ring are a Noetherian category themselves. It is probable that the homotopical algebra performed in [PS14b] yields interesting results similar to this one. More generally, it seems interesting to revisit modular representation theory from the point of view of spectra in $\mathbf{F}_p$-modules. The reason for this is the fact that whenever the monoid R is generated by the monoidal unit 1, such as in the case of FI-modules above, there is a Quillen equivalence

$$\mathcal{C} \xrightarrow{\sim} \mathbf{Sp}_1^+(\mathcal{C}).$$

In other words, homological calculations can just as well be done in spectra. This answers Question 3.4.

We apply these excellent model-theoretic properties of spectra in several directions:

- We show that the axioms of Toën and Vezzosi needed to do homotopical algebra in their sense (i.e., the foundations of derived algebraic geometry on a model categorical level) are satisfied for spectra.

- We show that axioms of Goerss-Hopkins obstruction theory (adressing the liftability of commutative monoids in the stable homotopy category to E_∞-spectra) are satisfied for spectra in the above generality.

- We show how to construct strictly commutative motivic ring spectra. We highlight one application of this, namely to Deligne cohomology.

This latter application closes, in a sense, the circle to my earlier work.

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ARAKELOV MOTIVIC COHOMOLOGY I

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Abstract

This paper introduces a new cohomology theory for schemes of finite type over an arithmetic ring. The main motivation for this Arakelov-theoretic version of motivic cohomology is the conjecture on special values of L-functions and zeta functions formulated by the second author. Taking advantage of the six functors formalism in motivic stable homotopy theory, we establish a number of formal properties, including pullbacks for arbitrary morphisms, pushforwards for projective morphisms between regular schemes, localization sequences, h-descent. We round off the picture with a purity result and a higher arithmetic Riemann-Roch theorem.

In a sequel to this paper, we relate Arakelov motivic cohomology to classical constructions such as arithmetic K and Chow groups and the height pairing.

1. Introduction

For varieties over finite fields, we have very good cohomological tools for understanding the associated zeta functions. These tools include ℓ-adic cohomology, explaining the functional equation and the Riemann hypothesis, and Weil-étale cohomology, which allows for precise conjectures and some partial results regarding the "special values", i.e., the vanishing orders and leading Taylor coefficients at integer values. The conjectural picture for zeta functions of schemes X of finite type over $\operatorname{Spec}\mathbb{Z}$ is less complete. Deninger envisioned a cohomology theory explaining the Riemann hypothesis, and Flach and Morin have developed the Weil-étale cohomology describing special values of zeta functions of regular projective schemes over $\mathbb{Z}$ at $s = 0$ [Den94, FM12, Mor11].

In [Sch13], the second author proposed a new conjecture, which describes the special values of all zeta functions and L-functions of geometric origin, up to a rational factor. It is essentially a unification of classical conjectures of Beilinson, Soulé and Tate, formulated in terms of the recent Cisinski-Déglise

theory of triangulated categories of motives over $\mathbb{Z}$. This conjecture is formulated in terms of a new cohomology theory for schemes of finite type over $\mathbb{Z}$. The purpose of this paper is to construct this cohomology theory and establish many of its properties.

This cohomology theory, which we call *Arakelov motivic cohomology*, is related to motivic cohomology, roughly in the same way as arithmetic Chow groups relate to ordinary Chow groups or as arithmetic K-theory relates to algebraic K-theory. The key principle for cohomology theories of this type has always been to connect some algebraic data, such as the algebraic K-theory, with an analytical piece of information, chiefly Deligne cohomology, in the sense of long exact sequences featuring the Beilinson regulator map between the two and a third kind of group measuring the failure of the regulator to be an isomorphism. This was suggested by Deligne and Soulé in the 1980s. Beilinson also expressed the idea that the "boundary" of an algebraic cycle on a scheme over $\mathbb{Z}$ should be a Deligne cohomology class [Beĭ87]. Gillet, Roessler, and Soulé then started developing a theory of arithmetic Chow groups [GS90b, GS90c, GS90a, Sou92], arithmetic K_0-theory and an arithmetic Riemann-Roch theorem [Roe99, GRS08]. Burgos and Wang [Bur94, Bur97, BW98] extended some of this to not necessarily projective schemes and gave an explicit representation of the Beilinson regulator. More recently, Goncharov gave a candidate for higher arithmetic Chow groups for complex varieties, Takeda developed higher arithmetic K-theory, while Burgos and Feliu constructed higher arithmetic Chow groups for varieties over arithmetic fields [Gon05, Tak05, BGF12]. The analogous amalgamation of topological K-theory and Deligne cohomology of smooth manifolds is known as smooth K-theory [BS09].

In a nutshell, these constructions proceed by representing the regulator as a map of appropriate complexes. Then one defines, say, arithmetic K-theory to be the cohomology of the cone of this map. Doing so, however, requires a good command of the necessary complexes, which so far has prevented extending higher arithmetic Chow groups to schemes over $\mathbb{Z}$ and also requires one to manually construct homotopies whenever a geometric construction is to be done, for example the pushforward. The idea of this work is to both overcome these hurdles and enhance the scope of these techniques by introducing a *spectrum*, i.e., an object in the stable homotopy category of schemes, representing the sought cohomology theory.

This paper can be summarized as follows: let S be a regular scheme of finite type over a number field F, a number ring $\mathcal{O}_F$, $\mathbb{R}$, or $\mathbb{C}$. In the stable homotopy category $\mathbf{SH}(S)$ (cf. Section 2.1) there is a ring spectrum H_D representing Deligne cohomology with real coefficients of smooth schemes X/S

(Theorem 3.6). We *define* (cf. Definition 4.1) the Arakelov motivic cohomology spectrum $\widehat{\mathrm{H}_{\mathrm{B}}}$ as the homotopy fiber of the map

$$\mathrm{H}_{\mathrm{B}} \overset{\mathrm{id} \wedge 1_{\mathrm{H}_{\mathrm{D}}}}{\longrightarrow} \mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{D}}.$$

Here, H_{B} is Riou's spectrum representing the Adams eigenspaces in algebraic K-theory (tensored by $\mathbb{Q}$). Étale descent for H_{D} implies that the canonical map $\mathrm{H}_{\mathrm{D}} \to \mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{D}}$ is an isomorphism (Theorem 3.6), so there is a distinguished triangle

$$\widehat{\mathrm{H}_{\mathrm{B}}} \to \mathrm{H}_{\mathrm{B}} \to \mathrm{H}_{\mathrm{D}} \to \widehat{\mathrm{H}_{\mathrm{B}}}[1].$$

We define *Arakelov motivic cohomology* to be the theory represented by this spectrum, that is to say,

$$\widehat{\mathrm{H}}^n(M, p) := \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(M, \widehat{\mathrm{H}_{\mathrm{B}}}(p)[n])$$

for any $M \in \mathbf{SH}(S)$. Thus, there is a long exact sequence involving Arakelov motivic cohomology, motivic cohomology and Deligne cohomology (Theorem 4.5). Moreover, Arakelov motivic cohomology shares the structural properties known for motivic cohomology, for example a projective bundle formula, a localization sequence, and h-descent (Theorem 4.14). It also has the expected *functoriality*: pullback for arbitrary morphisms of schemes (or motives, Lemma 4.9) and pushforward along projective maps between regular schemes (Definition and Lemma 4.10). All of this can be modified by replacing H_{B} by BGL, the spectrum representing algebraic K-theory. The resulting Arakelov version is denoted $\widehat{\mathrm{BGL}}$ and the cohomology theory so obtained is denoted $\widehat{\mathrm{H}}^n(M)$.

We extend the motivic Riemann-Roch theorem given by Riou to arbitrary projective maps between regular schemes (Theorem 2.5), a statement that is of independent interest. We deduce a *higher arithmetic Riemann-Roch theorem* (Theorem 4.13) for the cohomology theories $\widehat{\mathrm{H}}^*(M, -)$ vs. $\widehat{\mathrm{H}}^*(M)$. It applies to smooth projective morphisms and for projective morphisms between schemes that are smooth over the base.

In the second part of this paper [Sch12], we will show how to relate the homotopy-theoretic construction of Arakelov motivic cohomology to the classical definitions of arithmetic K- and Chow groups. For example, the arithmetic K_0-groups $\widehat{\mathrm{K}}_0^T(X)$ defined by Gillet and Soulé [GS90c, Section 6] for a regular projective variety X (over a base S as above) sit in an exact sequence

$$K_1(X) \to \bigoplus_{p \geq 0} A^{p,p}(X)/(\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}) \to \widehat{\mathrm{K}}_0^T(X) \to K_0(X) \to 0,$$

where $A^{p,p}(X)$ is the group of real-valued (p, p)-forms ω on $X(\mathbb{C})$ such that $\mathrm{Fr}_\infty^* \omega = (-1)^p \omega$. The full arithmetic K-groups $\widehat{\mathrm{K}}_0^T(X)$ are not homotopy

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invariant and can therefore not be addressed using $\mathbb{A}^1$-homotopy theory. Instead, we consider the subgroup

$$\widehat{\mathrm{K}}_0(X) := \ker \left(\mathrm{ch} : \widehat{\mathrm{K}}_0^T(X) \to \bigoplus_{p \geq 0} A^{p,p}(X) \right).$$

For smooth schemes X/S, we show a canonical isomorphism

$$(1.1) \qquad\qquad \widehat{\mathrm{H}}^0(\mathrm{M}(X)) \cong \widehat{\mathrm{K}}_0(X)$$

and similarly for higher arithmetic K-theory, as defined by Takeda. The homotopy-theoretic approach taken yields a considerable simplification since it is no longer necessary to construct explicit homotopies between the complexes representing arithmetic K-groups, say. For example, the Adams operations on $\widehat{\mathrm{K}}_i(X)$ defined by Feliu [Fel10] were not known to induce a decomposition $\widehat{\mathrm{K}}_*(X)_\mathbb{Q} \cong \bigoplus_p \widehat{\mathrm{K}}_*(X)_\mathbb{Q}^{(p)}$. Using that the isomorphism (1.1) is compatible with Adams operations, this statement follows from the essentially formal analogue for $\widehat{\mathrm{H}}^*$. Moreover, (1.1) is shown to be compatible with the pushforwards on both sides in an important case. This implies that the height pairing on a smooth projective scheme X/S, $S \subset \mathrm{Spec}\,\mathbb{Z}$, is expressible in terms of the natural pairing of motivic homology and Arakelov motivic cohomology of the motive of X. According to the second author's conjecture, the L-values of schemes (or motives) over $\mathbb{Z}$ are given by the determinant of this pairing.

2. Preliminaries

In this section, we provide the motivic framework that we are going to work with in Sections 3 and 4: we recall the construction of the stable homotopy category $\mathbf{SH}(S)$ and some properties of the Cisinski-Déglise triangulated category of motives. In Section 2.3, we generalize Riou's formulation of the Riemann-Roch theorem to regular projective morphisms. This will then be used to derive a higher arithmetic Riemann-Roch theorem (Theorem 4.13). Finally, we recall the definition and basic properties of Deligne cohomology that are needed in Section 3 to construct a spectrum representing Deligne cohomology.

2.1. The stable homotopy category. This section sets the notation and recalls some results pertaining to the homotopy theory of schemes due to Morel and Voevodsky [MV99].

Let S be a Noetherian scheme. We only use schemes which are of finite type over $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. Unless explicitly mentioned otherwise, all morphisms of schemes are understood to be separated and of finite type. Let $\mathbf{Sm}/S$ be the

category of smooth schemes over S. The category of presheaves of pointed sets on this category is denoted $\mathbf{PSh}_\bullet := \mathbf{PSh}_\bullet(\mathbf{Sm}/S)$. We often regard a scheme $X \in \mathbf{Sm}/S$ as the presheaf (of sets) represented by X, and we write $X_+ := X \sqcup \{*\}$ for the associated pointed version. The projective line $\mathbb{P}^1_S$ is always viewed as pointed by ∞. The prefix $\Delta^{\mathrm{op}}-$ indicates simplicial objects in a category. The simplicial n-sphere is denoted S^n; this should not cause confusion with the base scheme S.

We consider the *pointwise* and the *motivic model structure* on the category $\Delta^{\mathrm{op}}(\mathbf{PSh}_\bullet)$ [Jar00, Section 1.1]. The latter is obtained by considering objects that are local with respect to projections $U \times \mathbb{A}^1 \to U$ and the Nisnevich topology. The corresponding homotopy categories will be denoted by $\mathbf{Ho}_{\mathrm{sect},\bullet}$ and $\mathbf{Ho}_\bullet$, respectively. The identity functor is a Quillen adjunction with respect to these two model structures.

The category $\mathbf{Spt} := \mathbf{Spt}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$ consists of symmetric $\mathbb{P}^1_S$-spectra, that is, sequences $E = (E_n)_{n \geq 0}$ of simplicial presheaves which are equipped with an action of the symmetric group S_n and bonding maps $\mathbb{P}^1 \wedge E_n \to E_{n+1}$ such that $(\mathbb{P}^1)^{\wedge m} \wedge E_n \to E_{n+m}$ is $S_n \times S_m$-equivariant (and the obvious morphisms). The functor $\Sigma^\infty_{\mathbb{P}^1} : \Delta^{\mathrm{op}}(\mathbf{PSh}_\bullet) \ni F \mapsto \left((\mathbb{P}^1)^{\wedge n} \wedge F\right)_{n \geq 0}$ (bonding maps are identity maps; S_n acts by permuting the factors $\mathbb{P}^1$) is left adjoint to $\Omega^\infty : (E_n) \mapsto E_0$. Often, we will not distinguish between a simplicial presheaf F and $\Sigma^\infty_{\mathbb{P}^1}(F)$.

The category $\mathbf{Spt}$ is endowed with the *stable model structure* [Jar00, Theorems 2.9, 4.15]. The corresponding homotopy category is denoted $\mathbf{SH}$ (or $\mathbf{SH}(S)$) and referred to as the *stable homotopy category* of smooth schemes over S. The pair $(\Sigma^\infty_{\mathbb{P}^1}, \Omega^\infty)$ is a Quillen adjunction with respect to the motivic model structure on $\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet$ and the stable model structures on $\mathbf{Spt}$. We sum up this discussion by saying that there are adjunctions of homotopy categories

(2.1) $$\mathbf{Ho}_{\mathrm{sect},\bullet} \leftrightarrows \mathbf{Ho}_\bullet \leftrightarrows \mathbf{SH}.$$

The stable homotopy categories are triangulated categories. We will use both the notation $M[p]$ and $M \wedge (S^1)^{\wedge p}$, $p \in \mathbb{Z}$, for the shift functor. Moreover, in $\mathbf{Ho}(S)$ there is an isomorphism $\mathbb{P}^1_S \cong S^1 \wedge (\mathbb{G}_{m,S}, 1)$. Thus, in $\mathbf{SH}(S)$, wedging with $\mathbb{G}_{m,S}$ is invertible as well, and we write $M(p)$ for $M \wedge (\mathbb{G}_{m,S})^{\wedge p}[-p]$, $p \in \mathbb{Z}$, for the *Tate twist*. For brevity, we also put

$$M\{p\} := M(p)[2p].$$

For any triangulated, compactly generated category $\mathcal{C}$ that is closed under coproducts, we let $\mathcal{C}_\mathbb{Q}$ be the full triangulated subcategory of those objects Y such that $\mathrm{Hom}_\mathcal{C}(-, Y)$ is a $\mathbb{Q}$-vector space. The inclusion $i : \mathcal{C}_\mathbb{Q} \subset \mathcal{C}$ has a right

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adjoint which will be denoted by $(-)_{\mathbb{Q}}$. The natural map $\operatorname{Hom}_{\mathcal{C}}(X, Y) \otimes \mathbb{Q} \to \operatorname{Hom}_{\mathcal{C}}(X, i(Y_{\mathbb{Q}})) = \operatorname{Hom}_{\mathcal{C}_{\mathbb{Q}}}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$ is an isomorphism if X is compact; see e.g. [Rio07, Appendix A.2]. In particular, we will use $\mathbf{SH}(S)_{\mathbb{Q}}$. Wherever convenient, we use the equivalence of this category with $\mathbf{D}_{\mathbb{A}^1}(S, \mathbb{Q})$, the homotopy category of symmetric $\mathbb{P}^1$-spectra of complexes of Nisnevich sheaves of $\mathbb{Q}$-vector spaces (with the Tate twist and $\mathbb{A}^1$ inverted) [CD09, 5.3.22, 5.3.37].

Given a morphism $f : T \to S$, the stable homotopy categories are connected by adjunctions:

$$(2.2) \qquad f^* : \mathbf{SH}(S) \rightleftarrows \mathbf{SH}(T) : f_*,$$

$$(2.3) \qquad f_! : \mathbf{SH}(T) \rightleftarrows \mathbf{SH}(S) : f^!,$$

$$(2.4) \qquad f_\sharp : \mathbf{SH}(T) \rightleftarrows \mathbf{SH}(S) : f^*.$$

For the last adjunction, f is required to be smooth. (2.2) also applies to morphisms which are not necessarily of finite type ([Ayo07, Scholie 1.4.2]; see also [CD09, 1.1.11, 1.1.13; 2.4.4., 2.4.10]).

2.2. Beilinson motives. Let S be a Noetherian scheme of finite dimension. The key to Beilinson motives (in the sense of Cisinski and Déglise) is the motivic cohomology spectrum $\mathrm{H}_{\mathrm{B},S}$ due to Riou [Rio07, IV.46, IV.72]. There is an object $\mathrm{BGL}_S \in \mathbf{SH}(S)$ representing algebraic K-theory in the sense that

$$(2.5) \qquad \operatorname{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, \mathrm{BGL}_S) = K_n(X)$$

for any regular scheme S and any smooth scheme X/S, functorially (with respect to pullback) in X. The $\mathbb{Q}$-localization $\mathrm{BGL}_{S,\mathbb{Q}}$ decomposes as

$$\mathrm{BGL}_{S,\mathbb{Q}} = \bigoplus_{p \in \mathbb{Z}} \mathrm{BGL}_S^{(p)}$$

such that the pieces $\mathrm{BGL}_S^{(p)}$ represent the graded pieces of the γ-filtration on K-theory:

$$(2.6) \qquad \operatorname{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, \mathrm{BGL}_S^{(p)}) \cong \operatorname{gr}_\gamma^p K_n(X)_{\mathbb{Q}}.$$

The *Beilinson motivic cohomology spectrum* H_{B} is defined by

$$(2.7) \qquad \mathrm{H}_{\mathrm{B},S} := \mathrm{BGL}_S^{(0)}$$

and the resulting Chern character map $\mathrm{BGL}_{S,\mathbb{Q}} \to \bigoplus_p \mathrm{H}_{\mathrm{B},S}\{p\}$ is denoted ch. The parts of the K-theory spectrum are related by periodicity isomorphisms

$$(2.8) \qquad \mathrm{BGL}_S^{(p)} = \mathrm{H}_{\mathrm{B},S}\{p\}.$$

For any map $f : T \to S$, not necessarily of finite type, there are natural isomorphisms

$$(2.9) \qquad f^* \mathrm{BGL}_S = \mathrm{BGL}_T, \quad f^* \mathrm{H}_{\mathrm{B},S} = \mathrm{H}_{\mathrm{B},T}.$$

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The following definition and facts are due to Cisinski and Déglise [CD09, Sections 12.3, 13.2]. By a result of Röndigs, Spitzweck and Ostvaer [RSØ10], $\mathrm{BGL}_S \in \mathbf{SH}(S)$ is weakly equivalent to a certain cofibrant strict ring spectrum BGL'_S, that is to say, a monoid object in the underlying model category $\mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$. In the same vein, $\mathrm{H}_{\mathrm{B},S}$ can be represented by a strict commutative monoid object $\mathrm{H}'_{\mathrm{B},S}$ [CD09, Cor. 14.2.6]. The model structures on the subcategory of $\mathbf{Spt}^{\mathbb{P}^1}$ of BGL'_S- and $\mathrm{H}'_{\mathrm{B},S}$-modules are endowed with model structures such that the forgetful functor is Quillen right adjoint to smashing with BGL'_S and $\mathrm{H}'_{\mathrm{B},S}$, respectively. The homotopy categories are denoted $\mathbf{DM}_{\mathrm{BGL}}(S)$ and $\mathbf{DM}_{\mathrm{B}}(S)$, respectively. Objects in $\mathbf{DM}_{\mathrm{B}}(S)$ will be referred to as *motives* over S. We have adjunctions

$$(2.10) \qquad - \wedge \mathrm{BGL}_S : \mathbf{SH}(S) \leftrightarrows \mathbf{DM}_{\mathrm{BGL}}(S) : \mathrm{forget}$$

$$(2.11) \qquad - \wedge \mathrm{H}_{\mathrm{B},S} : \mathbf{SH}(S)_{\mathbb{Q}} \leftrightarrows \mathbf{DM}_{\mathrm{B}}(S) : \mathrm{forget}.$$

There is a canonical functor from the localization of $\mathbf{SH}(S)_{\mathbb{Q}}$ by all H_{B}-acyclic objects E (i.e., those satisfying $E \otimes \mathrm{H}_{\mathrm{B},S} = 0$) to $\mathbf{DM}_{\mathrm{B}}(S)$. This functor is an equivalence of categories, which shows that the above definition is independent of the choice of $\mathrm{H}'_{\mathrm{B},S}$. This also has the consequence that the forgetful functor $\mathbf{DM}_{\mathrm{B}}(S) \to \mathbf{SH}(S)_{\mathbb{Q}}$ is fully faithful [CD09, Prop. 14.2.8], which will be used in Section 4.1. All this stems from the miraculous fact that the multiplication map $\mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{B}} \to \mathrm{H}_{\mathrm{B}}$ is an isomorphism.

Motivic cohomology of any object M in $\mathbf{SH}(S)_{\mathbb{Q}}$ is defined as

$$(2.12) \qquad \mathrm{H}^n(M,p) := \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(M, \mathrm{H}_{\mathrm{B}}(p)[n])$$

$$\overset{(2.11)}{=} \mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(S)}(M \wedge \mathrm{H}_{\mathrm{B},S}, \mathrm{H}_{\mathrm{B},S}(p)[n]).$$

The adjunctions (2.10), (2.11) are morphisms of *motivic categories* [CD09, Def. 2.4.45], which means in particular that the functors $f_\sharp$, f_*, f^*, $f_!$ and $f^!$ of (2.2), (2.3), (2.4) on $\mathbf{SH}(-)$ can be extended to ones on $\mathbf{DM}_{\mathrm{BGL}}(-)$ and $\mathbf{DM}_{\mathrm{B}}(-)$ in a way that is compatible with these adjunctions [CD09, 13.3.3, 14.2.11]. For $\mathbf{DM}_{\mathrm{B}}(S)$ this can be rephrased by saying that these functors preserve the subcategories $\mathbf{DM}_{\mathrm{B}}(-) \subset \mathbf{SH}(-)_{\mathbb{Q}}$.

For any smooth quasi-projective morphism $f : X \to Y$ of constant relative dimension n and any $M \in \mathbf{DM}_{\mathrm{B}}(Y)$, we have the *relative purity* isomorphism (functorial in M and f)

$$(2.13) \qquad f^! M \cong f^* M \{n\}.$$

For example, $f^! \mathrm{H}_{\mathrm{B},Y} \cong \mathrm{H}_{\mathrm{B},X}\{n\}$. This is due to Ayoub; see e.g. [CD09, 2.4.21].

For any closed immersion $i : X \to Y$ between two regular schemes X and Y with constant relative codimension n, there are *absolute purity* isomorphisms [CD09, 13.6.3, 14.4.1]

$$(2.14) \qquad i^! \mathrm{H}_{\mathrm{B},Y} \cong \mathrm{H}_{\mathrm{B},X}\{-n\}, \quad i^! \mathrm{BGL}_Y \cong \mathrm{BGL}_X.$$

Definition 2.1. Let $f : X \to S$ be any map of finite type. We define the *motive* of X over S to be

$$\mathrm{M}(X) := \mathrm{M}_S(X) := f_! f^! \mathrm{H}_{\mathrm{B},S} \in \mathbf{DM}_{\mathrm{B}}(S).$$

Remark 2.2. In [CD09, 1.1.34] the motive of a smooth scheme $f : X \to S$ is defined as $f_\sharp f^* \mathrm{H}_{\mathrm{B},S}$. These two definitions agree up to functorial isomorphism: we can assume that f is of constant relative dimension d. By relative purity, the functors $f^!$ and $f^*\{d\}$ are isomorphic. Thus their left adjoints, namely $f_!$ and $f_\sharp\{-d\}$, agree too. Therefore, $f_! f^! \mathrm{H}_{\mathrm{B},S} = f_! f^* \mathrm{H}_{\mathrm{B},S}\{d\} = f_\sharp f^* \mathrm{H}_{\mathrm{B},S}$.

Definition 2.3. A map $f : X \to Y$ of S-schemes is a *locally complete intersection (l.c.i.) morphism* if both X and Y are regular and, for simplicity of notation, of constant dimension and if

$$f = p \circ i : X \xrightarrow{i} X' \xrightarrow{p} Y$$

where i is a closed immersion and p is smooth. Note that this implies that X' is regular. If there is such a factorization with $p : X' = \mathbb{P}_Y^n \to Y$ the projection, we call f a *regular projective* map.

We shall write $\dim f := \dim X - \dim Y$ for any map $f : X \to Y$ of finite-dimensional schemes.

Example 2.4. Let $f = p \circ i$ be an l.c.i. morphism. Absolute purity for i (2.14), relative purity for p, and the periodicity isomorphism $\mathrm{BGL} \cong \mathrm{BGL}\{1\}$ give rise to isomorphisms

$$f^! \mathrm{H}_{\mathrm{B},S} \cong f^* \mathrm{H}_{\mathrm{B},S}\{\dim(f)\}, \quad f^! \mathrm{BGL}_S \cong f^* \mathrm{BGL}_S.$$

Let $f : X \to Y$ be a projective regular map. Recall the *trace map* in $\mathbf{SH}(Y)$:

$$(2.15) \qquad \mathrm{tr}_f^{\mathrm{BGL}} : f_* \mathrm{BGL}_X = p_* i_* i^* \mathrm{BGL}_{X'} \overset{(2.14)}{\to} p_* \mathrm{BGL}_{X'} \to \mathrm{BGL}_Y,$$

constructed in [CD09, 13.7.3]. This is not an abuse of notation insofar as $\mathrm{tr}_f^{\mathrm{BGL}}$ is independent of the choice of the factorization. This is shown by adapting [Dég08, Lemma 5.11] to the case where all schemes in question are merely regular.

The trace map for H_B is defined as the composition

$$(2.16) \qquad \mathrm{tr}_f^B : f_* f^* H_{B,Y}\{\dim f\} \rightarrowtail f_* f^* \mathrm{BGL}_{\mathbb{Q},Y} \xrightarrow{\mathrm{tr}_f^{\mathrm{BGL}}} \mathrm{BGL}_{\mathbb{Q},Y} \twoheadrightarrow H_{B,Y}.$$

In case $f = i$, this is the definition of [CD09, Section 14.4].

Given another regular projective map g, the composition $g \circ f$ is also of this type. The trace maps are functorial: the composition

$$f^* g^* \mathrm{BGL} \xrightarrow{\mathrm{tr}_f^{\mathrm{BGL}}} f^! g^* \mathrm{BGL} \xrightarrow{f^! \mathrm{tr}_g^{\mathrm{BGL}}} f^! g^! \mathrm{BGL}$$

agrees with $\mathrm{tr}_{g \circ f}^{\mathrm{BGL}}$ and similarly with $\mathrm{tr}_?^B$. This can be deduced from the independence of the factorization; cf. [Dég08, Prop. 5.14].

By construction, for any smooth map $f : Y' \to Y$, the induced map $\mathrm{Hom}(f_\sharp f^* S^0, \mathrm{tr}_f^{\mathrm{BGL}}[-n]) : K_n(X') \to K_n(Y')$ is the K-theoretic pushforward along $f' : X' := X \times_Y Y' \to Y'$ [CD09, 13.7.3]. Similarly, $\mathrm{Hom}(f_\sharp f^* S^0, \mathrm{tr}_f^B[-n](p))$ is the pushforward $K_n(X')_{\mathbb{Q}}^{(p)} \to K_n(Y')_{\mathbb{Q}}^{(p)}$. Indeed, the pushforward on the Adams graded pieces of K-theory is defined as the induced map of the graded homomorphism f'_* on K-theory [FL85, V.6.4]. The adjoint maps

$$\mathrm{BGL}_X = f^* \mathrm{BGL}_Y \to f^! \mathrm{BGL}_Y, \quad f_* f^* \mathrm{BGL}_Y \to \mathrm{BGL}_Y$$

will also be denoted $\mathrm{tr}_f^{\mathrm{BGL}}$ and similarly with tr_f^B.

2.3. The Riemann-Roch theorem. We now turn to a motivic Riemann-Roch theorem, which will imply an arithmetic Riemann-Roch theorem for Arakelov motivic cohomology (Theorem 4.13). It generalizes the statement given by Riou for smooth morphisms [Rio10, Theorem 6.3.1] to regular projective maps. Independently, F. Déglise has obtained a similar result [Dég11]. Recall the *virtual tangent bundle* of a regular projective map $f = p \circ i : X \xrightarrow{i} X' \xrightarrow{p} Y$, $T_f := i^* T_p - C_{X/X'} \in K_0(X)$ (see e.g. [FL85, V.7]). Here $T_p := \Omega_{X'/Y}^\vee$ is the tangent bundle of p and $C_{X/X'} := (I/I^2)^\vee$ is the conormal sheaf associated to the ideal I defining i. As an element of $K_0(X)$, T_f does not depend on the factorization. Its Todd class $\mathrm{Td}(T_f)$ is an element of $\bigoplus_{p \in \mathbb{Z}} K_0(X)_{\mathbb{Q}}^{(p)}$ (see e.g. [FL85, p. 20] for the general definition of Td; this is applied to the Chern character $\mathrm{ch} : K_0(-) \to \bigoplus_p K_0(-)_{\mathbb{Q}}^{(p)}$ [FL85, pp. 127, 146]). It is regarded as an endomorphism of $\bigoplus_{p \in \mathbb{Z}} H_{B,X}\{p\}$ via the natural identification $\bigoplus_{p \in \mathbb{Z}} K_0(X)_{\mathbb{Q}}^{(p)} = \mathrm{End}_{\mathbf{DM}_{\mathrm{BGL}}(X)_{\mathbb{Q}}}(\bigoplus_{p \in \mathbb{Z}} H_{B,X}\{p\})$.

Theorem 2.5 (Riemann-Roch). *Let $f : X \to Y$ be a regular projective map. The following diagram is a commutative diagram in $\mathbf{SH}(Y)_{\mathbb{Q}}$ (or, equivalently, in $\mathbf{DM}_{\mathrm{B}}(Y)$):*

$$
\begin{array}{ccc}
f_* f^* \mathrm{BGL}_{\mathbb{Q},Y} & \xrightarrow{\ \mathrm{tr}_f^{\mathrm{BGL}}\ } & \mathrm{BGL}_{\mathbb{Q},Y} \\
{\scriptstyle \mathrm{ch}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \mathrm{ch}} \downarrow {\scriptstyle \cong} \\
f_* f^* \mathrm{B}_Y \xrightarrow{\ f_* \,\mathrm{Td}(T_f)\ } f_* f^* \mathrm{B}_Y & \xrightarrow{\ \mathrm{tr}_f^{\mathrm{B}}\ } & \mathrm{B}_Y .
\end{array}
$$

Here, B_Y is shorthand for $\bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{B},Y}\{p\}$.

Proof. The statement is easily seen to be stable under composition of regular projective maps, so it suffices to treat the cases $f = p : \mathbb{P}_Y^n \to Y$ and $f = i : X \to \mathbb{P}_Y^n$ separately. The former case has been shown by Riou, so we can assume $f : X \to Y$ is a closed embedding of regular schemes. The classical Riemann-Roch theorem says that the map

$$
K_0(X)_{\mathbb{Q}} \to \bigoplus_p K_0(Y)_{\mathbb{Q}}^{(p)}, x \mapsto \mathrm{ch}\, f_*(x) - f_*(\mathrm{Td}(T_f) \cup \mathrm{ch}(x))
$$

vanishes. Viewing x as an element of $\mathrm{Hom}_{\mathbf{SH}(Y)_{\mathbb{Q}}}(S^0, f_* f^* \mathrm{BGL}_{\mathbb{Q},Y})$, this can be rephrased by saying that $x \mapsto \alpha_f \circ x$ is zero, where

$$
\alpha_f := \mathrm{ch}_X \circ \mathrm{tr}_f^{\mathrm{BGL}} - \mathrm{tr}_f^{\mathrm{B}} \circ f_* \,\mathrm{Td}(T_f) \circ f_* f^* \,\mathrm{ch}_Y \in \mathrm{Hom}(f_* f^* \mathrm{BGL}_{\mathbb{Q},Y}, \mathrm{B}_Y).
$$

To show $\alpha_f = 0$, we first reduce to the case where $f : X \to Y$ has a retraction, that is, a map $p : Y \to X$ such that $p \circ f = \mathrm{id}_X$. Then, we prove the theorem by reducing it to the classical Riemann-Roch theorem.

For the first step, recall the deformation to the normal bundle [FL85, IV.5]:

(2.17)

We have written $M := \mathrm{Bl}_{X \times \infty}(\mathbb{P}_Y^1)$ and $Y' := \mathbb{P}(C_{X/Y} \oplus \mathcal{O}_X)$, $\tilde{Y} := \mathrm{Bl}_X Y$ and $Y' + \tilde{Y}$ for the scheme defined by the sum of the two divisors. All schemes except $Y' + \tilde{Y}$ are regular; all maps except π and pr are closed immersions.

The diagram is commutative and every square in it is cartesian. The map f' has a retraction. We show

$$\alpha_{f'} = 0 \Rightarrow \alpha_f = 0$$

by indicating how to replace each argument in [FL85, proof of Theorem II.1.3], which shows $\alpha_{f'} \circ x = 0 \Rightarrow \alpha_f \circ x = 0$ for any x as above, in a manner that is independent of x.

The identity $f_*(x) = f_* i_0^* pr^*(x) = g^* F_* pr^*(x)$ is replaced by the commutativity of the following diagram of maps of (BGL-)motives, where $v := g \circ f = F \circ i_0$:

$$\begin{array}{ccc}
F_! F^! \mathrm{BGL}_M & \xrightarrow{\mathcal{O}_{\mathbb{P}^1_X} \in K_0(\mathbb{P}^1_X)} & \mathrm{BGL}_M \\
{\scriptstyle \mathcal{O}_X \in K_0(X)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{O}_Y \in K_0(Y)} \\
v_! v^! \mathrm{BGL}_M & \xrightarrow[\mathcal{O}_X \in K_0(X)]{} & g_! g^! \mathrm{BGL}_M
\end{array}$$

The maps are given by the indicated structural sheaves in $K_0(?)$, via the identifications of Hom-groups in $\mathbf{DM}_{\mathrm{BGL}}(Y)$ with K-theory. For example, the upper horizontal map is the adjoint map to the inverse of the trace map isomorphism $\mathrm{tr}_F^{\mathrm{BGL}} : F^* \mathrm{BGL} \to F^! \mathrm{BGL}$, which corresponds via absolute purity to $\mathcal{O}_{\mathbb{P}^1_X} \in K_0(\mathbb{P}^1_X) = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{BGL}}(Y)}(F_! F^! \mathrm{BGL}, \mathrm{BGL})$. The composition of the map given by $\mathcal{O}_{\mathbb{P}^1_X}$ and $\mathcal{O}_Y$ is given by their tensor product (viewed as $\mathcal{O}_M$-modules), that is, $\mathcal{O}_X$, so the diagram commutes. The same argument applies to $f_*'(x) = g'^* F_* pr^*(x)$.

The projection formula is [CD09, Theorem 2.4.50(v)]. The divisors Y and $Y' + \tilde{Y} \subset M$ are linearly equivalent, which implies $g_*(1) = g_*'(1) + s_*(1) \in K_0(M)_{\mathbb{Q}}^{(1)}$ [FL85, IV.(5.11), Prop. V.4.4]. This in turn is equivalent to the agreement of the following two elements of $\mathrm{Hom}(\mathrm{H}_{\mathrm{B},M}, \mathrm{H}_{\mathrm{B},M}\{-1\})$:

$$\mathrm{H}_{\mathrm{B},M} \overset{\mathrm{adj.}}{\to} g_* g^* \mathrm{H}_{\mathrm{B},M} \xrightarrow{g_! \mathrm{tr}_g^{\mathrm{B}}} g_! g^! \mathrm{H}_{\mathrm{B},M}\{-1\} \overset{\mathrm{adj.}}{\to} \mathrm{H}_{\mathrm{B},M}\{-1\}$$

and

$$\mathrm{H}_{\mathrm{B},M} \overset{\mathrm{adj.}}{\to} g_*' g'^* \mathrm{H}_{\mathrm{B},M} \oplus s_* s^* \mathrm{H}_{\mathrm{B},M} \xrightarrow{g_!' \mathrm{tr}_{g'}^{\mathrm{B}} \oplus s_! \mathrm{tr}_s^{\mathrm{B}}} g_!' g'^! \mathrm{H}_{\mathrm{B},M}\{-1\} \oplus s_! s^! \mathrm{H}_{\mathrm{B},M}\{-1\}$$
$$\overset{\mathrm{adj.}}{\to} \mathrm{H}_{\mathrm{B},M}\{-1\}.$$

Finally, the identity $s^* F_* pr^*(x) = 0$ is formulated independently of x using again base-change (and using that the motive of the empty scheme is zero). This finishes the first step.

Thus, we can assume that f has a retraction $p : Y \to X$. By [Rio10, Section 5, esp. 5.3.6; cf. the proof of 6.1.3.2], the obvious "evaluation" maps $\mathrm{Hom}(\mathrm{BGL}_{X,\mathbb{Q}}, \mathrm{BGL}_{X,\mathbb{Q}})$ injectively to

$$\prod_{i \in \mathbb{Z}, T \in \mathbf{Sm}/X} \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, \mathrm{BGL}_{X,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, \mathrm{BGL}_{X,\mathbb{Q}})\right).$$

The outer Hom denotes $\mathbb{Q}$-linear maps; the inner ones are morphisms in $\mathbf{SH}(X)_{\mathbb{Q}}$. There is an isomorphism $u : f^*\mathrm{BGL}_{\mathbb{Q},Y} \to f^!\text{Б}_Y$, for example the Chern class followed by the absolute purity isomorphism (Example 2.4). Appending u on both sides, we conclude that the evaluation maps $\mathrm{Hom}(f^*\mathrm{BGL}_{Y,\mathbb{Q}}, f^!\text{Б}_Y)$ into

$$\prod_{i,T} \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, f^*\mathrm{BGL}_{Y,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i} \wedge T_+, f^!\text{Б}_Y)\right).$$

For any $T \in \mathbf{Sm}/X$, consider the following cartesian diagram:

$$\begin{array}{ccccc}
T & \xrightarrow{f_T} & U & \xrightarrow{p_T} & T \\
\downarrow{\scriptstyle t} & & \downarrow & & \downarrow{\scriptstyle t} \\
X & \xrightarrow{f} & Y & \xrightarrow{p} & X.
\end{array}$$

Recall that $T \in \mathbf{SH}(X)$ is given by $t_\sharp t^* S^0$. Here $t_\sharp$ is left adjoint to t^*; cf. (2.4). Thus, the term simplifies to

$$\prod_{i,T} \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, t^*f^*\mathrm{BGL}_{Y,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, t^*f^!\text{Б}_Y)\right).$$

The diagram $X \to Y \to X$ is stable with respect to smooth pullback: f_T is also an embedding of regular schemes; p_T is a retract of f_T. Moreover, the trace map $\mathrm{tr}_f^{\mathrm{BGL}}$ behaves well with respect to smooth pullback, i.e., $t^*\mathrm{tr}_f^{\mathrm{BGL}} = \mathrm{tr}_{f_T}^{\mathrm{BGL}}$ and similarly for $\mathrm{tr}_{f_T}^{\text{Б}}$, $\mathrm{ch}_?$ and $\mathrm{Td}(T_?)$. Thus, it is sufficient to consider the case $T = X$. That is, we have to show that β_f, the image of α_f in

$$\prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, f^*\mathrm{BGL}_{Y,\mathbb{Q}}), \mathrm{Hom}((\mathbb{P}^1)^{\wedge i}, f^!\text{Б}_Y)\right)$$

$$= \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{Q}}\left(\mathrm{Hom}_{\mathbf{SH}(X)_{\mathbb{Q}}}((\mathbb{P}^1_X)^{\wedge i}, \mathrm{BGL}_{X,\mathbb{Q}}), \mathrm{Hom}_{\mathbf{SH}(Y)_{\mathbb{Q}}}((\mathbb{P}^1_Y)^{\wedge i}, f_*f^!\text{Б}_Y)\right)$$

is zero. The composition

$$\mathrm{Hom}((\mathbb{P}^1_Y)^{\wedge i}, f_!f^*\text{Б}_Y) \xrightarrow{\mathrm{tr}_f^{\text{Б}},\cong} \mathrm{Hom}((\mathbb{P}^1_Y)^{\wedge i}, f_!f^!\text{Б}_Y) \xrightarrow{\gamma_f} \mathrm{Hom}((\mathbb{P}^1_Y)^{\wedge i}, \text{Б}_Y)$$

is the pushforward $f_* : \bigoplus_{p \in \mathbb{Z}} K_0(X)_{\mathbb{Q}}^{(p)} \to \oplus K_0(Y)_{\mathbb{Q}}^{(p)}$, which is injective since $p_*f_* = \mathrm{id}$. Thus, the right hand adjunction map γ_f is also injective, and it is

sufficient to show $\gamma_f \circ \beta_f = 0$. For any $i \in \mathbb{Z}$,

$$
\begin{aligned}
\gamma_f \circ \beta_f \quad & \stackrel{\text{by def.}}{=} \quad (f_* \circ (- \cup \mathrm{Td}(T_f)) \circ \mathrm{ch}_X) - (\mathrm{ch}_Y \circ f_*) \\
& \stackrel{\text{RR}}{=} \quad 0 \\
& \in \quad \mathrm{Hom}_{\mathbb{Q}} \left(K_0(X)_{\mathbb{Q}}, \oplus K_0(Y)_{\mathbb{Q}}^{(p)} \right) \\
& = \quad \mathrm{Hom}_{\mathbb{Q}} \left(\mathrm{Hom}_{\mathbf{SH}(X)_{\mathbb{Q}}}((\mathbb{P}^1)^{\wedge i}, f^* \mathrm{BGL}_{Y,\mathbb{Q}}), \mathrm{Hom}_{\mathbf{SH}(Y)_{\mathbb{Q}}}((\mathbb{P}^1)^{\wedge i}, \text{Б}_Y) \right).
\end{aligned}
$$

The vanishing labeled RR is the classical Riemann-Roch theorem for f. $\qquad \square$

2.4. Deligne cohomology.

Definition 2.6 ([GS90a, 3.1.1.]). An *arithmetic ring* is a datum $(S, \Sigma, \mathrm{Fr}_\infty)$, where S is a ring, $\Sigma = \{\sigma_1, \ldots, \sigma_n : S \to \mathbb{C}\}$ is a set of embeddings of S into $\mathbb{C}$ and $\mathrm{Fr}_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ is a $\mathbb{C}$-antilinear involution (called *infinite Frobenius*) such that $\mathrm{Fr}_\infty \circ \sigma = \sigma$, where $\sigma = (\sigma_i)_i : S \to \mathbb{C}^\Sigma$. For simplicity, we suppose that $S_\eta := S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Q}$ is a field. If S happens to be a field itself, we refer to it as an *arithmetic field*. For any scheme X over an arithmetic ring S, we write

$$X_{\mathbb{C}} := X \times_{S,\sigma} \mathbb{C}^\Sigma$$

and $X(\mathbb{C})$ for the associated complex-analytic space (with its classical topology). We also write $\mathrm{Fr}_\infty : X_{\mathbb{C}} \to X_{\mathbb{C}}$ for the pullback of infinite Frobenius on the base.

The examples to have in mind are the spectra of number rings, number fields, $\mathbb{R}$ or $\mathbb{C}$, equipped with the usual finite set Σ of complex embeddings and $\mathrm{Fr}_\infty : (z_v)_{v \in \Sigma} \mapsto (\overline{z_{\overline{v}}})_v$.

We recall the properties of Deligne cohomology that we need in the sequel. In order to construct a spectrum representing Deligne cohomology in Section 3 we recall Burgos' explicit complex whose cohomology groups identify with Deligne cohomology. In the remainder of this subsection, X/S is a smooth scheme (of finite type) over an arithmetic field.

Definition 2.7 ([Bur97, Def. 1.2, Thm. 2.6]). Let $E^*(X(\mathbb{C}))$ be the following complex:

$$(2.18) \qquad E^*(X(\mathbb{C})) := \varinjlim E^*_{\overline{X}(\mathbb{C})}(\log D(\mathbb{C})),$$

where the colimit is over the (directed) category of smooth compactifications $\overline{X}$ of X such that $D := \overline{X} \backslash X$ is a divisor with normal crossings. The complex $E^*_{\overline{X}(\mathbb{C})}(\log D(\mathbb{C}))$ is the complex of C^∞-differential forms on $\overline{X}(\mathbb{C})$ that have at most logarithmic poles along the divisor (see [Bur97] for details). We write $E^*(X) \subset E^*(X(\mathbb{C}))$ for the subcomplex of elements fixed under the $\overline{\mathrm{Fr}_\infty^*}$-action. Forms in $E^*(X)$ that are fixed under complex conjugation are

referred to as real forms and denoted $E_{\mathbb{R}}^*(X)$. As usual, a twist is written as $E_{\mathbb{R}}^*(X)(p) := (2\pi i)^p E_{\mathbb{R}}^*(X) \subset E^*(X)$. The complex $E^*(X)$ is filtered by

$$F^p E^*(X) := \bigoplus_{a \geq p, a+b=*} E^{a,b}(X).$$

Let $\mathrm{D}^*(X, p)$ be the complex defined by

$$\mathrm{D}^n(X, p) := \begin{cases} E_{\mathbb{R}}^{2p+n-1}(X)(p-1) \cap \bigoplus_{a+b=2p+n-1, a, b<p} E^{a,b}(X), & n < 0, \\ E_{\mathbb{R}}^{2p+n}(X)(p) \cap \bigoplus_{a+b=2p+n, a, b \geq p} E^{a,b}(X), & n \geq 0. \end{cases}$$

The differential $d_{\mathrm{D}}(x)$, $x \in \mathrm{D}^n(X, p)$, is defined as $-\mathrm{proj}(dx)$ $(n < -1)$, $-2\partial\bar{\partial}x$ $(n = -1)$, and dx $(n \geq 0)$. Here d is the standard exterior derivative, and proj denotes the projection onto the space of forms of the appropriate bidegrees. We also set

$$\mathrm{D} := \bigoplus_{p \in \mathbb{Z}} \mathrm{D}(p).$$

The pullback of differential forms turns D into complexes of presheaves on $\mathbf{Sm}/S$. *Deligne cohomology* (with real coefficients) of X is defined as

$$\mathrm{H}_{\mathrm{D}}^n(X, p) := \mathrm{H}^{n-2p}(\mathrm{D}(p)(X)).$$

For a scheme X over an arithmetic ring such that $X_\eta := X \times_S S_\eta$ is smooth (possibly empty), we set $\mathrm{H}_{\mathrm{D}}^n(X, p) := \mathrm{H}_{\mathrm{D}}^n(X_\eta)$.

Recall that a complex of presheaves $X \mapsto F_*(X)$ on $\mathbf{Sm}/S$ is said to have *étale descent* if for any $X \in \mathbf{Sm}/S$ and any étale cover $f : Y \to X$ the canonical map

$$F_*(X) \to \mathrm{Tot}(F_*(\dots \to Y \times_X Y \to Y))$$

is a quasi-isomorphism. The right hand side is the total complex defined by means of the direct product. (Below we apply it to $F_*(X) = \mathrm{D}(p)(X)$, which is a complex bounded by the dimension of X, so that it agrees with the total complex defined using the direct sum in this case.) The total complex is applied to the Čech nerve. At least if F is a complex of presheaves of $\mathbb{Q}$-vector spaces, this is equivalent to the requirement that

$$F_*(X) \to \mathrm{Tot}(F_*(\mathcal{Y}))$$

is a quasi-isomorphism for any étale hypercover $\mathcal{Y} \to X$. Indeed the latter is equivalent to F_* satisfying Galois descent (as in (2.26)) and Nisnevich descent in the sense of hypercovers. The latter is equivalent to the one in the sense of Čech nerves by the Morel-Voevodsky criterion (see e.g. [CD09, Theorem 3.3.2]).

Theorem 2.8.

(i) *The previous definition of Deligne cohomology agrees with the classical one (for which see e.g.* [EV88]*). In particular, there is a long exact sequence*

(2.19) $\quad \mathrm{H}^n_{\mathrm{D}}(X, p) \to \mathrm{H}^n(X(\mathbb{C}), \mathbb{R}(p))^{(-1)^p} \to (\mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}})/F^p \mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{C}}))^{\mathrm{Fr}_\infty}$

$$\to \mathrm{H}^{n+1}_{\mathrm{D}}(X, p)$$

involving Deligne cohomology, the $(-1)^p$-*eigenspace of the* $\overline{\mathrm{Fr}^*_\infty}$ *action on Betti cohomology, and the* Fr_∞-*invariant subspace of de Rham cohomology modulo the Hodge filtration.*

(ii) *The complex* $\mathrm{D}(p)$ *is* homotopy invariant *in the sense that the projection map* $X \times \mathbb{A}^1 \to X$ *induces a quasi-isomorphism* $\mathrm{D}(\mathbb{A}^1 \times X) \to \mathrm{D}(X)$ *for any* $X \in \mathbf{Sm}/S$.

(iii) *There is a functorial* first Chern class *map*

(2.20) $$c_1 : \mathrm{Pic}(X) \to \mathrm{H}^2_{\mathrm{D}}(X, 1).$$

(iv) *The complex* D *is a unital differential bigraded* $\mathbb{Q}$-*algebra which is associative and commutative up to homotopy. The product of two sections will be denoted by* $a \cdot_{\mathrm{D}} b$. *The induced product on Deligne cohomology agrees with the classical product* $\cup$ *on these groups* [EV88, Section 3]. *Moreover, for a section* $x \in \mathrm{D}^0(X)$ *satisfying* $d_{\mathrm{D}}(x)(= dx) = 0$ *and any two sections* $y, z \in \mathrm{D}^*(X)$, *we have*

(2.21) $$x \cdot_{\mathrm{D}} (y \cdot_{\mathrm{D}} z) = (x \cdot_{\mathrm{D}} y) \cdot_{\mathrm{D}} z$$

and

(2.22) $$x \cdot_{\mathrm{D}} y = y \cdot_{\mathrm{D}} x.$$

(v) *Let* E *be a vector bundle of rank* r *over* X. *Let* $p : P := \mathbf{P}(E) \to X$ *be the projectivization of* E *with tautological bundle* $\mathcal{O}_P(-1)$. *Then there is an isomorphism*

(2.23) $$p^*(-) \cup c_1(\mathcal{O}_P(1))^{\cup i} : \bigoplus_{i=0}^{r-1} \mathrm{H}^{n-2i}_{\mathrm{D}}(X, p-i) \to \mathrm{H}^n_{\mathrm{D}}(P, p).$$

In particular the following Künneth-type formula holds:

(2.24) $$\mathrm{H}^n_{\mathrm{D}}(\mathbb{P}^1 \times X, p) \cong \mathrm{H}^{n-2}_{\mathrm{D}}(X, p-1) \oplus \mathrm{H}^n_{\mathrm{D}}(X, p).$$

(vi) *The complex of presheaves* $\mathrm{D}(p)$ *satisfies étale descent.*

Proof. (i) This explicit presentation of Deligne cohomology is due to Burgos [Bur97, Prop. 1.3.]. The sequence (2.19) is a consequence of this and the degeneration of the Hodge to de Rham spectral sequence. See e.g. [EV88, Cor. 2.10]. (ii) follows from (2.19) and the homotopy invariance of

Betti cohomology, de Rham cohomology, and, by functoriality of the Hodge filtration, homotopy invariance of $F^p \mathrm{H}^n_{\mathrm{dR}}(-)$. For (iii), see [BGKK07, Section 5.1.] (or [EV88, Section 7] for the case of a proper variety). (iv) is [Bur97, Theorem 3.3.].[1]

For (v), see e.g. [EV88, Prop. 8.5.].

(vi) This statement can be read off the existence of the absolute Hodge realization functor [Hub00, Cor. 2.3.5] (and also seems to be folklore). Since it is crucial for us in Theorem 3.6, we give a proof here. Let

$$\tilde{\mathrm{D}}^*(X, p) := \mathrm{cone}(E^*_{\mathbb{R}}(X)(p) \oplus F^p E^*(X) \overset{(+1, -1)}{\longrightarrow} E^*(X))[-1 + 2p].$$

By [Bur97, Theorem 2.6.], there is a natural (fairly concrete) homotopy equivalence between the complexes of presheaves $\tilde{\mathrm{D}}(p)$ and $\mathrm{D}(p)$. The descent statement is stable under quasi-isomorphisms of complexes of presheaves and cones of maps of such complexes. Therefore it is sufficient to show descent for the complexes $\underline{E^*_{\mathbb{R}}(-)(p)}$, $F^p E^*(-)$, $E^*(-)$. Taking invariants of these complexes under the $\overline{\mathrm{Fr}^*_\infty}$-action is an exact functor, so we can disregard that operation in the sequel. From now on, everything refers to the analytic topology; in particular we just write X for $X(\mathbb{C})$, etc. Let $j : X \to \overline{X}$ be an open immersion into a smooth compactification such that $D := \overline{X} \backslash X$ is a divisor with normal crossings. The inclusion

$$\Omega^*_{\overline{X}}(\log D) \subset E^*_X(\log D)$$

of holomorphic forms into C^∞-forms (both with logarithmic poles) yields quasi-isomorphisms of complexes of vector spaces

$$\mathrm{R}\Gamma \mathrm{R}j_*\mathbb{C} \to \mathrm{R}\Gamma \mathrm{R}j_*\Omega^*_X \leftarrow \mathrm{R}\Gamma \Omega^*_{\overline{X}}(\log D) \to \Gamma E^*_X(\log D)$$

that are compatible with both the real structure and the Hodge filtration [Bur94, Theorem 2.1.], [Del71, 3.1.7, 3.1.8]. Here $(\mathrm{R})\Gamma$ denotes the (total derived functor of the) global section functor on $\overline{X}$. The complex $E^*(X)$, whose cohomology is $\mathrm{H}^*(X, \mathbb{C})$, is known to satisfy étale descent [Hub00, Prop. 2.1.7]. This also applies to $E^*_{\mathbb{R}}(X)(p)$ instead of $E^*(X)$. (Alternatively for the former, see also [CD12, 3.1.3] for the étale descent of the algebraic de Rham complex Ω^*_X.)

---

[1]Actually, the product on $\mathrm{D}(X)$ is commutative on the nose. We shall only use the commutativity in the case stated in (2.22) and the associativity as in (2.21); cf. Definition and Lemma 3.3.

It remains to show the descent for $X \mapsto F^p E^*(X)$. Consider a distinguished square in $\mathbf{Sm}/S$,

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y,
\end{array}
$$

i.e., cartesian such that $Y' \to Y$ is an open immersion, X/Y is étale and induces an isomorphism $(X \backslash X')_{\mathrm{red}} \to (Y \backslash Y')_{\mathrm{red}}$. Then the sequence

$$(2.25) \quad \mathrm{H}^n(F^p E^*(Y)) \to \mathrm{H}^n(F^p E^*(Y')) \oplus \mathrm{H}^n(F^p E^*(X)) \to \mathrm{H}^n(F^p E^*(X'))$$
$$\to \mathrm{H}^{n+1}(F^p E^*(Y))$$

is exact: firstly, the direct limit in (2.18) is exact. Moreover, $\mathrm{H}^n(\Gamma(F^p E_{\overline{X}}(\log D)))$ maps injectively into $\mathrm{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D))$, and the image is precisely the p-th filtration step of the Hodge filtration on $\mathrm{H}^n(\overline{X}, \Omega^*_{\overline{X}}(\log D))$ $= \mathrm{H}^n(X, \mathbb{C})$. Similarly for X', etc., so that the exactness of (2.25) results from the sequence featuring the Betti cohomology groups of Y, $Y' \sqcup X$ and X', together with the strictness of the Hodge filtration [Del71, Th. 1.2.10]. This shows Nisnevich descent for the Hodge filtration. Secondly, for any scheme X and a Galois cover $Y \to X$ with group G, the pullback map into the G-invariant subspace

$$(2.26) \qquad\qquad \mathrm{H}^n(F^p E^*(X)) \to \mathrm{H}^n(F^p E^*(Y)^G)$$

is an isomorphism. Indeed, a similar statement holds for $E^*(-)$ instead of $F^p E^*(-)$. We work with $\mathbb{Q}$-coefficients, so taking G-invariants is an exact functor; hence $\mathrm{H}^n(F^p E^*(Y)^G) = (\mathrm{H}^n(F^p E^*(Y)))^G = (F^p \mathrm{H}^n_{\mathrm{dR}}(Y))^G = F^p(\mathrm{H}^n_{\mathrm{dR}}(Y)^G)$, the last equality by functoriality of the Hodge filtration. Then, again using the strictness of the Hodge filtration, the claim follows. Hence the presheaf $X \mapsto F^p E^*(X)$ has étale descent. $\qquad\square$

3. The Deligne cohomology spectrum

Let S be a smooth scheme (of finite type) over an arithmetic field (Definition 2.6). The aim of this section is to construct a ring spectrum in $\mathbf{SH}(S)$ which represents Deligne cohomology for smooth schemes X over S. The method is a slight variation of the method of Cisinski and Déglise used in [CD12] to construct a spectrum for any mixed Weil cohomology, such as algebraic or analytic de Rham cohomology, Betti cohomology, and (geometric) étale cohomology. The difference compared to their setting is that the Tate twist on Deligne cohomology groups is not an isomorphism of vector spaces.

In this section, all complexes of (presheaves of) abelian groups are considered with homological indexing: the degree of the differential is -1, and $C[1]$ is the complex whose n-th group is C_{n+1}. As usual, any cohomological complex is understood as a homological one by relabeling the indices. In particular, we apply this to (the restriction to $\mathbf{Sm}/S$ of) the complexes $\mathrm{D}(p)$, D defined in Definition 2.7, and let

$$(3.1) \qquad \mathrm{D}_n := \mathrm{D}^{-n} = \bigoplus_{p \in \mathbb{Z}} \mathrm{D}^{-n}(p).$$

In order to have a complex of simplicial presheaves (as opposed to a complex of abelian groups), we use the Dold-Kan equivalence

$$\mathcal{K} : \mathbf{Com}_{\geq 0}(\mathbf{Ab}) \rightleftarrows \Delta^{\mathrm{op}}(\mathbf{Ab}) : \mathcal{N}$$

between homological complexes concentrated in degrees ≥ 0 and simplicial abelian groups. We write $\tau_{\geq n}$ for the good truncation of a complex.

Definition 3.1. We write

$$\mathrm{D}_s := \mathcal{K}(\tau_{\geq 0}\mathrm{D}),$$

$$\mathrm{D}_s(p) := \mathcal{K}(\tau_{\geq 0}\mathrm{D}(p)).$$

Via the Alexander-Whitney map, the product on D transfers to a map

$$(3.2) \qquad \mathrm{D}_s(p) \wedge \mathrm{D}_s(p') \to \mathrm{D}_s(p + p').$$

Lemma 3.2. *For X smooth over S and any $k \geq 0$, $p \in \mathbb{Z}$ we have:*

$$(3.3) \qquad \mathrm{Hom}_{\mathbf{Ho}_\bullet}(S^k \wedge X_+, \mathrm{D}_s(p)) = \mathrm{H}_{\mathrm{D}}^{2p-k}(X, p)$$

and similarly for D_s.

Proof. In $\mathbf{Ho}_{\mathrm{sect},\bullet}$ (cf. Section 2.1 for the notation), the Hom-group reads

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{Ho}_{\mathrm{sect},\bullet}}(S^k \wedge X_+, \mathcal{K}(\tau_{\geq 0}(\mathrm{D}))) &= \pi_k \mathcal{K}(\tau_{\geq 0}(\mathrm{D}(X))) \\
&= \mathrm{H}_k(\tau_{\geq 0}(\mathrm{D}(X))) \\
&= \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{D}}^{2p-k}(X, p).
\end{aligned}
$$

We have used the fact that any simplicial abelian group is a fibrant simplicial set and the identification $\pi_n(A, 0) = \mathrm{H}_n(\mathcal{N}(A))$ for any simplicial abelian group.

The presheaf D_s is fibrant with respect to the motivic model structure, since Deligne cohomology satisfies Nisnevich descent and is $\mathbb{A}^1$-invariant by Theorem 2.8 (vi) and (ii). Thus the Hom-groups agree when taken in $\mathbf{Ho}_{\mathrm{sect},\bullet}$ and $\mathbf{Ho}$, respectively. $\qquad \square$

Definition and Lemma 3.3. *The* Deligne cohomology spectrum H_D *is the spectrum consisting of the* $D_s(p)$ *($p \geq 0$), equipped with the trivial action of the symmetric group* Σ_p. *We define the bonding maps to be the composition*

$$\sigma_p : \mathbb{P}^1_S \wedge D_s(p) \overset{c^* \wedge \mathrm{id}}{\to} D_s(1) \wedge D_s(p) \overset{(3.2)}{\to} D_s(p+1).$$

Here c^* is the map induced by $c := c_1(\mathcal{O}_{\mathbb{P}^1}(1), FS) \in D_0(1)(\mathbb{P}^1)$, the first Chern form of the bundle $\mathcal{O}(1)$ equipped with the Fubini-Study metric. This defines a symmetric $\mathbb{P}^1$-spectrum.

Define the unit map $1_D : \Sigma^\infty_{\mathbb{P}^1} S_+ \to H_D$ in degree zero by the unit of the DGA $D(0)$. In higher degrees, we put

$$(3.4) \qquad (1_D)_p : (\mathbb{P}^1)^{\wedge p} \overset{(c^*)^{\wedge p}}{\longrightarrow} D_s(1)^{\wedge p} \overset{\mu}{\longrightarrow} D_s(p).$$

Equivalently, $(1_D)_p := \sigma_{p-1} \circ (\mathrm{id}_{\mathbb{P}^1} \wedge (1_D)_{p-1})$. This map and the product map $\mu : H_D \wedge H_D \to H_D$ induced by (3.2) turn H_D into a commutative monoid object of $\mathbf{SH}(S)$, i.e., a commutative ring spectrum.

Proof. Recall that c is a $(1,1)$-form which is invariant under Fr^*_∞ and under complex conjugation, so c is indeed an element of $D_0(1)(\mathbb{P}^1)$. Its restriction to the point ∞ is zero for dimension reasons, so c is a pointed map $(\mathbb{P}^1, \infty) \to (D_0(1), 0)$. It remains to show that the map

$$
\begin{aligned}
(\mathbb{P}^1)^{\wedge m} \wedge D_s(n) &\overset{\mathrm{id}^{\wedge m-1} \wedge c^* \wedge \mathrm{id}}{\longrightarrow} (\mathbb{P}^1)^{\wedge m-1} \wedge D_s(1) \wedge D_s(n) \\
&\overset{(3.2)}{\to} (\mathbb{P}^1)^{\wedge m-1} \wedge D_s(n+1) \\
&\to \cdots \\
&\to D_s(m+n)
\end{aligned}
$$

is $\Sigma_m \times \Sigma_n$-equivariant, i.e., invariant under permuting the m wedge factors $\mathbb{P}^1$. Given some map $f : U \to (\mathbb{P}^1)^{\times m}$ with $U \in \mathbf{Sm}/S$, let $f_i : U \to \mathbb{P}^1$ be the i-th projection of f and $c_i := f_i^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Given some form $\omega \in D(n)(U)_*$, we have to check that the expression

$$(3.5) \qquad c_1 \cdot_D (c_2 \cdot_D (\ldots (c_m \cdot_D \omega)\ldots))$$

is invariant under permutation of the c_i. Here $\cdot_D$ stands for the product map (3.2). This holds before applying the Dold-Kan functor $\mathcal{K}$ (i.e. $(\mathbb{P}^1)^{\times m} \times D(n) \to D(n+m)$ is Σ_m-invariant) since the forms $c_i \in D_0(1)(U)$ are closed, so by Theorem 2.8(iv) the expression (3.5) is associative and commutative. The Alexander-Whitney map is symmetric in (simplicial) degree 0, i.e. $\mathcal{K}(D(p)) \wedge \mathcal{K}(D(p')) \to \mathcal{K}(D(p) \otimes D(p'))$ commutes with the permutation of the two factors when restricting to elements of degree 0. Moreover, it is associative in all degrees. As $c_i \in D_0(1)$, the previous argument carries over to the product on $D_s(-)$ instead of $D(-)$. This shows that H_D is a symmetric spectrum.

By Theorem 2.8(iv), the product on D is (graded) commutative and associative up to homotopy; thus the diagrams checking, say, the commutativity of $H_D \wedge H_D \to H_D$ do hold in $\mathbf{SH}(S)$. The details of that verification are omitted. $\square$

Remark 3.4.   (1) Consider the spectrum D' obtained in the same way as H_D, but replacing $D_s(p)$ by H_D. Then the obvious map $\phi : \bigoplus_{p \in \mathbb{Z}} H_D\{p\} \to D'$ is an isomorphism. To see that, it is enough to check that $\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma^\infty_{\mathbb{P}^1} X_+, -)$ yields an isomorphism when applied to ϕ. By the compactness of $S^n \wedge \Sigma^\infty_{\mathbb{P}^1} X_+$ in $\mathbf{SH}(S)$, this Hom-group commutes with the direct sum. Then the claim is trivial.

(2) Choosing another metric λ on $\mathcal{O}(1)$ in the above definition, the resulting Deligne cohomology spectrum would be weakly equivalent to H_D since the difference of the Chern forms $c_1(\mathcal{O}(1), FS) - c_1(\mathcal{O}(1), \lambda)$ lies in the image of $d_D : D_1(1) \to D_0(1)$; see e.g. [Jos06, Lemma 5.6.1].

Lemma 3.5. *The Deligne cohomology spectrum* H_D *is an* Ω-*spectrum (with respect to smashing with* $\mathbb{P}^1$).

Proof. We have to check that the adjoint map to σ_p (Definition and Lemma 3.3),

$$b_p : D_s(p) \to R\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, D_s(p+1)),$$

is a motivic weak equivalence. As $\mathbb{P}^1$ is cofibrant and $D_s(p+1)$ is fibrant, the non-derived $\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, D_s(p))$ is fibrant and agrees with $R\underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, D_s(p))$. The map is actually a sectionwise weak equivalence, i.e., an isomorphism in $\mathbf{Ho}_{\mathrm{sect},\bullet}(S)$. To see this, it is enough to check that the map

$$D_s(p)(U) \to \underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, D_s(p+1)(U))$$

is a weak equivalence of simplicial sets for all $U \in \mathbf{Sm}/S$ [MV99, 1.8., 1.10, p. 50]. The m-th homotopy group of the left hand side is $H_D^{2p-m}(U, p)$ (Lemma 3.2), while π_m of the right hand simplicial set identifies with those elements of $\pi_m(\underline{\mathrm{Hom}}(\mathbb{P}^1 \times U, D_s(p+1))) = H_D^{2(p+1)-m}(\mathbb{P}^1 \times U, p+1)$ which restrict to zero when applying the restriction to the point $\infty \to \mathbb{P}^1$. By the projective bundle formula (2.24), the two terms agree. $\square$

Theorem 3.6.

(i) *The ring spectrum* H_D *represents Deligne cohomology in* $\mathbf{SH}(S)$*: for any smooth scheme X over S and any $n, m \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathbf{SH}(S)}((S^1)^{\wedge n} \wedge (\mathbb{P}^1_S)^{\wedge m} \wedge \Sigma^\infty_{\mathbb{P}^1} X_+, H_D) = H_D^{-n-2m}(X, -m).$$

(See Section 2.1 for the meaning of $(S^1)^{\wedge n}$, $(\mathbb{P}^1_S)^{\wedge m}$ with negative exponents.)

(ii) *The Deligne cohomology spectrum* H_D *has a unique structure of an* $H_{B,S}$-*algebra, and* $\bigoplus_{p \in \mathbb{Z}} H_D\{p\}$ *has a unique structure of a* BGL_S-*algebra. In particular,* H_D *is an object in* $\mathbf{DM}_B(S)$, *so that (*i*) and (2.11) yield a natural isomorphism* $\mathrm{Hom}_{\mathbf{DM}_B(S)}(M_S(X), H_D(p)[n]) = H_D^n(X, p)$ *for any smooth* X/S.

(iii) *The map* $\mathrm{id}_D \wedge 1_{H_B} : H_D \to H_D \wedge H_B$ *is an isomorphism in* $\mathbf{SH}(S)_{\mathbb{Q}}$.

Definition 3.7. The maps induced by the unit of H_D are denoted $\rho_D : H_B \to H_D$ and $\mathrm{ch}_D : BGL \to \bigoplus_p H_D\{p\}$, respectively.

Proof. By Lemma 3.5, H_D is an Ω-spectrum. Thus (i) follows from Lemma 3.2.

(ii) By 3.3, H_D is a commutative ring spectrum. Recall the definition of étale descent for spectra and that for this it is sufficient that the individual pieces of the spectrum have étale descent [CD09, Def. 3.2.5, Cor. 3.2.18]. Thus, H_D satisfies étale descent by Theorem 2.8(vi). Moreover, H_D is orientable since $\mathrm{Hom}_{\mathbf{SH}(S)}(\mathbb{P}_S^\infty, H_D\{1\}) = \varprojlim_n \mathrm{Hom}(\mathbb{P}^n, H_D\{1\})$ by the Milnor short exact sequence (see e.g. [CD12, Cor. 2.2.8] for a similar situation). This term equals $H_D^2(\mathbb{P}^1, 1)$ by (2.23). Any object in $\mathbf{SH}(S)_{\mathbb{Q}}$ satisfying étale descent is an object of $\mathbf{DM}_B(S)$, i.e., an $H_{B,S}$-module [CD09, proof of 16.2.18]. If it is in addition an orientable ring spectrum, there is a unique $H_{B,S}$-algebra structure on it [CD09, Cor. 14.2.16]. This settles the claim for H_D. Secondly, the natural map (in $\mathbf{SH}(S)$)

$$BGL \to BGL_{\mathbb{Q}} \overset{(2.8)}{\cong} \bigoplus_{p \in \mathbb{Z}} H_B\{p\} \overset{\rho_D\{p\}}{\longrightarrow} \bigoplus_p H_D\{p\}$$

and the ring structure of $\oplus H_D\{p\}$ defines a BGL-algebra structure on $\oplus H_D\{p\}$. This uses that the isomorphism (2.8) is an isomorphism of monoid objects [CD09, 14.2.17]. The unicity of that structure follows from the unicity of the one on H_D and $\mathrm{Hom}_{\mathbf{SH}(S)}(BGL_{\mathbb{Q}}, \oplus H_D\{p\}) = \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(BGL_{\mathbb{Q}}, \oplus H_D\{p\})$, since H_D is a spectrum of $\mathbb{R}$- (a fortiori: $\mathbb{Q}$-) vector spaces.

(iii) follows from (ii), using [CD09, 14.2.16]. $\square$

4. Arakelov motivic cohomology

Let S be a regular scheme of finite type over an arithmetic ring B. The generic fiber $S_\eta := S \times_{\mathbb{Z}} \mathbb{Q} \to B_\eta := B \times_{\mathbb{Z}} \mathbb{Q}$ is smooth, since B_η is a field (by Definition 2.6). We now define the Arakelov motivic cohomology spectrum $\widehat{H_{B,S}}$ which glues, in a sense, the Deligne cohomology spectrum $H_D \in \mathbf{SH}(S_\eta)$ (Section 3) with the Beilinson motivic cohomology spectrum $H_{B,S}$ (2.7). Parallelly, we do a similar construction with BGL_S instead of $H_{B,S}$. Once this

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is done, the framework of the stable homotopy category and motives readily imply the existence of functorial pullbacks and pushforwards for Arakelov motivic cohomology (Section 4.2). We also prove a higher arithmetic Riemann-Roch theorem (Theorem 4.13) and deduce further standard properties, such as the projective bundle formula in Section 4.4.

4.1. Definition. Recall from Section 2.1 the category $\mathbf{Spt}(S) := \mathbf{Spt}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$ with the stable model structure. The resulting homotopy category is $\mathbf{SH}(S)$.

Definition 4.1. For any $A \in \mathbf{Spt}(S)$, we put

$$(4.1) \qquad \widehat{A} := \mathrm{hofib}_{\mathbf{Spt}(S)}\left(A \wedge \mathrm{QR}(S^0) \overset{\mathrm{id}\wedge\mathrm{QR}(1_\mathrm{D})}{\longrightarrow} A \wedge \mathrm{QR}\eta_*\mathrm{H_D} \right) \in \mathbf{Spt}(S).$$

Here, hofib stands for the homotopy fiber, $1_\mathrm{D} : S^0 \to \mathrm{H_D}$ is the unit map given in (3.4), and Q and R are the cofibrant and fibrant replacement functors in $\mathbf{Spt}(S)$. The map 1_D is a map in $\mathbf{Spt}(S_\eta)$, as opposed to a map in the homotopy category $\mathbf{SH}(S_\eta)$. Hence so is the map used in (4.1). We wrote QR here for clarity, but drop these below, given that the fibrant-cofibrant replacement of any spectrum is weakly equivalent to the original one.

We write $[\widehat{A}]$ for the image of $\widehat{A}$ in $\mathbf{SH}(S)$ (or $\mathbf{SH}(S)_\mathbb{Q}$) under the localization functor. Using the strict ring spectra $\mathrm{H}'_{\mathrm{B},S}$ and BGL'_S (Section 2.2), we define the *Arakelov motivic cohomology spectrum* $\widehat{\mathrm{H}_{\mathrm{B},S}}$ as

$$\widehat{\mathrm{H}_{\mathrm{B},S}} := [\widehat{\mathrm{H}'_{\mathrm{B},S}}] \in \mathbf{SH}(S)_\mathbb{Q}$$

and similarly

$$\widehat{\mathrm{BGL}_S} := [\widehat{\mathrm{BGL}'}] \in \mathbf{SH}(S).$$

Theorem 4.2.

(i) *Given a morphism $f : A \to A'$ in $\mathbf{SH}$, there is a canonical morphism $[\widehat{f}] : [\widehat{A}] \to [\widehat{A'}]$ in $\mathbf{SH}$ which is an isomorphism if f is. In particular, the Chern character isomorphism $\mathrm{ch} : \mathrm{BGL}_{S,\mathbb{Q}} \cong \bigoplus_{p\in\mathbb{Z}} \mathrm{H}_{\mathrm{B},S}\{p\}$ gives rise to an isomorphism called* Arakelov Chern character,

$$(4.2) \qquad\qquad \widehat{\mathrm{ch}} : \widehat{\mathrm{BGL}}_{S,\mathbb{Q}} \cong \oplus\widehat{\mathrm{H}_{\mathrm{B},S}}\{p\}$$

in $\mathbf{SH}(S)_\mathbb{Q}$.

(ii) *If A is a strict ring spectrum, then $[\widehat{A}]$ is an A-module in a canonical way. In particular, $\widehat{\mathrm{H}_{\mathrm{B},S}}$ is in $\mathbf{DM}_\mathrm{B}(S)$ and $\widehat{\mathrm{BGL}_S}$ is an object of $\mathbf{DM}_{\mathrm{BGL}}(S)$.*

Proof. (i) We can represent f by a zig-zag of maps f_i and define $\widehat{f}$ to be the zig-zag of $\widehat{f_i} := \mathrm{hofib}(f_i \wedge (S^0 \overset{1_{H_D}}{\to} H_D))$. As any choice of the zig-zag represents the same given map $[f] : [A] \to [A']$ in $\mathbf{SH}(S)$, the resulting map $[\widehat{f}] : [\widehat{A}] \to [\widehat{A'}]$ is also independent of the choice of the zig-zag.

(ii) The map in (4.1) is a map of A-modules. Its homotopy fiber in the category of A-modules is an object $\widehat{A^{\mathbf{Mod}}} \in A - \mathbf{Mod}$. By the Quillen adjunction (2.11) and [Hir03, Theorem 19.4.5], $\widehat{A^{\mathbf{Mod}}}$ is weakly equivalent (in $\mathbf{Spt}$) to $\widehat{A}$. Therefore, the image of $[\widehat{A^{\mathbf{Mod}}}]$ in $\mathbf{SH}$ under the forgetful functor $\mathbf{Ho}(A - \mathbf{Mod}) \to \mathbf{SH}$ is isomorphic to $[\widehat{A}]$; i.e., the latter is canonically an A-module.   □

Remark 4.3.   (i) Theorem 4.2(i) shows that $\widehat{\mathrm{BGL}}$ does not depend on the choice of the spectrum representing BGL. In a similar vein, one can show that given a map $A \to A'$ of strict ring spectra (respecting the ring structure) that is also a weak equivalence, $[\widehat{A}]$ is mapped to $[\widehat{A'}]$ under the canonical equivalence of categories $- : \otimes_A^L A' : \mathbf{Ho}(A - \mathbf{Mod}) \to \mathbf{Ho}(A' - \mathbf{Mod})$. In this sense, the BGL-module structure on $\widehat{\mathrm{BGL}}$ does not depend on the choice of the strict ring spectrum. We will not use this fact, though.

(ii) We are mainly interested in gluing motivic cohomology with Deligne cohomology. However, nothing is special about Deligne cohomology. In fact, given some scheme $f : T \to S$ (not necessarily of finite type) and complexes of presheaves of $\mathbb{Q}$-vector spaces $D(p)$ on $\mathbf{Sm}/T$ satisfying the conclusion of Theorem 2.8(ii), (iii), (iv), (v) (actually (2.24) suffices), and (vi), everything could be done with $f_* D(p)$ instead of $\eta_* \mathrm{D}(p)$.

Definition 4.4. For any $M \in \mathbf{SH}(S)$, we define
$$\widehat{\mathrm{H}}^n(M) := \mathrm{Hom}_{\mathbf{SH}(S)}(M, \widehat{\mathrm{BGL}_S}[n]),$$
$$\widehat{\mathrm{H}}^n(M, p) := \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(M_{\mathbb{Q}}, \widehat{\mathrm{H}_{\mathrm{B}}}(p)[n]).$$
The latter is called *Arakelov motivic cohomology* of M. For any finite type scheme $f : X \to S$, we define Arakelov motivic cohomology of X as
$$\widehat{\mathrm{H}}^n(X/S, p) := \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(f_! f^! \Sigma_{\mathbb{P}^1}^{\infty} S^0, \widehat{\mathrm{H}_{\mathrm{B},S}}(p)[n])$$
and likewise
$$\widehat{\mathrm{H}}^n(X/S) := \mathrm{Hom}_{\mathbf{SH}(S)}(f_! f^! \Sigma_{\mathbb{P}^1}^{\infty} S^0, \widehat{\mathrm{BGL}_S}[n]).$$
Here $\Sigma_{\mathbb{P}^1}^{\infty} S^0$ is the infinite $\mathbb{P}^1$-suspension of the 0-sphere, i.e., the unit of the monoidal structure in $\mathbf{SH}$. When the base S is clear from the context, we will just write $\widehat{\mathrm{H}}^n(X, p)$ and $\widehat{\mathrm{H}}^n(X)$. See Theorem 4.13(i) for a statement concerning the independence of the base scheme S of the groups $\widehat{\mathrm{H}}^n(X/S)$.

Theorem 4.5.

(i) *For any $M \in \mathbf{SH}(S)$ there are long exact sequences relating Arakelov motivic cohomology to (usual) motivic cohomology (2.12) and, for appropriate motives, Deligne cohomology (Definition 2.7):*

(4.3)
$$\ldots \to \widehat{\mathrm{H}}^n(M,p) \to \mathrm{H}^n(M,p) \xrightarrow{\rho} \mathrm{Hom}_{\mathbf{SH}(S)}(M, \eta_* \mathrm{H}_{\mathrm{D}}(p)[n]) \to \widehat{\mathrm{H}}^{n+1}(M,p)$$

(4.4)
$$\ldots \to \widehat{\mathrm{H}}^n(M) \to \mathrm{H}^n(M) \xrightarrow{\mathrm{ch}} \mathrm{Hom}_{\mathbf{SH}(S)}(M, \oplus \eta_* \mathrm{H}_{\mathrm{D}}\{p\}[n]) \to \widehat{\mathrm{H}}^{n+1}(M) \ldots.$$

The maps ρ and ch agree with the one induced by ρ_{D} and ch_{D} (Definition 3.7).

(ii) *For any l.c.i. scheme X/S (Definition 2.3, for example $X = S$) we get exact sequences*

$$\cdots \to \widehat{\mathrm{H}}^n(X,p) \to K_{2p-n}(X)_{\mathbb{Q}}^{(p)} \to \mathrm{H}_{\mathrm{D}}^n(X,p) \to \widehat{\mathrm{H}}^{n+1}(X,p) \to \cdots,$$

(4.5)
$$\cdots \to \widehat{\mathrm{H}}^n(X) \to K_{-n}(X) \to \bigoplus_p \mathrm{H}_{\mathrm{D}}^{2p-n}(X,p) \to \widehat{\mathrm{H}}^{n+1}(X) \to \cdots.$$

(iii) *If $S' \xrightarrow{f} S$ is a scheme of positive characteristic over S, the obvious map $\widehat{\mathrm{H}}^n(f_* M, p) \to \mathrm{H}^n(f_* M, p)$ is an isomorphism for any $M \in \mathbf{SH}(S')$.*

(iv) *There is a functorial isomorphism*

(4.6)
$$\widehat{\mathrm{H}}^n(M) = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{BGL}}(S)}(\mathrm{BGL}_S \wedge M, \widehat{\mathrm{BGL}_S}),$$

where we view $\widehat{\mathrm{BGL}_S}$ as a BGL-module using Theorem 4.2(ii). A similar statement holds for $\mathrm{H}_{\mathrm{B},S}$. In addition, there is a canonical isomorphism $\widehat{\mathrm{H}}^n(M,p) = \widehat{\mathrm{H}}^n(M \wedge \mathrm{H}_{\mathrm{B},S}, p)$. For example, $\widehat{\mathrm{H}}^n(X,p) = \widehat{\mathrm{H}}^n(\mathrm{M}_S(X), p)$ for any X/S of finite type. For any compact object $M \in \mathbf{SH}(S)$, there is an isomorphism called the Arakelov Chern character*:*

(4.7)
$$\widehat{\mathrm{ch}} : \widehat{\mathrm{H}}^n(M) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{p \in \mathbb{Z}} \widehat{\mathrm{H}}^{n+2p}(M,p).$$

Proof. The long exact sequence in (i) follows from Theorem 3.6(iii), the projection formula $\mathrm{H}_{\mathrm{B}} \wedge \eta_* \mathrm{H}_{\mathrm{D}} = \eta_*(\mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{D}})$, and generalities on the homotopy fiber in stable model categories. Similarly, $\mathrm{BGL} \wedge \mathrm{H}_{\mathrm{D}}$ is canonically isomorphic, via the Chern class, to $\oplus \mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{D}}\{p\} \cong \oplus \mathrm{H}_{\mathrm{D}}\{p\}$. The agreement of ρ and ρ_{D} is also clear, since the H_{B}-module structure map $\mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{D}} \to \mathrm{H}_{\mathrm{D}}$ is inverse to $1_{\mathrm{B}} \wedge \mathrm{id}_{\mathrm{D}} : \mathrm{H}_{\mathrm{D}} \to \mathrm{H}_{\mathrm{B}} \wedge \mathrm{H}_{\mathrm{D}}$.

For (ii), we use (iv) and apply (i) to $\mathrm{M}_S(X)$ and $f_! f^! \mathrm{BGL}_S$, respectively where $f : X \to S$ is the structural map. In order to identify the motivic cohomology with the Adams eigenspace in K-theory, we use the adjunction (2.3) and the purity isomorphism for f (Example 2.4). To calculate

$\mathrm{Hom}(f_! f^! \mathrm{H}_{\mathrm{B},S}, \mathrm{H}_{\mathrm{D}})$, we can replace B by the arithmetic field $B_\eta := B \times_{\mathbb{Z}} \mathbb{Q}$. The scheme S is regular; thus $s : S \to B$ is smooth (of finite type). The same is true for the structural map $x : X \to B$. Now, combining the relative purity isomorphisms for x and for s, we get an isomorphism

$$
\begin{aligned}
f^! \mathrm{H}_{\mathrm{D}} &= f^! s^* \mathrm{H}_{\mathrm{D}} = f^! s^! \mathrm{H}_{\mathrm{D}}\{-\dim s\} \\
&= x^! \mathrm{H}_{\mathrm{D}}\{-\dim s\} = x^* \mathrm{H}_{\mathrm{D}}\{-\dim s + \dim x\} = f^* \mathrm{H}_{\mathrm{D}}\{\dim f\}.
\end{aligned}
$$

We conclude

$$
\begin{aligned}
\mathrm{Hom}_{\mathbf{SH}(S)}(f_! f^! \mathrm{H}_{\mathrm{B},S}, \mathrm{H}_{\mathrm{D}}(p)[n]) &= \mathrm{Hom}(f^! \mathrm{H}_{\mathrm{B},S}, f^! \mathrm{H}_{\mathrm{D}}(p)[n]) \\
&= \mathrm{Hom}(f^* \mathrm{H}_{\mathrm{B},S}\{\dim f\}, f^* \mathrm{H}_{\mathrm{D}}(p)[n]\{\dim f\}) \\
&= \mathrm{Hom}(\mathrm{H}_{\mathrm{B},X}, \mathrm{H}_{\mathrm{D}}(p)[n]) \\
&\overset{3.6}{=} \mathrm{H}_{\mathrm{D}}^{2p-n}(X, p).
\end{aligned}
$$

(iii) follows from localization. The first isomorphism in (iv) follows from (2.11). The second one uses in addition the full faithfulness of the forgetful functor $\mathbf{DM}_{\mathrm{B}} \to \mathbf{SH}_{\mathbb{Q}}$ (Section 2.2). The map $\widehat{\mathrm{ch}}$ is induced by (4.2). □

Remark 4.6. By (4.3), each group $\widehat{\mathrm{H}}^n(M)$ is an extension of a $\mathbb{Z}$-module by a quotient of a finite-dimensional $\mathbb{R}$-vector space by some $\mathbb{Z}$-module. Both $\mathbb{Z}$-modules are conjectured to be finitely generated in case $S = \mathrm{Spec}\,\mathbb{Z}$ and M compact (Bass conjecture). Similarly, the groups $\widehat{\mathrm{H}}^n(M, p)$ are extensions of $\mathbb{Q}$-vector spaces by groups of the form $\mathbb{R}^k$/some $\mathbb{Q}$-subspace. In particular, we note that the Arakelov motivic cohomology groups $\widehat{\mathrm{H}}^n(M, p)$ are typically infinite-dimensional (as $\mathbb{Q}$-vector spaces). However, one can redo the above construction using the spectrum $\mathrm{H}_{\mathrm{B}} \otimes \mathbb{R}$ instead of H_{B} to obtain *Arakelov motivic cohomology groups with real coefficients*, $\widehat{\mathrm{H}}^n(M, \mathbb{R}(p))$. These groups are real vector spaces of conjecturally finite dimension, with formal properties similar to those of $\widehat{\mathrm{H}}^n(M, p)$, and these are the groups needed in the second author's conjecture on ζ and L-values [Sch13].

Remark 4.7. In [Sch12, Theorem 6.1], we show that $\widehat{\mathrm{H}}^n(X)$ agrees with $\widehat{\mathrm{K}}_{-n}^T(X)$ for $n \leq -1$ and is a subgroup of the latter for $n = 0$. The group $\widehat{\mathrm{H}}^1(X) = \mathrm{coker}(K_0(X) \to \oplus \mathrm{H}_{\mathrm{D}}^{2p}(X, p))$ is related to the Hodge conjecture, which for any smooth projective $X/\mathbb{C}$ asserts the surjectivity of $K_0(X)_{\mathbb{Q}} \to \mathrm{H}_{\mathrm{D}}^{2p}(X, \mathbb{Q}(p))$ (Deligne cohomology with rational coefficients). For $n \geq 2$, $\widehat{\mathrm{H}}^n(X) = \oplus \mathrm{H}_{\mathrm{D}}^{2p+n-1}(X, p)$.

Example 4.8. We list the groups $\widehat{\mathrm{H}}^{-n} := \widehat{\mathrm{H}}^{-n}(\mathrm{Spec}\,\mathcal{O}_F)$ of a number ring $\mathcal{O}_F$. These groups and their relation to the Dedekind ζ-function are well-known; cf. [Sou92, III.4], [Tak05, p. 623]. For any $n \in \mathbb{Z}$, (4.5) reads

$$
\mathrm{H}_{\mathrm{D}}^0(X, n+1) \to \widehat{\mathrm{H}}^{-2n-1} \to K_{2n+1} \overset{\rho_*}{\to} \mathrm{H}_{\mathrm{D}}^1(X, n+1) \to \widehat{\mathrm{H}}^{-2n} \to K_{2n} \overset{\rho_*}{\to} \mathrm{H}_{\mathrm{D}}^0(X, n).
$$

In [Sch12, Theorem 5.7], we show that the map ρ_* induced by the BGL-module structure of $\oplus H_D\{p\}$ agrees with the Beilinson regulator. We conclude by Borel's work that $\widehat{H}^{-2n-1}$ is an extension of $(K_{2n+1})_{\mathrm{tor}}(= \mu_F$ if $n = 0)$ by $H_D^0(X, n+1)$ for $n \geq 0$. Moreover, for $n > 0$, $\widehat{H}^{-2n}$ is an extension of the finite group K_{2n} by a torus, i.e., a group of the form $\mathbb{R}^{s_n}/\mathbb{Z}^{s_n}$ for some s_n that can be read off (2.19). Finally, $\widehat{H}^0$ is an extension of the class group of F by a group $\mathbb{R}^{r_1+r_2-1}/\mathbb{Z}^{r_1+r_2-1} \oplus \mathbb{R}$.

For higher-dimensional varieties, the situation is less well-understood. For example, by Beilinson's, Bloch's, and Deninger's work we know that

$$K_{2n+2}(E)_{\mathbb{R}}^{(n+2)} \to H_D^2(E, n+2)$$

is surjective for $n \geq 0$, where E is a regular proper model of certain elliptic curves over a number field (for example a curve over $\mathbb{Q}$ with complex multiplication in case $n = 0$). We refer to [Nek94, Section 8] for references and further examples.

4.2. Functoriality. Let $f : X \to Y$ be a map of S-schemes. The structural maps of X/S and Y/S are denoted x and y, respectively. We establish the expected functoriality properties of Arakelov motivic cohomology. To define pullback and pushforward, we apply $\mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}}(-, \widehat{H_{\mathrm{B},S}})$ to appropriate maps, using (4.6).

Lemma 4.9. *There is a functorial pullback*

$$f^* : \widehat{H}^n(Y, p) \to \widehat{H}^n(X, p), \ f^* : \widehat{H}^n(Y) \to \widehat{H}^n(X).$$

More generally, for any map $\phi : M \to M'$ in $\mathbf{SH}(S)$ there is a functorial pullback

$$\phi^* : \widehat{H}^n(M', p) \to \widehat{H}^n(M, p), \ \ \phi^* : \widehat{H}^n(M') \to \widehat{H}^n(M).$$

This pullback is compatible with the long exact sequence (4.3) and, for compact objects M and M', with the Arakelov-Chern class (4.7).

Proof. The second statement is clear from the definition. The first claim follows by applying the natural transformation

$$x_! x^! = y_! f_! f^! y^! \xrightarrow{(2.3)} y_! y^!$$

to BGL_S or $H_{\mathrm{B},S}$, respectively. The last statement is also clear since (2.3) is functorial; in particular it respects the isomorphism $\widehat{\mathrm{ch}} : \widehat{\mathrm{BGL}}_{\mathbb{Q},S} \cong \oplus \widehat{H_{\mathrm{B},S}}\{p\}$. $\square$

In the remainder of this section, we assume that f and y (hence also x) are regular projective maps (Definition 2.3). Recall that $\dim f = \dim X - \dim Y$.

Definition and Lemma 4.10. *We define the pushforward*

$$f_* : \widehat{\mathrm{H}}^n(X, p) \to \widehat{\mathrm{H}}^{n-2\dim(f)}(Y, p - \dim(f))$$

on Arakelov motivic cohomology to be the map induced by the composition

$$
\begin{aligned}
\mathrm{M}_S(Y) = y_! y^! \mathrm{H}_{\mathrm{B},S} \quad &\overset{(\mathrm{tr}_y^{\mathrm{B}})^{-1}}{\longrightarrow} \quad y_! y^* \mathrm{H}_{\mathrm{B},S}\{\dim(y)\} \\
&\overset{(2.2)}{\longrightarrow} \quad y_! f_* f^* y^* \mathrm{H}_{\mathrm{B},S}\{\dim(y)\} \\
&= \quad x_! x^* \mathrm{H}_{\mathrm{B},S}\{\dim(y)\} \\
&\overset{\mathrm{tr}_x^{\mathrm{B}}}{\longrightarrow} \quad x_! x^! \mathrm{H}_{\mathrm{B},S}\{\dim(y) - \dim(x)\} \\
&= \quad \mathrm{M}_S(X)\{-\dim(f)\}.
\end{aligned}
$$

Similarly,

$$f_* : \widehat{\mathrm{H}}^n(X) \to \widehat{\mathrm{H}}^n(Y)$$

is defined using the trace maps on BGL *instead of the ones for* H_{B} *(2.15), (2.16).*

This definition is functorial (with respect to the composition of regular projective maps).

Proof. Let $g : Y \to Z$ be a second map of S-schemes such that both g (hence $h := g \circ f$) and the structural map $z : Z \to S$ are regular projective. The functoriality of the pushforward is implied by the fact that the following two compositions agree (we do not write $\mathrm{H}_{\mathrm{B},-}\{-\}$ or BGL_- for space reasons):

$$z_! z^! \overset{\mathrm{tr}_z^{-1}}{\to} z_! z^* \to z_! h_* h^* z^* = x_! x^* \overset{\mathrm{tr}_x}{\to} x_! x^!,$$

$$z_! z^! \overset{\mathrm{tr}_z^{-1}}{\to} z_! z^* \to z_! g_* g^* z^* = y_! y^* \overset{\mathrm{tr}_y}{\to} y_! y^! \overset{\mathrm{tr}_y^{-1}}{\to} y_! y^* \to y_! f_* f^* y^* = x_! x^* \overset{\mathrm{tr}_x}{\to} x_! x^!.$$

This agreement is an instance of the identity $ad_h = y_* ad_f y^* \circ ad_g$. $\qquad \square$

4.3. Purity and an arithmetic Riemann-Roch theorem. In this subsection, we establish a purity isomorphism and a Riemann-Roch theorem for Arakelov motivic cohomology. We cannot prove it in the expected full generality of regular projective maps, but need some smoothness assumption.

Given any closed immersion $i : Z \to \operatorname{Spec}\mathbb{Z}$, we let $j : U \to \operatorname{Spec}\mathbb{Z}$ be its open complement. The generic point is denoted $\eta : \operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$. We also write i, j, η for the pullback of these maps to any scheme, e.g. $i : X_Z := X \times_{\operatorname{Spec}\mathbb{Z}} Z \to X$. Recall that B is an arithmetic ring whose generic fiber B_η is a field (Definition 2.6).

Let $f : X \to S$ be a map of regular B-schemes. For clarity, we write $\mathrm{D}(p)_{X_\eta}$ for the complex of presheaves on $\mathbf{Sm}/X_\eta$ that was denoted $\mathrm{D}(p)$ above and $\mathrm{H}_{\mathrm{D},X_\eta}$ for the resulting spectrum. Moreover, we write $\mathrm{H}_{\mathrm{D},X} := \eta_* \mathrm{H}_{\mathrm{D},X_\eta} \in \mathbf{SH}(X)$. The complex $\mathrm{D}(p)_{X_\eta}$ is the restriction of the complex

$\mathrm{D}(p)_{B_\eta}$. Therefore, there is a natural map $f^*\mathrm{D}(p)_S \to \mathrm{D}(p)_X$, which in turn gives rise to a map of spectra

$$\alpha_{\mathrm{D}}^f : f^*\mathrm{H}_{\mathrm{D},S} \to \mathrm{H}_{\mathrm{D},X}.$$

This map is an isomorphism if f is smooth, since $f^* : \mathbf{PSh}(\mathbf{Sm}/S) \to \mathbf{PSh}(\mathbf{Sm}/X)$ is just the restriction in this case. Is α_{D}^f an isomorphism for a closed immersion f between flat regular B-schemes? The corresponding fact for BGL, i.e., the isomorphism $f^*\mathrm{BGL}_S = \mathrm{BGL}_X$, ultimately relies on the fact that algebraic K-theory of smooth schemes over S is represented in $\mathbf{SH}(S)$ by the infinite Grassmannian, which is a smooth scheme over S. Therefore, it would be interesting to have a geometric description of the spectrum representing Deligne cohomology, as opposed to the merely cohomological representation given by the complexes $\mathrm{D}(p)$.

Lemma 4.11.

(i) *Given another map $g : Y \to X$ of regular B-schemes, there is a natural isomorphism of functors $\alpha_{\mathrm{D}}^g \circ g^*\alpha_{\mathrm{D}}^f = \alpha_{\mathrm{D}}^{f \circ g}$.*

(ii) *The following are equivalent:*
 - *α_{D}^f is an isomorphism in $\mathbf{SH}(X)$.*
 - *For any $i : Z \to \mathrm{Spec}\,\mathbb{Z}$, the object $i^! f^*\mathrm{H}_{\mathrm{D},S}$ is zero in $\mathbf{SH}(X \times_{\mathbb{Z}} Z)$.*
 - *For any sufficiently small $j : U \to \mathrm{Spec}\,\mathbb{Z}$, the adjunction morphism $f^*\mathrm{H}_{\mathrm{D},S} \to j_*j^*f^*\mathrm{H}_{\mathrm{D},S}$ is an isomorphism in $\mathbf{SH}(X)$.*

(iii) *The conditions in (*ii*) are satisfied if f fits into a diagram*

$$
\begin{array}{ccccc}
X & \xrightarrow{\ x\ } & B' & \longrightarrow & B \\
{\scriptstyle f}\downarrow & \nearrow{\scriptstyle s} & & & \\
S & & & &
\end{array}
$$

where B' is regular and of finite type over B, x and s are smooth. In particular, this applies when f is smooth or when both X and S are smooth over B.

Proof. (i) is easy to verify using the definition of the pullback functor. (iii) is a consequence of the above remark and (i) using the chain of natural isomorphisms $f^*\mathrm{H}_{\mathrm{D},S} = f^*s^*\mathrm{H}_{\mathrm{D},B'} = x^*\mathrm{H}_{\mathrm{D},B'} = \mathrm{H}_{\mathrm{D},X}$. For (ii), consider the map of distinguished localization triangles:

$$
\begin{array}{ccccc}
i_*i^! f^*\mathrm{H}_{\mathrm{D},S} & \longrightarrow & f^*\mathrm{H}_{\mathrm{D},S} & \longrightarrow & j_*j^*f^*\mathrm{H}_{\mathrm{D},S} \\
\downarrow & & {\scriptstyle \alpha_{\mathrm{D}}^f}\downarrow & & \downarrow{\scriptstyle j_*j^*\alpha_{\mathrm{D}}^f = j_*\alpha_{\mathrm{D}}^{f_U}} \\
0 = i_*i^!\mathrm{H}_{\mathrm{D},X} & \longrightarrow & \mathrm{H}_{\mathrm{D},X} & \longrightarrow & j_*j^*\mathrm{H}_{\mathrm{D},X}.
\end{array}
$$

The map $\alpha_{\mathrm{D}}^{f_U}$ is an isomorphism as soon as j is small enough so that X_U and S_U are smooth over B_U. Such a j exists by the regularity of X and S. This shows the equivalence of the three statements in (ii). $\qquad\square$

Below, we write $\mathrm{B} := \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{B}}\{p\}$ and $\widehat{\mathrm{B}}_X := \mathrm{hofib}(\mathrm{B}_X \to \mathrm{B}_X \wedge \mathrm{H}_{\mathrm{D},X})$. We define

$$f^? \widehat{\mathrm{BGL}}_S := \mathrm{hofib}(f^! \mathrm{BGL}_S \xrightarrow{\mathrm{id} \wedge 1} f^! \mathrm{BGL}_S \wedge f^* \mathrm{H}_{\mathrm{D},S})$$

and similarly for $f^? \widehat{\mathrm{B}}_S$. (The notation is not meant to suggest a functor $f^?$; it is just shorthand.) The Chern class $\mathrm{ch} : \mathrm{BGL}_S \to \mathrm{B}_S$ induces a map $f^? \widehat{\mathrm{ch}} : f^? \widehat{\mathrm{BGL}}_S \to f^? \widehat{\mathrm{B}}_S$.

Theorem 4.12. *Let $f : X \to S$ be a regular projective map (Definition 2.3) such that α_{D}^f is an isomorphism. (In particular (Lemma 4.11 (iii)) this applies when B is a field or when X and S are smooth over B or when f is smooth.) Then there is a commutative diagram in $\mathbf{SH}(X)_{\mathbb{Q}}$ as follows. Its top row horizontal maps are BGL_X-linear (i.e., induced by maps in $\mathbf{DM}_{\mathrm{BGL}}(X)$), and the bottom horizontal maps are B_X-linear. All maps in this diagram are isomorphisms (in $\mathbf{SH}(X)_{\mathbb{Q}}$).*

$$(4.8) \quad \begin{array}{ccccccc}
\widehat{\mathrm{BGL}}_X & \xleftarrow{\widehat{\alpha}} & f^* \widehat{\mathrm{BGL}}_S & \xrightarrow{\widehat{\mathrm{tr}_{\mathrm{BGL}}}} & f^? \widehat{\mathrm{BGL}}_S & \xrightarrow{\beta} & f^! \widehat{\mathrm{BGL}}_S \\
\downarrow{\scriptstyle \widehat{\mathrm{ch}}_X} & & \downarrow{\scriptstyle f^* \widehat{\mathrm{ch}}_S} & & \downarrow{\scriptstyle f^? \widehat{\mathrm{ch}}_S} & & \downarrow{\scriptstyle f^! \widehat{\mathrm{ch}}_S} \\
\widehat{\mathrm{B}}_X & \xleftarrow{\widehat{\alpha}} & f^* \widehat{\mathrm{B}}_S & \xrightarrow[\widehat{\mathrm{Td}(T_f)}]{} f^* \widehat{\mathrm{B}}_S & \xrightarrow{\widehat{\mathrm{tr}_{\mathrm{B}}}} f^? \widehat{\mathrm{B}}_S & \xrightarrow{\beta} & f^! \widehat{\mathrm{B}}_S.
\end{array}$$

Proof. To define the maps $\widehat{\alpha}$ in (4.8), we don't make use of the assumption on α_{D}^f. Pick fibrant-cofibrant representatives of BGL and H_{B}, and H_{D}. Thus, in the following diagram of spectra, f^* and $\wedge$ are the usual, non-derived functors for spectra:

$$\begin{array}{ccccc}
f^* \mathrm{BGL}_S & =\!=\!=\!= & f^* \mathrm{BGL}_S & \xrightarrow{\alpha_{\mathrm{BGL}}^f} & \mathrm{BGL}_X \\
\downarrow{\scriptstyle f^*(\mathrm{id} \wedge 1_{\mathrm{D}})} & & \downarrow{\scriptstyle \mathrm{id} \wedge f^* 1_{\mathrm{D}}} & & \downarrow{\scriptstyle \mathrm{id} \wedge 1_{\mathrm{D}}} \\
f^*(\mathrm{BGL}_S \wedge \mathrm{H}_{\mathrm{D},S}) & \longleftarrow & f^* \mathrm{BGL}_S \wedge f^* \mathrm{H}_{\mathrm{D},S} & \xrightarrow{\alpha_{\mathrm{BGL}}^f \wedge \alpha_{\mathrm{D}}^f} & \mathrm{BGL}_X \wedge \mathrm{H}_{\mathrm{D},X}.
\end{array}$$

As f^* is a monoidal functor (on the level of spectra), the canonical lower left hand map is an isomorphism of spectra and the left square commutes. The right square commutes because of $\alpha_{\mathrm{D}}^f(f^* 1_{\mathrm{D}}) = 1_{\mathrm{D}}$. This diagram induces a map between the homotopy fibers of the two vertical maps, which are $f^* \widehat{\mathrm{BGL}}_S$ and $\widehat{\mathrm{BGL}}_X$, respectively. This is the map $\widehat{\alpha}$ above. The one for $\widehat{\mathrm{B}}$ is constructed the same way by replacing BGL by B throughout. Using $f^* \mathrm{ch}_S = \mathrm{ch}_X$, this shows the commutativity of the left hand square in (4.8).

By definition of BGL, $\alpha_{\mathrm{BGL}}^f : f^* \mathrm{BGL}_S \to \mathrm{BGL}_X$ is a weak equivalence. Thus, both maps $\widehat{\alpha}$ are isomorphisms in $\mathbf{SH}(X)$ when α_{D}^f is so. They are clearly BGL_X- and B_X-linear, respectively.

The horizontal maps in the middle quadrangle are defined as in Theorem 4.2(i): for example, the map $\mathrm{tr}_{\mathrm{BGL}} : f^* \mathrm{BGL} \to f^! \mathrm{BGL}$ gives rise to $\widehat{\mathrm{tr}_{\mathrm{BGL}}} : f^* \widehat{\mathrm{BGL}}_S \to f^? \widehat{\mathrm{BGL}}_S$. It is BGL_X-linear since $\mathrm{tr}_{\mathrm{BGL}}$ is so. Similarly, we define $\widehat{\mathrm{Td}(T_f)}$ (viewing $\mathrm{Td}(T_f)$ as a (B_X-linear) map $f^* \mathrm{B}_S \to f^* \mathrm{B}_S$) and $\widehat{\mathrm{tr}_{\mathrm{B}}}$. Picking representatives of all maps, the quadrangle will in general not commute in the category of spectra, but does so up to homotopy, by construction and by the Riemann-Roch Theorem 2.5. This settles the middle rectangle.

By the regularity of X and S, we can choose $j : U \subset \mathrm{Spec}\,\mathbb{Z}$ such that X_U and S_U are smooth over B_U. We will also write j for $X_U \to X$, etc.

By assumption, α_{D}^f is an isomorphism. Hence, the adjunction map $f^! \mathrm{BGL} \wedge f^* \mathrm{H}_{\mathrm{D}} \to j_* j^* (f^! \mathrm{BGL} \wedge f^* \mathrm{H}_{\mathrm{D}})$ is an isomorphism in $\mathbf{SH}$. In fact, both terms are isomorphic in $\mathbf{SH}$ to $\bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\}$, as one checks for example using the purity isomorphism $f^! \mathrm{BGL}_S \cong f^* \mathrm{BGL}_S = \mathrm{BGL}_X$. Thus, $f^? \widehat{\mathrm{BGL}}$ is canonically isomorphic to the homotopy fiber of $f^! \mathrm{BGL} \to j_* j^* f^! \mathrm{BGL} \to j_* j^* (f^! \mathrm{BGL} \wedge f^* \mathrm{H}_{\mathrm{D}}) = j_* (j^* f^! \mathrm{BGL} \wedge j^* f^* \mathrm{H}_{\mathrm{D}})$. Here, the last equality is a canonical isomorphism on the level of spectra, since j^* is just the restriction. By definition, $j^* f^! = j^! f^!$. We may therefore replace f by f_U. Now, $f_U^! M$ is *functorially* isomorphic (in $\mathbf{SH}$) to $f_U^* M\{n\}$, $n := \dim f_U$, by construction of the relative purity isomorphism by Ayoub [Ayo07, Section 1.6]. Indeed, a is a closed immersion, and p and every map in the diagram with codomain B_U are smooth:

$$\begin{array}{ccc} & \xrightarrow{\;\;f_U\;\;} & \\ X_U \underset{a}{\xrightarrow{\hspace{1cm}}} \mathbb{P}^n_{S_U} \xrightarrow{\;\;p\;\;} & S_U & \\ & \searrow \quad \downarrow & \\ & B_U. & \end{array}$$
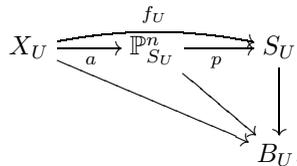
Hence to construct β, it is enough to construct a commutative diagram of spectra:

$$\begin{array}{ccc} f_U^* \mathrm{BGL}_{S_U}\{n\} & =\!=\!=\!=\!=\!=\!= & f_U^* \mathrm{BGL}_{S_U}\{n\} \\ \downarrow{\scriptstyle \mathrm{id} \wedge 1_{\mathrm{D}}} & & \downarrow{\scriptstyle f^*(\mathrm{id} \wedge 1_{\mathrm{D}})} \\ f_U^* \mathrm{BGL}_{S_U}\{n\} \wedge f_U^* \mathrm{H}_{\mathrm{D},S_U} & \xrightarrow{\;\;\gamma\;\;} & f_U^* (\mathrm{BGL}_{S_U} \wedge \mathrm{H}_{\mathrm{D},S_U})\{n\}. \end{array}$$

The map γ is the natural map of spectra $f_U^* x \wedge f_U^* y \to f_U^* (x \wedge y)$, which clearly makes the diagram commute in the category of spectra. We have constructed

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a map (in **SH**) $\beta : f^? \widehat{\mathrm{BGL}} \to f^! \widehat{\mathrm{BGL}}$ in a way that is functorial with respect to (a lift to the category of spectra of) the map ch : BGL → Б. Therefore, the analogous construction for $\widehat{\mathrm{Б}}$ produces the desired commutative square of isomorphisms (in **SH**). Again, the top row map β is BGL-linear and the bottom one is $\mathrm{Б}_X$-linear.

Finally, the vertical maps in (4.8) are isomorphisms using the Arakelov Chern character (4.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now conclude a higher arithmetic Riemann-Roch theorem. It controls the failure of $\widehat{\mathrm{ch}}$ to commute with the pushforward.

Theorem 4.13. *Let $f : X \to S$ be a regular projective map (Definition 2.3) of schemes of finite type over an arithmetic ring B (Definition 2.6). Moreover, we assume that f is such that*

$$\alpha_{\mathrm{D}}^f : f^* \mathrm{H}_{\mathrm{D},S} \to \mathrm{H}_{\mathrm{D},X}$$

is an isomorphism. This condition is satisfied, for example, when f is smooth or when X and S are smooth over B (Lemma 4.11). Then, the following hold:

(i) *(Purity) The absolute purity isomorphisms for* BGL *and* $\mathrm{H}_{\mathrm{Б}}$ *(2.14) induce isomorphisms (of* BGL_X*- and* $\mathrm{H}_{\mathrm{Б}\,X}$*-modules, respectively):*

$$\widehat{\mathrm{BGL}}_X \cong f^* \widehat{\mathrm{BGL}}_S \cong f^! \widehat{\mathrm{BGL}}_S, \quad \widehat{\mathrm{H}_{\mathrm{Б}}}_X \cong f^* \widehat{\mathrm{H}_{\mathrm{Б}}}_S \cong f^! \widehat{\mathrm{H}_{\mathrm{Б}}}_S \{-\dim f\}.$$

In particular, Arakelov motivic cohomology is independent of the base scheme in the sense that there are isomorphisms

$$\widehat{\mathrm{H}}^n(X/S) \cong \widehat{\mathrm{H}}^n(X/X), \quad \widehat{\mathrm{H}}^n(X/S,p) \cong \widehat{\mathrm{H}}^n(X/X,p).$$

(ii) *(Higher arithmetic Riemann-Roch theorem) There is a commutative diagram*

$$
\begin{array}{ccc}
\widehat{\mathrm{H}}^n(X/X) & \xrightarrow{\quad f_* \quad} & \widehat{\mathrm{H}}^n(S/S) \\
\downarrow{\scriptstyle \widehat{\mathrm{ch}}_X} & & \downarrow{\scriptstyle \widehat{\mathrm{ch}}_S} \\
\bigoplus_{p\in\mathbb{Z}} \widehat{\mathrm{H}}^{n+2p}(X,p) & \xrightarrow{f_* \circ \widehat{\mathrm{Td}}(T_f)} & \bigoplus_{p\in\mathbb{Z}} \widehat{\mathrm{H}}^{n+2p}(S,p).
\end{array}
$$

Here, the top line map f_* is given by

$$
\begin{aligned}
\widehat{\mathrm{H}}^n(X/X) \quad &:= \quad \mathrm{Hom}_{\mathbf{SH}(X)}(S^{-n}, \widehat{\mathrm{BGL}}_X) \\
&\overset{(4.8)}{\to} \quad \mathrm{Hom}_{\mathbf{SH}(X)}(S^{-n}, f^! \widehat{\mathrm{BGL}}_S) \\
&\overset{(2.2)}{\to} \quad \mathrm{Hom}_{\mathbf{SH}(S)}(S^{-n}, \widehat{\mathrm{BGL}}_S) = \widehat{\mathrm{H}}^n(S/S).
\end{aligned}
$$

Using the identifications $\widehat{\mathrm{H}}^n(X/X) \cong \widehat{\mathrm{H}}^n(X/S)$, this map agrees with the one defined in Lemma 4.10. The bottom map f_* is given similarly replacing BGL *with* Б.

Proof. The isomorphisms for $\widehat{\mathrm{BGL}}_?$ in (i) are a restatement of Theorem 4.12. The ones for $\widehat{\mathrm{H}}_{\mathrm{B}?}$ also follow from that by dropping the isomorphism $\widehat{\mathrm{Td}(T_f)}$ in the bottom row of (4.8) and noting that tr_{B}, hence $\widehat{\mathrm{tr}_{\mathrm{B}}}$, shifts the degree by $\dim f$. The isomorphisms in the second statement are given by the following identifications of morphisms in $\mathbf{DM}_{\mathrm{BGL}}(-)$, using (4.6):

$$\mathrm{Hom}(\mathrm{BGL}_X, \widehat{\mathrm{BGL}}_X) \xrightarrow{4.12} \mathrm{Hom}(\mathrm{BGL}_X, f^! \widehat{\mathrm{BGL}}_S)$$
$$\xrightarrow{(\mathrm{tr}_{\mathrm{BGL}})^{-1}} \mathrm{Hom}(f^! \mathrm{BGL}_S, f^! \widehat{\mathrm{BGL}}_S)$$
$$= \mathrm{Hom}(f_! f^! \mathrm{BGL}_S, \widehat{\mathrm{BGL}}_S)$$

and likewise for H_{B}.

(ii) is an immediate corollary of Theorem 4.12, given that the two isomorphisms (in $\mathbf{SH}(X)_{\mathbb{Q}}$) $\widehat{\mathrm{Td}(T_f)} \circ \widehat{\alpha}^{-1}$ and $\widehat{\alpha}^{-1} \circ \widehat{\mathrm{Td}(T_f)}$, where $\mathrm{Td}(T_f)$ is seen as an endomorphism of $f^* \mathrm{B}_S$ and of B_X, respectively, agree. This agreement follows from the definition of $\widehat{\alpha}$. The agreement of the two definitions of f_* is clear from the definition. $\square$

This also elucidates the behavior of (4.5) with respect to pushforward: in the situation of the theorem, the pushforward $f_* : \widehat{\mathrm{H}}^n(X) \to \widehat{\mathrm{H}}^n(S)$ sits between the usual K-theoretic pushforward and the pushforward on Deligne cohomology (which is given by integration of differential forms along the fibers in case $f(\mathbb{C})$ is smooth, and by pushing down currents in general), multiplied by the Todd class (in Deligne cohomology) of the relative tangent bundle.

4.4. Further properties.

Theorem 4.14.

(i) *Arakelov motivic cohomology satisfies h-descent (thus, a fortiori, Nisnevich, étale, cdh, qfh and proper descent). For example, there is an exact sequence*

$$\ldots \to \widehat{\mathrm{H}}^n(X, p) \to \widehat{\mathrm{H}}^n(U \sqcup V, p) \to \widehat{\mathrm{H}}^n(W, p) \to \widehat{\mathrm{H}}^{n+1}(X, p) \to \ldots$$

where

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{f} & X \end{array}$$

is a cartesian square of smooth schemes over S that is either a distinguished square for the cdh-topology (f is a closed immersion, p is proper such that p is an isomorphism over $X \backslash U$) or a distinguished square for the Nisnevich topology (f an open immersion, p étale inducing an isomorphism $(p^{-1}(X \backslash U)_{\mathrm{red}} \to (X \backslash U)_{\mathrm{red}})$) or such that $U \sqcup V \to X$ is an open cover.

(ii) *Arakelov motivic cohomology is homotopy invariant and satisfies a projective bundle formula:*

$$\widehat{\mathrm{H}}^n(X \times \mathbb{A}^1, p) \cong \widehat{\mathrm{H}}^n(X, p),$$

$$\widehat{\mathrm{H}}^n(\mathbf{P}(E), p) \cong \bigoplus_{i=0}^{d} \widehat{\mathrm{H}}^{n-2i}(X, p - i).$$

Here X/S is arbitrary (of finite type), $E \to X$ is a vector bundle of rank $d + 1$, and $\mathbf{P}(E)$ is its projectivization.

(iii) *Any distinguished triangle of motives induces long exact sequences of Arakelov motivic cohomology. For example, let X/S be an l.c.i. scheme (Example 2.4). Let $i : Z \subset X$ be a closed immersion of regular schemes of constant codimension c with open complement $j : U \subset X$. Then there is an exact sequence*

$$\widehat{\mathrm{H}}^{n-2c}(Z, p - c) \xrightarrow{i_*} \widehat{\mathrm{H}}^n(X, p) \xrightarrow{j^*} \widehat{\mathrm{H}}^n(U, p) \to \widehat{\mathrm{H}}^{n+1-2c}(Z, p - c).$$

(iv) *The cdh-descent and the properties (*ii*), (*iii*) hold* mutatis mutandis *for $\widehat{\mathrm{H}}^*(-)$.*

Proof. The h-descent is a general property of modules over $\mathrm{H}_{\mathrm{B},S}$ [CD09, Thm 16.1.3]. The $\mathbb{A}^1$-invariance and the bundle formula are immediate from $\mathrm{M}(X) \cong \mathrm{M}(X \times \mathbb{A}^1)$ and $\mathrm{M}(\mathbf{P}(E)) \cong \bigoplus_{i=0}^{d} \mathrm{M}(X)\{i\}$. For the last statement, we use the localization exact triangle [CD09, 2.3.5] for $U \xrightarrow{j} X \xleftarrow{i} Z$:

$$f_! j_! j^! f^! \mathrm{H}_{\mathrm{B},S} \to f_! f^! \mathrm{H}_{\mathrm{B},S} \to f_! i_* i^* f^! \mathrm{H}_{\mathrm{B},S}.$$

The purity isomorphism $f^* \mathrm{H}_{\mathrm{B},S}\{\dim f\} = f^! \mathrm{H}_{\mathrm{B},S}$ (Example 2.4) for the structural map $f : X \to S$ and the absolute purity isomorphism (2.14) for i imply that the rightmost term is isomorphic to $f_! i_! i^! f^! \mathrm{H}_{\mathrm{B},S}\{-\dim i\} = \mathrm{M}_S(Z)\{-c\}$. Mapping this triangle into $\widehat{\mathrm{H}_{\mathrm{B},S}}(p)[n]$ gives the desired long exact sequence.

The arguments for $\widehat{\mathrm{BGL}}_S$ are the same. The only difference is that descent for topologies exceeding the cdh-topology requires rational coefficients. $\square$

Acknowledgements

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ARAKELOV MOTIVIC COHOMOLOGY II

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Abstract

We show that the constructions done in part I generalize their classical counterparts: firstly, the classical Beilinson regulator is induced by the abstract Chern class map from BGL to the Deligne cohomology spectrum. Secondly, Arakelov motivic cohomology is a generalization of arithmetic K-theory and arithmetic Chow groups. For example, this implies a decomposition of higher arithmetic K-groups in its Adams eigenspaces. Finally, we give a conceptual explanation of the height pairing: it is the natural pairing of motivic homology and Arakelov motivic cohomology.

The purpose of this work is to compare the abstract constructions of the regulator map and the newly minted Arakelov motivic cohomology groups done in part I (in this issue) with their classical, more concrete counterparts. In a nutshell, Arakelov motivic generalizes and simplifies a number of classical constructions pertaining to arithmetic K- and Chow groups.

We show that the Chern class $\mathrm{ch_D} : \mathrm{BGL} \to \bigoplus_p \mathrm{H_D}\{p\}$ between the spectra representing K-theory and Deligne cohomology constructed in Definition 3.7[1] induces the Beilinson regulator

$$K_n(X) \to \bigoplus_p \mathrm{H_D^{2p-n}}(X, p)$$

for any smooth scheme X over an arithmetic field (Theorem 5.7).

Next, we turn to the relation of Arakelov motivic cohomology and arithmetic K- and Chow groups. Arithmetic K-groups were defined by Gillet-Soulé and generalized to higher K-theory by Takeda [GS90b, GS90c, Tak05]. We denote these groups by $\widehat{\mathrm{K}}_n^T(X)$. They fit into an exact sequence

$$K_{n+1}(X) \to \mathrm{D}_{n+1}(X)/\operatorname{im} d_\mathrm{D} \to \widehat{\mathrm{K}}_n^T(X) \to K_n(X) \to 0,$$

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[1]The numbering here continues from the end of part I.

where $D_*(X)$ is a certain complex of differential forms. The presence of the group $D_{n+1}(X)/\operatorname{im} d_D$, as opposed to the Deligne cohomology group $\ker d_D/\operatorname{im} d_D = \bigoplus_p H_D^{2p-n-1}(X, p)$ implies that the groups $\widehat{K}_n^T(X)$ are not homotopy invariant. Therefore they cannot be addressed using $\mathbb{A}^1$-homotopy theory. Instead, we focus on the subgroup (see Section 6)

$$\widehat{K}_n(X) := \ker\left(\operatorname{ch} : \widehat{K}_n^T(X) \to D_n(X)\right).$$

and show a canonical isomorphism

(*) $$\widehat{H}^{-n}(X) \cong \widehat{K}_n(X)$$

for smooth schemes X and $n \geq 0$. All our comparison results concern the groups $\widehat{K}_*(X)$ and, in a similar vein, the subgroup $\widehat{CH}^*(X)$ of Gillet-Soulé's group [GS90a] consisting of arithmetic cycles (Z, g) satisfying $\delta_Z = \partial\overline{\partial}g/(2\pi i)$; cf. (6.16). The homotopy-theoretic approach taken in this paper conceptually explains, improves, and generalizes classical constructions such as the arithmetic Riemann-Roch theorem, as far as these smaller groups are concerned. The simplification stems from the fact that it is no longer necessary to construct explicit homotopies between the complexes representing arithmetic K-groups, say. For example, the Adams operations on $\widehat{K}_n(X)$ defined by Feliu [Fel10] were not known to induce a decomposition

$$\widehat{K}_*(X)_{\mathbb{Q}} \cong \bigoplus_p \widehat{K}_*(X)_{\mathbb{Q}}^{(p)}.$$

Using that the isomorphism (*) is compatible with Adams operations, this statement follows from the entirely formal analogue for $\widehat{H}^*$, namely the Arakelov-Chern class isomorphism (4.7). We conclude a canonical isomorphism

$$\widehat{H}^{2p,p}(X, p) = \widehat{CH}^p(X)_{\mathbb{Q}}.$$

Moreover, the pushforward on Arakelov motivic cohomology established in Definition and Lemma 4.10 is shown to agree with the one on arithmetic Chow groups in two cases, namely for the map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ and for a smooth proper map $X \to S$, $S \subset \operatorname{Spec} \mathcal{O}_F$ for a number ring $\mathcal{O}_F$. The non-formal input in the second statement is the finiteness of the Chow group $CH^{\dim X}(X)$ proven by Kato and Saito [KS86]. In a similar vein, we identify the pushforward on $\widehat{K}_0$ with the one on $\widehat{H}^0$ (Theorem 6.4). In Section 7, it is shown that the height pairing

$$CH^m(X) \times \widehat{CH}^{\dim X - m}(X) \to \widehat{CH}^1(S)$$

coincides, after tensoring with $\mathbb{Q}$, with the *Arakelov intersection pairing* of the motive $M := M(X)(m - \dim X + 1)[2(m - \dim X + 1)]$ of any smooth

proper scheme X/S:

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^0, M) \times \mathrm{Hom}(M, \widehat{\mathrm{H}_{\mathrm{B},S}}(1)[2]) \;\to\; \widehat{\mathrm{H}}^2(S, 1),$$
$$(\alpha, \beta) \;\mapsto\; \beta \circ \alpha.$$

Conjecturally, the L-values of schemes (or motives) over $\mathbb{Z}$ are given by the determinant of this pairing [Sch13].

In the light of these results, stable homotopy theory offers a conceptual clarification of hitherto difficult or cumbersome explicit constructions of chain maps and homotopies representing the expected maps on arithmetic K-theory, such as the Adams operations. The bridge between these concrete constructions and the abstract path taken here is provided by a strong unicity theorem. Recall that there is a distinguished triangle

$$\bigoplus_{p \in \mathbb{Z}} \mathrm{H_D}\{p\}[-1] \to \widehat{\mathrm{BGL}} \to \mathrm{BGL} \xrightarrow{\mathrm{ch_D}} \bigoplus_{p \in \mathbb{Z}} \mathrm{H_D}\{p\}$$

in the stable homotopy category. Among other things we prove that $\widehat{\mathrm{BGL}}$ is unique, up to *unique* isomorphism fitting into the obvious map of distinguished triangles (see Theorem 6.1 for the precise statement). The proof of this theorem takes advantage of the motivic machinery, especially the computations of Riou pertaining to endomorphisms of BGL. Its only non-formal input is a mild condition involving the K-theory and Deligne cohomology of the base scheme. The unicity trickles down to the unstable homotopy category. It can therefore be paraphrased as: any construction for the groups $\widehat{\mathrm{K}}_*$ that is functorially representable by zig-zags of chain maps and compatible with its non-Arakelov counterpart is necessarily unique. The above-mentioned identification of the Adams operations and the K-theory module structure on $\widehat{\mathrm{K}}$ are consequences of this principle. In order to show that the arithmetic Riemann-Roch theorem by Gillet, Roessler and Soulé [GRS08], when restricted to $\widehat{\mathrm{K}}_0(X) \subset \widehat{\mathrm{K}}_0^T(X)$ (!), is a formal consequence of the motivic framework it remains to show that their arithmetic Chern class [GS90c, cf. Thm. 7.2.1],

$$\widehat{\mathrm{K}}_0(X)_{\mathbb{Q}} \cong \bigoplus_p \widehat{\mathrm{K}}_0(X)_{\mathbb{Q}}^{(p)},$$

agrees with the Arakelov Chern class established in (4.7). This will be a consequence of the above unicity result, once the arithmetic Chern class has been extended to higher arithmetic K-theory by means of a canonical (i.e., functorial) zig-zag of appropriate chain complexes.

5. Comparison of the regulator

After recalling some details of the construction of BGL in Section 5.1, we construct a Chern class map ch : BGL $\to \bigoplus_p \mathrm{H_D}\{p\}$ that induces the Beilinson regulator. This is done in Section 5.2. The strategy is to take Burgos' and Wang's representation of the Beilinson regulator as a map of simplicial presheaves and lift it to a map in $\mathbf{SH}(S)$. We finish this section by proving that this Chern class ch and the one obtained in Definition 3.7,

$$\mathrm{ch_D} : \mathrm{BGL} \xrightarrow{\mathrm{id}\wedge 1_D} \mathrm{BGL_{\mathbb{Q}}} \wedge \mathrm{H_D} \xrightarrow{\mathrm{ch}\wedge\mathrm{id}} \bigoplus_{p\in\mathbb{Z}} \mathrm{H_B}\{p\} \wedge \mathrm{H_D} \xleftarrow{1_B\wedge\mathrm{id_D},\cong} \bigoplus_p \mathrm{H_D}\{p\},$$

agree. In particular, $\mathrm{ch_D}$ also induces the Beilinson regulator. This result is certainly not surprising—after all, Beilinson's regulator is the Chern character map for Deligne cohomology.

Throughout, we will use the notation of part I. In particular, $\mathbf{Ho_\bullet}(S)$ and $\mathbf{SH}(S)$ are the unstable and the stable homotopy category of smooth schemes over some Noetherian base scheme S (Sections 2.1, 2.2).

5.1. Reminders on the object BGL **representing** K-**theory.** In order to prove our comparison results, we need some more details concerning the object BGL representing algebraic K-theory. This is due to Riou [Rio].

Let $\mathrm{Gr}_{d,r}$ be the Grassmannian whose T-points, for any $T \in \mathbf{Sm}/S$, are given by locally free subsheaves of $\mathcal{O}_T^{d+r}$ of rank d. As usual, we regard this (smooth projective) scheme as a presheaf on $\mathbf{Sm}/S$. For $d \le d'$, $r \le r'$, the transition map

$$(5.1) \qquad\qquad \mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d',r'}$$

is given on the level of T-points by mapping $M \subset \mathcal{O}_T^{d+r}$ to $\mathcal{O}_T^{d'-d}\oplus M\oplus 0^{r'-r} \subset \mathcal{O}^{d'+r'}$. Put $\mathrm{Gr} := \varinjlim_{\mathbb{N}^2} \mathrm{Gr}_{*,*}$, where the colimit is taken in $\mathbf{PSh}(\mathbf{Sm}/S)$. It is pointed by the image of $\mathrm{Gr}_{0,0}$. Write $\mathbb{Z}\times \mathrm{Gr}$ for the product of the constant sheaf $\mathbb{Z}$ (pointed by zero) and this presheaf, and also for its image in $\mathbf{Ho_\bullet}(S)$. For a regular base scheme S, there is a functorial (with respect to pullback) isomorphism

$$(5.2) \qquad\qquad \mathrm{Hom}_{\mathbf{Ho_\bullet}(S)}(S^n \wedge X_+, \mathbb{Z}\times \mathrm{Gr}) \cong K_n(X),$$

for any $X \in \mathbf{Sm}/S$ [MV99, Props. 3.7, 3.9, page 138].

Definition 5.1 ([Rio, I.124, IV.3])**.** The category $\mathbf{SH}^{\mathrm{naive}}(S)$ is the category of Ω-spectra (with respect to $-\wedge \mathbb{P}^1$) in $\mathbf{Ho_\bullet}(S)$: its objects are sequences $E_n \in \mathbf{Ho_\bullet}(S)$, $n \in \mathbb{N}$, with bonding maps $\mathbb{P}^1 \wedge E_n \to E_{n+1}$ inducing isomorphisms $E_n \to \underline{\mathrm{Hom}}_\bullet(\mathbb{P}^1, E_{n+1})$.[2] Its morphisms are sequences of maps

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[2] We will not write L or R for derived functors. For example, f^* stands for what is often denoted $\mathrm{L}f^*$ and similarly with right derived functors such as $\mathrm{R\underline{Hom}}$, $\mathrm{R}\Omega$, etc.

$f_n : E_n \to F_n$ (in $\mathbf{Ho}_\bullet(S)$) making the diagrams involving the bonding maps commute.

Remark 5.2. Recall the *projective Nisnevich-$\mathbb{A}^1$-model structure* on $\mathbb{P}^1$-spectra: a map $f : X \to Y$ is a weak equivalence (fibration), if all its levels $f_n : X_n \to Y_n$ form a weak equivalence (fibration, respectively) in the Nisnevich-$\mathbb{A}^1$-model structure on $\Delta^{\mathrm{op}}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$ (whose homotopy category is $\mathbf{Ho}_\bullet(S)$). The homotopy category of spectra with respect to the projective model structure is denoted $\mathbf{SH}_p(S)$. The composition of the inclusion of the full subcategory of Ω-spectra and the natural localization functor,

$$\{X \in \mathbf{SH}_p, X \text{ is an } \Omega\text{-spectrum}\} \subset \mathbf{SH}_p(S) \to \mathbf{SH}(S),$$

is an equivalence. This yields a natural "forgetful" functor $\mathbf{SH}(S) \to \mathbf{SH}^{\mathrm{naive}}(S)$.

Definition and Theorem 5.3 (Riou, [Rio, IV.46, IV.72])**.** The spectrum $\mathrm{BGL}^{\mathrm{naive}} \in \mathbf{SH}^{\mathrm{naive}}(S)$ consists of $\mathrm{BGL}_n^{\mathrm{naive}} := \mathbb{Z} \times \mathrm{Gr}$ (for each $n \geq 0$) with bonding maps

$$(5.3) \qquad \mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{Gr}) \xrightarrow{u_1^* \wedge \mathrm{id}} (\mathbb{Z} \times \mathrm{Gr}) \wedge (\mathbb{Z} \times \mathrm{Gr}) \xrightarrow{\mu} \mathbb{Z} \times \mathrm{Gr},$$

where u_1^* is the map corresponding to $u_1 = [\mathcal{O}(1)] - [\mathcal{O}(0)] \in K_0(\mathbb{P}^1) \overset{(5.2)}{=} \mathrm{Hom}_{\mathbf{Ho}}(\mathbb{P}^1, \mathbb{Z} \times \mathrm{Gr})$ and μ is the multiplication map, that is to say, the unique map [Rio, III.31], inducing the natural (i.e., tensor) product on $K_0(-)$.

For $S = \mathrm{Spec}\,\mathbb{Z}$, there is a lift $\mathrm{BGL}_\mathbb{Z} \in \mathbf{SH}(\mathrm{Spec}\,\mathbb{Z})$ of $\mathrm{BGL}^{\mathrm{naive}} \in \mathbf{SH}^{\mathrm{naive}}(\mathbb{Z})$ that is unique up to *unique* isomorphism. For any scheme $f : S \to \mathrm{Spec}\,\mathbb{Z}$, put $\mathrm{BGL}_S := f^*\mathrm{BGL}_\mathbb{Z}$. The unstable representability theorem (5.2) extends to an isomorphism

$$(5.4) \qquad \mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, \mathrm{BGL}_S) = K_n(X)$$

for any regular scheme S and any smooth scheme X/S. In $\mathbf{SH}(S)_\mathbb{Q}$, i.e., with rational coefficients, $\mathrm{BGL}_S \otimes \mathbb{Q}$ decomposes as

$$(5.5) \qquad \mathrm{BGL}_S \otimes \mathbb{Q} = \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{B},S}(p)[2p]$$

such that the pieces $\mathrm{H}_{\mathrm{B},S}(p)[2p]$ represent the graded pieces of the γ-filtration on K-theory:

$$\mathrm{Hom}_{\mathbf{SH}(S)}(S^n \wedge \Sigma_{\mathbb{P}^1}^\infty X_+, \mathrm{H}_{\mathrm{B},S}(p)[2p]) \cong \mathrm{gr}_\gamma^p K_n(X)_\mathbb{Q}.$$

Lemma 5.4. *For any d, r, the motive $\mathrm{M}(\mathrm{Gr}_{d,r})$ (cf. Section 2.2) is given by*

$$(5.6) \qquad \mathrm{M}(\mathrm{Gr}_{d,r}) = \bigoplus_\sigma \mathrm{M}(S)\left(\sum(\sigma_i - i)\right)\left[2\sum(\sigma_i - i)\right].$$

The sum runs over all Schubert symbols, i.e., sequences of integers satisfying $1 \leq \sigma_1 < \cdots < \sigma_d \leq d + r$. *For* $d \leq d'$, $r \leq r'$, *the transition maps* (5.1) $\mathrm{M}(\mathrm{Gr}_{d,r}) \to \mathrm{M}(\mathrm{Gr}_{d',r'})$ *exhibit the former motive as a direct summand of the latter.*

Proof. Formula (5.6) is well-known [Sem, 2.4]. The second statement follows from the same technique, namely the localization triangles for motives with compact support applied to the cell decomposition of the Grassmannian: for any field k, a d-space $V^{(d)}$ in k^{d+r} is uniquely described by a $(d, d+r)$-matrix A in echelon form such that $A_{\sigma_i, j} = \delta_{i,j}$ and $A_{i,j} = 0$ for $i > \sigma_j$ for some Schubert symbol σ. The constructible subscheme of $\mathrm{Gr}_{d,r}$ whose k-points are given by matrices with fixed σ is an affine space $\mathbb{A}_S^{(\sigma)}$. The transition map $V^{(d)} \mapsto k^{d'-d} \oplus V^{(d)} \oplus 0^{r'-r}$ corresponds to

$$A \mapsto \begin{bmatrix} \mathrm{Id}_{d'-d} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0^{r'-r} \end{bmatrix},$$

that is,

$$\sigma \mapsto (1, 2, \ldots, d'-d, \sigma_1 + (d'-d), \ldots, \sigma_d + (d'-d)) =: \sigma'.$$

In other words, the restriction of the transition maps (5.1) to the cells is the identity map $\mathbb{A}_S^{(\sigma)} \to \mathbb{A}_S^{(\sigma')}$, which shows the second statement. $\qquad \square$

5.2. Second construction of the regulator. In this subsection and the next one, S is an arithmetic field and X is a smooth scheme over S.

Let $\mathcal{K} : \mathbf{Com}^{\geq 0}(\mathbf{Ab}) \to \Delta^{\mathrm{op}}\mathbf{Ab}$ be the Dold-Kan equivalence on chain complexes concentrated in degrees ≥ 0 (with $\deg d = -1$ and shift given by $C[-1]_a = C_{a-1}$). Recall from Definitions 2.7 and 3.1 the abelian presheaf complex D and $\mathrm{D}_s := \mathcal{K}(\tau_{\geq 0}\mathrm{D})$. We have $\mathrm{H}_n(\mathrm{D}(X)) = \pi_n(\mathrm{D}_s(X)) = \bigoplus_p \mathrm{H}_{\mathrm{D}}^{2p-n}(X, p)$. We set $\mathrm{D}_s[-1] := \mathcal{K}((\tau_{\geq 0}\mathrm{D})[-1])$. Recall that for any chain complex of abelian groups C, there is a natural map $S^1 \wedge \mathcal{K}(C) = \mathrm{cone}(\mathcal{K}(C) \to \mathrm{point}) \to \mathcal{K}(\mathrm{cone}(C \to 0)) = \mathcal{K}(C[-1])$, hence a map $\mathcal{K}(C) \to \Omega_s \mathcal{K}(C[-1])$. (Here and elsewhere, Ω_s is the simplicial loop space; its $\mathbb{P}^1$-analogue is denoted $\Omega_{\mathbb{P}^1}$.) This map is a weak equivalence of simplicial abelian groups.

For any pointed simplicial presheaf $F \in \mathbf{Ho}_\bullet(S)$, let $\varphi(F)$ be the pointed presheaf

(5.7) $$\varphi(F) : \mathbf{Sm}/S \ni X \mapsto \mathrm{Hom}_{\mathbf{Ho}_\bullet(S)}(X_+, F).$$

According to (5.2) and Lemma 3.2, respectively,

$$
\begin{aligned}
(5.8) \qquad \varphi(\mathbb{Z}\times \mathrm{Gr}) &= K_0 : X \mapsto K_0(X), \\
\varphi(\Omega_s^n \mathrm{D}_s) &= \mathrm{H}_\mathrm{D}^{-n} : X \mapsto \bigoplus_p \mathrm{H}_\mathrm{D}^{2p-n}(X,p),\ n \geq 0.
\end{aligned}
$$

Let $\widehat{P}(X)$ be the (essentially small) Waldhausen category consisting of hermitian bundles $\overline{E} = (E,h)$ on X, i.e., a vector bundle E/X with a metric h on $E(\mathbb{C})/X(\mathbb{C})$ that is invariant under Fr_∞^* and smooth at infinity [BW98, Definition 2.5]. Morphisms are given by usual morphisms of bundles, ignoring the metric, so that $\widehat{P}(X)$ is equivalent to the usual category of vector bundles. Let

$$
(5.9) \qquad S_* : \mathbf{Sm}/S \ni X \mapsto \mathrm{Sing}|S_*\widehat{P}(X)|
$$

be the presheaf (pointed by the zero bundle) whose sections are given by the simplicial set of singular chains in the topological realization of the Waldhausen S-construction of $\widehat{P}(X)$. Its homotopy presheaves are
$$
(5.10)
$$
$$
\mathrm{Hom}_{\mathbf{Ho}_{\mathrm{sect},\bullet}(S)}(S^n \wedge X_+, S_*) = \pi_n S_*(X) = \pi_{n-1}\Omega_s S_*(X) \cong K_{n-1}(X),\ n \geq 1.
$$

Here, $\mathbf{Ho}_{\mathrm{sect},\bullet}$ denotes the homotopy category of $\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet(\mathbf{Sm}/S)$ (simplicial pointed presheaves), endowed with the section-wise model structure. K-theory (of regular schemes) is homotopy invariant and satisfies Nisnevich descent [TT90, Thm. 10.8]. Therefore, as is well-known, the left hand term agrees with $\mathrm{Hom}_{\mathbf{Ho}_\bullet(S)}(S^n \wedge X_+, S_*)$. That is, there is an isomorphism of pointed presheaves

$$
(5.11) \qquad \varphi(\Omega_s S_*) \cong K_0.
$$

According to [Rio, III.61], there is a unique isomorphism in $\mathbf{Ho}_\bullet(S)$,

$$
(5.12) \qquad \tau : \mathbb{Z}\times \mathrm{Gr} \to \Omega_s S_*,
$$

making the obvious triangle involving (5.11) and (5.8) commute.

The proof of our comparison of the regulator uses the following result due to Burgos and Wang [BW98, Prop. 3.11, Theorem 5.2., Prop. 6.13]:

Proposition 5.5. *There is a map in* $\Delta^{\mathrm{op}}(\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$,

$$
\mathrm{ch}_S : S_* \to \mathrm{D}_s[-1],
$$

such that the induced map

$$
\pi_n \mathrm{ch}_S : K_{n-1}(X) \to \bigoplus_{p\in\mathbb{Z}} \mathrm{H}_\mathrm{D}^{2p-(n-1)}(X,p)
$$

agrees with the Beilinson regulator for all $n \geq 1$.

By (5.12), we get a map in $\mathbf{Ho}_\bullet(S)$:

$$(5.13) \qquad \mathrm{ch} : \mathbb{Z} \times \mathrm{Gr} \xrightarrow{\tau, \cong} \Omega_s S_* \xrightarrow{\Omega_s \, \mathrm{ch}_S} \Omega_s(\mathrm{D}_s[-1]) \xrightarrow{\cong} \mathrm{D}_s.$$

The induced map

$$K_n(X) \stackrel{(5.4)}{\cong} \mathrm{Hom}_{\mathbf{Ho}_\bullet}(S^n \wedge X_+, \mathbb{Z} \times \mathrm{Gr}) \to \mathrm{Hom}_{\mathbf{Ho}_\bullet}(S^n \wedge X_+, \mathrm{D}_s)$$

$$(5.14) \qquad \stackrel{(3.3)}{\cong} \bigoplus_p \mathrm{H}_{\mathrm{D}}^{2p-n}(X, p)$$

agrees with the Beilinson regulator. In order to lift the map ch to a map in $\mathbf{SH}(S)$, we first check the compatibility with the $\mathbb{P}^1$-spectrum structures to get a map in $\mathbf{SH}^{\mathrm{naive}}(S)$. This means that the diagram involving the bonding maps only has to commute up to ($\mathbb{A}^1$-)homotopy. Then, we apply an argument of Riou to show that this map actually lifts uniquely to one in $\mathbf{SH}(S)$.

Recall the Deligne cohomology ($\mathbb{P}^1$-)spectrum H_{D} from Lemma 3.3. Its p-th level is given by $\mathrm{D}_s(p)$, for any $p \geq 0$.

Theorem 5.6.

(i) In $\mathbf{SH}^{\mathrm{naive}}(S)$, there is a unique map

$$\mathrm{ch}^{\mathrm{naive}} : \mathrm{BGL}_S^{\mathrm{naive}} \to \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{D}}(p)[2p] =: \bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\}$$

that is given by $\mathrm{ch} : \mathbb{Z} \times \mathrm{Gr} \xrightarrow{(5.13)} \mathrm{D}_s$ *in each level.*

(ii) In $\mathbf{SH}(S)$, there is a unique map

$$\mathrm{ch} : \mathrm{BGL}_S \to \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{D}}(p)[2p]$$

that maps to $\mathrm{ch}^{\mathrm{naive}}$ under the forgetful functor $\mathbf{SH}(S) \to \mathbf{SH}^{\mathrm{naive}}(S)$ (Remark 5.2).

(iii) There is a unique map

$$\rho : \mathrm{H}_{E,S} \to \mathrm{H}_{\mathrm{D}}$$

in $\mathbf{SH}(S)_\mathbb{Q}$ such that $\mathrm{ch} \otimes \mathbb{Q} = \bigoplus_{p \in \mathbb{Z}} \rho(p)[2p] : \mathrm{BGL}_\mathbb{Q} \to \oplus \mathrm{H}_{\mathrm{D}}(p)[2p]$, under the identification (5.5).

Proof. By Lemma 5.4, the transition maps (5.1) defining the infinite Grassmannian induce split monomorphisms $\mathrm{M}(\mathrm{Gr}_{d,r}) \to \mathrm{M}(\mathrm{Gr}_{d',r'})$ of motives and therefore (e.g. using Theorem 3.6) split surjections (for any $n \geq 0$, $d \leq d'$, $r \leq r'$)

$$(5.15) \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathbf{Ho}(S)}(\mathrm{Gr}_{d',r'}, \Omega_s^n \mathrm{D}_s) & \to & \mathrm{Hom}_{\mathbf{Ho}(S)}(\mathrm{Gr}_{d,r}, \Omega_s^n \mathrm{D}_s) \\ \| & & \| \\ \mathrm{H}_{\mathrm{D}}^{-n}(\mathrm{Gr}_{d',r'}) & & \mathrm{H}_{\mathrm{D}}^{-n}(\mathrm{Gr}_{d,r}). \end{array}$$

A similar surjectivity statement holds for the map of Deligne cohomology groups induced by transition maps defining the product $\mathrm{Gr} \times \mathrm{Gr}$, i.e.,

$$\mathrm{Gr}_{d_1,r_1} \times \mathrm{Gr}_{d_2,r_2} \to \mathrm{Gr}_{d_1',r_1'} \times \mathrm{Gr}_{d_2',r_2'} \, .$$

(i) the unicity of $\mathrm{ch}^{\mathrm{naive}}$ is obvious. Its existence amounts to the commutativity of the following diagram in $\mathbf{Ho}_\bullet(S)$:

$$(5.16) \qquad
\begin{array}{ccccc}
\mathbb{P}^1 \wedge \mathbb{Z}\times \mathrm{Gr} & \xrightarrow{u_1^* \wedge \mathrm{id}} & (\mathbb{Z}\times \mathrm{Gr}) \wedge (\mathbb{Z}\times \mathrm{Gr}) & \xrightarrow{\mu} & \mathbb{Z}\times \mathrm{Gr} \\
\downarrow{\scriptstyle \mathrm{id}\wedge\mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}\wedge\mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\
\mathbb{P}^1 \wedge \mathrm{D}_s & \xrightarrow{c^* \wedge \mathrm{id}} & \mathrm{D}_s \wedge \mathrm{D}_s & \xrightarrow{\mu} & \mathrm{D}_s.
\end{array}$$

The top and bottom lines are the bonding maps of $\mathrm{BGL}^{\mathrm{naive}}$ (cf. (5.3)) and $\bigoplus_p \mathrm{H}_\mathrm{D}\{p\}$ (cf. Definition and Lemma 3.3), respectively. The map c^* corresponds to the first Chern class $c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \in \mathrm{H}_\mathrm{D}^2(\mathbb{P}^1_S, 1)$. To see the commutativity of the right half, we use that the functor φ (5.7) induces an isomorphism

$$\mathrm{Hom}_{\mathbf{Ho}_\bullet(S)}((\mathbb{Z}\times \mathrm{Gr})^{\wedge 2}, \mathrm{D}_s) = \mathrm{Hom}_{\mathbf{PSh}_\bullet(\mathbf{Sm}/S)}(K_0(-) \wedge K_0, \mathrm{H}_\mathrm{D}^0).$$

This identification is shown exactly as [Rio, III.31], which treats $\mathbb{Z}\times \mathrm{Gr}$ instead of D_s. The point is a surjectivity argument in comparing cohomology groups of products of different Grassmannianns, which is applicable to Deligne cohomology by the remark above. By construction of the multiplication map on $\mathbb{Z}\times \mathrm{Gr}$, applying φ to the right half of (5.16) yields the diagram

$$
\begin{array}{ccc}
K_0 \wedge K_0 & \xrightarrow{\mu_{K_0}} & K_0 \\
\downarrow{\scriptstyle \mathrm{ch}\wedge\mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\
\mathrm{H}_\mathrm{D}^0 \wedge \mathrm{H}_\mathrm{D}^0 & \xrightarrow{\mu_\mathrm{D}} & \mathrm{H}_\mathrm{D}^0.
\end{array}
$$

Here μ_{K_0} is the usual (tensor) product on K_0 and μ_D is the classical product on Deligne cohomology [EV88]. The Beilinson regulator is multiplicative [Sch88, Cor., p. 28], so this diagram commutes.

For the commutativity of the left half, let $i_{m,n} : \mathbb{P}^m \to \mathbb{P}^n$ be the inclusion $[x_0 : \ldots : x_m] \mapsto [x_0 : \ldots : x_m : 0 : \ldots : 0]$, for $m \le n$, and $i_{m,\infty} := \mathrm{colim}_n i_{m,n} : \mathbb{P}^m \to \mathbb{P}^\infty := \mathrm{colim}_n \mathbb{P}^n$. The map u_1^* factors as

$$\mathbb{P}^1 \xrightarrow{i_{1,\infty}} \mathbb{P}^\infty \xrightarrow{u_\infty^*} \mathbb{Z}\times \mathrm{Gr}$$

where $u_\infty^* \in \mathrm{Hom}_{\mathbf{Ho}_\bullet(S)}(\mathbb{P}^\infty, \mathbb{Z}\times \mathrm{Gr})$ is induced by the compatible system $u_n = [\mathcal{O}_{\mathbb{P}^n}(1)] - [\mathcal{O}_{\mathbb{P}^n}] \in K_0(\mathbb{P}^n)$ simply because $i_{1,n}^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^1}(1)$. Similarly, $c^* = c_1(\mathcal{O}(1))$ is given by

$$c^* : \mathbb{P}^1 \xrightarrow{i_{1,\infty}} \mathbb{P}^\infty \xrightarrow{u_\infty^*} \mathbb{Z}\times \mathrm{Gr} \xrightarrow{\mathrm{ch}} \mathrm{D}_s,$$

because $\mathrm{ch}(\mathcal{O}(1)) - \mathrm{ch}(\mathcal{O}) = \exp(c_1(\mathcal{O}(1))) - 1$ which on $\mathbb{P}^1$ equals $c_1(\mathcal{O}(1)) \in \mathrm{H}^2_\mathrm{D}(\mathbb{P}^\infty, 1)$. Then the commutativity of the diagram in question is obvious.

(ii) For each $n \geq 0$ and $m = 0, -1$, put $V_n^m := \mathrm{Hom}_{\mathbf{PSh}(\mathbf{Sm}/S, \mathbf{Ab})}(K_0, \mathrm{H}^m_\mathrm{D})$. These groups form a projective system with transition maps

$$V_{n+1}^m \ni (f_n : K_0 \to \mathrm{H}^m_\mathrm{D}) \mapsto (\Omega_{\mathbb{P}^1} f_n : \Omega_{\mathbb{P}^1} K_0 \to \Omega_{\mathbb{P}^1} \mathrm{H}^m_\mathrm{D}) \in V_n^m,$$

where $\Omega_{\mathbb{P}^1}(F)$ is the presheaf $\mathbf{Sm}/S \ni U \mapsto \ker(F(\mathbb{P}^1_U) \overset{\infty^*}{\to} F(U))$. Indeed, the projective bundle formula (for $\mathbb{P}^1$) implies an isomorphism of presheaves $\Omega_{\mathbb{P}^1} K_0 \cong K_0$ and likewise with H^m_D.

The composition of functors

$$\mathbf{SH} \to \mathbf{SH}^{\mathrm{naive}} \overset{n}{\longrightarrow} \mathbf{Ho}_\bullet \overset{\varphi}{\longrightarrow} \mathbf{PSh}(\mathbf{Sm}/S)$$

actually takes values in $\mathbf{PSh}(\mathbf{Sm}/S, \mathbf{Ab})$. Here, n indicates taking the n-th level of a spectrum. By construction, BGL gets mapped to K_0, and H_D gets mapped to the presheaf $\mathrm{H}^0_\mathrm{D} = \bigoplus_p \mathrm{H}^{2p}_\mathrm{D}(-, p)$ for each $n \geq 0$. This gives rise to the following map (cf. [Rio, IV.11]):

$$\mathrm{Hom}_{\mathbf{SH}}(\mathrm{BGL}, \bigoplus_p \mathrm{H}_\mathrm{D}\{p\}) \to \mathrm{Hom}_{\mathbf{SH}^{\mathrm{naive}}(S)}(\mathrm{BGL}^{\mathrm{naive}}, \bigoplus_p \mathrm{H}_\mathrm{D}\{p\}) \cong \varprojlim_n V_n^0.$$

This map is part of the following Milnor-type short exact sequence [Rio, IV.48, III.26; see also IV.8] (it is applicable because of the surjectivity of (5.15) for $n = 1$ and $n = 2$):

$$(5.17) \qquad 0 \to \mathrm{R}^1 \varprojlim_n V_n^{-1} \to \mathrm{Hom}_{\mathbf{SH}}(\mathrm{BGL}, \bigoplus_p \mathrm{H}_\mathrm{D}\{p\}) \to \varprojlim_n V_n^0 \to 0.$$

The map $\mathrm{ch}^{\mathrm{naive}}$ thus corresponds to a unique element in the right-most term of (5.17). The natural map

$$\begin{aligned}
V_n^{-1} = \mathrm{Hom}_{\mathbf{PSh}(\mathbf{Ab})}(K_0, \mathrm{H}^{-1}_\mathrm{D}) &\to \varprojlim_e \bigoplus_p \mathrm{H}^{2p-1}_\mathrm{D}(\mathbb{P}^e_S, p) \\
&\cong \bigoplus_{p \in \mathbb{Z}} \bigoplus_{j=0}^p \mathrm{H}^{2p-2j-1}_\mathrm{D}(S, p - j) \\
f &\mapsto (f(\mathcal{O}_{\mathbb{P}^e}(1)))_e
\end{aligned}$$

is an isomorphism. Indeed, the proof of the analogous statement for motivic cohomology instead of Deligne cohomology [Rio, V.18] (essentially a splitting argument) only uses the calculation of motivic cohomology of $\mathbb{P}^e$. Thus it goes through by the projective bundle formula for Deligne cohomology.

Via this identification, the transition maps $\Omega_{\mathbb{P}^1} : V_{n+1}^{-1} \to V_n^{-1}$ are the direct sum over $p \in \mathbb{Z}$ of the maps

$$\bigoplus_{j=0}^p \mathrm{H}^{2p-2j-1}_\mathrm{D}(S, p - j) \to \bigoplus_{j=0}^{p-1} \mathrm{H}^{2(p-1)-2j-1}_\mathrm{D}(S, (p-1) - j),$$

which are the multiplication by j on the j-th summand at the left. Again, this is analogous to [Rio, V.24]. In particular $\Omega_{\mathbb{P}^1}$ is onto, since Deligne cohomology groups are divisible. Therefore $\mathrm{R}^1 \varprojlim V_n^{-1} = 0$, so (ii) is shown.

(iii) As in [Rio, V.36], one sees that $\mathrm{ch} \otimes \mathbb{Q}$ factors over the projections $\mathrm{BGL}_{\mathbb{Q}} \to \mathrm{H}_{\mathrm{B}}$ and $\bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{D}}(p)[2p] \to \mathrm{H}_{\mathrm{D}}$. $\qquad\square$

5.3. Comparison.

Theorem 5.7. *The regulator maps* ch, ρ *constructed in Theorem* 5.6 *and the regulator maps* ch_{D}, ρ_{D} *obtained in Definition* 3.7 *agree:*

$$
\begin{aligned}
\mathrm{ch}_{\mathrm{D}} &= \mathrm{ch} \in \mathrm{Hom}_{\mathbf{SH}(S)}(\mathrm{BGL}, \bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\}), \\
\rho_{\mathrm{D}} &= \rho \in \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(\mathrm{H}_B, \mathrm{H}_{\mathrm{D}}).
\end{aligned}
$$

In particular, ch_{D} *also induces the Beilinson regulator* $K_n(X) \to \bigoplus_p \mathrm{H}_{\mathrm{D}}^{2p-n}(X, p)$ *for any* $X \in \mathbf{Sm}/S$, $n \geq 0$.

Proof. The map ch is a map of ring spectra (i.e., monoid objects in $\mathbf{SH}(S)$): the multiplicativity, i.e., $\mathrm{ch} \circ \mu_{\mathrm{BGL}} = \mu_{\mathrm{D}} \circ (\mathrm{ch} \wedge \mathrm{ch})$ follows from the right half of the diagram (5.16). The unitality boils down to $\mathrm{ch}(\mathcal{O}) = 1 \in \mathrm{H}_{\mathrm{D}}^0(S, 0)$. We define a BGL-module structure on $\mathcal{D} := \bigoplus_{p \in \mathbb{Z}} \mathrm{H}_{\mathrm{D}}\{p\}$ in the usual manner:

$$
\mathrm{BGL} \wedge \mathcal{D} \xrightarrow{\mathrm{ch} \wedge \mathrm{id}} \mathcal{D} \wedge \mathcal{D} \xrightarrow{\mu} \mathcal{D}.
$$

It is indeed a BGL-module, as one sees using that ch is a ring morphism. By the unicity of the BGL-algebra structure on $\mathcal{D}$ (Theorem 3.6), this algebra structure agrees with the one established in Theorem 3.6. This implies $\mathrm{ch} = \mathrm{ch}_{\mathrm{D}}$. The proof for $\rho = \rho_{\mathrm{D}}$ is similar, replacing BGL with H_{B} throughout. $\qquad\square$

6. Comparison with arithmetic K-theory and arithmetic Chow groups

In this section, we show that the groups represented by $\widehat{\mathrm{BGL}}$ coincide with a certain subgroup of arithmetic K-theory as defined by Gillet-Soulé and Takeda for smooth schemes over appropriate bases (Theorem 6.1). This isomorphism is compatible with the Adams operations on both sides and with the module structure over K-theory (Corollary 6.2, Theorem 6.3). We also establish the compatibility of the comparison isomorphism with the pushforward in two cases (Theorem 6.4).

We consider the following situation: $X \to S \to B$, where B is a fixed arithmetic ring (Definition 2.6), S is a regular scheme (of finite type) over B (including the important case $S = B$), and $X \in \mathbf{Sm}/S$. Let $\eta : B_\eta := B \times_{\mathbb{Z}} \mathbb{Q} \to B$ be the "generic fiber". For any datum ? related to Deligne

cohomology, we write $? := \eta_* ?$ for simplicity of notation. That is, $D_s(X) := \eta_* D_s(X) = D_s(X \times_B B_\eta)$, $H_D := \eta_* H_D \in \mathbf{SH}(S)$, etc.

For a proper arithmetic variety X (i.e., X is regular and flat over an arithmetic ring B), Gillet and Soulé have defined the arithmetic K-group as the free abelian group generated by pairs $(\overline{E}, \alpha)$, where $\overline{E}/X$ is a hermitian vector bundle and $\alpha \in D_0(X)/\operatorname{im} d_D$, modulo the relation

$$(\overline{E}', \alpha') + (\overline{E}'', \alpha'') = (\overline{E}, \alpha' + \alpha'' + \widetilde{ch}(\overline{\mathcal{E}}))$$

for any extension

$$\overline{\mathcal{E}} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

of hermitian bundles. Here $\widetilde{ch}(\overline{\mathcal{E}})$ is a secondary Chern class of the extension (see [GS90c, Section 6] for details). We denote this group by $\widehat{K}_0^T(X)$. The superscript T stands for Takeda, who generalized this to higher n [Tak05, p. 621].[3] These higher arithmetic K-groups $\widehat{K}_n^T(X)$ fit into a commutative diagram with exact rows and columns, where $\widehat{K}_n(X) := \ker ch^T$ and $B_n^D(X) := \operatorname{im} d_D : D_{n+1}(X) \to D_n(X)$:
(6.1)

The full arithmetic K-groups $\widehat{K}_*^T$ are not accessible to homotopy theory since they fail to be $\mathbb{A}^1$-invariant. Moreover, due to the presence of $D_{n+1}/\operatorname{im} d_D$ the groups are usually very large. Therefore, we focus on the subgroups $\widehat{K}_* \subset \widehat{K}_*^T$ and refer to them as arithmetic K-theory.

By Theorem 5.7, the top exact sequence looks exactly like the one in Theorem 4.5. In order to show that $\widehat{K}_n(X)$ and $\widehat{H}^{-n}(X)$ are isomorphic, we use that there is a natural isomorphism (functorial with respect to pullback),

$$(6.2) \qquad \widehat{K}_n(X) \cong \pi_{n+1}(\operatorname{hofib}_{\Delta^{op}\mathbf{Sets}_\bullet} S_*(X) \xrightarrow{ch_S} D_s[-1](X)), \quad n \geq 0,$$

of the arithmetic K-group with the homotopy fiber in pointed simplicial sets (endowed with its standard model structure) [Tak05, Cor. 4.9]. We write

$$\widehat{S} := \operatorname{hofib}_{\Delta^{op}\mathbf{PSh}_\bullet(\mathbf{Sm}/S)}(S_* \to D_s[-1]),$$

---

[3] Gillet and Soulé use a slightly different normalization of the Chern class which differs from the one used by Burgos-Wang, Takeda (and this paper) by a factor of $2(2\pi i)^n$ for appropriate n. See [GS90c] for details.

for the homotopy fiber with respect to the section-wise model structure, so that $\pi_{n+1}\left(\widehat{S}(X)\right) = \widehat{\mathrm{K}}_n(X)$.

Recall from Section 4.1 the object $\widehat{\mathrm{BGL}}$. Its key property is the existence of a distinguished triangle (in $\mathbf{SH}(S)$):

$$(6.3) \qquad \bigoplus_p \mathrm{H}_\mathrm{D}\{p\}[-1] \to \widehat{\mathrm{BGL}} \to \mathrm{BGL} \overset{\mathrm{ch}}{\to} \bigoplus_p \mathrm{H}_\mathrm{D}\{p\}.$$

The cohomology groups represented by this object are denoted by $\widehat{\mathrm{H}}^*(-)$; cf. Definition 4.4.

The content of the following theorems and corollary (6.1, 6.2, 6.3, 6.4) can be paraphrased as follows: given a commutative diagram in some triangulated category,

$$
\begin{array}{ccccccc}
B[-1] & \longrightarrow & E & \longrightarrow & A & \longrightarrow & B \\
\downarrow{\scriptstyle b[-1]} & & \vdots{\scriptstyle e} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
B'[-1] & \longrightarrow & E' & \longrightarrow & A' & \longrightarrow & B',
\end{array}
$$

the map e (whose existence is granted by the axioms of a triangulated category) is in general not unique. The unicity of e is guaranteed if the map

$$(6.4) \qquad \mathrm{Hom}(E, A'[-1]) \to \mathrm{Hom}(E, B'[-1])$$

is onto. In our situation, we are aiming at a canonical comparison between, say, the groups $\widehat{\mathrm{H}}^*$ and $\widehat{\mathrm{K}}_*$. Both theories arise from distinguished triangles where two of the three vertices are the same, namely the one responsible for K-theory and the one for Deligne cohomology. Moreover, the map between them considered *up to homotopy*, i.e., in the triangulated category $\mathbf{SH}$, is the Chern class that is independent of choices—as opposed to the Chern form, which does depend on the choice of a hermitian metric on the vector bundle in question. As we shall see, the non-formal surjectivity of (6.4) is assured by conditions (a) and (b) of Theorem 6.1 (or condition (c) if one neglects torsion). Luckily, it only consists of an injectivity condition for the regulator on the base scheme S, not on all schemes $X \in \mathbf{Sm}/S$. This is one of the places where working with the objects representing the cohomology theories we are interested in is much more powerful than working with the individual cohomology groups.

Theorem 6.1. *Let S be a regular scheme over an arithmetic ring. We suppose that*

(a) $\mathrm{ch} : K_0(S) \to \mathrm{H}_\mathrm{D}^0(S) = \bigoplus_p \mathrm{H}_\mathrm{D}^{2p}(S, p)$ *is injective, and*
(b) $K_1(S)$ *is the direct sum of a finite and a divisible group.*

For example, these conditions are satisfied for $S = B = \mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$. Then the following hold:

(i) *Given any maps s, d in $\mathbf{Ho}_{\bullet}(S)$ such that the right square commutes, there is a unique $\widehat{s} \in \mathrm{End}_{\mathbf{Ho}(S)}(\widehat{S})$ making the diagram commute:*

$$
\begin{array}{ccccccc}
\mathrm{D}_s = \Omega_s \mathrm{D}_s[-1] & \longrightarrow & \widehat{S} & \longrightarrow & S_* & \xrightarrow{\mathrm{ch}_S} & \mathrm{D}_s[-1] \\
\downarrow{\scriptstyle \Omega_s d} & & \downarrow{\scriptstyle \widehat{s}} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle d} \\
\mathrm{D}_s = \Omega_s \mathrm{D}_s[-1] & \longrightarrow & \widehat{S} & \longrightarrow & S_* & \xrightarrow{\mathrm{ch}_S} & \mathrm{D}_s[-1].
\end{array}
$$

(ii) *Likewise, given any b and d making the right half commute in $\mathbf{SH}(S)$, there is a unique $\widehat{b} \in \mathrm{End}_{\mathbf{SH}(S)}(\widehat{\mathrm{BGL}})$ making everything commute:*

$$
\begin{array}{ccccccc}
\bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\}[-1] & \longrightarrow & \widehat{\mathrm{BGL}} & \longrightarrow & \mathrm{BGL} & \xrightarrow{\mathrm{ch}} & \bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\} \\
\downarrow{\scriptstyle d[-1]} & & \downarrow{\scriptstyle \widehat{b}} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\
\bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\}[-1] & \longrightarrow & \widehat{\mathrm{BGL}} & \longrightarrow & \mathrm{BGL} & \xrightarrow{\mathrm{ch}} & \bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\}.
\end{array}
$$

(iii) *The aforementioned unicity results give rise to a canonical isomorphism, functorial with respect to pullback,*

$$
(6.5) \qquad\qquad \widehat{\mathrm{K}}_n(X) \cong \widehat{\mathrm{H}}^{-n}(X/S),
$$

for any $X \in \mathbf{Sm}/S$, $n \geq 0$. (The definition of $\widehat{\mathrm{K}}_n(X)$ in [Tak05] is only done for X/B proper, but can be generalized to non-proper varieties using differential forms with logarithmic poles at infinity, as in Definition 2.7.)

Instead of (a) *and* (b), *let us suppose that*

(c) $\mathrm{ch} : K_0(S)_{\mathbb{Q}} \to \mathrm{H}_{\mathrm{D}}^0(S) = \bigoplus_p \mathrm{H}_{\mathrm{D}}^{2p}(S, p)$ *is injective. For example, this applies to arithmetic fields and open subschemes of $\mathrm{Spec}\,\mathcal{O}_F$ for a number ring $\mathcal{O}_F$.*

Then there is a canonical isomorphism

$$
(6.6) \qquad\qquad \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}} \cong \widehat{\mathrm{H}}^{-n}(X/S)_{\mathbb{Q}}.
$$

Proof of (ii). Let us write $(-,-) := \mathrm{Hom}_{\mathbf{SH}(S)}(-,-)$ and $R := \bigoplus_{p\in\mathbb{Z}} \mathrm{H_D}\{p\}$. Then we have exact sequences

(6.7)

$$
\begin{array}{ccc}
(R, R[-1]) & \xrightarrow{\ \alpha\ } & (\mathrm{BGL}, R[-1]) \\
\downarrow & & \downarrow{\scriptstyle\beta} \\
(R, \widehat{\mathrm{BGL}}) \longrightarrow (\mathrm{BGL}, \widehat{\mathrm{BGL}}) \longrightarrow (\widehat{\mathrm{BGL}}, \widehat{\mathrm{BGL}}) \xrightarrow{\ \delta\ } (R[-1], \widehat{\mathrm{BGL}}) \\
& & \downarrow \\
& & (\mathrm{BGL}, \mathrm{BGL}) \\
& & \downarrow{\scriptstyle\gamma} \\
& & (\mathrm{BGL}, R).
\end{array}
$$

We prove the injectivity of δ by showing that both α and β are surjective. For any Ω-spectrum $E \in \mathbf{SH}(S)$ whose levels E_n are H-groups such that the transition maps (5.1) induce surjections $\mathrm{Hom}_{\mathbf{Ho}}(\mathrm{Gr}_{d,r}, \Omega_s^m E_n) \to \mathrm{Hom}_{\mathbf{Ho}}(\mathrm{Gr}_{d',r'}, \Omega_s^m E_n)$ for $m = 1, 2$, $n \geq 0$, there is an exact sequence

$$0 \to \mathrm{R}^1 \varprojlim E_\Omega^1 \to \mathrm{Hom}_{\mathbf{SH}}(\mathrm{BGL}, E) \to \varprojlim E_\Omega^0 \to 0.$$

Here, for any group A, A_Ω is the projective system

$$A_\Omega : \quad \dots A[[t]] \to A[[t]] \to A[[t]] \to \dots \to A[[t]],$$

with transition maps $f \mapsto (1+t)df/dt$ and $E^r := \mathrm{Hom}_{\mathbf{SH}}(S^r, E)$ for $r = 0, 1$ [Rio, IV.48, 49]. This applies to $E = \mathrm{BGL}$ and $E = R$; cf. (5.15):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{R}^1 \varprojlim(K_1(S)_\Omega) & \longrightarrow & \mathrm{End}(\mathrm{BGL}) & \longrightarrow & \varprojlim(K_0(S)_\Omega) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_p \mathrm{R}^1 \varprojlim(\mathrm{H_D^{-1}}(S)_\Omega) & \longrightarrow & \mathrm{Hom}(\mathrm{BGL}, R) & \longrightarrow & \bigoplus_p \varprojlim(\mathrm{H_D^0}(S)_\Omega) & \longrightarrow & 0.
\end{array}
$$

The left hand upper term is 0 by assumption (b) and the vanishing of $\mathrm{R}^1 \varprojlim A_\Omega$ for a finite or a divisible group A [Rio, IV.40, IV.58]. The right hand vertical map $\varprojlim \mathrm{ch}$ is injective by assumption (a) and the left-exactness of $\varprojlim$. Hence γ is injective, so β is onto.

The surjectivity of α does not make use of the assumptions (a), (b). Indeed,

$$\mathrm{Hom}(\mathrm{BGL}, R[-1]) = \prod_{q\in\mathbb{Z}} \mathrm{Hom}(\mathrm{H_D}\{q\}, R[-1]) \overset{3.6(ii)}{=} \prod_q \mathrm{H_D^{-1}}(S).$$

Given some $x \in \mathrm{H_D^{-1}}(S)$, pick any representative $\xi \in \ker(\mathrm{D}_1(S) \to \mathrm{D}_0(S))$ and define $y : \mathrm{H_D}\{q\} \to R$ to be the cup product with ξ. Then $\alpha(y) = x$.

(i) We need to establish the injectivity of the map in the first row:
(6.8)

$$\begin{array}{ccc}
\mathrm{End}_{\mathbf{Ho}_\bullet(S)}(\widehat{S}) & \longrightarrow & \mathrm{Hom}_{\mathbf{Ho}_\bullet(S)}(\Omega_s \mathrm{D}_s[-1], \widehat{S}) \\
\| & & \| \\
\mathrm{End}_{\mathbf{Ho}_\bullet(S)}(\Omega_{\mathbb{P}^1}^\infty \widehat{\mathrm{BGL}}) & \longrightarrow & \mathrm{Hom}_{\mathbf{Ho}_\bullet(S)}(\Omega_{\mathbb{P}^1}^\infty \mathrm{H_D}[-1], \Omega_{\mathbb{P}^1}^\infty \widehat{\mathrm{BGL}}) \\
\| & & \| \\
\mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma_{\mathbb{P}^1}^\infty \Omega_{\mathbb{P}^1}^\infty \widehat{\mathrm{BGL}}, \widehat{\mathrm{BGL}}) & \longrightarrow & \mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma_{\mathbb{P}^1}^\infty \Omega_{\mathbb{P}^1}^\infty \mathrm{H_D}[-1], \widehat{\mathrm{BGL}}) \\
{\scriptstyle \Sigma_{\mathbb{P}^1}^\infty \rightleftarrows \Omega_{\mathbb{P}^1}^\infty} \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbf{SH}(S)}(\widehat{\mathrm{BGL}}, \widehat{\mathrm{BGL}}) & \xrightarrow{\ \delta\ } & \mathrm{Hom}_{\mathbf{SH}(S)}(\mathrm{H_D}[-1], \widehat{\mathrm{BGL}}).
\end{array}$$

The counit map $\Sigma_{\mathbb{P}^1}^\infty \Omega_{\mathbb{P}^1}^\infty \to \mathrm{id}$ is an isomorphism when applied to BGL and $\mathrm{H_D}[-1]$ (and thus $\mathrm{H_D}[-1]$), since these two spectra are Ω-spectra. Therefore, the same is true for $\widehat{\mathrm{BGL}}$. We are done by (ii).

(iii) We obtain the sought isomorphism as the following composition:

$$\begin{array}{rcl}
\widehat{\mathrm{H}}^{-n}(X/S) & := & \mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma_{\mathbb{P}^1}^\infty S^n \wedge X_+, \mathrm{hofib}(\mathrm{BGL} \xrightarrow{\mathrm{id}\wedge 1_{\mathrm{H_D}}} \mathrm{BGL} \wedge \mathrm{H_D})) \\
(6.9) & = & \mathrm{Hom}_{\mathbf{SH}(S)}(\Sigma_{\mathbb{P}^1}^\infty S^n \wedge X_+, \mathrm{hofib}(\mathrm{BGL} \xrightarrow{\mathrm{ch}} \bigoplus_p \mathrm{H_D}\{p\})) \\
(6.10) & = & \mathrm{Hom}_{\mathbf{Ho}(S)}(S^n \wedge X_+, \mathrm{hofib}(\mathbb{Z}\times \mathrm{Gr} \xrightarrow{\mathrm{ch}_0} \mathrm{D}_s)) \\
(6.11) & = & \mathrm{Hom}_{\mathbf{Ho}(S)}(S^n \wedge X_+, \mathrm{hofib}(\Omega_s S_* \xrightarrow{\mathrm{ch}_S} \mathrm{D}_s)) \\
& = & \mathrm{Hom}_{\mathbf{Ho}(S)}(S^n \wedge X_+, \mathrm{hofib}(\Omega_s S_* \xrightarrow{\mathrm{ch}_S} \mathrm{D}_s)) \\
(6.12) & = & \mathrm{Hom}_{\mathbf{Ho}_{\mathrm{sect},\bullet}(S)}(S^{n+1} \wedge X_+, \mathrm{hofib}(S_* \to \mathrm{D}_s[-1])) \\
& = & \pi_{n+1}\left(\mathrm{hofib}_{\Delta^{\mathrm{op}}\mathbf{Sets}_\bullet}(S_*(X) \xrightarrow{\mathrm{ch}_S} \mathrm{D}_s[-1](X))\right) \\
& \overset{(6.2)}{\cong} & \widehat{\mathrm{K}}_n(X).
\end{array}$$

The canonical isomorphism (6.9) follows from (ii): we can pick representatives of BGL and of $\mathrm{ch} : \mathrm{BGL} \to \oplus \mathrm{H_D}\{p\}$ (Theorem 5.6(ii)) in the underlying model category $\mathbf{Spt}$. We will denote them by the same symbols. We get a diagram of maps in $\mathbf{Spt} := \mathbf{Spt}^{\mathbb{P}^1}(\Delta^{\mathrm{op}}\mathbf{PSh}_\bullet(\mathbf{Sm}/S))$:

$$\begin{array}{ccc}
\text{hofib}(\text{id} \wedge 1_{H_D}) \longrightarrow \text{BGL} \xrightarrow{\text{id} \wedge 1_{H_D}} \text{BGL} \wedge H_D \\
\Big\downarrow \alpha \qquad\qquad \Big\| \qquad\qquad \Big\downarrow \text{ch} \\
\text{hofib}(\text{ch}) \longrightarrow \text{BGL} \xrightarrow{\text{ch}} \bigoplus_p H_D\{p\}.
\end{array}$$

The Chern character for motivic cohomology and Theorem 3.6(iii) induce an isomorphism ch : BGL $\wedge$ H$_D$ $\cong$ $\bigoplus_p$ H$_D\{p\}$ in $\mathbf{SH}(S)$. As $\mathbf{SH}(S)$ is triangulated, we get some (a priori non-unique) isomorphism α in $\mathbf{SH}(S)$. By (ii), however, it is unique.

Similarly, the isomorphism (6.11) follows from (i): still using the above lift of ch to $\mathbf{Spt}$, ch$_0 := \Omega^\infty_{\mathbb{P}^1}$ ch is a map of simplicial presheaves. The isomorphism $\tau : \mathbb{Z} \times \text{Gr} \cong \Omega_s S_*$ (5.12) can be lifted to a map $\tilde{\tau}$ of presheaves

$$\begin{array}{ccc}
\text{hofib ch}_0 \longrightarrow \mathbb{Z} \times \text{Gr} \xrightarrow{\text{ch}_0} D_s \\
\Big\downarrow \qquad\qquad \tilde{\tau}\Big\downarrow \qquad\qquad \Big\| \\
\text{hofib ch}_S \longrightarrow \Omega_s S_* \xrightarrow{\text{ch}_S} D_s.
\end{array}$$

The right hand square may not commute in $\Delta^{\text{op}}\mathbf{PSh}(\mathbf{Sm}/S)$, but it does in $\mathbf{Ho}_\bullet(S)$. By (i), the resulting isomorphism (in $\mathbf{Ho}_\bullet(S)$) between hofib$_{\Delta^{\text{op}}\mathbf{PSh}}$(ch$_0$) and hofib$_{\Delta^{\text{op}}\mathbf{PSh}}$(ch$_S$) is independent of the choice of $\tilde{\tau}$ and ch$_0$.

In order to explain the canonical isomorphisms (6.10), (6.12), recall the following generalities on model categories: let

$$F : \mathbf{C} \leftrightarrows \mathbf{D} : G$$

be a Quillen adjunction and let a diagram $\delta : d_1 \xrightarrow{f} d_2 \leftarrow *$ in $\mathbf{D}$ be given. The homotopy fiber of f is a fibrant replacement of the homotopy pullback of δ. If $\mathbf{C}$ and $\mathbf{D}$ are right proper and d_1 and d_2 are *fibrant*, then the homotopy pullback agrees with the homotopy limit and holim $G(\delta)$ is weakly equivalent to G holim(δ). Finally, replacing any object in δ by a fibrant replacement yields a weakly equivalent homotopy fiber [Hir03, 19.5.3, 19.4.5, 13.3.4]. Thus

(6.13) $\qquad \text{Hom}_{\mathbf{Ho}(\mathbf{D})}(F(c), \text{hofib } f) = \text{Hom}_{\mathbf{Ho}(\mathbf{C})}(c, \text{hofib } G(f)).$

We apply this to the Quillen adjunctions

$$\Delta^{\text{op}}(\mathbf{PSh}_\bullet(\mathbf{Sm}/X)) \overset{\text{id}}{\underset{\text{id}}{\leftrightarrows}} \Delta^{\text{op}}(\mathbf{PSh}_\bullet(\mathbf{Sm}/X)) \overset{\Omega^\infty_{\mathbb{P}^1}}{\underset{\Sigma^\infty_{\mathbb{P}^1}}{\leftrightarrows}} \mathbf{Spt}^{\mathbb{P}^1}(\mathbf{PSh}_\bullet(\mathbf{Sm}/X)).$$

The leftmost category is endowed with the section-wise model structure, then the Nisnevich-$\mathbb{A}^1$-local one, and the stable model structure at the right. These

model structures are proper [GJ99, II.9.6], [MV99, 3.2, p. 86], [Jar00, 4.15]. The simplicial presheaf D_s is fibrant with respect to the section-wise model structure, since it is a presheaf of simplicial abelian groups. Moreover, it is $\mathbb{A}^1$-invariant and has Nisnevich descent by Theorem 2.8(vi). Therefore, it is fibrant with respect to the Nisnevich-$\mathbb{A}^1$-local model structure. Moreover, H_D is an Ω-spectrum by Lemma 3.5, so it is a fibrant spectrum (any level-fibrant Ω-spectrum is stably fibrant [Jar00, 2.7]). For (6.10), we may pick a fibrant representative of BGL (still denoted BGL) such that $\Omega_{\mathbb{P}^1}^\infty \mathrm{BGL} =: V$ is weakly equivalent to $\mathbb{Z} \times \mathrm{Gr}$. Again using (i), the homotopy fibers of $\Omega_{\mathbb{P}^1}^\infty(\mathrm{ch}) : V \to D_s$ and of $\mathrm{ch}_0 : \mathbb{Z} \times \mathrm{Gr} \to D_s$ are canonically weakly equivalent. Finally, the S-construction presheaf S_* (cf. (5.9)) is $\mathbb{A}^1$-invariant (since $K_*(X) \cong K'_*(X)$ for all $X \in \mathbf{Sm}/S$ by the regularity of S) and Nisnevich local for all regular schemes [TT90, Thm. 10.8] and consists of Kan simplicial sets by definition. Hence S_* is a fibrant simplicial presheaf in the $\mathbb{A}^1$-model structure. Therefore, (6.10), (6.12) are fibrant, so these isomorphisms follow from (6.13).

The statement with rational coefficients is similar: one replaces S_*, which is given by simplicial chains in the topological realization of the S-construction, by its version with rational coefficients. Likewise, one replaces BGL by its $\mathbb{Q}$-localization (using the additive structure of $\mathbf{SH}(S)$) $\mathrm{BGL}_{\mathbb{Q}}$. Then condition (a) gets replaced by (c) and (b) becomes unnecessary, since the groups $\mathrm{R}^1 \varprojlim A^\Omega$ encountered above vanish for a divisible group A. $\square$

6.1. Adams operations.
Theorem 6.1 can colloquially be summarized by saying that any construction on $\widehat{\mathrm{K}}_*$, etc., that is both compatible with the classical constructions on K-theory and Deligne cohomology and canonical enough to be lifted to the category $\mathbf{SH}(S)$ (or $\mathbf{Ho}(S)$) is unique. We now use this to study Adams operations on arithmetic K-theory. In Section 6.2 below, this principle is used to identify the BGL-module structure on $\widehat{\mathrm{BGL}}$.

The arithmetic K-groups are endowed with Adams operations

$$(6.14) \qquad \Psi_{\widehat{\mathrm{K}}}^k : \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}} \to \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}.$$

This is due to Gillet and Soulé [GS90c, Section 7] for $n = 0$ and to Feliu in general [Fel10, Theorem 4.3]. Writing

$$\widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}^{(p)} := \{ x \in \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}, \Psi_{\widehat{\mathrm{K}}}^k(x) = k^p \cdot x \ \text{ for all } k \geq 1 \}$$

for the Adams eigenspaces, the obvious question

$$(6.15) \qquad \bigoplus_{p \geq 0} \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}^{(p)} \overset{?}{=} \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}$$

was answered positively for $n = 0$ in [GS90c], but could not be solved for $n > 0$ by Feliu since the management of explicit homotopies between the chain maps representing the Adams operations becomes increasingly difficult

for higher K-theory. In this section, we show that the above Adams operations agree with the natural ones on $\widehat{\mathrm{H}}^*(X)_{\mathbb{Q}}$ and thereby settle the question (6.15) affirmatively.

Feliu establishes a commutative diagram of presheaves of abelian groups:

$$
\begin{array}{ccc}
C_1 := N\widehat{C}_* & \xrightarrow{\text{ch}_1} & \mathrm{D}_* \\
\downarrow{\scriptstyle \Psi^k} & & \downarrow{\scriptstyle \Psi_{\mathrm{D}}^k} \\
C_2 := \tilde{\mathbb{Z}}\widehat{C}_*^{\widetilde{\mathbb{P}}} & \xrightarrow{\text{ch}_2} & \mathrm{D}_*.
\end{array}
$$

The Adams operation Ψ_{D}^k is the canonical one on a graded vector space:

$$
\Psi_{\mathrm{D}}^k : \mathrm{D}_* := \bigoplus_p \mathrm{D}_*(p) \to \bigoplus_p \mathrm{D}_*(p), \Psi^k = \bigoplus_p (k^p \cdot \mathrm{id}).
$$

The complexes C_i at the left hand side are certain complexes of abelian presheaves defined in [Fel10]. They come with maps $\Omega_s S_* \to \mathcal{K}(C_i)$ that induce isomorphisms $K_* \otimes \mathbb{Q} = \pi_*(\Omega_s S_*) \otimes \mathbb{Q} \to \mathrm{H}_*(C_i) \otimes \mathbb{Q}$, $i = 1, 2$. By means of these isomorphisms, Ψ^k corresponds to the usual Adams operation on K-theory (tensored with $\mathbb{Q}$). Moreover, both maps ch_i induce the Beilinson regulator from K-theory to Deligne cohomology.

Recall also the definition of the arithmetic Chow group from [GS90a, Section 3.3] in the proper case and [Bur97, Section 7] in general. In a nutshell, the group $\widehat{\mathrm{CH}}_{\mathrm{GS}}^p(X)$ is generated by arithmetic cycles (Z, g), where $Z \subset X$ is a cycle of codimension p and g is a Green current for Z, i.e., a real current satisfying $\mathrm{Fr}_\infty^* g = (-1)^{p-1} g$ such that $\omega(Z, g) := -\frac{1}{2\pi i} \partial \overline{\partial} g + \delta_Z$ is the current associated to a C^∞ differential form (and therefore an element of $\mathrm{D}_0(p)(X)$). Here δ_Z is the Dirac current of $Z(\mathbb{C}) \subset X(\mathbb{C})$. In analogy to the relation of $\widehat{\mathrm{K}}_0^T(X)$ vs. $\widehat{\mathrm{K}}_0(X)$, we put

$$
(6.16) \qquad \widehat{\mathrm{CH}}^p(X) := \ker(\omega : \widehat{\mathrm{CH}}_{\mathrm{GS}}^p(X) \to \mathrm{D}_0(p)(X)).^4
$$

Corollary 6.2. *Under the assumption of Theorem 6.1(c), the isomorphism $\widehat{\mathrm{K}}_n(X)_{\mathbb{Q}} \cong \widehat{\mathrm{H}}^{-n}(X)_{\mathbb{Q}}$ is compatible with the Adams operations $\Psi_{\widehat{\mathrm{K}}}^k$ on the left and, using the Arakelov-Chern class established in Theorem 4.2, the canonical ones on the graded vector space on $\widehat{\mathrm{H}}^{-n}(X)_{\mathbb{Q}} \cong \bigoplus_{p \in \mathbb{Z}} \widehat{\mathrm{H}}^{2p-n}(X, p)$. In particular, there are canonical isomorphisms*

$$
(6.17) \qquad\qquad\qquad \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}^{(p)} = \widehat{\mathrm{H}}^{2p-n}(X, p),
$$

$$
(6.18) \qquad \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}} = \widehat{\mathrm{K}}_0(X)_{\mathbb{Q}}^{(p)} = \widehat{\mathrm{H}}^{2p}(X, p),
$$

$$
(6.19) \qquad\qquad \bigoplus_{p \in \mathbb{Z}} \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}^{(p)} = \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}.
$$

---

[4] The group $\widehat{\mathrm{CH}}^p(X)$ is denoted $\widehat{\mathrm{CH}}^p(X)_0$ in [GS90a].

Proof. We write $\Omega_{s,\mathbb{Q}}A := \varinjlim C_*(\Omega|A|)$ for any pointed connected simplicial set A. Here, $|-| : \Delta^{\mathrm{op}}\mathbf{Sets} \rightleftarrows \mathbf{Top} : C_*$ is the usual Quillen adjunction, Ω is the (topological) loop space, the direct limit is indexed by $\mathbb{Z}^{>0}$ ordered by divisibility, and the transition maps $\Omega|A| \to \Omega|A|$ are the maps that correspond to the multiplication in $\pi_1(A)$. Then $\pi_n\Omega_{s,\mathbb{Q}}(A) = (\pi_n\Omega_s(A))\otimes_{\mathbb{Z}}\mathbb{Q}$ for all $n \geq 0$. The construction is functorial, so it applies to the simplicial presheaf S_* and gives us a $\mathbb{Q}$-rational variant denoted $S_{*,\mathbb{Q}}$. The map $\Psi^k : C_1 \to C_2$ yields an endomorphism $\Psi_S^k \in \mathrm{End}_{\mathbf{Ho}(S)}(S_{*,\mathbb{Q}})$. Moreover, the maps ch_i, $i = 1, 2$, mentioned above factor over $\mathrm{ch}_{i,\mathbb{Q}} : S_{*,\mathbb{Q}} \to \mathrm{D}_s[-1]$, and the obvious diagram ch_1, ch_2, Ψ_{D}^k and Ψ_S^k commutes up to simplicial homotopy, i.e., in $\mathbf{Ho}_{\mathrm{sect},\bullet}(S)$, a fortiori in $\mathbf{Ho}(S)$. By Theorem 6.1(i), therefore, we obtain a unique map $\Psi_{\widehat{S}}^k \in \mathrm{End}_{\mathbf{Ho}(S)}(\widehat{S}_{*,\mathbb{Q}})$, where $\widehat{S}_{*,\mathbb{Q}} := \mathrm{hofib}\,\mathrm{ch}_1 : S_{*,\mathbb{Q}} \to \mathrm{D}_s[-1]$. By construction, both $\Psi_{\widehat{S}}^k$ and the canonical Adams structure maps $\Psi_{\mathrm{D}}^k \in \mathrm{End}_{\mathbf{Ho}(S)}(\Omega_s\mathrm{D}_s[-1])$ map to the same element in $\mathrm{Hom}_{\mathbf{Ho}(S)}(\Omega_s\mathrm{D}_s[-1], (\widehat{S}_*)_{\mathbb{Q}})$. On the other hand, looking at

$$
\begin{array}{ccccccc}
\widehat{\mathrm{BGL}}_{\mathbb{Q}} & \longrightarrow & \mathrm{BGL}_{\mathbb{Q}} & \longrightarrow & \mathrm{BGL}_{\mathbb{Q}} \wedge \mathrm{H}_{\mathrm{D}} & \overset{\mathrm{ch}}{\underset{\cong}{\longrightarrow}} & R := \bigoplus_p \mathrm{H}_{\mathrm{D}}\{p\} \\
\Big\downarrow{\scriptstyle\Psi_{\widehat{\mathrm{BGL}}}^k} & & \Big\downarrow{\scriptstyle\Psi_{\mathrm{BGL}}^k} & & \Big\downarrow{\scriptstyle\Psi_{\mathrm{BGL}}^k\wedge\mathrm{id}} & & \Big\downarrow{\scriptstyle\Psi_{\mathrm{D}}^k} \\
\widehat{\mathrm{BGL}}_{\mathbb{Q}} & \longrightarrow & \mathrm{BGL}_{\mathbb{Q}} & \longrightarrow & \mathrm{BGL}_{\mathbb{Q}} \wedge \mathrm{H}_{\mathrm{D}} & \overset{\mathrm{ch}}{\underset{\cong}{\longrightarrow}} & R
\end{array}
$$

there is a unique $\Psi_{\widehat{\mathrm{BGL}}}^k \in \mathrm{End}_{\mathbf{SH}(S)_{\mathbb{Q}}}(\widehat{\mathrm{BGL}}_{\mathbb{Q}}) \overset{\delta}{\rightarrowtail} \mathrm{Hom}(R[-1], \widehat{\mathrm{BGL}}_{\mathbb{Q}})$ that maps to the image of the canonical Adams operation on the graded object $R[-1]$. Using $\mathrm{End}_{\mathbf{SH}}(R[-1]) = \mathrm{End}_{\mathbf{Ho}}(\Omega\mathrm{D}_s[-1])$ (compare the reasoning after (6.8)) we see that the Adams operations on $\widehat{\mathrm{BGL}}_{\mathbb{Q}}$ and on $\widehat{S}_{*,\mathbb{Q}}$ agree, which yields the compatibility statement using the definition of the comparison isomorphism (6.6). The isomorphism (6.17) is then clear, as is (6.19), using (4.7). (6.18) is a restatement of [GS90c, Theorem 7.3.4]. $\qquad\square$

6.2. The action of K-theory on $\widehat{K}$-theory. From Theorem 4.2(ii) recall that $\widehat{\mathrm{BGL}}$ is a BGL-module, i.e., there is a natural BGL-action

$$\mu : \mathrm{BGL} \wedge \widehat{\mathrm{BGL}} \to \widehat{\mathrm{BGL}}.$$

For any smooth scheme $f : X/S$, this induces a map called the *canonical* BGL-*action* on $\widehat{\mathrm{H}}$-groups:

$$
\begin{aligned}
\mathrm{H}^n(X)\times\widehat{\mathrm{H}}^m(X) &= \mathrm{Hom}_{\mathbf{SH}(S)}(X_+, \mathrm{BGL}[n])\times\mathrm{Hom}(X_+, \widehat{\mathrm{BGL}}[m]) \\
&\to \mathrm{Hom}(X_+ \wedge X_+, \mathrm{BGL} \wedge \widehat{\mathrm{BGL}}[n+m]) \\
&\overset{\Delta^*\circ\mu_*}{\longrightarrow} \mathrm{Hom}(X_+, \widehat{\mathrm{BGL}}[n+m]) = \widehat{\mathrm{H}}^{n+m}(X).
\end{aligned}
$$

Here $\Delta : X_+ \to X_+ \wedge X_+ = (X\times X)_+$ is the diagonal map.

Theorem 6.3. *Let S be a regular base scheme satisfying Condition* (c) *of Theorem* 6.1. *Then, at least up to torsion, the canonical comparison isomorphism $\widehat{\mathrm{K}}_n(X) \cong \widehat{\mathrm{H}}^{-n}(X)$ is compatible with the canonical* BGL-*action on the right hand side and the K_*-action*

$$K_*(X) \times \widehat{\mathrm{K}}_*(X) \to \widehat{\mathrm{K}}_*(X)$$

induced by the product structure on $\widehat{\mathrm{K}}_*^T(X)$ established by Gillet and Soulé (for $\widehat{\mathrm{K}}_0$) [GS90c, Theorem 7.3.2] *and Takeda (for higher $\widehat{\mathrm{K}}^T$-theory)* [Tak05, Section 6] *on the left hand side.*

Similarly, the pairing

$$\mathrm{CH}^n(X) \times \widehat{\mathrm{CH}}^m(X) \to \widehat{\mathrm{CH}}^{n+m}(X)$$

induced by the ring structure on $\widehat{\mathrm{CH}}_{\mathrm{GS}}^*(X)$ agrees, after tensoring with $\mathbb{Q}$, with the canonical pairing

$$\mathrm{H}^{2n}(X, n) \times \widehat{\mathrm{H}}^{2m}(X, m) \to \widehat{\mathrm{H}}^{2(n+m)}(X, n+m).$$

Proof. Before proving the theorem proper, we sketch the definition of the product on $\widehat{\mathrm{K}}_*^T$: instead of the S-construction, Takeda uses the Gillet-Grayson G-construction $G_*(-) := G_*(\widehat{P}(-))$ of the exact category of hermitian vector bundles on a scheme (see p. 761). There is a natural weak equivalence $G_*(T) \to \Omega_s S_*(T)$. In particular, $\pi_n(G_*(T)) = K_n(T)$ for any scheme T and $n \geq 0$. This gives rise to a canonical isomorphism

$$\widehat{\mathrm{K}}_n(X) = \pi_n \operatorname{hofib}_{\Delta^{\mathrm{op}}(\mathbf{Sets})}(G_*(X) \xrightarrow{\mathrm{ch}_G} \mathrm{D}_s(X))$$

(cf. [Tak05, Theorem 6.2]). The advantage of the G-construction is the existence of a bisimplicial version $G_*^{(2)}$ of G-theory together with a weak equivalence $R : G_* \to G_*^{(2)}$ and a map $\mu_G : G_*(X) \wedge G_*(X) \to G_*^{(2)}(X)$, so that the induced map $\pi_n(G_*(X)) \times \pi_m(G_*(X)) \to \pi_{n+m}(G_*(X))$ is the usual product on K-theory. Moreover, ch_G factors over R.

Consider the following diagram, where $\mu_{\mathrm{D}} : \mathrm{D}_s \wedge \mathrm{D}_s \to \mathrm{D}_s$ is the product (cf. Section 3) and the terms in the second column denote the homotopy fibers (with respect to the section-wise model structure) of the respective right-most

horizontal maps:

$$
\begin{array}{ccccccc}
\Omega_s(G \wedge \mathrm{D}_s) & \longrightarrow & G \wedge \widehat{G} & \longrightarrow & G \wedge G & \xrightarrow{\mathrm{id}\wedge\mathrm{ch}_G} & G \wedge \mathrm{D}_s \\
\downarrow{\scriptstyle \Omega_s\mu_\mathrm{D}\circ\mathrm{ch}_G} & & \vdots & & \downarrow{\scriptstyle \mu_G} & & \downarrow{\scriptstyle \mu_\mathrm{D}\circ\mathrm{ch}_G} \\
\Omega_s\mathrm{D}_s & \longrightarrow & \widehat{G^{(2)}} & \longrightarrow & G^{(2)} & \longrightarrow & \mathrm{D}_s \\
\| & & \uparrow & & \uparrow{\scriptstyle R} & & \| \\
\Omega_s\mathrm{D}_s & \longrightarrow & \widehat{G} & \longrightarrow & G & \xrightarrow{\mathrm{ch}_G} & \mathrm{D}_s.
\end{array}
$$

The lower right square is commutative (on the nose) according to [Tak05]. The upper right square is commutative up to (a certain) homotopy [Tak05, Theorem 5.2], so there is some dotted map such that the left-upper square commutes up to homotopy. This yields a map $\phi : G \wedge \widehat{G} \to \widehat{G}$ in $\mathbf{Ho}_\bullet(S)$ fitting into the following diagram (in $\mathbf{Ho}(S)$):

$$
(6.20) \qquad
\begin{array}{ccccccc}
G \wedge \Omega_s\mathrm{D}_s & \longrightarrow & G \wedge \widehat{G} & \longrightarrow & G \wedge G & \longrightarrow & G \wedge \mathrm{D}_s \\
\downarrow{\scriptstyle \mu_\mathrm{D}\circ\mathrm{ch}_G} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mu_G} & & \downarrow{\scriptstyle \mu_\mathrm{D}\circ\mathrm{ch}_G} \\
\Omega_s\mathrm{D}_s & \longrightarrow & \widehat{G} & \longrightarrow & G & \longrightarrow & \mathrm{D}_s.
\end{array}
$$

The K_*-action on $\widehat{\mathrm{K}}_*$ is induced by ϕ. Thus, to prove the theorem, it is sufficient to show that the diagram

$$
\begin{array}{ccc}
\Omega^\infty_{\mathbb{P}^1}(\mathrm{BGL} \wedge \widehat{\mathrm{BGL}}) & \xrightarrow{\cong} & G \wedge \widehat{G} \\
\downarrow{\scriptstyle \Omega^\infty_{\mathbb{P}^1}\mu} & & \downarrow{\scriptstyle \phi} \\
\Omega^\infty_{\mathbb{P}^1}(\widehat{\mathrm{BGL}}) & \xrightarrow{\cong} & \widehat{G}
\end{array}
$$

is commutative in $\mathbf{Ho}(S)$. Here the horizontal isomorphisms are the ones from Theorem 6.1. For this, it is sufficient to show that the dotted map in (6.20) is unique (in $\mathbf{Ho}_\bullet(S)$). The latter statement looks very much like Theorem 6.1(i). Indeed, it can be shown in the same manner, as we now sketch: again, one first does the stable analogue, namely the unicity of a map $\mathrm{BGL} \wedge \widehat{\mathrm{BGL}} \to \widehat{\mathrm{BGL}}$ in $\mathbf{SH}(S)$ making the diagram analogous to (6.20) commute. To do so, the sequences in (6.7) are altered by replacing $\mathrm{Hom}(?, *)$ by $\mathrm{Hom}(\mathrm{BGL}\wedge?, *)$ everywhere. For any $E \in \mathbf{DM}_\mathrm{B}(S)$, we have

$$
\mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Q}}(\mathrm{BGL}\wedge?, E) = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Q}}(\mathrm{H}_\mathrm{B}\{p\}\wedge?, E)
$$
$$
= \prod_p \mathrm{Hom}_{\mathbf{SH}(S)_\mathbb{Q}}(?\{p\}, E)
$$

since $\mathbf{DM}_{\mathrm{B}}(S) \subset \mathbf{SH}(S)_{\mathbb{Q}}$ is a full subcategory. This applies to $E = \mathrm{H}_{\mathrm{D}}$ and $E = \mathrm{BGL}_{\mathbb{Q}} = \bigoplus_p \mathrm{H}_{\mathrm{B}}\{p\}$. Therefore, both the surjectivity of α and the injectivity of γ in (6.7) carries over to the situation at hand.[5] Then, the unstable unicity statement mentioned above is deduced from the stable one.

The statement for the arithmetic Chow groups follows from this: $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$ is a direct factor of $\widehat{\mathrm{K}}_0(X)_{\mathbb{Q}}$ in a way that is compatible with the action of the direct factor $\mathrm{CH}^*(X)_{\mathbb{Q}} \subset K_0(X)_{\mathbb{Q}}$, by the multiplicativity of the arithmetic Chern class $\widehat{\mathrm{K}}_0^T(X)_{\mathbb{Q}} \cong \bigoplus_p \widehat{\mathrm{CH}}_{\mathrm{GS}}^p(X)_{\mathbb{Q}}$ [GS90c, Theorem 7.3.2(ii)]. Similarly, the H_{B}-action on $\widehat{\mathrm{H}_{\mathrm{B}}}$ is a direct factor of the $\mathrm{BGL}_{\mathbb{Q}}$-action on $\widehat{\mathrm{BGL}}_{\mathbb{Q}}$. $\qquad\square$

6.3. Pushforward. Let $f : X \to S$ be a smooth proper map. According to Definition and Lemma 4.10,

$$\mathrm{Hom}(\mathrm{BGL} \to f_* f^* \mathrm{BGL} \overset{\mathrm{tr}_f^{\mathrm{BGL}}, \cong}{\longrightarrow} f_! f^! \mathrm{BGL}, \ \widehat{\mathrm{BGL}})$$

defines a functorial pushforward

$$f_* : \widehat{\mathrm{H}}^n(X) \to \widehat{\mathrm{H}}^n(S)$$

and similarly

$$f_* : \widehat{\mathrm{H}}^n(X, p) \to \widehat{\mathrm{H}}^{n-2\dim f}(S - \dim f),$$

where $\dim f := \dim X - \dim S$ is the relative dimension of f. We now compare this with the classical pushforward on arithmetic K and Chow groups. Recall from [Roe99, Prop. 3.1] the pushforward $f_* : \widehat{\mathrm{K}}_0^T(X) \to \widehat{\mathrm{K}}_0^T(S)$. This pushforward depends on an auxiliary choice of a metric on the relative tangent bundle. It should be emphasized that the difficulty in the construction of f_* on the full groups $\widehat{\mathrm{K}}_0^T(X)$ is due to the presence of analytic torsion. We now show that its restriction to $\widehat{\mathrm{K}}_0(X)$ agrees with $f_* : \widehat{\mathrm{H}}^0(X) \to \widehat{\mathrm{H}}^0(S)$ in an important case. This shows that analytic torsion phenomena and the choice of metrics only concern the quotient $\widehat{\mathrm{K}}_0^T / \widehat{\mathrm{K}}_0$. See also [BFiML11] for similar independence results.

Theorem 6.4.

(i) *The pushforward* $i_* : \widehat{\mathrm{H}}^0(\mathbb{F}_p) = \mathrm{H}^0(\mathbb{F}_p) = \mathbb{Z} \to \widehat{\mathrm{H}}^0(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{R}$ *is given by* $(0, \log p)$.

(ii) *Let* $\mathcal{O}_F$ *be a number ring and* $S \subset \mathrm{Spec}\,\mathcal{O}_F$ *an open subscheme and let* $f : X \to S$ *be smooth projective. For any* $n \in \mathbb{Z}$, *the following diagram is commutative, where the right vertical map is the pushforward on Gillet-Soulé's arithmetic Chow groups* [GS90a, Theorem 3.6.1] *and the middle*

---

[5]We need to restrict to $\mathbb{Q}$-coefficients, since the author does not know how to compute $\mathrm{BGL} \wedge \mathrm{BGL}$.

map is its restriction:

$$\widehat{\mathrm{H}}^{2(\dim X+n)}(X,\dim X+n)\xrightarrow[6.1]{\cong}\widehat{\mathrm{CH}}^{\dim X+n}(X)_{\mathbb{Q}}\hookrightarrow\widehat{\mathrm{CH}}_{\mathrm{GS}}^{\dim X+n}(X)$$

$$\downarrow f_* \qquad\qquad \downarrow f_* \qquad\qquad \downarrow f_*$$

$$\widehat{\mathrm{H}}^{2+2n}(S,n+1)\xrightarrow{\cong,6.1}\widehat{\mathrm{CH}}^{n+1}(S)_{\mathbb{Q}}\hookrightarrow\widehat{\mathrm{CH}}_{\mathrm{GS}}^{n+1}(S).$$

(iii) *Under the same assumptions, the following diagram commutes, where the right vertical map is the pushforward mentioned above and the middle one is its restriction. In particular, the restriction of the $\widehat{\mathrm{K}}_0^T$-theoretic pushforward to the subgroups $\widehat{\mathrm{K}}_0$ does not depend on the choice of the metric on the tangent bundle T_f used in its definition:*

$$\widehat{\mathrm{H}}^0(X)_{\mathbb{Q}}\xrightarrow[6.1]{\cong}\widehat{\mathrm{K}}_0(X)_{\mathbb{Q}}\hookrightarrow\widehat{\mathrm{K}}_0^T(X)_{\mathbb{Q}}$$

$$\downarrow f_* \qquad\qquad \downarrow f_* \qquad\qquad \downarrow f_*$$

$$\widehat{\mathrm{H}}^0(S)\xrightarrow{\cong,6.1}\widehat{\mathrm{K}}_0(S)_{\mathbb{Q}}\hookrightarrow\widehat{\mathrm{K}}_0^T(S).$$

In order to prove (ii), we need some facts pertaining to the Betti realization due to Ayoub [Ayo10]: for any smooth scheme $B/\mathbb{C}$, let

$$-^{\mathrm{An}}:\mathbf{Sm}/B\to\mathbf{AnSm}/B^{\mathrm{An}}$$

be the functor which maps a smooth (algebraic) variety over B to the associated smooth analytic space (seen as a space over the analytic space attached to B), equipped with its usual topology. (This functor was denoted $-(\mathbb{C})$ above.) The adjunction

$$\mathrm{An}^*:\mathbf{PSh}(\mathbf{Sm}/B,\mathbb{C})\leftrightarrows\mathbf{PSh}(\mathbf{AnSm}/B^{\mathrm{An}},\mathbb{C}):\mathrm{An}_*$$

between the category of presheaves of complexes of $\mathbb{C}$-vector spaces on $\mathbf{Sm}/B$ and the similar category of presheaves on smooth analytic spaces over B^{An} carries over to an adjunction of stable homotopy categories:

(6.21) $$\mathrm{An}^*:\mathbf{SH}(B,\mathbb{C})\leftrightarrows\mathbf{SH}^{\mathrm{An}}(B^{\mathrm{An}},\mathbb{C}):\mathrm{An}_*.$$

We refer to [Ayo10, Section 2] for details and notation; we use $\mathbb{P}^1_{B^{\mathrm{An}}}$-spectra instead of $(\mathbb{A}^1_{B^{\mathrm{An}}}/\mathbb{G}_{m\,B^{\mathrm{An}}})$-spectra, which does not make a difference. Secondly, there is a natural equivalence

$$\phi_X:\mathbf{SH}^{\mathrm{An}}(X^{\mathrm{An}},\mathbb{C})\xrightarrow{\cong}\mathbf{D}(\mathbf{Shv}_{\mathrm{An}}(X^{\mathrm{An}},\mathbb{C}))$$

of the stable analytic homotopy category and the derived category of sheaves (of $\mathbb{C}$-vector spaces), for any smooth B-scheme X. Both this equivalence and

(6.21) are compatible with the exceptional inverse image and direct image with compact support in the sense that

$$f^{\mathrm{An}!}\phi_S \mathrm{An}^* = \phi_X \mathrm{An}^* f^!, \quad f_!^{\mathrm{An}}\phi_X \mathrm{An}^* = \phi_S \mathrm{An}^* f_!$$

for any smooth map $f : X \to S$ of smooth B-schemes [Ayo10, Th. 3.4]. Here $f_!$ and $f^!$ are the usual functors on the stable homotopy category, while $f^{\mathrm{An}!}$ and $f_!^{\mathrm{An}}$ are the classical ones on the derived category.

To show (i), we need the following auxiliary lemma. It is probably well-known, but we give a proof here for completeness.

Lemma 6.5. *In a triangulated category, let* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ *and* $A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \xrightarrow{\gamma'} A'[1]$ *be two distinguished triangles. Consider the maps of* Hom*-groups induced by* α, α', *etc. We suppose that* β^* *is onto and* γ^* *is bijective, as shown:*

$$\begin{array}{cccc}
\mathrm{Hom}(B, A') & \mathrm{Hom}(C, B') & \mathrm{Hom}(A[1], C') & \\
\downarrow \alpha^* \quad \searrow^{\alpha'_*} & \downarrow \beta^* \quad \searrow^{\beta'_*} & \downarrow \gamma^*,\cong \quad \searrow^{\gamma'_*} & \\
\mathrm{Hom}(A, A') & \mathrm{Hom}(B, B') & \mathrm{Hom}(C, C') & \mathrm{Hom}(A[1], A'[1]).
\end{array}$$

Then, for any $\xi \in \mathrm{Hom}(B, A')$, $(\alpha^*\xi)[1] = (\xi \circ \alpha)[1]$ *agrees with the image of any lift of* $\alpha'_*\xi$ *in* $\mathrm{Hom}(A[1], A'[1])$ *under the above maps.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] & \xrightarrow{\alpha[1]} & B[1] \\
\downarrow \xi \ (1) & & \downarrow v \ (2) & & \downarrow \zeta,\zeta' \ (3) & & \downarrow \xi[1] \\
A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & A[1].
\end{array}$$

By assumption, there is a map v making the square (1) commute. Next, there is a unique map ζ making the square (2) commute. On the other hand, by the axioms of a triangulated category, there is a (a priori non-unique) map ζ' making both (2) and (3) commute. Therefore, $\zeta = \zeta'$. This implies the claim. $\square$

Proof of Theorem 6.4. (i) Let $i : \mathrm{Spec}\,\mathbb{F}_p \to S := \mathrm{Spec}\,\mathbb{Z} \leftarrow U := \mathrm{Spec}\,\mathbb{Z}[1/p] : j$. Consider the triangles

$$S^0 \to i_* i^* S^0 \to j_! j^* S^0[1] \to S^0[1],$$

$$\widehat{\mathrm{BGL}} \to \mathrm{BGL} \xrightarrow{\mathrm{ch}} \bigoplus_p \mathrm{H_D}\{p\} \to \widehat{\mathrm{BGL}}[1].$$

The assumptions of Lemma 6.5 are satisfied, as can be checked using (6.1): the generator of $K_0(\mathbb{F}_p)$ lifts to $(p, \pm 1)$ under $K_1(U) = p^{\mathbb{Z}} \times \{\pm 1\} \twoheadrightarrow K_0(\mathbb{F}_p)$,

which in turn gets mapped to $\log p \in \mathrm{H}_\mathrm{D}^1(\mathbb{Q}, 1) = \mathbb{R}$ under the Beilinson (or Dirichlet) regulator, which agrees with the Chern class ch by Theorem 5.7. Therefore, the pushforward $i_* : \widehat{\mathrm{H}}^0(\mathbb{F}_p) = \mathrm{H}^0(\mathbb{F}_p) = K_0(\mathbb{F}_p) = \mathbb{Z} \to \widehat{\mathrm{H}}^0(\mathbb{Z}) = \widehat{\mathrm{K}}_0(S) = \mathbb{Z} \oplus \mathbb{R}$ is the map $(0, \log p)$, so it agrees with the classical $\widehat{\mathrm{K}}$-theoretic pushforward.

(ii) Put $d' := d + n$. We need to show the commutativity of the following diagram:

$$(6.22) \qquad \begin{array}{ccccc} (\mathrm{H}_{\mathrm{B}}, f^! \widehat{\mathrm{H}}_{\mathrm{B}}\{n+1\}) & \xrightarrow{\;\widehat{p}\;} & (\mathrm{H}_{\mathrm{B}}, \widehat{\mathrm{H}}_{\mathrm{B}}\{d'\}) & \xrightarrow{\;\cong\;} & \widehat{\mathrm{CH}}^{d'}(X)_{\mathbb{Q}} \\ \Big\| & & & & \Big\downarrow{\scriptstyle f_*} \\ (\mathrm{H}_{\mathrm{B}}, f_! f^! \widehat{\mathrm{H}}_{\mathrm{B}}\{n+1\}) & & & & \\ \Big\downarrow{\scriptstyle f_! f^! \to \mathrm{id}} & & & & \\ (\mathrm{H}_{\mathrm{B}}, \widehat{\mathrm{H}}_{\mathrm{B}}\{n+1\}) & & \xrightarrow{\hspace{3cm}\cong\hspace{3cm}} & & \widehat{\mathrm{CH}}^{n+1}(S)_{\mathbb{Q}}. \end{array}$$

Here $\widehat{p}$ is the relative purity isomorphism $f^! \widehat{\mathrm{H}}_{\mathrm{B}}\{1\} \cong f^* \widehat{\mathrm{H}}_{\mathrm{B}}\{d\}$.

We may assume $n \geq 0$ since $\widehat{\mathrm{CH}}^{\leq 0}(S) = 0$. The group $\mathrm{CH}^{d'}(X)$ is finite for $n = 0$ by class field theory [KS86, Theorem 6.1] and zero for $n > 0$. Hence $\mathrm{H}_\mathrm{D}^{2d'-1}(X, d') \to \widehat{\mathrm{K}}_0(X)_{\mathbb{Q}}^{(d')}$ is onto, by Theorem 4.5. On the other hand, for dimension reasons, $\mathrm{H}_\mathrm{D}^{2d'-1}(X, d') = \mathrm{H}_\mathrm{B}^{2d'-2}(X, \mathbb{R}(d'-1))$. By definition, the pushforward in arithmetic Chow groups [GS90a, Thm. 3.6.1] is compatible with

$$(6.23) \qquad f_* : \mathrm{H}_\mathrm{B}^{2d'-2}(X^{\mathrm{An}}, \mathbb{R}(d'-1)) \;\to\; \mathrm{H}_\mathrm{B}^{2n}(\mathbb{C}^{\mathrm{An}}, \mathbb{R}(n)) = \mathbb{R}$$
$$\omega \;\mapsto\; \frac{1}{(2\pi i)^{d-1}} \int_{X^{\mathrm{An}}} \omega.$$

Let C^* be the presheaf complex of real-valued C^∞-differential forms on smooth analytic spaces. This is a flasque complex, and its (presheaf) cohomology groups agree with Betti cohomology with real coefficients. The construction and properties of H_D (esp. Theorem 2.8) carry over and yield a spectrum $\mathrm{An}_*(\mathcal{B})$ representing Betti cohomology. The maps of complexes of sheaves on the analytic site,

$$[\mathbb{R}(p) \to \mathcal{O} \to \Omega^1 \to \ldots \to \Omega^{p-1}] \to \mathbb{R}(p) \xrightarrow{\sim} C^*(p),$$

give rise to a map of spectra $\mathrm{H}_\mathrm{D}(p) \to \mathrm{An}_* \mathcal{B}(p)$. The rectangle (6.22) is functorial with respect to maps of the target spectrum. Thus, we can replace $\widehat{\mathrm{H}}_{\mathrm{B}}\{n+1\}$ by $\mathrm{An}_* \mathcal{B}(n+1)[2n+1]$ and $f_* : \widehat{\mathrm{CH}}^{d'}(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^{n+1}(X)_{\mathbb{Q}}$ by $f_* : \mathrm{H}_\mathrm{B}^{2d'-2}(X^{\mathrm{An}}, \mathbb{R}(d'-1)) \to \mathrm{H}_\mathrm{B}^{2n}(\mathbb{C}, \mathbb{R}(n)) \overset{n=0}{=} \mathbb{R}$. This settles our claim,

since the adjointness map $f_!^{\mathrm{An}} f^{\mathrm{An}!}\mathbb{C} \to \mathbb{C}$ does induce the integration map (6.23) [KS90, Exercise III.20].

(iii) The diagram

$$
\begin{array}{ccccccc}
K_1(X) & \longrightarrow & \mathrm{H}_\mathrm{D}^{-1}(X) & \longrightarrow & \widehat{\mathrm{K}}_0(X) & \longrightarrow & K_0(X) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*\circ(-\cup\mathrm{Td}\,T_f)} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
K_1(S) & \longrightarrow & \mathrm{H}_\mathrm{D}^{-1}(S) & \longrightarrow & \widehat{\mathrm{K}}_0(S) & \longrightarrow & K_0(X)
\end{array}
$$

is commutative; see [Tak05, Section 7]. On the other hand, applying

$$
\mathrm{Hom}_{\mathrm{BGL}-\mathbf{Mod}}(f_! f^* \mathrm{BGL} \overset{\mathrm{tr}^{\mathrm{BGL}}}{\to} f_! f^! \mathrm{BGL} \to \mathrm{BGL}, -)
$$

to the triangle (6.3) yields a diagram which is the same, except that K_* is replaced by H^{-*} and $\widehat{\mathrm{K}}_*$ by $\widehat{\mathrm{H}}^{-*}$ (and their respective pushforwards established in Definition and Lemma 4.10). Indeed, the pushforward on Deligne cohomology induced by $\mathrm{tr}^{\mathrm{BGL}}$ (as opposed to tr^{B}) is the usual pushforward, modified by the Todd class. This is a consequence of Theorem 2.5.

Now, (iii) is shown exactly as (ii): the only non-trivial part is $\widehat{\mathrm{K}}_0(X)_\mathbb{Q}^{(d)}$, which is mapped onto by $\mathrm{H}_\mathrm{D}^{2d-1}(X, d)$, since $K_0(X)_\mathbb{Q}^{(d)} = \mathrm{CH}^d(X)_\mathbb{Q} = 0$. $\square$

Remark 6.6. The same proof works more generally for $f_* : \widehat{\mathrm{H}}^n(X, p) \to \widehat{\mathrm{H}}^{n-2\dim f}(S, p - \dim f)$, provided that $\mathrm{H}^n(X, p) = K_{2p-n}(X)_\mathbb{Q}^{(p)} \to \mathrm{H}_\mathrm{D}^n(X, p)$ is injective. For example, given a smooth projective complex variety X of dimension d, a conjecture of Voisin [Voi07, 11.23] generalizing Bloch's conjecture on surfaces satisfying $p_g = 0$ says that the cycle class map $K_0(X)_\mathbb{Q}^{(d-l)} \cong \mathrm{CH}^{d-l}(X)_\mathbb{Q} \to \mathrm{H}_\mathrm{B}^{2(d-l)}(X, \mathbb{Q})$ is injective (or, equivalently, that the cycle class map to Deligne cohomology is injective) for $l \leq k$ if the terms in the Hodge decomposition $\mathrm{H}^{p,q}(X)$ are zero for all $p \neq q$, $q \leq k$.

7. The Arakelov intersection pairing

Let $S = \mathrm{Spec}\,\mathbb{Z}[1/N]$ be an open, non-empty subscheme of $\mathrm{Spec}\,\mathbb{Z}$, where $N = p_1 \cdot \ldots \cdot p_n$ is a product of distinct primes. We write $Log(N) := \sum_i \mathbb{Z} \cdot \log p_i \subset \mathbb{R}$ for the subgroup ($\cong \mathbb{Z}^n$) spanned by the logarithms of the p_i.

In this section, we give a conceptual explanation of the height pairing by showing that it is the natural pairing between motivic homology and Arakelov motivic cohomology.

7.1. Definition.

Definition 7.1. For $M \in \mathbf{SH}(S)$, put

$$
\begin{aligned}
\mathrm{H}_0(M) &:= \mathrm{Hom}_{\mathbf{SH}(S)}(S^0, M) \\
\mathrm{H}_0(M, 0) &:= \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(S^0, M_{\mathbb{Q}}).
\end{aligned}
$$

The second group is called *motivic homology* of M (seen as an object of $\mathbf{SH}$ with rational coefficients): for $M \in \mathbf{DM}_{\mathrm{B}}(S)$, $\mathrm{H}_0(M, 0) \cong \mathrm{Hom}_{\mathbf{SH}(S)_{\mathbb{Q}}}(\mathrm{H}_{\mathrm{B}}, M_{\mathbb{Q}})$.

Definition 7.2. Fix some $M \in \mathbf{SH}(S)$. The *Arakelov intersection pairing* is either of the following two maps

$$
\begin{aligned}
: \mathrm{H}_0(M) \times \widehat{\mathrm{H}}^0(M) &\to \widehat{\mathrm{H}}^0(S^0) = \widehat{\mathrm{K}}_0(S) = \mathbb{Z} \oplus \mathbb{R}/Log(N), \\
\pi_M : \mathrm{H}_0(M, 0) \times \widehat{\mathrm{H}}^2(M, 1) &\to \widehat{\mathrm{H}}^2(S^0, 1) = \widehat{\mathrm{K}}_0(S)_{\mathbb{Q}}^{(1)} = (\mathbb{R}/Log(N)) \otimes \mathbb{Q}, \\
(\alpha, \beta) &\mapsto \beta \circ \alpha.
\end{aligned}
$$

Remark 7.3.

(i) The tensor structure on the category $\mathbf{DM}_{\mathrm{B}}^c(S)$, the subcategory of compact objects of $\mathbf{DM}_{\mathrm{B}}(S) \subset \mathbf{SH}(S)_{\mathbb{Q}}$, is rigid in the sense that the natural map $M \to M^{\vee\vee}$ is an isomorphism for any $M \in \mathbf{DM}_{\mathrm{B}}^c(S)$, where $M^{\vee} := \underline{\mathrm{Hom}}_{\mathbf{DM}_{\mathrm{B}}(S)}(M, \mathrm{H}_{\mathrm{B}})$ [CD09, 15.2.4]. This implies that the natural map $\mathrm{Hom}(M, N) \to \mathrm{Hom}(N^{\vee}, M^{\vee})$ is an isomorphism for any two such motives. In particular $\mathrm{H}_0(M, 0) \cong \mathrm{H}^0(M^{\vee}, 0)$, so the pairing can be rewritten as

(7.1) $\mathrm{H}^0(M^{\vee}, 0) \times \widehat{\mathrm{H}}^2(M, 1) \to \mathrm{H}^2(S, 1).$

This is the shape familiar from other dualities, such as Artin-Verdier duality,

$$
\mathrm{H}^0(\mathrm{Spec}\,\mathbb{Z}, \mathcal{F}^{\vee}) \times \mathrm{H}_c^3(\mathrm{Spec}\,\mathbb{Z}, \mathcal{F}(1)) \to \mathrm{H}^3(\mathrm{Spec}\,\mathbb{Z}, \mu_{\ell}) = \mathbb{Q}/\mathbb{Z}.
$$

In this analogy, an étale constructible ℓ-torsion sheaf $\mathcal{F}$ corresponds to a motive M and étale cohomology with compact support gets replaced by Arakelov motivic cohomology. The pairing (7.1) is conjecturally perfect when replacing $\widehat{\mathrm{H}_{\mathrm{B}}}$ by $\widehat{\mathrm{H}_{\mathrm{B},\mathbb{R}}}$, which is constructed in the same way, except that H_{B} gets replaced by $\mathrm{H}_{\mathrm{B},\mathbb{R}}$, a spectrum representing motivic cohomology tensored with $\mathbb{R}$. The implications of this conjecture and its relation to special L-values is the main topic of [Sch13].

(ii) By definition, the intersection pairing is functorial: given a map $f : M \to M'$, the following diagram commutes:

$$
\begin{array}{ccccc}
\pi_M : & \mathrm{H}^0(M, 0) & \times & \widehat{\mathrm{H}}^2(M^{\vee}, 1) & \longrightarrow & \mathbb{R} \\
 & \uparrow & & \downarrow & & \downarrow = \\
\pi_{M'} : & \mathrm{H}^0(M', 0) & \times & \widehat{\mathrm{H}}^2(M'^{\vee}, 1) & \longrightarrow & \mathbb{R}.
\end{array}
$$

7.2. Comparison with the height pairing. For a regular, flat, and projective scheme $X/\mathbb{Z}$ of absolute dimension d, Gillet and Soulé have defined the *height pairing* μ_{GS}:

$$
\begin{array}{ccccc}
\mathrm{CH}^m(X)_0 & \times & \mathrm{CH}^{d-m}(X)_0 & \xrightarrow{\mu_{\mathrm{B}}} & \widehat{\mathrm{CH}}^1(S) \\
\downarrow & & \uparrow & & \| \\
\mathrm{CH}^m(X) & \times & \widehat{\mathrm{CH}}^{d-m}(X) & \xrightarrow{\mu} & \widehat{\mathrm{CH}}^1(S) \\
\uparrow & & \downarrow & & \| \\
\widehat{\mathrm{CH}}^m_{GS}(X) & \times & \widehat{\mathrm{CH}}^{d-m}_{GS}(X) & \xrightarrow{\mu_{GS}} & \widehat{\mathrm{CH}}^1(S).
\end{array}
$$

Here, $\mathrm{CH}^m(X)_0 := \ker \mathrm{CH}^m(X) \to \mathrm{H}_{\mathrm{D}}^{2m}(X, m)$ is the subgroup of the Chow group consisting of cycles that are homologically trivial at the infinite place. The pairing μ is uniquely determined by μ_{GS}. It is given by

$$
(Z, (Z', g')) \mapsto (Z \cdot Z', \delta_Z \wedge g'),
$$

where Z and Z' are cycles of codimension m and $d - m$, δ_Z is the Dirac current, and g' is a Green current satisfying the differential equation

$$
\omega(Z', g') = -\frac{1}{2\pi i} \partial \overline{\partial} g' + \delta_{Z'} = 0.
$$

See [GS90a, Sections 4.2, 4.3] for details. The pairing μ_{B} is the height pairing defined by Beilinson [Beĭ87, 4.0.2]. More precisely, Beilinson considered the group of homologically trivial cycles on $X \times_S \mathbb{Q}$, but we will focus on the case where the variety in question is given over the one-dimensional base S.

We now give a very natural interpretation of the height pairing μ in terms of the Arakelov intersection pairing. Our statement applies to smooth schemes X only, essentially because of the construction of the stable homotopy category, which is built out of presheaves on $\mathbf{Sm}/S$ (as opposed to regular schemes, say).

Theorem 7.4. *Let $S \subset \mathrm{Spec}\,\mathbb{Z}$ be an open (non-empty) subscheme and let $f : X \to S$ be smooth and proper of absolute dimension d. For any m, let $n := m - \dim f = m - d + 1$ and let $M = \mathrm{M}(X)\{n\} = f_! f^! \mathrm{H}_B\{n\}$ be the motive of X (twisted and shifted). Then the height pairing μ (tensored with $\mathbb{Q}$) mentioned above agrees with the Arakelov intersection pairing in the sense*

that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathrm{CH}^m(X)_{\mathbb{Q}} & \times & \widehat{\mathrm{CH}}^{d-m}(X)_{\mathbb{Q}} & \xrightarrow{\ \mu\ } & \widehat{\mathrm{CH}}^1(S)_{\mathbb{Q}} \\
\cong \downarrow {\scriptstyle 2.2} & & \cong \downarrow {\scriptstyle 6.2} & & \downarrow \cong \\
\mathrm{H}_0(M,0) & \times & \widehat{\mathrm{H}}^2(M,1) & \xrightarrow{\ \pi_M\ } & \widehat{\mathrm{H}}^2(S,1).
\end{array}
$$

Proof. We need to show that the following diagram is commutative. Here $\mathbf{1} := \mathrm{H}_{\mathrm{B}}$ is the Beilinson motivic cohomology spectrum, $\widehat{\mathbf{1}} := \widehat{\mathrm{H}_{\mathrm{B}}}$ is its Arakelov counterpart (Definition 4.1), and $(-,-)$ stands for $\mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(?)}(-,-)$, where the base scheme ? is S or X, respectively. Every horizontal map is an isomorphism. The maps labelled p and $\widehat{p}$ are relative purity isomorphisms $f^! \cong f^*\{d-1\}$, applied to $\mathbf{1}$ and $\widehat{\mathbf{1}}$, respectively. The isomorphisms between the (arithmetic) Chow groups and (Arakelov) motivic cohomology are discussed in Section 2.2 and Corollary 6.2.

$$
\begin{array}{ccccccc}
(\mathbf{1}, f_! f^! \mathbf{1}\{n\}) & \xrightarrow{\ p\ } & (\mathbf{1}, \mathbf{1}\{m\}) & = & (\mathbf{1}, \mathbf{1}\{m\}) & \longrightarrow & \mathrm{CH}^m(X)_{\mathbb{Q}} \\
\times & & \times & & \times & & \times \\
(f_! f^! \mathbf{1}\{n\}, \widehat{\mathbf{1}}\{1\}) & \xrightarrow{\ p\ } & (\mathbf{1}\{m\}, f^! \widehat{\mathbf{1}}\{1\}) & \xrightarrow{\ \widehat{p}\ } & (\mathbf{1}\{m\}, \widehat{\mathbf{1}}\{d\}) & \longrightarrow & \widehat{\mathrm{CH}}^{d-m}(X)_{\mathbb{Q}}
\end{array}
$$

The commutativity of (1) is a routine exercise in adjoint functors. The commutativity of (2) is obvious. The commutativity of (3) and (4) is settled in Theorems 6.3 and 6.4. $\qquad\square$

Example 7.5. Using Remark 7.3(ii), we can also describe the baby example of the Arakelov intersection pairing for $M = \mathrm{M}(\mathbb{F}_p)$: according to Theorem 6.4(i), it is given by

$$
\begin{array}{ccccc}
\mathrm{H}_0(\mathbb{F}_p) & \times & \widehat{\mathrm{H}}^0(\mathbb{F}_p) = \mathbb{Z} & \xrightarrow{\ \pi_{\mathbb{F}_p}\ } & \widehat{\mathrm{H}}^0(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{R} \\
\cong \uparrow {\scriptstyle i^*} & & {\scriptstyle (0,\log p)} \downarrow {\scriptstyle i_*} & & \| \\
\mathrm{H}_0(\mathbb{Z}) = \mathbb{Z} & \times & \widehat{\mathrm{H}}^0(\mathbb{Z}) & \xrightarrow{\ \pi_{\mathbb{Z}}\ } & \widehat{\mathrm{H}}^0(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{R}.
\end{array}
$$

Using Theorem 6.3, the bottom row is the obvious multiplication map. Therefore, $\pi_{\mathbb{F}_p}$ is given by $(1,1) \mapsto (0, \log p)$.

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Special L-values of geometric motives

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Abstract

This paper proposes a conceptual unification of Beilinson's conjecture about special L-values for motives over $\mathbb{Q}$, the Tate conjecture over $\mathbb{F}_p$ and Soulé's conjecture about pole orders of ζ-functions of schemes over $\mathbb{Z}$. We conjecture the following: the order of $L(M,s)$ at $s=0$ is given by the negative Euler characteristic of motivic cohomology of $M^\vee(-1)$. Up to a nonzero rational factor, the L-value at $s=0$ is given by the determinant of the pairing of Arakelov motivic cohomology of M with the motivic homology of M:

$$L^*(M,0) \equiv \prod_{i \in \mathbb{Z}} \det(\mathrm{H}_{i-2}(M,-1) \otimes \widehat{\mathrm{H}}^i(M) \to \mathbb{R})^{(-1)^{i+1}} \pmod{\mathbb{Q}^\times}.$$

Under standard assumptions concerning mixed motives over $\mathbb{Q}$, $\mathbb{F}_p$, and $\mathbb{Z}$, this conjecture is equivalent to the conjunction of the above-mentioned conjectures of Beilinson, Tate, and Soulé. We use this to unconditionally prove the Beilinson conjecture for all Tate motives and, up to an n-th root of a rational number, for all Artin-Tate motives.

In this paper, we study special values of L-functions of geometric motives over $\mathbb{Z}$. This contains both L-functions over $\mathbb{Q}$ and Hasse-Weil ζ-functions of schemes X of finite type over $\mathbb{Z}$ (Propositions 3.5, 3.7):

$$L_{\mathbb{Q}}(M_\eta, s)^{-1} = L_{\mathbb{Z}}(\eta_{!*} M_\eta[1], s), \tag{0.1}$$

$$\zeta(X, s) = L(\mathrm{M}_{\mathrm{c}}(X), s).$$

Here M_η is a mixed motive over $\mathbb{Q}$, $\eta_{!*}$ is a generic intermediate extension functor similar to the one familiar in perverse sheaf theory, and $\mathrm{M}_{\mathrm{c}}(X)$ denotes the motive with compact support.

Our conjecture on special L-values is as follows:

Conjecture 0.1. (see Conjectures 4.1 and 5.2) Let M be any geometric motive over $\mathbb{Z}$. We conjecture that pole orders are given by the negative Euler characteristic of motivic cohomology of $M^\vee(-1)$:

$$\mathrm{ord}_{s=0} L(M, s) = -\chi(M^\vee(-1)). \tag{0.2}$$

We conjecture that the *Arakelov intersection pairing*, which is the natural pairing of $\mathbb{R}$-vector spaces

$$\pi_M : \underbrace{\mathrm{Hom}(\mathbf{1}(-1)[-2], M)}_{=:\mathrm{H}_{-2}(M,-1)} \times \underbrace{\mathrm{Hom}(M, \widehat{\mathbf{1}})}_{=:\widehat{\mathrm{H}}^0(M)} \overset{\circ}{\longrightarrow} \mathrm{Hom}(\mathbf{1}, \widehat{\mathbf{1}}(1)[2]) = \mathbb{R},$$

involving the motivic homology and the *Arakelov motivic cohomology* of M is a perfect pairing of finite-dimensional $\mathbb{R}$-vector spaces. This conjectural perfectness

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is very interesting in its own right. For example, special cases of it are equivalent to the Beilinson-Soulé vanishing conjecture (Theorem 4.5) and the Beilinson-Parshin conjecture (Theorem 4.3). It also allows to equivalently reformulate (0.2) using the Euler characteristic of Arakelov motivic cohomology:

$$\operatorname{ord}_{s=0} L(M, s) = -\widehat{\chi}(M).$$

Most importantly, though, it allows to express the following conjecture for the special L-value $L^*(M, 0)$ up to a nonzero rational factor, using the determinants of the pairings $\pi_{M[i]}$:

$$L^*(M, 0) \equiv \Pi_M^{-1} \pmod{\mathbb{Q}^\times},$$

where

$$\Pi_M := \prod_{i \in \mathbb{Z}} \det(\pi_{M[i]})^{(-1)^i} (\in \mathbb{R}^\times / \mathbb{Q}^\times).$$

The Arakelov motivic cohomology referred to above is a new cohomology established in [HS11, Sch12a] (or see Section 2.2). It can be thought of as a cohomology with compact support, where "compact" refers to the compactification of Spec $\mathbb{Z}$. More precisely, it is characterized by a long exact sequence

$$\dots \to \widehat{\mathrm{H}}^n(M) \to \mathrm{H}^n(M) \xrightarrow{\mathrm{ch}} \mathrm{H}_{\mathrm{D}}^n(M) \to \widehat{\mathrm{H}}^{n+1}(M) \to \dots$$

involving the Chern class map ch (also known as the Beilinson regulator) between motivic cohomology and Deligne cohomology.

This conjecture is related to existing conjectures on L-functions as follows:

Theorem 0.2. *(see Theorems 5.8, 5.9 for the precise statements) Assuming the existence of the category of mixed motives (see Axiom 1.2), Conjecture 0.1 is essentially equivalent to the conjunction of the conjectures 5.14, 5.11, 5.19 of Beilinson, Soulé and Tate on special L-values of motives over $\mathbb{Q}$ and ζ-functions à la Hasse-Weil of schemes over $\mathbb{Z}$ and over $\mathbb{F}_p$, respectively.*

Recall that the subcategory $\mathbf{DATM}(\mathbb{Z})$ of Artin-Tate motives is the triangulated subcategory generated by direct summands of motives of number rings $\mathcal{O}_F$ and finite fields $\mathbb{F}_q$. Only allowing $\mathbb{Q}$ and $\mathbb{F}_p$ instead of arbitrary $\mathcal{O}_F$ and $\mathbb{F}_q$, we get the triangulated category $\mathbf{DTM}(\mathbb{Z})$ of Tate motives. Note that these motives have rational coefficients. These categories do enjoy a motivic t-structure whose hearts are denoted $\mathbf{MATM}(\mathbb{Z})$ and $\mathbf{MTM}(\mathbb{Z})$, respectively [Sch11]. We get the following unconditional result:

Corollary 0.3. *The perfectness of the Arakelov intersection pairing, as well as the pole order formula (0.2) holds for any Artin-Tate motive over $\mathbb{Z}$. The formula for the special L-value holds for all motives in the triangulated category generated by motives $\mathrm{M}(\mathcal{O}_F)$ and $\mathrm{M}(\mathbb{F}_q)$, in particular for any Tate motive, i.e., any motive in $\mathbf{DTM}(\mathbb{Z})$. More generally, for any $M \in \mathbf{DATM}(\mathbb{Z})$,*

$$L^*(M, 0) \cdot \Pi_M$$

is a torsion element of $\mathbb{R}^\times / \mathbb{Q}^\times$.

In particular, Beilinson's conjecture holds for any smooth projective variety $X_\eta / \mathbb{Q}$ such that $\mathrm{h}^j(X_\eta)$ is a mixed Tate motive ($j \in \mathbb{Z}$). Examples of such varieties include linear varieties [Jan90, Section 14], [Tot14], such as toric varieties and Grassmannians. Similarly, Beilinson's conjecture holds up to the m-th root of a nonzero rational number if $\mathrm{h}^j(X_\eta)$ is a mixed Artin-Tate motive.

Proof: We first show that for any $M \in \mathbf{DATM}(\mathbb{Z})$, there is some $m > 0$ such that $mM := M^{\oplus m}$ lies in the triangulated subcategory $L \subset \mathbf{DATM}(\mathbb{Z})$ generated by motives of the form $\mathrm{M}(\mathcal{O}_F)(n)[1]$ and direct factors of $\mathrm{M}(\mathbb{F}_q)$, for any $q = p^r$, $n \in \mathbb{Z}$ and any number ring $\mathcal{O}_F$. This statement is unrelated to the Arakelov intersection pairing and L-functions. It is enough to show this for M being a direct summand of $\mathrm{M}(\mathcal{O}_F)(n)[1]$. By definition of $\eta_{!*}$, see [Sch12b, Section 5.4], $M' := \eta_{!*}\eta^* M$ lies in the triangulated category generated by M and motives of the form $i_* N$, where $N \in \mathbf{DATM}(\mathbb{F}_p)$ and $i : \mathrm{Spec}\ \mathbb{F}_p \to \mathrm{Spec}\ \mathbb{Z}$. As $i_* N \in L$ for all $N \in \mathbf{DATM}(\mathbb{F}_p)$, it is enough to show $mM' \in L$. Note that $M_\eta := \eta^* M[-1]$ is a direct summand of $\mathrm{M}(F)(n)$. After twisting by $-n$, these two motives are Artin motives over $\mathbb{Q}$ (with rational coefficients). This category is equivalent to continuous rational $\mathrm{Gal}(\mathbb{Q})$-representations. For some finite quotient $G = \mathrm{Gal}(E/\mathbb{Q})$ of $\mathrm{Gal}(\mathbb{Q})$, $\mathrm{M}(F)$ and M_η factor over G. By Artin induction [Ser78, II.13.1, Théorème 30], there is an equality in $K_0(\mathbb{Q}[G])$, the K_0-group of the group ring of G (with rational coefficients) $m[M_\eta(-n)] = \sum_i l_i [\mathrm{ind}_H^G \mathbb{Q}]$, where $m, l_i \in \mathbb{Z}$, $m > 0$, and H runs over the cyclic subgroups of G. The functor $\eta_{!*}[1]$ does not in general send a short exact sequence

$$E_\eta : \ 0 \to M_{\eta,1} \to M_{\eta,2} \to M_{\eta,3} \to 0$$

in $\mathbf{MATM}(\mathbb{Q})$ to a distinguished triangle in $\mathbf{DATM}(\mathbb{Q})$. However, for a sufficiently small open $j : U \subset \mathrm{Spec}\ \mathbb{Z}$, there is a similar short exact sequence E_U in $\mathbf{MATM}(U)$ such that $\eta^* E_U[-1] = E_\eta$ and such that $\eta_{!*} M_{\eta,n}[1] = j_{!*} M_{U,n}$ for all n. As $j_!$ is triangulated, $j_!(E_U)$ is a distinguished triangle in $\mathbf{DATM}(\mathbb{Z})$. Moreover, $j_{!*} M_{U,n}$ lies in a distinguished triangle whose other vertices are $j_! M_{U,n}$ and $i_* N$, where $i : Z \to \mathrm{Spec}\ \mathbb{Z}$ is the complement of j and $N \in \mathbf{DATM}(Z)$. Therefore, if $\eta_{!*} M_{\eta,j}[1] \in L$ for two out of the three $M_{\eta,j}$'s, it is true for the third. Noting that $\mathrm{ind}_H^G \mathbb{Q}$ corresponds to the motive $\mathrm{M}(E^H)$ of the subfield $E^H \subset E$ fixed by H and $\eta_{!*} \mathrm{M}(E^H)[1] = \mathrm{M}(\mathcal{O}_{E^H})[1] \in L$, we obtain $m\eta_{!*} M_\eta[1] \in L$.

For any number field F and number ring $\mathcal{O}_F$, the conjectured pole order formula, the special value and the perfectness of the Arakelov intersection pairings for $\mathrm{M}(\mathcal{O}_F)(n)[1]$ are (unconditionally, by Proposition 5.16, Remark 5.17, and Theorem 5.18) equivalent to Beilinson's conjecture for $\mathrm{M}(F)(n) \in \mathbf{MATM}(\mathbb{Q})$ which does hold by Borel's work [Bor77]. The three conjectures also hold for direct factors of $\mathrm{M}(\mathbb{F}_q)$ by Quillen's computation of K-theory of finite fields [Qui72]. By Theorem 5.5, the three conjectures therefore hold for any motive in $L \subset \mathbf{DATM}(\mathbb{Z})$.

Now, let $M \in \mathbf{DATM}(\mathbb{Z})$ be any Artin-Tate motive. There is an $m > 0$ such that $mM \in L$. Since the Arakelov intersection pairings are induced by the composition of morphisms in $\mathbf{DM}_{\text{B}}(\mathbb{Z})$, the map $r_{mM} : \mathrm{H}_{-2}(mM, -1) \to \widehat{\mathrm{H}}^0(mM)^\vee$ induced by π_{mM} is clearly the m-fold direct sum of the map r_M induced by π_M. Hence the perfectness of π_{mM}, i.e., r_{mM} being an isomorphism, implies the perfectness of π_M. Moreover, we have $(L^*(M, 0)\Pi_M)^m = L^*(mM, 0)\Pi_{mM} = 1 \in \mathbb{R}^\times/\mathbb{Q}^\times$, i.e., $L^*(M, 0)\Pi_M$ is torsion in $\mathbb{R}^\times/\mathbb{Q}^\times$. Similarly, $m(\mathrm{ord}_{s=0} L(M, s) + \chi(M^\vee(-1))) = \mathrm{ord}_{s=0} L(mM, s) + \chi(mM^\vee(-1)) = 0 \in \mathbb{Z}$, so that $\mathrm{ord}_{s=0} L(M, s) + \chi(M^\vee(-1)) = 0$, i.e., the pole order formula holds.

The last statement follows immediately. $\qquad \square$

Conjecture 0.1 is compatible with the functional equation of L-functions. It is also stable under distinguished triangles (Theorem 5.5). While the latter is a formal consequence of the setup, it is a key difference between our conjecture and Beilinson's conjecture for mixed motives over $\mathbb{Q}$. It allows to break up a motive into smaller pieces by means of distinguished triangles. This technique is unapplicable when working with Beilinson's original conjecture for motives over $\mathbb{Q}$. Moreover, Conjecture 0.1 gives more freedom because it allows to work in the larger category of all geometric motives, as opposed to just smooth and projective varieties. It should be noted, though, that the proof of the equivalence of Beilinson's L-value

formula and Conjecture 5.2 is formal, so that proving Beilinson's conjecture for any example not covered by techniques such as the ones in Corollary 0.3 will require new ideas.

The idea of reinterpreting the data in Beilinson's conjecture in terms of motives over $\mathbb{Z}$ is due to Huber. More precisely, a mixed motive M_η over $\mathbb{Q}$ corresponds to the mixed motive $\eta_{!*}M_\eta[1]$ over $\mathbb{Z}$. This is reified for L-functions by (0.1) and on the motivic side by an appropriate interpretation of f-cohomology [Sch12b]. The non-multiplicativity of L-functions (cf. Remark 3.2) is explained by the failure of $\eta_{!*}$ to be exact. L-functions of motives over $\mathbb{Z}$ *are* multiplicative, though.

This non-multiplicativity, which is a heavy technical burden, has been addressed by Scholl by introducing a category $\mathbf{MM}(\mathbb{Q}/\mathbb{Z})$ of mixed motives over $\mathbb{Z}$ [Sch91] (different from the one used here) by imposing non-ramification conditions. The (conjectural) value of the groups $\mathrm{Ext}^a_{\mathbf{MM}(\mathbb{Q}/\mathbb{Z})}(\mathbf{1}, \mathrm{h}^{b-1}(X_\eta, m))$ is closely related to the computation of $\mathrm{H}^*(\eta_{!*}\mathrm{h}^{-b+1}(X_\eta, -m)[1])$ (Theorem 1.3). As for the special L-values, a conjecture of Scholl [Sch91, Conj. C] says that some $M_\eta \in \mathbf{MM}(\mathbb{Q}/\mathbb{Z})$ is critical (i.e., its period map is an isomorphism, equivalently all weak Hodge cohomology groups $\mathrm{H}^*_{\mathrm{w}}(M_\eta)$ vanish) if

$$\mathrm{Ext}^a_{\mathbf{MM}(\mathbb{Q}/\mathbb{Z})}(M_\eta, \mathbf{1}(1)) = \mathrm{Ext}^a_{\mathbf{MM}(\mathbb{Q}/\mathbb{Z})}(\mathbf{1}, M_\eta) = 0 \text{ for } a = 0, 1.$$

Moreover, a reduction technique transforming any motive M_η into one satisfying these vanishings is given, so that Deligne's conjecture [Del79, Conj. 2.8.] concerning the L-value of critical motives can be applied. In similar spirit, the non-multiplicativity of L-functions of motives over $\mathbb{Q}$ has been addressed by Fontaine and Perrin-Riou by introducing the notion of f-exact sequences, which are ones where one does save multiplicativity [FPR94, III.3.1.4]. However, such exact sequences seem to be hard to characterize. The formulation of Conjecture 0.1 resembles their approach; for example the pole order in *op. cit.* is expressed as an Euler characteristic of f-cohomology. Using a "cohomology with compact support" to predict special L-values was already suggested by Beilinson [Beĭ87, 5.10.F]. The category of motives over $\mathbb{Z}$ is both the appropriate home for this idea and allows for the strikingly compact and beautiful formulation of the L-values conjecture by overcoming the technical obstacles related to motives over $\mathbb{Q}$.

The idea to recast special L-values of motives as determinants of appropriate pairings was explored by Deninger and Nart [DN95], who show that the motivic height pairing of [Sch94] can be represented by concatenating morphisms in the derived category of an appropriate category of motives.

Conjecture 0.1 is the first conjecture that predicts the special values of $\zeta(X)$ modulo $\mathbb{Q}^\times$ at all places ($X/\mathbb{Z}$ regular projective; see Example 5.13). A reformulation of the Tamagawa number conjecture in terms of the Weil-étale cohomology due to Flach and Morin predicts the special value of $\zeta(X)$ at $s = 0$ up to sign [FM12, Prop. 9.2]. It remains to explicitly compare the compatibility of the approach of *op. cit.* and Conjecture 0.1. I expect that similar techniques as the ones in this paper allow to refine Conjecture 0.1 to a conjectural L-values formula, up to sign, at all places. However, this remains to be done.

1 Preliminaries

1.1 Determinants and $\mathbb{Q}$-structures

For any ring R, let $\underline{R}$ be the category of finitely generated R-modules. Let K be a field. The *determinant* $\det V$ of $V \in \underline{K}$ is $\det V := \Lambda^{\dim V} V$. Its K-dual is denoted $\det^{-1} V$. For $V_* \in \mathbf{D}^{\mathrm{b}}(\underline{K})$, the derived category, we set $\det V_* := \bigotimes_i \det^{(-1)^i} \mathrm{H}^i(V_*)$. We abbreviate $\det H^* := \det^{(-1)^i} H^i$ for some $H^i \in \underline{K}$, $i \in \mathbb{Z}$.

Let $A, B \in \underline{\mathbb{Q}}$ and let $f : A_{\mathbb{R}} \to B_{\mathbb{R}}$ be an $\mathbb{R}$-linear map. We do not assume that it respects the rational subspaces. The "usual" determinant of f, which is well-defined up to a nonzero rational factor agrees, modulo $\mathbb{Q}^{\times}$ with the image of 1 under the map $\mathbb{Q} \cong \det A \otimes \det^{-1} B \to \det A_{\mathbb{R}} \otimes \det^{-1} B_{\mathbb{R}} \cong \mathbb{R}$. Here the right hand isomorphism is induced by f.

A complex with $\mathbb{Q}$-*structure* is a complex V_* of $\mathbb{R}$-vector spaces that is quasi-isomorphic to one in $\mathbf{D}^{\mathrm{b}}(\underline{\mathbb{R}})$ together with a non-zero map of $\mathbb{Q}$-vector spaces $d_{V_*} : \mathbb{Q} \to \det V_*$. In concrete situations, we usually have a distinguished identification $\det V_* \cong \mathbb{R}$. In that case, we may also call $\det V_*$ the real number that is the image of $1 \in \mathbb{Q}$ under d_{V_*} and the given identification.

Maps of complexes with $\mathbb{Q}$-structures are usual maps of complexes; they are *not* required to be compatible with the map d_{V_*}. For a map $f : V_* \to W_*$ of complexes with $\mathbb{Q}$-structures the cone of f is endowed with the following $\mathbb{Q}$-structure:

$$\mathbb{Q} \overset{d_W \otimes (d_V)^{-1}}{\longrightarrow} \det W_* \otimes \det^{-1} V_* \cong \det \mathrm{cone}(f).$$

Define a category $\mathbf{D}^{\mathrm{b}}(\underline{\mathbb{R}})^{\mathbb{Q}-\det}$ to consist of such complexes. Its morphisms are given by maps of complexes up to quasi-isomorphism (not necessarily respecting the $\mathbb{Q}$-structures). We say that a triangle $A \to B \to C$ of objects in $\mathbf{D}^{\mathrm{b}}(\underline{\mathbb{R}})^{\mathbb{Q}-\det}$ is *multiplicative* if it is distinguished in $\mathbf{D}^{\mathrm{b}}(\underline{\mathbb{R}})$ after forgetting the $\mathbb{Q}$-structure and $\det B = \det A \det C$ in the sense that the following diagram (whose right hand isomorphism stems from the triangle) is commutative:

$$
\begin{array}{ccc}
\mathbb{Q} & \xrightarrow{\ d_C\ } & \det C \\
& \llap{\scriptstyle (d_A)^{-1} \otimes d_B}\searrow & \big\downarrow{\scriptstyle \cong} \\
& & \det^{-1} A \otimes \det B.
\end{array}
$$

1.2 Motives

Our work takes place in the category $\mathbf{DM}_{\mathrm{B}}(S)$ of *Beilinson motives* over S, where S is either a finite field, a number ring $\mathcal{O}_F$, or a number field F. Cisinski and Déglise defined this category to be an appropriate subcategory of Morel and Voevodsky's stable homotopy category $\mathbf{SH}(S)_{\mathbb{Q}}$ (with rational coefficients) [CD09]. The category $\mathbf{DM}_{\mathrm{B}}(S)$ is tensor-triangulated, $\mathbb{Q}$-linear, and closed under arbitrary direct sums. Its tensor unit is denoted $\mathbf{1}_S$ or just $\mathbf{1}$. Given some scheme $f : X \to S$ (always tacitly supposed to be separated and of finite type), the *motive* of X and the *motive with compact support* are defined as

$$\mathrm{M}(X) := f_! f^! \mathbf{1}_S, \quad \mathrm{M_c}(X) := f_* f^! \mathbf{1}_S. \tag{1.1}$$

Here $f_! : \mathbf{DM}_{\mathrm{B}}(X) \to \mathbf{DM}_{\mathrm{B}}(S)$ etc. are the functors defined in *op. cit.* This determines a covariant functor $\mathrm{M} : \mathbf{Sch}/S \to \mathbf{DM}_{\mathrm{B}}(S)$ and likewise, but just for proper maps, with $\mathrm{M_c}$. The motive of the projective line decomposes as $\mathrm{M}(\mathbb{P}^1) = \mathbf{1} \oplus \mathbf{1}(1)[2]$. In $\mathbf{DM}_{\mathrm{B}}(S)$, tensoring with $\mathbf{1}(1)[2]$ is invertible and we write $M\{n\} :=$

$M\otimes(\mathbf{1}(1)[2])^{\otimes n}$ for any $n \in \mathbb{Z}$. For $M \in \mathbf{DM_B}(S)$, we put

$$
\begin{aligned}
\mathrm{H}_i(M,p) &:= \mathrm{Hom}_{\mathbf{DM_B}(S)}(\mathbf{1}(p)[i], M), \\
\mathrm{H}^i(M,p) &:= \mathrm{Hom}_{\mathbf{DM_B}(S)}(M, \mathbf{1}(p)[i]).
\end{aligned}
$$

For a regular base S and a regular, projective or affine (but not necessarily flat) scheme X over S and $G := \mathrm{M}(X)(-m)$, *motivic cohomology* of X is given by

$$
\mathrm{H}^i(X,m) := \mathrm{H}^i(G) = \mathrm{Hom}_S(f_!f^!\mathbf{1}(-m), \mathbf{1}[i]) = \mathrm{Hom}_X(\mathbf{1}, \mathbf{1}(m)[i]) = K_{2m-i}(X)_{\mathbb{Q}}^{(m)},
$$

using the purity isomorphism $f^!\mathbf{1}_S = f^*\mathbf{1}_S\{d\} = \mathbf{1}_X\{d\}$, where $d = \dim X - \dim S$. As a consequence of resolution of singularities, the full subcategory $\mathbf{DM}_{\mathbf{B},c}(S) \subset \mathbf{DM_B}(S)$ of compact objects agrees with the thick subcategory generated by such motives G, for any base S as above. We refer to objects of $\mathbf{DM}_{\mathbf{B},c}(S)$ as *geometric motives over S*. For a perfect field S, there is a natural equivalence of categories [CD09, Theorem 15.1.4]

$$
\mathbf{DM}_{\mathbf{B},c}(S) \xrightarrow{\cong} \mathbf{DM}_{\mathrm{gm}}(S)_{\mathbb{Q}} \tag{1.2}
$$

with Voevodsky's triangulated category of geometric motives (with rational coefficients) [Voe00]. It sends the motive $\mathrm{M}(X) \in \mathbf{DM}_{\mathbf{B},c}(\mathbb{Q})$ of a smooth S-scheme in the sense of (1.1) to the motive $\mathrm{M}_{\mathrm{gm}}(X)$ of X in Voevodsky's sense.

The category $\mathbf{DM}_{\mathbf{B},c}(S)$ is equipped with a notion of *weight*: there are full (non-triangulated) subcategories $\mathbf{DM}_{\mathbf{B},c}^{\mathrm{wt}\leq n}(S)$, $\mathbf{DM}_{\mathbf{B},c}^{\mathrm{wt}\geq n}(S)$ such that $f_!\mathbf{1}(a)[2a+n]$ lies in the subcategory

$$
\mathbf{DM}_{\mathbf{B},c}^{\mathrm{wt}=n}(S) := \mathbf{DM}_{\mathbf{B},c}^{\mathrm{wt}\leq n}(S) \cap \mathbf{DM}_{\mathbf{B},c}^{\mathrm{wt}\geq n}(S) \tag{1.3}
$$

of objects of pure weight n, for all $a, n \in \mathbb{Z}$ and all proper maps $f : X \to S$ with regular domain X [Bon10, Héb11]. For any map f (of finite type), the functors $f_!$, f^* preserve the subcategories $\mathbf{DM}_{\mathbf{B},c}^{\mathrm{wt}\leq n}(-)$ and dually for $f^!$, f_*.

The dual of any geometric motive M is defined as $M^\vee = \underline{\mathrm{Hom}}_{\mathbf{DM_B}(S)}(M, \mathbf{1})$. Dualizing exchanges ! and $*$: for example, for any map f, $(f_!f^!\mathbf{1})^\vee = f_*f^*(\mathbf{1}^\vee)$ which is canonically isomorphic $f_*f^*\mathbf{1}$. Therefore, the natural map

$$
M \to (M^\vee)^\vee \tag{1.4}
$$

is an isomorphism for any $M \in \mathbf{DM}_{\mathbf{B},c}(\mathbb{Z})$ [CD09, 14.3.31]. This yields a canonical isomorphism $\mathrm{H}_0(M^\vee, 0) = \mathrm{H}^0(M, 0)$.

Definition 1.1. Let $S \subset \mathrm{Spec}\,\mathcal{O}_F$ be an open subscheme. A motive $M \in \mathbf{DM}_{\mathbf{B},c}(S)$ is called *smooth* if the natural map [Ayo07, Section 2.3.2] [1]

$$
i^*M\{-1\} = i^*M \otimes i^!\mathbf{1} \to i^!(M\otimes\mathbf{1}) = i^!M
$$

is an isomorphism for all closed points $i : \mathrm{Spec}\,\mathbb{F}_{\mathfrak{p}} \to S$. A motive $M \in \mathbf{DM}_{\mathbf{B},c}(S)$ is *generically smooth* if j^*M is smooth for some open subscheme $j : U \subset S$.

Since $\mathrm{M}(X)(m)$ is smooth provided X/S is smooth and proper, every motive $M \in \mathbf{DM}_{\mathbf{B},c}(S)$ is generically smooth. We write $\eta : \mathrm{Spec}\,F \to \mathrm{Spec}\,\mathcal{O}_F$ for the generic point.

In order to interpret Beilinson's conjecture for mixed motives over $\mathbb{Q}$ in terms of motives over $\mathbb{Z}$ we need to assume the conjectural framework of mixed motives over F, $\mathrm{Spec}\,\mathcal{O}_F$ and $\mathbb{F}_q$. The precise axioms we are staking on are listed in [Sch12b,

---

[1] The use of this canonical map, as opposed to a mere noncanonical isomorphism, was suggested by Bruno Kahn.

Section 4], so we only summarize them briefly and refer to *loc. cit.* for a more complete discussion. [2] [3] Note that the corresponding statements for the triangulated category of *Artin-Tate motives* $\mathbf{DATM}(\mathcal{O}_F) \subset \mathbf{DM}_{\mathrm{B},c}(\mathcal{O}_F)$, which is the triangulated subcategory of $\mathbf{DM}_{\mathrm{B}}(\mathcal{O}_F)$ generated by $\mathrm{M}(V)(m)$ where $V \to \operatorname{Spec} \mathcal{O}_F$ is a quasi-finite, but not necessarily flat map and $m \in \mathbb{Z}$, have been shown in [Sch11].

Axiom 1.2. (i) [Sch12b, Axioms 4.1, 4.2] $\mathbf{DM}_{\mathrm{B},c}(S)$ is conjectured to enjoy a non-degenerate t-structure whose heart $\mathbf{MM}(S)$ is called the category of *mixed motives*. The cohomological dimension of $\mathbf{MM}(S)$ is conjectured to be 0 ($S = \mathbb{F}_q$) and 1 ($S = F$), respectively. The truncation with respect to the t-structure is denoted $^{\mathrm{p}}\mathrm{H}^*$. We write $\mathrm{h}^i(X, n)$ for $^{\mathrm{p}}\mathrm{H}^i(\mathrm{M}(X)(n))$. The t-structures are normalized by declaring $\mathbf{1} \in \mathbf{MM}(S)$ when $S = F, \mathbb{F}_q$ and $\mathbf{1}[1] \in \mathbf{MM}(\mathcal{O}_F)$, respectively. For example, $\mathrm{h}^{-1}(\mathbb{P}^1_{\mathcal{O}_F}) = \mathbf{1}_{\mathcal{O}_F}(1)[1]$, $\mathrm{h}^{-2}(\mathbb{P}^1_F) = \mathbf{1}_F(1)$. More generally, $\eta^*[-1]$ is t-exact and $\eta^* \mathrm{h}^{-b}(X, -m) = \mathrm{h}^{-b-1}(X_\eta, -m)$ for any scheme $X/\mathcal{O}_F$ with generic fiber X_η.

(ii) [Sch12b, Axiom 4.5.] The key requirement on the t-structure is that realization functors of the form $\mathbf{DM}_{\mathrm{B},c}(S) \to \mathbf{D}^{\mathrm{b}}(\mathcal{C})$ are to be exact (see *loc. cit.* and around (1.7) for the ℓ-adic realization over $\mathbb{Z}[1/\ell]$). In the guise of a spectrum representing the cohomology theory, the exactness requirement is to be understood as in (2.14).

(iii) [Sch12b, Axioms 4.4, 4.6, 4.11] Any mixed motive is conjectured to have a weight filtration which is compatible with the weight formalism mentioned around (1.3). The pure objects in $\mathbf{MM}(K)$ (for any field K) are conjectured to be identified with the category $\mathbf{M}_{\mathrm{num}}$ of pure motives with respect to numerical equivalence. This implies that the pure objects in $\mathbf{MM}(K)$ form an abelian semi-simple category [Jan92, Th. 1]. Moreover, homological and numerical equivalence are conjectured to agree. The cohomology functors $^{\mathrm{p}}\mathrm{H}^*$ belonging to the motivic t-structure are supposed to respect the weights, i.e., given some $M \in \mathbf{DM}_{\mathrm{B},c}^{\mathrm{wt}=w}$, $^{\mathrm{p}}\mathrm{H}^n(M) \in \mathbf{MM}$ is pure of weight $w + n$. For example, for a smooth projective scheme X/S, $\mathrm{M}(X)(-m) = f_! f^! \mathbf{1}(-m) \in \mathbf{DM}_{\mathrm{B},c}^{\mathrm{wt}=2m}$, so that $\mathrm{h}^{-b}(X, -m)$ is pure of weight $2m - b$. Morphisms of mixed motives are expected to respect weights strictly, thereby giving constraints on the existence of maps between motives.

In the remainder of this paper we assume that the axioms concerning mixed motives over open subschemes of $\operatorname{Spec} \mathcal{O}_F$, $\mathbb{F}_q$ *and* F *hold.*

Given a mixed motive over $\mathbb{Q}$, $M_\eta \in \mathbf{MM}(\mathbb{Q})$, pick any $M \in \mathbf{MM}(\mathbb{Z})$ satisfying $M_\eta = \eta^* M[-1]$ and some open subscheme $j : U \to \operatorname{Spec} \mathbb{Z}$ such that $j^* M$ is smooth. We call
$$\eta_{!*}(M_\eta[1]) := j_{!*} j^* M := \operatorname{im}(j_! j^* M \to j_* j^* M) \in \mathbf{MM}(\mathbb{Z})$$
the *generic intermediate extension* of $M_\eta[1]$. This is explained and shown to be well-defined in [Sch12b, Section 5.4]. We apply this to $M_\eta = \mathrm{h}^{-b-1}(X_\eta, -m)$ and $M = \mathrm{h}^{-b}(X, -m)$, where $X_\eta/\mathbb{Q}$ is smooth projective and $X/\mathbb{Z}$ is any projective (not necessarily regular) model of X_η of constant dimension d. Throughout this

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[2]Unlike this paper, *op. cit.* is written with a contravariant notation of motives. This induces a number of changes in notation: every $f_!$, $f^!$ gets replaced by a f_* and f^*, and vice versa. Moreover, a twist and shift $M(m)[n]$ corresponds to $M(-m)[-n]$ here. Both here and there, the normalization of the t-structure is such that $\mathbf{1}[1] \in \mathbf{MM}(\mathcal{O}_F)$, while $\mathbf{1} \in \mathbf{MM}(F)$.

[3] The decomposition axiom for smooth projective varieties formulated in [Sch12b, Axiom 4.13] is not needed: it is only used in [Sch12b, Lemma 5.10] to show that a certain motive is generically smooth, but this is inconditionally true for any motive by the remark after Definition 1.1.

paper, we write

$$E := \eta_{!*}\eta^* \, \mathrm{h}^{-b}(X, -m) = \eta_{!*}(\mathrm{h}^{-b-1}(X_\eta, -m)[1]) \in \mathbf{MM}(\mathbb{Z}). \qquad (1.5)$$

This motive is pure of weight $w := 2m - b$. Its motivic cohomology is given by the following theorem:

Theorem 1.3. *With the above notation, we write* $\mathrm{H}^b(X_\eta, m)_\mathbb{Z} := \mathrm{im}(\mathrm{H}^b(X, m) \to \mathrm{H}^b(X_\eta, m))$. *Moreover, let* $\mathrm{CH}^m(X_\eta)_{\mathbb{Q},\mathrm{hom}}$ *be the subgroup of the Chow group of cycles homologically equivalent to zero and* $\mathrm{CH}^m(X_\eta)_\mathbb{Q}/\mathrm{hom}$ *the group of cycles modulo homological equivalence (tensored with* $\mathbb{Q}$*). Then*

$$\mathrm{H}^a(E) = \mathrm{H}^a(\eta_{!*} \, \mathrm{h}^{-b-1}(X_\eta, -m)[1]) = \begin{cases} \mathrm{CH}^m(X_\eta)_\mathbb{Q}/\mathrm{hom} & a = 1, \, w = 1 \\ 0 & a = 1, \, w \neq 1 \\ 0 & a = 2, \, w \leq 1 \\ \mathrm{CH}^m(X_\eta)_{\mathbb{Q},\mathrm{hom}} & a = 2, \, w = 2 \\ \mathrm{H}^{b+2}(X_\eta, m)_\mathbb{Z} & a = 2, \, w \geq 3 \\ 0 & a = 3, \, w \leq 2 \\ ? & a = 3, \, w \geq 3 \\ 0 & a > 3, \, a < 1 \end{cases}$$

Proof: Everything except the cases $a = 2$, $w \leq 1$ and $a = 3$, $w \leq 2$ is shown in [Sch12b, Lemma 5.2, Theorem 6.11]. For $a = 2$ and $w \leq 1$, the map

$$\mathrm{H}^2(E) \to \mathrm{H}^2(\eta^* E) = \mathrm{H}^1(\mathrm{h}^{-b-1}(X_\eta, -m)) \to \mathrm{H}^{b+2}(X_\eta, m) = \mathrm{CH}^m(X_\eta, w - 2) = 0$$

is injective: for the first map this is [Sch12b, Lemma 6.9], the second one is because the cohomological dimension of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Q})$ is one [Sch12b, Axiom 4.1.]. For $a = 3$, $w \leq 2$, we use the exact localization sequence

$$\ldots \to \oplus_p \mathrm{H}^3(i_{p*}i_p^* E) \to \mathrm{H}^3(E) \to \mathrm{H}^3(\eta^* E) = \mathrm{H}^2(\eta^*[-1]E) = 0.$$

The right hand vanishing is again because the cohomological dimension of motives over $\mathbb{Q}$ being one. Also by cohomological dimension we have

$$\mathrm{H}^3(i_{p*}i_p^* E) = \mathrm{Hom}(i_p^* E, i_p^! \mathbf{1}[3]) = \mathrm{Hom}(i_p^* E, \mathbf{1}(-1)[1]) = \mathrm{Hom}_{\mathbf{MM}(\mathbb{F}_p)}({}^{\mathrm{P}}\mathrm{H}^{-1}i_p^* E(1), \mathbf{1}).$$

The functor i^* preserves negative weights, i.e., $\mathrm{wt}({}^{\mathrm{P}}\mathrm{H}^{-1}(i_p^* E(1))) \leq \mathrm{wt}(E) - 1 - 2 = w - 3$. By strictness of the weight filtration the group therefore vanishes for $w \leq 2$. $\qquad \square$

In accordance with Conjecture 4.1 (see the case $w \leq 1$ in the proof of Proposition 5.16) I expect $\mathrm{H}^3(E) = 0$ for arbitrary weight w. See the introduction for the relation of this to Scholl's notion of mixed motives over $\mathbb{Z}$. For Artin-Tate motives, the expected vanishing holds unconditionally for all weights:

Theorem 1.4. *Let* M_η *be an Artin-Tate motive over* F*, concentrated in cohomological degree* -1*. Then* $\mathrm{H}^3(\mathcal{O}_F, \eta_{!*}M_\eta) = 0$.

Proof: There is some $j : U \subset \mathrm{Spec}\,\mathcal{O}_F$ and a smooth Artin-Tate motive $M \in \mathbf{MATM}(U) = \mathbf{MM}(U) \cap \mathbf{DATM}(U)$ such that $M_\eta = \eta^*[-1]M$. Shrinking U further (using $j'_{!*}j'^* M \cong M$ for some $j' : U' \subset U$, as M is smooth), we may assume by the standard splitting routine [Sch11, Lemma 2.5] that there is an etale Galois cover $f' : V' \to U$ such that $f'^* M$ is a mixed Tate motive over V'. The map $M \to f'_* f'^* M \overset{\cong}{\leftarrow} f'_! f'^! M \to M$ is $\deg f' \cdot \mathrm{id}_M$, so M is a direct summand of $f'_* f'^* M$, since we use rational coefficients. The functor $f'_* = f'_!$ preserves Artin-Tate motives and is exact [Sch11, Theorem 4.2]. Hence $j_{!*} f'_* f'^* M = f_* j'_{!*} f'^* M$.

Here $f : V \to \operatorname{Spec} \mathcal{O}_F$ is the normalization of $\mathcal{O}_F$ in the function field of V' and $j' : V' \to V$ is the corresponding open immersion. Consequently,

$$\mathrm{H}^3(\mathcal{O}_F, \eta_{!*} M_\eta) = \mathrm{H}^3(\mathcal{O}_F, j_{!*} M) \subset \mathrm{H}^3(V, j'_{!*} f'^* M) = \operatorname{Hom}_V(j'_{!*} f'^* M, (\mathbf{1}[1])[2]) = 0,$$

since the cohomological dimension of mixed Tate motives over V is one, as opposed to two for Artin-Tate motives [Sch11, Proposition 4.4]. $\qquad\square$

The following conjecture will be needed to deal with motives over $\mathbb{F}_p$.

Conjecture 1.5. (Beilinson) Let $X/\mathbb{F}_q$ be smooth and projective. Up to torsion, numerical and rational equivalence agree on X.

Recall that homological equivalence lies between these two equivalence relations [And04, 3.2.1], so under 1.5, all three agree. The second important consequence of 1.5 is that the category of pure Chow motives over $\mathbb{F}_q$ is semisimple by Jannsen's theorem.

To study L-functions, we need some ℓ-adic realization functor. We use the machinery developed recently by Ayoub [Ayo12]. It allows the base scheme to be $\mathbb{Z}[1/\ell]$. Let ℓ be an odd prime number and S a scheme that is of finite type over $\mathbb{Z}$ or $\mathbb{Q}$ such that ℓ is invertible on S (cf. [Ayo12, Hyp. 6.5]). Define the ℓ-adic realization functor as the following composition

$$(-)_\ell : \mathbf{DM}_{\mathrm{B}}(S) \xrightarrow{F_1} \mathbf{SH}(S)_{\mathbb{Q}} \xrightarrow{F_2} DA^{\text{ét}}(S, \mathbb{Q}_\ell) \tag{1.6}$$
$$\xrightarrow{R_\ell} \mathbf{D}(\mathbf{Shv}_{\text{ét}}(S, \mathbb{Q}_\ell)) \xrightarrow{F_3} \mathbf{D}(\mathbf{Shv}_{\text{ét}}(S, \mathbb{Q}_\ell))$$

The functor F_1 is the inclusion of the category of $\mathbf{1}$-modules in $\mathbf{SH}(S)_{\mathbb{Q}}$. The category $DA^{\text{ét}}(S, \mathbb{Q}_\ell)$ is the homotopy category of the model category of symmetric $\mathbb{P}^1$-spectra of complexes of ℓ-adic presheaves on $\mathbf{Sm}/S$, endowed with the $\mathbb{A}^1$-étale-local model structure. The functor F_2 is obtained by combining the natural free abelian group functor $\Delta^{\mathrm{op}}\mathbf{Sets} \to \mathbf{Com}(\mathbf{Ab})$ and the sheafification (from Nisnevich sheaves to etale sheaves), see e.g. [CD09, 5.3.28, 5.3.37]. The functor R_ℓ is Ayoub's ℓ-adic realization functor. We append the contravariant functor $F_3 : M \mapsto \underline{\operatorname{Hom}}(M, f^! \mathbb{Q}_\ell)$, where $f : S \to \operatorname{Spec} \mathbb{Z}$ is the structural map (and $\underline{\operatorname{Hom}}$ denotes the derived inner Hom). For any map $g : X \to Y$ of quasi-projective S-schemes, the functors F_1, F_2, R_ℓ commute with $g_!$, g_*, g^* and $g^!$ and, when applied to compact objects, with $\underline{\operatorname{Hom}}$ [Ayo12, Thm. 6.6]. Finally, F_3 exchanges ! and *, e.g. $F_3(g^* M) = g^! F_3(M)$ for $M \in \mathbf{D}(\mathbf{Shv}_{\text{ét}}(S, \mathbb{Q}_\ell))$. Therefore, for some quasi-projective scheme $f : X \to S$, $(\mathrm{M}(X)(-m)[-n])_\ell = f_* f^* \mathbb{Q}_\ell(m)[n]$. This property is also satisfied for Huber's and Ivorra's realization functors provided S is a field [Hub00, Ivo07]. Thus, for the mere definition in 3.1, these realization functors are sufficient, but Lemma 3.4 relies on a realization functor over $\mathbb{Z}[1/\ell]$.

The exactness requirement for the functor $-_\ell$ mentioned in Axiom 1.2(ii) means that the restriction of $-_\ell$ to $\mathbf{DM}_{\mathrm{B},c}(S)$ is exact with respect to the (conjectural) motivic t-structure and the t-structure on $\mathbf{D}(\mathbf{Shv}_{\text{ét}}(S, \mathbb{Q}_\ell))$ (which is the obvious one if S is a field and the perverse t-structure for $S = \operatorname{Spec} \mathbb{Z}[1/\ell]$, see [Sch12b, Section 3]. For example, for a quasi-projective variety X over a field it implies

$$(\mathrm{h}^{-b}(X))_\ell = \mathrm{H}^b(X, \mathbb{Q}_\ell). \tag{1.7}$$

2 Arakelov motivic cohomology

2.1 Deligne cohomology

A key input to Beilinson's conjecture 5.14 is Deligne cohomology. We recall its classical definition and the well-known interpretation in terms of weak Hodge cohomology. Then, we recall from [HS11] the Deligne cohomology spectrum H_{D} which

is crucial for the definition of Arakelov motivic cohomology. In order to establish the $\mathbb{Q}$-structure on the groups represented by H_D, we explain how to apply the construction in *loc. cit.* to obtain spectra representing Betti and de Rham cohomology.

Let an : $\mathbf{Sm}/\mathbb{C} \to \mathbf{Sm}^{\mathrm{an}}$ be the functor that associates to any smooth $\mathbb{C}$-scheme the underlying complex analytic manifold. We also consider an : $\mathbf{Sm}/\mathbb{Q}$ (or $\mathbf{Sm}/\mathbb{R}$) $\to \mathbf{Sm}^{\mathrm{an},G}$, where the target category consists of complex analytic manifolds with a G-action, $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In this section, X is a smooth scheme over $\mathbb{Q}$. We usually write $X^{\mathrm{an}} := \mathrm{an}(X)$ and $\mathrm{Fr}_\infty : X^{\mathrm{an}} \to X^{\mathrm{an}}$ for the conjugation. We also pick a smooth proper compactification $j : X \to \overline{X}$ (over $\mathbb{Q}$) such that $D := \overline{X} \backslash X$ is a divisor with strict normal crossings. We write $\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}})$ for the complex of meromorphic forms on $\overline{X}$ that are holomorphic on $X \subset \overline{X}$, and have at worst logarithmic poles at the divisor D. This complex is endowed with the Hodge filtration $F^p := \sigma_{\geq p}$, which is simply the brutal truncation. The variant using algebraic (i.e., Kähler) differential forms is denoted $\Omega^{*,\mathrm{alg}}_{\overline{X}}(\log D)$. The C^∞-variant is denoted $E^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}})$. The subspace of real-valued forms is denoted $E^*_{\mathbb{R},\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}})$. These complexes are filtered by $F^p E^n_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}) = \oplus_{a+b=n,a\geq p} E^{a,b}_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}})$. To get rid of the choice of $\overline{X}$, put

$$E^*(X) := \varinjlim_{\overline{X}} E^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}),$$

and similarly for $E^*_{\mathbb{R}}(X)$, $\Omega^*(X)$, $\Omega^{*,\mathrm{alg}}(X)$. Here, the colimit runs over the directed category of all compactifications $\overline{X}$ as above. Finally, let $\mathbb{R}(p) := (2\pi i)^p \mathbb{R} \subset \mathbb{C}$ be the constant sheaf.

Definition 2.1. Set $\mathbb{R}_{\mathrm{D},D,\overline{X}}(p) := \mathrm{cone}(\mathrm{R}j_*\mathbb{R}(p) \oplus F^p\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}) \to \mathrm{R}j_*\Omega^*_{X^{\mathrm{an}}})$. For example, if X is proper, $\mathbb{R}_{\mathrm{D}}(p) \cong [\mathbb{R}(p) \to \Omega^0_{X^{\mathrm{an}}} \to \ldots \to \Omega^{p-1}_{X^{\mathrm{an}}}]$, with the terms lying in degrees 0 to p. *Deligne cohomology* of X is defined as the G-invariant subspace of a sheaf hypercohomology group,

$$\mathrm{H}^n_{\mathrm{D}}(X,p) := \mathbb{H}^n(\overline{X}^{\mathrm{an}}, \mathbb{R}_{\mathrm{D},D,\overline{X}}(p))^G.$$

(The G-action is obtained by letting G act on $\mathbb{R}(p)$ as $a \mapsto \overline{\mathrm{Fr}^*_\infty(a)}$ and on Ω^* by $\omega \mapsto \mathrm{Fr}^*_\infty(\omega)$. This group does not depend on the choice of $\overline{X}$ [EV88, Lemma 2.8].)

By definition, there is a long exact sequence

$$\ldots \to (\mathrm{H}^i_{\mathrm{dR}}(X^{\mathrm{an}})/F^p\mathrm{H}^i(X^{\mathrm{an}},\Omega^*_X))^G \to \mathrm{H}^{i+1}_{\mathrm{D}}(X,m) \to \mathrm{H}^{i+1}(X^{\mathrm{an}},\mathbb{R}(m))^{(-1)^m} \to \ldots.$$

Here the superscript denotes the $(-1)^m$-eigenspace of the Fr_∞-action on Betti cohomology of X^{an}. This sequence induces an isomorphism

$$\det \mathrm{H}^*_{\mathrm{D}}(X,m) = \det{}^{-1}(\mathrm{H}^*_{\mathrm{dR}}(X^{\mathrm{an}})/F^m)^G \otimes \det \mathrm{H}^*(X^{\mathrm{an}},\mathbb{R}(m))^{(-1)^m}. \qquad (2.1)$$

The right hand side carries a natural $\mathbb{Q}$-structure stemming from the isomorphisms $\mathrm{H}^*(X^{\mathrm{an}},\mathbb{R}(m)) = \mathrm{H}^*(X^{\mathrm{an}},\mathbb{Q}(m))\otimes_{\mathbb{Q}}\mathbb{R}$ and $\mathrm{H}^*(\overline{X}^{\mathrm{an}}, F^*\Omega^*_{\overline{X}^{\mathrm{an}}}(\log D^{\mathrm{an}}))^G \cong \mathrm{H}^*(\overline{X}_{\mathbb{R}}, F^*\Omega^{*,\mathrm{alg}}_{\overline{X}_{\mathbb{R}}}(\log D_{\mathbb{R}})) = \mathrm{H}^*(\overline{X}, F^*\Omega^{*,\mathrm{alg}}_{\overline{X}}(\log D))\otimes_{\mathbb{Q}}\mathbb{R}$ (GAGA). We use the above isomorphism to carry over the $\mathbb{Q}$-structure to the left hand side.

If X is (smooth and) proper, the degeneration of the Hodge-de Rham spectral sequence and weight reasons give us short exact sequences (*loc. cit.*)

$$0 \to \mathrm{H}^i(X^{\mathrm{an}},\mathbb{R}(m))^{(-1)^m} \to \mathrm{H}^i_{\mathrm{dR}}(X_{\mathbb{R}})/F^m \to \mathrm{H}^{i+1}_{\mathrm{D}}(X,m) \to 0 \qquad (2.2)$$

for $i - 2m \leq -2$ and, for $i - 2m \geq 0$,

$$0 \to \mathrm{H}^i_{\mathrm{D}}(X,m) \to \mathrm{H}^i(X^{\mathrm{an}},\mathbb{R}(m))^{(-1)^m} \to \mathrm{H}^i_{\mathrm{dR}}(X_{\mathbb{R}})/F^m \to 0, \qquad (2.3)$$

respectively. In this case, each individual Deligne cohomology group carries a $\mathbb{Q}$-structure, as opposed to the general case of a merely smooth $X/\mathbb{Q}$.

Now, we recall Beilinson's notion of weak absolute Hodge cohomology. It is relevant to us because of its relation to archimedean factors of L-functions, see (3.2). It is based on Deligne's abelian category $\mathbf{MHS}_{\mathbb{Q}}(\mathbb{R})$ of mixed Hodge structures [Del71, 2.3.1]. The subscript $\mathbb{Q}$ indicates that we are considering $\mathbb{Q}$-vector spaces, "$(\mathbb{R})$" means that the structure is endowed with an action of $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For example, $\mathbf{1}(n)$ is the one-dimensional $\mathbb{Q}$-space, such that it is pure of weight $-2n$ and the Hodge filtration is concentrated in degree $-n$, and the non-trivial element of G acts as multiplication by $(-1)^n$. Let

$$\mathbf{Com}_{\mathrm{H}}^{\mathrm{b}} = \{C = (C_{\mathrm{dR}}, C_{\mathrm{B}}, C_c, i_{\mathrm{dR}}, i_{\mathrm{B}})\}$$

be the category of bounded *Hodge complexes* [Beĭ86, 3.2]. Its objects consist of a bounded bifiltered complex of $\mathbb{Q}$-vector spaces $(C_{\mathrm{dR}}, W_*, F^*)$, a filtered complex of $\mathbb{Q}[G]$-modules (C_{B}, W_*) and a filtered complex of $\mathbb{C}$-modules with $\mathbb{C}$-antilinear G-action, (C_c, W_*), a filtered G-equivariant quasi-isomorphism $i_{\mathrm{B}} : (C_{\mathrm{B}}, W_*) \otimes_{\mathbb{Q}} \mathbb{C} \to (C_c, W_*)$ (G acts on the left hand term by the action on C_{B} and complex conjugation on $\mathbb{C}$) and finally a filtered G-equivariant quasi-isomorphism $i_{\mathrm{dR}} : (C_{\mathrm{dR}}, W_*) \otimes_{\mathbb{Q}} \mathbb{C} \to (C_c, W_*)$ (on the left, G acts by conjugation on $\mathbb{C}$). These data are subject to the requirement that the cohomology quintuple $\mathrm{H}^i(C)$ defined by the cohomologies of the various complexes and comparison maps has to be an object of $\mathbf{MHS}_{\mathbb{Q}}(\mathbb{R})$. Morphisms in the category $\mathbf{Com}_{\mathrm{H}}^{\mathrm{b}}$ are required to respect the filtrations and the comparison quasi-isomorphisms. To any Hodge complex, we can associate its *weak Hodge cohomology* [Beĭ86, 3.13]

$$\mathrm{R}\Gamma_{\mathrm{w}}(C) := \mathrm{cone}[-1]\left(C_{\mathrm{B}}^G \otimes \mathbb{R} \oplus F^0 C_{\mathrm{dR}} \otimes \mathbb{R} \xrightarrow{i_{\mathrm{B}} - i_{\mathrm{dR}}} C_c^G\right) \in \mathbf{Com}(\mathbb{R}).$$

This descends to a functor

$$\mathrm{R}\Gamma_{\mathrm{w}} : \mathbf{D}_{\mathrm{H}}^{\mathrm{b}} := \mathbf{Com}_{\mathrm{H}}^{\mathrm{b}}/\text{quasi-isomorphisms} \to \mathbf{D}^{\mathrm{b}}(\underline{\mathbb{R}})^{\mathbb{Q}-\det}.$$

Indeed, taking G-invariants and applying the Hodge filtration are exact operations, since morphisms of Hodge structures respect the Hodge filtration strictly [Del71, 2.3.5(iii)]. The $\mathbb{Q}$-structure on $\mathrm{R}\Gamma_{\mathrm{w}}(C)$ is the one stemming from the very definition, where C_c^G is endowed with a $\mathbb{Q}$-structure using the one on C_{dR} via i_{dR}. Set $\mathrm{H}_{\mathrm{w}}^i(C) := \mathrm{H}^i(\mathrm{R}\Gamma_{\mathrm{w}}(C))$. A spectral sequence argument yields an exact sequence:

$$0 \to \mathrm{H}_{\mathrm{w}}^1(\mathrm{H}^{i-1}C) \to \mathrm{H}_{\mathrm{w}}^i(C) \to \mathrm{H}_{\mathrm{w}}^0(\mathrm{H}^iC) \to 0. \tag{2.4}$$

Unlike absolute Hodge cohomology, i.e., the derived functor of $V \mapsto \Gamma_{\mathbf{MHS}}(V) := \mathrm{Hom}_{\mathbf{MHS}}(\mathbf{1}, V) = \mathrm{H}_{\mathrm{w}}^0(W_0 V)$, the weak variant has a duality: the natural pairing (induced by $A \times A^\vee \to \mathbb{R}$ for any $\mathbb{R}$-vector space A),

$$\mathrm{H}_{\mathrm{w}}^i(C) \times \mathrm{H}_{\mathrm{w}}^{1-i}(C^\vee(1)) \to \mathrm{H}_{\mathrm{w}}^1(\mathbf{1}(1)) = \mathbb{R}, \tag{2.5}$$

is perfect for all i [FPR94, Prop.III.1.2.3].

The following well-known lemma states that weak Hodge cohomology is the same as Deligne cohomology. Recall the Hodge complex $\underline{\mathrm{R}\Gamma}(X, m)$ of [Beĭ86, Section 4] whose cohomology objects are the Hodge structures $\mathrm{H}^i(X^{\mathrm{an}}, \mathbb{Q}(m))$.

Lemma 2.2. *For $X/\mathbb{Q}$ smooth and projective and any i, m, we have*

$$\mathrm{H}_{\mathrm{w}}^i(\underline{\mathrm{R}\Gamma}(X, m)) = \mathrm{H}_{\mathrm{D}}^i(X, m). \tag{2.6}$$

The induced isomorphism $\det \mathrm{H}_{\mathrm{w}}^*(\underline{\mathrm{R}\Gamma}(X, m)) = \det \mathrm{H}_{\mathrm{D}}^*(X, m)$ *respects the $\mathbb{Q}$-structure.*

Proof: The Hodge structures $L_i := \mathrm{H}^i(\underline{\mathrm{R\Gamma}}(X, m)) = \mathrm{H}^i(X^{\mathrm{an}}, \mathbb{Q}(m))$ are pure of weight $i - 2m$. For $i - 2m < 0$, $\mathrm{H}^0_{\mathrm{w}}(L_i) = \Gamma_{\mathbf{MHS}}(L_i) = 0$. By duality, $\mathrm{H}^1_{\mathrm{w}}(L_i) = \mathrm{H}^0_{\mathrm{w}}(L_i^\vee(1))^\vee = 0$ for $i - 2m > -2$. Hence, by (2.4),

$$\mathrm{H}^i_{\mathrm{w}}\underline{\mathrm{R\Gamma}}(X, m) = \left\{ \begin{array}{ll} \mathrm{H}^1_{\mathrm{w}}(L_{i-1}) & i - 2m < 0 \\ \mathrm{H}^0_{\mathrm{w}}(L_i) & i - 2m \geq 0 \end{array} \right.$$

The map in the exact sequences (2.2) between Betti and de Rham cohomology is the one from the definition of $\mathrm{R\Gamma}_{\mathrm{w}}(L_*)$. This shows (2.6). The identification of the $\mathbb{Q}$-structures follows similarly. $\qquad\square$

For archimedean factors of L-functions of arbitrary motives, we use the Hodge realization functor (see [Beĭ86, Section 3] for an early avatar):

$$\mathrm{R\Gamma}_{\mathrm{H}} : \mathbf{DM}_{\mathsf{B},c}(\mathbb{Q})^{\mathrm{op}} \stackrel{\cong, (1,2)}{\longrightarrow} \mathbf{DM}_{\mathrm{gm}}(\mathbb{Q})^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}_{\mathrm{H}}. \qquad (2.7)$$

The right hand functor is Huber's Hodge realization functor [Hub00, 2.3.5]. It maps $\mathrm{M}_{\mathrm{gm}}(X)(-m)$ to $\underline{\mathrm{R\Gamma}}(X, m)$. For any $M \in \mathbf{DM}_{\mathsf{B},c}(\mathbb{Q})$, the natural map $\mathrm{R\Gamma}_{\mathrm{H}}(\underline{\mathrm{Hom}}(M, \mathbf{1})) \to \underline{\mathrm{Hom}}(\mathrm{R\Gamma}_{\mathrm{H}}(M), \mathrm{R\Gamma}_{\mathrm{H}}(\mathbf{1}))$ is an isomorphism. It is enough to check this on generators $M = \mathrm{M}(X)$ with $X/\mathbb{Q}$ smooth and projective, where it follows from $(\mathrm{M}(X))^\vee = \mathrm{M}(X)\{\dim X\}$. We obtain $\mathrm{R\Gamma}_{\mathrm{H}}(M^\vee(1)) = (\mathrm{R\Gamma}_{\mathrm{H}}(M))^\vee(-1)$. We put

$$\mathrm{R\Gamma}_{\mathrm{wH}} := \mathrm{R\Gamma}_{\mathrm{w}} \circ \mathrm{R\Gamma}_{\mathrm{H}} : \mathbf{DM}_{\mathsf{B},c}(\mathbb{Q}) \to \mathbf{D}^{\mathrm{b}}(\underline{\mathbb{R}})^{\mathbb{Q}-\det}. \qquad (2.8)$$

The composition of these functors with $\eta^* : \mathbf{DM}_{\mathsf{B},c}(\mathbb{Z}) \to \mathbf{DM}_{\mathsf{B},c}(\mathbb{Q})$ will be denoted the same.

Finally, we recall the construction of the Deligne cohomology spectrum H_{D} [HS11]. We also sketch how to obtain similar spectra for Betti and de Rham cohomology. The aim is (2.13), the $\mathbb{Q}$-structure on Deligne cohomology groups of general motives.

Let $\mathcal{C}$ be either the category $\mathbf{Sm}^{G,\mathrm{an}}$ or $\mathbf{Sm}/\mathbb{Q}$. Consider simplicial presheaves $C(p)$ of pointed sets on $\mathcal{C}$, for each $p \geq 0$, together with a "product" map $\cdot_C : C(p) \wedge C(p') \to C(p + p')$. Moreover, we assume there is an element $c_1 \in C(1)(\mathbb{G}_m)$, that restricts to zero at the point $1 \in \mathbb{G}_m$ (equivalently, a pointed map $c_1 : (\mathbb{G}_m, 1) \to C(1)$) such that for any two maps $f_i : U \to \mathbb{G}_m$, $U \in \mathcal{C}$, $i = 1, 2$,

$$f_1^*(c_1) \cdot_C (f_2^*(c_1) \cdot_C c') = f_2^*(c_1) \cdot_C (f_1^*(c_1) \cdot_C c'). \qquad (2.9)$$

The element c_1 is referred to as a *bonding element*. Under these assumptions, the presheaves $C(p)$ with the bonding maps $\mathbb{G}_m \wedge C(p) \stackrel{c_1 \wedge \mathrm{id}}{\longrightarrow} C(1) \wedge C(p) \stackrel{\cdot_C}{\longrightarrow} C(p+1)$ form a symmetric $\mathbb{G}_m$-spectrum C (where the Σ_p-action on $C(p)$ is trivial). The category of such spectra is denoted $\mathbf{Spt}(\mathcal{C})$. It is endowed with a model structure whose homotopy category $\mathbf{SH}(\mathbb{Q})$ (or $\mathbf{SH}(\mathbb{R}^{\mathrm{an}})$) satisfies (cf. e.g. [Ayo10, Section 1] for the analytic version):

$$\mathrm{Hom}_{\mathbf{SH}}(\Sigma^\infty(X \sqcup \{*\}) \wedge S^n \wedge \mathbb{G}_m^{\wedge m}, C) = \pi_{n+m+N}(C(m+N)(X))$$

for any $X \in \mathcal{C}$, and $n, m \in \mathbb{Z}$ and $N \gg 0$, provided that

1. all levels $C(p)$ are homotopy invariant: $C(p)(-) \to C(p)(- \times \mathbb{A}^1)$ (respectively, $- \times (\mathbb{A}^1)^{\mathrm{an}}$) is a weak equivalence,

2. all levels $C(p)$ satisfy descent (with respect to the Nisnevich and the analytic topology, respectively), and

3. C is an Ω-spectrum. In the presence of the first two conditions, this is implied by the bundle formula, which says that

$$\oplus_{i=0}^{1} p_X^*(-) \cdot_C p_{\mathbb{G}_m}^*(c_1)^i : \oplus \pi_{i+*}(C(p-i)(X)) \to \pi_*(C(p)(X \times \mathbb{G}_m))$$

is an isomorphism, where p_X, $p_{\mathbb{G}_m} : X \times \mathbb{G}_m \to X$, $\mathbb{G}_m$ are the projections.

The spectra below are all obtained by putting $C(p) := DK(\tau_{\geq 0} A(p))$ for appropriate complexes of abelian groups $A(p)$. Here τ is the good truncation and DK the Dold-Kan equivalence.

We now define four different (but isomorphic) spectra representing Betti cohomology with real coefficients by specifying the levels $C(p)$ and the bonding elements in $C(1)(\mathbb{G}_m)$. The product structure map on the level complexes is obvious for these Betti cohomology spectra, and is strictly commutative and associative. For any presheaf of abelian groups F on $\mathbf{Sm}^{G,\mathrm{an}}$, we define the Čech-complex in degrees $n \geq 0$

$$\mathfrak{C}^n F(X) := \varprojlim F(U^{n+1}).$$

The limit runs over the directed category of all open covers $\{U_i\}$ of $X \in \mathbf{Sm}^{G,\mathrm{an}}$ and $U := \sqcup U_i$. Given some involution $\overline{?} : F \to F$, we write $\mathfrak{C}^G F$ for the subcomplex consisting of elements that are fixed by $\overline{\mathrm{Fr}_\infty^*}$.

Let $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(1)}$ be the spectrum whose levels are $\mathfrak{C}^G(\mathbb{R}(p)[p])$. To describe the bonding element, we replace $\mathbb{G}_m^{\mathrm{an}}$ by S^1 (equipped with its usual topology). The inclusion $S^1 \subset \mathbb{G}_m^{\mathrm{an}}$ is a homotopy equivalence, and an explicit description of a Čech cocycle generating $\mathrm{H}^1(\mathbb{G}_m^{\mathrm{an}}, \mathbb{C})$ is left to the reader. As for S^1, consider the standard covering by $U_\pm = \{z \in S^1, \pm \Re(z) > -0.5\}$. This covering is equivariant with respect to $z \mapsto \overline{z}$. Frobenius Fr_∞ acts on the Čech complex

$$\mathbb{R}(1)(U_+) \oplus \mathbb{R}(1)(U_-) \to \mathbb{R}(1)(U_+ \cap U_-) = \mathbb{R}(1)^2, (a,b) \mapsto (v,w) := (b-a, b-a)$$

as $(a,b) \mapsto (a,b)$ and $(v,w) \mapsto (w,v)$. Hence $(\pi i, -\pi i) \in \mathbb{R}(1)(U_+ \cap U_-)$ is a $\overline{\mathrm{Fr}_\infty}$-invariant element which generates $\mathrm{H}^1(\mathbb{G}_m^{\mathrm{an}}, \mathbb{R}(1))^G$. This determines the spectrum $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(1)}$. It is well-known that $\mathrm{H}^*(\mathfrak{C}^*(\mathbb{R})(X)) = \mathrm{H}^*(X, \mathbb{R})$. Thus

$$\mathrm{Hom}_{\mathbf{SH}(\mathbb{R}^{\mathrm{an}})}(\Sigma^\infty X, \mathrm{H}_{\mathrm{B},\mathbb{R}}^{(1)}(p)[n]) = \mathrm{H}_{\mathrm{B}}^n(X, \mathbb{R}(p))^{(-1)^p}, \qquad (2.10)$$

where the superscript at the right denotes the subgroup of elements a satisfying $\mathrm{Fr}_\infty^*(a) = (-1)^p a$. The complexes $\mathrm{Tot}(\mathfrak{C}^G(E_{\mathbb{R}}^*(p)[p]))$ and the bonding element induced by the previous one via the inclusion $\mathbb{R}(1)[1] \subset E_{\mathbb{R}}^0(1)[1]$ yield a spectrum $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(2)}$ that is naturally isomorphic to $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(1)}$, since $\mathbb{R} \to E_{\mathbb{R}}^*$ is a quasi-isomorphism of sheaf complexes. Consider the spectrum $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(3)}$ whose levels are the one of $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(2)}$, but the bonding element is the 1-form

$$dz/z \in E_{\mathbb{R},\mathbb{P}^1}^1(\log\{0,\infty\}) \to \mathfrak{C}^0 E_{\mathbb{R}}^1(1)(\mathbb{G}_m^{\mathrm{an}}) \subset \mathrm{Tot}(\mathfrak{C}^* E_{\mathbb{R}}^*(1))^1(\mathbb{G}_m^{\mathrm{an}}).$$

Both $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(2)}$ and $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(3)}$ are Ω-spectra (the above bonding element and dz/z give the same element in $\mathrm{H}^1(\mathbb{G}_m^{\mathrm{an}}, \mathbb{R}(1))$ by Cauchy's residue formula). The identity map between their level-0-complexes thus yields a canonical isomorphism of spectra (in $\mathbf{SH}(\mathbb{R}^{\mathrm{an}})$). The complexes $E_{\mathbb{R}}^{*,G}(p)[p]$ (again $?^G$ denotes invariants under $\overline{\mathrm{Fr}_\infty^*}$) together with the bonding element dz/z form a spectrum denoted $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(4)}$. The obvious quasi-isomorphism $E_{\mathbb{R}}^{*,G} = \mathfrak{C}^{0,G} E_{\mathbb{R}}^* \to \mathrm{Tot}(\mathfrak{C}^G E_{\mathbb{R}}^*)$ induces an isomorphism $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(4)} \to \mathrm{H}_{\mathrm{B},\mathbb{R}}^{(3)}$ in $\mathbf{SH}(\mathbb{R}^{\mathrm{an}})$. The purpose of the chain of isomorphisms $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(4)} \cong \mathrm{H}_{\mathrm{B},\mathbb{R}}^{(1)}$ is the existence of $\mathrm{H}_{\mathrm{B},\mathbb{Q}}^{(1)}$, the obvious $\mathbb{Q}$-linear variant of $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(1)}$. It induces a $\mathbb{Q}$-structure on the groups represented by $\mathrm{H}_{\mathrm{B},\mathbb{R}}^{(4)}$.

As for de Rham cohomology, consider the complexes $E^F(p)^G := \text{cone}(F^p E^* \to E^*)^G[p-1]$. The product

$$
\begin{aligned}
(F^p E^n \oplus E^{n-1}) \otimes (F^{p'} E^{n'} \oplus E^{n'-1}) &\to (F^{p+p'} E^{n+n'} \oplus E^{n+n'-1}), \\
(f_1, e_1) \otimes (f_2, e_2) &\mapsto (f_1 \wedge f_2, f_1 \wedge e_2)
\end{aligned}
$$

is strictly associative, but in general commutative only up to homotopy [EV88, Section 3]. However, putting $c_1 = (dz/z, 0) \in E^F(1)(\mathbb{G}_m) = (F^1 E^1 \oplus E^0)(\mathbb{G}_m)$, (2.9) clearly holds. We obtain a spectrum $\text{H}_{\text{dR}}^{/F,\text{an}} \in \mathbf{SH}(\mathbb{R}^{\text{an}})$. Using Fr_∞^*-invariant algebraic differential forms, i.e., $\Omega^{*,\text{alg},G}$ instead of $E^{*,G}$, we get a similar spectrum denoted $\text{H}_{\text{dR}}^{/F,\text{alg}} \in \mathbf{SH}(\mathbb{Q})$. For smooth $X/\mathbb{Q}$, the obvious maps

$$
\Omega^{*,\text{alg}}(X) \otimes_{\mathbb{Q}} \mathbb{R} \to \Omega^{*,\text{alg}}(X_{\mathbb{R}}) = \Omega^{*,\text{alg},G}(X_{\mathbb{C}}) \leftarrow \Omega^{*,G}(X) \to E^{*,G}(X)
$$

are filtered (with respect to the Hodge filtration) quasi-isomorphisms by flat base change for $\Omega^{*,\text{alg}}$, GAGA and [Bur94, Thm. 2.1]. We thus get an isomorphism $c_* \text{an}_* \text{H}_{\text{dR}}^{/F,\text{an}} = \text{H}_{\text{dR}}^{/F,\text{alg}} \otimes \mathbb{R}$ in $\mathbf{SH}(\mathbb{Q})$. Here $c : \text{Spec } \mathbb{R} \to \text{Spec } \mathbb{Q}$.

Finally, the complex

$$
\text{D}(p)^G := \text{cone}(E_{\mathbb{R}}^{*,G}(p)[p] \to E^F(p)^G)[-1] \tag{2.11}
$$

carries a product map $\cdot_{\text{D},\alpha}$ depending on some auxiliary parameter $\alpha \in \mathbb{R}$. It is only commutative and associative up to homotopy (for each α). Again, $c_1 = (dz/z, dz/z, 0) \in \text{D}(1)^0(\mathbb{G}_m) = (E_{\mathbb{R}}^{n+p}(p) \oplus F^p E^{n+p} \oplus E^{n+p-1})(\mathbb{G}_m)$ satisfies (2.9) (independently of α, see the multiplication table in *loc. cit.*). The resulting spectrum H_{D} sits in a distinguished triangle in $\mathbf{SH}(\mathbb{R}^{\text{an}})$, $\text{H}_{\text{D}} \to \text{H}_{\text{B},\mathbb{R}}^{(4)} \to \text{H}_{\text{dR}}^{/F,\text{an}}$ and thus, in $\mathbf{SH}(\mathbb{Q})$,

$$
c_* \text{an}_* \text{H}_{\text{D}} \to c_* \text{an}_* (\text{H}_{\text{B},\mathbb{Q}}^{(1)} \otimes_{\mathbb{Q}} \mathbb{R}) \to \text{H}_{\text{dR}}^{/F,\text{alg}} \otimes_{\mathbb{Q}} \mathbb{R} \to c_* \text{an}_* \text{H}_{\text{D}}[1]. \tag{2.12}
$$

From now on, we write H_{D} for $\text{an}_* c_* \text{H}_{\text{D}} \in \mathbf{SH}(\mathbb{Q})$. This is the spectrum established in [HS11, Section 3], except for two inessential differences: instead of $\text{D}(p)$, *loc. cit.* used other complexes that are homotopic (including the product structure, regardless of α) to $\text{D}(*)$. Secondly, the construction of *loc. cit.* builds a symmetric $\mathbb{P}^1$-spectrum, but again this is inessential at the level of the homotopy categories, since $- \wedge \mathbb{P}^1 = - \wedge \mathbb{G}_m \wedge S_s^1$, where S_s^1 is the simplicial sphere. By [HS11, Thm. 3.6],

$$
\text{Hom}_{\mathbf{DM}_{\mathbf{B}}(\mathbb{Q})}(\text{M}(X), \text{H}_{\text{D}}(p)[n]) = \text{Hom}_{\mathbf{SH}(\mathbb{Q})}(\text{M}(X), \text{H}_{\text{D}}(p)[n]) = \text{H}_{\text{D}}^n(X, p)
$$

for any $X \in \mathbf{Sm}/\mathbb{Q}$. For any $M \in \mathbf{DM}_{\mathbf{B},c}(\mathbb{Q})$, (2.12) induces an isomorphism

$$
\begin{aligned}
\det \text{H}_{\text{D}}^*(M) &= \det c_* \text{an}_* \text{H}_{\text{B},\mathbb{R}}^{(4),*}(M) \otimes \det^{-1} c_* \text{an}_* \text{H}_{\text{dR}}^{/F,\text{an}}(M) \\
&= \left(\det c_* \text{an}_* \text{H}_{\text{B},\mathbb{Q}}^{(1),*}(M) \otimes \det^{-1} \text{H}_{\text{dR}}^{/F,\text{alg},*}(M) \right) \otimes_{\mathbb{Q}} \mathbb{R}. \tag{2.13}
\end{aligned}
$$

Here $\det \text{H}_{\text{D}}^*(M) := \otimes_{n \in \mathbb{Z}} \det^{(-1)^n} \text{Hom}(M, \text{H}_{\text{D}}[n])$ etc. is well-defined since M is compact. This is the promised extension of (2.1) to Deligne cohomology groups of general geometric motives.

Applied to the Betti realization, the exactness axiom (see Axiom 1.2(ii)) means

$$
\text{Hom}(M, \text{H}_{\text{B},\mathbb{R}}) = \text{Hom}(^p\text{H}^0(M), \text{H}_{\text{B},\mathbb{R}}), \quad \text{for all } M \in \mathbf{DM}_{\mathbf{B},c}(\mathbb{Q}) \tag{2.14}
$$

and likewise for de Rham cohomology. This implies that for any smooth projective $X_\eta/\mathbb{Q}$,

$$
\text{H}_{\text{D}}^i(\text{h}^{-b-1}(X_\eta, -m)) = \begin{cases} \text{H}_{\text{D}}^{b+1}(X_\eta, m) & i = 0 \text{ and } b+1-2m \geq 0 \\ \text{H}_{\text{D}}^{b+2}(X_\eta, m) & i = 1 \text{ and } b+1-2m \leq -2 \\ 0 & \text{else.} \end{cases} \tag{2.15}
$$

2.2 Arakelov motivic cohomology

In order to formulate Conjecture 5.2 below, we need to recall some facts about Arakelov motivic cohomology.

Theorem 2.3. *[HS11, Sch12a] In* $\mathbf{DM}_{\mathrm{B}}(\mathbb{Z})$*, there is a unique map* $\mathrm{ch} : \mathbf{1} \to \eta_*\mathrm{H}_{\mathrm{D}}$ *representing the Chern class map from motivic cohomology to Deligne cohomology, i.e.*

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(\mathbb{Z})}(\mathrm{M}(X), \mathbf{1}(p)[n]) \xrightarrow{\mathrm{ch}(p)[n]} \mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(\mathbb{Z})}(\mathrm{M}(X), \eta_*\mathrm{H}_{\mathrm{D}}(p)[n])$$

agrees with the Chern class $K_{2p-n}(X)_{\mathbb{Q}}^{(p)} \to \mathrm{H}_{\mathrm{D}}^n(X_{\mathbb{Q}}, p)$ *(also known as Beilinson regulator) for all regular projective schemes* $X/\mathbb{Z}$*. There is a certain, canonically defined object* $\widehat{\mathbf{1}} \in \mathbf{DM}_{\mathrm{B}}(\mathbb{Z})$ *called* Arakelov motivic cohomology spectrum *such that there is a distinguished triangle*

$$\widehat{\mathbf{1}} \xrightarrow{f} \mathbf{1} \xrightarrow{\mathrm{ch}} \eta_*\mathrm{H}_{\mathrm{D}} \xrightarrow{\delta} \widehat{\mathbf{1}}[1]. \tag{2.16}$$

Moreover, given another object $\widehat{\mathbf{1}}'$ *in a similar triangle, there is a* unique *isomorphism* $\widehat{\mathbf{1}} \to \widehat{\mathbf{1}}'$ *in* $\mathbf{DM}_{\mathrm{B}}(\mathbb{Z})$ *fitting in the obvious commutative diagram of distinguished triangles.*

Definition 2.4. Given a motive $M \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$, its *Arakelov motivic cohomology* is defined as

$$\widehat{\mathrm{H}}^i(M, m) := \mathrm{Hom}_{\mathbf{DM}_{\mathrm{B}}(\mathbb{Z})}(M, \widehat{\mathbf{1}}(m)[i]).$$

We write $\widehat{\mathrm{H}}^i(X, m) := \widehat{\mathrm{H}}^i(\mathrm{M}(X), m)$. We also consider the $\mathbb{R}$-linear variant of these groups, denoted $\widehat{\mathrm{H}}_{\mathbb{R}}^i(X, m)$, obtained by replacing $\mathbf{1}$ by $\mathbf{1}_{\mathbb{R}}$ in (2.16). This amounts to tensoring the motivic cohomology groups with $\mathbb{R}$.

The triangle (2.16) induces long exact sequences

$$\widehat{\mathrm{H}}_{\mathbb{R}}^i(M, m) \to \mathrm{H}^i(M, m)_{\mathbb{R}} \to \mathrm{H}_{\mathrm{D}}^i(M, m) \to \widehat{\mathrm{H}}_{\mathbb{R}}^{i+1}(M, m). \tag{2.17}$$

On the other hand, we have the notion of arithmetic K-theory. For a regular and projective scheme X over $\mathbb{Z}$, such groups $\widehat{\mathrm{K}}_n^T(X)$ have been defined by Gillet and Soulé for $n = 0$ and for higher n by Takeda [GS90b, Section 6], [Tak05, p. 621]. These groups sit in an exact sequence

$$K_{n+1}(X) \to \oplus_{p\in\mathbb{Z}}\mathrm{D}(p)^{2p-n-1,G}(X)/\operatorname{im}d_{\mathrm{D}} \to \widehat{\mathrm{K}}_n^T(X) \to K_n(X) \to 0$$

where $\mathrm{D}(p)^G$ is the complex defined in (2.11). Moreover, they come with a Chern class map $\mathrm{ch} : \widehat{\mathrm{K}}_n^T(X) \to \oplus_{p\in\mathbb{Z}}\mathrm{D}(p)^{2p-n,G}(X)$. The group $\widehat{\mathrm{K}}_n(X) := \ker\mathrm{ch}$ fits in a long exact sequence

$$\ldots \to \oplus_{p\in\mathbb{Z}}\mathrm{H}_{\mathrm{D}}^{2p-n-1}(X, p) \to \widehat{\mathrm{K}}_n(X) \to K_n(X) \to \oplus\mathrm{H}_{\mathrm{D}}^{2p-n}(X, p) \to \ldots \tag{2.18}$$

The group $\widehat{\mathrm{K}}_0^T(X)_{\mathbb{Q}}$ is also isomorphic, via the arithmetic Chern class to $\oplus_p\widehat{\mathrm{CH}}_{\mathrm{GS}}^p(X)_{\mathbb{Q}}$, where $\widehat{\mathrm{CH}}_{\mathrm{GS}}$ denotes the arithmetic Chow group of Gillet and Soulé [GS90a, 3.3.4]. It is generated by arithmetic cycles (Z, g_Z), i.e., cycles $Z \subset X$ and Green currents, i.e., such that $\omega_Z := \delta_Z - 2\partial\overline{\partial}g_Z$ is a differential form. Here δ_Z is the Dirac current. Under the arithmetic Chern class, the subgroup $\widehat{\mathrm{K}}_0(X)_{\mathbb{Q}} \subset \widehat{\mathrm{K}}_0^T(X)_{\mathbb{Q}}$ corresponds to the subgroup $\widehat{\mathrm{CH}}^*(X) \subset \widehat{\mathrm{CH}}_{\mathrm{GS}}^*(X)$ generated by arithmetic cycles (Z, g_Z) such that $\omega_Z = 0$ [GS90b, Thm. 7.3.4].

For a smooth scheme X over $S \subset \operatorname{Spec}\mathbb{Z}$, the resulting decomposition of $\widehat{\mathrm{K}}_0(X)_{\mathbb{Q}}$ in Adams eigenspaces is extended to higher $\widehat{\mathrm{K}}$-theory [Sch12a, Cor. 6.2]: $\widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}$

decomposes as a direct sum of Adams eigenspaces $\oplus \widehat{\mathrm{K}}_n(X)_{\mathbb{Q}}^{(p)}$, compatibly with (2.18). In fact, this statement is derived from a canonical isomorphism

$$\widehat{\mathrm{H}}^i(X,m) = \widehat{\mathrm{K}}_{2m-i}(X)_{\mathbb{Q}}^{(m)} \quad (= \widehat{\mathrm{CH}}^m(X)_{\mathbb{Q}} \text{ for } i = 2m). \tag{2.19}$$

Definition 2.5. Let $S \subset \operatorname{Spec} \mathbb{Z}$ be an open subscheme and let $M \in \mathbf{DM}_{\mathrm{B}}(S)$ be any motive. The natural pairing of motivic homology (see (1.2)) and Arakelov motivic cohomology,

$$\pi_M : \mathrm{H}_{-2}(M,-1)_{\mathbb{R}} \times \widehat{\mathrm{H}}_{\mathbb{R}}^0(M) \to \widehat{\mathrm{H}}_{\mathbb{R}}^2(\mathbf{1}_S, 1)$$

given by the composition of morphisms in $\mathbf{DM}_{\mathrm{B}}(S)$ is called *Arakelov intersection pairing*.

Remark 2.6. (i) For $M \in \mathbf{DM}_{\mathrm{B},c}(S)$, we often tacitly identify $\mathrm{H}_{-2}(M,-1) \cong \mathrm{H}^2(M^\vee, 1)$, cf. (1.4).

(ii) The Arakelov intersection pairing is functorial in M in an obvious sense.

(iii) Let $M \in \mathbf{DM}_{\mathrm{B},c}(S)$. Consider

$$
\begin{array}{ccccc}
\widehat{\mathrm{H}}_{\mathbb{R}}^0(M) & \times & \mathrm{H}_{\mathbb{R}}^2(M^\vee, 1) & \longrightarrow & \widehat{\mathrm{H}}^2(\mathbf{1}, 1) \\
\downarrow & & \uparrow & & \downarrow= \\
\mathrm{H}_{\mathbb{R}}^0(M) & \times & \widehat{\mathrm{H}}_{\mathbb{R}}^2(M^\vee, 1) & \longrightarrow & \widehat{\mathrm{H}}^2(\mathbf{1}, 1) \\
\downarrow & & \uparrow & & \uparrow\cong \\
\mathrm{H}_{\mathrm{D}}^0(M) & \times & \mathrm{H}_{\mathrm{D}}^1(M^\vee, 1) & \longrightarrow & \mathrm{H}_{\mathrm{D}}^1(\mathbf{1}, 1),
\end{array}
\tag{2.20}
$$

where in the first row $(a : M \to \widehat{\mathbf{1}}, b : M^\vee \to \mathbf{1}\{-1\})$ is mapped to $\mu \circ (a \otimes b) \circ \operatorname{coev}$, where the coevaluation $\mathbf{1} \to M \otimes M^\vee$ is obtained from (1.4), $\mu : \mathbf{1} \otimes \widehat{\mathbf{1}} \to \widehat{\mathbf{1}}$ is the $\mathbf{1}$-module structure map for $\widehat{\mathbf{1}}$. This is just another way to write π_M. Likewise, the second row pairing is $\pi_{M^\vee\{-1\}}$. The pairing in the third row is defined similarly using the product of the ring spectrum $\mu_{\mathrm{D}} : \mathrm{H}_{\mathrm{D}} \otimes \mathrm{H}_{\mathrm{D}} \to \mathrm{H}_{\mathrm{D}}$ instead. This diagram is commutative. This follows from the commutativity of the following diagram, which in turn is a rephrasing of the fact that (2.16) is a distinguished triangle of $\mathbf{1}$-modules.

$$
\begin{array}{ccccc}
\widehat{\mathbf{1}} \otimes \widehat{\mathbf{1}} & \xrightarrow{f \otimes \mathrm{id}} & \mathbf{1} \otimes \widehat{\mathbf{1}} & \xleftarrow{\quad\delta\quad} & \mathbf{1} \otimes \mathrm{H}_{\mathrm{D}}[-1] \\
\downarrow{\scriptstyle \mathrm{id} \otimes f} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{ch} \otimes \mathrm{id}} \\
\widehat{\mathbf{1}} \otimes \mathbf{1} & \xrightarrow{\quad\mu\quad} & \widehat{\mathbf{1}} & \xleftarrow[\delta]{} \mathrm{H}_{\mathrm{D}}[-1] \xleftarrow[\mu_{\mathrm{D}}]{} & \mathrm{H}_{\mathrm{D}} \otimes \mathrm{H}_{\mathrm{D}}[-1].
\end{array}
$$

(iv) The pairing $\mathrm{H}_{\mathrm{D}}^0(M) \times \mathrm{H}_{\mathrm{D}}^1(M^\vee, 1) \to \mathbb{R}$ is a perfect pairing for any M. It suffices to see this for $M = \mathrm{M}(X)(p)[n]$ for $X/\mathbb{Z}$ regular and projective, in which case it follows from the identification of Deligne cohomology with weak Hodge cohomology (Lemma 2.2) and the duality of weak Hodge cohomology, (2.5). This plays an important role in the compatibility of our L-values conjecture with respect to the functional equation, see Theorem 5.5(ii).

Example 2.7. Consider a motive $M = i_* N$, where $i : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ and $N \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$ (for example $M = \mathrm{M}(\mathbb{F}_p) = i_* i^* \mathbf{1}\{-1\}$). The forgetful map $f : \widehat{\mathrm{H}}_{\mathbb{R}}^0(M) \to \mathrm{H}^0(M)_{\mathbb{R}} = \mathrm{H}^{-2}(N,-1)_{\mathbb{R}}$ is an isomorphism and the pairing π_M coincides with the natural pairing $\mathrm{H}_{-2}(N,-1)_{\mathbb{R}} \times \mathrm{H}^{-2}(N,-1)_{\mathbb{R}} \to \mathrm{H}^0(\mathbf{1}_{\mathbb{F}_p}, 0)_{\mathbb{R}} = \mathbb{R}$ followed by the pushforward $i_* : \widehat{\mathrm{H}}^0(\mathbf{1}_{\mathbb{F}_p}, 0)_{\mathbb{R}} \to \widehat{\mathrm{H}}_{\mathbb{R}}^2(\mathbf{1}_{\mathbb{Z}}, 1)$, which is $\log p : \mathbb{R} \to \mathbb{R}$ [Sch12a, Theorem 6.4.(i)].

Example 2.8. Let X be a regular projective scheme over $S \subset \mathrm{Spec}\,\mathbb{Z}$ of constant dimension d. We pick some open $j : U \subset S$ such that X_U is smooth over U. Let $M := \mathrm{M}(X)\{m - d\}[i] \in \mathbf{DM}_{\mathrm{B},c}(S)$. Then $\mathrm{H}_{-2}(M, -1) = K_i(X)_{\mathbb{Q}}^{(m)}$ by absolute purity. Let $M_U := j^*M \in \mathbf{DM}_{\mathrm{B},c}(U)$. Consider

$$
\begin{array}{ccccc}
K_i(X_U)_{\mathbb{Q}}^{(m)} \times \widehat{\mathrm{K}}_{-i}(X_U)_{\mathbb{Q}}^{(d-m)} & \xrightarrow{\ \cup\ } & \widehat{\mathrm{K}}_0(X_U)_{\mathbb{Q}}^{(d)} & \xrightarrow{\ f_{U*}\ } & \widehat{\mathrm{K}}_0(U)_{\mathbb{Q}}^{(1)} \\
\cong \big\uparrow & & & & \cong \big\uparrow \\
\mathrm{H}_{-2}(M_U, -1) \times \widehat{\mathrm{H}}^0(M_U) & \xrightarrow{\ \pi_{M_U[-i]}\ } & & & \widehat{\mathrm{H}}^2(\mathbf{1}_U, 1) = \mathbb{R}/\sum_{p \notin U} \log p\,\mathbb{Q} \\
j^* \big\uparrow & & & & j^* \big\uparrow \\
\mathrm{H}_{-2}(M, -1) \times \widehat{\mathrm{H}}^0(M) & \xrightarrow{\ \pi_{M[-i]}\ } & & & \widehat{\mathrm{H}}^2(\mathbf{1}_S, 1) = \mathbb{R}/\sum_{p \notin S} \log p\,\mathbb{Q}
\end{array}
$$

In the first row, the pushforward f_{U*} is not the pushforward on arithmetic K-theory, but the one on arithmetic Chow groups using the arithmetic Chern class isomorphism (2.19). The top square is commutative by [Sch12a, Thm. 7.4.]. The bottom square is commutative by definition. See also Remark 5.7.

3 L-functions of motives over number rings

Let F be a number field and $\mathcal{O}_F$ its ring of integers. For every finite prime $\mathfrak{p}$ of $\mathcal{O}_F$ we fix a rational prime ℓ that does not lie under $\mathfrak{p}$. Moreover, fix for every ℓ an embedding $\sigma_\ell : \mathbb{Q}_\ell \to \mathbb{C}$. All subsequent definitions of L-functions are taken with respect to these choices.

Definition 3.1. The L-series of a mixed motive M_η over F is defined by

$$
L_F(M_\eta, s) := \prod_{\mathfrak{p} < \infty} \det\left(\mathrm{Id} - \mathrm{Fr}^{-1} \cdot N(\mathfrak{p})^{-s} | (M_{\eta\ell} \otimes_{\mathbb{Q}_\ell, \sigma_\ell} \mathbb{C})^{I_\mathfrak{p}}\right)^{-1}.
$$

The L-series of a geometric motive M over $\mathcal{O}_F$ is given by

$$
L_{\mathrm{Spec}\,\mathcal{O}_F}(M, s) := L(M, s) := \prod_{\mathfrak{p} < \infty} \det\left(\mathrm{Id} - \mathrm{Fr}^{-1} \cdot N(\mathfrak{p})^{-s} | (i_\mathfrak{p}^! M)_\ell \otimes_{\mathbb{Q}_\ell, \sigma_\ell} \mathbb{C}\right)^{-1}.
$$

The first definition is classical, the second is a natural adaptation to motives over $\mathcal{O}_F$. The products run over all finite primes of $\mathcal{O}_F$, Fr is the arithmetic Frobenius map (given on residue fields by $a \mapsto a^{N(\mathfrak{p})}$), $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$, $i_\mathfrak{p}$ denotes the immersion of the corresponding closed point and $-_\ell$ denotes the ℓ-adic realization functor, see (1.6). The determinants are understood in the sense of Section 1.1. The superscript $I_\mathfrak{p}$ denotes the invariants under the action of the inertia group.

Remark 3.2. By [Sch12b, Axiom 4.5.], the ℓ-adic realization $M_{\eta\ell}$ is in fact an ℓ-adic sheaf. For example, $(\mathrm{h}^{-b-1}(X_\eta, -m))_\ell = \mathrm{H}^{b+1}(X_\eta, \mathbb{Q}_\ell(m))$ for some scheme X_η over F.

The independence of the choices of ℓ and the embeddings σ_ℓ is discussed around Lemma 3.10. See also Theorem 4.3.

The L-series for motives over $\mathcal{O}_F$ is multiplicative, i.e., given a triangle $M \to M' \to M''$ in $\mathbf{DM}_{\mathrm{B},c}(\mathcal{O}_F)$, one gets

$$
L(M', s) = L(M, s) \cdot L(M'', s).
$$

A similar property does *not* hold for L-functions of motives over F [Sch91]. See also [FPR94, 1.3.3].

By definition and the calculation of ℓ-adic cohomology of $\mathbb{P}^1_{\mathbb{F}_p}$, one has

$$L(M(-m), s) = L(M, m + s), \quad m \in \mathbb{Z}. \tag{3.1}$$

For an open subscheme $j : \operatorname{Spec} \mathcal{O}_F \backslash Z \to \operatorname{Spec} \mathcal{O}_F$ with complement $i : Z \to \operatorname{Spec} \mathcal{O}_F$, the L-function of $j_* j^* M$ is the one of M, but the Euler factors for the points in Z are omitted. This follows from $i^! j_* = 0$.

The following lemma is well-known, see [Del73, Prop. 3.8.(ii)] or [Neu92, VII.10.4.(iv)] for similar statements. It permits to replace any number ring $\mathcal{O}_F$ by $\mathbb{Z}$ and to study L-values of motives over $\mathbb{Z}$, only.

Lemma 3.3. *The L-series is an absolute invariant of a motive, i.e., for any geometric motive M over $\operatorname{Spec} \mathcal{O}_F$ we have $L_{\operatorname{Spec} \mathcal{O}_F}(M, s) = L_{\operatorname{Spec} \mathbb{Z}}(f_* M, s)$, where $f : \operatorname{Spec} \mathcal{O}_F \to \operatorname{Spec} \mathbb{Z}$ denotes the structural map.*

We now relate L-series of motives over $\mathbb{Q}$ to ones over $\mathbb{Z}$. Recall the notion of smooth motives from Definition 1.1. The following lemma is proven in [Sch12b, Section 5.5] as a corollary of the exactness axiom for ℓ-realization functors (see around (1.7)).

Lemma 3.4. *Let M be a mixed smooth motive over U, where $j : U \to \operatorname{Spec} \mathbb{Z}[1/\ell]$ is an open subscheme. Let i be the complementary closed immersion to j and let η' and η be the generic point of U and $\operatorname{Spec} \mathbb{Z}[1/\ell]$, respectively. Then $(i^! j_{!*} M)_\ell = i^*(\mathrm{R}^0 \eta_* \eta'^* M_\ell[1])[-1]$.*

The following proposition relates L-series of motives over $\mathbb{Q}$ and $\mathbb{Z}$. Our main example is $M_\eta = \mathrm{h}^{-b-1}(X_\eta, -m)$ and $M = \mathrm{h}^{-b}(X, -m)$ where $X/\mathbb{Z}$ is some projective scheme whose generic fiber $X_\eta/\mathbb{Q}$ is smooth.

Proposition 3.5. *Let $M_\eta \in \mathbf{MM}(\mathbb{Q})$. Pick some $M \in \mathbf{MM}(\mathbb{Z})$ with $M_\eta = \eta^*[-1]M$. Then*

$$L_{\mathbb{Q}}(M_\eta, s)^{-1} = L_{\mathbb{Z}}(\eta_{!*} \eta^* M, s).$$

Proof: For sufficiently small $j : U \to \operatorname{Spec} \mathbb{Z}$, the right hand side is equal to

$$
\begin{aligned}
L_{\mathbb{Z}}(j_{!*} j^* M, s) &\overset{3.4}{=} \left(\prod_p \det \left(\mathrm{Id} - \mathrm{Fr}^{-1} p^{-s} | i_p^*(\mathrm{R}^0 \eta_* \eta^* M_\ell[1])[-1] \right) \right)^{-1} \\
&= \prod_p \det \left(\mathrm{Id} - \mathrm{Fr}^{-1} p^{-s} | i_p^* \mathrm{R}^0 \eta_* M_{\eta \ell} \right) \\
&= \prod_p \det \left(\mathrm{Id} - \mathrm{Fr}^{-1} p^{-s} | (M_{\eta \ell})^{I_p} \right) = L_{\mathbb{Q}}(M_\eta, s)^{-1}.
\end{aligned}
$$

$\square$

3.1 Hasse-Weil ζ-functions – Motives with compact support

Definition 3.6. (see e.g. [Ser65]) The *Hasse-Weil zeta function* of a scheme $X/\mathbb{Z}$ (always separated and of finite type) is defined as $\zeta(X, s) := \prod_x (1 - N(x)^{-s})^{-1}$. The product is over all closed points x of X, and $N(x)$ denotes the cardinality of the (finite) residue field of x.

Recall from (1.1) the motive with compact support $\mathrm{M}_c(X)$ of some scheme X.

Proposition 3.7. *For any scheme $X/\mathbb{Z}$, we have*

$$\zeta(X, s) = L(\mathrm{M}_c(X), s).$$

Proof: Writing $i_p : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ and $X_p := X \times \mathbb{F}_p$, base-change implies $i_{p*}i_p^! \operatorname{M_c}(X) = \operatorname{M_c}(X_p)$. (At the right hand side, X_p is seen as a $\mathbb{Z}$-scheme.) Therefore, $L(\operatorname{M_c}(X), s) = \prod_p L(\operatorname{M_c}(X_p), s)$. A similar decomposition for the ζ-function allows us to assume that X is an $\mathbb{F}_p$-scheme. The ℓ-adic realization functor satisfies $(f_* f^! \mathbf{1})_\ell = f_! f^* \mathbb{Q}_\ell$. Grothendieck's trace formula (see e.g. [Mil80, Sections VI.12, 13]) says

$$
\begin{aligned}
\zeta(X, s) &= \prod_{i=0}^{2 \dim X} \left(\det \left(\operatorname{Id} - \operatorname{Fr}^{-1} \cdot p^{-s} | \operatorname{H}_c^i(X \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \right) \right)^{(-1)^{i+1}} \\
&= \det(\operatorname{Id} - \operatorname{Fr}^{-1} \cdot p^{-s} | f_! f^* \mathbb{Q}_\ell)^{-1},
\end{aligned}
$$

where $\operatorname{H}_c^i(X \times \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) = \operatorname{H}^i(f_! f^* \mathbb{Q}_\ell)$ denotes ℓ-adic cohomology with compact support. $\square$

The L-series of a motive over $\mathbb{Q}$ is conjectured to be independent of the choice of ℓ and σ_ℓ in every factor (assuming $p \neq \ell$). This is known for the individual Euler factors at p if the motive is $\operatorname{h}^i(X_\eta, n)$, where X_η is a variety with good reduction at p, by Deligne's work on the Weil conjectures [Del74, Th. 1.6]. From Proposition 3.7 we now immediately obtain another statement concerning independence of ℓ.

Definition 3.8. The smallest triangulated subcategory of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ containing the motives $\operatorname{M}(X)(n)$ ($n \in \mathbb{Z}$) of all regular schemes X which are projective and flat over $\mathbb{Z}$, and the image of $i_* : \mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p) \to \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ for all primes p, is called $\mathbf{DM}_{\mathrm{B},\mathrm{tr}}(\mathbb{Z})$ and called category of *accessible motives*. Its triangulated subcategory generated by $\operatorname{M}(X)(n)$ where X is regular and projective, but not necessarily flat over $\mathbb{Z}$ (such as a smooth projective $X/\mathbb{F}_p$) is called the category of *easily accessible motives*.

Remark 3.9.   (i) By de Jong's resolution of singularities using alterations, the thick closure (i.e., closure under direct summands and triangles) of the category of easily accessible motives contains the motives $\operatorname{M}(X)(n)$ of all X schemes (of finite type) over $\mathbb{Z}$. Therefore, this thick closure is the entire category $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ of geometric motives.

(ii) By the proof of [Sch12b, Prop. 5.6], $\mathbf{DM}_{\mathrm{B},\mathrm{tr}}(\mathbb{Z})$ is contained in the triangulated category generated by $i_* \mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$ and motives of the form $E := \eta_{!*} \eta^* \operatorname{h}^{-b}(X, -m)$, where $X/\mathbb{Z}$ is regular, flat and projective.

The following lemma shows that the question of independence of L-functions of ℓ is solely about the behavior of L-functions under direct summands.

Lemma 3.10. *For any easily accessible motive M over $\mathbb{Z}$, the L-series $L(M, s)$ does not depend on the choices of ℓ (provided $p \nmid \ell$) and σ_ℓ.*

Proof: Using (3.1), we may assume $M = \operatorname{M}(X) = \operatorname{M_c}(X)$ for some X which is projective over $\mathbb{Z}$ (and regular). Then the claim immediately follows from Proposition 3.7. $\square$

3.2   Meromorphic continuation and functional equation

Properties of L-series for motives over $\mathbb{Q}$ tend to generalize to ones over $\mathbb{Z}$, given that the property in question is known for motives over $\mathbb{F}_p$. We illustrate this by the absolute convergence, meromorphic continuation, and the functional equation. Recall from [Del79, 5.2.] or [Sch88, p. 4] the definition of the archimedean Euler factor $L_\infty(V, s)$ for a mixed Hodge structure V. Essentially, $L_\infty(V, s)$ is a product

of Γ-functions. The pole order at $s = 0$ is given by ([Beĭ86, Lemma 7.1.] or [FPR94, III.1.2.5 + III.1.2.3]):

$$\operatorname{ord}_{s=0} L_\infty(V, s) = -\dim_\mathbb{R} \mathrm{H}^1_\mathrm{w}(V^\vee(1)). \tag{3.2}$$

For $V_* \in \mathbf{D}^\mathrm{b}_\mathrm{H}$, we put $L_\infty(V_*, s) := \prod_{i \in \mathbb{Z}} L_\infty(\mathrm{H}^i(V_*), s)^{(-1)^i}$. Here $\mathrm{H}^i(V_*)$ denotes the i-th cohomology Hodge structure of the complex V_*.

Definition 3.11. Let M be a geometric motive over $\mathbb{Z}$ or a mixed motive over $\mathbb{Q}$. The function

$$L_\infty(M, s) := L_\infty(\mathrm{R}\Gamma_\mathrm{H}(M), s)$$

is called the *archimedean factor* of the L-function of M. Here $\mathrm{R}\Gamma_\mathrm{H}$ is the Hodge realization functor (2.7). The *completed L-function* of M is defined as

$$\Lambda(M, s) := L(M, s) L_\infty(M, s).$$

Much the same way as L-functions of motives over $\mathbb{Q}$, archimedean factors are not multiplicative with respect to short exact sequences of Hodge structures. (See [FPR94, 1.1.9, 1.2.5] for a necessary and sufficient criterion for multiplicativity.)

The following is a long-standing conjecture concerning L-functions [Del73], [Del79, 5.2, 5.3] or [FPR94, p. 610, 699]:

Conjecture 3.12. Let M_η be a mixed motive over $\mathbb{Q}$. The L-series $L_\mathbb{Q}(M_\eta, s)$ converges absolutely for $\Re(s) \gg 0$ and has a meromorphic continuation to the complex plane. There is a functional equation relating the Λ-functions of M_η and $M_\eta^\vee(-1)$:

$$\Lambda(M_\eta, s) = \epsilon(M, s) \Lambda(M_\eta^\vee(-1), -s),$$

where $\epsilon(M, s)$ is of the form ab^s, with nonzero constants a and b depending on M.

Lemma 3.13. *Conjecture 3.12 implies the following: for any accessible motive M over $\mathbb{Z}$ (Definition 3.8), the L-series $L(M, s)$ converges absolutely for $\Re(s) \gg 0$, has a meromorphic continuation to the complex plane, and there is a functional equation $\Lambda(M, s) = \epsilon(M, s) \Lambda(M^\vee(-1), -s)$, where $\epsilon(M, s)$ is of the form ab^s, with nonzero constants a and b depending on M.*

Proof: The claim is triangulated, since the assignments $M \mapsto L(M, s)$, and $M \mapsto L_\infty(M, s) / L_\infty(M^\vee(-1), -s)$ are multiplicative for $M \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$, the latter up to sign [FPR94, Prop. III.1.2.8]. By Remark 3.9(ii), it is enough to show the claim for $M = i_* N$, $N \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$ and $M = E := \eta_! \eta^* \mathrm{h}^{-b}(X, -m)$, where $X/\mathbb{Z}$ is regular, flat and projective. For $M = E$, we have $L(M, s) = L_\mathbb{Q}(\mathrm{h}^{-b-1}(X_\eta), s)^{-1}$. This and the formula (5.2) for $M^\vee(-1)$ in this case shows that the conjectural (see 3.12) properties of $L_\mathbb{Q}(\mathrm{h}^{-b-1}(X_\eta), s)$ implies the same properties for $L(M, s)$. The L-series of $M = i_* N$ is a rational function in p^{-s} (a priori with complex coefficients), which immediately yields the convergence for $\Re(s) \gg 0$ and the meromorphicity. Noting that $(i_* N)^\vee\{-1\} = i_*(N^\vee)$, the functional equation also holds unconditionally, as is well-known. $\square$

Remark 3.14. Under Conjecture 1.5 the constant a above is rational for $M = i_* N$, where $i : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. To see this, we may assume by triangulatedness that N is a pure motive with respect to numerical or homological equivalence, so that its L-function is a rational function in p^{-s} with rational coefficients (see the reference in the proof of Theorem 5.20).

4 Is the Arakelov intersection pairing perfect?

Conjecture 4.1. For any geometric motive M over $\mathbb{Z}$, $M \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ (see Section 1.2 for the notation), the Arakelov intersection pairing between motivic homology and Arakelov motivic cohomology (Definition 2.5)

$$\pi_M : \mathrm{H}_{-2}(M,-1)_{\mathbb{R}} \times \widehat{\mathrm{H}}^0_{\mathbb{R}}(M) \to \mathbb{R} \tag{4.1}$$

is a perfect pairing of finite-dimensional $\mathbb{R}$-vector spaces.

Remark 4.2. (i) The shape of (4.1) is similar to the situation of étale constructible sheaves over $\mathrm{Spec}\,\mathbb{Z}$: thinking of $M \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ as being analogous to a complex of constructible sheaves $\mathcal{F}$ over $\mathbb{Z}$, the groups $\mathrm{H}^*_{\mathrm{D}}(M)$ correspond (in spirit) to the Tate cohomology groups $\mathrm{H}^*_{\mathrm{Tate}}(\mathbb{R}, \mathcal{F}|_{\mathbb{R}})$ at the archimedean place. Given that, $\widehat{\mathrm{H}}^i(M)$ parallels $\mathrm{H}^i_{\mathrm{c}}(\mathcal{F}) := \mathrm{H}^i \mathrm{R}\Gamma_{\mathrm{c}}(\mathbb{Z}, \mathcal{F})$, that is to say, cohomology with compact support, which is defined via $\mathrm{R}\Gamma_{\mathrm{c}} := \mathrm{cone}[-1]\,(\mathrm{R}\Gamma(\mathbb{Z}, \mathcal{F}) \to \mathrm{R}\Gamma_{\mathrm{Tate}}(\mathbb{R}, \mathcal{F}|_{\mathbb{R}}))$, much the same way as (2.16), (2.17). Finally, the Arakelov intersection pairing corresponds to the perfect pairing of *Artin-Verdier* duality, see e.g. [Mil06, Ch. II.3]

$$\mathrm{H}^i_{\mathrm{c}}(\mathbb{Z}, \mathcal{F}) \times \mathrm{Ext}^{3-i}_{\mathbb{Z}}(\mathcal{F}, \mathbb{G}_m) \to \mathrm{H}^3_{\mathrm{c}}(\mathbb{Z}, \mathbb{G}_m).$$

A higher-dimensional extension was conjectured by Milne [Mil06, Conjecture II.7.17] and proven by Geisser [Gei10].

(ii) For any fixed $M \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$, Conjecture 4.1 for all $M[k]$ ($k \in \mathbb{Z}$) is equivalent to the one for $M^\vee\{-1\}[k]$. This follows from Remark 2.6(iii), (iv) and the five lemma.

(iii) Gillet and Soulé conjecture that the intersection product

$$\widehat{\mathrm{CH}}^m_{\mathrm{GS}}(X)_{\mathbb{R}} \times \widehat{\mathrm{CH}}^{d-m}_{\mathrm{GS}}(X)_{\mathbb{R}} \to \mathbb{R} \tag{4.2}$$

is non-degenerate for any regular scheme X that is projective and flat over $\mathbb{Z}$ of constant dimension d [GS94, Conjecture 1]. By Example 2.8, at least for X smooth, this pairing is compatible with the Arakelov intersection pairing $\pi_{\mathrm{M}(X)\{m-d\}}$, i.e., there is a commutative diagram of pairings,

$$
\begin{array}{ccccccccc}
0 & \to & \widehat{\mathrm{H}}^0(M) = \widehat{\mathrm{CH}}^m(X)_{\mathbb{R}} & \to & \widehat{\mathrm{CH}}^m_{\mathrm{GS}}(X)_{\mathbb{R}} & \xrightarrow{\omega} & \mathrm{im}\,\omega & \to & 0 \\
& & \times & & \times & & \times & & \\
0 & \leftarrow & \mathrm{H}_{-2}(M,-1) = \mathrm{CH}^{d-m}(X)_{\mathbb{R}} & \leftarrow & \widehat{\mathrm{CH}}^{d-m}_{\mathrm{GS}}(X)_{\mathbb{R}} & \leftarrow & \mathrm{im}\,a & \leftarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & &
\end{array}
$$

where $\omega : \widehat{\mathrm{CH}}^m_{\mathrm{GS}}(X) \to A^{m,m}(X)$ and $a : A^{d-m-1,d-m-1}(X)/(\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}) \to \widehat{\mathrm{CH}}^{d-m}_{\mathrm{GS}}(X)$ are defined in [GS90a, Section 3.3.4]. I don't know whether the pairing on the right is a non-degenerate pairing, so the relation of Gillet-Soulé's conjecture and 5.2 is unclear. Note that $\mathrm{im}\,\omega$ and $\mathrm{im}\,a$ are infinite-dimensional $\mathbb{R}$-vector spaces.

Next, we show that Conjecture 4.1 recovers all the axioms on mixed motives over $\mathbb{F}_p$ we were willing to assume. Previously, it was known that Tate's conjecture about the pole order of ζ-functions over finite fields and Conjecture 1.5 together imply the Beilinson-Parshin conjecture [Gei98, Thm. 1.2.], and that the Beilinson-Parshin conjecture is equivalent to Bondarko's weight functor $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(\mathbb{F}_p) \to \mathbf{K}^{\mathrm{b}}(\mathrm{M}^{\mathrm{eff}}_{\mathrm{rat}})$ between the triangulated category of effective motives with the bounded homotopy category of effective Chow motives (with rational coefficients) being an equivalence of categories [Bon09, Section 8.3.2].

Theorem 4.3. *Conjecture 4.1 for motives of the form $M = i_*N$ (N any geometric motive over $\mathbb{F}_p$, $i : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$) is equivalent to the conjunction of Conjecture 1.5 and the* Beilinson-Parshin conjecture *stating*

$$K_r(X)_{\mathbb{Q}} = 0 \tag{4.3}$$

for any smooth projective variety X over $\mathbb{F}_p$ and all $r > 0$.

Under the axioms concerning the existence and cohomological dimension of mixed motives over $\mathbb{F}_p$ and the weight formalism (see Axiom 1.2), Conjecture 4.1 for all motives i_*N is equivalent to Conjecture 1.5.

Proof: Using the axioms about mixed motives, we first show that Conjecture 1.5 implies the perfectness. By construction, cf. (2.17), $\widehat{\mathrm{H}}^*_{\mathbb{R}}(i_*N) = \mathrm{H}^*(N)_{\mathbb{R}}$. By [Sch12b, Axiom 4.1.], the cohomological dimension of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$ is zero, so that $\mathrm{H}^j(N) = \mathrm{H}^0({}^{\mathrm{p}}\mathrm{H}^j N)$ and similarly for $N^\vee$. By the same axiom, only finitely many j yield a non-zero term. Therefore, we may replace N by ${}^{\mathrm{p}}\mathrm{H}^j N$ and assume that N is a mixed motive. Using the weight filtration we reduce to the case where N is a pure motive. Under Conjecture 1.5, all adequate equivalence relations agree, so we may regard N as a Chow motive or as a pure motive with respect to numerical equivalence. By the semi-simplicity of pure numerical motives there is a decomposition $N = \mathbf{1}^r \oplus R$, where R satisfies $\mathrm{H}^0_{\mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)}(R^\vee) = \mathrm{H}^0_{\mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)}(R) = 0$. By functoriality of the pairing we get a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^0(N)_{\mathbb{R}} & \times & \mathrm{H}^0(N^\vee)_{\mathbb{R}} & \longrightarrow & \mathbb{R} \\
\downarrow \cong & & \uparrow \cong & & \downarrow = \\
\mathrm{H}^0(\mathbf{1}^r)_{\mathbb{R}} & \times & \mathrm{H}^0(\mathbf{1}^r)_{\mathbb{R}} & \longrightarrow & \mathbb{R}
\end{array}
$$

The lower line is a perfect pairing, since the one for $\mathbf{1}_{\mathbb{F}_p}$ is by Example 2.7.

We now show the second statement. Let X be a smooth equidimensional projective variety over $\mathbb{F}_q$. We regard it as a $\mathbb{Z}$-scheme. By Example 2.7, the Arakelov intersection pairing

$$\widehat{\mathrm{K}}_{2m-k}(X)^{(m)} \times K_{k-2m}(X)^{(\dim X - m)}_{\mathbb{Q}} = K_{2m-k}(X)^{(m)}_{\mathbb{R}} \times K_{k-2m}(X)^{(\dim X - m)}_{\mathbb{R}} \to \mathbb{R}$$

is the usual multiplication on Adams eigenspaces in K-theory, followed by the multiplication with $\log p$ (which is irrelevant for the question of the perfectness). For $2m - k > 0$ the second factors vanishes, hence the perfectness is equivalent to (4.3). For $2m = k$ is perfectness is equivalent, by definition, to the agreement of numerical and rational equivalence (up to torsion). This shows one implication of the second statement. By resolution of singularities, the category $\mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$ is generated as a thick category by motives $\mathrm{M}(X)(m)$ as above. Since the perfectness only has to be checked on such generators, we are done with the converse implication as well. $\square$

The following corollary was pointed out to me by Bruno Kahn.

Corollary 4.4. *The perfectness of π_M for all motives $M = i_*N$ implies a canonical equivalence $\mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p) = \mathbf{D}^{\mathrm{b}}(\mathbf{M}_{\mathrm{rat}}(\mathbb{F}_p))$, which in turn implies among other things the independence of L-functions of ℓ.*

Proof: That description of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$ is a consequence of $\sim_{\mathrm{num}} = \sim_{\mathrm{rat}}$ and the Beilinson-Parshin conjecture [Kah05, proof of Theorem 56]. $\square$

We now give some interesting consequences of Conjecture 4.1 for motives which are truly motives over $\mathbb{Z}$, i.e., not coming from a motive over $\mathbb{F}_p$. It would be interesting to know whether other axioms on mixed motives over $\mathbb{Q}$, such as the agreement of homological and numerical equivalence on smooth projective varieties $X_\eta/\mathbb{Q}$ can be derived from Conjecture 4.1.

Theorem 4.5. *As in Example 2.8, consider the motive $M = \mathrm{M}(X)\{m-d\}[p-2m]$, $X/\mathbb{Z}$ regular, flat, projective and of equidimension d. Then Conjecture 4.1 for M is equivalent to the* Beilinson-Soulé vanishing conjecture

$$K_{2m-p}(X)_{\mathbb{Q}}^{(m)} = 0 \quad (\text{for } p < 0 \text{ and for } p = 0, \ m > 0).$$

Proof: The group $\widehat{\mathrm{H}}^0(M)$ appears in the long exact sequence

$$\dots \to \mathrm{H}_{\mathrm{D}}^{-1}(M) = \mathrm{H}_{\mathrm{D}}^{2d-p-1}(X, d-m) \to \widehat{\mathrm{H}}^0(M) \to \mathrm{H}^0(M) = \underbrace{K_{p-2m}(X)^{(d-m)}}_{=0} \to \dots$$

where the right hand vanishing is because $p - 2m < 0$ for $p < 0$ and $p = 0$, $m > 0$. The left hand vector space is dual to $\mathrm{H}_{\mathrm{D}}^p(X, m)$ by (2.5) (note that $d = \dim X_{\mathbb{C}} + 1$). It vanishes for $p < 0$ for trivial reasons. For $p = 0$, the short exact sequence (2.3) gives $\mathrm{H}_{\mathrm{D}}^0(X, m) = 0$ for $m > 0$. Indeed, the Hodge structure on $\mathrm{H}_{\mathrm{dR}}^0(X)$ only lies in the $(0,0)$-part of the Hodge diamond, i.e., $F^m = 0$ for $m > 0$. Hence the injectivity of $\mathrm{H}_{\mathrm{B}}^0(X, \mathbb{R}(m)) \to \mathrm{H}_{\mathrm{B}}^0(X, \mathbb{C}) \xrightarrow{\cong} \mathrm{H}_{\mathrm{dR}}^0(X_{\mathbb{C}})$ gives the claim. Therefore Conjecture 4.1 for M is equivalent to $\mathrm{H}_{-2}(M, -1) = \mathrm{H}_{2m-p-2}(\mathrm{M}(X)\{m-d\}, -1) = K_{2m-p}(X)_{\mathbb{R}}^{(m)} = 0$. $\qquad\square$

Example 4.6. Using the notation of Theorem 4.5, the group $\mathrm{H}_{-2}(M, -1)$ vanishes for $2m - p < 0$. Therefore, 4.1 asserts that the Chern class map

$$\mathrm{H}^0(M)_{\mathbb{R}} = K_{p-2m}(X)_{\mathbb{R}}^{(d-m)} \to \mathrm{H}_{\mathrm{D}}^0(M) = \mathrm{H}_{\mathrm{D}}^{2d-p}(X, d-m) \qquad (4.4)$$

is injective for $p - 2m > 0$ and an isomorphism for $p - 2m > 1$. In particular, the non-torsion part of higher K-theory of X is finitely generated—a weakening of Conjecture 5.1.

Proposition 4.7. *Assuming the existence of motivic t-structure on $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ such that Betti and de Rham realization are exact (see Axiom 1.2(ii) and (2.14)), the perfectness of the Arakelov intersection pairing for all motives $M \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ implies that the cohomological dimension of mixed motives over $\mathbb{Z}$ is two, i.e., $\mathrm{Hom}(\mathbf{1}[1], M[n]) = 0$ for any $n > 2$ and $M \in \mathbf{MM}(\mathbb{Z})$.*

Proof: Let M be a mixed motive. The group $\mathrm{H}_{1-n}(M, -1)_{\mathbb{R}} = \mathrm{Hom}(\mathbf{1}[1 - n](-1), M)_{\mathbb{R}}$ is zero for $n < 0$: in this case $\mathbf{1}[1-n]$ lies in degree $n < 0$ (with respect to the motivic t-structure). On the other hand this group is dual, via $\pi_{M[n-3]}$, to $\widehat{\mathrm{H}}_{\mathbb{R}}^{3-n}(M)$. In (2.17), this group lies between $\mathrm{H}_{\mathbb{R}}^{3-n}(M)$ which vanishes for $n > 2$ for the same reason and the Deligne cohomology group $\mathrm{H}_{\mathrm{D}}^{3-n}(M) = \mathrm{Hom}(M, \mathrm{H}_{\mathrm{D}}[3-n])$ which in turn vanishes by exactness of Betti and de Rham realization, except for $n = 1, 2$, as in (2.15). Consequently, $\mathrm{H}_{1-n}(M) = 0$ except for $n = 0, 1, 2$. $\qquad\square$

Lemma 4.8. *Under Conjecture 4.1, $\mathrm{H}^i(M)$ is nonzero only for finitely many $i \in \mathbb{Z}$.*

This is a consequence of the spectral sequence $\mathrm{H}^a({}^p\mathrm{H}^b(M)) \Rightarrow \mathrm{H}^{a+b}(M)$, the boundedness of the motivic t-structure and of the cohomological dimension [Sch12b, Axiom 4.1.]. It also follows from the perfectness of the Arakelov intersection pairing (not using the axioms on mixed motives):

Proof: It suffices to check the claim for $M = \mathrm{M}(X)(m)$, where X is as in Example 4.6 and $m \in \mathbb{Z}$, since these objects generate $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ as a thick category by resolution of singularities. Now, the claim follows as in Proposition 4.7 using the vanishing $K_k(X)$ for $k < 0$ and the vanishing of almost all Deligne cohomology groups of X. $\qquad\square$

5 Are special L-values given by the Arakelov intersection pairing?

Throughout this section, let M be any geometric motive over $\mathbb{Z}$. In this chapter, wherever ranks of motivic cohomology groups are involved, we assume that the Bass conjecture holds up to torsion:

Conjecture 5.1. For any regular scheme $X/\mathbb{Z}$, $\dim_{\mathbb{Q}} K_i(X)_{\mathbb{Q}} < \infty$.

We need the following consequence (by resolution of singularities): motivic cohomology of all geometric motives over $\mathbb{Z}$ is finitely generated.

By [Sch12b, Axiom 4.1.] (see also Lemma 4.8) and (2.17) only finitely many $\mathrm{H}^i(M)$ and $\widehat{\mathrm{H}}^i(M)$ are nonzero as $i \in \mathbb{Z}$ varies. Thus, the Euler characteristic

$$\chi(M) := \sum_i (-1)^i \dim \mathrm{H}^i(M) \tag{5.1}$$

and similarly $\widehat{\chi}(M)$, $\chi_{\mathrm{D}}(M)$ are well-defined. We write $\det \mathrm{H}^* := \otimes_{i \in \mathbb{Z}} \det^{(-1)^i} H^i$ for any bounded family H^i of finite-dimensional vector spaces, such as $\mathrm{H}^i(M)$ etc. The determinant of Arakelov motivic cohomology groups carries a $\mathbb{Q}$-structure by the isomorphism induced by (2.13) and (2.17),

$$\det \widehat{\mathrm{H}}^*_{\mathbb{R}}(M) = \left(\det \mathrm{H}^*(M) \otimes \det^{-1} \mathrm{H}^*_{\mathrm{B},\mathbb{Q}}(M) \otimes \det \mathrm{H}^{/F,\mathrm{alg},*}_{\mathrm{dR}}(M) \right) \otimes_{\mathbb{Q}} \mathbb{R}.$$

Conjecture 5.2. The order of the L-function of M (Definition 3.1) is given by

$$\mathrm{ord}_{s=0} L(M, s) = -\chi(M^\vee(-1)).$$

As usual, negative orders mean a pole, positive ones a zero of the L-function. Moreover, assuming the perfectness of the Arakelov intersection pairings $\pi_{M[k]}$ (Definition 2.5) for all $k \in \mathbb{Z}$ asserted by Conjecture 4.1, the special L-value is given by

$$L^*(M, 0) \equiv 1/\Pi_M \pmod{\mathbb{Q}^\times}.$$

Here Π_M means the following: the perfectness of the Arakelov intersection pairing yields a map

$$\det \mathrm{H}_{-2+*}(M, -1)_{\mathbb{R}} \otimes \det \widehat{\mathrm{H}}^*_{\mathbb{R}}(M) \to \mathbb{R}.$$

The $\mathbb{Q}$-structure on the left maps to a real number denoted Π_M. Note that Π_M is well-defined up to multiplication by a non-zero rational number.

Notation 5.3. For a projective flat scheme $X/\mathbb{Z}$ with smooth generic fiber $X_\eta/\mathbb{Q}$, we write $E := \eta_{!*}\eta^* \mathrm{h}^{-b}(X, -m) \in \mathbf{MM}(\mathbb{Z})$ and $M_\eta = \eta^*[-1]E = \mathrm{h}^{-b-1}(X_\eta, -m) \in \mathbf{MM}(\mathbb{Q})$. The definition of E is recalled in Section 1.2. In particular, whenever E is considered, we need to assume the axioms on mixed motives mentioned in Section 1.2. The motive E only depends on X_η, not on X. It is pure of weight $w := \mathrm{wt}(E) = 2m - b$. Putting $d := \dim X$ and $d_\eta = \dim X_\eta$, the dual

$$E^\vee = (\eta_{!*}\eta^* \mathrm{h}^{-2d+4+b}(X, 1 - d + m))[-2] \tag{5.2}$$

is pure of weight $-w$, while M_η is pure of weight $w - 1$.

Under Conjecture 4.1, the pole order conjecture is equivalent to

$$\mathrm{ord}_{s=0} L(M, s) = -\widehat{\chi}(M).$$

We expound some structural properties of the conjecture. In order to state the compatibility with the functional equation, we shall need the following conjecture due to Deligne. It implies the compatibility of the L-values conjecture for critical pure motives M_η over $\mathbb{Q}$ (i.e., motives such that $\mathrm{H}^i_{\mathrm{w}}(M_\eta) = 0$, $i = 0, 1$) with the functional equation [Del79, Theorem 5.6].

Conjecture 5.4. [Del79, Conjecture 6.6] Let M be a pure motive over $\mathbb{Q}$ with respect to homological equivalence, i.e., a direct summand in $\mathbf{M}_{\mathrm{hom}}(\mathbb{Q})$ of $h(X_\eta, m)$ where $X_\eta/\mathbb{Q}$ is smooth projective. Assume that M is of rank one, that is to say, its Betti realization (or, equivalently, de Rham or ℓ-adic realization) is one-dimensional. Then M is of the form $M(\epsilon)(n)$, where n is an integer and $\epsilon : \mathrm{Gal}(\mathbb{Q}) \to \mathbb{Q}^\times$ is a finite character and $M(\epsilon)$ denotes the Dirichlet motive attached to the one-dimensional representation, ϵ, of $\mathrm{Gal}(\mathbb{Q})$ (*loc. cit.*).

Theorem 5.5. *(i) Conjecture 5.2 is triangulated: given a distinguished triangle $M_1 \to M_2 \to M_3$ in $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$, the conjecture predicts*

$$L^*(M_1, 0)L^*(M_3, 0) = L^*(M_2, 0)$$

and additively with the pole orders. In particular, the subcategory of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ of motives for which the conjecture holds is triangulated.

(ii) Assume Deligne's Conjecture 5.4, Conjecture 1.5 ($\sim_{\mathrm{rat}}=\sim_{\mathrm{num}}$), the functional equation for completed L-functions over $\mathbb{Q}$ (Conjecture 3.12) and 4.1. Then Conjecture 5.2 for any accessible motive M (Definition 3.8) is equivalent to the one for $M^\vee\{-1\}$.

Note that accessible motives generate $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ as a thick category (Remark 3.9).

Proof: (i): The pole order additivity is clear. The multiplicativity of the special values formula follows easily by considering the long exact sequences made of $\widehat{\mathrm{H}}^*_\mathbb{R}(M_i)$ and $\mathrm{H}_*(M, -1)_\mathbb{R}$. By construction, the $\mathbb{Q}$-structure on Arakelov motivic cohomology is triangulated, i.e., there is a canonical isomorphism $\det \widehat{\mathrm{H}}^*_\mathbb{R}(M_2) = \det \widehat{\mathrm{H}}^*_\mathbb{R}(M_1) \otimes \det \widehat{\mathrm{H}}^*_\mathbb{R}(M_3)$ of $\mathbb{R}$-vector spaces, respecting the $\mathbb{Q}$-structure.

(ii): By Remark 3.9(ii), it is enough to show the claim for all M contained in the triangulated subcategory of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ generated by the image of $i_* : \mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p) \to \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ for all primes p and motives E as in Notation 5.3.

We put $\mathrm{ord} := \mathrm{ord}_{s=0}$ and $\chi^a_w(M) := \sum_{i \in \mathbb{Z}}(-1)^i \dim \mathrm{H}^a_w(\mathrm{H}^i(\mathrm{R}\Gamma_\mathrm{H}(M))$ for $a = 0, 1$, where $\mathrm{R}\Gamma_\mathrm{H}$ denotes the Hodge realization functor defined in (2.7). Conjecture 5.2 for M, $\mathrm{ord}\, L(M, s) = -\chi(M^\vee(-1))$, is equivalent to

$$
\begin{aligned}
\mathrm{ord}\,\Lambda(M) &\overset{(3.2)}{=} \mathrm{ord}\, L(M) - \chi^1_w(M^\vee(-1)) \\
&= -\chi(M^\vee(-1)) - \chi^1_w(M^\vee(-1)) \\
&\overset{4.1}{=} -\widehat{\chi}(M) - \chi^1_w(M^\vee(-1)) \\
&\overset{(2.5)}{=} -\chi(M) + \chi_\mathrm{D}(M) - \chi^0_w(M) \\
&= -\chi(M) - \chi^1_w(M)
\end{aligned}
$$

Indeed, $\chi_\mathrm{D}(M) = \chi^0_w(M) - \chi^1_w(M)$ (at least) for all M as in the claim: for $M = E$, this follows from (2.4), (2.6), and (2.15), while for $M = i_*N$, these terms are zero. By Lemma 3.13, the functional equation for mixed motives over $\mathbb{Q}$ implies the one for motives over $\mathbb{Z}$, so that $\mathrm{ord}\,\Lambda(M^\vee(-1)) = \mathrm{ord}\,\Lambda(M)$. Again invoking the pole order calculation of L_∞-functions we get $\mathrm{ord}\, L(M^\vee(-1)) = -\chi(M)$, that is, the conjectural prediction of the pole order of $L(M^\vee(-1))$. This settles the compatibility of the pole order prediction with the functional equation.

As for the special L-values, the claim is again triangulated. For motives $M = i_*N$, where $i : \mathrm{Spec}\,\mathbb{F}_p \to \mathrm{Spec}\,\mathbb{Z}$ and N is any geometric motive over $\mathbb{F}_p$ we have $M^\vee\{-1\} = i_*N^\vee$. The functional equation reads $L(i_*N, s) = ab^s L(i_*N^\vee, -s)$, with a and b in $\mathbb{Q}^\times$ (Remark 3.14; this uses the agreement of numerical and homological

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equivalence, implied by Conjecture 1.5.) On the other hand $\Pi_{i_*N} \equiv \Pi_{i_*N^\vee} \mod \mathbb{Q}^\times$ by Example 2.7.

To check the claim for $M = E$ as above, we may assume X is of equidimension d. Each individual Deligne cohomology group $\mathrm{H}_{\mathrm{D}}^r(E)$ carries the $\mathbb{Q}$-structure mentioned in (2.2). We can assume $w := \mathrm{wt}(E) = 2m - b \leq 2$, since otherwise we can replace E by $E^\vee(-1)[2]$. Let $M_\eta := \eta^* M[-1] = \mathrm{h}^{-b-1}(X_\eta, m)$, where X_η is the generic fiber of X. For $w = 2$, the hard Lefschetz axiom implies an isomorphism $E \cong E^\vee(-1)[2]$ (see (5.10)), so that there is nothing to show in that case. Let now $w \leq 1$. We write $L^*(-) := L^*(-, 0)$. Deligne's conjecture 5.4 implies (see *loc. cit.*)

$$\frac{L^*(E)}{L^*(E^\vee(-1))} \overset{3.5}{=} \frac{L^*(M_\eta^\vee(-1))}{L^*(M_\eta)} \overset{5.4}{\cong} \frac{a_1}{a_2} \mod \mathbb{Q}^\times$$

where a_1 is an element in the $\mathbb{Q}$-lattice of $\det \mathrm{H}_{\mathrm{D}}^0(E^\vee(-1))(= \mathbb{R})$ given by the $\mathbb{Q}$-structure on this Deligne cohomology group, and a_2 is an element in the $\mathbb{Q}$-lattice of $\det^{-1} \mathrm{H}_{\mathrm{D}}^1(E)$, regarded as an element of $\det \mathrm{H}_{\mathrm{D}}^0(E^\vee(-1))$ using the isomorphism $\mathrm{H}_{\mathrm{D}}^1(M_\eta^\vee(-1))^\vee \to \mathrm{H}_{\mathrm{D}}^0(M_\eta)$, cf. (2.5). In other words, the isomorphism $\det \mathrm{H}_{\mathrm{D}}^1(E) \to \det \mathrm{H}_{\mathrm{D}}^0(E^\vee(-1))$ is multiplication by a_1/a_2 with respect to the $\mathbb{Q}$-structures on both sides.

For $r \neq 1$, the group $\mathrm{H}_{\mathrm{D}}^r(E)$ and its $\mathbb{Q}$-structure is trivial, since the corresponding Betti and (truncated) de Rham cohomology groups vanish. Therefore there is a canonical isomorphism

$$\det \mathrm{H}_{\mathrm{D}}^*(E) \cong \det^{-1} \mathrm{H}_{\mathrm{D}}^1(E) \tag{5.3}$$

(including the $\mathbb{Q}$-structure). Thus

$$\begin{aligned}
\det \widehat{\mathrm{H}}_{\mathbb{R}}^*(E) \otimes \det \mathrm{H}_*(E, -1)_{\mathbb{R}} &= \det \mathrm{H}^*(E)_{\mathbb{R}} \otimes \det^{-1} \mathrm{H}_{\mathrm{D}}^*(E) \otimes \det \mathrm{H}^*(E^\vee, 1)_{\mathbb{R}} \\
&\cong \det \mathrm{H}^*(E)_{\mathbb{R}} \otimes \det^{-1} \mathrm{H}_{\mathrm{D}}(E^\vee(-1)) \otimes \det \mathrm{H}^*(E^\vee, 1)_{\mathbb{R}} \\
&= \det \mathrm{H}_*(E^\vee)_{\mathbb{R}} \otimes \det \widehat{\mathrm{H}}_{\mathbb{R}}^*(E^\vee, 1)
\end{aligned}$$

Both the left hand side and the last term on the right hand side map to $\mathbb{R}$ via the Arakelov intersection pairings for E and $E^\vee\{-1\}$, respectively. The two pairings are compatible with the isomorphism by the commutativity of (2.20). By Conjecture 5.2 for E, the image of the $\mathbb{Q}$-structure on the left hand side is $L^*(E)^{-1}$, while the one from the right hand side is, by 5.2, just $1/L^*(E^\vee(-1))$. Hence the two cases of the conjecture are equivalent. □

In the remainder of this paper, we show how certain special cases of 5.2 are related to conjectures of Beilinson, Soulé, and Tate. In order to formulate our main result as succinctly as possible, we formulate the following

Conjecture 5.6. For the motive E defined in Notation 5.3 with $w := \mathrm{wt}(E) = 2$, the Arakelov intersection pairing $\pi_{E[-2]} : \mathrm{H}_0(E, -1) \times \widehat{\mathrm{H}}^2(E) \to \mathbb{R}$ agrees with Beilinson's height pairing (5.6).

Remark 5.7. By Theorem 1.3 and (2.15), we know $\mathrm{H}_0(E, -1) = \mathrm{CH}^{d-m}(X_\eta)_{\mathbb{Q}}$ and $\widehat{\mathrm{H}}^2(E) = \mathrm{H}^2(E) = \mathrm{CH}^m(X_\eta)_{\mathbb{Q}}$ (cf. the proof of Proposition 5.16), so this conjecture only concerns the pairing itself. Moreover, (5.6) is induced by the Gillet-Soulé intersection pairing

$$\mathrm{CH}^{d-m}(X)_{\mathbb{Q}} \times \widehat{\mathrm{CH}}^m(X)_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^d(X)_{\mathbb{Q}} \overset{f_*}{\longrightarrow} \widehat{\mathrm{CH}}^1(\mathbb{Z}) = \mathbb{R},$$

which in turn is induced by (4.2). As mentioned in Example 2.8, this pairing agrees with the Arakelov intersection pairing for $\mathrm{M}(X)\{-m\}$ at least up a $\mathbb{Q}$-linear combination of $\log p_i$, where p_i are the primes such that the restriction of X is

smooth over $\mathbb{Z}[1/\prod p_i]$. It is worth mentioning that this comparison is an entirely formal consequence of the use of stable homotopy category. Its definition as the homotopy category of spectra of simplicial presheaves on *smooth* schemes yields immediate comparison results such as [Sch12a, Thm. 7.4] for smooth schemes, but not easily for other schemes. Therefore, it is a natural idea to overcome this hurdle by studying (Arakelov) motivic cohomology for log-smooth schemes. By de Jong's resolution of singularities, motives of all log-smooth $\mathbb{Z}$-schemes should generate a category of motives of logarithmic schemes over $\mathbb{Z}$. This would allow to bypass Conjecture 5.6. I plan to return to this question in a subsequent paper.

The following two theorems summarize the remainder of this paper: under standard assumptions on motives and their L-functions, it shows that Beilinson's, Soulé's, and Tate's conjectures are essentially equivalent to the conceptual reformulation made possible by the use of the Arakelov intersection pairing.

Theorem 5.8. *The following are equivalent:*

(i) *The conjecture of Soulé (5.11), restricted to regular, projective (but not necessarily flat) schemes.*

(ii) *The restriction of the pole order formula (Conjecture 5.2) to the category of easily accessible motives (Definition 3.8).*

Proof: This follows immediately from Theorem 5.12 by Theorem 5.5.   □

By Remark 3.9, the thick closure of the category of easily accessible motives is the entire category $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$. Thus, the pole order formula of Conjecture 5.2 can be regarded as an extension of Soulé's conjecture to direct summands.

Theorem 5.9. *We assume the existence of mixed motives as formulated in Axiom 1.2 and the agreement of Beilinson's height pairing with the Arakelov intersection pairing (Conjecture 5.6). Moreover, in order to incorporate the compatibility of L-values with respect to the functional equation, we assume Deligne's conjecture 5.4 on rank one motives, and the functional equation for completed L-functions over $\mathbb{Q}$ (Conjecture 3.12). Finally, we assume that the pole order formula of Conjecture 5.2 holds for all motives in $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$.*

Then, the following are equivalent:

(i) *The conjunction of the conjectures of Beilinson (L-values and $\sim_{\mathrm{num}} = \sim_{\mathrm{rat}}$, 5.14, 1.5), and Tate (5.19).*

(ii) *The restriction of the conjunction of the perfectness of the Arakelov intersection pairings (Conjecture 4.1) and the special L-values formula (Conjecture 5.2) to the subcategory $\mathbf{DM}_{\mathrm{B},\mathrm{tr}}(\mathbb{Z}) \subset \mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ of accessible motives (Definition 3.8).*

Proof: By Remark 3.9(ii), $\mathbf{DM}_{\mathrm{B},\mathrm{tr}}(\mathbb{Z})$ is contained in the triangulated category generated by motives $M = E$ as in Notation 5.3, and motives of the form $M = i_*N$, $N \in \mathbf{DM}_{\mathrm{B},c}(\mathbb{F}_p)$, $i : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. For the latter type of motives, Conjecture 4.1 is equivalent to Conjecture 1.5 by Theorem 4.3 and 5.2 is equivalent to the Tate conjecture by Theorem 5.20.

The subcategory of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ of motives M for which all pairings $\pi_{M[k]}$ are perfect is triangulated since motivic and Arakelov motivic cohomology behave well under triangles. Moreover, 4.1 for $M(\in \mathbf{DM}_{\mathrm{B},\mathrm{tr}}(\mathbb{Z}))$ is equivalent to 4.1 for $M^{\vee}\{-1\}$ by Remark 4.2(ii). In a similar vein, Conjecture 5.2 is stable under distinguished triangles, and 5.2 for M is equivalent to 5.2 for $M^{\vee}\{-1\}$ (Theorem 5.5).

To finish (i) $\Rightarrow$ (ii), using the calculation of $E^\vee\{-1\}$ in (5.2), we therefore only need to consider $M = E$ with $w := \mathrm{wt}(E) = 2m - b \leq 2$. Beilinson's pole order conjecture for M_η, 5.14(A), is equivalent (see (5.14)) to

$$\mathrm{ord}_{s=0} L(E, s) = -\chi(E^\vee(-1)) + \dim \mathrm{H}^1(E^\vee(-1)). \qquad (5.4)$$

By assumption, $L(E, s) = -\chi(E^\vee\{-1\}) = -\chi(E^\vee(-1))$, so that we get $\mathrm{H}^1(E^\vee(-1)) = 0$. Using this vanishing, part (B) of Beilinson's conjecture is equivalent to the perfectness of the intersection pairings $\pi_{E[k]}$, $k \in \mathbb{Z}$ (with $w = \mathrm{wt}(E) \leq 2$), by Proposition 5.16. This shows that 1.5, 5.11, and 5.14(B) together imply 4.1 for all $M \in \mathbf{DM}_{\mathrm{B,tr}}(\mathbb{Z})$. Then parts (A), (C) of Beilinson's conjecture are equivalent to 5.2 for all motives of the form E (of weight ≤ 2), by Theorem 5.18.

The converse implication (ii) $\Rightarrow$ (i) is shown using the same arguments. $\qquad\square$

Remark 5.10. It is natural to ask for the equivalence of the following two statements:

(i) The conjectures of Beilinson, Soulé, and Tate (5.14, 1.5, 5.11, 5.19).

(ii) The restriction of Conjectures 4.1 and 5.2 to the category of accessible motives.

Under the assumptions of 5.9, *except* for the pole order formula assumption, the above proof does show (ii) $\Rightarrow$ (i). The latter addditional assumption is only needed to prove the converse, and is actually only needed for motives of the form $M = E$ as above. Moreover, it holds unconditionally if $\mathrm{M}(X_\eta)$ is an Artin-Tate motive (Theorem 1.4). The vanishing $\mathrm{H}^1(E^\vee(-1)) = 0$ also follows from the Soulé+Tate conjecture if one can show $E \in \mathbf{DM}_{\mathrm{B,tr}}(\mathbb{Z})$, which in its turn would follow if the motivic t-structure on $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$ restricts to a t-structure on $\mathbf{DM}_{\mathrm{B,tr}}(\mathbb{Z})$. In this case, the proof of [Sch12b, Prop. 5.6] referred to in Remark 3.9(ii) could be adapted to $\mathbf{DM}_{\mathrm{B,tr}}(\mathbb{Z})$.

5.1 Relation to a conjecture of Soulé

Conjecture 5.11. (Soulé, [Sou84, Conjecture 2.2.]) Let $Y/\mathbb{Z}$ be quasiprojective. Let $m \in \mathbb{Z}$ be arbitrary. Then

$$\mathrm{ord}_{s=m} \zeta(Y, s) = \sum_{i \geq 0} (-1)^{i+1} \dim_{\mathbb{Q}} K_i'(Y)_{(m),\mathbb{Q}} \qquad (5.5)$$

We refer to *loc. cit.* for the definition of the Adams eigenspace $K_i'(Y)_{(m),\mathbb{Q}}$. For Y regular, it agrees with $K_i(Y)_{\mathbb{Q}}^{(\dim Y - m)}$.

Soulé's conjecture extends a previous conjecture of Tate [Tat65, p. 105]. A formally similar conjecture was also expressed by Lichtenbaum [Lic84]. The right hand side of (5.5) makes sense under the Bass conjecture 5.1 and the vanishing of almost all K'-groups, which in turn is a consequence of [Sch12b, Axiom 4.1.]. See also Lemma 4.8. As the thick closure of $\mathbf{DM}_{\mathrm{B,tr}}(\mathbb{Z})$ is all of $\mathbf{DM}_{\mathrm{B},c}(\mathbb{Z})$, the following statement can be paraphrased by saying that Soulé's conjecture is essentially equivalent to the pole order part of Conjecture 5.2. This proof does not make use of mixed motives.

Theorem 5.12. *Conjecture 5.11 for Y and m is equivalent to the pole order prediction of Conjecture 5.2 for $M = \mathrm{M}_{\mathrm{c}}(Y)(-m)$.*

Proof: Proposition 3.7 says $\zeta(Y, s + m) = L(\mathrm{M}_{\mathrm{c}}(Y)(-m), s)$. The statement for Y is implied by the conjunction of the one for some open subscheme U of Y and $Z := Y \backslash U$, since Adams eigenspaces in K'-theory have a localization sequence [Sou84,

1.3.], and motives with compact support behave well: $M_c(Z) \to M_c(Y) \to M_c(U)$ is a distinguished triangle. In particular we may assume that Y is integral. Thus, there is an open affine subscheme U of Y that is either smooth over $\mathbb{Z}$ or over some $\mathbb{F}_p$: if $Y/\mathbb{Z}$ is flat, one can take an open neighborhood of a smooth point of the generic fiber of Y, otherwise Y lies over some $\operatorname{Spec} \mathbb{F}_p$ and one can take a neighborhood of a smooth point of Y. Let $f : Y \to \mathbb{Z}$ be the projection. By Noetherian induction, we may replace Y by U and hence assume Y is regular and affine of dimension d, so that $(M_c(Y)(-m))^\vee\{-1\} = f_! f^* \mathbf{1}(m)\{-1\} = f_! f^! \mathbf{1}(m)\{-d\}$ by purity. Hence

$$
\begin{aligned}
\chi((M_c(Y)(-m))^\vee\{-1\}) &= \chi(M(Y)(m-d)[-2d]) \\
&= \sum_{i\in\mathbb{Z}}(-1)^i \dim H^{i+2d}(Y, m-d) \\
&= \sum_{i\in\mathbb{Z}}(-1)^i \dim K_i(Y)_{\mathbb{Q}}^{(d-m)}.
\end{aligned}
$$

$\square$

Example 5.13. We continue Examples 2.8 and 4.6 and look at the special values of the ζ-function of X: by Proposition 3.7 we have $L(M, s) = \zeta(X, s+d-m)$. The Arakelov intersection pairing $\pi_{M[i]}$ concerns the following groups

$$
\begin{array}{ccccc}
0 & \times & 0 & & i \le -1 \\
K_0(X)_{\mathbb{R}}^{(m)} & \times & \widehat{H}_{\mathbb{R}}^0(M) & & i = 0 \\
K_1(X)_{\mathbb{R}}^{(m)} & \times & \operatorname{coker} K_0(X)_{\mathbb{R}}^{(d-m)} \to H_D^{2(d-m)}(X, d-m) & & i = 1 \\
K_i(X)_{\mathbb{R}}^{(m)} & \times & H_D^{2(d-m)+i-1}(X, d-m) & & i > 1.
\end{array}
$$

The pairing for $i \ge 1$ is given by the Chern class $K_i(X)_{\mathbb{R}}^{(m)} \to H_D^{2m-i}(X, m)$ together with the cup product on Deligne cohomology, followed by the push-forward $f_* : H_D^{2d-1}(X, d) \to H_D^1(U, 1) = \mathbb{R}$. I expect that the group $\widehat{H}_{\mathbb{R}}^0(M)$ is isomorphic to $\operatorname{CH}^{d-m}(X)_{\mathbb{R}}$ and that the pairing π_M is the natural pairing of (arithmetic) Chow groups (cf. Remark 5.7). We do know that these two pairings agree up to a $\mathbb{Q}$-linear combination of $\log \prod p_i$, where p_i are the primes such that the restriction of X is smooth over $\mathbb{Z}[1/\prod p_i]$.

These pairings assemble to a map

$$
\bigotimes \pi_{M[i]} : \bigotimes_i \det^{(-1)^i}(H_{-2-i}(M, -1)_{\mathbb{R}} \otimes \widehat{H}_{\mathbb{R}}^i(M)) \overset{\cong}{\to} \mathbb{R}.
$$

(Even though the groups $\widehat{H}^i(M)$ vanish for $i < 0$, the determinant carries a non-trivial information related to these groups, namely the determinants of the Chern class map, see (4.4).) Conjecture 5.2 asserts that—modulo $\mathbb{Q}^\times$—$L^*(M, 0)$ is the reciprocal of the image of 1 in $\mathbb{R}$ via the $\mathbb{Q}$-structure map of the left hand term. The class number formula has been interpreted in terms similar to the one above, see [Sou92, III.4.3].

5.2 Relation to Beilinson's conjecture

In this section, we use the notation of 5.3. The following is Beilinson's conjecture [Beĭ84, Beĭ86]. Part (A) concerns the pole order of L-functions, part (B) is about the relation of Deligne cohomology and motivic cohomology, and (C) expresses the special L-value up to $\mathbb{Q}^\times$ in terms of determinants of the isomorphisms asserted in (B). The pole order conjecture in case $w = 3$ is due to Tate [Tat65].

Conjecture 5.14. Let $X_\eta/\mathbb{Q}$ be smooth and projective.

(A)

$$\operatorname{ord}_{s=m} L_{\mathbb{Q}}(\mathrm{h}^{-b-1}(X_\eta), s) = \operatorname{ord}_{s=0} L_{\mathbb{Q}}(M_\eta, s) = \begin{cases} 0 & w \geq 4 \\ -\dim \mathrm{CH}^n(X_\eta)_{\mathbb{Q}}/\mathrm{hom} & w = 3 \\ \dim \mathrm{CH}^n(X_\eta)_{\mathbb{Q},\mathrm{hom}} & w = 2 \\ \dim \mathrm{H}^{b+2}(X_\eta, n)_{\mathbb{Z}} & w \leq 1 \end{cases}$$

Here $n := b+2-m = m+2-w$, and the groups at the right have been defined in Section 1.2.

(B) For $w = 2$, the *height pairing*

$$\mathrm{CH}^m(X_\eta)_{\mathbb{Q},\mathrm{hom}} \otimes \mathrm{CH}^{d-m}(X_\eta)_{\mathbb{Q},\mathrm{hom}} \to \mathbb{R} \tag{5.6}$$

is perfect.

For $w = 1$, the map

$$r_\infty : (\mathrm{CH}^m(X_\eta)_{\mathbb{Q}}/\mathrm{hom} \oplus \mathrm{H}^{2m+1}(X_\eta, n)_{\mathbb{Z}}) \otimes_{\mathbb{Q}} \mathbb{R} \to \mathrm{H}_{\mathrm{D}}^{2m+1}(X, n). \tag{5.7}$$

obtained by the composition

$$\mathrm{CH}^m(X_\eta)_{\mathbb{Q}}/\mathrm{hom} \otimes \mathbb{R} \to \mathrm{H}_{\mathrm{dR}}^{2m}(X_{\mathbb{R}}) \to \mathrm{H}_{\mathrm{D}}^{2m+1}(X, n)$$

(see (2.2) for the right hand map) and the realization map, is an isomorphism.

For $w \leq 0$, the statement is the same, except that (5.7) gets replaced by

$$r_\infty : \mathrm{H}^{b+2}(X_\eta, n)_{\mathbb{Z}} \otimes_{\mathbb{Q}} \mathbb{R} \to \mathrm{H}_{\mathrm{D}}^{b+2}(X_\eta, n). \tag{5.8}$$

(C) The special L-value $L^*(M_\eta, 0)$ is conjecturally given up to a nonzero rational multiple by the following:

For $w = 2$, by the determinant of the height pairing (5.6) multiplied with the period of M_η, that is to say, the determinant of the isomorphism

$$\alpha_{M_\eta} : \mathrm{H}^{2m-1}(X_\eta(\mathbb{C}), \mathbb{R}(m))^{(-1)^m} \to \mathrm{H}_{\mathrm{dR}}^{2m-1}((X_\eta)_{\mathbb{R}})/F^m$$

with respect to the usual $\mathbb{Q}$-structures on both sides (compare (2.2)).

For $w = 1$, the L-value is given, mod $\mathbb{Q}^\times$, by $d_\infty(1)$, where

$$d_\infty := \det r_\infty : \det(\mathrm{H}^b(X_\eta, m)_{\mathbb{Z}} \oplus \mathrm{CH}^m(X_\eta)/\mathrm{hom})_{\mathbb{R}} = \mathbb{R} \to \det \mathrm{H}_{\mathrm{D}}^b(X, n) = \mathbb{R},$$

the left hand term is endowed with the obvious $\mathbb{Q}$-structure, the right one gets the one stemming from the identification of $\mathrm{H}_{\mathrm{D}}^b(X_\eta, n) = \mathrm{H}_{\mathrm{w}}^1(\mathrm{H}^{b-1}(X, \mathbb{Q}(n)))$ with the dual of $\mathrm{H}_{\mathrm{w}}^0(\mathrm{H}^{b-1}(X, \mathbb{Q}(n))^\vee(1))$.

For $w \leq 0$, the statement is the same, except that the term $\mathrm{CH}^m(X_\eta)/\mathrm{hom}$ is omitted.

This concludes the statement of Beilinson's conjecture. It predicts L-values of motives $\mathrm{h}^{-b-1}(X_\eta, m)$ with $w = 2m - b \leq 2$, up to a nonzero rational factor. The remaining weights are adressed by the functional equation (Conjecture 3.12).

We compare Beilinson's conjecture with Conjecture 4.1 and 5.2 applied to the generic intermediate extension $E := \eta_{!*}\eta^* \mathrm{h}^{-b}(X, -m)$, where X is any projective model of X_η (see Notation 5.3).

Recall from [And04, 5.4.2.1] that the agreement of homological and numerical equivalence (which is part of Axiom 1.2) implies the hard Lefschetz isomorphism:

$$\mathrm{h}^{-b-1}(X_\eta, m-b-2) \xrightarrow{\cong} \mathrm{h}^{-2d_\eta+b+1}(X_\eta, -d_\eta+m-1) = M_\eta^\vee(-1). \tag{5.9}$$

For $b+1 \leq d_\eta$ the map is given by the $(d_\eta - b - 1)$-st power of cup product with a hyperplane section, with respect to some embedding $X_\eta \subset \mathbb{P}_{\mathbb{Q}}^N$. The right hand term of (5.9) is $M_\eta^\vee(-1)$ by relative purity, applied to the smooth map $X_\eta/\mathbb{Q}$.

Lemma 5.15. *The hard Lefschetz isomorphism (5.9) yields an isomorphism*

$$E^\vee(-1)[2] = E(m - n) = E(w - 2). \qquad (5.10)$$

It induces isomorphisms of motivic and Deligne cohomology groups (respecting the $\mathbb{Q}$-structure of the latter):

$$
\begin{aligned}
\mathrm{CH}^m(X_\eta)_\mathbb{Q}/\mathrm{hom} &\cong \mathrm{CH}^{d-m-1}(X_\eta)_\mathbb{Q}/\mathrm{hom}, \\
\mathrm{CH}^m(X_\eta)_{\mathbb{Q},\mathrm{hom}} &\cong \mathrm{CH}^{d-m}(X_\eta)_{\mathbb{Q},\mathrm{hom}} \text{ [Beĭ87, Conj. 5.3.(a)]}, \\
\mathrm{H}^b(X_\eta, b - m)_\mathbb{Z} &\cong \mathrm{H}^{2d-b}(X_\eta, d - m)_\mathbb{Z} \text{ for } w = 2m - b \le 1 \\
\mathrm{H}^b_\mathrm{D}(X_\eta, b - m) &\cong \mathrm{H}^{2d-b}_\mathrm{D}(X_\eta, d - m) \text{ for } w \le 1.
\end{aligned}
$$

Proof: (5.10) is obtained from (5.9) by applying $\eta_{!*}[1]$. Now apply Theorem 1.3 and the calculation of Deligne cohomology in (2.15). □

The following proposition compares the perfectness of certain Arakelov intersection pairings with the statements in part (B) in Beilinson's conjecture.

Proposition 5.16. *Let E be as in Notation 5.3 with weight $w = \mathrm{wt}(E) \le 2$. If the weight of E is 2, we assume Conjecture 5.6. The following are equivalent:*

(i) *The pairings $\pi_{E[i]}$ and $\pi_{E^\vee\{-1\}[i]}$ $(i \in \mathbb{Z})$ are perfect.*

(ii) *Part (B) of Beilinson's conjecture and $\mathrm{H}^3(E^\vee(-1)[2]) = \mathrm{H}^1(E^\vee, 1) = 0$ (only needed if $w \le 1$).*

Remark 5.17. The group $\mathrm{H}^1(E^\vee, 1)$ vanishes unconditionally if X_η is such that M_η is a mixed Artin-Tate motive over $\mathbb{Q}$ (as opposed to a general mixed motive) by Theorem 1.4. Recall from Theorem 1.3 that $\mathrm{H}^3(E) = 0$ for $w := \mathrm{wt}(E) \le 2$.

Proof: The proof combines the hard Lefschetz isomorphism (Lemma 5.15) and the calculation of motivic and Deligne cohomology of E and $E^\vee(-1)$ (Theorem 1.3, (2.15)).

The map $\mathrm{H}^{b+2}(X_\eta, n)_\mathbb{Z} \to \mathrm{H}^{b+2}_\mathrm{D}(X_\eta, n)$ featuring in (5.7), (5.8) in the cases $w \le 1$ of Conjecture 5.14 is the Chern class map $\mathrm{H}^2(E(m-n)) \to \mathrm{H}^2_\mathrm{D}(E(m-n))$. Via hard Lefschetz, this is the same as the Chern class map

$$\mathrm{ch}(E^\vee(-1)) : \mathrm{H}^0(E^\vee(-1)) \to \mathrm{H}^0_\mathrm{D}(E^\vee(-1)). \qquad (5.11)$$

Consider the case $w = 1$. By Fontaine's reformulation [Fon92, 9.5], the map (5.7) being an isomorphism is equivalent to the existence of an exact sequence whose right hand map is the composition of the Poincaré duality isomorphism ϕ stemming from (2.5), the hard Lefschetz isomorphism and the Chern class map.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CH}^m(X_\eta)_\mathbb{R}/\mathrm{hom} & \xrightarrow{\mathrm{ch}} & \mathrm{H}^{2m}_\mathrm{D}(X_\eta, m) & \longrightarrow & \mathrm{H}^{2m+1}(X_\eta, m+1)^\vee_\mathbb{Z} \otimes \mathbb{R} & \longrightarrow & 0 \\
& & & & \phi \downarrow \cong & & \uparrow \mathrm{ch}^\vee & & \\
& & & & \mathrm{H}^{2d_\eta - 2m+1}_\mathrm{D}(X_\eta, d_\eta + 1 - m)^\vee & \xrightarrow[\cong]{(5.9)} & \mathrm{H}^{2m+1}_\mathrm{D}(X_\eta, m+1)^\vee. & &
\end{array}
$$

In terms of motivic and Deligne cohomology groups, it reads

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^1(E)_\mathbb{R} & \xrightarrow{\mathrm{ch}^1(E)} & \mathrm{H}^1_\mathrm{D}(E) & \longrightarrow & \mathrm{H}^0(E^\vee(-1))^\vee_\mathbb{R} & \longrightarrow & 0 \qquad (5.12) \\
& & & & \phi \downarrow \cong & \nearrow \mathrm{ch}^0(E^\vee(-1))^\vee & & & \\
& & & & \mathrm{H}^0_\mathrm{D}(E^\vee(-1))^\vee. & & & &
\end{array}
$$

These groups also occur in the following exact sequences, whose terms are paired by the pairings indicated on top:

$$
\begin{array}{ccccccccc}
\pi_{E[-1]}: & & \pi_{E^\vee(-1)[-1]}: & & & & \pi_{E[-2]}: & & \pi_{E^\vee(-1)}: \\
\widehat{\mathrm{H}}^1_{\mathbb{R}}(E) & \to & \mathrm{H}^1(E)_{\mathbb{R}} & \to & \mathrm{H}^1_{\mathrm{D}}(E) & \to & \widehat{\mathrm{H}}^2_{\mathbb{R}}(E) & \to & \mathrm{H}^2(E)_{\mathbb{R}} \\
\times & & \times & & \times & & \times & & \times \\
\mathrm{H}^1(E^\vee,1)_{\mathbb{R}} & \leftarrow & \widehat{\mathrm{H}}^1_{\mathbb{R}}(E^\vee,1) & \leftarrow & \mathrm{H}^0_{\mathrm{D}}(E^\vee,1) & \leftarrow & \mathrm{H}^0(E^\vee,1)_{\mathbb{R}} & \leftarrow & \widehat{\mathrm{H}}^0_{\mathbb{R}}(E^\vee,1)
\end{array}
$$
$$(5.13)$$

We have $\mathrm{H}^2(E) = 0$, so the injectivity of $\mathrm{ch}^0(E^\vee(-1))$ is equivalent to $\pi_{E^\vee(-1)}$ being perfect. The identification of $\operatorname{coker}\mathrm{ch}^1(E)$ with $\mathrm{H}^0(E^\vee,1)^\vee_{\mathbb{R}}$ of (5.12) is equivalent to $\pi_{E[-2]}$ being perfect. The Chern class map $\mathrm{H}^1(E)_{\mathbb{R}} = \mathrm{CH}^m(X_\eta)/\mathrm{hom} \to \mathrm{H}^1_{\mathrm{D}}(E) = \mathrm{H}^{2m}_{\mathrm{D}}(X_\eta, m) \subset \mathrm{H}^{2m}_{\mathrm{B}}(X_\eta, \mathbb{R}(m))$ is injective by definition of homological equivalence. Hence $\widehat{\mathrm{H}}^1_{\mathbb{R}}(E) = 0$ so that $\mathrm{H}^1(E^\vee,1) = 0$ is equivalent to $\pi_{E[-1]}$ being perfect. By the five lemma, $\pi_{E^\vee(-1)[-1]}$ is then perfect, too. All other Deligne, motivic, and hence Arakelov motivic cohomology groups of $E^\vee(-1)$ and E, except for the ones displayed above, vanish.

The case $w < 1$ is done similarly: in addition to the above, we have $\mathrm{H}^1(E) = 0$. Accordingly, (5.12) reduces to an isomorphism $\mathrm{H}^1_{\mathrm{D}}(E) \xrightarrow{\cong} \mathrm{H}^0(E^\vee(-1))^\vee_{\mathbb{R}}$. The details are omitted.

For $w = 2$, *all* Deligne cohomology $\mathrm{H}^*_{\mathrm{D}}(E)$ and $\mathrm{H}^*_{\mathrm{D}}(E^\vee,1)$ vanish for weight reasons. Moreover $\mathrm{H}^a(E) = \mathrm{H}^{a-2}(E^\vee(-1)) = 0$ for $a \neq 2$, so that $\pi_{E^\vee(-1)[-1]}$ and $\pi_{E[-1]}$ are perfect. The height pairing (5.6) is just $\pi_{E[-2]}$ according to Conjecture 5.6. Its perfectness is equivalent to the one of $\pi_{E^\vee(-1)}$. □

Theorem 5.18. *We assume the perfectness of the Arakelov intersection pairing for motives of the form $M = E[n]$, with E as in Notation 5.3 and $n \in \mathbb{Z}$. We also assume Conjecture 5.6 if E is of weight 2. Then Beilinson's conjecture (parts (A), (C)) for M_η is equivalent to Conjecture 5.2 for E.*

Proof: By hard Lefschetz (Lemma 5.15) and calculation of motivic cohomology of E, Theorem 1.3, part (A) of Beilinson's conjecture reads

$$
\operatorname{ord}_{s=0} L_{\mathbb{Q}}(M_\eta, s) \overset{3.5}{=} -\operatorname{ord}_{s=0} L_{\mathbb{Q}}(E,s) = \sum_{a \neq 1} (-1)^a \dim \mathrm{H}^a(E^\vee(-1)). \qquad (5.14)
$$

In fact, $\mathrm{H}^a(E^\vee(-1)) \overset{(5.10)}{=} \mathrm{H}^{a+2}(\eta_{!*}\eta^* \mathrm{h}^{-b}(X, -n))$. For example, in case $w = 2m - b \leq 1$, this equals $\mathrm{H}^{b+2}(X_\eta, n)_{\mathbb{Z}}$ for $a = 0$ and vanishes for $a \neq 0, 1$. As was mentioned above, the perfectness of $\pi_{E[-1]}$ conjectured in 4.1 implies $\mathrm{H}^1(E^\vee(-1)) = 0$. (In case $w \geq 2$, we know this vanishing without invoking 4.1.) This settles the pole order part (A) of Beilinson's conjecture.

For the special L-values, we revisit the proof of Proposition 5.16 and look at the involved $\mathbb{Q}$-structures. Again using hard Lefschetz, we replace the map $\mathrm{H}^{b+2}(X_\eta, n)_{\mathbb{Z}} \otimes \mathbb{R} \to \mathrm{H}^{b+2}_{\mathrm{D}}(X_\eta, n)$ occurring in (5.7), (5.8) by $\mathrm{ch}(E^\vee(-1))$, see (5.11). The involved $\mathbb{Q}$-structures remain unchanged.

We first treat the case $w = 1$. By [Fon92, 9.5], [FPR94, Conj. III.4.4.3], Beilinson's conjecture is equivalent to saying that the L-value of M_η is given by the reciprocal of the image (in $\mathbb{R}$) of the $\mathbb{Q}$-structure on the right hand side:

$$
\begin{aligned}
\mathbb{R} &\cong \operatorname{det}^{-1} \mathrm{H}^0(E^\vee(-1))_{\mathbb{R}} \otimes \operatorname{det}^{-1} \mathrm{H}^1_{\mathrm{D}}(E) \otimes \operatorname{det} \mathrm{H}^1(E)_{\mathbb{R}} \\
&\overset{(5.3)}{=} \operatorname{det}^{-1} \mathrm{H}^*(E^\vee(-1))_{\mathbb{R}} \otimes \operatorname{det} \mathrm{H}^*_{\mathrm{D}}(E) \otimes \operatorname{det}^{-1} \mathrm{H}^*(E)_{\mathbb{R}}.
\end{aligned}
$$

Here the isomorphism stems from the exact sequence (5.12) and the $\mathbb{Q}$-structure on $\mathrm{H}^1_{\mathrm{D}}(E)$ is the natural one defined in Section 2.1. (This $\mathbb{Q}$-structure is distinct from the one on the isomorphic group $\mathrm{H}^0_{\mathrm{D}}(E^\vee(-1))^\vee$, as is apparent from the discussion of

the functional equation in Theorem 5.5.) Moreover, all groups $\mathrm{H}^*(E^\vee(-1))$ except H^0 and $\mathrm{H}^*(E)$ except H^1 vanish. By construction of Arakelov motivic cohomology the above is isomorphic, including the $\mathbb{Q}$-structure, to

$$\det{}^{-1} \mathrm{H}^*(E^\vee(-1))_\mathbb{R} \otimes \det{}^{-1} \widehat{\mathrm{H}}_\mathbb{R}^*(E).$$

The above identification with $\mathbb{R}$ agrees with the dual of the Arakelov intersection pairing for E, so that $L^*(E,0) = L^*(M_\eta, 0)^{-1}$ is indeed the inverse of Π_E. This accomplishes the case $w = 1$.

Again, the case $w \leq 0$ is similar but simpler, since in addition $\mathrm{H}^1(E) = 0$. Correspondingly, only the determinant of the realization map $\mathrm{ch}(E^\vee(-1))$ (5.11), as opposed to the one of (5.12), appears in Beilinson's conjecture.

Finally, consider the special value at the central point, i.e., $w = 2$. In this case all groups $\mathrm{H}_\mathrm{D}^*(E)$ are trivial, but the $\mathbb{Q}$-structure on

$$\det{}^{-1} \mathrm{H}_\mathrm{D}^*(E) = \det \mathrm{H}_\mathrm{D}^*(M_\eta) = \det \underbrace{\mathrm{H}_\mathrm{B}^{b+1}(X_\eta, \mathbb{R}(m))^{(-1)^m}}_{=:B} \otimes \det{}^{-1} \underbrace{\mathrm{H}_\mathrm{dR}^{b+1}(X_\eta \times \mathbb{R})/F^m)}_{=:dR}$$

is non-trivial since the period isomorphism $\alpha : B \to dR$ does not respect the natural $\mathbb{Q}$-structures. By linear algebra, $\det \alpha$ agrees (modulo $\mathbb{Q}^\times$) with the image (in $\mathbb{R}$) of the $\mathbb{Q}$-lattice under the natural isomorphism induced by α: $\det B \otimes \det{}^{-1} dR \overset{\cong}{\to} \mathbb{R}$. Except for $\mathrm{H}^2(E)$ and $\mathrm{H}^0(E^\vee(-1))$, all motivic cohomology groups of E and $E^\vee(-1)$ vanish (Theorem 1.3). The Arakelov intersection pairing $\pi_{E[-2]}$ agrees with the height pairing under Conjecture 5.6. By (2.17), we have an isomorphism of $\mathbb{R}$-vector spaces respecting the $\mathbb{Q}$-structure

$$\det \widehat{\mathrm{H}}_\mathbb{R}^*(E) = \det \mathrm{H}^*(E)_\mathbb{R} \otimes \det{}^{-1} \mathrm{H}_\mathrm{D}^*(E),$$

so Beilinson's conjecture is indeed equivalent to saying that $L^*(E,0) = L^*(M_\eta, 0)^{-1}$ is the reciprocal of the image of the $\mathbb{Q}$-lattice under $\det \widehat{\mathrm{H}}_\mathbb{R}^*(E) \otimes \det \mathrm{H}^*(E^\vee(-1))_\mathbb{R} \to \mathbb{R}$. □

5.3 Relation to the Tate conjecture over $\mathbb{F}_p$

Conjecture 5.19. (Tate conjecture over finite fields [Tat65]) Let $X/\mathbb{F}_q$ be smooth and projective. Let ℓ be a prime such that $\ell \nmid q$. Any $\mathrm{Gal}(\mathbb{F}_q)$-invariant element of $\mathrm{H}^{2i}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell(i))$ is a $\mathbb{Q}_\ell$-linear combination of algebraic elements, i.e., elements in the image of the cycle class map $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell(i))$.

Theorem 5.20. *In addition to the general assumptions on mixed motives over $\mathbb{F}_p$ (Section 1.2), we assume Conjecture 1.5. Then the Tate conjecture 5.19 is equivalent to Conjecture 5.2 for motives $M = i_* N$, where N is any geometric motive over $\mathbb{F}_p$, $i : \mathrm{Spec}\, \mathbb{F}_p \to \mathrm{Spec}\, \mathbb{Z}$. More precisely, the special value prediction of 5.2 in this case is*

$$L^*(i_* N, 0) \equiv (\log p)^{-\chi(N^\vee(-1))} \pmod{\mathbb{Q}^\times}, \tag{5.15}$$

where $\chi(N^\vee(-1))$ is the Euler characteristic of motivic cohomology (see (5.1), computed in the category $\mathbf{DM}_{\mathbf{B},c}(\mathbb{F}_p)$).

Proof: $\Rightarrow$: to show Conjecture 5.2 and (5.15) for $i_* N$, we may replace N by $\mathrm{gr}_*^W {}^\mathrm{p}\mathrm{H}^* N$, the weight graded pieces of the truncations with respect to the motivic t-structure, since both the weight filtration and the t-structure are bounded [Sch12b, Axiom 4.1.]. The subcategory of $\mathbf{MM}(\mathbb{F}_p)$ consisting of pure objects is, by [Sch12b, Axiom 4.11], the category of pure motives with respect to numerical equivalence, $\mathbf{M}_{\mathrm{num}}(\mathbb{F}_p)$. Under Conjecture 1.5, this agrees with Chow motives $\mathbf{M}_{\mathrm{rat}}(\mathbb{F}_p)$. Finally, $\chi_{\mathbf{DM}_\mathbf{B}(\mathbb{Z})}((i_* N)^\vee(-1)) = \chi_{\mathbf{DM}_\mathbf{B}(\mathbb{F}_p)}(N^\vee(-1))$, so we have

to show $\mathrm{ord}_{s=0} L(i_* N) = -\dim \mathrm{H}^0(N^\vee, 1) = -\dim \mathrm{H}_0(N, -1)$ and $L^*(i_* N) \equiv (\log p)^{-\dim \mathrm{H}_0(N,-1)} \pmod{\mathbb{Q}^\times}$.

Consider first $N = H := \mathrm{M}_{\mathbb{F}_p}(X)\{-n\}$ with $X/\mathbb{F}_p$ smooth and projective. Then $L(i_* H) = L(\mathrm{M}_{\mathbb{Z}}(X)\{1-n\})$. Let $Z^n(X)/\mathrm{num}$ be the group of codimension n cycles on X modulo numerical equivalence. Then

$$\dim \mathrm{H}^0(H) = \mathrm{rk}\, \mathrm{CH}^n(X) \overset{1.5}{=} \mathrm{rk}\, Z^n(X)/\mathrm{num} = -\mathrm{ord}_{s=n} \zeta(X, s),$$

so the pole order claim holds for H by assumption: the Tate conjecture and the agreement of the ℓ-adic homological and numerical equivalence relations on X (up to torsion) together are equivalent to the rightmost equality [Tat94, Thm. 2.9].

In general, N is a direct summand of H as above. Let $N \oplus N' = H$, which as an object in $\mathbf{M}_{\mathrm{rat}}(\mathbb{F}_p)$ is denoted $h(X)(n)$. By the previous case,

$$\dim \mathrm{H}^0 N + \dim \mathrm{H}^0 N' = -\mathrm{ord}\, L(N) - \mathrm{ord}\, L(N'). \qquad (5.16)$$

Let $-_\ell : \mathbf{M}_{\mathrm{rat}}(\mathbb{F}_p) \to \oplus \underline{\mathbb{Q}_\ell[\mathrm{Gal}(\mathbb{F}_p)]}$, $\pi h(X)(n) \mapsto \oplus_a \pi^* \mathrm{H}^a(X, \mathbb{Q}_\ell(n))$ be the ℓ-adic realization functor taking values in graded continuous ℓ-adic $\mathrm{Gal}(\mathbb{F}_p)$-representations. We write $\mathrm{H}^0(N_\ell) := N_\ell^{\mathrm{Gal}(\mathbb{F}_p)}$, the Galois cohomology of the ℓ-adic Galois module N_ℓ. The following way of reasoning is borrowed from *loc. cit.* We have the following chain of inequalities:

$$
\begin{aligned}
-\mathrm{ord}_{s=0} L(N, s) &\geq \dim_{\mathbb{Q}_\ell} \ker(\mathrm{Id} - \mathrm{Fr}^{-1})|N_\ell \\
&\geq \dim_{\mathbb{Q}_\ell} (N_\ell)^{\mathrm{Gal}(\mathbb{F}_p)} \\
&= \dim_{\mathbb{Q}_\ell} \mathrm{H}^0(N_\ell) \\
&\geq \dim_{\mathbb{Q}} \mathrm{H}^0(N)
\end{aligned}
$$

The last inequality is by the injectivity of the cycle class map $\mathrm{H}^0(N) \to \mathrm{H}^0(N_\ell)$, which follows from the injectivity of $\mathrm{H}^0(H) \to \mathrm{H}^0(H_\ell) = \mathrm{H}^{2n}(X, \mathbb{Q}_\ell(n))$, i.e., the agreement of homological and rational equivalence, which holds under Conjecture 1.5. Therefore, in (5.16) equality of dimensions must hold for the individual summands, so the pole order part is shown.

The claim (5.15) and the special values formula of 5.2 trivially hold for $N = \mathbf{1}(-1)$: the residue of $L(i_* \mathbf{1}(-1), s) = \zeta(\mathrm{Spec}\,\mathbb{F}_p, s) = (1 - p^{-s})^{-1}$ at $s = 0$ is $(\log p)^{-1}$, which is the inverse of the determinant of $\pi_{\mathrm{M}(\mathbb{F}_p)} = \pi_{i_* i^* \mathbf{1}\{-1\}}$ (Example 2.7). Jannsen's semisimplicity theorem for $\mathbf{M}_{\mathrm{num}}(\mathbb{F}_p)$ yields a decomposition $N = \mathbf{1}(-1)^r \oplus R$ with $\mathrm{Hom}_{\mathbf{M}_{\mathrm{num}}(\mathbb{F}_p)}(\mathbf{1}(-1), R) = \mathrm{Hom}_{\mathbf{M}_{\mathrm{num}}(\mathbb{F}_p)}(R, \mathbf{1}(-1)) = 0$. Hence we can assume $N = R$. By the Lefschetz trace formula, the L-function of any pure motive over $\mathbb{F}_p$ is a rational function in p^{-s} with rational coefficients that are independent of ℓ, see e.g. [And04, Section 7.1.4]. By the preceding part, the L-function of $i_* R$ does not have a pole at $s = 0$, therefore the leading term of the Laurent series $L(i_* R, s)$ is simply the value at this point, a nonzero rational number (as opposed to an ℓ-adic or, via σ_ℓ, a complex number).

$\Leftarrow$: we again use the theorem of Tate cited above: the Tate conjecture for $X/\mathbb{F}_p$ is implied by $\mathrm{ord}_{s=j} \zeta(X, s) = -\mathrm{rk}\, Z^j(X)/\mathrm{num}$. Under 1.5, that term is $-\mathrm{rk}\, \mathrm{CH}^j(X) = -\dim \mathrm{H}^{2j}(\mathrm{M}(X)(-j))$. Thus, Conjecture 5.2 for $i_* \mathrm{M}(X)(-j)$ implies the Tate conjecture on the j-th Chow group of X. $\qquad \square$

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f-COHOMOLOGY AND MOTIVES OVER NUMBER RINGS

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Abstract

This paper is concerned with an interpretation of *f*-cohomology, a modification of motivic cohomology of motives over number fields, in terms of motives over number rings. Under standard assumptions on mixed motives over finite fields, number fields and number rings, we show that the two extant definitions of *f*-cohomology of mixed motives M_η over a number field *F*—one via ramification conditions on ℓ-adic realizations, another one via the *K*-theory of proper regular models—both agree with motivic cohomology of $\eta_{!*} M_\eta[1]$. Here $\eta_{!*}$ is constructed by a limiting process in terms of intermediate extension functors $j_{!*}$ defined in analogy to perverse sheaves.

The aim of this paper is to give an interpretation of *f*-cohomology in terms of motives over number rings. The notion of *f*-cohomology goes back to Beilinson who used it to formulate a conjecture about special *L*-values [6, 7]. The most classical example is what is now called $H^1_f(F, \mathbf{1}(1))$, *f*-cohomology of $\mathbf{1}(1)$, the motive of a number field *F*, twisted by one. This group is $\mathcal{O}_F^\times \otimes_{\mathbf{Z}} \mathbf{Q}$, as opposed to the full motivic cohomology $H^1(F, \mathbf{1}(1)) = F^\times \otimes \mathbf{Q}$. Together with the Dirichlet regulator, it explains the residue of the Dedekind zeta function $\zeta_F(s)$ at $s = 1$. This idea has been generalized in many steps and many ways, for example to the notion of Selmer complexes [36]. This work is concerned with the *f*-cohomology of a mixed motive M_η over *F*. There are two independent yet conjecturally equivalent ways to define $H^1_f(F, M_\eta) \subset H^1(F, M_\eta)$. We interpret the two definitions of *f*-cohomology as motivic cohomology of suitable motives over $\mathcal{O}_F$. This idea is due to Huber.

There are two approaches to $H^1_f(M_\eta)$. The first is due to Beilinson [8, Remark 4.0.1.b], Bloch and Kato [11, Conj. 5.3.] and Fontaine [20, 22]. It is given by picking elements in motivic cohomology acted on by the local Galois groups in a prescribed way (Definition 6.1, Definition 6.4, Definition 6.6). The second definition of $H^1_f(M_\eta)$, due to Beilinson [7, Section 8], applies to $M_\eta = h^{i-1}(X_\eta)(n)$, with X_η smooth and projective over *F*, $i - 2n < 0$. It is given by the image of *K*-theory of a regular proper model *X* of X_η (Definition 6.10). Such a model may not exist, but there is a unique meaningful extension of this definition to all Chow motives over *F* due to Scholl [44].

Our main results (Theorems 6.8, 6.11, 6.13) show that both definitions of $H^1_f(M_\eta)$ agree with $H^0(\eta_{!*}h^{i-1}(X_\eta, n)[1])$. Here $\eta_{!*}$ is a functor that attaches to any suitable mixed motive over F one over $\mathcal{O}_F$. It is defined by a limiting process using the intermediate extension $j_{!*}$ familiar from perverse sheaves [10] along open immersions $j : U \to \operatorname{Spec} \mathcal{O}_F$. Even to formulate such a definition, one has to rely on profound conjectures, namely the existence of mixed motives over (open subschemes of) $\operatorname{Spec} \mathcal{O}_F$. The proof of the main theorems also requires us to assume a number of properties related to weights of motives.

We point out that previously Jannsen and Scholl have shown the agreement of these two notions (in the case $M_\eta = h^i(X_\eta)(n)$, X_η/F smooth and proper) under weaker hypotheses than the ones considered here [42]. Also Scholl unconditionally proved the agreement for products of smooth projective curves over F (*op. cit.*). Our motivation for studying and employing this stronger set of assumptions about motives lies in an application to special L-values conjectures [40]. Very briefly, Beilinson's conjecture concerning special L-values for mixed motives M_η over $\mathbf{Q}$ has f-cohomology as motivic input. L-functions of such motives can be generalized to motives over $\mathbf{Z}$ such that the classical L-function of M_η agrees with the L-function (over $\mathbf{Z}$) of $\eta_{!*}M_\eta[1]$. Thereby the L-function and the motivic data in Beilinson's conjecture belong to the same motive over $\mathbf{Z}$, thus giving content to a more general conjecture about special L-values for motives over $\mathbf{Z}$. In this light it is noteworthy that $H^0(\eta_{!*}h^{2n-1}(X_\eta, n)[1])$ identifies with the group that occurs in the part of Beilinson's conjecture that describes special values at the central point.

The contents of the paper are as follows: Section 1 is the basis of the remainder; it lists a number of axioms on triangulated categories of motives. Such categories $\mathbf{DM}_{\mathrm{gm}}(S)$ have been constructed by Voevodsky [45] and Hanamura [24] (over fields) and Levine [33] (over bases S over a field). The various approaches are known to be (anti-)equivalent, at least for rational coefficients [33, Section VI.2.5], [12, Section 4]. Over more general bases S, the category $\mathbf{DM}(S)$ has been constructed by Ivorra [30] and Cisinski and Déglise [13]. We sum up the properties of this construction by specifying a number of axioms concerning triangulated categories of motives that will be used in the sequel. They are concerned with the "core" behavior of $\mathbf{DM}(S)$, that is: functoriality, compacity, the monoidal structure and the relation to algebraic K-theory, as well as localization, purity, base-change and resolution of singularities. We work with motives with rational coefficients only, since this is sufficient for all our purposes. We use a contravariant notation for motives, that is to say the functor that maps any scheme X to its motive $\mathrm{M}(X)$ shall be contravariant. This is in line with most pre-Voevodsky papers.

Section 2 is a very brief reminder on realizations. The existence of various realizations, due to Huber and Ivorra [25, 27, 30], is pinning down the intuition that motives should be universal among (reasonable) cohomology theories.

After Section 3, a brief intermezzo on perverse ℓ-adic sheaves over $\mathcal{O}_F$, Section 4 spells out a number of conjectural properties (also called axioms in the sequel) of $\mathbf{DM}_{\mathrm{gm}}(S)$, where S is either a finite field $\mathbf{F}_{\mathfrak{p}}$, a number field F or a

number ring $\mathcal{O}_F$. The first group of these properties centers around the existence of a category of mixed motives $\mathbf{MM}(S)$, which is to be the heart of the so-called motivic t-structure. The link between mixed motives over $\mathcal{O}_F$ and $\mathbf{F}_\mathfrak{p}$ or F is axiomatized by mimicking the exactness properties familiar from perverse sheaves (Axiom 4.2). A key requirement on mixed motives is that the realization functors on motives should be exact (Axiom 4.5). For the ℓ-adic realization over Spec $\mathcal{O}_F[1/\ell]$, this requires a notion of perverse sheaves over that base (Section 3). Another important conjectural facet of mixed motives are weights. Weights are an additional structure encountered in both Hodge structures and ℓ-adic cohomology of algebraic varieties over finite fields, both due to Deligne [16, 17]. They are important in that morphisms between Hodge structures or ℓ-adic cohomology groups are known to be strictly compatible with weights, moreover, they are respected to a certain extent by smooth maps and proper maps. It is commonly assumed that this should be the case for mixed motives, too. We show in a separate work that the t-structure axioms and the needed weight properties hold in the triangulated subcategory $\mathbf{DATM}(\mathcal{O}_F) \subset \mathbf{DM}_{\mathrm{gm}}(\mathcal{O}_F)$ of Artin-Tate motives (as far as they are applicable) [41].

The remaining two sections assume the validity of the axiomatic framework set up so far. The first key notion in Section 5 is the intermediate extension $j_{!*}M$ of a mixed motive M along some open embedding j inside Spec $\mathcal{O}_F$. This is done as in the case of perverse sheaves, due to Beilinson, Bernstein and Deligne [10]. Quite generally, much of this paper is built on the idea that the abstract properties of mixed perverse sheaves (should) give a good model for mixed motives over number rings. Next we develop a notion of smooth motives, which is an analog of lisse étale sheaves. This is needed to use a limiting technique to get the extension functor $\eta_{!*}$ that extends motives over F to ones over $\mathcal{O}_F$. Finally, we apply the axiom on the exactness of ℓ-adic realization to show that intermediate extensions commute with the realization functors. This will be a stepstone in a separate work on L-functions of motives [40].

Section 6 gives the comparison theorems on f-cohomology mentioned above. The two definitions of f-cohomology being quite different, the proofs of the comparison statements are different, too: the first is essentially based on the Hochschild-Serre spectral sequence. The crystalline case of that definition of f-cohomology is disregarded throughout. The second proof is a purely formal, if occasionally intricate bookkeeping of cohomological degrees and weights.

The problem of finding a motivic interpretation of terms such as $\mathrm{H}^1_f(M_\eta)$ underlying the formulation of Beilinson's conjecture has been studied by Scholl [43, 44, 42], who develops an abelian category $\mathbf{MM}(F/\mathcal{O}_F)$ of mixed motives over $\mathcal{O}_F$ by taking mixed motives over F, and imposing additional non-ramification conditions. Conjecturally, the group $\mathrm{Ext}^i_{\mathbf{MM}(F/\mathcal{O}_F)}(\mathbf{1}, \mathrm{h}^i(X_\eta, n))$ for X_η/F smooth and projective, $i = 0, 1$, agrees with what amounts to $\mathrm{H}^{i-1}(\eta_{!*}\mathrm{h}^{2n-1}(X_\eta, n)[1])$.

No originality is claimed for Sections 1, 2, and 4, except perhaps for the formulation of the relation of mixed motives over $\mathcal{O}_F$ and F and the residue fields $\mathbf{F}_\mathfrak{p}$, which however is a natural and immediate translation of the theory of perverse sheaves. I would like to thank Denis-Charles Cisinski and Frédéric

Déglise for communicating to me their work on $\mathbf{DM}_{\mathrm{gm}}(S)$ over general bases [13] and Baptiste Morin for explaining me a point in étale cohomology. Most of all, I gratefully acknowledge Annette Huber's advice in writing my thesis, of which this paper is a part.

1. Geometric motives

Throughout this paper, F is a number field, $\mathscr{O}_F$ its ring of integers, $\mathfrak{p}$ stands for a place of F. For finite places, the residue field is denoted $\mathbf{F}_{\mathfrak{p}}$. By scheme we mean a Noetherian separated scheme. Actually, it suffices to think of schemes of finite type over one of the rings just mentioned. In this section S denotes a fixed base scheme.

This section is setting up a number of axioms describing a triangulated category $\mathbf{DM}_{\mathrm{gm}}(S)$ of geometric motives over S. They will be used throughout this work. As pointed out in the introduction, the material of this section is due to Cisinski and Déglise [13], who build such a category of motives using Ayoub's base change formalism [4].

> AXIOM 1.1 (Motivic complexes and functoriality).
> · *There is a triangulated $\mathbf{Q}$-linear category $\mathbf{DM}(S)$. It is called category of* motivic complexes *over S (with rational coefficients). It has all limits and colimits.*
> · (*Tensor structure*) *The category $\mathbf{DM}(S)$ is a triangulated symmetric mono-idal category (see e.g. [33, Part 2, II.2.1.3]). Tensor products commute with direct sums. The unit of the tensor structure is denoted $\mathbf{1}_S$ or $\mathbf{1}$. Also, there are internal* Hom-*objects in* $\mathbf{DM}$, *denoted* Hom. *The dual $M^{\vee}$ of an object $M \in \mathbf{DM}(S)$ is defined by $M^{\vee} := \underline{\mathrm{Hom}}(M, \mathbf{1})$.*
> · *For any map $f : X \rightarrow Y$ of schemes, there are pairs of adjoint functors*
>
> (1) $f^{*} : \mathbf{DM}(Y) \leftrightarrows \mathbf{DM}(X) : f_{*}$
>
> *such that $f^{*}\mathbf{1}_Y = \mathbf{1}_X$ and, if f is quasi-projective,*
>
> $$f_{!} : \mathbf{DM}(X) \leftrightarrows \mathbf{DM}(Y) : f^{!}.$$

The existence of $f_{!}$ and $f^{!}$ is restricted to quasi-projective maps since the abstract construction of these functors in Ayoub's work [4, Section 1.6.5], on which Cisinski's and Déglise's construction of motives over general bases [13] relies, has a similar restriction.

Recall that an object X in a triangulated category $\mathscr{T}$ closed under arbitrary direct sums is *compact* if $\mathrm{Hom}(X, -)$ commutes with direct sums. The sub-category of $\mathscr{T}$ of compact objects is triangulated and closed under direct summands (a.k.a. a *thick subcategory*) [35, Lemma 4.2.4]. The category $\mathscr{T}$ is called *compactly generated* if the smallest triangulated subcategory closed under arbitrary sums containing the compact objects is the whole category $\mathscr{T}$.

AXIOM 1.2 (Compact objects). *The motive $\mathbf{1} \in \mathbf{DM}(S)$ is compact. The functors f^* and $f^!$, whenever defined, and $\otimes$ and $\underline{\mathrm{Hom}}$ preserve compact objects. The same is true for f_* and $f_!$ if f is of finite type. The canonical map $M \to (M^\vee)^\vee$ is an isomorphism for any compact object M.*

DEFINITION 1.3. The subcategory of compact objects of $\mathbf{DM}(S)$ is denoted $\mathbf{DM}_{\mathrm{gm}}(S)$ and called the category of *geometric motives* over S.

For any map $f : X \to S$ of finite type, the object $\mathrm{M}_S(X) := \mathrm{M}(X) := f_* f^* \mathbf{1} \in \mathbf{DM}_{\mathrm{gm}}(S)$ is called the *motive* of X over S. By adjunction, M is a contravariant functor from schemes of finite type over S to $\mathbf{DM}_{\mathrm{gm}}(S)$. For any quasi-projective $f : X \to S$, the *motive with compact support* of X, $\mathrm{M}_{\mathrm{c}}(X)$, is defined as $f_! f^* \mathbf{1} \in \mathbf{DM}_{\mathrm{gm}}(S)$.

The smallest thick subcategory of $\mathbf{DM}(S)$ containing the image of M is denoted $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S)$ and called the category of *effective geometric motives*. The closure of that subcategory under all direct sums is called the category of *effective motives*, $\mathbf{DM}_{\mathrm{eff}}(S)$.

AXIOM 1.4 (Tensor product vs. fiber product). *The functor M is an additive tensor functor, i.e., maps disjoint unions of schemes over S to direct sums and fiber products of schemes over S to tensor products in $\mathbf{DM}_{\mathrm{gm}}(S)$.*

AXIOM 1.5 (Compact generation). *The categories $\mathbf{DM}(S)$ and $\mathbf{DM}_{\mathrm{eff}}(S)$ are compactly generated.*

The category $\mathbf{DM}(S)$, being closed under countable direct sums is pseudo-abelian [33, Lemma II.2.2.4.8.1], i.e., it contains kernels of projectors. In particular, the projector $\mathrm{M}(\mathbf{P}_S^1) \to \mathrm{M}(S) \to \mathrm{M}(\mathbf{P}_S^1)$ has a kernel K (the first map is induced by the projection onto the base, the second map stems from the rational point $0 \in \mathbf{P}_S^1$). The object

$$\mathbf{1}(-1) := K[2],$$

is called *Tate object* or Tate motive. The resulting decomposition $\mathrm{M}(\mathbf{P}_S^1) = \mathbf{1} \oplus \mathbf{1}(-1)[-2]$ implies $\mathbf{1}(-1) \in \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S)$.

AXIOM 1.6 (Cancellation and Effectivity). *In $\mathbf{DM}_{\mathrm{gm}}(S)$ (and thus in $\mathbf{DM}(S)$), the Tate object $\mathbf{1}(-1)$ has a tensor-inverse denoted $\mathbf{1}(1)$. For any $M \in \mathbf{DM}(S)$, $n \in \mathbf{Z}$, set $M(n) := M \otimes \mathbf{1}(1)^{\otimes n}$. Then there is a canonical isomorphism called* cancellation *isomorphism ($n \in \mathbf{Z}$, $M, N \in \mathbf{DM}(S)$):*

$$\mathrm{Hom}_{\mathbf{DM}(S)}(M, N) \cong \mathrm{Hom}_{\mathbf{DM}(S)}(M(n), N(n)).$$

The smallest tensor subcategory of $\mathbf{DM}_{\mathrm{gm}}(S)$ that contains $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S)$ and $\mathbf{1}(1)$ is $\mathbf{DM}_{\mathrm{gm}}(S)$. In other words, $\mathbf{DM}_{\mathrm{gm}}(S)$ is obtained from $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S)$ by tensor-inverting $\mathbf{1}(-1)$.

Definition 1.7. Let M be any geometric motive over S. We write $\mathrm{H}^i(M) := \mathrm{H}^i(S, M) := \mathrm{Hom}_{\mathbf{DM}(S)}(\mathbf{1}, M[i])$. For $M = \mathrm{M}(X)(n)$ for any X over S we also write $\mathrm{H}^i(X, n) := \mathrm{H}^i(\mathrm{M}(X)(n)) = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(S)}(\mathbf{1}, \mathrm{M}(X)(n)[i]) \overset{(1)}{=} \mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(X)}(\mathbf{1}, \mathbf{1}(n)[i])$. This is called *motivic cohomology* of M and X, respectively.

Axiom 1.8 (Motivic cohomology vs. K-theory). *For any regular scheme X, there is an isomorphism* $\mathrm{H}^i(X, n) \cong K_{2n-i}(X)_{\mathbf{Q}}^{(n)}$, *where the right hand term denotes the Adams eigenspace of algebraic K-theory tensored with* $\mathbf{Q}$ [39].

This is a key property of motives, since algebraic K-theory is a universal cohomology theory in the sense that Chern characters map from algebraic K-theory to any other (reasonable) cohomology theory of algebraic varieties [23]. For S a perfect field, this axiom is given by [45, Prop. 4.2.9] and its non-effective analogue. See also [33, Theorem I.III.3.6.12.].

Recall Grothendieck's category of pure motives $\mathbf{M}_{\sim}(K)$ with respect to an adequate equivalence relation $\sim$, see e.g. [3, Section 4]. For rational equivalence they are also called Chow motives, since, for any smooth projective variety X over a field K,

$$(2) \qquad\qquad \mathrm{Hom}_{\mathbf{M}_{\mathrm{rat}}(K)}(\mathbf{1}(-n), h(X)) = \mathrm{CH}^n(X),$$

where $h(X)$ denotes the Chow motive of X and the right hand term is the *Chow group* of cycles of codimension n in X. This way, the above axiom models the fact [45, 2.1.4] that Chow motives are a full subcategory of $\mathbf{DM}_{\mathrm{gm}}(K)$. Under the embedding $\mathbf{M}_{\mathrm{rat}}(K) \subset \mathbf{DM}_{\mathrm{gm}}(K)$, $h(X, n)$ maps to $\mathrm{M}(X)(n)[2n]$.

Remark 1.9. We do not need to assume *expressis verbis* homotopy invariance (i.e., $\mathbf{1} \overset{\cong}{\to} pr_* pr^* \mathbf{1} \in \mathbf{DM}_{\mathrm{gm}}(S)$ for $pr : S \times \mathbf{A}^1 \to S$) nor the projective bundle formula [45, Prop. 3.5.1]. (Note, however, that K'-theory does have such properties.)

Axiom 1.10 (Localization). *Let $i : Z \to S$ be any closed immersion and $j : V \to S$ the open complement. The adjointness maps give rise to the following distinguished triangles in* $\mathbf{DM}(S)$:

$$j_! j^* \to \mathrm{id} \to i_* i^*,$$
$$i_* i^! \to \mathrm{id} \to j_* j^*.$$

(*In particular, $f_* f^* \cong \mathrm{id}$, where $f : X_{\mathrm{red}} \to X$ denotes the canonical map of the reduced subscheme structure.*) *In addition, one has $j^* j_* = \mathrm{id}$ and $i^* i_* = \mathrm{id}$, equivalently $j^* i_* = i^* j_! = 0$.*

Axiom 1.11 (Purity and base change).
· *For any quasi-projective map f, there is a functorial transformation of functors $f_! \to f_*$. It is an isomorphism if f is projective.*

- (Relative purity): *If f is quasi-projective and smooth of constant relative dimension d, there is a functorial (in f) isomorphism $f^! \cong f^*(d)[2d]$.*
- (Absolute purity): *If $i : Z \to U$ is a closed immersion of codimension c of two regular schemes Z and U, there is a natural isomorphism $i^! \mathbf{1} \cong \mathbf{1}(-c)[-2c]$.*
- (Base change): *For any two quasi-projective maps f and g let f' and g' denote the pullback maps*:

$$
(3) \qquad
\begin{array}{ccc}
X' \times_X Y & \xrightarrow{\;g'\;} & Y \\
\downarrow{f'} & & \downarrow{f} \\
X' & \xrightarrow{\;g\;} & X
\end{array}
$$

Then there is canonical isomorphism of functors

$$
f^* g_! \xrightarrow{\;\cong\;} g'_! f'^*.
$$

This axiom is proven by Cisinski & Déglise using Ayoub's general base change formalism. See in particular [4, 1.4.11, 12] for the construction of the base change map. See also [33, Theorem I.I.2.4.9] for a similar statement in Levine's category of motives.

DEFINITION 1.12. Let $f : S \to \mathrm{Spec}\,\mathbf{Z}$ be the structural map. Assume f is quasi-projective. Then $D(M) := \underline{\mathrm{Hom}}(M, f^! \mathbf{1}(1)[2])$ is called *Verdier dual* of M.

By the preceding axioms, D induces a contravariant endofunctor of $\mathbf{DM}_{\mathrm{gm}}(S)$. The shift and twist in the definition is motivated as follows: given some complex analytic space X, the Verdier dual of a sheaf $\mathscr{F}$ on X is defined by

$$
D(\mathscr{F}) := \underline{\mathrm{RHom}}_{\mathbf{D}(\mathbf{Shv}(X))}(\mathscr{F}, f^! \mathbf{Z}),
$$

where f denotes the projection to a point, see e.g. [29, Ch. VI]. When X is smooth of dimension d, one has $f^! \mathbf{Z} = f^* \mathbf{Z}(d)[2d] = \mathbf{Z}(d)[2d]$. A similar fact holds for ℓ-adic sheaves (see e.g. [31, Section II.7–8]). The above definition mimics this situation insofar as $\mathrm{Spec}\,\mathbf{Z}$ is seen as an analogue of a smooth affine curve.

Let us give a number of consequences of the preceding axioms, in particular purity, base change and localization: in (3), suppose that f is smooth and $g : X' \to X$ is a codimension one closed immersion between regular schemes. Then there is a canonical isomorphism

$$
(4) \qquad g^! \mathrm{M}_X(Y) = \mathrm{M}_{X'}(X' \times_X Y)(-1)[-2].
$$

Let $Z \subset X$ be a closed immersion of quasiprojective schemes over S. Then there is a distinguished triangle of motives with compact support

$$
\mathrm{M}_{\mathrm{c}}(Z) \to \mathrm{M}_{\mathrm{c}}(X) \to \mathrm{M}_{\mathrm{c}}(X \backslash Z).
$$

Let S be a scheme of equidimension d such that the structural map $f : S \to \operatorname{Spec} \mathbf{Z}$ factors as

$$S \xrightarrow{j} S' \xrightarrow{i} \mathbf{A}_{\mathbf{Z}}^n \quad \text{or} \quad \mathbf{P}_{\mathbf{Z}}^n \xrightarrow{p} \operatorname{Spec} \mathbf{Z},$$

where j is an open immersion into a regular scheme S', i is a closed immersion and p is the projection map. Then $f^! \mathbf{1} = \mathbf{1}(d-1)[2d-2]$, as one sees by applying relative purity to p and to j, and absolute purity to i. In particular, the Verdier duality functor on any open subscheme S of $\operatorname{Spec} \mathcal{O}_F$ is given by $D_{\mathbf{DM}_{\mathrm{gm}}(S)}(?) = \underline{\operatorname{Hom}}(?, \mathbf{1}(1)[2])$ while on $\mathbf{DM}_{\mathrm{gm}}(\mathbf{F}_{\mathfrak{p}})$ it is given by $\underline{\operatorname{Hom}}(?, \mathbf{1}) = ?^{\vee}$.

AXIOM 1.13 (Verdier dual). *The Verdier dual functor D exchanges "!" and "$*$" throughout, e.g., there are natural isomorphisms $D(f^! M) \cong f^* D(M)$ for any quasi-projective map $f : X \to Y$ and $M \in \mathbf{DM}(Y)$ and similarly with $f_!$ and f_*.*

LEMMA 1.14. *Let S be such that $f^! \mathbf{1} = f^* \mathbf{1}(d)[2d]$ for some integer d, where $f : S \to \operatorname{Spec} \mathbf{Z}$ is the structural map. For example, S might be regular and affine or projective over $\mathbf{Z}$ (see above), or smooth over $\operatorname{Spec} \mathbf{Z}$ (purity). Then, for any compact object $M \in \mathbf{DM}_{\mathrm{gm}}(S)$, the canonical map $M \to D(D(M))$ is an isomorphism. This will be referred to as* reflexivity *of Verdier duality.*

Proof. By Axiom 1.5, it suffices to check it for $M = \pi_* \pi^* \mathbf{1}$, where $\pi : X \to S$ is some map of finite type. In this case it follows for adjointness reasons and the assumption. $\square$

AXIOM 1.15 (Resolution of singularities). *Let K be a field. As a triangulated additive tensor category (i.e., closed under triangles, arbitrary direct sums and tensor product), $\mathbf{DM}(K)$ is generated by $\mathbf{1}(-1)$ and all $\mathrm{M}(X)$, where X/K is a smooth projective variety.*

When S is an open subscheme of $\operatorname{Spec} \mathcal{O}_F$, the generators of $\mathbf{DM}(S)$ are $\mathbf{1}(-1)$, $i_{\mathfrak{p}_*} \mathrm{M}(X_{\mathfrak{p}})$, and $\mathrm{M}(X)$, instead, where $X_{\mathfrak{p}}$ is any projective and smooth variety over $\mathbf{F}_{\mathfrak{p}}$, $i_{\mathfrak{p}}$ denotes the immersion of any closed point $\mathbf{F}_{\mathfrak{p}}$ of S, and X is any regular, flat projective scheme over $\mathcal{O}_F$.

Consequently, the subcategories of compact objects $\mathbf{DM}_{\mathrm{gm}}(-)$ are generated as a thick tensor subcategory by the mentioned objects. In Voevodsky's theory of motives over a field of characteristic zero, this is [45, Section 4.1]. This uses Hironaka's resolution of singularities. Over a field of positive characteristic and number rings, one has to use de Jong's resolution result, see [28, Lemma B.4].

We also need a limit property of the generic point. Let S be an open subscheme of $\operatorname{Spec} \mathcal{O}_F$, let $\eta : \operatorname{Spec} F \to S$ be the generic point.

AXIOM 1.16 (Generic point). *Let M be any geometric motive over S. The natural maps $j_* j^* M \to \eta_* \eta^* M$ give rise to an isomorphism $\varinjlim j_* j^* M = \eta_* \eta^* M$,*

where the colimit is over all open subschemes $j : S' \to S$.   It induces a distinguished triangle in $\mathbf{DM}(S)$

$$(5) \qquad \bigoplus_{\mathfrak{p} \in S} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! M \to \mathrm{id} \to \eta_* \eta^* M,$$

where the sum runs over all closed points $\mathfrak{p} \in S$ *and* $i_{\mathfrak{p}}$ *is the closed immersion.*

2.   Realizations

One of the main interests in motives lies in the fact that they are explaining (or are supposed to explain) common phenomena in various cohomology theories. These cohomology functors are commonly referred to as *realization* functors. They typically have the form $\mathbf{DM}_{\mathrm{gm}}(S) \to \mathbf{D}^b(\mathscr{C})$, where $\mathscr{C}$ is an abelian category whose objects are amenable with the methods of (linear) algebra, such as finite-dimensional vector spaces or finite-dimensional continuous group representations or constructible sheaves.

For example, let ℓ be a prime and let S be either a field of characteristic different from ℓ or a scheme of finite type over $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$.   The ℓ-adic cohomology maps any scheme X of finite type over S to

$$\mathrm{R}\Gamma_\ell(X) := \mathrm{R}\pi_* \pi^* \mathbf{Q}_\ell \in \mathbf{D}_c^b(S, \mathbf{Q}_\ell),$$

where $\pi : X \to S$ is the structural map and the right hand category denotes the "derived" category of constructible $\mathbf{Q}_\ell$-sheaves on S (committing the standard abuse of notation, see e.g. [31, II.6., II.7.]).   This functor factors over the *ℓ-adic realization functor* ([27, p. 772], [30]) $\mathrm{R}\Gamma_\ell : \mathbf{DM}_{\mathrm{gm}}(S) \to \mathbf{D}_c^b(S, \mathbf{Q}_\ell)$.   When S is of finite type over $\mathbf{F}_\mathfrak{p}$, the realization functor actually maps to $\mathbf{D}_{c,m}^b(S, \overline{\mathbf{Q}_\ell})$, the full subcategory of complexes C in $\mathbf{D}_c^b(S, \overline{\mathbf{Q}_\ell})$ such that all $\mathrm{H}^n(C)$ are mixed sheaves [17, 1.2].

Further realization functors include Betti, de Rham and Hodge realization. See e.g. [27, 2.3.5].   The following axiom says (in particular) that the ℓ-adic realization of $\mathrm{M}(X)$ does give the ℓ-adic cohomology groups.

AxIOM 2.1 (Functoriality and realizations).   *The ℓ-adic realization functor commutes with the six Grothendieck functors f_*, $f_!$, $f^!$, f^*, $\otimes$ and* <u>Hom</u> *(where applicable).   For example, for any map $f : S' \to S$ and any geometric motive M over S':*

$$(f_* M)_\ell = f_*(M_\ell).$$

3.   Interlude: Perverse sheaves over number rings

This section is devoted to a modest extension of ℓ-adic perverse sheaves [10] to the situation where the base S is an open subscheme of $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$.   It is needed to formulate Axiom 4.5 for the ℓ-adic realization of motives over number rings.   This section may be considered a reformulation in "perverse language" of

the well-known duality for cohomology of the inertia group. In a nutshell, the theory of perverse sheaves on varieties over $\mathbf{F}_q$ stakes on relative purity, that is $f^!\mathbf{Z}_\ell = f^*\mathbf{Z}_\ell(n)[2n]$ for a smooth map f of relative dimension n. The analogous identity for a closed immersion $i : \operatorname{Spec} \mathbf{F}_\mathfrak{p} \to S$ reads

$$(6) \qquad\qquad i^!\mathbf{Z}_\ell = i^*\mathbf{Z}_\ell(-1)[-2].$$

It is a reformulation of well-known cohomological properties of the inertia group: $\mathrm{H}^1(I_\mathfrak{p}, V) = (V(-1))_{I_\mathfrak{p}}$ for any ℓ-adic module with continuous $I_\mathfrak{p}$-action $(\mathfrak{p} \nmid \ell)$. All higher group cohomologies of $I_\mathfrak{p}$ vanish.

Let $\mathbf{D}^{\mathrm{b}}(S, \mathbf{Z}_\ell)$ be the bounded "derived" category of $\mathbf{Z}_\ell$-sheaves on S as constructed by Ekedahl [19]. All following constructions can be done for $\mathbf{Q}_\ell$ instead of $\mathbf{Z}_\ell$, as well. We keep writing j_* for the total derived functor, commonly denoted $\mathrm{R}j_*$ etc. However, $\mathrm{R}^n j_*$ etc. keep their original meaning.

As in *loc. cit.*, see especially [2.2.10, 2.1.2, 2.1.3, 1.4.10][1], one first defines a notion of stratification, and secondly obtains a t-structure on the subcategory $\mathbf{D}^{\mathrm{b}}_{(\Sigma, L)}(S, \mathbf{Z}_\ell)$ that are constructible with respect to a given stratification $\Sigma = \{\Sigma_i\}$ and a set L of irreducible lisse sheaves on the strata. Thirdly, one takes the "limit" over the stratifications. The union of all $\mathbf{D}^{\mathrm{b}}_{(\Sigma, L)}(S, \mathbf{Z}_\ell)$ is the "derived" category $\mathbf{D}^{\mathrm{b}}_c(S, \mathbf{Z}_\ell)$ of constructible sheaves. In order to extend the t-structure on the subcategories to one on $\mathbf{D}^{\mathrm{b}}_c(S, \mathbf{Z}_\ell)$, one has to check that the inclusion $\mathbf{D}^{\mathrm{b}}_{(\Sigma', L')}(S, \mathbf{Z}_\ell) \to \mathbf{D}^{\mathrm{b}}_{(\Sigma, L)}(S, \mathbf{Z}_\ell)$ is t-exact for any refinement of stratifications. Here we employ a different argument. The proof of [2.1.14, 2.2.11] relies on relative purity for ℓ-adic sheaves [2, Exp. XVI, 3.7]. As in the proof of [2.1.14] we have to check the following: let $\Sigma_i \xrightarrow{a} \Sigma'_i \xrightarrow{b} S$ be the inclusions of some strata and let $C \in \mathbf{D}^{\mathrm{b}, \geq 0}_{(\Sigma', L')}(S, \mathbf{Z}_\ell)$. Then $C \in \mathbf{D}^{\mathrm{b}, \geq 0}_{(\Sigma, L)}(S, \mathbf{Z}_\ell)$. We can assume $\dim \Sigma_i = 0$, $\dim \Sigma'_i = 1$, since all other cases are clear. Thus, b is an open immersion. We may also assume for notational simplicity that $\Sigma_i = \operatorname{Spec} \mathbf{F}_\mathfrak{p}$. Let j be the complementary open immersion to a. By definition, $\mathrm{H}^n b^! C = b^! \mathrm{H}^n C = b^* \mathrm{H}^n C$ is locally constant and vanishes for $n < -1$. In the parlance of Galois modules this means that, viewed as a $\pi_1(\Sigma'_i)$-representation, the action of the inertia group $I_\mathfrak{p} \subset \pi_1(\Sigma'_i)$ on that sheaf is trivial. Thus

$$a^! \mathrm{H}^n b^* C = a^*(\mathrm{R}^1 j_* j^* \mathrm{H}^n b^* C)[-2] = \mathrm{H}^1(I_\mathfrak{p}, \mathrm{H}^n b^* C)[-2] = a^* \mathrm{H}^n b^* C(-1)[-2].$$

(We have used $\mathfrak{p} \nmid \ell$ at this point.) The spectral sequence

$$\mathrm{H}^{p-2} a^* \mathrm{H}^q b^! C(-1) = \mathrm{H}^p a^! \mathrm{H}^q b^! C \Rightarrow \mathrm{H}^n a^! b^! C$$

is such that the left hand term vanishes for $p \neq 2$ since a^* is exact w.r.t. the standard t-structure. It also vanishes for $q < -1$ by the above. Hence the right hand term vanishes for $n = p + q < 1$. A fortiori it vanishes for $n < -\dim \mathbf{F}_\mathfrak{p} = 0$.

---

[1] In the sequel, any reference in brackets refers to [10].

Objects in the heart of this t-structure on $\mathbf{D}_c^b(S, \mathbf{Z}_\ell)$ are called *perverse sheaves* on S. For example $\mathbf{Z}_\ell[1]$ and $i_* \mathbf{Z}_\ell$ for any immersion i of a closed point are perverse sheaves on S. The *Verdier dual* of any $\mathscr{C} \in \mathbf{D}_c^b(S, \mathbf{Z}_\ell)$ is defined by $D(C) := \underline{\mathrm{Hom}}(C, \mathbf{Z}_\ell(1)[2])$. As above, we have dropped "R" from the notation, so that this $\underline{\mathrm{Hom}}$ means what is usually denoted $\underline{\mathrm{RHom}}$.

LEMMA 3.1. *Let* $j : S' \to S$ *be an open immersion and* $i : Z \to S$ *a closed immersion. Let* $\eta : \mathrm{Spec}\, F \to S$ *be the generic point. Then* j_*, $j_!$, i_*, $\eta^*[-1]$, j^* *and* D *are* t-*exact, while* i^* ($i^!$) *is of cohomological amplitude* $[-1, 0]$ ($[0, 1]$), *in particular right-exact* (*left-exact, respectively*)*. Finally, the* t-*structure on* $\mathbf{D}_c^b(S, \mathbf{Z}_\ell)$ *is non-degenerate* [10, p. 32].

Proof. The only non-formal statement is the exactness of j_*. The corresponding precursor result [4.1.10] is a reformulation of [1, Th. 3.1., Exp. XIV], which says for any affine map $j : X \to Y$ over schemes over a field K, and any (honest) sheaf $\mathscr{F}$ which is torsion (prime to char K)

$$d(\mathrm{R}^q j_* \mathscr{F}) \leq d(\mathscr{F}) - q$$

where $d(\mathscr{G}) := \sup\{\dim \overline{\{x\}}, \mathscr{G}_{\bar{x}} \neq 0\}$ for any sheaf $\mathscr{G}$. In our situation, we are given a locally constant sheaf $\mathscr{F}$ on S' whose torsion is prime to all characteristics of S. The conclusion of the theorem also holds for j, as follows from the cohomological dimension of $I_{\mathfrak{p}}$, which is one. $\qquad\square$

Let $\mathscr{F}$ be any perverse sheaf on S'. Following [1.4.22], let the *intermediate extension* $j_{!*} \mathscr{F}$ be the image of the map $j_! \mathscr{F} \to j_* \mathscr{F}$ of perverse sheaves on S. As in [2.1.11] one sees that it can be calculated in terms of the good truncation with respect to the standard t-structure: $j_{!*} \mathscr{F} = \tau_{\leq -1}^{can} j_* \mathscr{F}$. If $\mathscr{F} = \mathscr{G}[1]$, where $\mathscr{G}$ is a lisse (honest) sheaf on S', this gives $(\mathrm{R}^0 j_* \mathscr{G})[1]$.

4. Mixed motives

Throughout this section, let $S = \mathrm{Spec}\, F$ or $\mathrm{Spec}\, \mathbf{F}_{\mathfrak{p}}$ or an open subscheme of $\mathrm{Spec}\, \mathscr{O}_F$.

This section formulates a number of axioms concerning weights and the motivic t-structure on triangulated categories of motives over S. In contrast to the axioms listed in Section 1, the axioms mentioned in this section are wide open, so it might be more appropriate to call them conjectures instead.

AXIOM 4.1 (Motivic t-structure and cohomological dimension). *The category of geometric motives* $\mathbf{DM}_{\mathrm{gm}}(S)$ *has a non-degenerate* t-*structure* [10, *Def.* 1.3.1] *called* motivic t-structure*. Its heart is denoted* $\mathbf{MM}(S)$*. Objects of* $\mathbf{MM}(S)$ *are called* mixed motives *over S.*

For any $M \in \mathbf{DM}_{\mathrm{gm}}(S)$*, there are* $a, b \in \mathbf{Z}$ *such that* $\tau_{\leq a} M = \tau_{\geq b} M = 0$*. Here and in the sequel,* $\tau_{\leq -}$ *and* $\tau_{\geq -}$ *denote the truncation functors with respect to the motivic* t-*structure.*

The cohomological dimension *of* $\mathbf{DM}_{\mathrm{gm}}(\mathbf{F}_{\mathfrak{p}})$ *and* $\mathbf{DM}_{\mathrm{gm}}(F)$ *is* 0 *and* 1, *respectively, in the sense that*

$$\mathrm{Hom}_{\mathbf{DM}(\mathbf{F}_{\mathfrak{p}})}(M, N[i]) = 0$$

for all mixed motives M, N *over* $\mathbf{F}_{\mathfrak{p}}$ *and* $i > 0$ *and similarly for mixed motives over* F *and* $i > 1$. *(For* $i < 0$ *the term vanishes by the* t*-structure axioms.)*

The t*-structures are such that over* $S = \mathrm{Spec}\, F$ *or* $\mathrm{Spec}\, \mathbf{F}_{\mathfrak{p}}$, $\mathbf{1} \in \mathbf{MM}(S)$, *while for an open subscheme* $S \subset \mathrm{Spec}\, \mathcal{O}_F$, $\mathbf{1}[1] \in \mathbf{MM}(S)$.

The existence of the motivic *t*-structure on $\mathbf{DM}_{\mathrm{gm}}(K)$ satisfying the axioms listed in this section is part of the general motivic conjectural framework, see e.g. [8, App. A], [3, Ch. 21]. The idea of building a triangulated category of motives and descending to mixed motives by means of a *t*-structure is due to Deligne. The existence of a motivic *t*-structure on $\mathbf{DM}_{\mathrm{gm}}(K)$ is only known in low dimensions: the subcategory of Artin motives, i.e., motives of zero-dimensional varieties, carries such a *t*-structure [45, Section 3.4.]. By *loc. cit.*, [37], the subcategory of $\mathbf{DM}_{\mathrm{gm}}(K)$ generated by motives of smooth varieties of dimension ≤ 1 is equivalent to the bounded derived category of 1-motives [16, Section 10] up to isogeny. Finally, if K is a field satisfying the Beilinson-Soulé vanishing conjecture, such as a finite field or a number field, the category of Artin-Tate motives over K enjoys a motivic *t*-structure [32, 46]. The results on Artin-Tate motives are generalized to bases S which are open subschemes of $\mathrm{Spec}\, \mathcal{O}_F$ in [41].

The conjecture about the cohomological dimension is due to Beilinson. A (fairly weak) evidence for this conjecture is the cohomological dimension of Tate motives over F and $\mathbf{F}_{\mathfrak{p}}$, which is one and zero, respectively. This follows from vanishing properties of K-theory of these fields.

The normalization in the last item is merely a matter of bookkeeping, but is motivated by similar shifts in perverse sheaves (Section 3). The existence of a motivic *t*-structure is not expected to hold for motives with integral coefficients.

We do not (need to) assume that the canonical functor $\mathbf{D}^{\mathrm{b}}(\mathbf{MM}(S)) \to \mathbf{DM}_{\mathrm{gm}}(S)$ is an equivalence of categories or, equivalently [9, Lemma 1.4.], $\mathrm{Ext}^i_{\mathbf{MM}(S)}(A, B) = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(S)}(A, B[i])$ for all mixed motives A and B.

AXIOM 4.2 (Exactness properties). *Let* $S \subset \mathrm{Spec}\, \mathcal{O}_F$ *be an open subscheme, let* $i : \mathrm{Spec}\, \mathbf{F}_{\mathfrak{p}} \to \mathrm{Spec}\, \mathcal{O}_F$ *be a closed point,* $j : U \to S$ *an open immersion and* $\eta : \mathrm{Spec}\, F \to S$ *the generic point.*

Then $j^* = j^!$, $\eta^*[-1]$, i_*, j_* *and* $j_!$ *are exact with respect to the motivic* t*-structures on the involved categories of geometric motives. Further,* i^* *is right-exact, more precisely it maps objects in cohomological degree* 0 *to degrees* $[-1, 0]$. *Dually,* $i^!$ *has cohomological amplitude* $[0, 1]$. *Verdier duality* D *is "anti-exact", i.e., maps objects in positive degrees to ones in negative degrees and vice versa.*

The axiom is motivated by the same exactness properties in the situation of perverse sheaves over $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$ (Section 3). The corresponding exactness

properties of the above functors on Artin-Tate motives, where the motivic t-structure is available, are established in [41].

DEFINITION 4.3. The cohomology functor with respect to the motivic t-structure on $\mathbf{DM}_{\mathrm{gm}}(S)$ is denoted ${}^{\mathrm{p}}\mathrm{H}^*$. For any scheme X/S, we write

$$\mathrm{h}^i(X,n) := {}^{\mathrm{p}}\mathrm{H}^i\mathbf{M}_S(X)(n).$$

AXIOM 4.4. *Let X_η/F be any smooth projective variety. Then numerical equivalence and homological equivalence* (*with respect to any Weil cohomology*) *agree on X_η.*

Let either S be a field and let C stand for the ℓ-adic realization (in case char $S \neq \ell$), Betti, de Rham or absolute Hodge realization or let $S \subset$ Spec $\mathcal{O}_F[1/\ell]$ be an open subscheme and let C be the ℓ-adic realization. We write $\mathrm{R}\Gamma_C : \mathbf{DM}_{\mathrm{gm}}(S) \to \mathbf{D}^{\mathrm{b}}(\mathscr{C})$ for the realization functor, where $\mathbf{D}^{\mathrm{b}}(\mathscr{C})$ is understood as a placeholder of the target category of C. (We abuse the notation insofar as that target category is not a derived category in the strict sense when C is the ℓ-adic realization.) For all realizations over a field, this category is endowed with the usual t-structure on the derived category of an exact category, e.g. on $\mathbf{D}^{\mathrm{b}}_c(K,\mathbf{Q}_\ell)$ for ℓ-adic realization. When C is the ℓ-adic realization over an open subscheme S of Spec $\mathcal{O}_F[1/\ell]$, we take the perverse t-structure on $\mathbf{D}^{\mathrm{b}}_c(S,\mathbf{Q}_\ell)$ defined in Section 3. Using this, we have the following axiom:

AXIOM 4.5 (Exactness of realization functors). *Realization functors $\mathrm{R}\Gamma_C$ are exact with respect to the motivic t-structure on $\mathbf{DM}_{\mathrm{gm}}(S)$. Equivalently, as the t-structure on $\mathbf{D}^{\mathrm{b}}(\mathscr{C})$ is non-degenerate, $\mathrm{R}\Gamma_C({}^{\mathrm{p}}\mathrm{H}^0 M) = {}^{\mathrm{p}}\mathrm{H}^0\mathrm{R}\Gamma_C(M)$ for any geometric motive M over S. On the left, ${}^{\mathrm{p}}\mathrm{H}^0$ denotes the cohomology functor belonging to the motivic t-structure on $\mathbf{DM}_{\mathrm{gm}}(S)$, while on the right hand side it is the one belonging to the t-structure on $\mathbf{D}^{\mathrm{b}}(\mathscr{C})$.*

This axiom is, if fairly loosely, motivated by a similar fact in the theory of mixed Hodge modules: let X be any complex algebraic variety. Then, under the faithful "forgetful functor" from the derived category of mixed Hodge modules to the derived category of constructible sheaves with rational coefficients

$$\mathbf{D}^{\mathrm{b}}(\mathbf{MHM}(X)) \to \mathbf{D}^{\mathrm{b}}_c(X,\mathbf{Q})$$

the category $\mathbf{MHM}(X)$ corresponds to perverse sheaves on X.

Recall that in an abelian category $\mathscr{C}$, a morphism $f : (X,W^*) \to (Y,W^*)$ between filtered objects is called *strict* if $f(W^nX) = f(X) \cap W^nY$ for all n.

AXIOM 4.6 (Weights). *Any mixed motive M over S has a functorial finite exhaustive separated filtration W_*M called* weight filtration, *i.e., a sequence of subobjects in the abelian category $\mathbf{MM}(S)$*

$$0 = W_aM \subset W_{a+1}M \subset \cdots \subset W_bM = M.$$

Any morphism between mixed motives is strict with respect to the weight filtration.

Tensoring any motive with $\mathbf{1}(n)$ shifts its weights by $-2n$.

Let $\mathrm{R}\Gamma_C : \mathbf{DM}_{\mathrm{gm}}(S) \to \mathbf{D}^{\mathrm{b}}(\mathscr{C})$ be any realization functor that has a notion of weights (such as the ℓ-adic realization when $S = \mathrm{Spec}\,\mathbf{F}_{\mathfrak{p}}$ or the Hodge realization when $S = \mathrm{Spec}\,\mathbf{Q}$). Then

$$\mathrm{gr}_n^W \, \mathrm{R}\Gamma_C(M) = \mathrm{R}\Gamma_C(\mathrm{gr}_n^W \, M)$$

for any mixed motive M over S.

DEFINITION 4.7. For any $M \in \mathbf{MM}(S)$, we write $\mathrm{wt}(M)$ for the (finite) set of integers n such that $\mathrm{gr}_n^W M \neq 0$. For $M \in \mathbf{DM}_{\mathrm{gm}}(S)$, define $\mathrm{wt}(M) := \bigcup_{i \in \mathbf{Z}} \mathrm{wt}({}^{\mathrm{p}}\mathrm{H}^i(M)) - i$.

AXIOM 4.8 (Preservation of weights). *Let $f : X \to S$ be a quasi-projective map. Then the functors $f_! f^*$ preserve negativity of weights, i.e., given a geometric motive M over S with weights ≤ 0, $f_! f^* M$ also has weights ≤ 0. Dually, $f_* f^!$ preserves positive weights.*

In the particular case $S \subset \mathrm{Spec}\,\mathscr{O}_F$ (open), let $j : U \to S$ and $\eta : \mathrm{Spec}\,F \to S$ be an open immersion into S and the generic point of S, respectively. Let $i : \mathrm{Spec}\,\mathbf{F}_{\mathfrak{p}} \to S$ be a closed point. Then, i^* and $j_!$ preserve negativity of weights and dually for $i^!$ and j_*. Finally, j^* and η^* both preserve both positivity and negativity of weights.

The preceding weight axioms are motivated by the very same properties of ℓ-adic perverse sheaves on schemes over $\mathbf{C}$ or finite fields [10, 5.1.14], number fields [26] as well as Hodge structures [16, Th. 8.2.4] and Hodge modules (see [38, Chapter 14.1] for a synopsis). In these settings, actually $f_!$ and f^* preserve negative weights, but we do not need weights for motives over more general bases than the ones above. The weight formalism we require is stronger than the one provided by the differential-graded interpretation of $\mathbf{DM}_{\mathrm{gm}}$ over a field [12] or [5, 6.7.4].

Remark 4.9. Over $S = \mathrm{Spec}\,\mathscr{O}_F$, we actually only use the following weight properties: for any $M \in \mathbf{DM}_{\mathrm{gm}}(S)$, the interval $\mathrm{wt}(M)$ containing the weights of M satisfies the following two properties: first, it is compatible under functoriality as in 4.8 and, second, $j_{!*}$ preserves weights of pure smooth motives. (See Definitions 5.3, 5.7 for these two notions and the proof of Theorem 6.11.)

Example 4.10. For any projective (smooth) scheme X of finite type over S, the weights of $\mathrm{h}^i(X)(n)$ are $\leq i - 2n$ ($\geq i - 2n$, respectively).

AXIOM 4.11 (Mixed vs. pure motives). *For any field K, the subcategory of pure objects in $\mathbf{MM}(K)$ identifies with $\mathbf{M}_{\mathrm{num}}(K)$, the category of numerical pure motives over K.*

By Axiom 4.1, there is an exact sequence

$$0 \to \mathrm{H}^1(\mathrm{h}^{2n-1}(X_\eta, n)) \to \mathrm{H}^{2n}(X_\eta, n) \to \mathrm{H}^0(\mathrm{h}^{2n}(X_\eta, n)) \to 0.$$

By Axioms 4.4 and 4.11, it reads

$$(7) \qquad 0 \to \mathrm{CH}^n(X_\eta)_{\mathbf{Q},\mathrm{hom}} \to \mathrm{CH}^n(X_\eta)_{\mathbf{Q}} \to \mathrm{CH}^n(X_\eta)_{\mathbf{Q}}/\mathrm{hom} \to 0.$$

Here $\mathrm{CH}^m(X_\eta)_{\mathbf{Q},\mathrm{hom}}$ and $\mathrm{CH}^m(X_\eta)_{\mathbf{Q}}/\mathrm{hom}$ are by definition the kernel and the image (seen as a quotient of the Chow group) of the cycle class map from the m-th Chow group to ℓ-adic cohomology of X_η, $\mathrm{CH}^m(X_\eta)_{\mathbf{Q}} \to \mathrm{H}^{2m}(X_\eta, \mathbf{Q}_\ell(m))$ [34, VI.9].

As a consequence of the weight filtration, every mixed motive is obtained in finitely many steps by taking extensions of motives in $\mathbf{M}_{\mathrm{num}}(K)$. Recall also that for any $X/\mathbf{F}_q$ which is smooth and projective the spectral sequence

$$\mathrm{Ext}^p_{\mathbf{MM}(\mathbf{F}_q)}(\mathbf{1}, \mathrm{h}^q(X)) \Rightarrow \mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(\mathbf{F}_q)}(\mathbf{1}, \mathrm{M}(X)[p+q])$$

degenerates by Axiom 4.1 and yields an agreement

$$(8) \qquad \mathrm{CH}^q(X)/\mathrm{num} = \mathrm{Hom}_{\mathbf{M}_{\mathrm{num}}(\mathbf{F}_q)}(\mathbf{1}, \mathrm{h}^q_{\mathrm{num}}(X))$$

$$\overset{4.11}{=} \mathrm{Hom}_{\mathbf{MM}(\mathbf{F}_q)}(\mathbf{1}, \mathrm{h}^q(X)) = \mathrm{CH}^q(X),$$

i.e., the agreement of rational and numerical equivalence (and thus, of all adequate equivalence relations).

Remark 4.12. Recall that the agreement of numerical and homological equivalence on all smooth projective varieties over F implies the motivic hard Lefschetz [3, 5.4.2.1]: for such a variety X_η/F of constant dimension d_η, let $i \le d_\eta$ and a any integer. Then, taking the $(d_\eta - i)$-fold cup product with the cycle class of a hyperplane section with respect to an embedding of X_η into some projective space over F yields an isomorphism ("hard Lefschetz isomorphism")

$$(9) \qquad \mathrm{h}^i(X_\eta, a) \overset{\cong}{\to} \mathrm{h}^{2d_\eta - i}(X_\eta, d_\eta - i + a).$$

The hard Lefschetz is known to imply a non-canonical decomposition [18]

$$\mathrm{M}(X_\eta) \cong \bigoplus \mathrm{h}^n(X_\eta)[-n].$$

We need to assume the following generalization of this. It will be used in Lemma 5.10, which in turn is crucial in Section 6. Note that the index shift in the second part is due to the normalization in Axiom 4.1: for $S = \mathrm{Spec}\,\mathcal{O}_F$ and a closed point i as above, take for example $X = S$, $\mathrm{M}(S) = \mathbf{1} = \mathrm{h}^1(S)[-1]$ (sic) and $i^*\mathrm{M}(S) = \mathbf{1}_{\mathbf{F}_{\mathfrak{p}}} = \mathrm{h}^0(\mathrm{Spec}\,\mathbf{F}_{\mathfrak{p}})$.

AXIOM 4.13 (Decomposition of smooth projective varieties). *Let X/S be smooth and projective. In $\mathbf{DM}_{\mathrm{gm}}(S)$, there is a non-canonical isomorphism*

$$\phi_X : \mathrm{M}(X) \cong \bigoplus_n \mathrm{h}^n(X)[-n].$$

For open subschemes $S \subset \mathrm{Spec} \, \mathcal{O}_F$, this isomorphism is compatible with pullbacks along all closed points $i : \mathrm{Spec} \, \mathbf{F}_{\mathfrak{p}} \to S$ in the following sense: let $X_{\mathfrak{p}}$ be the fiber of X over $\mathbf{F}_{\mathfrak{p}}$, and let ψ be the isomorphism making the following diagram commutative. Its left hand isomorphism is an instance of base change.

$$
\begin{array}{ccc}
i^* \mathrm{M}(X) & \xrightarrow{\;i^*\phi_X\;} & \bigoplus_{n} i^* \mathrm{h}^n(X)[-n] \\
\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \psi} \\
\mathrm{M}(X_{\mathfrak{p}}) & \xrightarrow{\;\phi_{X_{\mathfrak{p}}}\;} & \bigoplus_{m} \mathrm{h}^m(X_{\mathfrak{p}})[-m]
\end{array}
$$

Then ψ respects the direct summands, i.e., induces isomorphisms

$$
i^* \mathrm{h}^n(X)[-n] \cong \mathrm{h}^{n-1}(X_{\mathfrak{p}})[-n+1].
$$

5. Motives over number rings

In the following sections we assume the axioms of Sections 1, 2, and 4. Unless explicitly mentioned otherwise, let S be an open subscheme of $\mathrm{Spec} \, \mathcal{O}_F$, let $i : \mathrm{Spec} \, \mathbf{F}_{\mathfrak{p}} \to \mathrm{Spec} \, \mathcal{O}_F$ be a closed point, $j : S' \to S$ an open subscheme and $\eta : \mathrm{Spec} \, F \to S$ the generic point.

This section derives a number of basic results about motives over S from the axioms spelled out above. We define and study the *intermediate extension* $j_{!*} : \mathbf{MM}(S') \to \mathbf{MM}(S)$ in analogy to perverse sheaves (Definition 5.3). An "explicit" set of generators of $\mathbf{DM}_{\mathrm{gm}}(S)$ (Proposition 5.6) is obtained using $j_{!*}$. We introduce a notion of *smooth motives* (Definition 5.7), which should be thought of as analogs of lisse sheaves. Using this notion, we extend the intermediate extension to a functor $\eta_{!*}$ spreading out motives over F with a certain smoothness property to motives over S, cf. Definition 5.13. This functor will be the main technical tool in dealing with f-cohomology in Section 6. In Lemma 5.15 we express the ℓ-adic realization of motives of the form $j_{!*}M$ in sheaf-theoretic language.

5.1. Cohomological dimension
The following is an immediate consequence of Axiom 4.2:

LEMMA 5.1. *For any scheme X over S we have $\eta^*[-1]\mathrm{h}^i(X,n) = \mathrm{h}^{i-1}(X \times_S F, n)$.*

The following lemma parallels (and is a consequence of) Axiom 4.1.

LEMMA 5.2. *The cohomological dimension of $\mathbf{DM}_{\mathrm{gm}}(S)$ is two, that is to say, for any two mixed motives M, N over S,*

$$
\mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(S)}(M, N[i]) = 0
$$

for all $i > 2$. In particular $\mathrm{H}^i(M)$ vanishes for $|i| > 1$.

Proof. Apply $\mathrm{Hom}(M,-)$ to the localization triangle $\bigoplus_{\mathfrak{p} \in S} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! N \to N \to$ $\eta_* \eta^* N$ of Axiom 1.16, where $i_{\mathfrak{p}}$ are the immersions of the closed points of S. The terms adjacent to $\mathrm{Hom}(M, N[i])$ are $\mathrm{Hom}(M, \bigoplus_{\mathfrak{p}} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! N[i]) =$ $\bigoplus_{\mathfrak{p}} \mathrm{Hom}(i_{\mathfrak{p}}^* M, i_{\mathfrak{p}}^! N[i])$ (as M is compact) and $\mathrm{Hom}(M, \eta_* \eta^* N[i]) =$ $\mathrm{Hom}(\eta^* M, \eta^* N[i])$. The latter term vanishes for $i > 1$ since $\eta^*[-1]$ is exact and the cohomological dimension of $\mathbf{DM}_{\mathrm{gm}}(F)$ is one.

To deal with the former term, we have to take into account that $i_{\mathfrak{p}}^!$ and $i_{\mathfrak{p}}^*$ are not t-exact, but of cohomological amplitude $[0, 1]$ and $[-1, 0]$, respectively. By decomposing $i_{\mathfrak{p}}^! N$ into its $^{\mathrm{p}}\mathrm{H}^1$- and $^{\mathrm{p}}\mathrm{H}^0$-part and similarly with $i_{\mathfrak{p}}^* M$ and using that the cohomological dimension of $\mathbf{DM}_{\mathrm{gm}}(\mathbf{F}_{\mathfrak{p}})$ is zero, the term vanishes for $i > 2$. Using general t-structure properties, the second claim is a particular case of the first one. $\qquad\square$

5.2. Intermediate extension

DEFINITION 5.3 (Motivic analog of [10, Def. 1.4.22]). The *intermediate extension* $j_{!*}$ of some mixed motive M over S' is defined as

$$j_{!*} M := \mathrm{im}(j_! M \to j_* M).$$

The image is taken in the abelian category $\mathbf{MM}(S)$, using the exactness of $j_!$ and j_*, Axiom 4.2.

Remark 5.4. Let $i : Z \to S$ be the complement of j. The localization triangles (Axiom 1.10) and cohomological amplitude of i^* (Axiom 4.2) yield short exact sequences in $\mathbf{MM}(S)$

(10) $$0 \to i_* {}^{\mathrm{p}}\mathrm{H}^{-1} i^* j_* M \to j_! M \to j_{!*} M \to 0,$$

(11) $$0 \to j_{!*} M \to j_* M \to i_* {}^{\mathrm{p}}\mathrm{H}^0 i^* j_* M \to 0.$$

These triangles are the same as for perverse sheaves in the situation that the analog of Axiom 4.2, [10, 4.1.10], is applicable.

> LEMMA 5.5. · *Given any mixed motive M over S', $j_{!*} M$ is, up to a unique isomorphism, the unique mixed extension of M (i.e., an object X in $\mathbf{MM}(S)$ such that $j^* X = M$) not having nonzero subobjects or quotients of the form $i_* N$, where $i : Z \to S$ is the closed complement of j and N is a mixed motive on Z.*
> · *For any two composable open immersions j_1 and j_2 we have $j_{1!*} \circ j_{2!*} = (j_1 \circ j_2)_{!*}$.*
> · *$j_{!*}$ commutes with duals, i.e., $D(j_{!*}-) \cong j_{!*} D(-)$.*

Proof. The proofs of the same facts for perverse sheaves [10, Cor. 1.4.25, 2.1.7.1] carry over to this setting. The first statement easily implies the last one. $\qquad\square$

The following proposition makes precise the intuition that any motive M over S should be reconstructed by its generic fiber (over F) and a finite number of special fibers (over various $\mathbf{F_p}$).

PROPOSITION 5.6.   *As a thick subcategory of* $\mathbf{DM}(S)$, $\mathbf{DM}_{\mathrm{gm}}(S)$ *is generated by motives of the form*
  · $i_*\mathrm{M}(X_\mathfrak{p})(m)$, *where* $X_\mathfrak{p}/\mathbf{F_p}$ *is smooth projective,* $m \in \mathbf{Z}$ *and* $i : \mathrm{Spec}\,\mathbf{F_p} \to S$ *is any closed point and*
  · $j_{!*}j^*\mathrm{h}^k(X,m)$, *where* X *is regular, flat and projective over* S *with smooth generic fiber, and* $j : S' \to S$ *is such that* $X \times_S S'$ *is smooth over* S' *and* k *and* m *are arbitrary.*

Proof.   Let $\mathscr{D} \subset \mathbf{DM}_{\mathrm{gm}}(S)$ be the thick category generated by the objects in the statement.   By resolution of singularities over S (Axiom 1.15), $\mathbf{DM}_{\mathrm{gm}}(S)$ is the thick subcategory of $\mathbf{DM}(S)$ generated by objects $i_*\mathrm{M}(X_\mathfrak{p})(m)$ and $\mathrm{M}(X)(m)$, where $X_\mathfrak{p}$ and X are as in the statement and $m \in \mathbf{Z}$.

It is therefore sufficient to see $M := \mathrm{M}(X) \in \mathscr{D}$.   Let $j : S' \to S$ be such that $X_{S'}$ is smooth over S'.   By 1.10 it is enough to show $j_*j^*M \in \mathscr{D}$, since motives over finite fields are already covered.   Applying the truncations with respect to the motivic t-structure to j_*j^*M and exactness of j_*, j^* (Axiom 4.2) shows that we may deal with $j_*j^*\mathrm{h}^k(X,m)$ for all k instead of j_*j^*M.   (Only finitely many k yield a nonzero term by Axiom 4.1.)   By Remark 5.4, there is a short exact sequence of mixed motives

$$0 \to j_{!*}j^*\mathrm{h}^k(X,m) \to j_*j^*\mathrm{h}^k(X,m) \to i_*{}^{\mathrm{p}}\mathrm{H}^0 i^* j_*j^*\mathrm{h}^k(X,m) \to 0.$$

Here i is the complement of j.   The left and right hand motives are in $\mathscr{D}$, hence so is the middle one.   □

5.3. Smooth motives
The notion of smooth motives (a neologism leaning on lisse sheaves) is a technical stepstone for the definition of the generic intermediate extension $\eta_{!*}$, cf. Definition 5.13.   Roughly speaking, smoothness for mixed motives M means that i^*M and $i^!M$ do not intermingle in the sense that their cohomological degrees are disjoint.

DEFINITION 5.7.   Let M be a geometric motive over S.   It is called *smooth* if for any closed point $i : \mathrm{Spec}\,\mathbf{F_p} \to S$ there is an isomorphism

$$i^!M \cong i^*M(-1)[-2].$$

M is called *generically smooth* if there is an open (non-empty) immersion $j : S' \to S$ such that j^*M is smooth.

Let X/S be a scheme with smooth generic fiber X_η.   Then $\mathrm{M}_S(X)$ is a generically smooth motive.

The isomorphism in Definition 5.7 is not required to be canonical in any sense. Therefore, the subcategory of smooth motives is *not* triangulated in $\mathbf{DM}_{\mathrm{gm}}(S)$.

LEMMA 5.8. *Let M be a smooth mixed motive over S. Let $i : Z \to S$ be proper closed subscheme, let $j : S' \to S$ be its complement. Then $i^! M = (^{\mathrm{p}}\mathrm{H}^1 i^! M)[-1]$ and dually $i^* M = (^{\mathrm{p}}\mathrm{H}^{-1} i^* M)[1]$.*

Proof. By assumption $i^! M \cong i^* M(-1)[-2]$. By Axiom 4.2, the left hand side of that isomorphism is concentrated in degrees $[0, 1]$. The right hand side is in degrees $[1, 2]$. This shows $i^! M = {}^{\mathrm{p}}\mathrm{H}^1(i^! M)[-1]$ by Axiom 4.1 and similarly for $i^* M$. $\qquad\square$

The following is the key relation of smooth motives and the intermediate extension. Note the similarity with Lemma 5.14.

LEMMA 5.9. *Let M be a smooth mixed motive over S. Then M is canonically isomorphic to $j_{!*} j^* M$.*

Proof. Let $i : Z \to S$ be the complement of j. Given any $i_* N \subset M$ with $N \in \mathbf{MM}(Z)$, we apply the left-exact functor $i^!$ and see $N \subset {}^{\mathrm{p}}\mathrm{H}^0(i^! M) \overset{5.8}{=} 0$. Quotients of M of the form $i_* N$ are treated dually. We now invoke Lemma 5.5. $\qquad\square$

LEMMA 5.10. *Let X be any smooth projective scheme over S. Set $M := \mathrm{M}(X)$. Then all $\mathrm{h}^n X = {}^{\mathrm{p}}\mathrm{H}^n M$ are smooth.*

Proof. Let $f_{m,n}$ be the (m, n)-component of the bottom isomorphism making the following commutative:

$$
\begin{array}{ccc}
i^! M & \xrightarrow{\ \cong,\ \mathrm{see}\ (4)\ } & i^* M(-1)[-2] \\
\Big\downarrow{\scriptstyle \cong,\ 4.13} & & \Big\downarrow{\scriptstyle \cong,\ 4.13} \\
\bigoplus_m A_m := \bigoplus i^!(^{\mathrm{p}}\mathrm{H}^m M)[-m] & \xrightarrow{\ \cong\ } & \bigoplus_n B_n := \bigoplus i^*(^{\mathrm{p}}\mathrm{H}^n M)(-1)[-n-2].
\end{array}
$$

We claim $f_{m,n} = 0$ for all $m \neq n$, from which the lemma follows. By Axiom 4.13 we have $B_n \cong \mathrm{h}^{n-1}(X_{\mathfrak{p}})[-n-1](-1)$. Using this and the reflexivity of the Verdier dual functor, we obtain an isomorphism $A_m \cong (^{\mathrm{p}}\mathrm{H}^{m+1} i^! M)[-1-m]$. Hence B_n is concentrated in cohomological degree $n + 1$, while A_m is in degree $n + 2$. (The a priori bounds of Axiom 4.2 would be $[m, m+1]$ and $[n+1, n+2]$, respectively.) As the cohomological dimension of motives over $\mathbf{F}_{\mathfrak{p}}$ is zero (Axiom 4.1), the only way for $f_{m,n} \neq 0$ is $m = n$. $\qquad\square$

5.4. Generic intermediate extension

LEMMA 5.11 (Spreading out morphisms). *Given two geometric motives M and M' over S together with a map $\phi_\eta : \eta^* M \to \eta^* M'$, there is an open sub-scheme $j : S' \subset S$ and a map $\phi_{S'} : j^* M \to j^* M'$ which extends ϕ_η. Any two such extensions agree when restricted to a possibly smaller open subscheme. In particular, if ϕ_η is an isomorphism, then $\phi_{S'}$ is an isomorphism for sufficiently small S'.*

Proof. The adjunction map $M \to \eta_* \eta^* M$ and $\eta_* \phi_\eta$ give a map $M \to \eta_* \eta^* M'$, hence by (5) a map $M \to \bigoplus_{\mathfrak{p}} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! M'[1]$. As M is compact, it factors over a finite sum $\bigoplus_{\mathfrak{p} \in T} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! M'[1]$. Let $j : S' \to S$ be the complement of the points in T. The map $M \to \eta_* \eta^* M'$ factors over $j_* j^* M'$ and gives a map $j^* M \to j^* M'$ which extends ϕ_η. The first claim is shown.

For the unicity of the extension, we may assume that ϕ_η is zero, and show that $\phi_{S'}$ is zero for some suitable S'. This is the same argument as before: the map $M \to j_* j^* M'$ constructed in the previous step factors over $\bigoplus_{\mathfrak{p} \in S'} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! M'$, since $M \to \eta_* \eta^* M'$ is zero. By compacity of M, only finitely many primes in the sum contribute to the map, discarding these yields the claim.

If ϕ_η is an isomorphism, $\psi_\eta := \phi_\eta^{-1}$ can be extended to some $\psi_{S'}$. As both $\phi_{S'} \circ \psi_{S'}$ and $\mathrm{id}_{S'}$ extend id_η, they agree on some possibly smaller open subscheme of S and similarly with $\psi_{S'} \circ \phi_{S'}$. $\qquad\square$

Remark 5.12. The lemma shows the full faithfulness of the functor

$$\varinjlim_{S' \subset S} \mathbf{DM}_{\mathrm{gm}}(S') \xrightarrow{\eta^*} \mathbf{DM}_{\mathrm{gm}}(F).$$

Its essential surjectivity is a consequence of Axiom 1.5, so we have an equivalence. However, we will stick to the more basic language of colimits in $\mathbf{DM}(S)$ instead of colimits of the categories of geometric motives.

DEFINITION 5.13. Let $M_\eta \in \mathbf{DM}_{\mathrm{gm}}(F)$ be a motive such that there exists a generically smooth mixed motive M over S (Definition 5.7) with $\eta^* M \cong M_\eta$. Then the *generic intermediate extension* $\eta_{!*} M_\eta$ is defined as

$$\eta_{!*} M_\eta := j_{!*} j^* M$$

where $j : S' \to S$ is an open immersion such that $j^* M$ is smooth.

This is independent of the choices of j and M (Lemmas 5.9, 5.11) and functorial (5.11). For a mixed, non-smooth motive M, there need not be a map $j_{!*} j^* M \to M$. Therefore, $\varinjlim j_{!*} j^* M$ does not make sense unless there is some smoothness constraint on M_η.

5.5. Intermediate extension and ℓ-adic realization

This subsection deals with the interplay of the (generic) intermediate extension functor on mixed motives and the ℓ-adic realization. In this subsection, S is an open subscheme of $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$. The following lemma is well-known.

LEMMA 5.14. *Let $\mathscr{F}$ be an étale (honest) locally constant sheaf on S. Let $\eta : \operatorname{Spec} F \to \operatorname{Spec} \mathcal{O}_F[1/\ell]$ be the generic point. Then the canonical map $\mathscr{F} \to \mathrm{R}^0\eta_*\eta^*\mathscr{F}$ is an isomorphism.*

LEMMA 5.15. *Let M be a mixed motive over S'. Let $j : S' \to S$ be an open immersion. Then*

$$(j_{!*}M)_\ell = j_{!*}(M_\ell).$$

Let i be the complementary closed immersion to $j : S' \to S$ and let η' and η be the generic point of S' and S, respectively. If M is additionally smooth, one has

$$(i^*j_{!*}M)_\ell = i^*j_{!*}M_\ell = i^*(\mathrm{R}^0\eta_*\eta'^*M_\ell[-1])[1].$$

To clarify the statement, note that the ℓ-adic realization of M is a perverse sheaf on S' by Axiom 4.5. Thus, $j_{!*}$ (Section 3) can be applied to it.

Proof. The first statement follows from Axiom 2.1, the definition of $j_{!*}$ and the exactness of $\mathrm{R}\Gamma_\ell$ (Axiom 4.5).

Let now M be mixed and smooth over S'. As M_ℓ is a perverse sheaf by 4.5, there is an open immersion $j' : S'' \to S'$ such that $j'^*M_\ell[-1]$ is a locally constant (honest) sheaf on S''. As M is smooth we know from Lemmas 5.5 and 5.9

$$i^*j_{!*}M = i^*(j \circ j')_{!*}j'^*M.$$

By the interpretation of $(j \circ j')_{!*}$ in terms of $\mathrm{R}^0(j \circ j')_*$ (Section 3) we have

$$(i^*j_{!*}M)_\ell = i^*j_{!*}M_\ell = i^*(\mathrm{R}^0(j \circ j')_*j'^*M_\ell[-1])[1] \overset{5.14}{=} i^*(\mathrm{R}^0\eta_*\eta'^*M_\ell[-1])[1]. \qquad \square$$

6. f-cohomology

6.1. f-cohomology via non-ramification

Let F be a number field. For any place $\mathfrak{p}$ of F, let $F_\mathfrak{p}$ be the completion, $G_\mathfrak{p}$ the local Galois group. For finite places, $I_\mathfrak{p}$ denotes the inertia group. For brevity, we will usually write $\mathrm{H}^*(M)$ for $\mathrm{H}^*(S, M)$, where M is any motive over some base S.

DEFINITION 6.1 [11, Section 3]. Let V be a finite-dimensional ℓ-adic vector space, endowed with a continuous action of $G_\mathfrak{p}$, where $\mathfrak{p}$ is a finite place of F not over ℓ. Set

$$\mathrm{H}_f^i(F_\mathfrak{p}, V) := \begin{cases} \mathrm{H}^0(F_\mathfrak{p}, V) & i = 0 \\ \ker \mathrm{H}^1(F_\mathfrak{p}, V) \to \mathrm{H}^1(I_\mathfrak{p}, V) & i = 1 \\ 0 & \text{else.} \end{cases}$$

Remark 6.2.   If $\mathfrak{p}$ lies over ℓ, the definition is completed by $\mathrm{H}^1_f(F_\mathfrak{p}, V) :=$ ker $\mathrm{H}^1(F_\mathfrak{p}, V) \to \mathrm{H}^1(F_\mathfrak{p}, B_{crys} \otimes V)$, where B_{crys} denotes the ring of $\mathfrak{p}$-adic periods [21].   We will disregard this case throughout.

LEMMA 6.3.   *Let* $\eta_\mathfrak{p} : \mathrm{Spec}\, F_\mathfrak{p} \to \mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}$ *be the generic point of the completion of* $\mathcal{O}_F$ *at* $\mathfrak{p}$. *Using the above notation, for* $\mathfrak{p}$ *not over* ℓ, *there is a canonical isomorphism* $\mathrm{H}^1_f(F_\mathfrak{p}, V) \cong \mathrm{H}^1(\mathcal{O}_{F_\mathfrak{p}}, \mathrm{R}^0\eta_{\mathfrak{p}_*} V)$. *(The right hand side denotes* ℓ-*adic cohomology over* $\mathcal{O}_{F_\mathfrak{p}}$.*)*

Proof.   For any ℓ^n-torsion sheaf $\mathscr{F}$ on $F_\mathfrak{p}$ we write $A(\mathscr{F}) := \mathrm{ker}\, \mathrm{H}^1(F_\mathfrak{p}, \mathscr{F}) \to \mathrm{H}^1(I_\mathfrak{p}, \mathscr{F})$.   The $\mathbf{Q}_\ell$-sheaf V is, by definition, of the form $U \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$, where $U = (U_n)_n$ is a projective system of $\mathbf{Z}/\ell^n$-sheaves.   By definition

$$\mathrm{H}^1(F_\mathfrak{p}, V) = \varprojlim_{n \in \mathbf{N}} \mathrm{H}^1(F_\mathfrak{p}, U_n) \otimes \mathbf{Q}_\ell$$

and similarly for $\mathrm{H}^1(I_\mathfrak{p}, V)$.   Both $\varprojlim_n$ and $- \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ are left-exact functors, so one has

$$\mathrm{H}^1_f(F_\mathfrak{p}, V) = \left(\varprojlim_n A(U_n) \right) \otimes \mathbf{Q}_\ell.$$

Thus it is sufficient to show $A(U) = \mathrm{H}^1(\mathcal{O}_{F_\mathfrak{p}}, \mathrm{R}^0\eta_{\mathfrak{p}_*} U)$ for any ℓ^n-torsion sheaf U over $F_\mathfrak{p}$.

Recall the description of étale sheaves on $\mathcal{O}_{F_\mathfrak{p}}$ from [34, II.3.12, II.3.16]. Let $i : \mathrm{Spec}\, \mathbf{F}_\mathfrak{p} \to \mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}$ be the closed point.   As $\mathcal{O}_{F_\mathfrak{p}}$ is a henselian ring [34, Prop. I.4.5], for any sheaf $\mathscr{F}$ on $\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}$, the global sections depend only on the special fiber and

$$\Gamma_{\mathrm{Spec}\, F_\mathfrak{p}} = \Gamma_{\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}} \circ (\eta_{\mathfrak{p}_*}) = \Gamma_{\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}} \circ (i_* i^* \eta_{\mathfrak{p}_*}).$$

These functors can be interpreted using group cohomology: $\Gamma_{\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}} \circ i_* = \Gamma_{\mathbf{F}_\mathfrak{p}}$ and $(-)^{I_\mathfrak{p}} = i^* \eta_{\mathfrak{p}_*}$ (*loc. cit.*).   The Hochschild-Serre spectral sequence for $(-)^{G_\mathfrak{p}} = (-)^{\mathrm{Gal}(\mathbf{F}_\mathfrak{p})} \circ (-)^{I_\mathfrak{p}}$ can be translated to

$$\mathrm{H}^p(\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}, i_* i^* \mathrm{R}^q \eta_{\mathfrak{p}_*} U) \Rightarrow \mathrm{H}^n(F_\mathfrak{p}, U).$$

In addition we have the Leray spectral sequence

$$\mathrm{H}^p(\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}, \mathrm{R}^q \eta_{\mathfrak{p}_*} U) \Rightarrow \mathrm{H}^n(F_\mathfrak{p}, U).$$

The exact sequence of low degrees of the Hochschild-Serre sequence maps to the sequence below:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1(\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}, \mathrm{R}^0\eta_{\mathfrak{p}_*} U) & \longrightarrow & \mathrm{H}^1(F_\mathfrak{p}, U) & \longrightarrow & \mathrm{H}^0(\mathrm{Spec}\, \mathcal{O}_{F_\mathfrak{p}}, \mathrm{R}^1\eta_{\mathfrak{p}_*} U) \\
& & & & \downarrow = & & \downarrow \\
0 & \longrightarrow & A(U) & \longrightarrow & \mathrm{H}^1(F_\mathfrak{p}, U) & \longrightarrow & \mathrm{H}^1(I_\mathfrak{p}, U)
\end{array}
$$

As $\mathrm{H}^0(\mathrm{Gal}(\mathbf{F}_{\mathfrak{p}}), \mathrm{H}^1(I_{\mathfrak{p}}, U)) \subset \mathrm{H}^1(I_{\mathfrak{p}}, U)$ and $\Gamma_{\mathscr{O}_{F_{\mathfrak{p}}}} = \Gamma_{\mathscr{O}_{F_{\mathfrak{p}}}} \circ i_* i^*$, the right hand map is injective, therefore there is a unique isomorphism between the left hand terms making the diagram commutative. $\square$

In order to proceed to a global level, the following definition is done:

DEFINITION 6.4 [22, II.1.3].   Given an ℓ-adic continuous representation V of $G = \mathrm{Gal}(F)$, define $\mathrm{H}_f^i(F, V)$ to be such that the following diagram is cartesian. In the lower row, V is considered a $G_{\mathfrak{p}} = \mathrm{Gal}(F_{\mathfrak{p}})$-module by restriction.

$$
\begin{array}{ccc}
\mathrm{H}_f^i(F, V) & \longrightarrow & \mathrm{H}^i(F, V) \\
\downarrow & & \downarrow \\
\prod \mathrm{H}_f^i(F_{\mathfrak{p}}, V) & \longrightarrow & \prod \mathrm{H}^i(F_{\mathfrak{p}}, V)
\end{array}
$$

The product ranges over all finite places $\mathfrak{p}$ of F.   We define $\mathrm{H}_{f, \backslash\mathrm{crys}}^i(F, V)$ similarly, except that in the lower row of the preceding diagram only places $\mathfrak{p}$ that do not lie over ℓ occur.

LEMMA 6.5.   *Let V be an ℓ-adic étale sheaf on* $\mathrm{Spec}\, F$.   *Then there is a natural isomorphism*

$$
\mathrm{H}_{f, \backslash\mathrm{crys}}^i(F, V) \cong \mathrm{H}^1(\mathscr{O}_F[1/\ell], \mathrm{R}^0 \eta_* V).
$$

Proof.   By the same argument as in the previous proof, we may assume that V is a sheaf of $\mathbf{Z}/\ell^n$-modules, since the isomorphism we are going to establish is natural in V and

$$
\mathrm{H}_{f, \backslash\mathrm{crys}}^i(F, V) = \ker \mathrm{H}^i(F, V) \to \prod_{\mathfrak{p} \nmid \ell} (\mathrm{H}^i(F_{\mathfrak{p}}, V)/\mathrm{H}_f^i(F_{\mathfrak{p}}, V)).
$$

Consider the following cartesian diagram ($\mathfrak{p} \nmid \ell$)

$$
\begin{array}{ccccc}
\mathrm{Spec}\, F_{\mathfrak{p}} & \xrightarrow{\eta_{\mathfrak{p}}} & \mathrm{Spec}\, \mathscr{O}_{F_{\mathfrak{p}}} & \xleftarrow{\quad i_{\mathfrak{p}} \quad} & \mathrm{Spec}\, \mathbf{F}_{\mathfrak{p}} \\
\downarrow b & & \downarrow a & & \downarrow = \\
\mathrm{Spec}\, F & \xrightarrow{\eta} & \mathrm{Spec}\, \mathscr{O}_F[1/\ell] & \xleftarrow{\quad i \quad} & \mathrm{Spec}\, \mathbf{F}_{\mathfrak{p}}
\end{array}
$$

In the derived category of $\mathbf{Z}/\ell^n$-sheaves on $\mathrm{Spec}\, \mathscr{O}_F[1/\ell]$, there is a triangle $\mathrm{R}^0 \eta_* V \to \mathrm{R}\eta_* V \to \mathrm{R}^1 \eta_*[-1]V$.   Likewise, $\mathrm{R}^0 \eta_{\mathfrak{p}_*} b^* V \to \mathrm{R}\eta_{\mathfrak{p}_*} b^* V \to \mathrm{R}^1 \eta_{\mathfrak{p}_*} b^* V[-1]$.   (We have used $\mathfrak{p} \nmid \ell$, since the inertia group has cohomological dimension bigger than one for $\mathfrak{p} \mid \ell$.)   This yields exact horizontal sequences, the vertical maps are adjunction maps

$$0 \longrightarrow \mathrm{H}^1(\mathrm{Spec}\ \mathscr{O}_F[1/\ell], \eta_* V) \longrightarrow \mathrm{H}^1(F, V) \longrightarrow \mathrm{H}^0(\mathrm{Spec}\ \mathscr{O}_F[1/\ell], \mathrm{R}^1\eta_* V)$$

$$0 \longrightarrow \prod_{\mathfrak{p} \nmid \ell} \mathrm{H}^1(\mathscr{O}_{F_\mathfrak{p}}, \mathrm{R}^0\eta_{\mathfrak{p}_*} b^* V) \longrightarrow \prod_{\mathfrak{p} \nmid \ell} \mathrm{H}^1(F_\mathfrak{p}, b^* V) \longrightarrow \prod_{\mathfrak{p} \nmid \ell} \mathrm{H}^0(\mathscr{O}_{F_\mathfrak{p}}, \mathrm{R}^1\eta_{\mathfrak{p}_*} b^* V)$$

We will show that α is injective. Hence, the left square is cartesian and by definition and Lemma 6.3 the claim is shown. Indeed, α factors as

$$\mathrm{H}^0(\mathscr{O}_F[1/\ell], \mathrm{R}^1\eta_* V) \subset \prod_{\mathfrak{p} \nmid \ell} \mathrm{H}^0(\mathbf{F}_\mathfrak{p}, i_\mathfrak{p}^* \mathrm{R}^1\eta_* V)$$

$$\to \prod_{\mathfrak{p} \nmid \ell} \mathrm{H}^0(\mathscr{O}_{F_\mathfrak{p}}, \mathrm{R}^1\eta_{\mathfrak{p}_*} b^* V) \left(\overset{\cong}{} \prod_{\mathfrak{p} \nmid \ell} \mathrm{H}^0(\mathbf{F}_\mathfrak{p}, i_\mathfrak{p}^* \mathrm{R}^1\eta_{\mathfrak{p}_*} b^* V) \right).$$

using $i^* \mathrm{R}^1\eta_* V = i_\mathfrak{p}^* a^* \mathrm{R}^1\eta_* V = i_\mathfrak{p}^* \mathrm{R}^1\eta_{\mathfrak{p}_*} b^* V$. $\qquad\qquad\square$

DEFINITION 6.6 [8, Remark 4.0.1.b], [11, Conj. 5.3], [20, Section 6.5], [22, III.3.1.3]. Let M_η be a mixed motive over F. Let, similarly to Definition 6.4, $\mathrm{H}_f^i(M_\eta)$ be defined such that the following diagram, in which the bottom products are taken over all primes ℓ, is cartesian. As usual, $M_{\eta\ell}$ is the ℓ-adic realization, seen as a G-module.

$$\begin{array}{ccc} \mathrm{H}_f^i(F, M_\eta) & \longrightarrow & \mathrm{H}^i(F, M_\eta) \\ \downarrow & & \downarrow \\ \prod_\ell \mathrm{H}_f^i(F, M_{\eta\ell}) & \longrightarrow & \prod_\ell \mathrm{H}^i(F, M_{\eta\ell}) \end{array}$$

Again, to rid ourselves from crystalline questions at $\mathfrak{p} \mid \ell$, we define $\mathrm{H}_{f,\backslash\mathrm{crys}}^i(F, M_\eta)$ by replacing $\prod_\ell \mathrm{H}_f^i(F, M_{\eta\ell})$ in the bottom row by $\prod_\ell \mathrm{H}_{f,\backslash\mathrm{crys}}^i(F, M_{\eta\ell})$.

We are now going to exhibit an interpretation of f-cohomology thus defined in terms of the generic intermediate extension $\eta_{!*}$. Recall that we are assuming in this section the axioms of Sections 1, 2, and 4. Mixed motives are needed to even define $\eta_{!*}$. Moreover, for the comparison result, we need to assume the following conjecture.

LEMMA 6.7. *Let N be any mixed motive over $\mathbf{F}_\mathfrak{p}$. The ℓ-adic realization map $\mathrm{H}^0(\mathbf{F}_\mathfrak{p}, N) \to \mathrm{H}^0(\mathbf{F}_\mathfrak{p}, N_\ell) := N_\ell^{\mathrm{Gal}(\mathbf{F}_\mathfrak{p})}$ is injective.*

Proof. By the strictness of the weight filtration, the canonical maps

$$\mathrm{H}^0(\mathrm{gr}_0^W N) \leftarrow \mathrm{H}^0(W_0 N) \to \mathrm{H}^0(N)$$

are both isomorphisms. Moreover, the ℓ-adic realization functor commutes with gr_0^W by Axiom 4.6, so that we can replace N by gr_0^W and assume that N is pure of weight 0. In view of our assumptions on motives, cf. (8), all adequate equivalence relations agree, so that we may regard N as a pure motive with respect to any adequate equivalence relation. As the injectivity is stable under taking direct summands, we may assume $N = h(X, n)$ for X smooth and projective over $\mathbf{F}_\mathfrak{p}$, by definition of pure motives and Axiom 4.11. The left hand side is given by $\mathrm{CH}^n(X)$, so the map is injective by (8). $\qquad\square$

THEOREM 6.8. *Let M be a generically smooth mixed motive over $\mathcal{O}_F$ (Definition 5.7). Set $\eta^* M[-1] =: M_\eta$. There is a natural isomorphism*

$$\mathrm{H}^0(\mathcal{O}_F, \eta_{!*}\eta^* M) \xrightarrow{\cong} \mathrm{H}^1_{f, \backslash \mathrm{crys}}(F, M_\eta).$$

Proof. Notice that $\eta_{!*}\eta^* M$ is well-defined by the assumptions. We want to show that there is a cartesian commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^0(\eta_{!*}\eta^* M) & \longrightarrow & \mathrm{H}^0(\eta_*\eta^* M) = \mathrm{H}^1(M_\eta) \\
\vdots & & \downarrow \\
\prod_\ell \mathrm{H}^1_{f, \backslash \ell}(F, M_{\eta\ell}) & \longrightarrow & \prod_\ell \mathrm{H}^1(F, M_{\eta\ell})
\end{array}$$

Let $j: U \to \mathrm{Spec}\, \mathcal{O}_F$ be any open immersion such that $j^* M$ is smooth. We have $\eta_{!*}\eta^* M = j_{!*}j^* M$. The left hand term of the exact sequence

$$\bigoplus_{\mathfrak{p} \in U} \mathrm{H}^0(i_{\mathfrak{p}*}i_\mathfrak{p}^! M) \to \mathrm{H}^0(j_* j^* M) \to \mathrm{H}^0(\eta_*\eta^* M) \to \bigoplus_{\mathfrak{p} \in U} \mathrm{H}^1(i_{\mathfrak{p}*}i_\mathfrak{p}^! M)$$

induced by (5) vanishes as $i_\mathfrak{p}^! M$ is concentrated in cohomological degree 1 for $\mathfrak{p} \in U$ (Lemma 5.8). Any $a \in \mathrm{H}^0(\eta^* M)$ maps to a finite sub-sum of $\bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_F} \mathrm{H}^1(i_{\mathfrak{p}*}i_\mathfrak{p}^! M)$, so letting j be the open complement of these points, a lies in (the image of) $\mathrm{H}^0(j_* j^* M)$:

$$\mathrm{H}^0(\eta^* M) = \varinjlim_{\substack{j: U \to \mathrm{Spec}\, \mathcal{O}_F \\ j^* M \text{ smooth}}} \mathrm{H}^0(j_* j^* M).$$

By Lemma 6.9 below, the map $\mathrm{H}^0(j_{!*}j^* M) \to \mathrm{H}^0(j_* j^* M) \to \mathrm{H}^0(\eta^* M)$ is injective. Therefore, taking the colimit over all U such that $M|_U$ is smooth, the exact localization sequence

$$0 \to \mathrm{H}^0(j_{!*}j^* M) \to \mathrm{H}^0(j_* j^* M) \to \bigoplus_{\mathfrak{p} \notin U} \mathrm{H}^0(^p\mathrm{H}^0 i_\mathfrak{p}^* j_* j^* M)$$

stemming from (11) gives

$$0 \to \mathrm{H}^0(j_{!*}j^* M) \to \mathrm{H}^0(\eta_*\eta^* M) \to \bigoplus_\mathfrak{p} \mathrm{H}^0(^p\mathrm{H}^0 i_\mathfrak{p}^* j_{\mathfrak{p}*}j_\mathfrak{p}^* M).$$

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Here $j_{\mathfrak{p}}$ is the complementary open immersion to $i_{\mathfrak{p}}$ and the direct sum is over all (finite) places $\mathfrak{p}$ of $\mathcal{O}_F$. We have $i_{\mathfrak{p}}^* \eta_* \eta^* M = i_{\mathfrak{p}}^* j_{\mathfrak{p}*} j_{\mathfrak{p}}^* M$, so the top sequence in the following commutative diagram is exact:

(12)
$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^0(j_! j^* M) & \longrightarrow & \mathrm{H}^1(M_\eta) & \longrightarrow & \bigoplus_{\mathfrak{p}} \mathrm{H}^0(^{\mathrm{p}}\mathrm{H}^0 i_{\mathfrak{p}}^* \eta_* \eta^* M) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_\ell \mathrm{H}^0((j_\ell^* j_! j^* M)_\ell) & \longrightarrow & \prod_\ell \mathrm{H}^1(M_{\eta\ell}) & \longrightarrow & \prod_\ell \bigoplus_{\mathfrak{p} \nmid \ell} \mathrm{H}^0((^{\mathrm{p}}\mathrm{H}^0 i_{\mathfrak{p}}^* \eta_* \eta^* M)_\ell)
\end{array}
$$

The lower row denotes ℓ-adic cohomology over $\mathcal{O}_F[1/\ell]$, F, and the various $\mathbf{F}_{\mathfrak{p}}$, respectively. Moreover, $j_\ell : \mathrm{Spec}\, \mathcal{O}_F[1/\ell] \to \mathrm{Spec}\, \mathcal{O}_F$ is the open immersion. The remainder of the proof consists in the following steps: we show that the diagram is commutative, that the second row is exact, identify its lower leftmost term and show that the rightmost vertical map is injective. This implies that the left square is cartesian, hence the theorem follows.

We write ι and ι_ℓ for the open immersions of $U \cap \mathrm{Spec}\, \mathcal{O}_F[1/\ell]$ into $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$ and U, respectively. By Lemma 5.15 and the exactness of j_ℓ^* we have

$$
(j_\ell^* j_! j^* M)_\ell = (\iota_! \iota^* j_\ell^* M)_\ell = \iota_! \iota^* (j_\ell^* M)_\ell.
$$

Thus (12) is commutative since every term at the bottom just involves the ℓ-adic realization of the motive above it, restricted to $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$.

The exactness of the bottom row is shown separately for each ℓ, so ℓ is fixed for this argument. By the characterization just mentioned, $\iota_! \iota^* (j_\ell^* M)_\ell$ does not change when shrinking U, since $j_! j^* M$ is independent of U (as soon as M is smooth over U). On the other hand, by the exactness of the ℓ-adic realization functor (Axiom 4.5) $(j_\ell^* M)_\ell$ is a perverse sheaf on $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$, so is a locally constant sheaf (shifted into degree -1) on a suitable small open subscheme. Hence we may assume that $\iota^* (j_\ell^* M)_\ell$ is a locally constant sheaf in degree -1. By Section 3, $\iota_! \iota^* (j_\ell^* M)_\ell = (\mathrm{R}^0 \iota_! \iota^* (j_\ell^* M)_\ell[-1])[+1]$, so the lower row is the exact cohomology sequence belonging to the distinguished triangle of sheaves on $\mathrm{Spec}\, \mathcal{O}_F[1/\ell]$

$$
\mathrm{R}^0 \eta_{\ell*}(M_\eta)_\ell \to \mathrm{R}\eta_{\ell*}(M_\eta)_\ell \to (\mathrm{R}^1 \eta_{\ell*}(M_\eta)_\ell)[-1].
$$

Here $\eta_\ell : \mathrm{Spec}\, F \to \mathrm{Spec}\, \mathcal{O}_F[1/\ell]$ is the generic point. As is well-known, there is an isomorphism

(13)
$$
D := \mathrm{R}^1 \eta_{\ell*} \eta_\ell^* A \xrightarrow{\cong} \bigoplus_{\mathfrak{p} \nmid \ell} i_{\mathfrak{p}*} i_{\mathfrak{p}}^* \mathrm{R}^1 \eta_{\ell*} \eta_\ell^* A =: \bigoplus B_{\mathfrak{p}}
$$

for any generically locally constant constructible ℓ-adic sheaf A, such as $M_\ell[-1]$. Indeed, the adjunction map $a : D \to \prod_{\mathfrak{p} \nmid \ell} B_{\mathfrak{p}}$ factors over the direct sum: note that $(\bigoplus B_{\mathfrak{p}})/\ell^n = \bigoplus(B_{\mathfrak{p}}/\ell^n)$ and likewise with the product. Then

$$
\mathrm{Hom}(D, \bigoplus B_{\mathfrak{p}}) = \varprojlim_n \mathrm{Hom}(D/\ell^n, \bigoplus(B_{\mathfrak{p}}/\ell^n)) \subset \varprojlim_n \mathrm{Hom}\left(D/\ell^n, \prod(B_{\mathfrak{p}}/\ell^n)\right)
$$

and to see that a lies in the left hand subgroup, it is enough to consider the $\mathbf{Z}/\ell^n$-sheaves D/ℓ^n etc. The corresponding map $\mathrm{H}^1(\mathrm{Gal}(F), A/\ell^n) \to \prod \mathrm{H}^1(I_{\mathfrak{p}}, A/\ell^n)$ (Galois cohomology of the inertia groups) factors over the direct sum, since the left hand term agrees with $\mathrm{H}^1(\mathrm{Gal}(F'/F), A)$ for some finite extension F'/F. This uses that A/ℓ^n is constructible. The extension F'/F is ramified in finitely many places (only), so the claimed factorization follows. This implies (13) and thus the exactness of the lower row of (12). By Lemma 6.5 and Lemma 5.14, the factors in the lower left-hand term of (12) agree with $\mathrm{H}^1_{f,\backslash\mathrm{crys}}(F, \eta^* M_\ell[-1])$.

To show that the rightmost vertical map of (12) is an injection, let $a = (a_{\mathfrak{p}})_{\mathfrak{p} \in \mathrm{Spec}\,\mathcal{O}_F}$ be a nonzero element of the rightmost upper term. Only finitely many $a_{\mathfrak{p}}$ are nonzero. Pick some ℓ not lying under any of these prime ideals $\mathfrak{p}$. Then the image of a in $\bigoplus_{\mathfrak{p} \nmid \ell} \mathrm{H}^0(({}^{\mathfrak{p}}\mathrm{H}^0 i_{\mathfrak{p}}^* \eta_* \eta^* M)_\ell)$ is nonzero by Lemma 6.7. $\qquad\square$

LEMMA 6.9. *Let M be a mixed motive over S such that $j^* M$ is smooth for some open immersion $j : U \to S$. Then both maps $\mathrm{H}^0(j_{!*} j^* M) \to \mathrm{H}^0(j_* j^* M) \to \mathrm{H}^0(\eta^* M)$ are injective.*

Proof. Indeed the kernels are $\mathrm{H}^{-1}({}^{\mathfrak{p}}\mathrm{H}^0 i^* j_* j^* M) = 0$ and $\bigoplus_{\mathfrak{p} \in U} \mathrm{H}^0(i_{\mathfrak{p}}^! M)$, which vanishes since $i_{\mathfrak{p}}^! M$ sits in cohomological degree $+1$, for M is smooth around $\mathfrak{p} \in U$ (Lemma 5.8). $\qquad\square$

6.2. f-cohomology via K-theory of regular models

DEFINITION 6.10. Let X_η be a smooth and projective variety over F. Let $X/\mathcal{O}_F$ be any projective model, i.e., $X \times_{\mathcal{O}_F} F = X_\eta$. Then we define

$$\mathrm{H}^i(X_\eta, n)_{\mathcal{O}_F} := \mathrm{im}(\mathrm{H}^i(X, n) \to \mathrm{H}^i(X_\eta, n)).$$

Recall that we are assuming the axioms of Sections 1, 2, and 4; the full force of mixed motives will be made use of in the sequel.

THEOREM 6.11. *The above is well-defined, i.e., independent of the choice of the model X. More precisely we have natural isomorphisms*:

$$\mathrm{H}^0(\eta_{!*} \mathrm{h}^{i-1}(X_\eta, n)[1]) = \begin{cases} \mathrm{H}^i(X_\eta, n)_{\mathcal{O}_F} & i < 2n \\ \mathrm{CH}^n(X_\eta)_{\mathbf{Q}, \mathrm{hom}} & i = 2n \end{cases}$$

Moreover

$$\mathrm{H}^{-1}(\eta_{!*} \mathrm{h}^{i-1}(X_\eta, n)[1]) = \mathrm{H}^0(\mathrm{h}^{i-1}(X_\eta, n)).$$

When X is regular, the definition and the statement are due to Beilinson [7, Lemma 8.3.1]. In this case one has $\mathrm{H}^i(X_\eta, n)_{\mathcal{O}_F} = \mathrm{im}\, K'_{2n-i}(X)^{(n)}_{\mathbf{Q}} \to K'_{2n-i}(X_\eta)^{(n)}_{\mathbf{Q}}$, but that expression does in general depend on the choice of the model [14, 15].

An extension of Beilinson's definition to all Chow motives over F due to Scholl is discussed in the theorem below. We first provide a preparatory lemma.

LEMMA 6.12. *Let* $M \in \mathbf{MM}(\mathrm{Spec}\ \mathcal{O}_F)$ *be a mixed, generically smooth motive with strictly negative weights* (*Definition* 5.7). *Let* $j : U \to \mathrm{Spec}\ \mathcal{O}_F$ *be an open non-empty immersion such that* $M|_U$ *is smooth. The natural map* $j_{!*}j^*M \to \eta_*\eta^*M$ *gives rise to an isomorphism*

$$\mathrm{H}^0(j_{!*}j^*M) = \mathrm{im}(\mathrm{H}^0(M) \to \mathrm{H}^0(\eta_*\eta^*M)).$$

Proof. By Lemma 6.9, $\mathrm{H}^0(j_*j^*M) \to \mathrm{H}^0(\eta_*\eta^*M)$ is injective. Hence it suffices to show $\mathrm{H}^0(j_{!*}j^*M) = \mathrm{im}(\mathrm{H}^0M \to \mathrm{H}^0(j_*j^*M))$. Let i be the complement of j. From (10), (11), we have a commutative exact diagram

$$
\begin{array}{ccccc}
\mathrm{H}^0(j_!j^*M) & \xrightarrow{\ \alpha\ } & \mathrm{H}^0(M) & \longrightarrow & \mathrm{H}^0(i_*i^*M) \\
\downarrow & & \downarrow & & \\
0 = \mathrm{H}^{-1}(i_*{}^p\mathrm{H}^0 i^* j_* j^*M) \longrightarrow \mathrm{H}^0(j_{!*}j^*M) & \rightarrowtail & \mathrm{H}^0(j_*j^*M) \\
& & \downarrow & & \\
& & \mathrm{H}^1(i_*{}^p\mathrm{H}^{-1} i^* j_* j^*M) = 0 & &
\end{array}
$$

The indicated vanishings are because of t-structure reasons and Axiom 4.1, respectively. It remains to show that α is surjective. As i^*M is concentrated in cohomological degrees $[-1, 0]$ (Axiom 4.2), there is an exact sequence

$$0 = \mathrm{H}^1({}^p\mathrm{H}^{-1}i^*M) \to \mathrm{H}^0(i^*M) \to \mathrm{H}^0({}^p\mathrm{H}^0 i^*M).$$

However $\mathrm{H}^0({}^p\mathrm{H}^0 i^*M) = 0$ as i^* preserves negative weights (Axiom 4.8) and by strictness of the weight filtration and compatibility with the t-structure (Axiom 4.6). $\qquad\square$

Proof. Let $j : U \to \mathrm{Spec}\ \mathcal{O}_F$ be an open nonempty immersion (which exists by smoothness of X_η) such that X_U is smooth over U. By definition of $\eta_{!*}$ and Lemmas 5.1 and 5.10, the left hand term in the theorem agrees with $\mathrm{H}^0(j_{!*}\mathrm{h}^i(X_U, n))$. In the sequel, we write $M := \mathrm{h}^i(X, n)$ and $M_\eta := \eta^*[-1]M = \mathrm{h}^{i-1}(X_\eta, n)$.

We first do the case $i < 2n$. The spectral sequences

$$\mathrm{H}^a(\mathrm{h}^b(X, n)) \Rightarrow \mathrm{H}^{a+b}(X, n), \quad \mathrm{H}^a(\mathrm{h}^b(X_\eta, n)) \Rightarrow \mathrm{H}^{a+b}(X_\eta, n)$$

resulting from repeatedly applying truncation functors of the motivic t-structure converge since the cohomological dimension is finite (Axiom 4.1 over F, Lemma 5.2 over $\mathcal{O}_F$). By Lemma 5.2, $\mathrm{H}^i(-)$, applied to mixed motives over $\mathcal{O}_F$, is non-zero for $i \in \{-1, 0, 1\}$ only. We thus have to consider two exact sequences. The exact functor $\eta^*[-1]$ maps to similar exact sequences for motivic cohomology over F (the indices work out properly, see Lemma 5.1):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{H}^i(X,n) & \longrightarrow & \mathrm{H}^{-1}(\mathrm{h}^{i+1}(X,n)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_\eta & \longrightarrow & \mathrm{H}^i(X_\eta,n) & \longrightarrow & \mathrm{H}^0(\mathrm{h}^i(X_\eta,n)) & \overset{4.6,4.10}{=\!=} 0 \longrightarrow & 0
\end{array}
$$

(14) appears to the left of the first array; (15) to the left of the second.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1(\mathrm{h}^{i-1}(X,n)) & \longrightarrow & K & \longrightarrow & \mathrm{H}^0(M) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^2(\mathrm{h}^{i-2}(X_\eta,n)) & \overset{4.1}{=\!=} 0 \longrightarrow & K_\eta & \longrightarrow & \mathrm{H}^1(M_\eta) & \longrightarrow & 0
\end{array}
$$

Here, K and K_η are certain E_3-terms of the spectral sequences above. The rightmost vertical map in (14) is injective as one sees by combining (5) with the left-exactness of $i_{\mathfrak{p}}^!$. Hence

$$
\mathrm{H}^i(X_\eta,n)_{\mathcal{O}_F} = \mathrm{im}(\mathrm{H}^i(X,n) \to \mathrm{H}^i(X_\eta,n)) = \mathrm{im}(K \to K_\eta)
$$

$$
= \mathrm{im}(\mathrm{H}^0(M) \to \mathrm{H}^1(M_\eta))
$$

The motive $M = \mathrm{h}^i(X,n)$ is a generically smooth (mixed) motive by Lemma 5.10. (Recall that this uses the decomposition axiom 4.13 for smooth projective varieties.) By Example 4.10, its weights are strictly negative. Thus Lemma 6.12 applies and the case $i < 2n$ is shown.

We now do the case $i = 2n$. The motive j^*M is pure of weight zero (Example 4.10), hence by strictness of the weight filtration for motives over $\mathcal{O}_F$ and (10), (11) the same is true for $E := j_{!*}j^*M$. (This is an avatar of [10, Cor. 5.3.2].) Thus $^{\mathrm{p}}\mathrm{H}^1 i^! E$ has strictly positive weights because of Axiom 4.8 and the compatibility of weights and the motivic t-structure, i.e., $\mathrm{wt}\,^{\mathrm{p}}\mathrm{H}^1(-) \subset \mathrm{wt}(-) + 1$. Therefore $\mathrm{H}^0(^{\mathrm{p}}\mathrm{H}^1 i^! E) = 0$. Here i is any closed immersion. The localization triangle (5) yields

$$
\mathrm{H}^0(E) \overset{\alpha}{\to} \mathrm{H}^0(\eta_*\eta^* E) \overset{(7)}{=\!=} \mathrm{CH}^n(X_\eta)_{\mathbf{Q},\mathrm{hom}} \to \bigoplus_{\mathfrak{p}} \mathrm{H}^1(i_{\mathfrak{p}}^! E) = \bigoplus \mathrm{H}^0(^{\mathrm{p}}\mathrm{H}^1(i^! E)) = 0.
$$

Therefore, α is surjective. The injectivity of α is Lemma 6.9.

To calculate $\mathrm{H}^{-1}(\eta_{!*}M_\eta[1])$, let $j : U \to \mathrm{Spec}\,\mathcal{O}_F$ be as above. The natural map $\mathrm{H}^{-1}(\mathrm{Spec}\,\mathcal{O}_F, j_{!*}j^*M) \to \mathrm{H}^{-1}(U, j^*M)$ is an isomorphism by the exact cohomology sequence belonging to (11). Thus we have to show

$$
\mathrm{H}^{-1}(\mathrm{Spec}\,\mathcal{O}_F, j_*j^*M) = \mathrm{H}^{-1}(\mathrm{Spec}\,\mathcal{O}_F, \eta_*\eta^*M).
$$

This follows from the localization axiom 1.10 and $i_{\mathfrak{p}}^!M$ being in cohomological degree $+1$ for all points $\mathfrak{p}$ in U (Lemma 5.8), so that $\mathrm{H}^0(\mathbf{F}_{\mathfrak{p}}, i_{\mathfrak{p}}^!M) = \mathrm{H}^{-1}(\mathbf{F}_{\mathfrak{p}}, i_{\mathfrak{p}}^!M) = 0$. $\qquad \square$

By a theorem of Scholl [44, Thm. 1.1.6], there is a unique functorial and additive (i.e., converting finite disjoint unions into direct sums) way to extend the definition of $\mathrm{H}^i(X_\eta,n)_{\mathcal{O}_F}$ as the image of the K-theory of a *regular* proper

flat model (Definition 6.10) to all Chow motives over F, in particular to ones of smooth projective varieties X_η/F that do not possess a regular proper model X. The following theorem compares this definition with the one via intermediate extensions.

THEOREM 6.13. *Let h_η be a direct summand in the category of Chow motives of $h(X_\eta, n)$ where X_η/F is smooth projective. Let $i \in \mathbf{Z}$ be such that $i - 2n < 0$. Let $\iota : \mathbf{M}_{\mathrm{rat}}(F) \to \mathbf{DM}_{\mathrm{gm}}(F)$ be the embedding. Then, the group*

$$\mathrm{H}^i(h_\eta)_{\mathcal{O}_F} := \mathrm{H}^0(\eta_{!*}(^{\mathrm{p}}\mathrm{H}^{i-2n-1}(\iota(h_\eta))[1])).$$

is well-defined and agrees with the aforementioned definition by Scholl.

Proof. Recall $\iota(h(X_\eta, n)) = \mathrm{M}(X_\eta, n)[2n] \in \mathbf{DM}_{\mathrm{gm}}(F)$. We first check that the group is well-defined: let $X/\mathcal{O}_F$ be a projective model of X_η. By Lemma 5.11, there is some model $M \in \mathbf{MM}(\mathcal{O}_F)$ of $^{\mathrm{p}}\mathrm{H}^{i-2n-1}\iota(h_\eta)[1]$ and an open sub-scheme U of Spec $\mathcal{O}_F$ such that M is a direct summand of $^{\mathrm{p}}\mathrm{H}^{i-1}\mathrm{M}(X)(n)$ and such that $X \times U$ is smooth over U. Then $\mathrm{h}^{i-1}(X, n)$ is a smooth motive when restricted to U (Lemma 5.10). Hence so is M. Thus $\eta_{!*}$ can be applied to $(^{\mathrm{p}}\mathrm{H}^{i-2n-1}\iota(h_\eta))[1]$.

The assignment $h_\eta \mapsto \mathrm{H}^0(\eta_{!*}(^{\mathrm{p}}\mathrm{H}^{i-2n-1}\iota(h_\eta))[1])$ is functorial and additive and $h(X_\eta)(n)$ maps to

$$\mathrm{H}^0(\eta_{!*}(^{\mathrm{p}}\mathrm{H}^{i-1}\mathrm{M}(X_\eta, n))[1]) \overset{6.11}{=} \mathrm{H}^i(X_\eta, n)_{\mathcal{O}_F}.$$

Thus the two definitions agree by Scholl's theorem. $\qquad\square$

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Mixed Artin–Tate motives over number rings

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A B S T R A C T

This paper studies Artin–Tate motives over bases $S \subset \operatorname{Spec} \mathcal{O}_F$, for a number field F. As a subcategory of motives over S, the triangulated category of Artin–Tate motives **DATM**(S) is generated by motives $\phi_* \mathbf{1}(n)$, where ϕ is any finite map. After establishing the stability of these subcategories under pullback and pushforward along open and closed immersions, a motivic t-structure is constructed. Exactness properties of these functors familiar from perverse sheaves are shown to hold in this context. The cohomological dimension of mixed Artin–Tate motives (**MATM**(S)) is two, and there is an equivalence **DATM**$(S) \cong \mathbf{D}^{\mathrm{b}}(\mathbf{MATM}(S))$.

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Geometric motives, as developed by Hanamura [5], Levine [8] and Voevodsky [14], are established as a valuable tool in understanding geometric and arithmetic aspects of algebraic varieties over fields. However, the stupefying ambiance inherent to motives, exemplified by Grothendieck's motivic proof idea of the Weil conjectures, remains largely conjectural—especially what concerns the existence of mixed motives **MM**(K) over some field K. That category should be the heart of the so-called motivic t-structure on **DM**$_{\mathrm{gm}}(K)$, the category of geometric motives. Much the same way as the cohomology groups of a variety X over K, e.g. $\mathrm{H}^n_{\text{ét}}(X \times_K \overline{K}, \mathbb{Q}_\ell)$, ℓ-adic cohomology for $\ell \neq \operatorname{char} K$ are commonly realized as cohomology groups of a complex, e.g. $\mathrm{R}\Gamma_\ell(X, \mathbb{Q}_\ell)$, there should be mixed motives $\mathrm{h}^n(X)$ that are obtained by applying truncation functors belonging to the t-structure to $\mathrm{M}(X)$, the motive of X. However, progress on mixed motives has proved hard to come by. To date, such a formalism has been developed for motives of zero- and one-dimensional varieties, only. This is due to Levine [7], Voevodsky [14], Orgogozo [9] and Wildeshaus [16].

Building upon Voevodsky's work, Ivorra [6] and recently Cisinski and Déglise [3] developed a theory of geometric motives **DM**$_{\mathrm{gm}}(S)$ over more general bases. The purpose of this work is to join the ideas of Beilinson et al. on perverse sheaves [2] with the ones on Artin–Tate motives over fields to obtain a workable category of mixed Tate and Artin–Tate motives over bases S which are open subschemes of $\operatorname{Spec} \mathcal{O}_F$, the ring of integers in a number field F. As over a field, this provides some evidence for the existence and properties of the conjectural category of mixed motives over S.

The triangulated category **DTM**(S) (**DATM**(S)) of Tate (Artin–Tate) motives is defined 2.2 to be the triangulated subcategory of **DM**$_{\mathrm{gm}}(S)$ (with rational coefficients) generated by direct summands of $\mathbf{1}(n)$ and $i_* \mathbf{1}(n)$ ($\phi_* \mathbf{1}(n)$, respectively). Here, $\mathbf{1}$ is a shorthand for the motive of the base scheme, (n) denotes the Tate twist, $i : \operatorname{Spec} \mathbb{F}_{\mathfrak{p}} \to S$ is a closed point, $\phi : V \to S$ is any finite map and $\phi_* : \mathbf{DM}_{\mathrm{gm}}(V) \to \mathbf{DM}_{\mathrm{gm}}(S)$ etc. denotes the pushforward functor on geometric motives. In case S is a finite disjoint union of $\operatorname{Spec} \mathbb{F}_{\mathfrak{p}}$, the usual definition of (Artin–)Tate motives over S is recalled in Definition 2.1.

The following theorem and its "proof" is an overview of the paper.

Theorem 0.1. *The categories **DTM**(S) and **DATM**(S) are stable under standard functoriality operations such as $i^!$, j_* etc. for open and closed embeddings j and i, respectively.*

Both categories enjoy a non-degenerate t-structure called motivic t-structure. *Its heart is denoted **MTM**(S) or **MATM**(S), respectively and called category of* mixed (Artin–)Tate motives.

The functors i^*, j_* etc. feature exactness properties familiar from the corresponding situation of perverse sheaves. For example, $i^!$ is left-exact, and j_* is exact with respect to the motivic t-structure.

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The cohomological dimension of **MTM**(*S*) *and* **MATM**(*S*) *is one and two, respectively. We have an equivalence of categories*

$$\mathbf{D}^{\mathrm{b}}(\mathbf{MATM}(S)) \cong \mathbf{DATM}(S)$$

and likewise for Tate motives.

The "site" of mixed Artin–Tate motives over S has enough points in the sense that a mixed Artin–Tate motive over S is zero if and only if its restrictions to all closed points of S vanish.

Proof. The first statement is Theorem 2.4. It is proven using the localization, purity and base-change properties of geometric motives.

We will write $T(S)$ for either **DTM**(*S*) or **DATM**(*S*). The existence of the motivic t-structure on $T(S)$ is proven in three steps. The first ingredient is the well-known motivic t-structure on Artin–Tate motives over finite fields (Lemma 3.6). The second step is the study of a subcategory $\tilde{T}(S) \subset T(S)$ generated by $\phi_* \mathbf{1}(n)$, where ϕ is finite and étale (Artin–Tate motives), or just by $\mathbf{1}(n)$ (Tate motives). This category is first equipped with an auxiliary t-structure. Using the cohomology functor for the auxiliary t-structure, a motivic t-structure on $\tilde{T}(S)$ is defined in Section 3. This statement uses (and its proof imitates) the corresponding situation for Artin–Tate motives over number fields due to Levine and Wildeshaus. Thirdly, the t-structure on $\tilde{T}(S)$ is glued with the one over finite fields, using the general gluing procedure of t-structures of [2], see Theorem 3.8. Much the same way as with perverse sheaves, there are shifts accounting for dim $S = 1$, that is to say, $i_* \mathbf{1}(n)$ and $\mathbf{1}(n)[1]$ are mixed Tate motives. Beyond the formalism of geometric motives, the only non-formal ingredient of the motivic t-structure are vanishing properties of the algebraic K-theory of number rings, number fields and finite fields due to Quillen, Borel and Soulé.

The exactness statements are shown in Theorem 4.2. This theorem gives some content to the exactness axioms for general mixed motives over S [11, Section 4]. The key step stone is the following: for any immersion of a closed point $i : \mathrm{Spec}\,\mathbb{F}_\mathfrak{p} \to S$, the functor i^* maps the heart $T^0(S)$ of $T(S)$ to $T^{[-1,0]}(\mathrm{Spec}\,\mathbb{F}_\mathfrak{p})$, that is, the category of (Artin–)Tate motives over $\mathbb{F}_\mathfrak{p}$ whose only nonzero cohomology terms are in degrees -1 and 0. The proof is a careful reduction to basic calculations relying on facts gathered in Section 3 about the heart of $\tilde{T}(S)$.

The cohomological dimensions are calculated in Proposition 4.4. The Artin–Tate case is a special (but non-conjectural) case of a similar fact for general mixed motives over S. The difference in the Tate case is because the generators of **DTM**(*S*) have a good reduction at all places.

By an argument of Wildeshaus, the identity on $T^0(S)$ extends to a functor $\mathbf{D}^{\mathrm{b}}(T^0(S)) \to T(S)$ (Theorem 4.5). While it is an equivalence in the case of Tate motives for formal reasons, the Artin–Tate case requires some localization arguments.

The last statement is Proposition 4.6. It might be seen as a first step into motivic sheaves. □

Deligne and Goncharov define a category of mixed Tate motives over rings $\mathcal{O}_S$ of S-integers of a number field F [4, 1.4., 1.7.]. Unlike the mixed Tate motives we study, their category is a *sub*category of mixed Tate motives over F, consisting of motives subject to certain non-ramification constraints, akin to Scholl's notion of mixed motives over $\mathcal{O}_F$ [12].

This paper is an outgrowth of part of my thesis. I owe many thanks to Annette Huber for her advice during that time. I am also grateful to Denis-Charles Cisinski and Frédéric Déglise for teaching me their work on motives over general bases.

1. Geometric motives

This section briefly recalls some properties of the triangulated categories of geometric motives **DM**$_{\mathrm{gm}}$(*X*), where X is either a number field F or an open or closed subscheme of Spec $\mathcal{O}_F$. All of this is due to Cisinski and Déglise [3]. In this section, all references in brackets refer to *op. cit.*, e.g. [Section 14.1].

Let X be any of the afore-mentioned bases. There is the triangulated category **DM**(*X*) of Beilinson motives and its subcategory **DM**$_{\mathrm{gm}}$(*X*) of compact objects.[1] Objects of the latter category will be referred to as *geometric motives*. The categories are related by adjoint functors

$$f^* : \mathbf{DM}(X) \leftrightarrows \mathbf{DM}(Y) : f_*, \tag{1}$$

where $f : Y \to X$ is any map [13.2.11, 1.1.11]. If f is separated and of finite type this adjunction restricts to an adjunction between the subcategories of compact objects [14.1.5, 14.1.26] and there is an adjunction [13.2.11, 2.4.2]

$$f_! : \mathbf{DM}_{\mathrm{gm}}(Y) \leftrightarrows \mathbf{DM}_{\mathrm{gm}}(X) : f^!. \tag{2}$$

If f is smooth in addition, $f^* : \mathbf{DM}_{\mathrm{gm}}(X) \to \mathbf{DM}_{\mathrm{gm}}(Y)$ also has a left adjoint $f_\sharp$ [13.2.11, 1.1.2]. These five functors respect composition of morphisms in the sense that there are natural isomorphisms

$$f_* \circ g_* = (f \circ g)_*, \qquad f^* \circ g^* = (g \circ f)^* \text{ etc.} \tag{3}$$

for any two composable maps f and g [Section 1.1, 2.4.21]. The category **DM**$_{\mathrm{gm}}$(*X*) enjoys inner Hom's, denoted $\underline{\mathrm{Hom}}$, and a tensor structure such that pullback functors f^* are monoidal [13.2.11, 1.1.28]. The unit of the tensor structure is denoted $\mathbf{1}$.

---

[1] **DM**(*X*) is denoted **DM**$_{\mathrm{B}}$(*X*) in [3, Sections 13.2, 14.1].

In particular $f^* \mathbf{1}_X = \mathbf{1}_Y$ for $f : Y \to X$. The *motive* of any separated scheme $f : Y \to X$ of finite type is defined as $f_! f^! \mathbf{1}$ and denoted $\mathrm{M}(Y)$. (For f smooth, [Section 1.1.] puts $\mathrm{M}(Y) := f_\sharp f^* \mathbf{1}$. The two agree, see Lemma 1.2.) The tensor structure in $\mathbf{DM}_{\mathrm{gm}}(X)$ is such that

$$\mathrm{M}(Y) \otimes \mathrm{M}(Y') = \mathrm{M}(Y \times_X Y') \tag{4}$$

for any two smooth schemes Y and Y' over X [1.1.35]. There is a distinguished object $\mathbf{1}(1)$ such that $\mathrm{M}(\mathbb{P}^1_X) = \mathbf{1} \oplus \mathbf{1}(1)[2]$. Tensoring with $\mathbf{1}(1)$ is an equivalence on $\mathbf{DM}_{\mathrm{gm}}(X)$ [2.1.5], and $\mathbf{1}(n)$ is defined in the usual way in terms of tensor powers of $\mathbf{1}(1)$. We exclusively work with rational coefficients, i.e., all morphism groups are $\mathbb{Q}$-vector spaces. If X is regular, morphisms in $\mathbf{DM}_{\mathrm{gm}}(X)$ are given by

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}(X)}(\mathbf{1}, \mathbf{1}(q)[p]) \cong K_{2q-p}(X)^{(q)}_{\mathbb{Q}}, \tag{5}$$

the q-th Adams eigenspace in algebraic K-theory of X, tensored with $\mathbb{Q}$ [Section 13.2]. Having rational coefficients (or coefficients in a bigger number field) is vital when it comes to vanishing properties of Hom-groups in $\mathbf{DM}_{\mathrm{gm}}(X)$. (With integral coefficients, the existence of a t-structure is unclear even in the case of Artin motives over a field.)

For any closed immersion $i : Z \to X$ with open complement j we have the following functorial distinguished *localization* triangles in $\mathbf{DM}_{\mathrm{gm}}(X)$ [2.2.14, 2.3.3]:

$$j_! j^* \to \mathrm{id} \to i_* i^*. \tag{6}$$

Moreover $i^* i_* = \mathrm{id}$ [2.3.1], so that

$$i^* j_! = 0, \tag{7}$$

and i_* is fully faithful. There is an isomorphism of functors

$$f_! \xrightarrow{\cong} f_* \tag{8}$$

for any proper map f [2.2.14, 2.2.16]. For smooth and quasi-projective maps f of constant relative dimension d there is a *relative purity* isomorphism [Theorem 1, p. 5]

$$f^! \cong f^*(d)[2d]. \tag{9}$$

Moreover, when $i : Z \to X$ is a closed immersion of constant relative codimension c and Z and X are regular, we have an isomorphism

$$i^! \mathbf{1} \cong i^* \mathbf{1}(-c)[-2c]. \tag{10}$$

This is called *absolute purity* [Sections 2.4, 13.4]. Finally, for $f : Y \to X, g : X' \to X, f' : Y' := X' \times_X Y \to X'$ and $g' : Y' \to Y$, there is a natural *base-change* isomorphism of functors [Section 2.2]

$$f^* g_! \cong g'_! f'^*. \tag{11}$$

The *Verdier dual* functor $D_X : \mathbf{DM}_{\mathrm{gm}}(X)^{\mathrm{op}} \to \mathbf{DM}_{\mathrm{gm}}(X)$ is defined by $D_X(M) := \underline{\mathrm{Hom}}(M, \pi^! \mathbf{1}(1)[2])$ for any $M \in \mathbf{DM}_{\mathrm{gm}}(X)$, where $\pi : X \to \mathrm{Spec}\,\mathbb{Z}$ denotes the structural map.

Lemma 1.1. *For an open subscheme X of $\mathrm{Spec}\,\mathcal{O}_F$ we have*

$$D_X(-) = \underline{\mathrm{Hom}}(-, \mathbf{1}(1)[2]).$$

Secondly, we have $D_{\mathrm{Spec}\,\mathbb{F}_q}(-) = \underline{\mathrm{Hom}}(-, \mathbf{1})$.

Proof. The structural map $\pi : X \to \mathrm{Spec}\,\mathbb{Z}$ factors as

$$X \xrightarrow{j} \mathrm{Spec}\,\mathcal{O}_F \xrightarrow{i} \mathbb{A}^n_{\mathbb{Z}} \xrightarrow{p} \mathrm{Spec}\,\mathbb{Z},$$

where j is an open immersion, i is a closed immersion and p is the projection. Thus we have $\pi^! \mathbf{1} = \pi^* \mathbf{1}$ by absolute purity (10), applied to i, and relative purity (9), applied to j and p. Using (10) we get the second statement. $\square$

The Verdier dual functor exchanges "!" and "∗", that is, there are natural isomorphisms [Section 14.3]

$$D(f^! M) \cong f^* D(M), \qquad f_! D(M) \cong D(f_* M). \tag{12}$$

For example, the Verdier dual of (6) yields a distinguished triangle

$$i_* i^! \to \mathrm{id} \to j_* j^*. \tag{13}$$

Lemma 1.2. *For $f : X \to Y$ smooth, we have a natural isomorphism $f_! f^! \mathbf{1} = f_\sharp f^* \mathbf{1}$.*

Proof. This is well known. We can assume f is of constant relative dimension d. Then the claim follows from the adjunctions $f_\sharp \leftrightharpoons f^* \overset{(9)}{=} f^!(-d)[-2d]$ and $f_!(d)[2d] \leftrightharpoons f^!(-d)[-2d]$. $\square$

Let $X = \mathrm{Spec}\,\mathcal{O}_F$. The colimit over the triangles (13) over increasingly small open subschemes $j : U \subset X$ is still a distinguished triangle. For any geometric motive M over X we get the following distinguished triangle in $\mathbf{DM}(X)$:

$$\oplus_{\mathfrak{p}} i_{\mathfrak{p}*} i_{\mathfrak{p}}^! M \to M \to \eta_* \eta^* M, \tag{14}$$

where $\eta : \mathrm{Spec}\,F \to \mathrm{Spec}\,\mathcal{O}_F$ is the generic point, the sum runs over all closed points $\mathfrak{p} \in X$, $i_{\mathfrak{p}}$ is the closed immersion. Indeed $\mathrm{colim}\, j_* j^* M = \eta_* \eta^* M$ for any $M \in \mathbf{DM}_{\mathrm{gm}}(X)$ [Section 14.2].

2. Triangulated Artin–Tate motives

Recall the following classical definition. We apply it to a number field or a finite field:

Definition 2.1. Let K be a field. The category of Tate motives **DTM**(K) over K is by definition the triangulated subcategory of **DM**$_{gm}(K)$ generated by $\mathbf{1}(n)$ where $n \in \mathbb{Z}$. The smallest full triangulated subcategory **DATM**(K) stable under tensoring with $\mathbf{1}(n)$ and containing direct summands of motives $f_*\mathbf{1}$, where $f : K' \to K$ is any finite map, is called a category of Artin–Tate motives over K. For a scheme S of the form $S = \sqcup \text{Spec } K_i$, a finite disjoint union of spectra of fields, we put **DATM**$(S) := \oplus_i$**DATM**(K_i) and likewise for **DTM**.

This section gives a generalization of that definition to bases S which are open subschemes of Spec $\mathcal{O}_F$ based on the idea that Artin–Tate motives over S should be compatible with the ones over F and $\mathbb{F}_\mathfrak{p}$ under standard functoriality.

Definition 2.2. The categories **DTM**$(S) \subset$ **DM**$_{gm}(S)$ of *Tate motives* and **DATM**$(S) \subset$ **DM**$_{gm}(S)$ of *Artin–Tate motives* over S are the triangulated subcategories generated by the direct summands of

$$\mathbf{1}(n), \; i_*\mathbf{1}(n) \quad \text{(Tate motives)}$$

and

$$\phi_*\mathbf{1}(n), \quad \text{(Artin–Tate motives)}$$

respectively, where $n \in \mathbb{Z}$, $\phi : V \to S$ is any finite map (including those that factor over a closed point) and $i : \text{Spec } \mathbb{F}_\mathfrak{p} \to S$ is the immersion of any closed point of S.

Remark 2.3. • We can assume by localization (see (6), (13)) that the domain of ϕ is a reduced scheme.
• The category of Tate motives **DTM**(S) agrees with the triangulated category generated by the above generators (without taking direct summands). Indeed, by (5), the endomorphism rings End$(\mathbf{1}(n))$, End$(i_*\mathbf{1}(n))$ identify with $K_0(S)_{\mathbb{Q}}^{(0)}$ and $K_0(\mathbb{F}_\mathfrak{p})_{\mathbb{Q}}^{(0)}$, respectively, which are both one-dimensional over $\mathbb{Q}$. Hence these objects do not have any proper direct summands.

For brevity, we write $T(S)$ or T for **DATM**(S) *or* **DTM**(S) *in the sequel. In most proofs, we will only spell out the case of Artin–Tate motives.*

Theorem 2.4. *Let $j : S' \to S$ be any open immersion, $i : Z \to S$ be any closed immersion and $f : V \to S$ any finite map such that V is regular. Let $\eta : \text{Spec } F \to S$ be the generic point. Then the functors $f_* \overset{(8)}{=} f_!, f^*$ and $f^!$ preserve Artin–Tate motives. Similar statements hold for Artin–Tate and Tate motives for j and i. Moreover, η^*, the Verdier dual functor D and the tensor product on* **DM**$_{gm}(S)$ *respect the subcategories of (Artin–)Tate motives.*

The functor η_* does not respect Artin–Tate motives: we will see in Proposition 4.6 that any Artin–Tate motive M of the form $M = \eta_* M_\eta$, where M_η is a geometric motive over F, necessarily satisfies $M = 0$.

Proof. The stability of (Artin–)Tate motives under j^*, η^*, i_* and i^*, f^* and — for Artin–Tate motives, under f_* — is immediate from the definition, (8), and (11). For example, $i^*\phi_*\mathbf{1}(n) = \phi''_*\mathbf{1}(n)$. Here $\phi : S' \to S$ is any finite map and $\phi'' : Z' \to Z$ is its pullback along i. Let $i' : Z' \to S'$ be the pullback of i. For the stability under $i^!$ we use $i^!\phi_*\mathbf{1} \overset{(11)}{=} \phi''_* i'^! \mathbf{1}$. We can assume S' is reduced and, since the zero-dimensional case is easy, one-dimensional. Let $n : S'' \to S'$ be the normalization map; let $v : Y' \subset S'$ be the "exceptional divisor", i.e., the smallest (zero-dimensional) closed reduced subscheme such that $n^{-1}(S'\backslash Y') \to S'\backslash Y'$ is an isomorphism. Moreover, put $z : Y'' := Y' \times_{S'} S'' \to S'' \to S'$. Consider the the distinguished triangle

$$\mathbf{1}_{S'} \to v_*\mathbf{1}_{Y'} \oplus n_*\mathbf{1}_{S''} \to z_*\mathbf{1}_{Y''}.$$

It is a special case of [3, Theorem 4, p. 5] or can alternatively be derived from localization. Note that $i^! n_*\mathbf{1}_{S''} \overset{(11)}{=} n'_* i''^! \mathbf{1}_{S''} \overset{(10)}{=} n'_*\mathbf{1}(-1)[-2]$ by the regularity of S''. Here, again, n' and i'' denote the pullback maps. Similar considerations for $i^! v_*\mathbf{1}_{Y'}$ and $i^! z_*\mathbf{1}_{Y''}$ show that $i^!\mathbf{1}_{S'}$ is an Artin–Tate motive.

For the stability under j_* it is sufficient to show $j_*\phi'_*\mathbf{1}$ is an Artin–Tate motive over S for any finite flat map $\phi' : V' \to S'$. Choose some finite flat (possibly non-regular) model $\phi : V \to S$ of ϕ', i.e., $V \times_S S' = V'$, so that $j^*\phi_*\mathbf{1} = \phi'_*\mathbf{1}$ is an Artin–Tate motive over S'. The localization triangle (13)

$$i_* i^! \phi_*\mathbf{1} \to \phi_*\mathbf{1} \to j_* j^* \phi_*\mathbf{1}$$

and the above steps show that $j_*\phi'_*\mathbf{1}$ is an Artin–Tate motive over S.

To see the stability under the Verdier dual functor D, it is enough to see that

$$D(\phi_*\phi^*\mathbf{1}) \overset{(12)}{=} \phi_!\phi^! D(\mathbf{1}) \overset{1.1}{=} \phi_*\phi^!\mathbf{1}(1)[2]$$

is an Artin–Tate motive for any finite map $\phi : V \to S$ with reduced domain (Remark 2.3). If V is zero-dimensional, this follows from purity (10), (9) and the regularity of S. If not, there is an open (non-empty) immersion $j : S' \to S$ such that $V' := V \times_S S'$ is regular (for example, take S' such that V'/S' is étale). Let i be the complement of j. We apply the localization triangle (13) to $\phi_* \phi^! \mathbf{1}$. By base-change (11) we obtain

$$i_* \phi''_* \phi''^! i^! \mathbf{1} \to \phi_* \phi^! \mathbf{1} \to j_* \phi'_* \phi'^! j^* \mathbf{1}.$$

Here ϕ'' and ϕ' is the pullback of ϕ along i and j, respectively. By the regularity of S and purity we have $i^! \mathbf{1} = \mathbf{1}(-1)[-2]$, so the left hand term is an Artin–Tate motive. The right one also is by purity. This shows the claim for D.

The stability under $f^!$, $i^!$, and $j_!$ now follow for duality reasons.

As for the stability under tensor products we note that $\phi_* \mathbf{1} \otimes \phi'_* \mathbf{1} \overset{(4)}{=} (\phi \times \phi')_* \mathbf{1}$ if ϕ and ϕ' are (finite and) smooth, cf. (4). Using the localization triangle (6), it is easy to reduce the general case of merely finite maps ϕ, ϕ' to this case. $\quad\square$

Remark 2.5. Theorem 2.4 also holds for a similarly defined category of Artin–Tate motives over open subschemes S of a smooth curve over a field.

Proposition 2.6. *Let $M \in \mathbf{DATM}(S)$ be any Artin–Tate motive. Then there is a finite map $f : V \to S$ such that $f^* M \in \mathbf{DTM}(S) \subset \mathbf{DATM}(S)$. We describe this by saying that f* splits *M.*

Proof. As f^* is triangulated, this statement is stable under triangles (with respect to M), and also under direct sums and summands. Therefore, we only have to check the generators, i.e., $M = \phi_* \mathbf{1}(n)$ with $\phi : S' \to S$ a finite map with reduced domain. The corresponding splitting statement for Artin–Tate motives over finite fields is well-known. Therefore, by localization (6), (13), it is sufficient to find a splitting map f after replacing S by a suitable small open subscheme, so we may assume ϕ étale. We first assume that ϕ is moreover Galois of degree d, i.e., $S' \times_S S' \cong S'^{\sqcup d}$, a disjoint union of d copies of S'. In that case one has $\phi^* \phi_* \mathbf{1} = \mathbf{1}^{\oplus d}$ by base-change (11), so the claim is clear. In general ϕ need not be Galois, so let S'' be the normalization of S in some normal closure of the function field extension $k(S')/k(S)$. Both $\mu : S'' \to S$ and $\psi : S'' \to S'$ are generically Galois. By shrinking S we may assume both are Galois. From $\mathrm{Hom}(\mathbf{1}_{S'}, \psi_* \mathbf{1}_{S''}) = \mathrm{Hom}(\mathbf{1}_{S''}, \mathbf{1}_{S''}) = \mathbb{Q}$ and $\mathrm{Hom}(\psi_* \mathbf{1}_{S''}, \mathbf{1}_{S'}) = \mathrm{Hom}(\mathbf{1}_{S''}, \psi^! \mathbf{1}_{S'}) = \mathrm{Hom}(\mathbf{1}_{S''}, \mathbf{1}_{S''}) = \mathbb{Q}$ we see that $\mathbf{1}_{S'}$ is a direct summand of $\psi_* \mathbf{1}_{S''}$. Therefore $\mu^* \phi_* \mathbf{1}_{S'}$ is a summand of $\mu^* \phi_* \psi_* \mathbf{1}_{S''} = \mu^* \mu_* \mathbf{1}_{S''} = \mathbf{1}^{\oplus \deg S''/S}$, a Tate motive. $\quad\square$

3. The motivic t-structure

In this section, we establish the motivic t-structure on the category of Artin–Tate motives over S (Theorem 3.8). It is obtained by the standard gluing procedure, applied to the t-structures on Artin–Tate motives over finite fields and on a subcategory $\tilde{T}(S') \subset T(S')$ for open subschemes $S' \subset S$. Under the analogy of mixed (Artin–Tate) motives with perverse sheaves, the objects in the heart of the t-structure on $\tilde{T}(S')$ correspond to sheaves that are locally constant, i.e., have good reduction. We refer to [2, Section 1.3.] for generalities on t-structures.

Definition 3.1 (*Compare [7, Def. 1.1]*). For $-\infty \le a \le b \le \infty$, let $\tilde{T}_{[a,b]}$ denote the smallest triangulated subcategory of $T(S)$ containing direct factors of $\phi_* \mathbf{1}(n), a \le -2n \le b$, where $\phi : S' \to S$ is a finite *étale* map. For Tate motives, ϕ is required to be the identity map. (We will not specify this restriction *expressis verbis* in the sequel.) Furthermore, $\tilde{T}_{[a,a]}$ and $\tilde{T}_{[-\infty,\infty]}$ are denoted $\tilde{T}_a$ and $\tilde{T}$. If it is necessary to specify the base, we write $\tilde{T}_{[a,b]}(S)$ etc.

We need the following vanishing properties of the K-theory of number fields, related Dedekind rings and finite fields up to torsion. In order to weigh the material appropriately, it should be said that the content of the theorem below is the only non-formal part of the proofs in this paper, and all complexity occurring with Artin–Tate motives ultimately lies in these computations.

Theorem 3.2 (*Borel, Quillen, Soulé*). *Let $\phi : S' \to S$ and $\psi : V \to S$ be two finite maps with zero-dimensional domains.*

$$\mathrm{Hom}_S(\phi_* \mathbf{1}, \psi_* \mathbf{1}(n)[m]) = \begin{cases} \textit{finite-dimensional} & n = m = 0 \\ 0 & \textit{else.} \end{cases}$$

Now let $\phi : S' \to S$ and $\psi : V \to S$ be two finite étale maps over S. Then

$$\mathrm{Hom}_S(\phi_* \mathbf{1}, \psi_* \mathbf{1}(n)[m]) = \begin{cases} \textit{finite-dimensional} & n = m = 0 \\ \textit{finite-dimensional} & m = 1, \ n \textit{ odd and positive} \\ 0 & \textit{else.} \end{cases}$$

Proof. By (5)

$$\mathrm{Hom}_V(\mathbf{1}, \mathbf{1}(q)[p]) \cong K_{2q-p}(V)^{(q)}_{\mathbb{Q}},$$

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for a regular scheme V. For the first statement, we may assume that S' and V are finite fields. Then the statement follows from adjunction, base-change, purity and

$$K_n(\mathbb{F}_q) = \begin{cases} \mu_{q^i-1} & n = 2i - 1, i > 0 \\ 0 & n = 2i, i > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

[10]. K-theory of Dedekind rings R whose quotient field is a number field is known (up to torsion) by Borel's work. The relation to the K-theory of number fields is given by an exact sequence (due to Soulé [13, Th. 3]; up to two-torsion) for $n > 1$

$$0 \to K_n(R) \xrightarrow{\eta^*} K_n(F) \to \oplus_{\mathfrak{p}} K_{n-1}(\mathbb{F}_{\mathfrak{p}}) \to 0.$$

Here $\eta : \operatorname{Spec} F \to \operatorname{Spec} R$ is the generic point and the direct sum runs over all (finite) primes in R. Also, $K_0(R) = \mathbb{Z} \oplus Pic(R)$ and $K_1(R) = R^\times$. In particular, for all n and m, $K_n(R)_{\mathbb{Q}}^{(m)}$ vanishes when $K_n(F)_{\mathbb{Q}}^{(m)}$ vanishes, since η^* respects the Adams grading. One has the following list (see e.g. [15])

$$K_{2q-p}(F)_{\mathbb{Q}}^{(q)} = \begin{cases} 0 & q < 0 \\ 0 & q = 0, p \neq 0 \\ \mathbb{Q} & q = p = 0 \\ 0^{BS} & q > 0, p \leq 0 \\ 0 & q > 0, \text{ even}, p = 1 \\ F^\times \otimes_{\mathbb{Z}} \mathbb{Q} & q = p = 1 \\ \mathbb{Q}^{r_1+r_2} & q > 1, q \equiv 1 \,(\mathrm{mod}\,4), p = 1 \\ \mathbb{Q}^{r_2} & q > 0, q \equiv 3 \,(\mathrm{mod}\,4), p = 1 \\ 0 & q > 0, p > 1. \end{cases}$$

As usual, r_1 and r_2 are the numbers of real and pairs of complex embeddings of F, respectively. (The agreement of $K_{2q-1}(F)$ and $K_{2q-1}(F)^{(q)}$ for odd positive q is not mentioned in [15].) The spot marked 0^{BS} is referred to as *Beĭlinson–Soulé vanishing* (see e.g. [7]). As first realized by Levine [7], this translates into the non-existence of morphisms in the "wrong" direction with respect to the motivic t-structure.

For the last claim, put $V' = V \times_S S'$:

$$\begin{array}{ccc} V' & \xrightarrow{\phi'} & V \\ \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi} \\ S' & \xrightarrow{\phi} & S. \end{array}$$

To save space, we omit the twist and the shift in writing the Hom-groups. By (2), (11), and (1) we have

$$\operatorname{Hom}_S(\phi_* \mathbf{1}, \psi_* \mathbf{1}) = \operatorname{Hom}_{S'}(\mathbf{1}, \phi^! \psi_* \mathbf{1}) = \operatorname{Hom}_{S'}(\mathbf{1}, \psi'_* \phi'^! \mathbf{1}) = \operatorname{Hom}_{V'}(\mathbf{1}, \phi'^! \mathbf{1}).$$

Now, V' is (affine and) étale over V, so $\phi'^! \mathbf{1} \overset{(9)}{=} \phi'^* \mathbf{1} = \mathbf{1}$ by (9) and we are done in that case by the above vanishings of the K-theory up to torsion. $\square$

The following lemma is a variant of [7, Lemma 1.2], [16, Lemma 1.9] and can be proven by faithfully imitating the technique in *loc. cit.*

Lemma 3.3. *For any $-\infty \leq a < b \leq c \leq \infty$, $(\tilde{T}_{[a,b-1]}, \tilde{T}_{[b,c]})$ is a t-structure on $\tilde{T}_{[a,c]}$.*

Definition 3.4. The resulting truncation and cohomology functors are denoted $F_{\leq b}$ and $F_{>b}$ and gr_b^F, respectively.

The following definition is modeled on [7, Def. 1.4]. We also refer to [1, Section 2.1.3] for a general way (due to Morel) of constructing a t-structure starting from a given set of generators. For any odd integer n set $\mathbf{1}(n/2) := 0$, for notational convenience.

Definition 3.5. Let S be an open subscheme of $\operatorname{Spec} \mathcal{O}_F$. Let $\tilde{T}_a^{\geq 0}(S)$ $(\tilde{T}_a^{\leq 0}(S))$ be the full subcategory of $\tilde{T}_a(S)$ (Definition 3.1) generated by direct summands of

$$\phi_* \mathbf{1}\left(-\frac{a}{2}\right)[n+1]$$

for any $n \leq 0$ ($n \geq 0$, respectively), and any finite étale map ϕ. "Generated" means the smallest subcategory containing the given generators stable under isomorphism, finite direct sums, and $\mathrm{cone}(f)[-1]$ ($\mathrm{cone}(f)$, resp.) for any morphism f in $\tilde{T}_a^{\geq 0}(S)$ ($\tilde{T}_a^{\leq 0}(S)$, respectively).

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For any $-\infty \leq a \leq b \leq \infty$, let $\tilde{T}^{\geq 0}_{[a,b]}(S)$ be the triangulated subcategory generated by objects X, such that for all $a \leq c \leq b$, $\mathrm{gr}^F_c(X) \in \tilde{T}^{\geq 0}_c(S)$ and similarly for $\tilde{T}^{\leq 0}_{[a,b]}(S)$. For $a = -\infty$ and $b = \infty$ we simply write $\tilde{T}^{\leq 0}(S)$, $\tilde{T}^{\geq 0}(S)$. We may omit S in the notation, if no confusion arises.

In particular $\mathbf{1}(-a/2)[1] \in \tilde{T}^0_a(S)$. This shift is as in the situation of perverse sheaves [2], [11, Section 3]. Before stating and proving the existence of the motivic t-structure, we need some preparatory steps. Levine has established the existence of the motivic t-structure on Tate motives over number fields and finite fields [7, Theorem 1.4.]. This has been generalized to Artin–Tate motives by Wildeshaus [16, Theorem 3.1]. We briefly recall these precursor statements. Let K be either a finite field or a number field. For any $-\infty \leq a \leq b \leq \infty$, let $T_{[a,b]}(K)$ be the triangulated subcategory of $T(K)$ generated by $\mathbf{1}(n)$ with $a \leq -2n \leq b$ (Tate motives) and direct summands of $\phi_* \mathbf{1}(n)$, $\phi : \mathrm{Spec}\, K' \to \mathrm{Spec}\, K$ a finite map (Artin–Tate motives, respectively). For any $a \leq c < b$, the datum $\left(T_{[a,c]}, T_{[c+1,b]} \right)$ forms a t-structure on $T_{[a,b]}$. Let gr^F_* be the cohomology functor corresponding to that t-structure. Write $T_a(K)$ for $T_{[a,a]}(K)$ and let $T^{\geq 0}_a(K)$ and $T^{\leq 0}_a(K)$ be the subcategories of $T_a(K)$ generated by $\mathbf{1}(-a/2)[n]$ with $n \leq 0$ and $n \geq 0$, respectively. Here, "generated" has the same meaning as in Definition 3.5. Let $T^{\geq 0}_{[a,b]}$ and $T^{\leq 0}_{[a,b]}$ be the subcategories of $T_{[a,b]}$ of objects X such that all $\mathrm{gr}^F_c X \in T^{\geq 0}_c$ ($\mathrm{gr}^F_c X \in T^{\leq 0}_c$, respectively) for all $a \leq c \leq b$. Then, $\left(T^{\leq 0}_{[a,b]}(K), T^{\geq 0}_{[a,b]}(K) \right)$ is a non-degenerate t-structure on $T_{[a,b]}$.

The following well-known fact is a consequence of vanishing of all K-theory groups of finite fields except for $K_0(\mathbb{F}_{\mathfrak{p}})^{(0)}_{\mathbb{Q}}$, see Theorem 3.2.

Lemma 3.6. *Let $\mathfrak{p}$ be a closed point in S with residue field $\mathbb{F}_{\mathfrak{p}}$. The inclusions $T_a(\mathbb{F}_{\mathfrak{p}}) \subset T(\mathbb{F}_{\mathfrak{p}})$ induce an equivalence of categories*

$$\bigoplus_{a \in \mathbb{Z}} T_a(\mathbb{F}_{\mathfrak{p}}) = T(\mathbb{F}_{\mathfrak{p}}).$$

There are canonical equivalences of categories

$$T(Z) := \bigoplus_{\mathfrak{p} \in Z, a \in \mathbb{Z}} T_a(\mathbb{F}_{\mathfrak{p}}) = \bigoplus_{\mathfrak{p}, a} \mathbf{D}^{\mathrm{b}}(\underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}})]) = \bigoplus_{\mathfrak{p}, a} \underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}})]^{\mathbb{Z}-\mathrm{graded}}.$$

Here and in the sequel, $\underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}})]$ denotes finite-dimensional rational permutation representations of the absolute Galois group. By means of that equivalence, $T(Z)$ is endowed with the obvious t-structure. The heart $T^0_a(\mathbb{F}_{\mathfrak{p}}) = T^{\leq 0}_a(\mathbb{F}_{\mathfrak{p}}) \cap T^{\geq 0}_a(\mathbb{F}_{\mathfrak{p}})$ is semisimple and consists of direct sums of summands of $\phi_* \mathbf{1}(a)$, ϕ finite.

We now provide the motivic t-structure on $\tilde{T}(S)$, which stems from the one on $T(F)$. The two together will then be glued to give the t-structure on $T(S)$. Recognizably, the following is again an adaptation of Levine's proof of the t-structure on Tate motives over number fields.

Proposition 3.7. *For any $-\infty \leq a \leq b \leq \infty$, $\left(\tilde{T}^{\leq 0}_{[a,b]}, \tilde{T}^{\geq 0}_{[a,b]} \right)$ is a non-degenerate t-structure on $\tilde{T}_{[a,b]}(S)$ (Definitions 3.1 and 3.5). The cohomology functors associated to it are denoted ${}^p\mathrm{H}^*$. The functor $\eta^*[-1] : \tilde{T}_{[a,b]}(S) \to T_{[a,b]}(F)$ is t-exact.*

Any motive in $\tilde{T}^0_a(S)$ is a finite direct sum of summands of motives $\phi_* \mathbf{1}(-a/2)[1]$ with ϕ finite étale. The closure of the direct sum of the $\tilde{T}^0_a(S)$, $a \in \mathbb{Z}$, under extensions (in the abelian category $\tilde{T}^0(S)$) is $\tilde{T}^0(S)$.

Proof. We may assume that a and b are finite, since

$$\tilde{T}(S) = \bigcup_{-\infty < a \leq b < \infty} \tilde{T}_{[a,b]}(S)$$

and the inclusion functors given by the identity between the various $T_{[-,-]}$ are exact.

The proof proceeds by induction on $b - a$. The case $b = a$ is treated as follows: the category $\tilde{T}_a := \tilde{T}_a(S)$ is generated by $\phi_* \mathbf{1}(-a/2)[n]$, $n \in \mathbb{Z}$, ϕ étale and finite. The functor $\eta^*[-1](a/2) : \tilde{T}_a(S) \to T_0(F)$ is fully faithful. To see this it suffices to remark $\mathrm{Hom}_S(\phi_* \mathbf{1}(-a/2)[n+1], \psi_* \mathbf{1}(-a/2)[n'+1]) = \mathrm{Hom}_F(\phi_{\eta_*} \mathbf{1}[n], \psi_{\eta_*} \mathbf{1}[n'])$, for any finite étale maps ϕ and ψ with generic fiber ϕ_η and ψ_η. This equality follows from the K-theory computations, see the proof of Theorem 3.2. Therefore, the image of $\eta^*[-1](a/2)$ is a triangulated subcategory of $T_0(F)$ which contains the generators of $T_0(F)$, so the functor establishes an equivalence between $\tilde{T}_a(S)$ with the derived category of finite-dimensional rational permutation representations of $\mathrm{Gal}(F)$ by [14, 3.4.1]. Hence $\tilde{T}_a(S)$ carries a non-degenerate t-structure.

The remainder of the proof is done as in Levine's proof. One shows

$$\mathrm{Hom}\left(\tilde{T}^{\leq 0}_{[a+1,b]}, \tilde{T}^{\geq 0}_c \right) = 0 \tag{15}$$

for any $c \leq a$. This reduces to the Beĭlinson–Soulé vanishing. Then the t-structure axioms follow for formal reasons.

The exactness of $\eta^*[-1]$ is obvious from the definitions. The statement about the heart $\tilde{T}^0_a$ is done as follows: the exact functor $\eta^*[-1](a/2)$ identifies $\tilde{T}^0_a(S) = \tilde{T}^{\geq 0}_a(S) \cap \tilde{T}^{\leq 0}_a(S)$ with the semi-simple category $T^0_0(F) = \underline{\mathbb{Q}}[\mathrm{Perm}, \mathrm{Gal}(F)]$. We claim that for any object $X \in \tilde{T}_a(S)$, all ${}^p\mathrm{H}^n(X)$ are direct summands of sums of motives $\phi_* \mathbf{1}(-a/2)[1]$, ϕ finite and étale. This claim

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does hold for the generators of $\tilde{T}_a(S)$. We now show that the condition is stable under triangles, which accomplishes the proof of the claim and thus the proof of the statement. Let $A \to X \to B$ be a triangle in $\tilde{T}_a(S)$ such that A and B satisfy the claim. The long exact cohomology sequence

$$\cdots \to {}^{\mathrm{p}}\mathrm{H}^{n-1}B \xrightarrow{\delta^{n-1}} {}^{\mathrm{p}}\mathrm{H}^n A \to {}^{\mathrm{p}}\mathrm{H}^n X \to {}^{\mathrm{p}}\mathrm{H}^n B \xrightarrow{\delta^n} {}^{\mathrm{p}}\mathrm{H}^{n+1}A \to \cdots$$

yields the short exact sequence in $\tilde{T}_a^0(S)$

$$0 \to \operatorname{coker}\delta^{n-1} \to {}^{\mathrm{p}}\mathrm{H}^n X \to \ker\delta^n \to 0.$$

By the semi-simplicity of $\tilde{T}_a^0(S)$ (this is the key point!), the sequence splits and there is a non-canonical isomorphism ${}^{\mathrm{p}}\mathrm{H}^n X \cong \operatorname{coker}\delta^{n-1} \oplus \ker\delta^n$ and $\operatorname{coker}\delta^{n-1}$ and $\ker\delta^n$ are direct summands of ${}^{\mathrm{p}}\mathrm{H}^n A$ and ${}^{\mathrm{p}}\mathrm{H}^n B$, respectively.

For the statement concerning $\tilde{T}^0(S)$ one uses the finite exhaustive F-filtration of any $X \in \tilde{T}^0(S)$:

$$0 = F_a X \subset F_{[a,a+1]}X \subset \cdots \subset F_{[a,b]}X = X.$$

The successive quotients $\operatorname{gr}_*^F X$ of that chain are in $\tilde{T}_*^0(S)$, since truncations with respect to the t-structure related to F are exact with respect to the motivic t-structure, by definition. Thus the claim about $\tilde{T}^0(S)$ follows. $\square$

Theorem 3.8. *The motivic t-structures on $T(Z)$ and $\tilde{T}(S')$ glue to a non-degenerate t-structure on the category $T(S)$ of (Artin–)Tate motives over S (Definition 2.2). It is called* motivic t*-structure. Here S' runs through open subschemes of S and $Z := S \backslash S'$.*

Proof. We apply the gluing procedure of t-structures of [2, Theorem 1.4.10]: for any open subscheme $j : S' \subset S$, we write $T_{S'}(S)$ for the full triangulated subcategory of objects $X \in T(S)$ such that $j^*X \in \tilde{T}(S') \subset T(S')$. Let $i : Z' \to S$ be the closed complement of j. Put

$$T_{S'}^{\leq 0}(S) := \{X \in T_{S'}(S), j^*X \in \tilde{T}^{\leq 0}(S'), i^*X \in T^{\leq 0}(Z')\},$$
$$T_{S'}^{\geq 0}(S) := \{X \in T_{S'}(S), j^*X \in \tilde{T}^{\geq 0}(S'), i^!X \in T^{\geq 0}(Z')\}.$$

The assumptions of the gluing theorem, [2, 1.4.3], namely the existence of $i_*, i^*, i^!, j_*, j_!, j^*$ satisfying the usual adjointness properties, $j^*i_* = 0$, localization sequences and full faithfulness of $i_*, j_!$ and j_* are met, since they are in the surrounding categories of geometric motives, cf. Section 1, and the stability of the subcategories of Artin–Tate motives under these functors (Theorem 2.4). Thus, the above defines a t-structure on $T_{S'}(S)$.

The field F is of characteristic zero, so any finite map $\phi : V \to S$ with V reduced and one-dimensional is generically étale. This implies $T(S) = \cup_{S' \subset S} T_{S'}(S)$. We set

$$T^{\geq 0}(S) := \bigcup_{S' \subset S} T_{S'}^{\geq 0}(S)$$

and dually for $T^{\leq 0}(S)$. The t-structure axioms on $T(S)$ and the non-degeneracy are implied by the exactness of the identical inclusion $T_{S'}(S) \to T_{S''}(S)$ for any $S'' \subset S'$.

To see the exactness of the identity, let $j'' : S'' \subset S$ and $i'' : Z'' \subset S$ be its complement. Let $X \in T_{S'}^{\leq 0}(S)$. It is clear that $j''^*X \in \tilde{T}^{\leq 0}(S'')$. Let us check $i''^*X \in T^{\leq 0}(Z'')$. The pullback i''^*X decomposes as a direct sum parametrized by the points of Z'' and we only have to deal with the points that are not contained in Z'. Let $p : \operatorname{Spec}\mathbb{F}_{\mathfrak{p}} \to S$ be such a point; it factors over S': $p = j \circ q$, where $q : \operatorname{Spec}\mathbb{F}_{\mathfrak{p}} \to S'$ is the same point as p. Thus $p^*X \overset{(3)}{=} q^*j^*X \in q^*\tilde{T}^{\leq 0}(S')$. The containment $q^*\tilde{T}^{\leq 0}(S') \subset T^{\leq 0}(\operatorname{Spec}\mathbb{F}_{\mathfrak{p}})$ follows from $q^*\tilde{T}_a^{\leq 0}(S') \subset T_a^{\leq 0}(\operatorname{Spec}\mathbb{F}_{\mathfrak{p}})$, since q^* clearly commutes with the F-truncation functors belonging to the auxiliary t-structure. To see the latter containment, it suffices to check the generators (in the sense of Definition 3.5) of $\tilde{T}_a^{\leq 0}(S')$, that is, it is sufficient to remark

$$q^*\phi_* \mathbf{1}(-a/2)[n+1] \overset{(11)}{=} \phi'_* \mathbf{1}(-a/2)[n+1] \in T_a^{\leq -1}(\operatorname{Spec}\mathbb{F}_{\mathfrak{p}}) \subset T_a^{\leq 0}(\operatorname{Spec}\mathbb{F}_{\mathfrak{p}}),$$

where $n \geq 0$ and ϕ is a finite étale map with pullback ϕ'. This shows that the identity is left-exact. The right-exactness is done dually. $\square$

4. Mixed Artin–Tate motives

Definition 4.1. The heart $T^0(S)$ of the motivic t-structure is called the category of *mixed (Artin–)Tate motives* over S, denoted **MTM**(S) and **MATM**(S), respectively. The cohomology functors belonging to the motivic t-structure are denoted ${}^{\mathrm{p}}\mathrm{H}^*$.

We now study the categories of mixed Tate motives over S in some detail. The key is Theorem 4.2 below, establishing exactness properties of pullback and pushforward functors along closed and open immersions. The exactness axioms for mixed motives over number rings (see [11, Section 4]) are modeled on this theorem. Of course, the theorem is an Artin–Tate

motivic analog of a similar fact about perverse sheaves [2, Prop. 1.4.16, 4.2.4.], suggesting that the theory of perverse sheaves is to some extent quite formal. Proposition 4.4 calculates the cohomological dimension of mixed (Artin–)Tate motives. We obtain an equivalence $\mathbf{DTM}(S) \cong \mathbf{D}^b(\mathbf{MTM}(S))$, using a result of Wildeshaus, and likewise for Artin–Tate motives. Finally, we do a first step into (Artin–Tate) motivic sheaves, in Proposition 4.6.

All exactness statements below are with respect to the motivic t-structure of Theorem 3.8. Recall from Theorem 2.4 that the functors discussed below do preserve (Artin–)Tate motives. For brevity, we write $T^{[a,b]}$ for the full subcategory of objects M satisfying ${}^p\mathrm{H}^n M = 0$ for all $n < a$ and $n > b$. We say that a triangulated functor F between categories of Artin–Tate motives has *cohomological amplitude* $[a, b]$ if $F(T^0)$ is contained in $T^{[a,b]}$. Note that F is right exact iff $b \leq 0$ and left exact iff $a \geq 0$.

Theorem 4.2. *Let $j : S' \to S$ be an open immersion, $i : Z \to S$ a closed immersion with $\dim Z = 0$. Finally, let $f : V \to S$ be a finite map with regular one-dimensional domain.*

(i) *The Verdier duality functor D is exact in the sense that it maps $T^{\geq 0}$ to $T^{\leq 0}$ and vice versa. Therefore, it induces an endofunctor on $T^0(S)$.*
(ii) *The functors $j_*, j_!, j^*$, as well as $i_* = i_!$ are exact.*
(iii) *The functor i^* has cohomological amplitude $[-1, 0]$. Dually, $i^!$ has cohomological amplitude $[0, 1]$.*
(iv) *The functor $f_* = f_!$ is exact. The cohomological amplitude of f^* and $f^!$ is $[-1, 0]$ and $[0, 1]$, respectively. If f is also étale, $f^* = f^!$ is exact.*
(v) *The functor $\eta^*[-1] : T(S) \to T(\mathrm{Spec}\, F)$ is exact.*

Proof. (i) This is clear from (12) and the definitions of the t-structures on $T(S)$, $\tilde{T}(S')$ and $T(Z)$, for open and closed subschemes S' and Z of S, respectively. Notice that this requires putting $\mathbf{1}[1]$ in degree 0.

(ii) The following exactness properties are immediate from the definition: j^* and i_* are exact, j_* and $i^!$ are left-exact and $j_!$ and i^* are right-exact. For example, let us show the left-exactness of j_*. Given some motive $M \in T^{\geq 0}(S')$, we have to show $j_*M \in T^{\geq 0}(S)$. Let $j_1 : S_1 \subset S'$ be an open immersion such that $j_1^*M \in \tilde{T}^{\geq 0}(S_1)$. Let i_1 be the immersion of $Z_1 := S'\backslash S_1$ into S', then $i_1^! M \in T^{\geq 0}(Z_1)$. The situation is as follows:
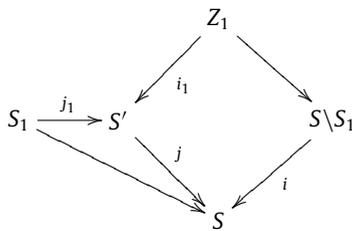
Now $(j \circ j_1)^* j_* M = j_1^* M \in T^{\geq 0}(S_1)$. Let $i : S\backslash S_1 \to S$ be the complement of $j \circ j_1$. By (7), $i^! j_* M$ is supported only in Z_1, where it agrees with $i_1^! M$. This shows $j_* M \in T^{\geq 0}(S)$.

To prove (iii) we first show

$$i^* j_* \tilde{T}^0(S') \subset T^{[-1,0]}(Z) \tag{16}$$

for any two complementary immersions $i : Z \to S$ (closed) and $j : S' \to S$ (open). By Proposition 3.7, $\tilde{T}^0(S)$ is generated by means of direct sums and extensions by summands of $\phi_* \mathbf{1}(n)[1]$, where $n \in \mathbb{Z}$ is arbitrary and ϕ is finite and étale. For any short exact sequence

$$0 \to A \to X \to B \to 0$$

in $\tilde{T}^0(S)$, such that $i^* j_* A \in T^{[-1,0]}(Z)$ and $i^* j_* B \in T^{[-1,0]}(Z)$, it follows $i^* j_* X \in T^{[-1,0]}(Z)$. This uses the non-degeneracy of the motivic t-structure on Z. A similar remark applies to direct summands and sums. Therefore we only have to check that the generators $X = \phi_* \mathbf{1}(n)[1]$ of $\tilde{T}^0(S')$ are mapped to $T^{[-1,0]}(Z)$ under $i^* j_*$. By (13), there is a distinguished triangle in $T(Z)$

$$i^* \phi_* \mathbf{1}(n)[1] \to i^* j_* j^* \phi_* \mathbf{1}(n)[1] \stackrel{(11)}{=} i^* j_* \phi'_* \mathbf{1}(n)[1] \to i^! \phi_* \mathbf{1}(n)[2] \to i^* \phi_* \mathbf{1}(n)[2].$$

Here ϕ' is the pullback of ϕ along j. The first term is in degree -1. The third term is in degree 0 by absolute purity (10), using the regularity of S. The claim (16) is shown.

We now show $i^* T^0(S) \subset T^{[-1,0]}(Z)$. Any $X \in T^0(S)$ is in some $T^0_{S'}(S)$ for sufficiently small S'. We shrink S' if necessary to ensure that $S' \cap Z = \emptyset$. Let $j : S' \to S$ be the open immersion and let $p : W \to S$ be its closed complement. There is a triangle

$$p^! X \to p^* X \to p^* j_* j^* X \to p^! X[1].$$

By the above, $p^!$ (p^*) is left-exact (right-exact), that is to say, the first (second) term is in degrees ≥ 0 (≤ 0, respectively). By assumption $j^* X \in \tilde{T}^0(S')$, so $p^* j_* j^* X \in T^{[-1,0]}(W)$ by (16). As the t-structure on W is non-degenerate $p^* X$ is in degrees $[-1, 0]$. As W is the disjoint union of Z and some more (finitely many) closed points, this also shows $i^* X \in T^{[-1,0]}(Z)$.

Now let $i : Z \to S$ and $j : S' \to S$ be complementary. We claim $i^* j_* T^0(S') \subset T^{[-1,0]}(Z)$. Given an object $X \in T^0(S')$, there is some open immersion $j' : S'' \to S'$ such that $j'^* X \in \tilde{T}^0(S'')$. We have $i^* j_* X = i^* j_* j'_* j'^* X$. The motive $i_* i^* j_* j'_* j'^* X$ is a direct summand of $p_* p^* (j \circ j')_* j'^* X$, where p is the complement of $j \circ j'$. By the above, $p^* (j \circ j')_* j'^* X \in T^{[-1,0]}(Z)$, so the full faithfulness and exactness of p_* implies the claim. Part (iii) is shown.

The cohomological amplitude of $i^* j_*$ implies the exactness of j_*: given a mixed (Artin–)Tate motive $M \in T^0(S')$, the terms in the localization triangle

$$j_! M \to j_* M \to i_* i^* j_* M$$

are in degrees ≤ 0, ≥ 0 and $[-1, 0]$, respectively, by the above. From the non-degeneracy of the t-structure we see that $j_* M$ is then in degree 0. This implies the exactness of j_* by the non-degeneracy of the t-structure. The exactness of $j_!$ follows by the Verdier duality, as does the cohomological amplitude of $i^!$. Thus, (ii) is shown.

(iv) It is easy to see that $f^* : \tilde{T}(S) \to \tilde{T}(V)$ is exact. Using this and (6), one sees that f^* has cohomological amplitude $[-1, 0]$ and dually for $f^!$. By a general criterion on t-exactness of adjoint functors [2, 1.3.17], the adjunctions $f^* \leftrightarrows f_* \overset{(8)}{=} f_! \leftrightarrows f^!$ imply that f_* is exact. If f is étale then $f^! \overset{(9)}{=} f^*$, so that their exactness is clear in that case, too.

(v) This follows from the exactness of $j^* : T(S) \to T(S')$ and the exactness of $\eta'^*[-1] : \tilde{T}(S') \to T(\mathrm{Spec}\, F)$ (Proposition 3.7), where η' is the generic point of S'. $\square$

Definition 4.3 (*Compare [2, 1.4.22]*)**.** Let $j : S' \to S$ be an open immersion. For any mixed (Artin–)Tate motive M over S', put

$$j_{!*} M := \mathrm{im}\, j_! M \to j_* M.$$

This is called the *intermediate extension* of M along j.

The image is taken in the (abelian) category of mixed (Artin–)Tate motives over S, using the exactness of $j_!$ and j_*. Thereby, $j_{!*}$ is a (non-exact) functor $T^0(S') \to T^0(S)$. Given any mixed motive M over S, such that $i^! M$ is concentrated in cohomological degree -1 (as opposed to the general range $[-1, 0]$), and such that $i^* M$ is in degree $+1$, there is a canonical isomorphism

$$j_{!*} j^* M = M. \tag{17}$$

In particular, this applies to $M \in \tilde{T}^0(S)$, such as $M = \mathbf{1}[1]$. Moreover, taking the intermediate extension commutes with compositions of open immersions. These features will be used below, see [11, Section 4] for a proof. The reader may want to check that that proof only uses the motivic t-structure and exactness properties of $i^!$ etc., which are established by Theorems 3.8, 4.2.

Proposition 4.4. *The cohomological dimension of* **DTM**(S) *and* **DATM**(S) *is one and two, respectively.*

Proof. We have to show $\mathrm{Hom}(M, M'[n]) = 0$ for any mixed motives M, M' over S and $n > 1$ (Tate) and $n > 2$ (Artin–Tate). Let $j : S' \to S$ be an open immersion such that $j^* M, j^* M' \in \tilde{T}^0(S')$. Let i be the complementary closed immersion of j. In the sequel we write $(-, -)^n$ for $\mathrm{Hom}(-, -[n])$ for brevity.

The case $n \geq 3$ is done as follows: the localization triangle (13) for M' and adjunction (1) gives a long exact sequence

$$(\underbrace{i^* M}_{[-1,0]}, \underbrace{i^! M'[n]}_{[-n,-n+1]})^0 \to (M, M')^n \to (M, j_* j^* M')^n \to (\underbrace{i^* M}_{[-1,0]}, \underbrace{i^! M'[n+1]}_{[-n-1,-n]})^0.$$

We have written the cohomological degrees of the motives underneath, using the cohomological range of i^* and $i^!$. The cohomological dimension zero of (Artin–)Tate motives over finite fields makes the outer terms vanish. Similar vanishings will be used below without further discussion. Hence we only have to look at $(j^* M, j^* M')^n$, i.e., we may assume M and $M' \in \tilde{T}^0(S)$. In that case one reduces (exactly as below) to $M = \phi_* \mathbf{1}(a)[1]$ and $M = \phi'_* \mathbf{1}(a')[1]$, where ϕ and ϕ' are finite and étale. In that case the vanishing is given by Theorem 3.2.

The vanishing in the case $n = 2$ for Tate motives needs a more involved localization argument. A similar reasoning for Artin–Tate motives fails—the difference is because the motives $\mathbf{1}(n)[1]$, which generate $\tilde{T}^0(S)$ in the case of Tate motives, have good reduction at all places by absolute purity.

The localization triangle (6) for M' gives an exact sequence

$$(M, j_! j^* M')^2 \to (M, M')^2 \to (M, i_* i^* M')^2 \overset{(1)}{=} (\underbrace{i^* M}_{[-1,0]}, \underbrace{i^* M'[2]}_{[-3,-2]})^0 = 0.$$

Therefore, in order to show that the middle term vanishes, we may replace M' by $j_! j^* M'$. Similarly, we may replace M by $j_* j^* M$. In particular $M \in j_* \tilde{T}^0(S')$, $M' \in j_! \tilde{T}^0(S')$. By Proposition 3.7 and Remark 2.3, $\tilde{T}^0(S')$ is generated by means of extensions by $\mathbf{1}(a)[1]$ where $a \in \mathbb{Z}$. The claim is stable under extensions so that we may assume $M = j_* A$, $A := \mathbf{1}(a)[1]$, $M' = j_! A'$, $A' := \mathbf{1}(a')[1]$. Let $\tilde{A} := \mathbf{1}(a)[1] \in \tilde{T}^0(S)$ and define $\tilde{A}'$ similarly. We have $j^* \tilde{A} = A$ and similarly with A'.

The localization triangle $j_*A' \to i_*i^*j_*A' \to j_!A'[1]$ maps to $j_*A' \to i_*{}^p\mathrm{H}^0i^*j_*A' \to (j_{!*}A')[1] = \tilde{A}[1]$. We apply $(\tilde{A}, -)^1$ to this map, which gives the last two exact rows in the diagram. The first exact row maps to the second via the map $\tilde{A} = j_{!*}A \to j_*A$.

$$
\begin{array}{ccccccc}
(j_*A, j_*A')^1 & \longrightarrow & (j_*A, i_*i^*j_*A')^1 & \longrightarrow & (j_*A, j_!A')^2 & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
(\tilde{A}, j_*A')^1 & \longrightarrow & (\tilde{A}, i_*i^*j_*A')^1 & \longrightarrow & (\tilde{A}, j_!A)^2 & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
(\tilde{A}, j_*A')^1 & \longrightarrow & (\tilde{A}, i_*{}^p\mathrm{H}^0i^*j_*A')^1 & \longrightarrow & (\tilde{A}, \tilde{A})^2 & \longrightarrow & 0.
\end{array}
$$

The $=$ signs in the leftmost column are by adjunction (1) and $j^*j_*A = j^*\tilde{A} = A$. The $=$ signs in the second column all use the adjunction $i^* \leftrightarrows i_*$ as well as the cohmological dimension zero of Tate motives over finite fields and cohomological amplitude of i^*, which imply

$$
(\underbrace{i^*j_*A}_{[-1,0]}, \underbrace{i^*j_*A'[1]}_{[-2,-1]})^0 = ({}^p\mathrm{H}^{-1}i^*j_*A, {}^p\mathrm{H}^0i^*j_*A')^0.
$$

Applying i^* to the triangle $i_*{}^p\mathrm{H}^{-1}i^*j_*A \to j_!A \to j_{!*}A$ and using $i^*j_! \overset{(7)}{=} 0$ we see $({}^p\mathrm{H}^{-1}i^*j_*A, {}^p\mathrm{H}^0i^*j_*A')^0 = (i^*j_{!*}A, {}^p\mathrm{H}^0i^*j_*A')^1$. This justifies the upper $=$ in the second column. The lower $=$ in that column follows by the same argument. However, $(\tilde{A}, \tilde{A}')^2 = 0$, by vanishing of the K-theory in the relevant range (see Theorem 3.2). $\square$

Theorem 4.5. *For both Tate and Artin–Tate motives, the inclusion $T^0(S) \subset T(S)$ extends to a triangulated functor*

$$
\mathbf{D}^b(T^0(S)) \to T(S). \tag{18}
$$

This functor is an equivalence of categories.

Proof. The category $\mathbf{DM}_{\mathrm{gm}}(S)$ and thus the subcategories of (Artin–)Tate motives embed into some unbounded derived category $\mathbf{D}(\mathscr{A})$, where $\mathscr{A}$ is an exact category. This implies the first statement by a general fact in homological algebra [17, Theorem 1.1.]. Indeed, the interpretation of $\mathbf{DM}_{\mathrm{gm}}(S)$ in terms of h-sheaves shows that (using the notation of [3] and abbreviating **Shv** for the category of $\mathbb{Q}$-linear sheaves with respect to the h-topology on the big site of schemes of finite type over S)

$$
\mathbf{DM}_{\mathrm{gm}}(S) \cong \mathbf{D}_{\mathbb{A}^1}(\mathbf{Shv}) \subset \mathbf{D}_{\mathbb{A}^1}^{\mathrm{eff}}(\mathbf{Sp}(\mathbf{Shv})) \subset \mathbf{D}(\mathbf{Sp}(\mathbf{Shv})).
$$

More precisely, $\mathbf{DM}_{\mathrm{gm}}(S)$ identifies with the subcategory of W_Ω-local objects in the middle category, which identifies with the subcategory of $W_{\mathbb{A}^1}$-local objects in the right hand category [3, Sections 5.2, 5.3].

The t-structure on $T(S)$ is bounded and non-degenerate, so it remains to show the full faithfulness of (18) or equivalently that the map

$$
f_n : \mathrm{Ext}_{T^0}^n(M, M') \to \mathrm{Hom}_T(M, M'[n])
$$

is an isomorphism for any $M, M' \in T^0(S)$. The general theory (see e.g. [4, 1.1.5]) shows that f_0 and f_1 are isomorphisms and that f_2 is injective for all M and M'. For Tate motives, f_2 is therefore an isomorphism, since the right hand side is zero by Proposition 4.4. We now show that f_2 is an isomorphism for Artin–Tate motives. The motives M and M' are fixed, so there is some open embedding $j : S' \to S$ such that j^*M and j^*M' are in $\tilde{T}^0(S')$. Let i be the complement of j. The following exact sequences are a consequence of (6) and Theorem 4.2:

$$
0 \to i_*{}^p\mathrm{H}^{-1}i^*M \overset{a}{\to} j_!j^*M \to K := \mathrm{coker}\, a \to 0 \tag{19}
$$

$$
0 \to K \to M \to i_*{}^p\mathrm{H}^0i^*M \to 0. \tag{20}
$$

We write $^n(-, -)$ for Ext^n and $_n(-, -)$ for $\mathrm{Hom}_T(-, -[n])$. (19) induces a commutative diagram with exact rows

$$
\begin{array}{ccccc}
{}^1(i_*{}^p\mathrm{H}^{-1}i^*M, M') & \longrightarrow & {}^2(K, M') & \longrightarrow & {}^2(j_!j^*M, M') \\
\| & & \downarrow & & \downarrow \\
{}_1(i_*{}^p\mathrm{H}^{-1}i^*M, M') & \longrightarrow & {}_2(K, M') & \longrightarrow & {}_2(j_!j^*M, M') = {}_2(j^*M, j^*M').
\end{array}
$$

The rightmost lower term is zero by the vanishing of the K-theory (cf. the argument in the proof of Proposition 4.4), so all vertical maps are isomorphisms. This and (20) yields a similar diagram:

$$\begin{array}{ccccccc}
{}^2(i_*{}^pH^0i^*M, M') & \longrightarrow & {}^2(M, M') & \longrightarrow & {}^2(K, M') & \longrightarrow & {}^3(i_*{}^pH^0i^*M, M') \\
\downarrow & & \downarrow{\scriptstyle r} & & \| & & \downarrow \\
{}_2(i_*{}^pH^0i^*M, M') & \longrightarrow & {}_2(M, M') & \longrightarrow & {}_2(K, M') & \longrightarrow & {}_3(i_*{}^pH^0i^*M, M').
\end{array}$$

The outer terms in the lower row vanish because the cohomological dimension of Artin–Tate motives over $\mathbb{F}_p$ is zero and $i^!$ has cohomological amplitude [0, 1]. We now show that the rightmost upper term is zero. Altogether, this implies that r is also surjective. We write $A := {}^pH^0i^*M$; it is a mixed motive over $\mathbb{F}_p$. Any element of the Yoneda-Ext-group in question is represented by an exact sequence

$$0 \to i_*A \to X_1 \xrightarrow{s} X_2 \to X_3 \to M' \to 0$$

in **MATM**(S). This extension is the image under the concatenation mapping

$${}^2(i_*A, \operatorname{coker} s) \times {}^1(\operatorname{coker} s, M') \to {}^3(i_*A, M').$$

The left hand factor is a subgroup of ${}_2(i_*A, \operatorname{coker} s) = {}_2(A, i^! \operatorname{coker} s) = 0$ (see above). Therefore, the extension above splits and we have shown that the second Ext-groups and Hom-groups agree.

This shows that the Hom$(M, M'[n])$ form an effaceable δ-functor, so they are universal and agree with Ext$^n(M, M')$ for all $n \geq 0$. Indeed, for $n \leq 2$ the groups are effaceable since they agree with Ext's by the above, for $n > 2$ the groups are zero by Proposition 4.4. $\square$

The functor $\eta_* : \mathbf{DM}(F) \to \mathbf{DM}(S)$ does not preserve Artin–Tate motives:

$$\operatorname{Hom}_{\mathbf{DM}(S)}(\mathbf{1}, \eta_*\mathbf{1}(1)[1]) \overset{(1)}{=} \operatorname{Hom}_{\mathbf{DM}(F)}(\mathbf{1}, \mathbf{1}(1)[1]) \overset{(5)}{=} K^1(F)_{\mathbb{Q}}^{(1)} = F^\times \otimes \mathbb{Q},$$

which is a countably infinite-dimensional $\mathbb{Q}$-vector space. However, the dimensions of all Hom-groups in $T(S)$ are finite (Theorem 3.2). This example is sharpened by the following proposition. It might be paraphrased by saying that the "site" of mixed Artin–Tate motives over S has enough points.

Proposition 4.6. *For any Artin–Tate motive M over $S \subset \operatorname{Spec} \mathcal{O}_F$, the following are equivalent:*

(i) $M = 0$.
(ii) $M = \eta_*M_\eta$, where M_η is some geometric motive over F.
(iii) $i_p^*M = 0$ for all closed points p of S.
(iv) $i_p^!M = 0$ for all closed points p of S.

Proof. The equivalence of (ii), (iii), and (iv) is an easy consequence of Verdier duality (12) and the limiting localization triangle (14). We now show (iii) $\Rightarrow$ (i). Using localization (6), the claim for M is implied by the one for j^*M for any open immersion j. Therefore we may assume $M \in \tilde{T}(S)$. Using the (-1)-exactness of $i_p^* : \tilde{T}(S) \to T(\mathbb{F}_p)$ we can even assume $M \in \tilde{T}^0(S)$. Given a short exact sequence in the abelian category $\tilde{T}^0(S)$

$$0 \to A \to M \to B \to 0$$

with $\eta_*\eta^*M = M$, it follows that $\eta_*\eta^*A = A$ and likewise for B. This is shown as follows: for all closed points $p \in S$, $i_{p*}i_p^!M = 0$ implies $i_p^!B = i_p^!A[1]$, by the full faithfulness of i_{p*}. The long exact ${}^pH^-$-sequence and the cohomological amplitude of $i_p^!$ (Theorem 4.2) shows ${}^pH^0i_p^!B = {}^pH^1i_p^!A$ and all other ${}^pH^*i_p^!B$, ${}^pH^*i_p^!A$ vanish. However, for any $B \in \tilde{T}^0(S)$, $i_p^!B$ is in cohomological degree 1 (as opposed to the general range [0, 1]): this may be checked on generators of $\tilde{T}_a^0(S)$ for all a, where it follows directly from the definitions (see the proof of Theorem 4.2). Thus ${}^pH^0i_p^!B = 0$, whence $i_p^!B = i_p^!A[1] = 0$ for all p.

Thus the statement for M is implied by the one for A and B. By the characterization of $\tilde{T}^0(S)$ of Proposition 3.7, we therefore only need to check the statement for generators of $\tilde{T}_{-2n}^0(S)$.

We first do this in the case of Tate motives. Then $\tilde{T}_{-2n}^0(S)$ consists of direct sums of motives $G := \mathbf{1}(n)[1]$. In that case the claim is clear, since none of the (nonzero) generators G satisfy $\eta_*\eta^*G = G$: we can twist it so that $n = 1$. Then $H^0(\eta_*\eta^*G)$ is infinite-dimensional, namely the group of units in some number field (tensored with $\mathbb{Q}$), but $H^0(G)$ is the group of units in some ring of S-integers, which are of finite rank.

In the case of Artin–Tate motives, the category $\tilde{T}_{-2n}^0(S)$ is generated by means of direct sums and summands by motives $G := \phi_*\mathbf{1}(n)[1]$, $\phi : V \to S$ finite and étale. Actually, we may assume ϕ is Galois: by the same argument as in the proof of Proposition 2.6, after shrinking S sufficiently, $\mathbf{1}_V$ is a direct summand of $\tilde{\phi}_*\mathbf{1}$ where $\tilde{\phi} : \tilde{V} \to V$ is the map corresponding to some normal closure of the function field extension $k(V)/k(S)$. Let M be a summand of G satisfying $\eta_*\eta^*M = M$. There

is a map $f : S' \to S$ such that f^*M is a Tate motive, Proposition 2.6. By base-change (11) and the preceding step, we get $f^*M = 0$. The map $\text{End}(M) \subset \text{End}(G) \xrightarrow{a} \text{End}(f^*G)$ factors over $\text{End}(f^*M) = 0$, so we have to show that a is injective. This is done with the same argument as in the proof of Proposition 2.6: we may shrink S so that f is étale. Since ϕ is Galois, we have

$$\text{End}(G) \stackrel{(1),(9)}{=} \text{Hom}(\mathbf{1}_V, \phi^*\phi_*\mathbf{1}_V) \stackrel{(11)}{=} \text{Hom}(\mathbf{1}_V, \mathbf{1}_V^{\oplus \deg \phi})$$

and

$$\text{End}(f^*G) = \text{Hom}(\mathbf{1}_{V'}, \phi'^*\phi'_*\mathbf{1}_{V'}) = \text{Hom}(\mathbf{1}_{V'}, \mathbf{1}_{V'}^{\oplus \deg \phi'}),$$

where $\phi' : V' := V \times_S S' \to S'$ is the pullback of ϕ along f. It is also Galois and $\deg \phi = \deg \phi'$. $\quad\square$

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Algebraic *K*-theory of the infinite place

Jakob Scholbach

Abstract We show that the algebraic *K*-theory of generalized archimedean valuation rings occurring in Durov's compactification of the spectrum of a number ring is given by stable homotopy groups of certain classifying spaces. We also show that the "residue field at infinity" is badly behaved from a *K*-theoretic point of view.

1 Introduction

In number theory, it is a universal principle that the spectrum of $\mathbb{Z}$ should be completed with an infinite prime. This is corroborated, for example, by Ostrowski's theorem, the product formula

$$\prod_{p \leq \infty} |x|_p = 1, \quad x \in \mathbb{Q}^{\times},$$

the Hasse principle, Artin–Verdier duality, and functional equations of *L*-functions.

This "compactification" $\operatorname{Spec}\widehat{\mathbb{Z}} := \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$ was just a philosophical device until recently: Durov has proposed a rigorous framework which allows for a discussion of, say, $\mathbb{Z}_{(\infty)}$, the local ring of $\operatorname{Spec}\widehat{\mathbb{Z}}$ at $p = \infty$ [1]. The purpose of this work is to study the *K*-theory of the so-called generalized rings intervening at the infinite place.

Algebraic *K*-theory is a well-established, if difficult, invariant of arithmetical schemes. For example, the pole orders of the Dedekind ζ-function $\zeta_F(s)$ of a number

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field F are expressible by the ranks of the K-theory groups of $\mathcal{O}_F$, the ring of integers. By definition, K-theory only depends on the category of projective modules over a ring. Therefore, this interacts nicely with Durov's theory of *generalized rings* which describes (actually: defines) such a ring R by defining its free modules. For example, the free $\mathbb{Z}_{(\infty)}$-module of rank n is defined as the n-dimensional octahedron, i.e.,

$$\mathbb{Z}_{(\infty)}(n) := \left\{ (x_1, \ldots, x_n) \in \mathbb{Q}^n, \sum_i |x_i| \leq 1 \right\}.$$

The abstract theory of such modules is a priori more complicated than in the classical case since $\mathbb{Z}_{(\infty)}$-modules fail to build an abelian category. Nonetheless, using Waldhausen's $S_\bullet$-construction it is possible to study the *algebraic K-theory* of $\mathbb{Z}_{(\infty)}$ and similar rings occurring for other number fields (Theorem 3.10, Definition 3.12).

Theorem 3.14. *The K-groups of $\mathbb{Z}_{(\infty)}$ are given by*

$$K_i(\mathbb{Z}_{(\infty)}) = \pi_i^s(B\mu_2 \sqcup \{*\}, *) = \begin{cases} \mathbb{Z} & i = 0 \ (Durov[Dur, 10.4.19]) \\ \mathbb{Z}/2 \oplus \mu_2 & i = 1 \\ \text{a finite group} & i > 1. \end{cases}$$

The $\mathbb{Z}/2$-part in K_1 stems from the first stable homotopy group π_1^s, while $\mu_2 = \{\pm 1\}$ arises as the subgroup of $\mathbb{Z}_{(\infty)}$ of elements of norm 1, i.e., the subgroup of (multiplicative) units of $\mathbb{Z}_{(\infty)}$. The finite K-group for $i > 1$ is the abutment of an Atiyah–Hirzebruch spectral sequence.

This theorem is proven for more general generalized valuation rings including $\mathcal{O}_{F(\sigma)}$, the ring corresponding to an infinite place σ of a number field F. In this case the group μ_2 above is replaced by the group $\{x \in F, |\sigma(x)| = 1\}$. The basic point is this: the only admissible monomorphisms (i.e., the ones occurring in the $S_\bullet$-construction of K-theory)

$$\mathbb{Z}_{(\infty)}(1) = [-1, 1] \cap \mathbb{Q} \to \mathbb{Z}_{(\infty)}(2)$$

are given by mapping the interval to one of the two diagonals of the lozenge. Thereby, the Waldhausen category structure on free $\mathbb{Z}_{(\infty)}$-modules turns out to be equivalent to the one of finitely generated pointed $\{\pm 1\}$-sets, whose K-theory is well-known. In the course of the proof we also show that other plausible definitions, such as the S^{-1} S-construction, the Q-construction, and the $+$-construction yield the same K-groups.

We finish this note by pointing out two K-theoretic differences of the infinite place: we show that $K_0(\mathbb{F}_\infty) = 0$ (Proposition 4.2), as opposed to $K_0(\mathbb{F}_p) = \mathbb{Z}$. Also, the completions at infinity are not well-behaved from a K-theoretic viewpoint. These remarks raise the question whether the "local" ring $\mathbb{Z}_{(\infty)}$ should be considered regular or, more precisely, whether

$$K_0(\mathbb{Z}_{(\infty)}) \to K_0'(\mathbb{Z}_{(\infty)}) := \mathbb{Z}[\text{finitely presented } \mathbb{Z}_{(\infty)} - \textbf{Mod}]/\text{short exact sequences}$$

is an isomorphism. Unlike in the classical case, there does not seem to be an easy resolution argument in the context of Waldhausen categories. Another natural question is whether there is a Mayer–Vietoris sequence of the form

$$K_i(\widehat{\mathbb{Z}}) \to K_i(\mathbb{Z}) \oplus K_i(\mathbb{Z}_{(\infty)}) \to K_i(\mathbb{Q}) \to K_{i-1}(\widehat{\mathbb{Z}}),$$

where $\widehat{\mathbb{Z}}$ is a generalized scheme obtained by glueing $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Z}_{(\infty)}$ along $\operatorname{Spec}\mathbb{Q}$. The usual proof of this sequence proceeds by the localization sequence, which is not available in our context.

Throughout the paper, we use the following *notation*: F is a number field with ring of integers $\mathcal{O}_F$. Finite primes of $\mathcal{O}_F$ are denoted by $\mathfrak{p}$. We write Σ_F for the set of real and pairs of complex embeddings of F. The letter σ usually denotes an element of Σ_F. It is referred to as an infinite prime of $\mathcal{O}_F$.

2 Generalized rings

In a few brushstrokes, we recall the definition of generalized rings and their modules and some basic properties. Everything in this section is due to Durov. All references in brackets refer to [1], where a much more detailed discussion is found.

A monad in the category of sets is a functor $R : \mathbf{Sets} \to \mathbf{Sets}$ together with natural transformations $\mu : R \circ R \to R$ and $\epsilon : \mathrm{Id} \to R$ required to satisfy an associativity and unitality axiom akin to the case of monoids. We will write $R(n) := R(\{1, \ldots, n\})$. An R-module is a set X together with a morphism of monads $R \to \mathrm{End}(X)$, where the endomorphism monad $\mathrm{End}(X)$ satisfies $\mathrm{End}(X)(n) = \mathrm{Hom}_{\mathbf{Sets}}(X^n, X)$. In other words, X is endowed with an action

$$R(n) \times X^n \to X$$

satisfying the usual associativity conditions. Thus, $R(n)$ can be thought of as the n-ary operations (acting on any R-module).

Definition 2.1 (*Durov* [*5.1.6*]) A *generalized ring* is a monad R in the category of sets satisfying two additional properties:

- R is *algebraic*, i.e., it commutes with filtered colimits. Since every set is the filtered colimit of its finite subsets, this implies that R is determined by $R(n)$ for $n \geq 0$ [4.1.3].
- R is *commutative*, i.e., for any $t \in R(n)$, $t' \in R(n')$, any R-module X (it suffices to take $X = R(n \times n')$) and $A \in X^{n \times n'}$, we have

$$t(t'(A)) = t'(t(A)),$$

where on the left hand side $t'(A) \in X^n$ is obtained by letting act t' on all rows of A and similarly (with columns) on the right hand side.

For a unital associative ring R (in the sense of usual abstract algebra), let

$$R(S) := \oplus_{s \in S} R$$

be the free R-module of rank $\sharp S$, where S is any set. The addition and multiplication on R turn this into an (algebraic) monad which is commutative iff $R = R(1)$ is [3.4.8]. Indeed, the required map

$$R(1) \times R(1) \to R(1) \tag{1}$$

is just the multiplication in R, while the addition is reformulated as

$$R(2) \times (R(1) \times R(1)) \to R(1), ((x_1, x_2), (y_1, y_2)) \mapsto \sum x_i y_i.$$

Note that (1) is required to exist for any monad, so multiplication is in a sense more fundamental than addition, which requires the particular element $(1, 1) \in R(2)$ [3.4.9].

Reinterpreting a ring as a monad in this way defines a functor from commutative rings to generalized rings, which is easily seen to be fully faithful: given two classical rings R, R', and a map of monads, i.e., a collection of maps $R(n) = R^n \to R'(n) = R'^n$, one checks that the maps for $n \geq 2$ are determined by $R \to R'$. In the same vein, R-modules in the classical sense are equivalent to R-modules (in the generalized sense). Henceforth, we will therefore not distinguish between classical commutative rings and their associated generalized rings.

The initial generalized ring is the monad $\mathbb{F}_0 : \mathbf{Sets} \to \mathbf{Sets}$, $M \mapsto M$. Its modules are just the same as sets. The monad $\mathbf{Sets} \ni M \mapsto M \sqcup \{*\}$ is denoted $\mathbb{F}_1$. Neither of these two generalized rings is induced by a classical ring. See Definition 3.2 for our main example of a non-classical ring.

Given a morphism $\phi : R \to S$ of generalized rings, the forgetful functor $\mathbf{Mod}(S) \to \mathbf{Mod}(R)$ between the module categories has a left adjoint $\phi^* : \mathbf{Mod}(R) \to \mathbf{Mod}(S)$ called *base change*. We also denote it by $- \otimes_R S$. Being a left adjoint, this functor preserves colimits [4.6.19]. For example, for a generalized ring R, the unique map $\mathbb{F}_0 \to R$ of generalized rings induces an adjunction

$$\mathbf{Sets} = \mathbf{Mod}(\mathbb{F}_0) \leftrightarrows \mathbf{Mod}(R) : \text{forget}$$

Its left adjoint is explicitly given by $X \mapsto R(X)$, the so-called *free R-module* on some set X. That is,

$$\text{Hom}_{\mathbf{Mod}(R)}(R(X), M) = \text{Hom}_{\mathbf{Sets}}(X, M),$$

as in the classical case.

Coequalizers and arbitrary coproducts exist in $\mathbf{Mod}(R)$, for any generalized ring R [4.6.17]. Therefore, arbitrary colimits exist. Base change functors ϕ^* commute with coequalizers. Moreover, arbitrary limits exist in $\mathbf{Mod}(R)$, and commute with the forgetful functor $\mathbf{Mod}(R) \to \mathbf{Sets}$ [4.6.1].

An R-module M is called *finitely generated* if there is a surjection $R(n) \twoheadrightarrow M$ for some $0 \leq n < \infty$ [4.6.9]. Unless the contrary is explicitly mentioned, all our modules are supposed to be finitely generated over the ground generalized ring in question. An R-module M is *projective* iff it is a retract of a free module, i.e., if there

are maps $M \xrightarrow{i} R(n) \xrightarrow{p} M$ with $pi = \mathrm{id}_M$. As in the classical case this is equivalent to the property that for any surjection of R-modules $N \twoheadrightarrow N'$, $\mathrm{Hom}_{\mathbf{Mod}(R)}(M, N)$ maps onto $\mathrm{Hom}_{\mathbf{Mod}(R)}(M, N')$ [4.6.23]. The categories of (finitely generated) free and projective R-modules are denoted $\mathbf{Free}(R)$ and $\mathbf{Proj}(R)$, respectively.

As usual, an *ideal I* of R is a submodule of $R(1)$. A proper ideal $I \subsetneq R(1)$ is called *prime* if $R(1) \backslash I$ is multiplicatively closed [6.2.2].

3 Archimedean valuation rings

3.1 Definitions

Let K be an integral domain equipped with a norm $| - | : K \to \mathbb{R}^{\geq 0}$. We will write Q for the quotient field of K. We put $E := \{x \in K, |x| = 1\}$. We also write $|x|$ for the L^1-norm on K^n, i.e., $|x| = \sum_i |x_i|$. Throughout, we assume:

Assumption 3.1 (A) $|K^\times| = \{|k|, k \in K^\times\} \subset \mathbb{R}^{\geq 0}$ is dense.
(B) $E \subset K^\times$.

Definition 3.2 The *(generalized) valuation ring* associated to $(K, | - |)$ is the submonad $\mathcal{O}$ of K given by

$$\mathcal{O}(S) := \left\{ x = (x_s) \in \bigoplus_{s \in S} K, |x| := \sum_{s \in S} |x_s| \leq 1 \right\}.$$

This is clearly algebraic. Moreover, the multiplication of the monad, i.e., $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ is well-defined by restricting the one of K (and is therefore commutative):

$$\mathcal{O}(\mathcal{O}(n)) = \left\{ (y_x) \in \bigoplus_{x \in \mathcal{O}(n)} K, \sum_x |y_x| \leq 1 \right\} \to \mathcal{O}(n)$$

sends (y_x) to (the finite sum) $\sum_x y_x \cdot x$. A priori, this expression is an element of K^n, only, but is actually contained in $\mathcal{O}(n)$ since

$$\left| \sum_x y_x \cdot x \right| \leq \left(\sum_x |y_x| \right) \cdot \sup |x| \leq 1.$$

In the case of an archimedean valuation, this definition of $\mathcal{O}$ is the one of Durov [1, 5.7.13]. For non-archimedean valuations, Durov's original definition gives back the (generalized ring corresponding to the) ordinary ring $\{x \in K, |x| \leq 1\}$ which is different from Definition 3.2 (see Example 3.4).

By definition, an $\mathcal{O}$-module M is therefore a set such that an expression $\sum_{i=1}^n \lambda_i m_i$ is defined for $n \geq 0$, $m_i \in M$, $\lambda_i \in K$ such that $\sum |\lambda_i| \leq 1$, obeying the usual laws of commutativity, associativity and distributivity. Maps $f : M \to N$ of $\mathcal{O}$-modules are described similarly: they satisfy $f(\sum_i \lambda_i m_i) = \sum_i \lambda_i f(m_i)$. The set $\{0\}$, with its

obvious $\mathcal{O}$-module structure is both an initial and terminal $\mathcal{O}$-module. Given a map $f : M' \to M$ of $\mathcal{O}$-modules, the (co)kernel is defined to be the (co)equalizer of the two morphisms f and $M' \to 0 \to M$. As was noted above, the forgetful functor $\mathcal{O} - \mathbf{Mod} \to \mathbf{Sets}$ preserves limits, so the kernel $\ker f$ is just $f^{-1}(0)$. The cokernel is described by the following proposition. Also see Remark 3.11 for an explicit example of a cokernel computation.

Proposition 3.3 *Given a map* $f : M' \to M$ *of* $\mathcal{O}$-modules, the cokernel is given by

$$coker(f) = M/ \sim, \tag{2}$$

where $\sim$ *is the equivalence relation generated by* $\sum_{i \in I} \lambda_i m_i \sim \sum_{i \in I} \lambda_i \tilde{m}_i$, *where* I *is any finite set,* $\lambda = (\lambda_i) \in \mathcal{O}(\sharp I)$ *and* $m_i, \tilde{m}_i \in M$ *are such that either* $m_i = \tilde{m}_i$ *or both* $m_i, \tilde{m}_i \in f(M') \subset M$. *This set is endowed with the* $\mathcal{O}$-action via the natural *projection* $\pi : M \to coker(f)$.

Proof This follows from the description of cokernels given in [1, 4.6.13]. It is also easy to check the universal property directly: we clearly have $\pi \circ f = 0$. Given a map $t : M \to T$ of $\mathcal{O}$-modules such that $tf = 0$, we need to see that t factors uniquely through $\text{coker} f$. The unicity of the factorization is clear since $M \to \text{coker} f$ is onto. The existence is equivalent to $t(m_1) = t(m_2)$ whenever $\pi(m_1) = \pi(m_2)$. This is obvious from the definition of the equivalence relation $\sim$ above. □

The base change functor resulting from the monomorphism $\mathcal{O} \subset K$ of generalized rings is denoted

$$(-)_K : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Mod}(K).$$

Actually, using Assumption 3.1, we may pick $t \in K^{\times}$ such that $|t| < 1$. Then, K is the unary localization $K = \mathcal{O}[1/t]$. This is shown in [1, 6.1.23] for $K = \mathbb{R}$. The proof for a general domain is the same. Therefore K is flat over $\mathcal{O}$, so $(-)_K$ preserves finite limits, in particular kernels [1, 6.1.2, 6.1.8]. Recall from p. 4 that $(-)_K$ also preserves colimits, such as cokernels.

Let $E(n) := \{x \in K(n) = K^n, |x| = 1\}$ be the "boundary" of $\mathcal{O}(n)$. (This is merely a collection of sets, not a monad.) We write $\mathcal{O}$ for $\mathcal{O}(1)$ and E for $E(1)$, if no confusion arises. In particular, $x \in \mathcal{O}$ means $x \in \mathcal{O}(1)$. The i-th standard coordinate vector $e_i = (0, ..., 1, ..., 0)$ is called a *basis vector* of $\mathcal{O}(n)$ ($1 \le i \le n$).

Example 3.4 Let F be a number field with ring of integers $\mathcal{O}_F$. We fix a complex embedding $\sigma : F \to \mathbb{C}$ and take the norm $| - |$ induced by σ. Let K be either $\mathcal{O}_F[1/N]$ where $N \in \mathbb{Z}$ has at least two distinct prime divisors, or F, or $\widehat{F}^{\sigma}$, the completion of F with respect to σ. The respective generalized valuation rings will be denoted $\mathcal{O}_{F,1/N,(\sigma)}$, $\mathcal{O}_{F,(\sigma)}$, and $\mathcal{O}_{F,\sigma}$, respectively. For example, $\mathcal{O}_{F,(\sigma)} = \mathcal{O}_{F,(\overline{\sigma})}$. Assumption 3.1(A) is satisfied: for $\mathcal{O}_F[1/N]$, pick two distinct prime divisors $p_1 \ne p_2$ of N. The elements $p_1^{n_1} p_2^{n_2} \in K$ are invertible for any $n_1, n_2 \in \mathbb{Z}$. The subgroup $\{\log(|p_1^{n_1} p_2^{n_2}|), n_i \in \mathbb{Z}\} \subset \mathbb{R}$ is dense: otherwise it was cyclic, in contradiction to the $\mathbb{Q}$-linear independence of $\log p_1$ and $\log p_2$ (Gelfand's theorem).

As for Assumption 3.1(B), let $x \in K$ with $|x| = 1$. If σ is a real embedding, $x = \pm|x| = \pm 1$. If σ is a complex embedding, let $\overline{\sigma}$ be its complex conjugate and $\overline{x} \in K$ be such that $\sigma(\overline{x}) = \overline{\sigma}(x)$. Then $\sigma(x)\sigma(\overline{x}) = \sigma(x)\overline{\sigma(x)} = |\sigma(x)|^2 = 1$ implies $x \in K^\times$.

According to Durov, $\mathcal{O}_{F,(\sigma)}$ is the replacement for infinite places of the local rings $\mathcal{O}_{F(\mathfrak{p})}$ at finite places. However, the analogy is relatively loose, as is shown by the following two remarks: first, for $p < \infty$, let $|x|_p := p^{-v_p(x)}$ for $x \in \mathbb{Q}^\times$. Then the generalized ring $\mathbb{Z}_{|-|_p}$ (in the sense of Definition 3.2) maps injectively to the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal p, but the map is a bijection only in degrees $\le p$. (Less importantly, Assumption 3.1(A) is not satisfied for $\mathbb{Z}_{|-|_p}$.)

Secondly, recall that the semilocalization $\mathcal{O}_{F(\mathfrak{p}_1,\mathfrak{p}_2)} = \mathcal{O}_{F(\mathfrak{p}_1)} \cap \mathcal{O}_{F(\mathfrak{p}_2)}$ at two finite primes is one-dimensional. In analogy, pick two $\sigma_1, \sigma_2 \in \Sigma_F$ and consider $\mathcal{O} := \mathcal{O}_{(\sigma_1)} \cap \mathcal{O}_{(\sigma_2)} \subset F$, i.e.,

$$\mathcal{O}(n) := \left\{ (x_1, \ldots, x_n) \in F^n, \sum_k |\sigma_i(x_k)| \le 1 \quad \text{for } i = 1, 2 \right\}.$$

Let $\mathfrak{p}_i = \{x \in \mathcal{O}, |\sigma_i(x)| < 1\}$ and $\mathfrak{p} := \{x \in \mathcal{O}, |\sigma_1(x)\sigma_2(x)| < 1\}$. These are ideals: for example, for $x = (x_j) \in \mathcal{O}(n)$, $s_1, \ldots, s_n \in \mathfrak{p}$, we need to check $\sum s_j x_j \in \mathfrak{p}$: if, say, $|\sigma_1(s_1)| < 1$ then

$$\left| \sigma_1 \left(\sum_j s_j x_j \right) \right| \le \sum |\sigma_1(s_j)||\sigma_1(x_j)| < \sum |\sigma_1(x_j)| \le 1.$$

The complement $\mathcal{O} \backslash \mathfrak{p} = \{x, |\sigma_1(x)| = |\sigma_2(x)| = 1\}$ is multiplicatively closed (and contains 1). We get a chain of prime ideals

$$0 \subsetneq \mathfrak{p}_1 \subset \mathfrak{p} \subsetneq \mathcal{O}.$$

The middle inclusion is, in general, strict, namely when $F = \mathbb{Q}[t]/p(t)$ with some irreducible polynomial $p(t)$ having zeros $a_1, a_2 \in \mathbb{C}$ with $|a_1| = 1$, $|a_2| < 1$. That is, $\operatorname{Spec} \mathcal{O}$ is not one-dimensional.

3.2 Projective and free $\mathcal{O}$-modules

In this section we gather a few facts about projective and free $\mathcal{O}$-modules. We begin with a handy criterion for monomorphisms of certain $\mathcal{O}$-modules (Lemma 3.5). Lemma 3.6 concerns a particular unicity property of the basis vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathcal{O}(n)$. This is used to prove Theorem 3.7: every projective $\mathcal{O}$-module is free, provided that the norm is archimedean. This improves a result of Durov which treats only the cases where $\mathcal{O}$ is either the "unclompeted local ring" of a number ring at an infinite place σ, $\mathcal{O}_{F,(\sigma)}$, in the case where σ is a real embedding or the "completed local ring" $\mathcal{O}_{F,\sigma}$ for both real and complex places. Therefore, we only study the K-theory of free $\mathcal{O}$-modules in this paper (but see Remark 3.18). We also

use Lemma 3.6 to establish a highly combinatorial flavor of automorphisms of free $\mathcal{O}$-modules (Proposition 3.9), which will later give rise to the computation of higher K-theory of $\mathcal{O}$.

Lemma 3.5 (compare [1, 2.8.3.]) *Let* $f : M' \to M$ *be a map of* $\mathcal{O}$-*modules. We suppose both* M' *and* M *are submodules of free* $\mathcal{O}$-*modules. (For example, they might be projective.) Then the following are equivalent:*

a) $f_Q : M'_Q \to M_Q$ *is injective, where* Q *is the quotient field of* K,
b) $f_K : M'_K \to M_K$ *is injective,*
c) f *is injective (as a map of sets),*
d) f *is a monomorphism of* $\mathcal{O}$-*modules,*

Proof Consider the diagram

$$
\begin{array}{ccccc}
M' & \hookrightarrow & M'_K & \hookrightarrow & M'_Q \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle f_Q} \\
M & \hookrightarrow & M_K & \hookrightarrow & M'_Q.
\end{array}
$$

Its horizontal maps are injective since both modules are submodules of free modules and, for these, $\mathcal{O}(n) \subset K(n) = K^n \subset Q(n) = Q^n$. This shows (a) $\Rightarrow$ (b) $\Rightarrow$ (c). (c) implies (d) since the forgetful functor $\mathbf{Mod}(\mathcal{O}) \to \mathbf{Sets}$ is faithful. (d) $\Rightarrow$ (b): by Assumption 3.1, we may pick $t \in K^\times$ with $|t| < 1$. Any two element of M'_K are of the form m'_1/t^n, m'_2/t^n, where $m'_1, m'_2 \in M'$ and $n \geq 0$. Suppose that $f_K(m'_1/t^n) = f(m'_1)/t^n$ agrees with $f_K(m'_2/t^n)$. The multiplication with t^{-n} is injective on M'_K, since M' (M'_K) is a submodule of a free $\mathcal{O}$- (K-, respectively) module. Thus $f(m'_1) = f(m'_2)$ so the assumption (d) implies our claim. Finally (b) $\Rightarrow$ (a) follows from the flatness of Q over K. □

The following lemma can be paraphrased by saying that the basis vectors $e_i = (0, \ldots, 1, \ldots 0) \in \mathcal{O}(n)$ cannot be generated as a nontrivial $\mathcal{O}$-linear combination of other elements of $\mathcal{O}(n)$.

Lemma 3.6 *Suppose that K is a field (as opposed to a domain). Suppose further that*

$$
e_i = \sum_{j=1}^{m} \lambda_j f_j \tag{3}
$$

with $f_j \in \mathcal{O}(n)$ *and* $(\lambda_j)_j \in \mathcal{O}(m)$, $\lambda_j \neq 0$. *Then for each* j, $f_j = \mu_j \cdot e_i$ *with* $\mu_j \in E$.

Proof The proof proceeds by induction on m, the case $m = 1$ being trivial.
Each f_j can be written as $f_j = \sum_{l=1}^{n} \kappa_{jl} e_l$ with $(\kappa_{jl})_l \in \mathcal{O}(n)$. We get

$$
1 = |e_i| \overset{(3)}{=} |\sum \lambda_j f_j| \leq \sum |\lambda_j||f_j| \leq \sum |\lambda_j| \leq 1. \tag{4}
$$

Therefore equality holds throughout. We have $e_i = \sum_{j,l} \lambda_j \kappa_{jl} e_l$. This K-linear relation between the basis vectors of K^n yields $1 = \sum_j \lambda_j \kappa_{ji}$. Hence

$$1 \le \sum_j |\lambda_j \kappa_{ji}| \le \underbrace{\left(\sum_j |\lambda_j|\right)}_{\overset{(4)}{=} 1} \cdot \max_j |\kappa_{ji}|.$$

On the other hand, $|\kappa_{ji}| \le 1$, so there is some j_0 such that $|\kappa_{j_0 i}| = 1$. Using $\sum_l |\kappa_{j_0 l}| \le 1$ we see $\kappa_{j_0 l} = 0$ for all $l \ne i$, thus $f_{j_0} = \kappa_{j_0 i} e_i$. Put $\mu_{j_0} := \kappa_{j_0 i} (\in E)$, so

$$(1 - \lambda_{j_0} \mu_{j_0}) e_i = \sum_{j \ne j_0} \lambda_j f_j$$

holds. If $|\lambda_{j_0} \mu_{j_0}| = 1$, we are done since all other λ_j, $j \ne j_0$ must vanish in this case. If $|\lambda_{j_0} \mu_{j_0}| < 1$, then

$$e_i = \sum_{j \ne j_0} \frac{\lambda_j}{1 - \lambda_{j_0} \mu_{j_0}} f_j.$$

This finishes the induction step since the right hand side is actually an $\mathcal{O}$-linear combination of the f_j, for

$$\sum_{j \ne j_0} |\lambda_j| \overset{(4)}{=} 1 - |\lambda_{j_0}| = 1 - |\lambda_{j_0} \mu_{j_0}| \le |1 - \lambda_{j_0} \mu_{j_0}|.$$

$\square$

Theorem 3.7 *Suppose that the norm $|-|$ giving rise to the generalized valuation ring $\mathcal{O}$ is archimedean. Then every projective $\mathcal{O}$-module M is free.*

Proof Let K' be the completion (with respect to the norm $|-|$) of Q, the quotient field of K. By Ostrowski's theorem, we have either $K' = \mathbb{R}$ or $K' = \mathbb{C}$ (with their usual norms). Let us write $-' := - \otimes_{\mathcal{O}} \mathcal{O}'$, where $\mathcal{O}' := \mathcal{O}_{K'}$ is the generalized valuation ring belonging to K'. We consider the following maps of $\mathcal{O}'$-modules, where O_i are certain free $\mathcal{O}$-modules that are defined in the course of the proof:

$$O_3' \to O_2' \to O_1' \xrightarrow{p'} M' \xrightarrow{\phi, \cong} O_0'.$$

First, M' is a projective $\mathcal{O}'$-module: given a projector $p : O_1 := \mathcal{O}(n_1) \to \mathcal{O}(n_1)$ with $M = \mathrm{im}\, p$, we get $M' = \mathrm{im}\, p'$. By the afore-mentioned result of Durov [1, 10.4.2], there is an isomorphism of $\mathcal{O}'$-modules, $\phi : M' \xrightarrow{\cong} O_0' := \mathcal{O}'(n_0)$. The composition $\phi \circ p'$ is surjective, so for any basis vector $e_i \in O_0'$ ($1 \le i \le n_0$), there is some $\mathcal{O}'$-linear combination $\sum_{j \le n_1} \lambda_{ij} e_j$ mapping to e_i under $\phi p'$. Thus, $\sum_j \lambda_{ij} \phi p'(e_j) = e_i$. Therefore, by Lemma 3.6, $\phi p'(e_j) \in E' \cdot e_i$ for each j. Here

$E' = \{x \in \mathcal{O}', |x| = 1\}$ (which is $S^1 \subset \mathbb{C}$ or $\{\pm 1\} \subset \mathbb{R}$ depending on K'). We put $O_2 := \sqcup_{j_2 \in J_2} e_{j_2} \mathcal{O} = \mathcal{O}(J_2)$, where the coproduct runs over

$$J_2 := \{1 \le j_2 \le n_1, \phi p'(e_{j_2}) \in E' e_i \text{ for some } i \le n_0\}.$$

The inclusion $J_2 \subset \{1, \ldots, n_1\}$ induces a ($\mathcal{O}$-linear!) injection $f_{21} : O_2 \to O_1$. According to the previous remark, $O_2' \overset{\phi p' f_{21}'}{\twoheadrightarrow} O_1'$ is surjective. Consider the map $J_2 \to \{1, \ldots, n_0\}$ which maps j_2 to the (unique) i with $e_i \in E' \phi p'(e_{j_2})$. This map is onto. By Assumption 3.1, we may pick some $J_3 \subset J_2$ on which it is a bijection. Let $f_{32} : O_3 := \sqcup_{j_3 \in J_3} e_{j_3} \mathcal{O} = \mathcal{O}(J_3) \to O_2 = \mathcal{O}(J_2)$ be the map induced by $J_3 \subset J_2$. Set $f_{31} = f_{21} \circ f_{32}$. Then the composition $O_3' \overset{f_{31}'}{\rightarrowtail} O_1' \overset{p'}{\to} M' \overset{\phi, \cong}{\to} O_0'$ is an isomorphism of $\mathcal{O}'$-modules. Note that f_{31} and p are $\mathcal{O}$-linear maps, but ϕ is defined over $\mathcal{O}'$, only. Writing $v := p \circ f_{31}$, we must show the implication

$$v' \text{ isomorphism} \Rightarrow v \text{ isomorphism}.$$

The elements $m_j := p(e_j) \in M$, $j \le n_1$, generate M. The map $v' \otimes_{\mathcal{O}'} K' = v_Q \otimes_Q K'$ is an isomorphism of K'-vector spaces. The inclusion of the quotient field $Q \to K'$ is fully faithful, so that v_Q is also an isomorphism. Hence there is some $k_j = a_j/b_j \in Q \backslash \{0\}$ such that $k_j m_j \in \mathrm{im} v$. According to Assumption 3.1, we can pick some $N \in K^\times$ such that $|a_j/N|, |b_j/N| \le 1$ for all j. Then $m_j a_j/N \in \mathrm{im} v$. Similarly, pick some $t \in \mathcal{O}$ with $0 < |t| \le \min_j |a_j/N|$. Then $tM \subset \mathrm{im} v$.

To show the surjectivity of v, we fix $m \in M$ and pick some $o_3 \in O_3$ with $tm = v(o_3)$. Since $M \subset M'$ and v' is an isomorphism, there is a unique $\tilde{o}_3' \in O_3'$ with $v'(\tilde{o}_3') = m$. Hence $v(o_3) = v'(o_3) = v'(t\tilde{o}_3')$, so that $t\tilde{o}_3' = o_3$. In other words, $o_3' = t^{-1}o_3 \in O_3' \cap (O_3)_K = O_3$. This shows the surjectivity of v. The injectivity of v is clear, since $O_3 \subset O_3'$ and v' is injective. Consequently, v is an isomorphism. $\square$

Definition 3.8 Recall that **Free**($\mathcal{O}$) is the category of (finitely generated) free $\mathcal{O}$-modules. In **Free**($\mathcal{O}$) let *cofibrations* ($\rightarrowtail$) be the monomorphisms whose cokernel (in the category of all $\mathcal{O}$-modules) lies in **Free**($\mathcal{O}$). Morphisms which are obtained as cokernels of cofibrations are called *fibrations* and denoted $\twoheadrightarrow$. Let *weak equivalences* $\overset{\sim}{\to}$ be the isomorphisms.

Proposition 3.9 *Let $f : M' \to M$ be a monomorphism of free $\mathcal{O}$-modules with projective cokernel M'' (for example, a cofibration). Then there is a unique isomorphism $\phi : M \cong M' \sqcup M''$ such that the following diagram is commutative*

$$
\begin{array}{ccccc}
M' & \overset{f}{\rightarrowtail} & M & \overset{\pi}{\twoheadrightarrow} & M'' \\
\| & & \downarrow{\scriptstyle \phi} & & \| \\
M' & \overset{incl}{\rightarrowtail} & M' \sqcup M'' & \overset{proj}{\twoheadrightarrow} & M''.
\end{array}
\tag{5}
$$

Proof Let $M' = \mathcal{O}(n')$, $M = \mathcal{O}(n)$ and let $f_i := f(e_i) \in M$, $1 \le i \le n'$ be the images of the basis vectors.

We claim that f factors through $\sqcup_{i \leq n, e_i \in f(M')} e_i \mathcal{O} = \mathcal{O}(\tilde{n}') \subset M = \mathcal{O}(n)$, where $\tilde{n}' := \sharp\{i \leq n, e_i \in f(M')\}$. To show this, write $f(M') \ni m' = \sum_{i \in I} \lambda_i e_i$, where all $\lambda_i \neq 0$ and the e_i are the basis vectors of M. Put

$$m' = \underbrace{\sum_{e_i \notin f(M')} \lambda_i e_i}_{=:m'_1} + \underbrace{\sum_{e_i \in f(M')} \lambda_i e_i}_{=:m'_2}.$$

By Assumption 3.1, we can pick some $t \in K^\times$ such that $|t| \leq 1/2$. Then $tm'_1 = tm' - tm'_2 \in f(M')$. Let i be such that $e_i \notin f(M')$. We need to see $\lambda_i = 0$.

We write $(-)_Q$ for the functor $- \otimes_\mathcal{O} \mathcal{O}_Q$, where $\mathcal{O}_Q$ is the generalized valuation ring associated to the unique extension of the norm $|-|$ in K to the quotient field Q of K. The functor $(-)_Q$ preserves colimits, in particular $\mathrm{coker}(f_Q) = (\mathrm{coker}\, f)_Q$. In addition, f_Q is a monomorphism by Lemma 3.5. The assumption $e_i \notin f(M')$ implies $e_i \notin f_Q(M'_Q)$: suppose that $e_i = \sum_{i' \leq n'} \kappa_{i'} f_{i'}$ where $(\kappa_{i'}) \in \mathcal{O}_Q(n')$ and $f_{i'} := f(e_{i'})$ are the images of the basis vectors of M'. By Lemma 3.6, we have $f_{i'} = \epsilon_{i'} e_i$ for all i', with some $\epsilon_{i'} \in \mathcal{O}_Q$, $|\epsilon_{i'}| = 1$. But $f_{i'}$ also lies in M (as opposed to M_Q). Thus, $\epsilon_{i'}$ must lie in $\mathcal{O}$, that is, $e_i \in f(M')$. Therefore, to prove the claim we may assume K is a field.

Now, by Lemma 3.6, e_i is not a non-trivial $\mathcal{O}$-linear combination of other elements of M. As $e_i \notin f(M')$, Proposition 3.3 implies

$$\pi^{-1}(\pi(e_i)) = \{e_i\}. \tag{6}$$

Fix a section $\sigma : M'' \to M$ of π, which exists by the assumption that M'' be projective. We obtain $\sigma(\pi(e_i)) = e_i$. Hence,

$$0 = \sigma(0_{M''}) = \sigma(\pi(tm'_1)) = \sum_{e_i \notin f(M')} t\lambda_i \sigma(\pi(e_i)) = \sum_{e_i \notin f(M')} t\lambda_i e_i,$$

so that $\lambda_i = 0$. The claim is shown.

By the claim, f induces a bijection $\tilde{f} : M' = \mathcal{O}(n') \to \mathcal{O}(\tilde{n}')$, which gives rise to a bijection $K^{n'} \to K^{\tilde{n}'}$. This shows $\tilde{n}' = n'$. We conclude that the basis vectors $e_i \in M'$ get mapped under f to $\epsilon_i e_{J(i)}$ where $\epsilon_i \in E$ and $J : \{1, \ldots, n'\} \to \{1, \ldots, n\}$ is an injective set map. In fact, suppose $\tilde{f}^{-1}(e_i) = \sum_{j \in J} \lambda_{ij} e_j$ with $(\lambda_{ij}) \in \mathcal{O}(J)$ with all $\lambda_{ij} \neq 0$. Equivalently, $\sum \lambda_{ij} \tilde{f}(e_j) = e_i$. Therefore, by Lemma 3.6 (applied with Q instead of K), $\tilde{f}_Q(e_j) \in E_Q \cdot e_i$ for all j, where $E_Q = \{q \in Q, |q| = 1\}$. Since $\tilde{f}$ and therefore, by Lemma 3.5, $\tilde{f}_Q$ is injective, this implies that only one summand appears in this sum, i.e., $\tilde{f}(e_j) = \lambda_{ij}^{-1} e_i$ for some $j \in J$. A priori, λ_{ij}^{-1} only lies in Q, but $\tilde{f}(e_j) \in \mathcal{O}(n')$ shows that $\epsilon_i := \lambda_{ij}^{-1} \in \mathcal{O}$, hence in E.

By Assumption 3.1, $\epsilon_i \in E$ is a unit in K. We can therefore define $\phi' : \mathcal{O}(n') \to M'$ by mapping the basis vectors e_i of $\mathcal{O}(n')$ (which correspond, in the above notation, to the basis vectors $e_{J(i)}$ of M) to $\epsilon_i^{-1} e_i$. Also, let $\phi'' : \mathcal{O}(n - n') \subset M \to M''$ be the

map which sends the remaining basis vectors $e_{j'}$ for $j' \notin \mathrm{im}\, J$ to $\pi(e_{j'})$. Put

$$\phi := \phi' \sqcup \phi'' : M = \mathcal{O}(n) = \mathcal{O}(n') \sqcup \mathcal{O}(n - n') \to M' \sqcup M''.$$

Both ϕ' and ϕ'' are onto, hence so is ϕ. This follows from the construction of coproducts of modules over generalized rings [1, 4.6.15]. (Also see [1, 10.4.7] for an explicit description of the coproduct for modules over archimedean valuation rings.) Alternatively, the surjective maps ϕ' and ϕ'' are epimorphisms of $\mathcal{O}$-modules. Hence their coproduct ϕ is an epimorphism. As $M' \sqcup M''$ is projective, ϕ has a section, so it is also surjective. The map ϕ is injective, as can be seen by checking the definition or using Lemma 3.5(b) $\Rightarrow$ (c). Hence ϕ is an isomorphism.

We finally show the unicity of ϕ or, in other words, that there are no non-trivial automorphism of cofiber sequences

$$0 \to M' \rightarrowtail M \twoheadrightarrow M'' \to 0.$$

Suppose $\tilde{\phi}$ is another isomorphism fitting into (5). We replace ϕ by $\tilde{\phi}\phi^{-1}$ and $\tilde{\phi}$ by id_M and assume f is the standard inclusion $M' \to M = M' \sqcup M''$ and π is the standard projection onto M''. Applying the base change functor $(-)_Q$ (see above), we may assume that K is a field. Then M''_K is a free K-module, so the endomorphism $\phi_K : M_K \to M_K$ is given by a matrix

$$B = \begin{pmatrix} \mathrm{Id}_{M'} & A \\ 0 & \mathrm{Id}_{M''} \end{pmatrix},$$

where A is the matrix corresponding to the map $M''_K \to M'_K$ (of free K-modules). On the other hand, ϕ is a map of free $\mathcal{O}$-modules, so every column in B is in $\mathcal{O}(n)$. This forces $A = 0$, so that $\phi = \mathrm{id}_M$. $\qquad\square$

Theorem 3.10 *The category* $(\mathbf{Free}(\mathcal{O}), \rightarrowtail, \overset{\sim}{\to})$ *defined in 3.8 is a Waldhausen category.*

Proof The only non-trivial thing to show is the stability of cofibrations under cobase-change. By Proposition 3.9, a cofibration sequence $M' \overset{\iota}{\rightarrowtail} M \overset{\pi}{\twoheadrightarrow} M''$ in $\mathbf{Free}(\mathcal{O})$ is isomorphic to $M' \rightarrowtail M' \sqcup M'' \twoheadrightarrow M''$. Hence, given any map $f : M' \to \tilde{M}'$, the pushout of ι along f, $\tilde{M}' \to \tilde{M}' \sqcup_{M'} M$ is isomorphic to $\tilde{M}' \to \tilde{M}' \sqcup M''$ which is a monomorphism with cokernel M''. $\qquad\square$

Remark 3.11 Mahanta uses split monomorphisms as cofibrations in the category of finitely generated modules over a fixed $\mathbb{F}_1$-algebra (i.e., pointed monoid) to define G-(a.k.a. K'-)theory of such algebras [3]. In $\mathbf{Free}(\mathcal{O})$, we have seen that all cofibrations are split, but not conversely: the cokernel of the split monomorphism $\varphi : \mathbb{Z}_\infty(1) \to \mathbb{Z}_\infty(2)$, $e_1 \mapsto \frac{e_1}{2} + \frac{e_2}{2}$ is not free. This follows either from Proposition 3.9 or by an explicit computation, using Proposition 3.3. Indeed, two elements $x_i e_1 + y_i e_2 \in \mathbb{Z}_\infty(2)$ $(i = 1, 2)$ are identified in $\mathrm{coker}\, \varphi$ iff $|y_1 - x_1| = |y_2 - x_2| < 1$. On $\mathrm{coker}\, \varphi$, multiplication with $1/2$ is therefore not injective. Thus $\mathrm{coker}\, \varphi$ is not a submodule of a free $\mathbb{Z}_\infty$-module, in particular it is not projective.

3.3 K-theory

In this subsection, we compute the K-theory of the generalized valuation ring $\mathcal{O}$ (Definition 3.2) or, more precisely, of the category of free $\mathcal{O}$-modules. By Theorem 3.7, every projective $\mathcal{O}$-module is free, provided that the norm is archimedean.

We define the K-theory using Waldhausen's $S_\bullet$-construction, which has the advantage of being immediately applicable (Theorem 3.10). Other constructions, such as Quillen's Q-construction can also be applied (slightly modified, since $\mathcal{O}$-modules do not form an exact category). The resulting K-groups do not depend on the choice of the construction.

Recall the definition of K-theory of a Waldhausen category $\mathcal{C}$ (see e.g. [7, Section IV.8] for more details). We always assume that the weak equivalences of $\mathcal{C}$ are its isomorphisms. The category $S_n\mathcal{C}$ consists of diagrams

$$
\begin{array}{ccccccccc}
0 = A_{00} & \rightarrowtail & A_{01} & \rightarrowtail & A_{02} & \rightarrowtail & \cdots \rightarrowtail & A_{0n} & \quad (7)\\
& & \downarrow & & \downarrow & & & \downarrow & \\
& & 0 = A_{11} & \rightarrowtail & A_{12} & \rightarrowtail & \cdots \rightarrowtail & A_{1n} & \\
& & & & \downarrow & & & \downarrow & \\
& & & & 0 = A_{22} & \rightarrowtail & \cdots \rightarrowtail & A_{2n} & \\
& & & & & & & \downarrow & \\
& & & & & \ddots & & \vdots & \\
& & & & & & & \downarrow & \\
& & & & & & & A_{n-1,n} &
\end{array}
$$

such that $A_{i,j} \rightarrowtail A_{i,k} \twoheadrightarrow A_{j,k}$ is a cofibration sequence. Varying n yields a simplicial category $S_\bullet\mathcal{C}$. The subcategory of isomorphisms is denoted $wS_\bullet\mathcal{C}$. Applying the classifying space construction of a category yields a pointed bisimplicial set $S(\mathcal{C})_{n,m} := B_m w S_n\mathcal{C}$. For example, $S(\mathcal{C})_{n,0} = \mathrm{Obj}(S_n\mathcal{C})$. The K-theory of $\mathcal{C}$ is defined as

$$
K_i(\mathcal{C}) := \pi_{i+1}d(B_* w S_\bullet\mathcal{C}),
$$

where $d(-)$ is the diagonal of a bisimplical set.

By Theorem 3.10, we are ready to define the *algebraic K-theory* of $\mathcal{O}$. More precisely, we consider the Waldhausen category of (finitely generated) free $\mathcal{O}$-modules, which is the same as projective $\mathcal{O}$-modules in all cases of interest by Theorem 3.7.

Definition 3.12

$$
K_i(\mathcal{O}) := K_i(\mathbf{Free}(\mathcal{O})) = \pi_{i+1}(dBwS_\bullet\mathbf{Free}(\mathcal{O})), \quad i \geq 0.
$$

Lemma 3.13 *Given two normed domains and a ring homomorphism $f : K \to K'$ between them satisfying $|f(x)| = |x|$ (so that f restricts to a map $f : \mathcal{O} \to \mathcal{O}'$), the functor $f^* : \mathbf{Free}(\mathcal{O}) \to \mathbf{Free}(\mathcal{O}')$, $M \mapsto M \otimes_{\mathcal{O}} \mathcal{O}'$ is (Waldhausen-)exact and therefore induces a functorial map*

$$f^* : K_i(\mathcal{O}) \to K_i(\mathcal{O}').$$

Proof As pointed out at p. 4, $f^* : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Mod}(\mathcal{O}')$ preserves cokernels. Secondly, tensoring with $\mathcal{O}'$ preserves cofibrations since a map $M \to M'$ of free (or projective) $\mathcal{O}$-modules is a monomorphism iff $M_Q \to M'_Q$ is one (where Q is the quotient field of K, Lemma 3.5) and the statement is true for Q-modules: the map $Q \to Q'$ is injective since $|f(1)| = |1| = 1$ and therefore flat. □

The group $K_0(\mathcal{O})$ is the free abelian group generated by the isomorphisms classes of free $\mathcal{O}$-modules modulo the relations

$$[\mathcal{O}(n') \sqcup \mathcal{O}(n'')] = [\mathcal{O}(n')] + [\mathcal{O}(n'')].$$

Indeed, any cofiber sequence satisfies additivity of the ranks of the involved free modules, as one sees by tensoring the sequence with the quotient field Q of K. Therefore, $K_0(\mathcal{O}) = \mathbb{Z}$.

We now turn to higher K-theory of $\mathcal{O}$. Recall that $E := \{x \in \mathcal{O}, |x| = 1\}$ is the subgroup of norm one elements. Let us write $\mathrm{GL}_n(\mathcal{O}) := \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}(n))$. According to Proposition 3.9,

$$\mathrm{GL}_n(\mathcal{O}) = E \wr S_n = E^n \rtimes S_n, \tag{8}$$

where the symmetric group S_n acts on E^n by permutations. For $E = \mu_2 = \{\pm 1\}$, this group is known as the *hyperoctahedral group*. As usual, we write

$$\mathrm{GL}(\mathcal{O}) := \varinjlim_n \mathrm{GL}_n(\mathcal{O})$$

for the infinite linear group, where the transition maps are induced by $\mathrm{GL}_n(\mathcal{O}(n)) \ni f \mapsto f \sqcup \mathrm{id}_{\mathcal{O}}$. For any group G, let $G_{\mathrm{ab}} = G/[G, G]$ be its abelianization. We write $\pi_i^s(-)$ for the stable homotopy groups of a space and abbreviate $\pi_i^s := \pi_i^s(S^0)$.

Theorem 3.14 *Let $\mathcal{O}$ be a generalized valuation ring as defined in 3.2. Then for $i \geq 0$, there is an isomorphism*

$$K_i(\mathcal{O}) \cong \pi_i^s(BE_+, *),$$

where the right hand side denotes the i-th stable homotopy group of the classifying space of E (viewed as a discrete group), with a disjoint base point $*$. For a map f as in Lemma 3.13, this isomorphism identifies f^* in K-theory with the map on stable homotopy groups induced by $E(\mathcal{O}) \to E(\mathcal{O}')$.

For $i = 1, 2$ we get

$$K_1(\mathcal{O}) = \mathrm{GL}(\mathcal{O})_{\mathrm{ab}} = E \times \mathbb{Z}/2$$
$$K_2(\mathcal{O}) = \varinjlim_n \mathrm{H}_2([\mathrm{GL}_n(\mathcal{O}), \mathrm{GL}_n(\mathcal{O})], \mathbb{Z}) \qquad (9)$$

where the right hand side in (9) is group homology with $\mathbb{Z}$-coefficients.

Before proving the theorem, we first discuss our main example, when $\mathcal{O}$ comes from an infinite place of a number field, as in Example 3.4. Then, we prove a preliminary lemma.

Example 3.15 Let us consider a number field F with the norm induced by some complex embedding $\sigma \in \Sigma_F$ (see p. 3 for notation). The torsion subgroup E_{tor} of $E := \{x \in F^\times, |x| = 1\}$ agrees with the finite group μ_F of roots of unity. The exact localization sequence involving all finite primes of $\mathcal{O}_F$,

$$1 \to \mathcal{O}_F^\times \to F^\times \to L := \ker(\oplus_{\mathfrak{p} < \infty} \mathbb{Z} \to \mathrm{cl}(F)) \to 0,$$

shows $F^\times/\mu_F \cong \mathcal{O}_F^\times/\mu_F \oplus L$. Hence it is free abelian by Dirichlet's unit theorem. Thus

$$E \subset \mu_F \oplus \mathbb{Z}^{r_1+r_2-1} \oplus L,$$

where r_1 and r_2 are the numbers of real and pairs of complex embeddings. Therefore, $E = \mu_F \oplus \mathbb{Z}^S$, where $S := \mathrm{rk}\, E$ is at most countably infinite. Of course, $E = \{\pm 1\}$ whenever σ is a real embedding, but also, for example, for any complex embedding of $F = \mathbb{Q}[\sqrt[3]{2}]$. For $F = \mathbb{Q}[\sqrt{-1}]$, E is the (countably) infinitely generated group of pythagorean triples [2] (see also [8] for a description of the group structure of pythagorean triples in more general number fields).

The group μ_F is cyclic of order w, so the long exact sequence of group homology,

$$\mathrm{H}_i(\mu_F, \mathbb{Z}) \xrightarrow{\cdot n} \mathrm{H}_i(\mu_F, \mathbb{Z}) \to \mathrm{H}_i(\mu_F, \mathbb{Z}/n) \to \mathrm{H}_{i-1}(\mu_F, \mathbb{Z}),$$

together with the Atiyah–Hirzebruch spectral sequence

$$\mathrm{H}_p(\mu_F, \pi_q^{\mathrm{s}}) = \mathrm{H}_p(B\mu_F, \pi_q^{\mathrm{s}}) \Rightarrow \pi_{p+q}^{\mathrm{s}}(B\mu_F) = \pi_{p+q}^{\mathrm{s}}((B\mu_F)_+, *)$$

yield at least for small p and q explicit bounds on $\pi_{p+q}^{\mathrm{s}}((B\mu_F)_+, *)$: the E^2-page reads

$q \uparrow$				
2	$\pi_2^{\mathrm{s}} = \mathbb{Z}/2$	$\mathbb{Z}/w'$		$\mathbb{Z}/w'$
1	$\pi_1^{\mathrm{s}} = \mathbb{Z}/2$	$\mu_F/2 = \mathbb{Z}/w'$		$\mathbb{Z}/w'$
0	$\mathbb{Z}$	$\mu_F = \mathbb{Z}/w$		0
	0	1		2 $\quad p \rightarrow$

where $w' = (2, w)$. In general, $\pi^s_{p+q}((B\mu_F)_+, *)$ is finite for $p + q > 0$. For $i > 0$,

$$
\begin{aligned}
K_i(\mathcal{O}_{F\sigma}) &= K_i(\mathcal{O}_{F(\sigma)}) \\
&= \pi^s_i(B(\mu_F \oplus \mathbb{Z}^{\oplus S})_+, *) \\
&= \pi^s_i\left((B\mu_F)_+ \vee \bigvee_S S^1, *\right) \\
&= \pi^s_i(B\mu_F) \oplus \bigoplus_S \pi^s_{i-1}.
\end{aligned}
$$

In particular

$$
\begin{aligned}
K_1(\mathcal{O}_{F(\sigma)}) &= \mathbb{Z}/2 \oplus \mu_F \oplus \mathbb{Z}^{\oplus S}, \\
K_2(\mathcal{O}_{F(\sigma)}) &= G \oplus (\mathbb{Z}/2)^{\oplus S},
\end{aligned}
$$

where G is a finite (abelian) group which is filtered by a filtration whose graded pieces are subquotients of $\mathbb{Z}/2$ and $\mathbb{Z}/w'$. (Determining G would require studying the differentials of the spectral sequence).

Lemma 3.16 *The map*

$$
\mathrm{GL}(\mathcal{O})_{\mathrm{ab}} \to E \times \mathbb{Z}/2, \ (\epsilon, \sigma) \mapsto \left(\prod_{i=1}^{\infty} \epsilon_i, parity(\sigma)\right)
$$

is an isomorphism. Here the representation of elements of $\mathrm{GL}(\mathcal{O})$ *is as in (8). The group* $[\mathrm{GL}(\mathcal{O}), \mathrm{GL}(\mathcal{O})]$ *is perfect.*

Proof For $i \geq 1$ and $\epsilon \in E$, let $\epsilon_i = (1, \ldots, 1, \epsilon, 1, \ldots) \in E \times E \times \ldots$ be the vector with ϵ at the i-th spot. Let $\sigma_i = (i, i+1) \in S_n$ be the permutation swapping the i-th and $i+1$-st letter. The ϵ_i and σ_i, for $i \geq 1$ and $\epsilon \in E$, generate $G := \mathrm{GL}(\mathcal{O})$ as we have seen in the proof of Proposition 3.9. In G, we have relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, which implies $\sigma_i = \sigma_{i+1}$ in G_{ab}. Moreover, in G we have the relation $\epsilon_i \sigma_i = \sigma_{i+1} \epsilon_{i+1}$, so that we get $\epsilon_i = \epsilon_{i+1}$ in G_{ab}. This shows the first claim.

The perfectness of $[\mathrm{GL}(\mathcal{O}), \mathrm{GL}(\mathcal{O})]$ is a special case of [6, Prop. 3], for example. Alternatively, the above implies that $H := [\mathrm{Aut}(\mathcal{O}(n)), \mathrm{Aut}(\mathcal{O}(n))]$ is given by $H = L \rtimes A_n$, where the alternating group A_n acts on $L := \ker(\prod_{i=1}^n E \to E, (\epsilon^1, \ldots, \epsilon^n) \mapsto \prod \epsilon^i)(\cong E^{n-1})$ by restricting the S_n-action on E^n. Now, the perfectness of A_n for $n \geq 5$ and a simple explicit computation shows $H_{\mathrm{ab}} = 1$ for $n \geq 5$. $\square$

We now prove Theorem 3.14. This theorem is actually an immediate consequence of Proposition 3.9, together with well-known facts about K-theory of G-sets, where G is some group [7, Ex. IV.8.9]. For example, the K-theory of the Waldhausen category of finite pointed sets (which would correspond to the impossible case $E = 1$) is

$$
K_i(\mathbb{F}_1) := K_i((\text{finite pointed sets, injections, bijections})) = \pi^s_i,
$$

the stable homotopy groups of spheres. More generally, for some (discrete) group G, the K-theory of the category $\mathbf{Free}(G)$ of finitely generated (i.e., only finitely many orbits) pointed G-sets on which the G-action is fixed-point free, together with bijections as weak equivalences and injections as cofibrations, is known to be the stable homotopy group of $(BG)_+$. By Proposition 3.9, the canonical functor

$$\mathbf{Free}(E) \to \mathbf{Free}(\mathcal{O}), (E^X) \sqcup \{*\} \mapsto \mathcal{O}(X)$$

induces an equivalence of the categories of cofibrations and therefore an isomorphism of K-theory. For the convenience of the reader, we recall the necessary arguments, which also includes showing that other definitions of higher K-theory (of free $\mathcal{O}$-modules) yield the same K-groups.

Proof Let $Q\mathbf{Free}(\mathcal{O})$ be Quillen's Q-construction, i.e., the category whose objects are the ones of $\mathbf{Free}(\mathcal{O})$ and

$$\mathrm{Hom}_{Q\mathbf{Free}(\mathcal{O})}(A, B) := \{A \leftarrow A' \rightarrowtail B\}/\sim,$$

where two such roofs are identified if there is an isomorphism between them which is the identity on A and B. It forms a category whose composition is given by the composite roof defined by the cartesian diagram
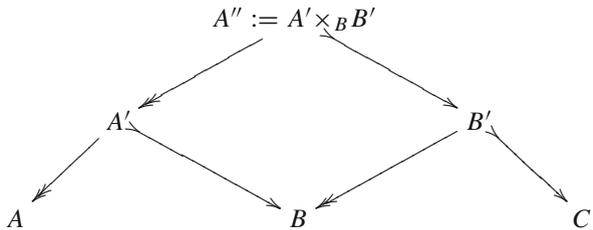
Here, we use that A'' exists (in $\mathbf{Free}(\mathcal{O})$) since it is the kernel of the composite $B' \twoheadrightarrow B \twoheadrightarrow B/A'$, which is split by Proposition 3.9. The subcategory $S := \mathrm{Iso}(\mathbf{Free}(\mathcal{O}))$ of $\mathbf{Free}(\mathcal{O})$ consisting of isomorphisms only is a monoidal category under the coproduct. Hence $S^{-1}S$ is defined. We claim

$$\Omega B Q\mathbf{Free}(\mathcal{O}) = B(S^{-1}S).$$

Indeed, the proof of [7, Theorem IV.7.1] carries over: the extension category $\mathcal{E}\mathbf{Free}(\mathcal{O})$ is defined as in *loc. cit.* and comes with a functor $t : \mathcal{E}\mathbf{Free}(\mathcal{O}) \to Q\mathbf{Free}(\mathcal{O}), (A \rightarrowtail B \twoheadrightarrow C) \mapsto C$. The fiber $\mathcal{E}_C := t^{-1}C$ ($C \in \mathbf{Free}(\mathcal{O})$) consists of sequences $A \rightarrowtail B \twoheadrightarrow C$. The functor

$$\phi : S \to \mathcal{E}_C, \quad A \mapsto A \rightarrowtail A \sqcup C \twoheadrightarrow C$$

induces a homotopy equivalence $B(S^{-1}S) \to B(S^{-1}\mathcal{E}_C)$ in the classical case of an exact category (instead of $\mathbf{Free}(\mathcal{O})$). In our situation, ϕ is an equivalence of categories

since any extension in **Free**$(\mathcal{O})$ splits *uniquely* (Proposition 3.9). Thus [7, Theorem IV.4.10] gives

$$BQ\mathbf{Free}(\mathcal{O}) = K_0(S) \times BGL(\mathcal{O})^+,$$

where the right hand side is the $+$-construction with respect to the perfect normal subgroup $[GL(\mathcal{O}), GL(\mathcal{O})]$ (Lemma 3.16). In the same vein, Waldhausen's comparison of the Q-construction and his $S_\bullet$-construction carries over: $d(BwS_\bullet\mathbf{Free}(\mathcal{O}))$ is weakly equivalent to $BQ\mathbf{Free}(\mathcal{O})$.

Finally, by the Barratt–Priddy theorem (see e.g. [5, Th. 3.6])

$$\pi_i(BGL(\mathcal{O})^+) \cong \pi_i^s(BE_+, *).$$

The identification of the low-degree K-groups is the standard calculation of the S^{-1} S-construction [7, IV.4.8.1, IV.4.10]. $\qquad\square$

Remark 3.17 The calculation of $K_1(\mathcal{O})$ could also be done using the description of K_1 of a Waldhausen category due to Muro and Tonks [4].

Remark 3.18 Recall that for an (ordinary) ring R the following two properties of an R-module M are equivalent: (i) it is projective, (ii) there is another projective module M' such that $M \sqcup M'$ is free. I have not been able to show the corresponding statement for projective $\mathcal{O}$-modules. For example, for a projector $p : \mathcal{O}(n) \to \mathcal{O}(n)$ with $M = \operatorname{im} p$, it is *not* true that the canonical map

$$\phi : M \sqcup \ker p \to \mathcal{O}(n)$$

is an isomorphism of $\mathcal{O}$-modules: for $n = 2$ and the projector p given by the matrix

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

$\ker p$ is the free $\mathcal{O}$-module of rank 1, generated by $(e_1 - e_2)/2 \in \mathcal{O}(2)$. In this case, ϕ induces an isomorphism of $M \sqcup \ker p$ with the free $\mathcal{O}$-module of rank 2 generated by $(e_1 \pm e_2)/2$, but not with $\mathcal{O}(2) = (e_1, e_2)$. The analogous statement of Proposition 3.9 for cofibrations of projective $\mathcal{O}$-modules, as well as the computation of $K_i(\mathbf{Proj}(\mathcal{O}))$ for $i > 0$ (using Waldhausen's cofinality theorem) would carry over verbatim if the above statement about projective $\mathcal{O}$-modules holds. However, the distinction between projective and free modules is only relevant for non-archimedean valuations, by Theorem 3.7.

4 The residue field at infinity

We finish this work by noting two differences (as far as K-theory is concerned) to the case of classical rings, namely the K-theory of the residue "field" at infinity, and the

behavior with respect to completion. For simplicity, we restrict our attention to the case $F = \mathbb{Q}$.

Let $p < \infty$ be a (rational) prime with residue field $\mathbb{F}_p$. There is a long exact sequence

$$K_n(\mathbb{F}_p) \to K_n(\mathbb{Z}_{(p)}) \to K_n(\mathbb{Q}) \xrightarrow{\delta} K_{n-1}(\mathbb{F}_p)$$

which stems from the fact that $\mathbb{Z}_{(p)}$ (the localization of $\mathbb{Z}$ at the prime ideal (p)) is a Noetherian regular local ring of dimension one. Moreover, for $n = 1$ the map δ is the p-adic valuation $v_p : \mathbb{Q}^\times \to \mathbb{Z}$. The situation is less formidable at the infinite places, as we will now see. The (generalized) valuation ring $\mathbb{Z}_{(\infty)}$ (Definition 3.2) is *not* Noetherian: ascending chains of ideals need not terminate. Indeed, consider a finitely generated ideal $I = (m_1, \ldots, m_n) \subset \mathbb{Z}_{(\infty)}$. Then $|I| = \{|m|, m \in I\} = [0, \max_i |m_i|] \cap |\mathbb{Z}_{(\infty)}|$. In particular, an ideal of the form $\{x \in \mathbb{Z}_{(\infty)}, |x| < \lambda\}, \lambda \leq 1$ is not finitely generated, since $|\mathbb{Z}_{(\infty)}|$ is dense in $[0, 1]$. This should be compared with the well-known fact that the valuation ring of a non-archimedian field is noetherian iff the field is trivially or discretely valued.

Definition 4.1 [1, 4.8.13] Put $\mathbb{F}_\infty := \mathbb{Z}_{(\infty)}/\widetilde{\mathbb{Z}_{(\infty)}}$, where $\widetilde{\mathbb{Z}_{(\infty)}}$ is the submonad given by

$$\widetilde{\mathbb{Z}_{(\infty)}}(n) = \{x \in \mathbb{Q}^n, |x| < 1\}.$$

We refer to *loc. cit.* for the general definition of strict quotients of generalized rings by appropriate relations. For us, it is enough to note that every element of $\mathbb{Z}_{(\infty)}(n)$ is uniquely represented by $z = \sum_{i \in I} \lambda_i \epsilon_i e_i$, where $I \subset \{1, \ldots, n\}$, $0 < \lambda_i \leq 1$, $\sum \lambda_i \leq 1$, $\epsilon_i \in E_{\mathbb{Z}_{(\infty)}} = \{\pm 1\}$, and e_i is the standard basis vector. Two elements $z, z' \in \mathbb{Z}_{(\infty)}(n)$ get identified in $\mathbb{F}_\infty(n)$ (Notation: $z \equiv z'$) iff

$$|z| < 1 \quad \text{and} \quad |z'| < 1 \tag{10}$$

or

$$|z| = |z'| = 1, \ I_z = I_{z'}, \quad \text{and} \quad \epsilon_{i,z} = \epsilon_{i,z'} \quad \text{for all } i \in I_z. \tag{11}$$

That is, as a set $\mathbb{F}_\infty(n)$ consists of the faces of the n-dimensional octahedron. Again, 0 is the initial and terminal $\mathbb{F}_\infty$-module, so we can speak about (co)kernels.

As usual, we put

$$K_0(\mathbb{F}_\infty) := \left(\bigoplus_{M \in \mathbf{Free}(\mathbb{F}_\infty)/Iso} \mathbb{Z} \right) /[M] = [M'] + [M''],$$

with a relation for each monomorphism $M' \to M$ in $\mathbf{Free}(\mathbb{F}_\infty)$ such that its cokernel M'' (computed in $\mathbf{Mod}(\mathbb{F}_\infty)$) lies in $\mathbf{Free}(\mathbb{F}_\infty)$. Similarly, we define $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ using projective $\mathbb{F}_\infty$-modules. Using the above, one sees that $\mathbb{F}_\infty$ is not finitely presented as

a $\mathbb{Z}_{(\infty)}$-module. Thus, one should not expect a natural map $i_* : K_0(\mathbb{F}_\infty) \to K_0(\mathbb{Z}_{(\infty)})$. Actually, K-theory of $\mathbb{F}_\infty$-modules behaves badly in the sense of the following proposition:

Proposition 4.2 $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty) = 0$, $K_0(\mathbb{F}_\infty) = \mathbb{Z}$. *In particular, there is no exact* localization sequence *(regardless of the maps involved)*

$$K_1(\mathbb{Z}_{(\infty)}) = \mathbb{Z}/2 \times \{\pm 1\} \to K_1(\mathbb{Q}) = \mathbb{Q}^\times \to K_0(\mathbb{F}_\infty) \to K_0(\mathbb{Z}_{(\infty)})$$
$$= \mathbb{Z} \to K_0(\mathbb{Q}) = \mathbb{Z},$$

or similarly with $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ instead.

Proof We first show that any projective $\mathbb{F}_\infty$-module M which is generated by n elements contains $\mathbb{F}_\infty$ as a submodule, such that the cokernel is a projective $\mathbb{F}_\infty$-module generated by $n-1$ elements. This implies that $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ is generated by $[\mathbb{F}_\infty]$ (which is obvious for $K_0(\mathbb{F}_\infty)$).

The projective module M is specified by a projector $\pi : \mathbb{F}_\infty(n) \to \mathbb{F}_\infty(n)$ with $M = \pi(\mathbb{F}_\infty(n))$. Let $a_i := \pi(e_i) \in \mathbb{F}_\infty(n)$. We pick $a_{ij} \in [-1, 1] \subset \mathbb{R}$ such that $a_i \equiv \sum_{j \in J_i} a_{ij} e_j$ with $a_{ij} \neq 0$ for all $j \in J_i$. Set $A := (a_{ij}) \in \mathbb{R}^{n \times n}$. We may assume that the number n of generators of M is minimal, i.e., there is no surjection $p' : \mathbb{F}_\infty(n') \to M$ with $n' < n$. Indeed, if there is such a surjection, it has a section σ' since M is projective, and $\pi' := \sigma' p'$ would again be a projector.

The minimality of n implies that $a_i \not\equiv a_j$ for all $i \neq j$. Otherwise, the restriction of π to $\mathbb{F}_\infty(n\backslash\{i\}) \subset \mathbb{F}_\infty(n)$ would be surjective. Similarly, the minimality implies $a_i \not\equiv 0 \in \mathbb{F}_\infty(n)$ for all i. Also, put $B = (b_{ij}) := A^2 \in \mathbb{R}^{n \times n}$. Using $(b_{ij})_j \equiv \pi(a_i) \equiv a_i \not\equiv 0 \in \mathbb{F}_\infty(n)$, we obtain $\sum_j |b_{ij}| = 1$ and $\sum_j |a_{ij}| = 1$ by (10).

The minimality of n implies $i \in J_i$ or equivalently, $a_{ii} \neq 0$: otherwise $a_i \equiv \pi(a_i) \equiv \sum_{j \in J_i \backslash \{i\}} a_{ij} a_j$ would be an $\mathbb{F}_\infty$-linear combination of the remaining columns of A.

For every $i \leq n$,

$$1 = \sum_j |b_{ij}| = \sum_j |\sum_k a_{ik} a_{kj}|$$

$$\leq \sum_j \sum_k |a_{ik}||a_{kj}| = \sum_k |a_{ik}| \underbrace{\left(\sum_j |a_{kj}|\right)}_{=1}$$

$$= 1,$$

so equality holds. In particular, the terms $\mathrm{sgn}(a_{ik} a_{kj})$ are either all (for arbitrary $i, j, k \leq n$) non-negative or non-positive. Picking $k = j := i$, we see that they are non-negative, since $\mathrm{sgn}(a_{ii}^2) > 0$, for $a_{ii} \neq 0$.

Let $I^> := \{i, a_{ii} > 0\}$ and likewise with $I^<$. Then $I^> \sqcup I^- = \{1, \ldots, n\}$. Moreover, for $i \in I^>$ and $j \in I^<$, $a_{ii} a_{ij} \geq 0$ and $a_{ij} a_{jj} \geq 0$ imply $a_{ij} = 0$. In other words, the matrix A decomposes as a direct sum matrix $A^> \sqcup A^<$, where $A^>$ and $A^<$ are the submatrices of A consisting of the rows and columns with indices in $I^>$ and

$I^<$, respectively. We may therefore assume $A = A^>$, say. For $i (\in I^>)$, and any j, $a_{ii} a_{ij} \geq 0$ implies $a_{ij} \geq 0$, i.e., the entries of A are all non-negative.

Fix some $i \leq n$. As π is a projector, $a_i \equiv \pi(a_i)$, i.e.,

$$a_i \equiv \sum_{j \in J_i} a_{ij} e_j \equiv \sum a_{ij} \pi(e_j) \equiv \sum_{j \in J_i, k \in J_j} a_{ij} a_{jk} e_k \in \mathbb{F}_\infty(n).$$

By (10), (11), this implies $\text{sgn}(a_{ik}) = \text{sgn}(\sum_j a_{ij} a_{jk})$, which gives

$$J_i = \cup_{j \in J_i} J_j. \tag{12}$$

Indeed, "$\subset$" is easy to see without using the non-negativity of the entries. Conversely, for $k \notin J_i$, $\sum_j a_{ij} a_{jk} = 0$. Since all $a_{**} \geq 0$, this implies $a_{jk} = 0$ for all $j \in J_i$, i.e., $k \notin \cup_{j \in J_i} J_j$.

Now, pick some $i \leq n$ such that J_i is maximal, i.e., not contained in any other J_j, $i \neq j$. Then $i \notin J_j$ for any $i \neq j$ by (12). In other words, the i-th row only contains a single non-zero entry. For simplicity of notation, we may suppose $i = 1$.

Consider the diagram

where ρ is the projection onto the last $n - 1$ coordinates, ι is the injection in the first coordinate. The lower left-hand map is a monomorphism since the first row of A is nonzero. Its cokernel M' is the projective module determined by the matrix $(a_{ij})_{2 \leq i, j \leq n}$. This exact sequence shows that $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ is generated by $[\mathbb{F}_\infty]$.

On the other hand, consider the projective $\mathbb{F}_\infty$-module P defined by the projector $\begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$ [1, 10.4.20]. It consists of 5 elements and can be visualized as
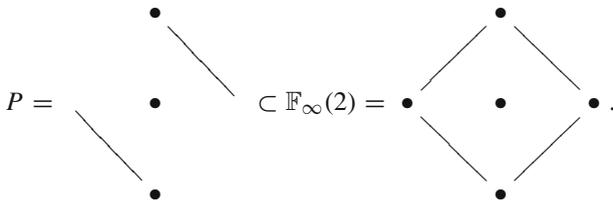
The composition $\mathbb{F}_\infty \xrightarrow{(1/2, 1/2)} \mathbb{F}_\infty(2) \twoheadrightarrow P$ is a monomorphism with cokernel $\mathbb{F}_\infty$. The pictured inclusion $P \rightarrow \mathbb{F}_\infty(2)$ has cokernel $\mathbb{F}_\infty$, spanned by e_1. This shows that $[\mathbb{F}_\infty(2)] = 2[\mathbb{F}_\infty] = [P] + [\mathbb{F}_\infty] = 3[\mathbb{F}_\infty]$. Hence $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty) = 0$.

Finally, we have to show $K_0(\mathbb{F}_\infty) = \mathbb{Z}$. For this, consider a cofiber sequence

$$\mathbb{F}_\infty(n') \overset{i}{\rightarrowtail} \mathbb{F}_\infty(n) \overset{p}{\twoheadrightarrow} \mathbb{F}_\infty(n'').$$

We have to show $n = n' + n''$. Pick a section σ of p. The natural map $i \sqcup \sigma : \mathbb{F}_\infty(n') \sqcup \mathbb{F}_\infty(n'') \to \mathbb{F}_\infty(n)$ is injective, as one easily shows. Thus $n' + n'' \le n$ for cardinality reasons. Conversely, for any basis vector $e_i \in \mathbb{F}_\infty(n) \backslash \mathrm{im}\, i$, $p^{-1}(p(e_i)) = \{e_i\}$, as one shows in the same way as for $\mathbb{Z}_\infty$-modules, cf. (6). Thus $\sigma(p(e_i)) = e_i$, so there are at most n'' such basis vectors by the injectivity of σ. Moreover, at most n' of the basis vectors e_i of $\mathbb{F}_\infty(n)$ are in $\mathrm{im}\, i$ by the injectivity of i. This shows $n' + n'' \ge n$. □

Remark 4.3 For $p \le \infty$, let Fib be the homotopy fiber of $\Omega K(\mathbb{Z}_{(p)}) \to \Omega K(\mathbb{Q})$ and $\widehat{Fib}$ the one of $\Omega K(\mathbb{Z}_p) \to \Omega K(\mathbb{Q}_p)$. The localization sequence for K-theory shows in case $p < \infty$ that Fib and $\widehat{Fib}$ are homotopy equivalent (and given by $K(\mathbb{F}_p)$). Here Ω is the loop space and $K(-)$ is a space (or spectrum) computing K-theory, for example the $S_\bullet$-construction. However, for $p = \infty$, we have

$$
\begin{array}{ccccccccc}
\pi_1(Fib) & \longrightarrow & K_1(\mathbb{Z}_{(\infty)}) & \longrightarrow & K_1(\mathbb{Q}) = \mathbb{Q}^\times & \longrightarrow & \pi_0(Fib) & \longrightarrow & 0 \\
& & \| & & \cap & & & & \\
\pi_1(\widehat{Fib}) & \longrightarrow & \underbrace{K_1(\mathbb{Z}_\infty)}_{(\mathbb{Z}/2)^{\oplus 2}} & \longrightarrow & K_1(\mathbb{R}) = \mathbb{R}^\times & \longrightarrow & \pi_0(\widehat{Fib}) & \longrightarrow & 0,
\end{array}
$$

so that $\pi_0(Fib) \subsetneq \pi_0(\widehat{Fib})$.

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Admissibility and rectification of colored symmetric operads

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1. INTRODUCTION

This paper is devoted to the model-categorical study of operads and their algebras. The concept of an algebra over a colored symmetric operad allows for a uniform treatment of algebraic structures which produce an output from multiple inputs, subject to some symmetry constraints. For example, a commutative monoid X in a symmetric monoidal category $\mathcal{C}$ is specified by Σ_n-equivariant maps $X^{\otimes n} \to X$, subject to the usual associativity and unitality constraints. In a seemingly artificial way, this can be rewritten as

$$\mathrm{Comm}_n \otimes_{\Sigma_n} X^{\otimes n} \to X,$$

where Comm is the so-called commutative operad, which satisfies $\mathrm{Comm}_n = 1$, the monoidal unit. More generally, an algebra of a single-colored operad O is an object $A \in \mathcal{C}$ together with maps

$$O_n \otimes_{\Sigma_n} A^{\otimes n} \to A,$$

which are compatible with the multiplication in O in a suitable sense. Colored symmetric operads, also known as symmetric multicategories, are a many-objects version of ordinary operads. They allow input from more than one object. For example, there is a two-colored operad whose algebras are pairs (R, M), where R is a commutative monoid in $\mathcal{C}$ and M is an R-module. Interestingly, operads themselves are algebras over a certain operad.

Symmetric operads and their algebras, which were first introduced by May, are ubiquitous in homotopy theory and beyond. A prototypical example is the m-fold loop space $\Omega^m X$ of some topological space X: concatenation of paths yields a multiplication map

$$\mu_n \colon (\Omega^m X)^n \to \Omega^m X,$$

which is neither associative nor commutative, but only associative and commutative up to homotopy. This and the compatibility of these homotopies for various n is concisely encoded in the fact that $\Omega^m X$ is an algebra over some operad O, meaning that there are maps (for all n, and compatible with each other):

$$O_n \times_{\Sigma_n} (\Omega^m X)^n \to \Omega^m X.$$

If O_n was just a point, then this would mean that the multiplication on $\Omega^m X$ is strictly commutative and associative, which it is not. However, O can be chosen to be the little disks operad E_m. For $m = \infty$ these levels O_n are contractible spaces, which can be interpreted as saying that infinite loop spaces are homotopy coherent commutative monoids. Recently, E_n-algebras have been attracting a lot of attention in questions related to factorization homology (also known as topological chiral homology) and Goodwillie calculus of functors.

Our first main theorem is a highly flexible existence criterion for a model structure on algebras over operads in a model category. This is a powerful tool for homotopical computations related to algebras over operads, such as the loop space.

Theorem 1.1. *(See Theorems 5.10, 6.6.) Suppose $\mathcal{C}$ is a symmetric monoidal model category which is symmetric h-monoidal and satisfies some minor technical assumptions. Then any symmetric W-colored operad O is admissible, i.e., the category $\mathbf{Alg}_O(\mathcal{C})$ of O-algebras carries a model structure whose weak equivalences and fibrations are inherited from $\mathcal{C}$. Moreover, the forgetful functor $\mathbf{Alg}_O(\mathcal{C}) \to \mathcal{C}^W$ preserves cofibrant objects and cofibrations between them if $\mathcal{C}$ is symmetroidal.*

<div style="text-align:center">1</div>

This admissibility result is widely applicable because its assumptions are satisfied for many basic model categories such as simplicial sets, topological spaces, simplicial presheaves, chain complexes of rational vector spaces. It does not apply to chain complexes of abelian groups, and in fact the commutative operad is provably not admissible in this category. Moreover, as was shown in [PS15], symmetric h-monoidality (and similarly with symmetroidality and symmetric flatness) are stable under transfer and monoidal left Bousfield localizations, which allows to easily promote these properties from basic model categories to more advanced model categories, such as spectra. The latter are shown in [PS14] to be symmetric h-monoidal, symmetroidal, and symmetric flat.

The key condition of symmetric h-monoidality is a symmetric strengthening of the h-monoidality condition. The latter was introduced by Batanin and Berger in [BB13] and is closely related to the monoid axiom. Essentially, it means that for any object Y in $\Sigma_n \mathcal{C}$ (objects of $\mathcal{C}$ with a Σ_n-action) and any cofibration f, the map

$$Y \otimes_{\Sigma_n} s^{\square n} := (Y \otimes s^{\square n})_{\Sigma_n}$$

is an h-cofibration, which is a weak equivalence if f is an acyclic cofibration. Here $f^{\square n}$ is the n-fold pushout product of f. Symmetroidality is a related condition, obtained by replacing "h-cofibration" above by "cofibration" and $Y \otimes -$ by $y \square -$ for some map y.

In practice, a frequent question is how to replace algebras over some operad by those over a weakly equivalent operad. For example, the little disks operad is such that O_n is a contractible space and has a free Σ_n-action. It is therefore called an E_∞-operad. One can therefore ask whether $\Omega^\infty X$, together with the multiplications μ_n, is weakly equivalent to some space with a strictly commutative and associative multiplication. In this example, it is well-known that connected E_∞-spaces with nontrivial Postnikov invariants, e.g., the identity component of the space $\Omega^\infty \Sigma^\infty S^0$, can *not* be strictified to a simplicial abelian group. Indeed by a classical result of Moore [Moo58, Theorem 3.29], connected simplicial abelian groups have trivial Postnikov invariants.

The following rectification theorem identifies a criterion when a rectification of operadic algebras is possible.

Theorem 1.2. *(See Theorem 7.5.) For any map of admissible operads $O \to P$ in a symmetric monoidal model category, there is a Quillen adjunction*

$$\mathbf{Alg}_O(\mathcal{C}) \rightleftarrows \mathbf{Alg}_P(\mathcal{C}).$$

Provided that $\mathcal{C}$ satisfies some minor technical assumptions, it is a Quillen equivalence if and only if $O \to P$ is a symmetric flat *map in $\mathcal{C}$.*

The symmetric flatness condition essentially requires that the map

$$O_n \otimes_{\Sigma_n} X^{\otimes n} \to P_n \otimes_{\Sigma_n} X^{\otimes n}$$

is a weak equivalence for all cofibrant objects X and all $n \geq 0$. If $\mathcal{C}$ is the model category of rational chain complexes, this condition holds for all weak equivalences $O \to P$. In [PS14], we show that the same is true for symmetric spectra in an abstract model category. However, this condition does not hold for all maps in simplicial sets, in particular, it fails for the components of $E_\infty \to \mathrm{Comm}$. This matches the above observation of the nonrectifiability of E_∞-algebras to strictly commutative simplicial monoids. Nevertheless, it is satisfied for any pair of E_∞ operads in simplicial sets, which shows that the algebras over such operads are all Quillen equivalent to each other.

As a consequence of this rectification result, we obtain Theorem 7.10 which relates algebras over operads in the strict sense, as above, and algebras over quasicategorical operads as introduced by Lurie.

Operads and their algebras in different model categories also behave as nicely as possible. Such a result allows to replace $\mathcal{C}$ by a more convenient model category, which is often necessary in practice.

Theorem 1.3. *(See Theorem 8.10). For any Quillen equivalence*

$$F: \mathcal{C} \rightleftarrows \mathcal{D} : G$$

between symmetric monoidal model categories as above, where F is symmetric oplax monoidal such that the canonical maps $FQ(1_\mathcal{C}) \to 1_\mathcal{D}$ and $F(C \otimes C') \to F(C) \otimes F(C')$ are weak equivalences for all cofibrant objects $C, C' \in \mathcal{C}$ there is a Quillen equivalence of the categories of W-colored (symmetric) operads

$$F^{(\mathbf{s})\mathbf{Oper}} : (\mathbf{s})\mathbf{Oper}(\mathcal{C}) \rightleftarrows (\mathbf{s})\mathbf{Oper}(\mathcal{C}') : G.$$

Moreover, there is a Quillen equivalence for any cofibrant (symmetric) operad O,

$$F^{\mathbf{Alg}} : \mathbf{Alg}_O(\mathcal{C}) \rightleftarrows \mathbf{Alg}_{F^{(\mathbf{s})\mathbf{Oper}}(O)}(\mathcal{D}) : G.$$

The admissibility and rectification of nonsymmetric and symmetric operads is a topic that was addressed by various authors. Spitzweck has shown the existence of a semi-model structure for special symmetric operads, namely those whose underlying symmetric sequence is projectively cofibrant (which roughly means that Σ_n acts freely on O_n) [Spi01, Theorem 4.7]. This rules out the commutative operad, whose algebras are commutative monoids. The admissibility of the commutative operad was shown by Lurie under the assumption of symmetroidality of the commutative operad, see Lemma 4.5.4.11.(1) and Proposition 4.5.4.6 of [Lur]. An independent account of this result was later given by White [Whi14, Theorem 3.2]. The admissibility of all

operads was shown by Elmendorf and Mandell for $\mathcal{C} = \mathbf{sSet}$ [EM06, Theorem 1.3], Berger and Moerdijk [BM03] and Caviglia [Cav14] for colored operads. The latter two results use an assumption on the path object which serves to cut short a certain homotopical analysis of pushouts, which is performed in this paper. The path object argument was also used by Johnson and Yau to establish a model structure on colored PROPs [JY09]. PROPs are more general than symmetric operads in that not only multiple inputs, but also multiple outputs are allowed. Harper showed the admissibility of all symmetric operads in simplicial symmetric spectra [Har09]. This was generalized by Hornbostel to spectra in simplicial presheaves [Hor13]. Finally, Muro has shown the admissibility of all nonsymmetric operads [Mur11, Mur15]. A more detailed review of these results is found in §5.

Harper also established a rectification result under the assumption that every symmetric sequence is projectively cofibrant [Har10, Theorem 1.4]. This strong assumption applies to categories such as rational chain complexes. In this case, rectification is due to Hinich [Hin97]. Lurie [Lur] established rectification of E_∞-algebras in the context of ∞-operads, again under a strong assumption that only applies to special model categories such as rational chain complexes. These and further results are reviewed in §7.

Thus all previous results have either restrictions on the operad and/or on the category in which the operad lives. Our results are applicable to all operads and to a very broad range of model categories. This wide applicability results from the fact that conditions of symmetric h-monoidality, symmetroidality and symmetric flatness occurring above are stable under transfer and left Bousfield localization. Thus, they are easily promoted from simplicial sets to simplicial presheaves, say.

In §2, we recall the *symmetricity properties* introduced in [PS15]: symmetric h-monoidality, symmetroidality, and symmetric flatness, and a few other basic notions on model categories. As was shown in [PS15, 5.2.1, 5.2.6, 6.2.1, 6.2.2], these properties are stable transfer and monoidal Bousfield localizations. Given that these two methods are the most commonly used tools to construct model structures, the admissibility and rectification results in this paper are applicable to a wide range of model categories.

In §3, we start with a brief review of colored symmetric collections and the substitution product. Symmetric operads are defined as monoids in this category $\mathbf{sColl}_W(\mathcal{C})$. In §5, we show that symmetric h-monoidality is the key condition needed to ensure the admissibility of arbitrary symmetric operads O, i.e., the existence of the transferred model structure on O-algebras. In §6, we show that symmetroidality is needed to additionally guarantee the strong admissibility of O, i.e., the functor forgetting the O-algebra structure preserves cofibrations with cofibrant source. In §7, we show the rectification of algebras of weakly equivalent symmetric operads. In §8, we establish Quillen equivalences of operads and their algebras in different model categories.

We obtain the above-mentioned theorems by systematically using the symmetricity properties above. In addition to that, this section uses Spitzweck's and Berger–Moerdijk's description of certain pushouts of operads [Spi01, BM09]. In §9, we finish this paper with examples and applications ranging from low-dimensional category theory to prefactorization algebras.

2. Symmetricity properties

Let $\mathcal{C}$ be a symmetric monoidal model category in the sense of [Hov99, Definitions 4.1.6, 4.2.6], except that we do not require the unit axiom. In this section we briefly recall from [PS15, §4] the symmetricity properties which are the key conditions in the admissibility, strong admissibility and rectification results of this paper (see Theorems 5.10, 6.6, 7.5).

We use the notation of [PS15, especially §3.1, Definition 4.2.1]. In particular, in the definitions below, $n = (n_1, \ldots, n_e)$ is an arbitrary finite multiindex. For a family $s = (s_1, \ldots, s_e)$ of maps in $\mathcal{C}$, $\Sigma_n := \prod_i \Sigma_{n_i}$ acts on the pushout product $s^{\square n} := \square_i s_i^{\square n_i}$. A subscript Σ_n denotes the coinvariants of the Σ_n-action, such as $- \otimes_{\Sigma_n} -$.

The concept of h-monoidality in Part (iii) is due to Batanin and Berger [BB13, Definition 1.7]. Recall from op. cit. that an *h-cofibration* $f : X \to Y$ is a map such that in any pushout diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & A & \xrightarrow{g} & B \\
\downarrow f & & \downarrow & & \downarrow \\
X' & \longrightarrow & A' & \xrightarrow{g'} & B'.
\end{array}
$$

the map g' is a weak equivalence if g is one. If, in addition, f is a weak equivalence, it is an *acyclic h-cofibration*.

Definition 2.1. Suppose $\mathcal{C}$ is a symmetric monoidal model category.

(i) $\mathcal{C}$ is *admissibly generated* if it is cofibrantly generated and if the (co)domains of a set I of generating cofibrations (equivalently, by [Hir03, Corollary 10.4.9], all cofibrant objects) are small with respect to the subcategory

$$(2.2) \qquad\qquad \mathrm{cell}(Y \otimes_{\Sigma_n} s^{\square n})$$

for any finite family s of cofibrations, and any object $Y \in \Sigma_n \mathcal{C}$. As usual, cell denotes the closure of a class of maps under pushouts and transfinite composition.

(ii) $\mathcal{C}$ is *strongly admissibly generated* if it is cofibrantly generated and if (co)dom(I) are ($\aleph_0$-)compact (also known as finite) relative to (2.2) [Hir03, Definition 10.8.1].

(iii) $\mathcal{C}$ is *h-monoidal* if the map $Y \otimes s$ is an (acyclic) h-cofibration for any (acyclic) cofibration s, and any object $Y \in \mathcal{C}$.

(iv) $\mathcal{C}$ is *symmetric h-monoidal* if $Y \otimes_{\Sigma_n} s^{\square n}$ is an (acyclic) h-cofibration for any finite family s of (acyclic) cofibrations, and any $Y \in \Sigma_n \mathcal{C}$.

(v) Let $\mathcal{Y} = (\mathcal{Y}_n)_{n \geq 1}$ be a collection of classes $\mathcal{Y}_n$ of morphisms in $\Sigma_n \mathcal{C}$, where $n \geq 1$ is any finite multi-index. We suppose that for $y \in \mathcal{Y}_n$, $y \square -$ preserves *injective* (acyclic) cofibrations in $\Sigma_n \mathcal{C}$, i.e., those maps which are (acyclic) cofibrations in $\mathcal{C}$. Then $\mathcal{C}$ is *$\mathcal{Y}$-symmetroidal* if the morphism

$$y \square_{\Sigma_n} s^{\square n}$$

is an (acyclic) cofibration in $\mathcal{C}$ for all finite families s of (acyclic) cofibrations and all maps $y \in \mathcal{Y}_n$. If $\mathcal{Y}_n = C_{\Sigma_n^{\text{in}} \mathcal{C}}$ (injective cofibrations), we say that $\mathcal{C}$ is *(acyclic) symmetroidal*.

(vi) A weak equivalence y is flat if $y \square s$ is a weak equivalence in $\mathcal{C}$ for any cofibration s. $\mathcal{C}$ is *flat* if all weak equivalences are flat.

(vii) $\mathcal{C}$ is *symmetric flat with respect to a class $\mathcal{Y} = (\mathcal{Y}_n)$ of weak equivalences* $\mathcal{Y}_n \subset \Sigma_n \mathcal{C}$ if $y \square_{\Sigma_n} s^{\square n}$ is a weak equivalence (in $\mathcal{C}$) for any family s of cofibrations and any $y \in \mathcal{Y}_n$. For $\mathcal{Y} = (W_{\Sigma_n \mathcal{C}})$, we just say $\mathcal{C}$ is *symmetric flat*.

These conditions are usually stable under weak saturation, i.e., they only have to be checked for *generating* (acyclic) cofibrations s. Simplicial sets with their standard model structure are symmetroidal, symmetric h-monoidal, and flat (but not symmetric flat). The same is true for simplicial presheaves with the projective, injective, or local (with respect to some topology) model structures, and also for simplicial modules.

For any commutative ring R, chain complexes of R-modules with their projective model structure are flat and h-monoidal. They are symmetroidal, symmetric h-monoidal, and symmetric flat if and only if R contains $\mathbf{Q}$.

The admissible generation is automatic if $\mathcal{C}$ is combinatorial [Lur09, Definition A.2.6.1]. Moreover, topological spaces are admissibly generated, symmetric h-monoidal, and symmetroidal.

To check symmetricity properties of more involved model categories, one can use the fact that the properties above are stable under transfer (appropriately compatible with the monoidal structure), and monoidal Bousfield localizations. Combining these principles, we show in [PS14, Theorem 3.3.4] that spectra with values in a flat, h-monoidal (but not necessarily symmetric flat nor symmetric h-monoidal) category $\mathcal{C}$, with the positive stable model structure, are symmetric flat, symmetroidal, and symmetric h-monoidal. In particular, this allows to replace $\mathcal{C}$ by a Quillen equivalent, symmetric flat and symmetric h-monoidal model category.

The reader is referred to [PS15, Theorem 4.3.9, Theorem 5.2.6, Theorem 6.2.2, §7] for precise statements of the above facts and further examples.

Many results below include a condition that weak equivalences in $\mathcal{C}$ are stable under transfinite compositions or filtered colimits. This condition is satisfied if $\mathcal{C}$ is cofibrantly generated and its generating cofibrations I have compact domain and codomain or, slightly more generally, if $\mathcal{C}$ is pretty small in the sense of [PS15, Definition 2.0.2]. This condition is satisfied for $\mathbf{sSet}$, $\mathbf{Ch}(\mathbf{Mod}_R)$, and many other basic model categories, but not for $\mathbf{Top}$. However, $\mathbf{Top}$ is strongly admissibly generated, which is enough to conclude that the filtered colimits of the weak equivalences that actually occur (as a result of a cellular presentation of cofibrant objects) are indeed again weak equivalences. We call $\mathcal{C}$ *quasi-tractable* if its (acyclic) cofibrations are contained in the weak saturation of (acyclic) cofibrations with cofibrant source (and target). Again, this holds for $\mathbf{sSet}$, $\mathbf{Top}$, $\mathbf{Ch}(\mathbf{Mod}_R)$. All three conditions are stable under localization and transfer, turning them into viable and effectively checkable conditions.

3. Colored collections

In §3–4, let $\mathcal{C}$ be a closed symmetric monoidal category. In this section we give a very brief overview of W-colored (symmetric) operads and colored modules over them (e.g., algebras over operads). The reader can consult Gambino and Joyal [GJ14] for more details. Constructions in this section involve a set W, whose elements are called *colors*. The reader may assume that W has exactly one element, which yields ordinary operads.

W-colored symmetric operads in $\mathcal{C}$ are defined as monoids in a certain monoidal category $(\mathbf{sColl}_W(\mathcal{C}), \circ)$ and V-colored modules over a given W-colored (symmetric) operad O are defined as left modules over O in the category $(\mathbf{sColl}_{V,W}(\mathcal{C}), \circ)$, which itself is a left module over the monoidal category $(\mathbf{sColl}_W(\mathcal{C}), \circ)$. The idea behind $(\mathbf{sColl}_W(\mathcal{C}), \circ)$ is that an object in $\mathbf{sColl}_W(\mathcal{C})$ encodes all possible operations, whereas the monoid structure encodes the composition of operations. Operations have a multisource, consisting of a finite family of colors, and a target, which is a single color. Furthermore, for any operation we can permute elements in its source and obtain another operation. Operations with a fixed multisource and target form an object of $\mathbf{sColl}_W(\mathcal{C})$. Likewise, an object in $\mathbf{sColl}_{V,W}(\mathcal{C})$ encodes operands that can be acted upon from the left by operations in a W-colored operad and the left module structure encodes these actions. The operands are encoded by a V-valued

multisource and a target in W. Thus the data of all operations can be encoded as a $\mathcal{C}$-valued presheaf on a certain groupoid $\mathbf{sSeq}_W$ or $\mathbf{sSeq}_{V,W}$, which we define first.

We simultaneously treat symmetric and nonsymmetric W-colored operads with values in a symmetric monoidal category $\mathcal{C}$, indicating the modifications necessary for the symmetric case in parentheses. I.e., we write $(\mathbf{s})\mathbf{Oper}$ to mean either $\mathbf{sOper}$ (symmetric operads) or $\mathbf{Oper}$ (nonsymmetric operads) etc.

Definition 3.1. Given two sets V, W, define the groupoid of *(symmetric) V,W-sequences* as

$$(\mathbf{s})\mathbf{Seq}_{V,W} := (\mathbf{s})\mathbf{Seq}_V^\times \times W,$$

where W denotes a category with objects W and identities as morphisms and $(\mathbf{s})\mathbf{Seq}_V^\times$ is the category of functions $s\colon I \to V$, where I is a finite ordered set (respectively, finite unordered set, in the symmetric case) set and morphisms $s \to s'$ are isomorphisms of ordered (respectively unordered) sets $f\colon I \to I'$ such that $s = s'f$. We abbreviate $(\mathbf{s})\mathbf{Seq}_W := (\mathbf{s})\mathbf{Seq}_{W,W}$.

The idea is that an object (s,t) in $(\mathbf{s})\mathbf{Seq}_W^\times \times W$ encodes multisource s and target $t \in W$. Morphisms in $\mathbf{sSeq}_W^\times$ account for the fact that one can permute sources in the symmetric case. In the nonsymmetric variant $\mathbf{Seq}_W^s$, no permutation of multisources is allowed. If $W = \{*\}$, then $(\mathbf{s})\mathbf{Seq}_W$ is the category $\mathbf{N}$ of finite ordered sets and identity morphisms (respectively, the category Σ of symmetric sequences, i.e., finite sets and bijections). Their objects can be interpreted as arities. For some $s\colon I \to W$, we write $\Sigma_s := \mathrm{Aut}_{(\mathbf{s})\mathbf{Seq}_W^\times}(s)$. In the nonsymmetric case this group is trivial. In the symmetric case, there is an isomorphism

$$(3.2) \qquad \Sigma_s = \prod_{w \in W} \Sigma_{s^{-1}(w)}.$$

For example, if $W = \{*\}$, then $\Sigma_s = \Sigma_{\sharp I}$.

Given a (symmetric) sequence $X \in (\mathbf{s})\mathbf{Seq}_W$, we write $X_0 \in \mathcal{C}^W$ for the restriction to objects with empty multisource, i.e., $s\colon \emptyset \to W$. We refer to this by saying that X_0 is concentrated in *degree 0*. We refer to the $X_{s,w}$ with $s\colon I \to W$ satisfying $\sharp I = 1$, $s(i) = w$ as the *unit degrees* and will write $X_{w,w}$ in this case. The remaining components are called the *nonunit degrees*.

Definition 3.3. Given symmetric monoidal categories $\mathcal{V}$ and $\mathcal{C}$ such that $\mathcal{C}$ is enriched over $\mathcal{V}$, for a given pair of sets V and W define the categories

$$(\mathbf{s})\mathbf{Coll}_{V,W}(\mathcal{C}) := \mathbf{Fun}((\mathbf{s})\mathbf{Seq}_{V,W}^{\mathrm{op}}, \mathcal{C})$$

where $\mathbf{Fun}$ denotes the $\mathcal{V}$-enriched category of functors. Set

$$(\mathbf{s})\mathbf{Coll}_W(\mathcal{C}) = (\mathbf{s})\mathbf{Coll}_{W,W}(\mathcal{C}),$$

which we call the category of *W-colored (symmetric) collections in $\mathcal{C}$*. The category $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$ is a monoidal category and the category $(\mathbf{s})\mathbf{Coll}_{V,W}(\mathcal{C})$ is a left module over $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$ via the *substitution product*

$$(3.4) \qquad \circ\colon (\mathbf{s})\mathbf{Coll}_{V,W}(\mathcal{C}) \times (\mathbf{s})\mathbf{Coll}_{U,V}(\mathcal{C}) \to (\mathbf{s})\mathbf{Coll}_{U,W}(\mathcal{C}).$$

The substitution product of $F \in \mathbf{sColl}_{V,W}(\mathcal{C})$ and $G \in \mathbf{sColl}_{U,V}(\mathcal{C})$ can be computed as the left Kan extension

where $T_{U,V,W}$ is the category whose objects are quadruples $(u\colon I \to U, v\colon J \to V, w\colon 1 \to W, f\colon I \to J)$, where I and J are finite sets, and morphisms are commutative diagrams

where i and j are isomorphisms and $w = w'$. The functor $F * G$ sends an object (u,v,w,f) to $F(v,w) \otimes \bigotimes_{p \in J} G(u|_{f^{-1}(p)}, p)$ and a morphism (i,j) to the isomorphism $F(j) \otimes \bigotimes_{p \in J} G(i|_{f^{-1}(p)})$.

The monoidal unit of $(\mathbf{s})\mathbf{Coll}_W$ is the W-colored collection that assigns the monoidal unit $1 \in \mathcal{C}$ to all unit degrees (w,w), $w \in W$ and the initial object of $\mathcal{C}$ to anything else. We denote it by $1[1]$.

See Theorem 10.2 and Remark 11.7 in Gambino and Joyal [GJ14] for additional details. In the notation of Gambino and Joyal $\mathcal{R}$ stands for $\mathcal{C}$.

Example 3.5. For example, for $U = \emptyset$ which is the special case relevant for algebras over colored operads,

$$(F * G)(v, w) = F(v, w) \otimes \bigotimes_{p \in J} G(p)$$

and $(F * G)(j) = F(j) \otimes \mathrm{id}$.

In the case $W = \{*\}$ the substitution product in **sColl** can be expressed concisely using the symmetric smash product $\otimes$ on symmetric sequences, see Kelly [Kel05, §3 and §4]:

$$F \circ G = \int^{m \in \Sigma} F(m) \otimes G^{\otimes m} = \coprod_{m \geq 0} F(m) \otimes_{\Sigma_m} G^{\otimes m}.$$

Recall that a category I is *sifted* if for all finite sets k the diagonal functor $I \to I^k$ is cofinal. Filtered categories are sifted. An example of a sifted category that is not filtered is given by the walking reflexive pair category, consisting of two objects 0 and 1 with two parallel arrows $f, g \colon 0 \to 1$ and another arrow $h \colon 1 \to 0$ such that $fh = gh = \mathrm{id}_1$. Sifted colimits of this type are precisely reflexive coequalizers. Any colimit can be expressed using reflexive coequalizers and coproducts, which explains why reflexive coequalizers appear constantly in constructions involving monoids and algebras over monoids.

Proposition 3.6. *The substitution product (3.4) is associative and unital. Moreover, it is cocontinuous in the first variable and preserves sifted colimits in the second variable. In particular, the substitution product is right closed, i.e., the functor $- \circ G$ has a right adjoint for any G.*

Proof. See [GJ14, Proposition 10.9 and Theorem 14.8]. The bicategory of distributors used there is the opposite of the bicategory of finite sets, symmetric collections (with $\circ$ as the composition) and morphisms of collections. $\square$

We emphasize that the substitution product does not preserve nonsifted colimits in the second variable, for example, coproducts, because the functor $X \mapsto X^{\otimes k}$ in general does not preserve nonsifted colimits. In particular, the substitution product is not left closed. The substitution product is also not braided (in particular, not symmetric). Note that the definition of the associator of $\circ$ in the nonsymmetric case needs $\mathcal{C}$ to be *symmetric* monoidal, see Muro [Mur11, Remark 2.2].

Definition 3.7. The category $(\mathbf{s})\mathbf{Oper} := (\mathbf{s})\mathbf{Oper}_W(\mathcal{C})$ of W-colored (symmetric) operads in $\mathcal{C}$ is the category of monoids in $((\mathbf{s})\mathbf{Coll}_W\mathcal{C}, \circ)$, i.e., $O \in (\mathbf{s})\mathbf{Coll}_W\mathcal{C}$ together with a unit map $1[1] \to O$ and a multiplication map $O \circ O \to O$ satisfying the associativity and unitality conditions. For any set V the category of V-colored (symmetric) modules over a (symmetric) W-colored operad O is the category of left modules over O in $(\mathbf{s})\mathbf{Coll}_{V,W}(\mathcal{C})$. It is denoted by $\mathbf{Mod}_O^V$. Explicitly, its objects are given by $M \in (\mathbf{s})\mathbf{Coll}_{V,W}(\mathcal{C})$ together with a map $O \circ M \to M$ subject to the standard associativity and unitality requirements. For $V = \emptyset$ and $V = W$, we speak of O-*algebras* and O-*modules*, respectively and denote them by $\mathbf{Alg}_O$ and $\mathbf{Mod}_O$. Note that any O-algebra is naturally an O-module whose non-zero degrees are $\emptyset$.

The following result describes the categorical properties of colored modules over colored operads.

Theorem 3.8. *Suppose $(\mathcal{C}, \otimes)$ is a symmetric monoidal category that is enriched over a symmetric monoidal category $\mathcal{V}$. Fix two sets V and W, and a W-colored (symmetric) operad O in $\mathcal{C}$.*

 (i) *If $\mathcal{C}$ is complete then so is $\mathbf{Mod}_O^V$ and the forgetful functor $U \colon \mathbf{Mod}_O^V \to (\mathbf{s})\mathbf{Coll}_{V,W}$ creates limits.*

 (ii) *If $\mathcal{C}$ admits sifted colimits (respectively filtered colimits or reflexive coequalizers), which are preserved in each variable by the monoidal product in $\mathcal{C}$, then $\mathbf{Mod}_O^V$ admits sifted colimits, which are created by U.*

 (iii) *If $\mathcal{C}$ admits reflexive coequalizers, which are preserved in each variable by the monoidal product in $\mathcal{C}$, then $\mathbf{Mod}_O^V$ is cocomplete.*

 (iv) *If $\mathcal{C}$ is locally presentable and $\otimes$ preserves filtered colimits in each variable, then $\mathbf{Mod}_O^V$ is locally presentable.*

 (v) *Suppose $f \colon O \to P$ is a morphism of W-colored (symmetric) operads in $\mathcal{C}$. If $\mathcal{C}$ admits reflexive coequalizers that are preserved in each variable by the monoidal product in $\mathcal{C}$, then the pullback functor $f^* \colon \mathbf{Mod}_P^V \to \mathbf{Mod}_O^V$ admits a left adjoint f_*.*

Proof. Via Proposition 3.6, these statements are reduced to similar statements about modules in (nonsymmetric, nonbraided) monoidal categories. (i), (iv), and (v) are then special cases of [BW05, Theorem 3.4.1], [Bor94b, Theorem 5.5.9], and [Lin69, Corollary 1], respectively.

(ii): [Bor94b, Proposition 4.3.2] implies that $\mathbf{Mod}_O^V$ has sifted colimits, which are preserved by U. Reflection of sifted colimits by U is then implied by [Bor94a, Proposition 2.9.7] applied to the opposite functor $U^{\mathrm{op}} \colon (\mathbf{Mod}_O^V)^{\mathrm{op}} \to (\mathbf{Mod}_O^V)^{\mathrm{op}}$. The cases of filtered colimits and reflexive coequalizers are treated identically.

(iii): By (ii), $\mathbf{Mod}_O^V$ admits reflexive coequalizers, which are created by U. Now apply [Lin69, Corollary 2], which in our case says that $\mathbf{Mod}_O^V$ has small colimits if it has reflexive coequalizers and $(\mathbf{s})\mathbf{Coll}_{V,W}(\mathcal{C})$ has small coproducts. $\square$

4. THE ENVELOPING OPERAD

The enveloping operad (see for example [BM09, Propositions 1.5], [BM03, Proposition 5.4]) turns a module or algebra over an operad back into an operad. This is used to relate properties of operadic algebras to those of operads, for example pushouts (Proposition 5.7) and transports along weak monoidal Quillen adjunctions (see Theorem 8.10(ii) and its proof). We continue using the notation of §3.

Definition 4.1. The category **Pairs** consists of pairs (O, A), where $O \in (\mathbf{s})\mathbf{Oper}_W$ is a (symmetric) W-colored operad in $\mathcal{C}$ and $A \in (\mathbf{s})\mathbf{Coll}_W$ is an O-module, and a morphism of pairs $(O, A) \to (P, B)$ is a morphism $f \colon O \to P$ of operads together with a morphism $g \colon A \to f^*B$ of O-modules, where f^* is the restriction functor from P-modules to O-modules.

Lemma 4.2. *There are adjunctions*

$$(4.3) \qquad (\mathbf{s})\mathbf{Coll}_W \underset{U}{\overset{1[1]\times\mathrm{id}}{\rightleftarrows}} \mathbf{Pairs} \underset{\mathrm{id}\times U}{\overset{\mathrm{Env}}{\rightleftarrows}} (\mathbf{s})\mathbf{Oper}_W$$

The functor $\mathrm{id} \times U$ *sends an operad* O *to* $(O, U(O))$, *where* $U(O)$ *is regarded as an* O-module in the obvious way. The functor $1[1] \times \mathrm{id}$ sends X to $(1[1], X)$, where $1[1]$ is the initial operad. The functor U at the left sends (O, M) to $U(M)$, i.e., it forgets the O-module structure on M. The functor Env is called the enveloping operad. It satisfies $\mathrm{Env}(1[1], X) = \mathrm{Free}(X)$, where $\mathrm{Free}\colon (\mathbf{s})\mathbf{Coll}_W \rightleftarrows (\mathbf{s})\mathbf{Oper}_W : U$ is the free-forgetful adjunction.

Proof. The left adjunction holds since

$$\mathbf{Pairs}((1[1], X), (O, M)) = (\mathbf{s})\mathbf{Coll}_W(X, \eta^*M) = (\mathbf{s})\mathbf{Coll}_W(X, U(M)).$$

Here $\eta\colon 1[1] \to O$ is the unit of O, which is the unique morphism of operads $1[1] \to O$. The right adjunction is a special case of Theorem 3.8(v) since **Pairs** are algebras over an operad similar to the operad of operads (§9.4). The last statement follows from the two adjunctions. $\square$

Proposition 4.4. *Fix a (symmetric) operad* O *and consider the functor* $\mathrm{Env}(O, -)\colon \mathbf{Mod}_O \to (\mathbf{s})\mathbf{Oper}_W$. *(We also apply this functor to* O-algebras.)

(i) *The enveloping monoid of the initial* O-algebra is given by $\mathrm{Env}(O, O \circ \emptyset) = O$.

(ii) *The enveloping operad functor* $\mathrm{Env}(O, -)$ *preserves connected colimits of* O-algebras, in particular transfinite compositions.

(iii) *Given a map* $x\colon X \to X'$ in $(\mathbf{s})\mathbf{Coll}_W$, an O-module A, and a map $X \to U(A)$ in $(\mathbf{s})\mathbf{Coll}_W$, we form the pushout square in $\mathbf{Mod}_O$,

$$(4.5) \qquad \begin{array}{ccc} O \circ X & \xrightarrow{\ f\ } & A \\ {\scriptstyle O\circ x}\downarrow & & \downarrow{\scriptstyle a} \\ O \circ X' & \longrightarrow & A'. \end{array}$$

Then the following diagram is cocartesian in $(\mathbf{s})\mathbf{Oper}_W$, *where the top horizontal map is* $\mathrm{Free}(X) \overset{4.2}{=} \mathrm{Env}(1[1], X) \xrightarrow{\mathrm{Env}(\eta, f)} \mathrm{Env}(O, A)$:

$$(4.6) \qquad \begin{array}{ccccc} \mathrm{Free}(X) & \longrightarrow & \mathrm{Free}(U(A)) & \xrightarrow{\ \mathrm{counit}\ } & \mathrm{Env}(O, A) \\ {\scriptstyle \mathrm{Free}(x)}\downarrow & & {\scriptstyle \mathrm{Free}(u)}\downarrow & & \downarrow \\ \mathrm{Free}(X') & \longrightarrow & \mathrm{Free}(U(A) \sqcup_X X') & \longrightarrow & \mathrm{Env}(O, A'). \end{array}$$

(iv) *For any* $A \in \mathbf{Alg}_O$, *there is an equivalence of categories with the undercategory of* A *in* $\mathbf{Alg}_O$:

$$\mathbf{Alg}_{\mathrm{Env}(O, A)} = A \downarrow \mathbf{Alg}_O.$$

In particular $\mathrm{Env}(O, A)_0 = A$.

Proof. (i): For any operad T, we have by adjunction

$$(\mathbf{s})\mathbf{Oper}_W(\mathrm{Env}(O, O \otimes \emptyset), T) = \{(f \in (\mathbf{s})\mathbf{Oper}(O, T), g\colon O \circ \emptyset \to f^*U(T) \in \mathbf{Alg}_O)\}.$$

As $O \circ \emptyset$ is initial in $\mathbf{Alg}_O$, g is unique, so that this Hom-set is isomorphic to $(\mathbf{s})\mathbf{Oper}_W(O, T)$. Hence our claim.

(ii): For a connected index category I, O is the colimit of the constant diagram $i \mapsto O$. Therefore,

$$(O, \mathrm{colim}\, A_i) = \mathrm{colim}(O, A_i).$$

Now apply the cocontinuity of the enveloping operad functor $\mathbf{Pairs} \to (\mathbf{s})\mathbf{Oper}$.

(iii): By Lemma 4.2, the diagram (4.6) is obtained by applying Env to the following diagram of pairs, which is easily seen to be cocartesian. We conclude using that Env preserves all colimits, in particular pushouts.

$$
\begin{array}{ccccc}
(1[1], X) & \xrightarrow{(1[1],f)} & (1[1], U(A)) & \xrightarrow{(\eta,\mathrm{id})} & (O, A) \\
\downarrow{\scriptstyle(1[1],x)} & & \downarrow & & \downarrow \\
(1[1], X') & \longrightarrow & (1[1], U(A) \sqcup_X X')) & \longrightarrow & (O, A' = A \sqcup_{O \circ X} O \circ X').
\end{array}
$$

(iv): Since the monoidal product in $(\mathbf{s})\mathbf{Coll}_W \mathcal{C}$ is right closed, an $\mathrm{Env}(O, A)$-module structure on some $X \in (\mathbf{s})\mathbf{Coll}_W$ is the same as a morphism of operads $\mathrm{Env}(O, A) \to \mathrm{End}(X)$, where $\mathrm{End}(X) := \underline{\mathrm{Hom}}(X, X) \in (\mathbf{s})\mathbf{Oper}_W$ is the endomorphism operad. The adjunction (4.3) tells us that morphisms $\mathrm{Env}(O, A) \to \mathrm{End}(X)$ correspond to morphisms of pairs $(O, A) \to (\mathrm{End}(X), U(\mathrm{End}(X)))$. This is the same as an O-module structure on X and a map $A \to \mathrm{End}(X)$ of O-modules, where $\mathrm{End}(X)$ is regarded as an O-module via the chosen O-module structure on X. Giving $A \to \mathrm{End}(X)$ is the same as $A = A \circ X \to X$. The last equality uses that A is an algebra, i.e., concentrated in degree 0.

The second claim holds since $\mathrm{Env}(O, A)_0 = \mathrm{Env}(O, A) \circ \emptyset$ is the initial $\mathrm{Env}(O, A)$-module, which by the previous step is A. $\qquad\qquad\square$

5. Admissibility of operads

The following definition of admissibility of operads is standard, see, e.g., [BM09, §2]. Strong admissibility does not seem to have been studied before as an independent notion. See [Man01, Lemma 13.6], [Shi04, Proposition 4.1], and [HH13, Proposition 5.17] for strong admissibility statements for operads in chain complexes, simplicial symmetric spectra, and arbitrary model categories, though.

Definition 5.1. A W-colored (symmetric) operad O in a symmetric monoidal model category $\mathcal{C}$ is *admissible* if the product model structure on $\mathcal{C}^W$ transfers to $\mathbf{Alg}_O$ via the forgetful functor

$$
\mathcal{C}^W \leftarrow \mathbf{Alg}_O : U,
$$

i.e., if the classes $\mathrm{W}_{\mathbf{Alg}_O} = U^{-1}(\mathrm{W}_{\mathcal{C}^W})$ of weak equivalences and $\mathrm{F}_{\mathbf{Alg}_O} = U^{-1}(\mathrm{F}_{\mathcal{C}^W})$ of fibrations define a model category structure on $\mathbf{Alg}_O$. Moreover, O is *strongly admissible* if it is admissible and if in addition U preserves cofibrations with cofibrant source, i.e., for a cofibration $a \colon A \to A'$ of O-algebras, $U(a)$ is a cofibration and $U(A)$ is cofibrant in $\mathcal{C}^W$.

The admissibility of symmetric operads is a central problem in homotopical algebra. It was addressed by Berger and Moerdijk [BM03, Theorem 3.2] using the path object argument. Their theorem requires the existence of a symmetric monoidal fibrant replacement functor and the monoidal unit to be cofibrant. A well-known result due to Lewis [Lew91, Theorem 1.1] precludes the existence of such data for a stable monoidal model category of spectra. The conditions of their theorem were weakened by Kro [Kro07, Corollary 2.7], whose version does not require the monoidal unit to be cofibrant. Previously, Spitzweck had shown the existence of a semi-model structure for operads whose underlying symmetric sequence is projectively cofibrant (which roughly means that Σ_n acts freely on O_n) [Spi01, Theorem 4.7]. This covers the Barratt-Eccles operad, for example, which satisfies $O_n = E\Sigma_n$, but excludes, say, the commutative operad Comm which is given by $\mathrm{Comm}_n = 1$, the monoidal unit. This is one of the most important examples of a symmetric operad, since its algebras are commutative monoid objects.

The admissibility of Comm, i.e., the model structure on commutative monoid objects in $\mathcal{C}$, was established by Harper [Har09, Proposition 4.20] and Lurie [Lur, Proposition 4.5.4.6] if $\mathcal{C}$ is freely powered. Their proofs actually only use the weaker condition that the map $f_{\Sigma_n}^{\square n}$ is an acyclic cofibration whenever f is. This property was later called the *commutative monoid axiom* by White, who also suggested a weakening similar to the one discussed in Remark 5.12 [Whi14, Theorem 3.2, Remark 3.3].

The admissibility of arbitrary operads was also shown by Harper under the hypothesis that all objects in $\Sigma_n \mathcal{C}$ are *projectively* cofibrant. Again this is much stronger than being symmetric h-monoidal (see [PS15, Remark 4.2.10, §7]). Subsequently to the present paper, White and Yau reproduced the admissibility of arbitrary operads under the condition that $X \otimes_{\Sigma_n} f^{\square n}$ is an (acyclic) cofibration when f is [WY15, Theorem 6.1.1]. This is a stronger assumption than symmetric h-monoidality, and is inapplicable to various flavors of spectra (e.g., symmetric, orthogonal, etc.) and other constructions used in stable homotopy theory, e.g., L-spaces.

For *nonsymmetric operads*, the situation is quite a bit simpler, since no modding out by Σ_n occurs in the definition of the circle product on nonsymmetric sequences. Muro has shown the admissibility of all nonsymmetric operads under assumptions on $\mathcal{C}$ [Mur11, Theorem 1.2], [Mur15], which by [PS15, Lemma 3.2.6] are very closely related to the nonsymmetric part of Theorem 5.10 below. See Remark 5.12.

A technical key part in all proofs below is the analysis of pushouts of free O-algebra maps and free operad maps. We will start with pushouts of operads and then deduce the pushouts of algebras from this. The following description of pushouts of free (symmetric) operads is due to Spitzweck [Spi01, Proposition 3.5] and, in the slightly different formulation given below, to Berger and Moerdijk [BM09, Lemma 3.1], [BM03, §5.11].

The description of such pushouts is based on the groupoid $(\mathbf{s})\mathbf{Tree}_W$ of W-*colored (symmetric) marked trees*. These are finite planar trees whose edges are labeled with colors $w \in W$. The root vertex has a half-open (i.e., having only one boundary vertex) outgoing edge without called the *root edge*. It also has a (finite) number of vertices having half-open ingoing edges called the *input edges*. Any edge that is not a root edge nor an input edge is called an internal edge. Their boundary consists of two vertices. Moreover, a (finite) number of vertices of the tree is *marked*, the others are not marked. The markings is required to be such that every internal edge has at least one marked vertex at its boundary. *Automorphisms* of symmetric trees are isomorphisms of trees which don't respect the planar structure, but do respect the markings, the colors of the edges and send input edges to input edges. Automorphisms of nonsymmetric trees are only identity morphisms. For a vertex r in a tree, the *valency* $\mathrm{val}(r) \in (\mathbf{s})\mathbf{Seq}_W$ is given by (s, w), where the multisource $s\colon I \to W$ is given by the set I of the incoming edges of r, ordered according to the planar structure (which is only needed to make this notion unambiguous) and their corresponding colors, and target w given by the color of the outgoing edge. In a similar vein, the valency $\mathrm{val}(T)$ of the tree is given by the colors of the input edges and the root edge. The subgroupoid of trees with k marked vertices and valency $(s, w) \in (\mathbf{s})\mathbf{Seq}_W$ is denoted $(\mathbf{s})\mathbf{Tree}_{s,w}^{(k)}$.

Using the notation of Proposition 5.2, the intuitive meaning of these notions is that a tree T with valency (s, w) stands for an operation in O' with inputs given by the multi-source s and target w. Such operations are nested applications of the more elementary operations given by vertices. If T contains no marked vertices, i.e., $k = 0$, then T is just a corolla consisting of a root edge and finitely many input edges, corresponding to the operations that are present in O. More generally, for $k \geq 0$, k operations coming from $\mathrm{Free}(X)$ have been identified by their image in $\mathrm{Free}(X')$.

Proposition 5.2. *(Spitzweck, Berger–Moerdijk) Let $\mathcal{C}$ be a symmetric monoidal model category. For any map $x\colon X \to X'$ in $(\mathbf{s})\mathbf{Coll}_W$ and any pushout diagram in $(\mathbf{s})\mathbf{Oper}_W$,*

(5.3)
$$
\begin{array}{ccc}
\mathrm{Free}(X) & \longrightarrow & O \\
{\scriptstyle \mathrm{Free}(x)}\downarrow & & \downarrow{\scriptstyle o} \\
\mathrm{Free}(X') & \longrightarrow & O'
\end{array}
$$

the map $U(o)_{s,w} \in \Sigma_s \mathcal{C}$ is the transfinite composition of maps $O_{s,w}^{(k)} \to O_{s,w}^{(k+1)}$, for $k \geq 0$, which arise as the following pushouts in $\Sigma_s \mathcal{C}$:

(5.4)
$$
\begin{array}{ccc}
\coprod_T \Sigma_s \cdot_{\mathrm{Aut}\, T}\, x^*(T) & \longrightarrow & O_{s,w}^{(k)} \\
{\scriptstyle \coprod_T \Sigma_s \cdot_{\mathrm{Aut}\, T}\, \epsilon(T)}\downarrow & & \downarrow \\
\coprod_T \Sigma_s \cdot_{\mathrm{Aut}\, T}\, x(T) & \longrightarrow & O_{s,w}^{(k+1)}.
\end{array}
$$

The coproducts run over all isomorphism classes of $(\mathbf{s})\mathbf{Tree}_{s,w}^{(k)}$ as defined above. For such a tree T, the map $\epsilon(T)\colon x^*(T) \to x(T)$ is inductively defined as

$$
\epsilon(T) := \epsilon(r(T)) \,\square\, \underbrace{\bigsquare_i \epsilon(T_i)^{\square t_i}}_{=: \epsilon'(T)},
$$

where $\epsilon(r(T)) \in \Sigma_{\mathrm{val}(r(T))}\mathcal{C}$ is defined as

(5.5)
$$
\epsilon(r(T)) := \begin{cases} x_{\mathrm{val}(r(T))}, & \text{if } r(T) \text{ is marked;} \\ (\eta_O)_{\mathrm{val}(r(T))}, & \text{if } r(T) \text{ is not marked.} \end{cases}
$$

where $\eta_O\colon 1[1] \to U(O)$ is the unit map of O and $\mathrm{val}(r(T))$ is the valency of the root $r(T)$ of T. Isomorphic subtrees (with markings, colors, and input edges induced from T) of the root are grouped together and denoted by T_i, $1 \leq i \leq k$. The number of subtrees isomorphic to T_i is denoted t_i, so that $\sum_{i=1}^k t_i$ equals the cardinality of the multisource of $r(T)$. The group

$$
\mathrm{Aut}(T) = \prod_{i=1}^k \mathrm{Aut}(T_i)^{t_i} \rtimes \prod_{i=1}^k \Sigma_{t_i}
$$

acts on $\epsilon(r(T))$ via the quotient $\prod \Sigma_{t_i}$ and in the natural way on $\epsilon'(T) \in (\prod \mathrm{Aut}(T_i)^{t_i})\mathcal{C}$.

Proof. This is exactly the statement of Berger and Moerdijk cited above, if we replace $\epsilon(T)$ by $\epsilon_u(T)$, which is defined as above, except that $\epsilon(r(T)) := u_{\mathrm{val}(r(T))}$ if the vertex $r(T)$ is marked, where $u\colon U(O) \to U(O) \sqcup_X X'$ is the pushout of x. We conclude using the pushout square $\Sigma_s \cdot_{\mathrm{Aut}\, T} \epsilon(T) \to \Sigma_s \cdot_{\mathrm{Aut}\, T} \epsilon_u(T)$ and [PS15, Proposition 3.1.6]. $\qquad\square$

Proposition 5.2 has the following model-categorical consequence, which again is due to Spitzweck [Spi01, Lemma 3.6] and, in the form below, to Berger-Moerdijk [BM03, Proposition 5.1]. We will show in Lemma 6.1(i) that $U(\eta_O)$ is a cofibration for any cofibrant operad O, so the corollary is applicable to such pushouts. This will be important in the study of strong admissibility. Recall that $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$ is equipped with the projective model structure. Unless the contrary is explicitly stated, all cofibrations in categories of the form $G\mathcal{C}$, for a finite group G, are understood as *projective* cofibrations. (The distinction between injective and projective model structures only matters in the symmetric case, for the category of nonsymmetric collections $\mathbf{Coll}_W(\mathcal{C})$ is just a product of copies of $\mathcal{C}$.)

Corollary 5.6. *In the situation of Proposition 5.2, suppose that $U(\eta_O)$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W$. Also suppose that x is a cofibration in $(\mathbf{s})\mathbf{Coll}_W$. Then the vertical maps in (5.4) are cofibrations in $\Sigma_s\mathcal{C}$. Therefore, $U(o)$ is also a cofibration in $(\mathbf{s})\mathbf{Coll}_W$.*

The following description of pushouts of free O-algebras is due to Fresse [Fre09, Proposition 18.2.11], Elmendorf and Mandell [EM06, §12], Harper [Har09, Proposition 7.12].

Proposition 5.7. *Let $\mathcal{C}$ be a symmetric monoidal model category and O a (symmetric) operad. Let*

$$(5.8) \qquad \begin{array}{ccc} O \circ X & \longrightarrow & A \\ {\scriptstyle O \circ x}\downarrow & & \downarrow{\scriptstyle a} \\ O \circ X' & \longrightarrow & A' \end{array}$$

be a pushout diagram of O-algebras, where $x\colon X \to X'$ is a map in $\mathcal{C}^W$. For any color $w \in W$, the map $U(a)_w \in \mathcal{C}$ lies in the weak saturation of morphisms of the form

$$(5.9) \qquad \mathrm{Env}(O,A)_{s,w} \otimes_{\Sigma_s} \bigsqcup_{r \in W} x_r^{\square s^{-1}(r)}, \quad s\colon I \to W \in (\mathbf{s})\mathbf{Seq}_W^\times, \quad I \neq \emptyset.$$

(The pushout product is finite, since I is a finite set.) For example, if W consists of a single color and we consider symmetric operads, $U(a)$ lies in

$$\mathrm{cof}(\{\mathrm{Env}(O,A)_n \otimes_{\Sigma_n} x^{\square n}, \ n \geq 1\}).$$

Proof. By Proposition 4.4(iv), the map $U(a)_w$ is the level $(\emptyset, w)$ of $\mathrm{Env}(O,A) \to \mathrm{Env}(O,A')$ which by the pushout diagram (4.6) and the description of pushouts in Proposition 5.2 is a transfinite composition of pushouts of the maps (5.4) (where the O there is now $\mathrm{Env}(O,A)$). The map x is concentrated in degree 0, so the only trees T such that the map $\epsilon(T)$ defined in (5.5) is not an isomorphism are the trees (with valence $(\emptyset, w)$) whose marked vertices have valency 0, i.e., are stumps. Since any internal edge has at least one marked vertex, the only such trees T are corollas whose root is not marked and has valence $(t\colon I \to W, w)$ and whose leaves are marked. We get $\epsilon(T) = \mathrm{Env}(O,A)_{t,w} \otimes \square_{i\in I} x_{t(i}$ and $\mathrm{Aut}(T) = \Sigma_t$. Hence the left hand vertical map in (5.4) agrees with (5.9). $\square$

The next result identifies (symmetric) h-monoidality as the key condition for admissibility of all (symmetric) operads. We emphasize that symmetric h-monoidality requirement is stable under weak saturation, transfer of model structures and left Bousfield localization (see [PS15, Theorem 4.3.9, Theorem 5.2.6 and Theorem 6.2.2] for the precise statements). Basic examples of symmetric h-monoidal model categories include simplicial sets, simplicial presheaves, topological spaces, chain complexes of rational vector spaces, and symmetric spectra. See [PS15, §7]. Chain complexes of abelian groups are not symmetric h-monoidal and, in fact, the commutative operad is provably not admissible in chain complexes of abelian groups. Recall the definitions of the terms below from Definition 2.1.

Theorem 5.10. *Suppose $\mathcal{C}$ is a symmetric monoidal model category and W is a set. Furthermore, suppose that either (a) $\mathcal{C}$ is combinatorial and weak equivalences are closed under transfinite compositions or (b) $\mathcal{C}$ is strongly admissibly generated and quasi-tractable. If $\mathcal{C}$ is (symmetric) h-monoidal (the acyclic part is sufficient), then any W-colored (symmetric) operad O in $\mathcal{C}$ is admissible.*

Proof. We apply [Hir03, Theorem 11.3.2] to the adjunction $O \circ - \colon \mathcal{C}^W \rightleftarrows \mathbf{Alg}_O : U$. By Theorem 3.8, U preserves sifted colimits and $\mathbf{Alg}_O$ is complete and cocomplete.

We now show that transfinite compositions of the images under U of cobase changes of elements in $F(J)$ are weak equivalences in $\mathcal{C}^W$. Consider a cocartesian diagram of O-algebras as in (5.8), where $x\colon X \to X'$ is generating acyclic cofibration in $\mathcal{C}^W$ which is also an acyclic (symmetric) h-cofibration. By Proposition 5.7, the morphism $U(a)$ is the (countable) transfinite composition of cobase changes of morphisms

$$(5.11) \qquad \mathrm{Env}(O,A)_{s,w} \otimes_{\Sigma_s} \bigsqcup_{r \in W} x_r^{\square s^{-1}(r)}, \qquad s\colon I \to W \in (\mathbf{s})\mathbf{Seq}_W^\times.$$

Here Env is the enveloping operad (Lemma 4.2) and Σ_s is the group of automorphisms of the multi-source s, which is trivial for nonsymmetric operads, and as in (3.2) for symmetric operads. Each of the above morphisms is

a couniversal weak equivalence or, equivalently [BB13, Lemmas 1.6 and 1.8], an acyclic h-cofibration since x is an acyclic (symmetric) h-cofibration, i.e., each x_r is one. Their transfinite composition is again a couniversal weak equivalence: in case (a) by [PS15, Lemma 2.0.6(iii)] and in case (b) since the above weak equivalences lie in the class (2.2), whose transfinite composition is again a weak equivalence as discussed in [PS15, Proposition 7.5.2].

We finally show that $F(I)$ and $F(J)$ permit the small object argument [Hir03, Definition 10.5.15]. If $\mathcal{C}$ is combinatorial, this is tautological since all objects are small. Suppose now that $\mathcal{C}$ is admissibly generated and quasi-tractable. By Definition 2.1, all cofibrant objects, in particular the (co)domains of I are small relative to cell$(-)$ applied to the maps in (5.11) where x is a cofibration. Therefore, they are small relative to $U(\text{cell}(O \circ I))$. By adjunction, the (co)domains of $O \circ I$ are therefore small relative to cell$(O \circ I)$. Again using the quasi-tractability, the same argument shows that $O \circ J$ is small relative to cell$(O \circ I)$, a fortiori relative to cell$(O \circ J)$. $\square$

Remark 5.12. The proof also shows the following statement: suppose $\mathcal{C}$ is a symmetric monoidal category, $\mathcal{C}'$ is a combinatorial (more generally, admissibly generated) and such that $\mathcal{C}'$ is a commutative $\mathcal{C}$-algebra. Finally suppose that for a finite family of generating cofibrations $x_{r_1}, \ldots, x_{r_k}$ in $\mathcal{C}'$, and $n_1, \ldots, n_k \geq 1$, any object $E \in (\prod_{j=1}^{k} \Sigma_{n_j})\mathcal{C}$, the map

$$(5.13) \qquad E \otimes_{\prod_j \Sigma_{n_j}} \bigsqcup_j x_{r_j}^{\square n_j}$$

lies in a class whose saturation under transfinite composition and pushouts consists of weak equivalences (in $\mathcal{C}'$). Then any W-colored symmetric operad O in $\mathcal{C}$ is admissible, i.e., the O-algebras in $\mathcal{C}'$ carry a transferred model structure. Since the differences are purely grammatical, we omit the proof of this assertion.

The same statement holds for nonsymmetric operads after dropping $\prod \Sigma_{n_j}$ in (5.13). If, in addition, the monoidal product of $\mathcal{C}'$ turns $\mathcal{C}'$ into a monoidal model category it can be further simplified to requiring the above condition only for the maps $E \otimes x$, where $E \in \mathcal{C}$ and $x \in \mathcal{C}'$ is a generating acyclic cofibration. This is exactly the monoid axiom [SS00, Definition 3.3], so the above proof reproduces the one of Muro's aforementioned admissibility result of nonsymmetric operads [Mur11, Theorem 1.2], [Mur15].

In particular, the nonacyclic part of (symmetric) h-monoidality is not necessary for the admissibility statement. We mention the nonacyclic part in the definition of (symmetric) h-monoidality, since the combination of the acyclic and the nonacyclic part of (symmetric) h-monoidality is easier to localize. Also, for concrete model categories, it is usually easier to establish both properties simultaneously. For the same reason, we have separated the saturation with respect to transfinite compositions and the one with respect to pushouts (governed by (symmetric) h-monoidality). See [PS15, Theorem 6.2.2(ii), §7] and the remarks at the end of §2.

6. Strong admissibility of operads

In addition to the admissibility of operads it is in practice desirable to know when the forgetful functor

$$\mathcal{C}^W \leftarrow \mathbf{Alg}_O : U$$

preserves cofibrant objects or even cofibrations with cofibrant source, i.e., when O is strongly admissible. We present two results in this direction: Proposition 6.2 is a result for levelwise projectively cofibrant operads. It works in any symmetric monoidal model category. Theorem 6.6 is a much more flexible criterion for levelwise *injectively* cofibrant operads. Here, the additional key condition is the symmetroidality of $\mathcal{C}$.

The following preparatory lemma captures the preservation of cofibrant objects under various forgetful functors. We don't claim originality for this lemma, for example Part (ii) is similar to [BM09, Proposition 2.3].

Lemma 6.1. *With $\mathcal{C}$ and W as before, the following claims hold:*

(i) *Let $f : O \to O'$ be a cofibration in $(\mathbf{s})\mathbf{Oper}_W$ such that $U(\eta_O)$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W$. Then $U(f)$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W$. In particular:*

   (1) *For any cofibrant operad O, the unit map $U(\eta_O) : 1[1] \to U(O)$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W$. In other words, the levels $O_{s,w}$ are cofibrant in $\Sigma_s^{\text{pro}}\mathcal{C}$ for all $s : I \to W$ if $\sharp I \neq 1$ or if $\sharp I = 1$ and $s(*) \neq w$ and the unit map $1 \to O_{w,w}$ is a cofibration in $\mathcal{C}$ for all $w \in W$.*

   (2) *The forgetful functor U sends cofibrations with cofibrant source to cofibrations.*

   (3) *If the unit $1 \in \mathcal{C}$ is cofibrant, U also preserves cofibrant objects, i.e., the underlying (symmetric) sequence $U(O) \in (\mathbf{s})\mathbf{Coll}_W$ of any cofibrant operad O is cofibrant.*

(ii) *For any (symmetric) operad O, the functor $\mathbf{Alg}_O \to (\mathbf{s})\mathbf{Oper}_W$, $A \mapsto \text{Env}(O, A)$ preserves cofibrations. For example, $O \to \text{Env}(O, A)$ is a cofibration for any cofibrant O-algebra A.*

Proof. (i): The map f is a retract of a transfinite composition of pushouts of maps Free(x) as in (5.3), where x is a cofibration in $(\mathbf{s})\mathbf{Coll}_W$ and, by assumption and cellular induction, O is such that $U(\eta_O)$ is a cofibration. The functor U commutes with retracts and transfinite compositions. Cofibrations (in $(\mathbf{s})\mathbf{Coll}_W$) are stable under these two types of saturation. Therefore the statement follows from Corollary 5.6, using that $U(\eta_O)$ is a cofibration.

The remaining statements are special cases: (i1) follows by applying the general statement to $\eta_O\colon 1[1] \to O$. (i2) follows by combining the general statement and (i1). Finally, (i3) holds since $1[1]$ is the initial operad, whose underlying symmetric sequence is cofibrant in $(\mathbf{s})\mathbf{Coll}_W$ if and only if 1 is cofibrant in $\mathcal{C}$.

(ii): The claim about $\mathrm{Env}(O,-)$ follows from Proposition 4.4: if a is a pushout of a free O-algebra map $O \circ x$ on a cofibration $x \in (\mathbf{s})\mathbf{Coll}_W$ as in (5.8), the map $\mathrm{Env}(O,a)\colon \mathrm{Env}(O,A) \to \mathrm{Env}(O,A')$ is the pushout of $\mathrm{Free}(x)$, which is a cofibration (in $(\mathbf{s})\mathbf{Oper}_W$). For a transfinite composition of cofibrations of O-algebras, we use that both U and $\mathrm{Env}(O,-)$ preserve filtered colimits. By Proposition 4.4(i), the last statement is the special case $a\colon O_0 = O \circ \emptyset \to A$. $\qquad\square$

The following result guarantees strong admissibility for those operads whose levels are projectively cofibrant (except for unit degrees, in which case the map from the monoidal unit to the level is required to be a cofibration). The condition that $U(\eta_O)$ be a cofibration has previously been referred to as well-pointedness or Σ-cofibrancy of O [BM03, BM09]. By [Spi01, Theorem 4], any cofibrant operad O is admissible if $\mathcal{C}$ satisfies the monoid axiom, so it is strongly admissible in this case by the result below.

Proposition 6.2. *Suppose $\mathcal{C}$ is a symmetric monoidal model category. Any admissible (symmetric) operad $O \in (\mathbf{s})\mathbf{Oper}_W(\mathcal{C})$ such that $U(\eta_O)$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$ is strongly admissible. For example, any admissible cofibrant operad is strongly admissible.*

Proof. Suppose A is a cofibrant O-algebra, i.e., $a\colon O_0 = O \circ \emptyset \to A$ is a cofibration in $\mathbf{Alg}_O$. The level 0 of the cofibration $U(\eta_{\mathrm{Env}(O,A)})$ is, by Proposition 4.4(iv), $\emptyset \to U(A)$. In other words $U(A)$ is cofibrant in $\mathcal{C}^W$. $\qquad\square$

The next theorem is a supplementary condition for strong admissibility of arbitrary symmetric operads. Recall from [PS15, §7] that rational chain complexes and symmetric spectra (with an appropriate stable positive model structure) are symmetroidal. The latter statement also shows that under very mild conditions, any monoidal model category is Quillen equivalent to a symmetroidal model category. Moreover, symmetroidality is stable under Bousfield localization and transfer, see [PS15, Theorem 5.2.6 and Theorem 6.2.2] for the precise statements. These results turn Theorem 6.6 into a powerful tool ensuring strong admissibility of operads.

The following lemma is the key stepstone for strong admissibility. In order to keep the exposition brief, we will again speak of "(symmetric) operads" in a symmetric monoidal category to simultaneously cover the case of symmetric and of nonsymmetric operads. Note that in the latter case all the groups Σ_s and $\mathrm{Aut}\,T$ appearing below are trivial by definition.

Lemma 6.3. *Let $\mathcal{C}$ be a symmetric monoidal model category. Let O be a (symmetric) W-colored operad and A any cofibrant O-algebra. For any $(s\colon I \to W, w) \in (\mathbf{s})\mathbf{Seq}_W$, the levels of the unit map*

$$(\eta_{\mathrm{Env}(O,A)})_{s,w}\colon 1[1]_{s,w} \to \mathrm{Env}(O,A)_{s,w}$$

in $\Sigma_s\mathcal{C}$ are contained in $\mathrm{cof}((\mathcal{Y}_O)_s)$, where $(\mathcal{Y}_O)_s$ is the smallest class of morphisms in $\Sigma_s\mathcal{C}$ that contains all isomorphisms, the generating cofibrations of $\mathcal{C}$ (for $\sharp I = 0$ only), and finally contains

$$(6.4) \qquad (\eta_O)_{s \sqcup t, w} \,\square_{\Sigma_t}\, x^{\square t} := (\eta_O)_{s \sqcup t, w} \,\square_{\Sigma_t}\, \bigsqcap_r x_r^{\square n_r} \qquad .$$

Here $t\colon J \to W$ is any multi-source and the multi-index n is given by $n_r = \sharp t^{-1}(r)$ for $r \in W$, and $x = (x_r)$ is a finite family of generating cofibrations in $\mathcal{C}$. (We use the convention that only the finitely many terms with $n_r \neq 0$ appear, unless $J = \emptyset$, in which case we interpret the above expression as $(\eta_O)_{s,w}$.)

In particular, for any cofibrant O-algebra A, the map $\emptyset \to U(A) \in \mathcal{C}^W$ is contained in

$$\mathrm{cof}(\mathrm{C}_{\mathcal{C}} \cup \{(\eta_O)_{t,w} \,\square_{\Sigma_t}\, x^{\square t}, (t,w) \in (\mathbf{s})\mathbf{Seq}_W\}).$$

Proof. We prove this by cellular induction on A, using the properties of the enveloping operad established in Proposition 4.4. We will write $\varphi\colon G\mathcal{C} \to \mathcal{C}$ for any functor that forgets the action of some finite group G, for example $G = \Sigma_s$. For $A = O \circ \emptyset = O_0$, $O = \mathrm{Env}(O,O_0)$ is an isomorphism, so the claim is clear by assumption For a pushout of O-algebras as in (5.8) where A is cofibrant and x is a cofibration, there is a pushout of operads

$$(6.5) \qquad \begin{array}{ccc} \mathrm{Free}(X) & \longrightarrow & \mathrm{Env}(O,A) \\ {\scriptstyle \mathrm{Free}(x)}\downarrow & & \downarrow{\scriptstyle o} \\ \mathrm{Free}(X') & \longrightarrow & \mathrm{Env}(O,A'). \end{array}$$

We now use Proposition 5.2, including the notation. We need to show

$$\Sigma_s \cdot_{\mathrm{Aut}\,T} \epsilon(T) \in (\mathcal{Y}_O)_s.$$

By induction on the tree T, one sees that

$$\varphi(\epsilon(T)) = \bigsqcap_{r \in T} \varphi(\epsilon(r)),$$

where the pushout product runs over all vertices r of T. Recall that $f \square g$ is an isomorphism for all maps g whenever f is an isomorphism. Hence, it is enough to prove our claim for those trees T such that none of the $\epsilon(r)$'s is an isomorphism.

If a vertex $r \in T$ is marked, then $\epsilon(r) = u_{\mathrm{val}(r)}$, where $u \colon U(\mathrm{Env}(O, A)) \to U(\mathrm{Env}(O, A)) \sqcup_X X'$ is the pushout of x along the map $X \to U(\mathrm{Env}(O, A))$ adjoint to the top horizontal map in (6.5). If r is marked and has positive valency, i.e., $(s, w) := \mathrm{val}(r)$ with a multisource $s \colon I \to W$ of arity $\sharp I > 0$, then $u_{s,w}$, which is a pushout of $x_{s,w} = \mathrm{id}_\emptyset$, is an isomorphism. Thus we may assume that the marked vertices have valency 0, i.e., no incoming edges. On the other hand, by definition of marked trees, any edge contains at most one nonmarked vertex. Therefore, the only trees we need to consider are:

(1) The tree denoted $^w+$ consisting of a single marked vertex with no incoming edge and the outgoing root edge colored by w.

(2) The trees denoted $^w-^{t+}_s$ consisting of a single nonmarked vertex which has a root edge of color w, some noninput edges whose other end is marked, and some input edges. The valency of the input edges is denoted s, the one of the noninput edges t.

Here is a picture of $^w+$ and of $^w-^{t+}_s$. The different dashing styles indicate different colors, the two rightmost lower arrows are input edges, the top arrows are the root edges, $\bullet^+$ is a marked vertex, $\bullet^-$ is not marked.
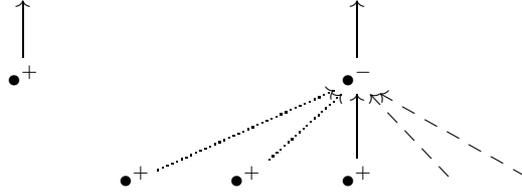
For $T = {}^w+$, we have $\Sigma_s = \mathrm{Aut}\, T = 1$ and $\epsilon(T) = x_w$, which is in $\mathcal{Y}_O$ being a cofibration. For $T = {}^w-^{t+}_s$, we have $\mathrm{Aut}(T) = \Sigma_s \times \Sigma_t$, where Σ_s and Σ_t are defined in (3.2). In the example above, $\Sigma_t = \Sigma_2 \times \Sigma_1$ and $\Sigma_s = \Sigma_2$. We group the noninput edges of $\bullet^-$ according to their color, say n_i noninput edges of color t_i. Then

$$\Sigma_s \cdot_{\mathrm{Aut}\, T} \epsilon(T) = (\eta_{\mathrm{Env}(O,A)})_{s \sqcup t, w} \square_{\prod \Sigma_{n_i}} \bigsqcup_i x_{t_i}^{\square n_i},$$

which is in $\mathcal{Y}_O$ by the inductive hypothesis. This finishes the pushout step.

The handling of retracts and transfinite compositions of cofibrant O-algebras is clear, noting that the functor $\mathbf{Alg}_O \to (\mathbf{s})\mathbf{Coll}_W$, $A \mapsto U(\mathrm{Env}(O, A))$ preserves filtered colimits and retracts.

The claim concerning $U(A)$ is the restriction of the statement about the levels of $\mathrm{Env}(O, A)$ to degree 0. $\qquad\square$

Theorem 6.6. *Suppose $\mathcal{C}$ is a symmetric monoidal model category and O is an admissible (symmetric) W-colored operad in $\mathcal{C}$.*

In the nonsymmetric case, suppose that $(\eta_O)_{s,w} \square - \colon \mathbf{Ar}(\mathcal{C}) \to \mathbf{Ar}(\mathcal{C})$ preserves (acyclic) cofibrations.

In the symmetric case, suppose that $\mathcal{C}$ is symmetroidal (Definition 2.1) with respect to the class $\mathcal{Y}_O = ((\mathcal{Y}_O)_n)$ consisting of

$$(\mathcal{Y}_O)_n := \bigcup_{(s,w)} (\mathcal{Y}_O)_s,$$

where as above s is such that $n_r = \sharp s^{-1}(r)$ (for $r \in W$), $w \in W$ is arbitrary, and $(\mathcal{Y}_O)_s$ is the class of morphisms in $\Sigma_s \mathcal{C}$ defined in Lemma 6.3.

Then O is strongly admissible.

For example, if $\mathcal{C}$ is symmetroidal (i.e., symmetroidal with respect to the injective cofibrations in $\Sigma_n \mathcal{C}$) and the unit map $U(\eta_O) \colon 1[1] \to U(O)$ is an injective cofibration (i.e., $O_{s,w}$ is cofibrant in $\mathcal{C}$ for all nonunit degrees (s, w) and $1 \to O_{w,w}$ is a cofibration in $\mathcal{C}$), then O is strongly admissible.

Proof. It is enough to show that the maps in (5.9) are cofibrations in $\mathcal{C}^W$ for any cofibrant O-algebra A and any cofibration x in $\mathcal{C}^W$.

To show this in the symmetric case, by the symmetroidality condition on $\mathcal{C}$ and [PS15, Lemma 4.3.2], which allows to weakly saturate the symmetroidality class, we have to show that the map

$$(\eta_{\mathrm{Env}(O,A)})_{s,w} \colon 1[1]_{s,w} = \begin{cases} 1, & \text{unit degrees;} \\ \emptyset, & \text{nonunit degrees.} \end{cases} \longrightarrow \mathrm{Env}(O, A)_{s,w}$$

lies (levelwise) in $(\mathcal{Y}_O)_s$. For unit degrees $(s, w) = (w, w)$, this guarantees that $\mathrm{Env}(O, A)_{w,w} \otimes x$ is a cofibration by Lemma 8.4(i). This is exactly the content of Lemma 6.3.

In the nonsymmetric case, the argument is similar, but considerably easier since Σ_s is trivial: if the pushout product with $(\eta_O)_{s,w}$ preserves (acyclic) cofibrations, then so does the pushout product with the maps in (6.4) and therefore also the pushout product with $(\eta_{\mathrm{Env}(O,A)})_{s,w}$. Again, this implies that the maps in (5.9) are cofibrations in $\mathcal{C}^W$.

The last statement is a special case: let $\mathcal{C}$ be symmetroidal, i.e., symmetroidal with respect to $\mathcal{Y}_n := \text{cofib}_{\Sigma_n^{\text{in}}\mathcal{C}}$. Then $\mathcal{Y}_n \supseteq (\mathcal{Y}_O)_n$: indeed, the maps in (6.4) are injective cofibrations by the symmetroidality of $\mathcal{C}$. $\qquad\square$

The following corollary illustrates how to transfer the strong admissibility of operads. Note that the symmetroidality of $\mathcal{C}$ does *not* imply the symmetroidality of $\mathcal{D}$, i.e., the symmetroidality with respect to $\text{cofib}_{\Sigma_n^{\text{in}}\mathcal{D}}$, but only the symmetroidality with respect to $F(\text{cofib}_{\Sigma_n^{\text{in}}\mathcal{C}})$. See [PS15, Theorem 5.2.6(iii) and Remark 5.2.7].

Corollary 6.7. *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be a Quillen adjunction of symmetric monoidal model categories such that the model structure on $\mathcal{D}$ is transferred from $\mathcal{C}$ and such that F is strong symmetric monoidal. Suppose $\mathcal{C}$ is symmetroidal (only required in the symmetric case) and let O be a (symmetric) operad in $\mathcal{C}$ such that $U(\eta_O)$ is an injective cofibration in $\mathbf{sColl}_W(\mathcal{C})$. Let P be the operad in $\mathcal{D}$ given by $P_{s,w} = F(O_{s,w})$. We assume P is admissible. Then P is strongly admissible.*

Proof. The strong monoidality of F gives the strong monoidality of the left adjoint in the adjunction $F : ((\mathbf{s})\mathbf{Coll}_W(\mathcal{C}), \circ) \rightleftarrows ((\mathbf{s})\mathbf{Coll}_W(\mathcal{D}), \circ) : G$. The resulting adjunction of monoids, i.e., W-colored operads (see also (8.11) below)

$$F^{(\mathbf{s})\mathbf{Oper}} : (\mathbf{s})\mathbf{Oper}_W(\mathcal{C}) \rightleftarrows (\mathbf{s})\mathbf{Oper}_W(\mathcal{D}) : G$$

is therefore such that $U_{\mathcal{D}}F^{(\mathbf{s})\mathbf{Oper}} = FU_{\mathcal{C}}$, where $U_? : \mathbf{sColl}_W(?) \to ?$ are the forgetful functors. Therefore, P as defined above, is indeed an operad.

As in the proof of Theorem 6.6, we have to show that $\mathcal{D}$ is $\mathcal{Y}_P$-symmetroidal. The generating cofibrations y of $\mathcal{D}$ are of the form $y = F(x)$, $x \in C_{\mathcal{C}}$. The (levels of) $U(\eta_P)$ are of the form $FU(\eta_O)$. Finally, using the notation of (6.4),

$$F((\eta_O)_{t,w} \,\square_{\Sigma_t} \,\boxed{}\, x^{\square t}) = (\eta_P)_{t,w} \,\square_{\Sigma_t}\, \boxed{}\, y^{\square t}$$

by the strong monoidality of F. Consequently, $\mathcal{Y}_P$ is contained in $F(\mathcal{Y}_O)$. By [PS15, Theorem 5.2.6(iii)], $\mathcal{D}$ is $F(\mathcal{Y}_O)$-symmetroidal, so we are done. $\qquad\square$

7. Rectification of algebras over operads

In this section we use the model structures on modules and algebras over colored operads constructed in the previous section to prove a general operadic rectification result. Rectification theorems address the following question: given a weak equivalence $P \to Q$ of admissible (symmetric) operads, when are their model categories of algebras Quillen equivalent?

An early rectification for symmetric operads is due to Hinich [Hin97] in the category $\mathbf{Ch}(\mathbf{Mod}_R)$, where R is a commutative ring containing $\mathbf{Q}$. In a similar vein, Harper [Har10, Theorem 1.4] showed rectification under the assumption that every symmetric sequence is projectively cofibrant. Lurie [Lur, Theorem 4.5.4.7] showed rectification of E_∞-algebras to commutative algebras (using the language of ∞-operads). All three results have in common that the model category is required to be freely powered [Lur, Definition 4.5.4.2].

Another class of rectification results applies to symmetric spectra with values in some model category $\mathcal{C}$. For individual model categories, such as $\mathcal{C} = \mathbf{Top}$, $\mathcal{C} = \mathbf{sSet}$ and motivic spaces, rectification is due to Elmendorf and Mandell [EM06, Theorem 1.3], Harper [Har09, Theorem 1.4], and Hornbostel [Hor13], respectively. For spectra in an abstract model category $\mathcal{C}$, Gorchinskiy and Guletskiǐ [GG11, Theorem 11] have shown an important special case of symmetric flatness. We show in [PS14, Theorem 3.3.4] that the stable positive model structure on symmetric spectra in (essentially) any model category $\mathcal{C}$ is symmetric flat and give several applications of this fact.

For nonsymmetric operads, Muro [Mur11, Theorem 1.3] has shown a rectification result for a weak equivalence between levelwise cofibrant operads, under similar assumptions to the ones of Theorem 7.5.

Our rectification result, Theorem 7.5, identifies (symmetric) flatness as a necessary and sufficient condition for the rectification of algebras over (symmetric) colored operads. It extends the first group of the above-mentioned results since being freely powered is a much stronger condition than being symmetric flat. It also covers the second group of results since the assumptions of 7.5 are satisfied for $\mathcal{C} = \mathbf{Top}$ etc., see [PS15, §7].

We finish this section with Theorem 7.10, a rectification result relating operadic algebras in the strict sense and in the ∞-categorical sense introduced by Lurie.

Theorem 7.1. *Assume that $\mathcal{C}$ is (symmetric) h-monoidal, symmetric monoidal model category which is (a) strongly admissibly generated, or (b) whose weak equivalences are stable under filtered colimits. Let g be a weak equivalence in $(\mathbf{s})\mathbf{Coll}_W$.*

  (i) *If g is (symmetric) flat in $\mathcal{C}$ (Definition 2.1), then g is pseudoflat on the $(\mathbf{s})\mathbf{Coll}_W$-module $\mathcal{C}^W$, meaning $g \,\square\, b$ is a weak equivalence for any cofibration with cofibrant domain $b: X \to Y$ in $\mathcal{C}^W$, where $\square$ denotes the pushout product of morphisms in $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$.*
  (ii) *If $g \circ X$ is a weak equivalence for any cofibrant object X in $\mathcal{C}^W$, then g is (symmetric) flat in $\mathcal{C}$, provided that the coproduct functor reflects weak equivalences and that $\mathcal{C}$ is quasi-tractable.*

Proof. Recall the multiindex conventions explained in §2. By definition,

$$(7.2) \qquad (g \,\square\, b)_w = \coprod_{s \in \pi_0((\mathbf{s})\mathbf{Seq}_W^\times)} g_{s,w} \,\square_{\Sigma_s}\, \underbrace{\bigotimes_{r \in W} b_r^{\otimes s^{-1}(r)}}_{=: \lambda_s}.$$

(sic, *not* $\square_{r \in W} \, b_r^{\square s^{-1}(r)}$). The coproduct is taken in the category $\mathbf{Ar}(\mathcal{C})$ of morphisms in $\mathcal{C}$ and runs over all isomorphism classes in $(\mathbf{s})\mathbf{Seq}_W^\times$ and Σ_s is the group of automorphisms of some representative of this isomorphism class. Recall that Σ_s is trivial in the nonsymmetric case. In the symmetric case, an isomorphism class amounts to specifying the number of occurrences of each color $r \in W$, and Σ_s is as in (3.2).

We define a multiindex n by $n_r := \sharp s^{-1}(r)$ and set $m_k := \Sigma_n \cdot_{\Sigma_{n-k} \times \Sigma_k} X^{\otimes n-k} \otimes b^{\square k}$ for $0 \le k \le n$. By [PS15, Lemma 4.3.6], applied to the composition $\emptyset \longrightarrow X \xrightarrow{b} Y$, the map $b^{\otimes n}$ is the (finite) composition of pushouts of the maps m_k, where $1 \le k < n$ and m_n (which is not pushed out). By [PS15, Proposition 3.1.6, Lemma 3.1.7], λ_s is therefore the composition of pushouts of

$$(7.3) \qquad\qquad g_{s,w} \,\square_{\Sigma_s}\, m_k.$$

(i): We claim that λ_s appearing in (7.2) is a weak equivalence with h-cofibrant (co)domains. Recall that an h-cofibrant object X is such that $\emptyset \to X$ is an h-cofibration. Weak equivalences with h-cofibrant (co)domains are stable under finite coproducts [BB13, Lemma 1.4(a)]. Presenting (7.2) as the filtered colimit over all finite subsets of the indexing set and using the assumption (b), the claim implies (i). For assumption (a), we use that the transition maps in the filtered diagram are cobase changes of morphisms of the form $\emptyset \to \lambda_s$, which in their own turn can be presented as a composition of maps of the form (2.2).

To show the claim, we focus on the symmetric case and briefly explain the simpler argument in the nonsymmetric case. By [PS15, Lemma 3.2.7] (more precisely, replace $\square$ by $\square_{\Sigma_s}$ there), for λ_s to be a weak equivalence it is enough to show that the maps in (7.3) are weak equivalences and that $(\text{co})\text{dom}(g_{s,w}) \otimes_{\Sigma_s} m_k$ is an h-cofibration. The former holds by symmetric flatness, the latter holds by symmetric h-monoidality, using in both cases the cofibrancy of the (co)domains of b_r.

We now show that $(\text{co})\text{dom}(\lambda_s)$ is an h-cofibrant object. Writing $g_{s,w} \colon A \to B$, this is clear for $\text{codom}(\lambda_s) = B \otimes_{\Sigma_s} Y^{\otimes n}$ which is h-cofibrant by symmetric h-monoidality, using the cofibrancy of Y_r. For the domain of λ_s we first observe that $B \otimes_{\Sigma_s} X^{\otimes n}$ is h-cofibrant. The map from this object to $\text{dom}(\lambda_s)$ is a cobase change of the map $A \otimes_{\Sigma_s} b^{\otimes n}$. Again using the above filtration, this map is a composition of pushouts of the maps $A \otimes_{\Sigma_s} m_k$, which are h-cofibrations by symmetric h-monoidality, using the cofibrancy of X. Since h-cofibrations are stable under pushout and composition [BB13, Lemma 1.3], this shows the claim.

(ii): First, observe that $g \,\square\, b$ is a weak equivalence for any cofibration with cofibrant source $b \colon X \to Y$ in $\mathcal{C}^W$. Indeed, it suffices to show that $A \circ b$ is an h-cofibration, where $A = \text{dom}(g)$, which follows from symmetric h-monoidality and stability of h-cofibrations under colimits of chains [PS15, Lemma 2.0.6(iii)]. Indeed, in this case the pushout of $A \circ b$ along $g \circ X$ is a homotopy pushout since $\mathcal{C}$ is left proper, so that $g \,\square\, b$ is a weak equivalence by the 2-out-of-3 axiom. The coproduct in (7.2) is a weak equivalence, hence so are the λ_s because the coproduct functor reflects weak equivalences. Now we use the filtration (7.3) and show by induction on n that the map $g_{s,w} \,\square_{\Sigma_m \times \Sigma_s}\, (X^{\otimes m} \otimes b^{\square n})$ in the definition of symmetric flatness is a weak equivalence for any cofibration b with cofibrant source X and any $m \ge 0$. The case $m = 0$ then gives the symmetric flatness of g relative to b.

The case $n = 0$ is true by assumption (recall that X is assumed to be cofibrant). For $n \ne 0$ consider the filtration (7.3) (tensored with $X^{\otimes m}$) of the map $g_{s,w} \,\square_{\Sigma_m \times \Sigma_s}\, X^{\otimes m} \otimes b^{\square n}$, which is a weak equivalence by assumption (extended to morphisms as explained in the previous paragraph). For $k \ne n$ the term $g_{s,w} \,\square_{\Sigma_m \times \Sigma_s}\, X^{\otimes m} \otimes m_k = g_{s,w} \,\square_{\Sigma_m \times \Sigma_{n-k} \times \Sigma_k}\, X^{\otimes m+(n-k)} \otimes b^{\otimes k}$ is a weak equivalence by the inductive assumption, and the argument in the previous part shows that its cobase change is a weak equivalence. Thus the remaining map in the filtration, $g_{s,w} \,\square_{\Sigma_m \times \Sigma_s}\, X^{\otimes m} \otimes b^{\square n}$ (we set $k = n$), is also a weak equivalence, as desired.

We have established the symmetric flatness property for the class of cofibrations with cofibrant source. Quasitractability and the weak saturation property for symmetric flatness [PS15, Theorem 4.3.9(ii)] imply the full symmetric flatness property. $\qquad\square$

Remark 7.4. In the situation of Theorem 7.1, similar arguments show that for any weak equivalence f in $\mathbf{sColl}_W(\mathcal{C})$ and any cofibrant object $B \in \mathbf{sColl}_W(\mathcal{C})$, $f \circ B$ is a weak equivalence. For simplicity of notation, we only consider the uncolored case: then $B = \coprod_{n \ge 0} G_n(A_n)$, where G_n places A_n in degree n. Using the fact that $\circ$ preserves filtered colimits in its second variable and the stability of weak equivalences in $\mathcal{C}$, hence $\mathbf{sColl}_W(\mathcal{C})$, under filtered colimits, we may assume that B is concentrated in finitely many degrees.

So let $B = \coprod_{i=1}^k G_{n_i}(A_i)$ (finite coproduct), where $A_i \in \Sigma_{n_i} \mathcal{C}$ is a projectively cofibrant object. The standard formula for multinomial coefficients takes the following form, where $A_i \in \Sigma_{n_i} \mathcal{C}$, $i = 1, \ldots, k$, $k \ge 0$.

$$G_m(f) \circ \left(\coprod_i G_{n_i}(A_i) \right) = \coprod G_{nm} \left(\Sigma_{nm} \cdot_{\Sigma_m \ltimes \Sigma_n^m} f \otimes A^{\otimes m} \right).$$

The coproduct runs over all partitions $m = \sum_{i=1}^{k} m_i$. The multi-index $(m_1, \ldots, m_k)$ will also be denoted by m and likewise for n. In line with the notation in §2, we write $mn = \sum m_i n_i$, and $\Sigma_n^m := \prod \Sigma_{n_i}^{\times m_i}$. The notation mn, Σ_m and Σ_n^m is understood as in [PS15, Definition 4.2.1]. Moreover, $A^{\otimes m}$ stands for $\bigotimes_i A_i^{\otimes m_i}$. By [PS15, Lemma 4.1.2], there is an isomorphism of objects in $\mathcal{C}$ (i.e., disregarding the action of Σ_{nm}),

$$\Sigma_{nm} \cdot_{\Sigma_m \rtimes \Sigma_n^m} f \otimes A^{\otimes m} \cong \left(f \otimes_{\Sigma_{m'}} A^{\otimes m'} \right) \otimes \left(\frac{\Sigma_{nm''}}{\prod \Sigma_{m''} \rtimes \Sigma_n^{m''}} \cdot A^{\otimes m''} \right).$$

Here m' is the subindex of m consisting of those indices m_i where $n_i = 0$ and m'' are the remaining ones. Similarly as above $\Sigma_{nm''} := \prod \Sigma_{n_j m''_j}$ etc. The right factor involving the A_j is cofibrant in $\mathcal{C}$ by the pushout product axiom. The left factor is a weak equivalence by the symmetric flatness of $\mathcal{C}$. Our claim now follows from the (nonsymmetric) flatness.

The following theorem addresses the question of *Quillen invariance* [SS03, Definition 3.11], also referred to as *rectification*, *rigidification*, or *strictification*, i.e., when a weak equivalence of (admissible) operads induces a Quillen equivalence of algebras.

Theorem 7.5. *Suppose $\mathcal{C}$ is a quasi-tractable symmetric monoidal model category such that (a) weak equivalences are stable under filtered colimits or (b) $\mathcal{C}$ is strongly admissibly generated. Given a weak equivalence $f : O \to P$ of admissible (symmetric) W-colored operads in $\mathcal{C}$, the induced Quillen adjunction*

$$f_* : \mathbf{Alg}_O \rightleftarrows \mathbf{Alg}_P : f^*$$

of the corresponding categories of algebras is a Quillen equivalence if and only if $f \circ A$ is a weak equivalence for any cofibrant object A in $\mathcal{C}^W$. This condition is satisfied if $\mathcal{C}$ is (symmetric) flat with respect to f and (symmetric) h-monoidal (Theorem 7.1). If the coproduct functor reflects weak equivalences (e.g., the model category is pointed, or we work with simplicial sets or topological spaces), then the opposite is true: if the above adjunction is a Quillen equivalence, then $\mathcal{C}$ is symmetric flat with respect to f.

Proof. The adjunction exists by Theorem 3.8(v). It is a Quillen adjunction since f^* preserves (acyclic) fibrations. By [Hir03, Definition 8.5.20] we have to show that a morphism $f_* A \xrightarrow{a} B$ is a weak equivalence if and only if its adjoint, i.e., the composition $A \xrightarrow{\eta} f^* f_* A \xrightarrow{f^* a} f^* B$, is a weak equivalence for any cofibrant object A in $\mathbf{Alg}_O$ and any fibrant object B in $\mathbf{Alg}_P$. The functor f^* preserves weak equivalences because both model structures are transferred from $\mathcal{C}^W$, thus it remains to prove that η is a weak equivalence or, equivalently, that the canonical morphism $U(A) \to U(f_* A)$ is a weak equivalence in $\mathcal{C}^W$.

As usual, we perform a cofibration induction for A. Cofibrant objects in $\mathbf{Alg}_O$ are retracts of cellular objects and the latter are obtained as codomains of transfinite compositions of cobase changes of generating cofibrations, starting with the initial O-algebra.

Given a transfinite composition $S = \operatorname{colim} S_i$ in $\mathbf{Alg}_O$, the map $U(S) \to U(f_* S)$ is a weak equivalence if all maps $U(S_i) \to U(f_* S_i)$ are weak equivalences because U creates filtered colimits and weak equivalences in $\mathcal{C}^W$ are stable under filtered colimits by assumption (a). In case (b), we additionally use that the transition maps $U(S_i) \to U(S_{i+1})$ and similarly with $f_* S_i$ are transfinite compositions of cobase changes of maps of the form in (2.2), as witnessed by the filtration (5.9).

To prove the induction step, we consider a cocartesian square of O-algebras as in (5.8) where $X \to X'$ is a cofibration between cofibrant (by quasitractability) objects in $\mathcal{C}^W$. The vertical maps in (5.8) are cofibrations in $\mathbf{Alg}_O$. Applying the left Quillen functor f_* to this square gives a cocartesian square of P-algebras whose vertical maps are again cofibrations and all three objects are cofibrant. Thus both cocartesian squares are also homotopy cocartesian [Lur09, Proposition A.2.4.4]. Furthermore, applying the functor U we obtain a natural transformation between the images of these squares, whose component $U(A) \to U(f_* A)$ is a weak equivalence by induction and the other two components are the maps $O \circ X \to P \circ X$ and $O \circ X' \to P \circ X'$, which are weak equivalences by assumption. Hence the three components of the original natural transformation are also weak equivalences because U creates weak equivalences. Thus the map $A' \to f_*(A')$ is also a weak equivalence because homotopy pushouts preserve weak equivalences.

Finally, the flatness condition is necessary because the map $f \circ A$ is the map $U(X) \to U(f_* X)$ for the cofibrant object $X = O \circ A$. The latter map is the underlying map of the (derived) unit map of X, which must be a weak equivalence for any Quillen equivalence. $\qquad\square$

Remark 7.6. Theorem 7.5 is also true for modules (as opposed to algebras) over weakly equivalent operads. This follows from Remark 7.4.

Remark 7.7. Rectification also holds in a slightly more general context (cf. Remark 5.12): $\mathcal{C}$ is a symmetric monoidal model category, $\mathcal{C}'$ is a quasi-tractable model category whose weak equivalences are stable under filtered colimits and that is a $\mathcal{C}$-algebra (in the symmetric case, a commutative $\mathcal{C}$-algebra). Finally suppose $\mathcal{C}'$ is (symmetric) flat as an algebra (respectively, commutative algebra) over $\mathcal{C}$ (again using an obvious extension of Definition 2.1). Then any weak equivalence of W-colored admissible operads $O \to P$ in $\mathcal{C}$ yields a Quillen equivalence of their algebras in $\mathcal{C}'$.

We finish this section by establishing a quasicategorical rectification result, which generalizes [Lur, Theorem 4.5.4.7] to the case of arbitrary symmetric quasicategorical operads (as opposed to just the commutative operad) and uses conditions that are significantly weaker than freely poweredness. The following proposition and theorem, as well as the fact that the former is relevant for the latter, were suggested to the first author by Thomas Nikolaus. Our proofs are quite similar to that of Lurie in [Lur], the most noticeable difference being the usage of notions of strong admissibility and symmetric flatness. In particular, strong admissibility allows us to give a rather concise proof of the preservation of cofibrant objects in the following proposition.

Proposition 7.8. *Suppose $\mathcal{C}$ is a $\mathcal{V}$-enriched cofibrantly generated symmetric monoidal model category and O is a symmetric colored operad in $\mathcal{V}$ that is admissible in $\mathcal{C}$. If the unit map $\eta_O \colon 1[1] \to O$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$ then the forgetful functor $U \colon \mathbf{Alg}_O(\mathcal{C}) \to \mathcal{C}$ creates (i.e., preserves and reflects) homotopy sifted colimits.*

Remark 7.9. We remind the reader that the notion of a sifted homotopy colimit is stronger than that of a sifted colimit. For example, the reflexive coequalizer diagram is sifted but not homotopy sifted [Ros07, Remark 4.5.(e)]. This is unlike the filtered case, where both notions coincide for ordinary categories.

Proof. The proof is similar to the proof of [Lur, Lemma 4.5.4.12]. The functor U creates weak equivalences, so the reflection property is implied by the preservation property. Denote by I an arbitrary homotopy sifted small category, such as Δ^{op}. We have a (strictly) commuting diagram

$$
\begin{array}{ccc}
\mathbf{Fun}(I, \mathbf{Alg}_O(\mathcal{C})) & \xrightarrow{\mathrm{colim}} & \mathbf{Alg}_O(\mathcal{C}) \\
\downarrow{\scriptstyle V} & & \downarrow{\scriptstyle U} \\
\mathbf{Fun}(I, \mathcal{C}) & \xrightarrow{\mathrm{colim}} & \mathcal{C},
\end{array}
$$

where V is also a forgetful functor. Preservation of homotopy colimits means that the diagram commutes up to a weak equivalence after we derive it. Both U and V are automatically derived because they preserve weak equivalences. We endow $\mathbf{Fun}(I, \mathbf{Alg}_O(\mathcal{C}))$ with the projective model structure (with respect to I) and the transferred model structure on $\mathbf{Alg}_O(\mathcal{C})$, which exists by assumption. Note that this model structure is the same as the model structure transferred from the projective model structure on $\mathbf{Fun}(I, \mathcal{C})$, if we regard O as an I-constant operad in $\mathbf{Fun}(I, \mathcal{C})$. Indeed, both model structures are transferred twice: once for the functor category, and the other time for operadic algebras, and it doesn't matter in which order to transfer.

The top colim (hence also $U \circ \mathrm{colim}$) can be derived by performing a cofibrant replacement in the source category. If V preserves cofibrant objects, then it can also be derived in this way, which proves the desired commutativity. To show that V preserves cofibrant objects, we observe that V can be rewritten as the forgetful functor $\mathbf{Alg}_O(\mathbf{Fun}(I, \mathcal{C})) \to \mathbf{Fun}(I, \mathcal{C})$. It preserves cofibrant objects since O is strongly admissible in $\mathbf{Fun}(I, \mathcal{C})$ by Proposition 6.2. $\qquad\square$

We are now ready to state the conditions under which every quasicategorical algebra over a quasicategorical operad corresponding to a strict colored symmetric operad can be rectified to a strict algebra over the strict operad. We state the theorem for the simplicial case, because a detailed writeup of quasicategorical operads is only available in this setting, however, the proof holds more generally as indicated in the remark below. This extends results of Lurie [Lur, Theorems 4.1.4.4, 4.5.4.7] for the associative operad and the commutative operad, Haugseng [Hau13, Theorem 2.16] for arbitrary nonsymmetric operads and Hinich [Hin13, Theorem 4.1.1] for symmetric operads in the case $\mathcal{C} = \mathbf{Ch}(\mathbf{Mod}_R)$.

Theorem 7.10. *Suppose $\mathcal{C}$ is a simplicial symmetric monoidal model category and O is a $\mathcal{C}$-admissible simplicial symmetric colored operad. Denote by $\mathrm{CO}_{\mathcal{C}}$ and $\mathrm{CO}_{\mathbf{Alg}_O(\mathcal{C})}$ the full subcategories spanned by the corresponding classes of cofibrant objects. The canonical comparison functor*

$$
\mathrm{N}(\mathrm{CO}_{\mathbf{Alg}_O(\mathcal{C})})[\mathrm{W}_{\mathbf{Alg}_O(\mathcal{C})}^{-1}] \to \mathbf{HAlg}_{\mathrm{N}^{\otimes}O}(\mathrm{N}(\mathrm{CO}_{\mathcal{C}})[\mathrm{W}_{\mathcal{C}}^{-1}])
$$

is an equivalence of quasicategories if and only if $\mathcal{C}$ is symmetric flat (Definition 2.1) with respect to $\mathrm{Q}O \to O$, the levelwise projective cofibrant replacement of the underlying symmetric sequence of O. Here $\mathbf{HAlg}$ is used in the sense of Definition 2.1.3.1 (denoted by $\mathbf{Alg}$ there) in Lurie [Lur] and $\mathrm{N}^{\otimes}O$ denote the operadic nerve of O, as explained in Definition 2.1.1.23 there.

Remark 7.11. If O is nonsymmetric, projective cofibrancy can be replaced by injective cofibrancy (tautologically true for simplicial sets) because we don't have to mod out symmetric group actions. Thus the condition of symmetric flatness can be dropped and every nonsymmetric simplicial colored operad admits quasicategorical rectification.

Proof. The symmetric sequence $\mathrm{Q}O$ can be constructed by taking the levelwise product of the Barratt—Eccles operad E_{∞} and O, which in fact gives us an operad and not just a symmetric sequence. The individual levels have a free action of the symmetric group and therefore are projectively cofibrant. (Note here that the levels

of O are injectively cofibrant, since any simplicial set is cofibrant.) They are weakly equivalent to those of O because simplicial sets are flat and every simplicial set is cofibrant.

The morphism $QO \to O$ induces an equivalence of the quasicategories of algebras over $N^{\otimes}QO$ and $N^{\otimes}O$, and below we will prove that the comparison functor is an equivalence of quasicategories for QO, so by the 2-out-of-3 property for equivalences of quasicategories the main statement is equivalent to $QO \to O$ inducing a Quillen equivalence, which by Theorem 7.5 is equivalent to symmetric flatness over $QO \to O$. It remains to show that the comparison map is an equivalence of quasicategories when O is levelwise projectively cofibrant.

The rest of the proof coincides with the proof of [Lur, Theorem 4.5.4.7] (modified in the obvious fashion for colored operads instead of the commutative operad), with the following modifications: for the part (d) (preservation of homotopy colimits of simplicial diagrams) we use Proposition 7.8, whereas for part (e) we have to establish that the free (strict) O-algebra on a cofibrant object $C \in \mathcal{C}^W$ is also the free quasicategorical O-algebra in the sense of [Lur, Definition 3.1.3.1]. Using Proposition 3.1.3.13 there this reduces to proving that the free O-algebra $O \circ C = \coprod_{n \geq 0} O_n \otimes_{\Sigma_n} C^{\otimes n}$ is also the derived free O-algebra. By assumption O is levelwise projectively cofibrant, so the individual terms in the coproduct are cofibrant in $\mathcal{C}^W$ and compute the corresponding derived tensor product. Coproducts of cofibrant objects are also homotopy coproducts, which concludes the proof. $\qquad\square$

Remark 7.12. The same proof works (and therefore the theorem holds) for enriched quasicategorical operads as soon as one has the obvious analog of [Lur, Proposition 3.1.3.13]. We refer the reader to the upcoming work of Haugseng on enriched quasicategorical operads for the case of an arbitrary enriching symmetric monoidal quasicategory.

8. Transport of operads and operadic algebras

This section gives an answer to the following important question: When does a Quillen equivalence $\mathcal{C} \rightleftarrows \mathcal{D}$ of symmetric monoidal model categories induce a Quillen equivalence of (symmetric) operads and their algebras? The first result in this direction, for monoids and modules over monoids, is due to Schwede and Shipley [SS03, Theorem 3.12]. This was generalized to nonsymmetric operads and their algebras by Muro [Mur14, Theorem 1.1, 1.5], [Mur15]. In both statements, the monoidal unit was assumed to be cofibrant. This assumption, however, is not satisfied in the very interesting stable positive model structure on symmetric spectra [PS14, Theorem 3.3.4], so we pay special attention to not assuming the cofibrancy of the monoidal unit 1. For example, Lemma 8.5, which governs certain cofibrant replacements, is trivial if 1 is cofibrant.

Definition 8.1. [SS03, Definition 3.6] An adjunction between symmetric monoidal categories

$$(8.2) \qquad\qquad F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

is a *(symmetric) oplax-lax adjunction* if G is symmetric lax monoidal (see, for example, [Bor94b, Definition 6.4.1]). It is a *weak symmetric monoidal Quillen adjunction* if in addition the oplax structural maps of F induced from the lax structure of G,

$$F(Q1_{\mathcal{C}}) \to 1_{\mathcal{D}},$$
$$F(C \otimes C') \to F(C) \otimes F(C').$$

are weak equivalences for all cofibrant objects $C, C' \in \mathcal{C}$.

Definition 8.3. An object A in a monoidal model category is *monoidally cofibrant* if there is a cofibration $1 \to A$ from the monoidal unit to A.

As far as their monoidal properties are concerned, monoidally cofibrant objects behave like cofibrant objects, as is illustrated by the following lemmas:

Lemma 8.4. *Let $\mathcal{C}$ be a monoidal model category.*

(i) *If B is monoidally cofibrant, then $- \otimes B \colon \mathcal{C} \to \mathcal{C}$ is a left Quillen functor. (Thus monoidally cofibrant objects are pseudocofibrant in the sense of Muro [Mur14, Appendix A].)*

(ii) *If $a \colon A \to A'$ and $b \colon B \to B'$ are two cofibrations with monoidally cofibrant source, then so is $a \,\square\, b$. If either A or B is cofibrant, then $a \,\square\, b$ is also cofibrant.*

Proof. (i): Pick a cofibration $\eta \colon 1 \to B$. For any (acyclic) cofibration a, the map $a \otimes B$ is the composition of a pushout of $a = a \otimes 1$ and $a \,\square\, \eta$. Both are (acyclic) cofibrations.

(ii): By (i), $A \otimes B$ is monoidally cofibrant and $a \otimes B$ and $A \otimes b$ are cofibrations. Hence $a \,\square\, b := \operatorname{dom}(a \,\square\, b)$ is monoidally cofibrant as well. If, say, A is cofibrant, then $\emptyset \to A \xrightarrow{A \otimes \eta} A \otimes B \to a \,\square\, b$ is a composition of cofibrations. $\qquad\square$

Lemma 8.5. *Let A and B be two cofibrant or monoidally cofibrant objects in a quasi-tractable monoidal model category satisfying the unit axiom, i.e., $Q(1) \otimes C \sim C$ for all cofibrant objects C. Also assume that (a) weak*

equivalences are stable under filtered colimits or (b) $\mathcal{C}$ is strongly admissibly generated. Then the following map is a weak equivalence:

$$Q(A) \otimes Q(B) \to A \otimes B.$$

Proof. If A and B are cofibrant, the claim is clear. We now show the statement if B is cofibrant and A is monoidally cofibrant.

The cofibration $1 \to A$ is a retract of a transfinite composition of maps $A_0 = 1 \to \cdots \to A_\infty = A$ where each $a_n \colon A_n \to A_{n+1}$ is the pushout of a generating cofibration $s \colon S \to S'$. We write $E_n \colon s \to a_n$ for the pushout square. The functor $- \otimes B$ is a left Quillen functor by Lemma 8.4(i). In particular, it preserves cofibrations, so that $E_n \otimes B$ is a pushout of a cofibration between cofibrant objects along a map with cofibrant target $A_n \otimes B$ (which holds by induction, starting with $A_0 \otimes B = B$). Hence it is a homotopy pushout square. Similarly, $Q(E_n)$ is a pushout one of whose legs is a cofibration, and all objects in the square are cofibrant. Hence $Q(E_n) \otimes Q(B)$ is also a homotopy pushout square. In the natural transformation of homotopy pushout squares

$$Q(E_n) \otimes Q(B) \longrightarrow E_n \otimes B$$

the two left maps in the depth direction are

(8.6) $$Q(S) \otimes Q(B) \xrightarrow{\sim} S \otimes B,$$

since $Q(S) \to S$ is a weak equivalence between cofibrant objects and similarly for B. (Only at this point we are using the cofibrancy of B.) The same works for S'. The third map is

(8.7) $$Q(A_n) \otimes Q(B) \to A_n \otimes B$$

which by induction on n is a weak equivalence, starting for $n = 0$ with the weak equivalence

$$Q(1) \otimes Q(B) \sim 1 \otimes Q(B) = Q(B) \sim B$$

given by the unit axiom. Thus, the fourth map in the cube, $Q(A_{n+1}) \otimes Q(B) \to A_{n+1} \otimes B$, is a weak equivalence. Thus, for all $n < \infty$, (8.7) is a weak equivalence. In other words, $Q(A_n) \otimes Q(B)$ is a cofibrant replacement of $A_n \otimes B$. Then $Q(A_\infty) \otimes Q(B) \sim \operatorname{colim} Q(A_n) \otimes Q(B) \sim \operatorname{colim} A_n \otimes B = A_\infty \otimes B$, using that weak equivalences are stable under filtered colimits by assumption and the preservation of filtered colimits by $\otimes$. In case (b) we additionally use that the transition maps are cobase changes of generating cofibrations tensored with a fixed object, hence in the class (2.2). We have shown the claim if B is cofibrant.

If B is merely monoidally cofibrant, we run the same argument again, noting that for a cofibrant object S, the weak equivalence $Q(S) \otimes Q(B) \sim S \otimes B$ used in (8.6) above is a weak equivalence by the previous step. □

The following variant can be proved using the same technique as Lemma 8.5. The left properness is used to ensure that the pushouts appearing in the cellular induction are homotopy pushouts. The details are left to the reader.

Lemma 8.8. *Let A be a cofibrant or monoidally cofibrant object in a flat left proper quasi-tractable monoidal model category $\mathcal{C}$ whose weak equivalences are stable under filtered colimits. Then $A \otimes -$ preserves weak equivalences.*

The following lemma of Berger and Moerdijk may be called an *equivariant pushout product axiom*.

Lemma 8.9. [BM06, Lemma 2.5.3] *Let $1 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 1$ be a short exact sequence of finite groups. Then, for a monoidal model category $\mathcal{C}$,*

$$\otimes \colon \Gamma_2^{\mathrm{pro}} \mathcal{C} \times \Gamma^{\mathrm{pro}'} \mathcal{C} \to \Gamma^{\mathrm{pro}} \mathcal{C}$$

is a left Quillen bifunctor. Here $\Gamma^{\mathrm{pro}'} \mathcal{C}$ denotes the model structure on $\Gamma \mathcal{C}$ whose cofibrations are Γ_1-projective cofibrations.

Theorem 8.10. *Suppose $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ is a weak symmetric monoidal Quillen adjunction (Definition 8.1) between quasi-tractable symmetric monoidal model categories such that (a) weak equivalences are stable under filtered colimits or (b) $\mathcal{C}$ is strongly admissibly generated. Also suppose that both $\mathcal{C}$ and $\mathcal{D}$ are either left proper or their monoidal unit is cofibrant.*

(i) *Suppose that the transferred model structures on the categories $(\mathbf{s})\mathbf{Oper}_W(\mathcal{C})$ and $(\mathbf{s})\mathbf{Oper}_W(\mathcal{D})$ exist. (See Corollary 9.4.1 for a sufficient condition.) Then there is a Quillen adjunction of the categories of (symmetric) operads*

(8.11) $$F^{(\mathbf{s})\mathbf{Oper}} : (\mathbf{s})\mathbf{Oper}_W(\mathcal{C}) \rightleftarrows (\mathbf{s})\mathbf{Oper}_W(\mathcal{D}) : G \qquad .$$

It is a Quillen equivalence if (F, G) is a Quillen equivalence.

(ii) *For any admissible (symmetric) operad O in $\mathcal{C}$, there is a Quillen adjunction*

(8.12) $$F^{\mathbf{Alg}} : \mathbf{Alg}_O(\mathcal{C}) \rightleftarrows \mathbf{Alg}_{F^{(\mathbf{s})\mathbf{Oper}}(O)}(\mathcal{D}) : G.$$

It is a Quillen equivalence if (F, G) is a Quillen equivalence and O is a cofibrant operad.

(iii) If P is an admissible (symmetric) operad in $\mathcal{D}$ such that $G(P)$ is also admissible, there is a Quillen adjunction

$$(8.13) \qquad F_{\mathbf{Alg}} : \mathbf{Alg}_{G(P)}^{\mathcal{C}} \rightleftarrows \mathbf{Alg}_P^{\mathcal{D}} : G.$$

It is a Quillen equivalence if (F, G) is a Quillen equivalence, P is fibrant, and $\mathcal{C}$ and $\mathcal{D}$ admit rectification of (symmetric) operads.

Proof. Since G is symmetric lax monoidal, it induces a lax monoidal adjunction

$$(8.14) \qquad F : ((\mathbf{s})\mathbf{Coll}_W\mathcal{D}, \circ) \to ((\mathbf{s})\mathbf{Coll}_W\mathcal{C}, \circ) : G \qquad .$$

In particular, G preserves monoids, i.e., (symmetric) operads. This defines the right adjoint in (8.11). The right adjoint in (8.12) sends an $F^{(\mathbf{s})\mathbf{Oper}}(O)$-algebra B to $G(B)$ which is an O-algebra via

$$O \circ G(B) \to GF^{(\mathbf{s})\mathbf{Oper}}(O) \circ G(B) \to G(F^{(\mathbf{s})\mathbf{Oper}}(O) \circ B) \to G(B).$$

The left adjoints exist by [Bor94b, Theorem 4.5.6]. Moreover, the right adjoints are Quillen right adjoints since (acyclic) fibrations are again created by the forgetful functors.

We now establish the advertised Quillen equivalences.

(i): We have to show that for any cofibrant operad O, the natural map

$$\phi_O : F(\mathrm{Q}(U(O))) \to U(F^{(\mathbf{s})\mathbf{Oper}}(O))$$

is a weak equivalence. In this case we have the following chain of equivalent statements for any cofibrant operad $O \in (\mathbf{s})\mathbf{Oper}_W(\mathcal{C})$ and any fibrant operad $P \in (\mathbf{s})\mathbf{Oper}_W(\mathcal{D})$ which implies the Quillen equivalence (8.11):

$$\begin{aligned}
F^{(\mathbf{s})\mathbf{Oper}}(O) \sim P &\Leftrightarrow UF^{(\mathbf{s})\mathbf{Oper}}(O) \sim U(P) \\
&\Leftrightarrow F(\mathrm{Q}(U(O))) \sim U(P) \\
&\Leftrightarrow \mathrm{Q}(U(O)) \sim G(U(P)) = U(G(P)) \\
&\Leftrightarrow U(O) \sim U(G(P)) \\
&\Leftrightarrow O \sim G(P).
\end{aligned}$$

The cellular induction starts with the initial operad $O = 1_{\mathcal{C}}[1]$, for which $F^{(\mathbf{s})\mathbf{Oper}}(O) = 1_{\mathcal{D}}[1]$. Thus $\phi_{1[1]}$ is a weak equivalence by the weak monoidality of F.

Using the notation of Proposition 5.2, we now consider a pushout of operads along a map $\mathrm{Free}(x)$ where x is a cofibration in $\mathbf{sColl}_W(\mathcal{C})$. We will show that $\phi_{O'}$ is a weak equivalence provided that ϕ_O is one.

Applying $F\mathrm{Q}$ to the filtration (see Proposition 5.2)

$$U(o) : O^{(0)} := U(O) \to \cdots \to O^{(\infty)} := U(O')$$

gives the front face of the following commutative cube in $\Sigma_s\mathcal{D}$. The back face is part of the filtration

$$U(\tilde{o}) : \tilde{O}^{(0)} := UF^{(\mathbf{s})\mathbf{Oper}}(O) \to \cdots \to \tilde{O}^{(\infty)} := UF^{(\mathbf{s})\mathbf{Oper}}(O')$$

associated to the pushout of operads in $\mathcal{D}$ which is obtained by applying the left adjoint $F^{(\mathbf{s})\mathbf{Oper}}$ to (5.3):

$$\begin{array}{ccc}
\mathrm{Free}(\tilde{X}) := F^{(\mathbf{s})\mathbf{Oper}}(\mathrm{Free}(X)) & \longrightarrow & \tilde{O} := F^{(\mathbf{s})\mathbf{Oper}}(O) \\
{\scriptstyle \mathrm{Free}(\tilde{x})} \big\downarrow & & \big\downarrow {\scriptstyle \tilde{o}} \\
\mathrm{Free}(\tilde{X}') := F^{(\mathbf{s})\mathbf{Oper}}(\mathrm{Free}(X')) & \longrightarrow & \tilde{O}' := F^{(\mathbf{s})\mathbf{Oper}}(O').
\end{array}$$

Here and below, the notation $\tilde{?}$ indicates the object or morphism that is obtained by considering the data in the filtration of $\tilde{o} := F^{(\mathbf{s})\mathbf{Oper}}(o)$. For example, $\tilde{X} := F(X)$ and similarly for X', x. The coproduct runs over all isomorphism classes of marked trees T in $(\mathbf{s})\mathbf{Tree}_{s,w}^{(k+1)}$.

At this point (and only here) we use the assumption that $\mathcal{D}$ is either left proper or its monoidal unit is cofibrant: in the former case any pushout along a cofibration is a homotopy pushout. In the latter case, $\tilde{O}_{s,w} = \tilde{O}_{s,w}^{(0)}$ is cofibrant for all (s,w) by Lemma 6.1(i3) and therefore by induction the same is true for $\tilde{O}_{s,w}^{(k)}$. Hence the pushout above is again a homotopy pushout. Likewise, the front square is a homotopy pushout, since $FQ(-)$ preserves those. Thus, $r^{(k+1)}$ is a weak equivalence if $r^{(k)}$, $*$ and $**$ are ones. The map $r^{(k)}$ is a weak equivalence by induction on k, starting with

$$r^{(0)} \colon FQ(O_{s,w}^{(0)}) = FQ(U(O)_{s,w}) \to \tilde{O}_{s,w}^{(0)} = UF^{(\mathbf{s})\mathbf{Oper}}(O)_{s,w}$$

which is the (s,w)-level of ϕ_O, which is a weak equivalence by the cellular induction on O. It remains to show that the maps $*$ and $**$ are weak equivalences.

Let $T \in (\mathbf{s})\mathbf{Tree}_{s,w}$ be any tree. By induction on the height of T, we prove the following claims:

(A) The map $\epsilon(T)$ is a cofibration in $(\operatorname{Aut} T)^{\mathrm{pro}}\mathcal{C}$ with cofibrant or monoidally cofibrant domain (Definition 8.3). The domain is cofibrant for all trees except (possibly) for the tree $T_w^- := (\overset{w}{\to}\bullet\overset{w}{\to}) \in (\mathbf{s})\mathbf{Tree}_{w,w}^{(0)}$ which consists of a single nonmarked vertex with input edge and root edge colored by w. In particular, $\epsilon(T)$ is a cofibration with cofibrant domain for all $T \in (\mathbf{s})\mathbf{Tree}_{s,w}^{(k+1)}$ with $k \geq 0$. (These are the trees appearing in the cubical diagram above. In order to perform the induction, we also need to consider $T \in (\mathbf{s})\mathbf{Tree}_{s,w}^{(0)}$.)

(B) There are weak equivalences in $\mathbf{Ar}(\mathcal{C})$ (i.e., both source and target of the morphisms are weakly equivalent)

$$FQ(\epsilon(T)) \to \tilde{\epsilon}(T).$$

Let $(t,w) := \operatorname{val}(r(T))$ be the valency of the root $r(T)$ of T. If T consists of a single vertex $r(T)$ (with an outgoing root edge and finitely many input edges), then $t = s$ and

$$\epsilon(T) = \epsilon(r(T)) = \begin{cases} (\eta_O)_{(t,w)}, & \text{if the root } r(T) \text{ is not marked;} \\ x_{(t,w)}, & \text{if the root } r(T) \text{ is marked.} \end{cases}$$

Both are cofibrations in $\Sigma_t(\mathcal{C})(= \operatorname{Aut}(T)\mathcal{C})$, the former by Lemma 6.1(i). Since $X = \operatorname{dom}(x)$ is cofibrant by quasitractability, the source of $\epsilon(T)$ is monoidally cofibrant for $(T =)r(T) = T_w^-$ and cofibrant else. This shows claim (A).

For claim (B), we note that $FQ(U(\eta_O))$ is weakly equivalent to $\eta_{\tilde{O}}$ by the unit part of the weak monoidality of F and the cellular induction on O. To show $FQ(u) \sim \tilde{u}$, we consider the pushout square in $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$, denoted E:

$$\begin{array}{ccc} X & \longrightarrow & U(O) \\ \downarrow x & & \downarrow u \\ X' & \longrightarrow & U(O) \sqcup_X X' \end{array}$$

It is a homotopy pushout square in all degrees: for unit degrees, the left vertical map is $\operatorname{id}_\emptyset$ and for nonunit degrees $O_{s,w}$ is (Σ_s-projectively) cofibrant (and $x_{s,w}$ is a cofibration). Applying FQ to E gives a homotopy pushout square in $(\mathbf{s})\mathbf{Coll}_W(\mathcal{D})$. The square $\tilde{E}$ in $\mathbf{sColl}_W(\mathcal{D})$ obtained by replacing X, X' and O by their $\tilde{?}$-counterparts is also a homotopy pushout square. By cellular induction $FQU(O) \sim U\tilde{O}$. Of course $FQ(X) \sim \tilde{X}(= F(X))$ by the cofibrancy of X (using the quasitractability of $\mathcal{C}$) and similarly for X'. We obtain the desired weak equivalence

$$F(Q(U(O) \sqcup_X X')) \sim U(\tilde{O}) \sqcup_{\tilde{X}} \tilde{X}'$$

and hence claim (B) for the tree T consisting of a single (marked or unmarked) vertex.

We now perform the induction step. We may assume that T has at least two vertices. By definition,

$$\epsilon(T) = \epsilon(r(T)) \,\Box\, \underbrace{\bigsqcup_i \epsilon(T_i)^{\Box t_i}}_{=:\epsilon'(T)}.$$

Recall that a map f in a model category $\mathcal{C}$ is a cofibration with cofibrant source if and only if it is a cofibrant object in $\mathbf{Ar}(\mathcal{C})$, i.e., $\operatorname{id}_\emptyset \to f$ is a cofibration. Likewise, f is a cofibration with monoidally cofibrant source if and only if there is a cofibration $\operatorname{id}_1 \to f$ in $\mathbf{Ar}(\mathcal{C})$.

We write $\epsilon(r(T)) \colon V \to W$ and $\epsilon'(T) \colon e^*(T) \to e(T)$. Let $\operatorname{val}(r(T)) = (s,w)$. As was noted above, $\epsilon(r(T))$ is a cofibration in $\mathbf{sColl}_W(\mathcal{C})$. Its domain $V_{s,w} := \operatorname{dom}(\epsilon(r(T))_{s,w})$ is monoidally cofibrant in $\Sigma_s\mathcal{C}$ if T is of the form $(T_1 \overset{w}{\to}\bullet\overset{w}{\to})$ where T_1 is the subtree of the root vertex. In this case, we abusively write $r(T) = T_w$. In all other cases, $V_{s,w}$ is cofibrant. Hence $\operatorname{id}_1 \to \epsilon(r(T))$ (respectively $\operatorname{id}_\emptyset \to \epsilon(r(T))$) is a cofibration in $\mathbf{Ar}(\Sigma_s\mathcal{C}) = \Sigma_s\mathbf{Ar}(\mathcal{C})$. By induction on T, $\epsilon(T_i)$ is an $\operatorname{Aut}(T_i)$-projective cofibration whose source is monoidally cofibrant (if $T_i = T_w$) and cofibrant (otherwise). Again, we reinterpret this in terms of cofibrations in $\mathbf{Ar}(\operatorname{Aut}(T_i)\mathcal{C})$.

We now consider four cases:

(1) $r(T) \neq T_w^-$, at least one $T_i \neq T_w^-$: By Lemma 8.9, applied to $\mathbf{Ar}(\mathcal{C})$ (with the pushout product), the map
$$(\mathrm{id}_\emptyset \to \epsilon(r(T))) \,\square\, (\mathrm{id}_\emptyset \to \epsilon'(T)) = (\mathrm{id}_\emptyset \to \epsilon(r(T)) \,\square\, \epsilon'(T)) = (\mathrm{id}_\emptyset \to \epsilon(T))$$
is a cofibration in $\mathbf{Ar}(\mathrm{Aut}(T)\mathcal{C})$ in this case, i.e., $\epsilon(T)$ is a cofibration with cofibrant source.

(2) $r(T) \neq T_w^-$, all $T_i = T_w^-$: Then
$$(\mathrm{id}_\emptyset \to \epsilon(r(T))) \,\square\, (\mathrm{id}_1 \to \epsilon'(T)) = (\mathrm{id}_\emptyset \to \epsilon(r(T)) \,\square\, \epsilon'(T)) = (\mathrm{id}_\emptyset \to \epsilon(T))$$
is a cofibration in $\mathbf{Ar}(\mathrm{Aut}(T)\mathcal{C})$.

(3) Similarly for $r(T) = T_w^-$, $T_1 \neq T_w^-$.

(4) $r(T) = T_w^-$, $T_1 = T_w^-$: By definition of the trees in $(\mathbf{s})\mathbf{Tree}_{s,w}$, any internal edge contains at least one marked vertex. Thus this tree does not lie in $(\mathbf{s})\mathbf{Tree}_{s,w}$ unless T_1 is empty, in which case we have shown the claim above.

This shows claim (A).

We now show (B). We may assume that T consists of at least two vertices. Consider the diagram E whose left square is by definition cocartesian,

$$(8.15)$$

$$\begin{array}{ccc}
V_{t,w} \otimes e^*(T) & \xrightarrow{V_{t,w} \otimes \epsilon'(T)} & V_{t,w} \otimes e(T) \\
{\scriptstyle \epsilon(r(T))_{t,w} \otimes e^*(T)} \downarrow & & \downarrow \\
W_{t,w} \otimes e^*(T) & \longrightarrow & P \xrightarrow[\epsilon(r(T))_{t,w} \,\square\, \epsilon'(T)]{} W_{t,w} \otimes e(T).
\end{array}$$

We claim that the left pushout square is a homotopy pushout. By Lemma 8.4(i), both the left vertical and the top horizontal maps are cofibrations (in $\mathcal{C}$, say), hence the claim is clear if $V_{t,w} \otimes e^*(T)$ is cofibrant, because in this case the above pushout diagram is cofibrant as a diagram. By the above, $V_{t,w}$ and $e^*(T)$ are either cofibrant or monoidally cofibrant. Again using Lemma 8.4, the only way that $V_{t,w} \otimes e^*(T)$ is only monoidally cofibrant is that both $V_{t,w}$ and $e^*(T)$ are monoidally cofibrant. By the above, the first only happens for $r(T) = T_w^-$ and the second happens only if all $T_i = T_w^-$. As was noted in Case (4), this means $T = (\overset{w}{\to}\bullet\overset{-}{\to}\overset{w}{\to}\bullet\overset{-}{\to}\overset{w}{\to})$, which is excluded.

We have weak equivalences
$$\begin{aligned}
FQ(V_{t,w} \otimes e^*(T)) &\sim F(QV_{t,w} \otimes Qe^*(T)) \\
&\sim FQ(V_{t,w}) \otimes FQ(e^*(T)) \\
&\sim Q(\tilde{V}_{t,w}) \otimes Q(\tilde{e}^*(T)) \\
&\sim \tilde{V}_{t,w} \otimes \tilde{e}^*(T).
\end{aligned}$$

The first equivalence holds by Lemma 8.5, which gives a weak equivalence between cofibrant objects
$$Q(V_{t,w} \otimes e^*(T)) \sim Q(V_{t,w}) \otimes Q(e^*(T))$$
since both $V_{t,w}$ and $e^*(T)$ are cofibrant or monoidally cofibrant. The second equivalence holds by weak monoidality of F. The third equivalence follows from Brown's lemma and the equivalences $FQ(V_{t,w}) \sim \tilde{V}_{t,w}$ and $FQ(e^*(T)) \sim \tilde{e}^*(T)$. The last weak equivalence holds by Lemma 8.5, again using the (monoidal) cofibrancy of $\tilde{V}_{t,w}$ and $\tilde{e}^*(T)$. The same is also true for $W_{t,w}$ and/or $e(T)$ instead.

We now apply FQ to the diagram E in (8.15). On the other hand, we consider the diagram $\tilde{E}$ obtained by replacing $V_{t,w}$ by $\tilde{V}_{t,w}$ etc. There is a map of diagrams $FQ(E) \to \tilde{E}$. By the above, all individual maps in this morphisms of diagrams are weak equivalences, except (a priori) for
$$FQ(P) \to \tilde{P}.$$

However, since the left squares of $FQ(E)$ and $\tilde{E}$ are homotopy pushout squares, this remaining map is also a weak equivalence. Therefore, $FQ(E) \sim \tilde{E}$. In particular we get the requested weak equivalence in $\mathbf{Ar}(\mathcal{C})$
$$FQ(\epsilon(T)) \sim \tilde{\epsilon}(T).$$

This finishes the induction step (with respect to the tree T). We have shown that the individual summands in the maps $*$ and $**$ are weak equivalences.

The coproducts appearing in the left face of the cube above are homotopy coproducts, since for all $T \in (\mathbf{s})\mathbf{Tree}_{s,w}^{(k+1)}$ ($k \geq 0$), the terms $\Sigma_t \cdot_{\mathrm{Aut}\, T} x^*(T)$ and similarly for $x(T)$ are Σ_t-projectively cofibrant by Claim (A). This implies that the maps $*$ and $**$ themselves are weak equivalences and therefore finishes the induction step with respect to the cellular induction by O.

For a cellular filtration of O_∞ by operads O_i such that ϕ_{O_i} is a weak equivalence for all $i < \infty$, the same is true for $i = \infty$ using that U preserves filtered colimits and assumption (a). In case (b), we also use that the transition maps $(\mathrm{co})\mathrm{dom}(\phi_{O_i}) \to (\mathrm{co})\mathrm{dom}(\phi_{O_{i+1}})$ lie in (2.2), by (5.4).

(ii): For any cofibrant O-algebra A, we have the following chain of canonical isomorphisms and weak equivalences, which as above shows the requested Quillen equivalence:

$$(8.16) \qquad U(F^{\mathbf{Alg}}(A)) = \mathrm{Env}(F^{(\mathbf{s})\mathbf{Oper}}(O), F^{\mathbf{Alg}}(A))_0$$
$$= F^{(\mathbf{s})\mathbf{Oper}}(\mathrm{Env}(O, A))_0$$
$$\sim F(\mathrm{Q}(\mathrm{Env}(O, A))_0)$$
$$\sim F(\mathrm{Env}(O, A)_0)$$
$$= F(U(A))$$
$$\sim F\mathrm{Q}(U(A)).$$

The last (and similarly the first) canonical isomorphism is Proposition 4.4(i). The second isomorphism comes from a natural isomorphism of functors

$$\mathrm{Env}(F^{(\mathbf{s})\mathbf{Oper}}(-), F^{\mathbf{Alg}}(*)) = F^{(\mathbf{s})\mathbf{Oper}}(\mathrm{Env}(-, *))$$

since both expressions are the left adjoint to $(\mathbf{s})\mathbf{Oper}_W(\mathcal{D}) \to \mathbf{Pairs}(\mathbf{sColl}_W(\mathcal{C}))$, $P \mapsto (G(P), G(P)_0)$. The first weak equivalence was shown in Part (i), which is applicable since $\mathrm{Env}(O, A)$ is a cofibrant operad by Lemma 6.1(ii). The second weak equivalence is given by Lemma 6.1(i). The last weak equivalence follows from Proposition 6.2.

(iii): Let $O \in (\mathbf{s})\mathbf{Oper}_W(\mathcal{C})$ be a cofibrant replacement of $G(P)$. Equivalently, by Part (i), $P \sim F^{(\mathbf{s})\mathbf{Oper}}(O)$. By rectification of operads for $\mathcal{D}$, (ii), and rectification of operads for $\mathcal{C}$, we have the following chain of Quillen equivalences

$$\mathbf{Alg}_P^{\mathcal{D}} \sim \mathbf{Alg}_{F^{(\mathbf{s})\mathbf{Oper}}(O)}^{\mathcal{D}} \sim \mathbf{Alg}_O^{\mathcal{C}} \sim \mathbf{Alg}_{G(P)}^{\mathcal{C}}.$$

$\square$

Remark 8.17. The condition in Theorem 8.10 that $\mathcal{C}$ and $\mathcal{D}$ have the property that they are either left proper or their monoidal unit is cofibrant is only used to show that pushouts of certain cofibrations with cofibrant domain are homotopy pushouts. Since being a homotopy pushout only depends on the class of weak equivalences, this also holds, for example, if $\mathcal{C}$ has another model structure with more cofibrations, and the same weak equivalences.

If the left adjoint F is in addition symmetric monoidal, we can relax the condition on O in Theorem 8.10(ii).

Corollary 8.18. *In the situation of Theorem 8.10, suppose in addition that the left adjoint F is strong symmetric oplax monoidal (i.e., the symmetric oplax structural maps $F(C \otimes C') \to F(C) \otimes F(C')$ are isomorphisms, so that F is also symmetric lax monoidal). Let O be any (symmetric) operad in $\mathcal{C}$ such that $U(\eta_O)$ is a cofibration in $(\mathbf{s})\mathbf{Coll}_W(\mathcal{C})$.*

Then there is a Quillen adjunction

$$F : \mathbf{Alg}_O(\mathcal{C}) \rightleftarrows \mathbf{Alg}_{F(O)}(\mathcal{D}) : G$$

which is a Quillen equivalence if (F, G) is a Quillen equivalence.

Proof. Since F is symmetric monoidal, $U \circ F^{\mathbf{Alg}} = F \circ U$, see, e.g., [AM10, Proposition 3.91]. Therefore, only the last weak equivalence in (8.16) requires proof. By Proposition 6.2 O is strongly admissible, i.e., $U(A)$ is cofibrant in $\mathcal{C}$, so that $F(U(A)) \sim F(Q(U(A)))$ by Brown's lemma. $\square$

9. Applications

This last section contains a few applications to the homotopy theory of enriched categories, ordinary categories, operads, and (monoidal) diagrams. The strategy is similar for all these applications: enriched categories, say, are algebras over a certain nonsymmetric operad. Therefore, the admissibility and rectification results of §5–7 can be applied.

The list presented here is by no means exhaustive, other potential applications include monads in model categories, internal categories (and higher internal categories), (higher) spans, etc. Symmetric operads in symmetric spectra and some applications are studied in [PS14].

In §9, let $\mathcal{V}$ be a symmetric monoidal model category and $\mathcal{C}$ be a $\mathcal{V}$-enriched model category whose weak equivalences are stable under filtered colimits. Moreover, assume that $\mathcal{C}$ is quasi-tractable and either combinatorial or $\mathcal{V}$-admissibly generated.

9.1. **Rectification of A_∞- and E_∞-monoids.** In this section we discuss rectification of homotopy coherent versions of monoids and commutative monoids. We start by giving explicit constructions of two important operads, A_∞ and E_∞.

The Barratt-Eccles operad E_∞ can be constructed by taking the associative symmetric operad in sets, applying the functor E to it (E sends a set to a groupoid with the same set of objects and a single morphism between any pair of objects), obtaining a symmetric operad in groupoids, and then applying the nerve functor, which gives a simplicial operad. See the paragraph after Corollary 3.5 in Elmendorf and Mandell [EM06].

An identical construction (apply E and then take the nerve) produces a model for the operad A_∞, but the original operad in sets is now the free operad on a single binary operation and a single nullary operation, so that $\mathcal{O}_n$ consists of planar rooted trees with n leaves, see, for example, [BM03, §5.8]. Alternatively, one can take the free operad generated by a single operation in each arity (which corresponds to the so-called *unbiased* monoids).

In what follows, we actually don't need to apply the nerve functor, because an operad in groupoids is sufficient for our purposes. We also note that any category enriched in simplicial sets is automatically enriched in groupoids by applying the nerve functor. The following propositions are mere specializations of the general theorems on admissibility and rectifiability. We give explicit statements here due to the importance of these examples.

Proposition 9.1.1. *If $\mathcal{C}$ is a symmetric h-monoidal and groupoid-enriched then the category of E_∞-algebras in $\mathcal{C}$ admits a transferred model structure. Furthermore, if $\mathcal{C}$ is symmetric flat with respect to the morphism $E_\infty \to \mathrm{Comm}$ (or simply symmetric flat), then the Quillen adjunction between commutative monoids and E_∞-monoids is a Quillen equivalence.*

A similar statement for A_∞ and As holds if $\mathcal{C}$ is merely h-monoidal and flat.

9.2. **Model structures on enriched categories.** For a small set W, Berger and Moerdijk [BM07, 1.5.4] have introduced a nonsymmetric $W \times W$-colored operad in $\mathcal{V}$ given by

$$\mathrm{Cat}_W^{\mathrm{As}}(((v_1, v_1'), \ldots, (v_n, v_n')), (v_0', v_{n+1}')) = \begin{cases} 1_\mathcal{V}, & v_i' = v_{i+1} \text{ for all } 0 \leq i \leq n; \\ \emptyset, & \text{otherwise.} \end{cases}$$

This defines a nonsymmetric operad in $\mathcal{V}$. Its algebras in $\mathcal{C}^{W \times W}$ are precisely $\mathcal{C}$-*enriched categories with W as objects*. More generally, given a nonsymmetric operad O in $\mathcal{V}$, one can also consider the nonsymmetric operad Cat_W^O, which is given by replacing $1_\mathcal{V} = \mathrm{As}_n$ in the previous formula by O_n. Algebras over this operad can be called $\mathcal{V}$-*enriched O-twisted categories*. Typically, O is taken to be A_∞. In this case we speak of $\mathcal{V}$-*enriched A_∞-categories*, i.e., composition is not strictly associative, but rather associative up to coherent higher homotopies.

The following lemma is an immediate application of the results on admissibility and rectification. Up to a minor expository difference (see Remark 5.12), the admissibility statement is the same as Muro's [Mur11, Corollaries 10.4, 10.5]. The rectification result in loc. cit. uses in addition the left properness of $\mathcal{C}$.

Corollary 9.2.1. *If $\mathcal{C}$ is h-monoidal, then all (nonsymmetric) operads in $\mathcal{V}$ are admissible. In particular, the operad Cat_W^O is admissible, so O-twisted $\mathcal{C}$-enriched categories with W as the set of objects and functors that induce identity on objects carry a model structure whose weak equivalences and fibrations are those $\mathcal{C}$-enriched functors $F \colon \mathcal{D} \to \mathcal{E}$ that induce weak equivalences, respectively fibrations in $\mathcal{C}$:*

$$\mathrm{Hom}_\mathcal{D}(D, D') \to \mathrm{Hom}_\mathrm{E}(C, C'),$$

for all objects $D = F(D)$ and $D' = F(D')$ in $\mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\mathcal{E}) = W$.

If $\mathcal{C}$ is in addition flat over the levels φ_n ($n \geq 0$) of some weak equivalence $\varphi \colon O \to P$ of nonsymmetric operads in $\mathcal{V}$, there is a Quillen equivalence of O- and P-twisted $\mathcal{C}$-enriched categories (both with W as objects):

$$\varphi_* \colon \mathbf{Cat}_W^O(\mathcal{C}) \leftrightarrows \mathbf{Cat}_W^P(\mathcal{C}) \colon \varphi^*.$$

For example, if $1_\mathcal{V}$ is cofibrant, then this condition is satisfied for any weak equivalence $A_\infty \to \mathrm{As}$, where A_∞ is a cofibrant replacement of As. It is satisfied for any weak equivalence if $\mathcal{C}$ is flat (Definition 2.1).

Proof. Admissibility follows from Theorem 5.10 and Remark 5.12 and rectification follows from Theorem 7.5. If $1_\mathcal{V}$ is cofibrant, then $\mathcal{C}$ is flat over the levels of $A_\infty \to \mathrm{As}$: $\mathrm{As}_n = 1_\mathcal{V}$ is cofibrant. Moreover, A_∞ is a cofibrant operad, so that its levels are cofibrant by Lemma 6.1. Any monoidal model category is flat over a weak equivalence between cofibrant objects by Brown's lemma. $\qquad\square$

These individual model structures on $\mathbf{Cat}_W(\mathcal{C})$ can be assembled into a single model structure on $\mathbf{Cat}(\mathcal{C})$. The following result is due to Muro [Mur12, Theorem 1.1]. Muro's work relaxes the assumptions of similar results of Stanculescu [Sta09] as well as Berger and Moerdijk [BM13, Theorem 1.9], which in turn generalizes results of Amrani ($\mathcal{V} = \mathbf{Top}$) [Ili15], Bergner (for $\mathcal{V} = \mathbf{sSet}$) [Ber07, Theorem 1.1], Lurie (every object of $\mathcal{V}$ is cofibrant) [Lur09, Proposition A.3.2.4], and Tabuada ($\mathcal{V} = \mathbf{Ch}(\mathbf{Mod}_R)$ for some ring R and $\mathcal{V}$ being symmetric spectra) [Tab05, Théorème 3.1], [Tab07], [Tab09, Theorem 5.10].

Given some property of objects or morphisms in $\mathcal{C}$ we say that a $\mathcal{C}$-enriched category or a $\mathcal{C}$-enriched functor has this property locally if it is true for the enriched objects of morphisms between each pair of objects. Given a $\mathcal{C}$-enriched category, its derived π_0 is an ordinary 1-category that is constructed by applying the derived internal hom from the monoidal unit of $\mathcal{C}$ to each object of morphisms.

Proposition 9.2.2. *(Muro) Suppose again that $\mathcal{C}$ is h-monoidal. Then $\mathbf{Cat}(\mathcal{C})$ is carries the Dwyer-Kan model structure whose weak equivalences are the Dwyer-Kan equivalences (i.e., local weak equivalences and their derived π_0 is an essentially surjective functor or, equivalently, an equivalence of categories) and whose acyclic fibrations are local acyclic fibrations that are surjective on objects.*

Proposition 9.2.3. *Fix $\mathcal{V}$ and $\mathcal{C}$ as in Corollary 9.2.1 and a weak equivalence $\varphi : O \to P$ of nonsymmetric oper-ads in $\mathcal{V}$. Assume that the Dwyer-Kan model structure on $\mathbf{Cat}^O(\mathcal{C})$ and $\mathbf{Cat}^P(\mathcal{C})$ exists, as in Proposition 9.2.2. If $\mathcal{C}$ is flat over a weak equivalence $\varphi : O \to P$ (more precisely, flat over the levels φ_n for all $n \geq 0$), then we have a Quillen equivalence*

$$\varphi_* : \mathbf{Cat}^O(\mathcal{C}) \leftrightarrows \mathbf{Cat}^P(\mathcal{C}) : \varphi^*.$$

For example, this holds for all weak equivalences φ if $\mathcal{C}$ is flat. It also holds for the weak equivalence $\varphi \colon \mathrm{A}_\infty \to \mathrm{As}$ if the monoidal unit $1_\mathcal{V}$ is cofibrant.

Remark 9.2.4. Under the above assumptions, we expect that the Dwyer-Kan model structure on $\mathbf{Cat}^O(\mathcal{C})$ exists for any operad O. The reader is encouraged to generalize Muro's result 9.2.2 to arbitrary operads.

Proof. For some cofibrant object $X \in \mathbf{Cat}^O(\mathcal{C})$ and a fibrant object $Y \in \mathbf{Cat}^P(\mathcal{C})$, the (co)unit morphism of the adjunction for X and Y can be computed in the corresponding slices $\mathbf{Cat}^O_{\mathrm{Obj}(X)}(\mathcal{C})$ and $\mathbf{Cat}^P_{\mathrm{Obj}(Y)}(\mathcal{C})$. Moreover, the (co)fibrancy of X and Y is equivalent to the one in the corresponding slice category. Now the Quillen equivalence immediately follows from the rectification of category structures with a fixed set of objects (Corollary 9.2.1). $\qquad\square$

An interesting question that arises in relation to these results is whether it is possible to define a monoidal structure on the category of enriched categories in such a way that the resulting model category is monoidal. The naive choice (take the product of sets of objects and the tensor product of enriched morphisms) already fails to satisfy the pushout product axiom in the case when $\mathcal{C}$ is the model category of small categories, as shown by Lack. The *Gray tensor product* does turn enriched categories in small categories (i.e., strict 2-categories) into a monoidal model category, however, it is unclear how one should generalize it to enriched categories. If such a monoidal product could be constructed, then one could iterate the construction of enriched categories and consider higher enriched categories (i.e., enriched categories in enriched categories etc.). Such a construction could explain how the traditional definitions of bicategories, tricategories, and tetracategories could be generalized in a systematic way to higher dimensions. Furthermore, for certain choices of the operad O (e.g., the categorical A_∞-operad) one would expect to get a model category that is Quillen equivalent to any of the usual model categories of (∞, n)-categories. (We cannot expect this for $O = \mathrm{As}$ because it is well-known that tricategories cannot in general be strictified to strict 3-categories.)

9.3. **Applications to category theory.** In this section we apply the results of §9.2 to some concrete examples of (low-dimensional) category theory.

Consider the category of sets equipped with the model structure whose weak equivalences are bijections and fibrations and cofibrations are arbitrary maps. Equip this model category with the monoidal structure given by the cartesian product. This model structure is tractable, proper, its weak equivalences are stable under filtered colimits (it is pretty small in the sense of [PS15, Definition 2.0.2] for the maps $\emptyset \to \{0\}$, $\{0, 1\} \to \{0\}$ generate the cofibrations, then use [PS15, Lemma 2.0.3]), symmetric h-monoidal and symmetroidal, and symmetric flat. By Proposition 9.2.2, the category $\mathbf{Cat}$ of categories admits a model structure whose weak equivalences are equivalences of categories and fibrations are the so-called isofibrations, i.e., functors $F \colon \mathcal{C} \to \mathcal{D}$ such that any isomorphism in $\mathcal{D}$, $F(C) \cong D$ (for $C \in \mathcal{C}$, $D \in \mathcal{D}$) has a lift to an isomorphism in $\mathcal{C}$. This is precisely the *canonical (folk) model structure on categories*, see, for example, Rezk [Rez]. The canonical model structure is tractable, pretty small, cartesian (i.e., monoidal with respect to the categorical product), simplicial, and all objects are fibrant and cofibrant, see Rezk [Rez] for details. Furthermore, it is symmetric h-monoidal and symmetroidal because cofibrations are precisely those functors which are injective on objects, and the latter property survives pushout products and coinvariants under Σ_n, the argument being similar to the one for simplicial sets, see [PS15, §7.1]. Finally, the canonical model structure is flat, which follows immediately from the definition of equivalences of categories, which are stable under products. However, symmetric flatness fails: the Σ_n-equivariant functor from the groupoid $\mathrm{E}\Sigma_n$ (objects are Σ_n and morphisms are $\Sigma_n \times \Sigma_n$) to the terminal groupoid is a weak equivalence, yet its Σ_n-coinvariants is the map $\mathrm{B}\Sigma_n \to 1$ ($\mathrm{B}\Sigma_n$ has one object whose endomorphisms are Σ_n), which is not an equivalence.

The results of §§5–§7 yield model structures on various types of monoidal categories and a strong form of *Mac Lane's coherence theorem.*

Proposition 9.3.1. *There is a model structure on strict monoidal categories, monoidal categories, strict sym-metric monoidal categories, and symmetric monoidal categories whose weak equivalences and fibrations are the ones of the underlying categories.*

Every monoidal category is equivalent (via a strong monoidal functor) to a strict monoidal category. This strict monoidal category is unique up to strict monoidal equivalence. Similarly, every monoidal functor is equivalent (via a strong monoidal natural transformation) to a strict monoidal functor which is again unique up to a strict monoidal natural transformation.

Proof. The above-mentioned categories are algebras (in **Cat**) over the associative operad As, the operad A_∞, the commutative operad Comm, and the operad E_∞, respectively. Hence the existence of the model structure follows from Theorem 5.10, whose assumptions have been verified above.

Furthermore, the nonsymmetric rectification theorem (Theorem 7.5) tells us that the canonical morphism from A_∞ to the associative operad induces a Quillen equivalence between As-algebras and A_∞-algebras. $\square$

Example 9.3.2. The morphism from E_∞ to the commutative operad is not symmetric flat, as explained above, which tells us that symmetric monoidal categories cannot always be strictified to strict symmetric monoidal categories. This is well-known because symmetric monoidal categories can have a nontrivial k-invariant whereas strict symmetric monoidal categories always have a trivial k-invariant.

Similarly, *Mac Lane's coherence theorem for bicategories* follows from the above, since strict 2-categories are $\mathbf{Cat}^{As}$-algebras and bicategories are $\mathbf{Cat}^{A\infty}$-algebras in **Cat**, respectively:

Proposition 9.3.3. *There is a Quillen equivalence between the model categories of strict 2-categories and bicategories.*

We conjecture that other strictification results of category theory, such as strictification of tricategories to Gray categories (Gordon, Power, and Street), partial strictification of symmetric monoidal bicategories, etc., can also be shown using the methods of this paper. However, considerations of volume prevent us from developing this topic further. Simpson's conjecture might also be amenable to the techniques explained above.

9.4. **The colored operad of colored operads.** Given a set W, there is a (symmetric) colored operad Oper_W whose category of algebras is equivalent to the category of (symmetric) W-colored operads in $\mathcal{C}$. It is due to Berger and Moerdijk [BM07, §1.5.6, §1.5.7]. See also [GV12, §3] for a detailed description of the multicolored case.

This operad is first constructed for $\mathcal{C} = \mathbf{Sets}$ as follows: the set of colors of (s)Oper_W is the set of objects of $(\mathbf{s})\mathbf{Seq}_{W,W}$, which we call *valencies*. Recall from §3 that the objects of $(\mathbf{s})\mathbf{Seq}_{W,W}$ are pairs $c = (s, w)$ where $s \colon I \to W$ is a map from a finite set I and $w \in W$. The operations

$$(\mathrm{s})\mathrm{Oper}_W(a_1, \ldots, a_k; b)$$

from a given sequence of valencies $(a_1, \ldots, a_k)$ to a valency b are given by isomorphism classes of triples (T, σ, τ) consisting of a W-colored (symmetric) tree T equipped with a bijection σ from $\{1, \ldots, k\}$ to the set of internal vertices of T such that the valency of $\sigma(i)$ equals a_i and a color-preserving bijection τ from $\{1, \ldots, m\}$, where m is the arity of b, to the input edges of T. Isomorphisms of such triples are isomorphisms of colored trees which are compatible with σ and τ. In the symmetric case the symmetric group Σ_k acts on such classes by precomposition with σ. The operadic unit sends each valency c to the corresponding corolla, interpreted as an operation from c to c. The operadic composition is given by grafting of trees, see [BM07, §1.5.6] in the uncolored case. One checks that this gives a (symmetric) operad, denoted (s)Oper_W, in **Sets**.

The functor $\mathbf{Sets} \to \mathcal{C}$, $X \mapsto \coprod_{x \in X} 1_\mathcal{C}$ is symmetric monoidal and therefore extends to a functor

$$(\mathbf{s})\mathbf{Oper}_{(\mathbf{s})\mathbf{Seq}_W}(\mathbf{Sets}) \to (\mathbf{s})\mathbf{Oper}_{(\mathbf{s})\mathbf{Seq}_W}(\mathcal{C}).$$

The image of (s)Oper_W under this functor is again denoted by (s)Oper_W.

The following admissibility statement unifies a few earlier results: the semi-model structure for symmetric operads established by Spitzweck [Spi01, Theorem 3.2], the model structure for nonsymmetric operads by Muro [Mur11, Theorem 1.1] and, the model structure on uncolored operads in orthogonal spectra with the positive stable model structure by Kro [Kro07, Theorem 1.1].

Corollary 9.4.1. *Let $\mathcal{C}$ be (symmetric) h-monoidal. Then the operad* (s)Oper_W *of (symmetric) W-colored operads is admissible, that is to say, the category* $(\mathbf{s})\mathbf{Oper}_W(\mathcal{C})$ *of (symmetric) W-colored operads in $\mathcal{C}$ has a model structure that is transferred along the adjunction*

$$\mathrm{Free} \colon \mathcal{C}^{(\mathbf{s})\mathbf{Seq}_{W,W}} \rightleftarrows \mathbf{Alg}_{\mathrm{Oper}_W}(\mathcal{C}) = (\mathbf{s})\mathbf{Oper}_W(\mathcal{C}) \colon U.$$

If $1_\mathcal{C}$ is cofibrant, then (s)Oper_W *is strongly admissible, i.e., the forgetful functor U preserves cofibrations with cofibrant domain.*

Proof. The admissibility follows from Theorem 5.10. The strong admissibility follows from Proposition 6.2 since (s)Oper is levelwise projectively cofibrant. $\square$

Operads can be generalized in the same way that enriched categories are generalized to enriched A_∞-categories. Fix a (symmetric) operad O. In practice, O is an A_∞-operad, i.e., we have a weak equivalence of operads $O \to$ As, where As denotes the associative operad. We define the colored (symmetric) operad Oper_W^O of O-twisted W-colored (symmetric) operads by the same construction as above, starting from a colored operads P in sets, except that we pass to a $\mathcal{C}$-valued operad in a modified fashion: instead of tensoring operations in degree k with $1_\mathcal{C}$ we tensor them with O_k. The intuitive idea behind this is that the composition of operadic operations is no longer strictly associative, but is rather governed by the operad O. An O-twisted W-colored operad is an O-algebra in

the monoidal category of W-colored (symmetric) sequences equipped with the substitution product, the latter being a left $\mathcal{C}$-module in the obvious way. Then Corollary 9.4.1 has an immediate generalization for the operad (s)Oper^O. For the strong admissibility, the requirement on $1_{\mathcal{C}}$ is replaced by the condition that the levels O_k be cofibrant as objects in $\mathcal{C}$. Moreover, Theorem 7.5 admits the following corollary.

Corollary 9.4.2. *If $\mathcal{C}$ is flat over a weak equivalence $O \to P$ of operads, then we have a Quillen equivalence* $\mathbf{sOper}_W^O(\mathcal{C}) \leftrightarrows \mathbf{sOper}_W^P(\mathcal{C})$ *of O-twisted and P-twisted (symmetric) W-colored operads in $\mathcal{C}$. For example, if $1_{\mathcal{C}}$ is cofibrant, then A_∞-twisted colored symmetric operads can be rectified to ordinary colored symmetric operads.*

Proof. This follows from Theorem 7.5 once we show the symmetric flatness of $\mathcal{C}$ with respect to $\mathbf{sOper}_W^O \to \mathbf{sOper}_W^P$. Every component of $\mathbf{sOper}_W^O$ is a coproduct of the corresponding components of O, and the relevant symmetric group acts freely on the components. Thus the symmetric flatness follows from the flatness of $\mathcal{C}$ over $O \to P$. $\qquad\square$

Remark 9.4.3. In fact, if $\mathcal{C}$ is a $\mathcal{V}$-enriched model category that is symmetric h-monoidal with respect to $\mathcal{V}$ only (and not necessarily with respect to itself), then the colored operad of colored operads can be defined with values in $\mathcal{V}$ and its algebras in $\mathcal{C}$ will still be W-colored operads in $\mathcal{C}$, so the above corollary holds in this more general setting. Gutiérrez and Vogt used such a setup (with a different set of conditions on $\mathcal{V}$) to construct a model structure on W-colored operads in symmetric spectra, see Corollary 4.1 in [GV12].

Starting from this point, further work is required to assemble the model structures on $\mathbf{sOper}_W(\mathcal{C})$ into one on the category $(\mathbf{s})\mathbf{Oper}(\mathcal{C})$ of (symmetric) operads with an arbitrary set of colors. This has been done for $\mathcal{C} = \mathbf{sSet}$ by Cisinski and Moerdijk [CM13, Theorem 1.14] and independently by Robertson [Rob11, Theorem 6] and was extended by Caviglia [Cav14] to more general model categories using similar arguments. We expect that the assumptions can be further relaxed to the ones stated in the above corollary.

9.5. **Diagrams.** In this section we construct a model structure on the category of enriched diagrams of some fixed shape and prove a rectification result. In particular, we recover the classical result of Vogt and its generalization by Cordier and Porter on homotopy coherent diagrams.

Proposition 9.5.1. *Assume that $\mathcal{C}$ is, in addition to the standing assumptions in this section, h-monoidal. For any $\mathcal{V}$-enriched, small category $\mathcal{D}$, the category of $\mathcal{V}$-enriched functors $\mathcal{D} \to \mathcal{C}$ admits a transferred model structure. Its weak equivalences and fibrations are those natural transformations of $\mathcal{V}$-enriched functors $F \to G$ such that for all objects $X \in \mathcal{D}$,*

$$F(X) \to G(X)$$

is a weak equivalence, respectively a fibration. Furthermore, if $\mathcal{V}$ has a model structure and $\mathcal{C}$ is flat over $\mathcal{V}$, then a componentwise weak equivalence of diagrams $\mathcal{D} \to \mathcal{D}'$ whose object map is the identity induces a Quillen equivalence of the two model categories of diagrams.

Remark 9.5.2. A more general version of the rectification result allows for a Dwyer-Kan equivalence $\mathcal{D} \to \mathcal{D}'$.

Proof. Following Berger and Moerdijk [BM07, §1.5.5], we consider the nonsymmetric colored operad $\mathrm{Diag}_{\mathcal{D}}$ that encodes diagrams in $\mathcal{C}$ indexed by a fixed $\mathcal{V}$-enriched category $\mathcal{D}$, i.e., $\mathcal{V}$-enriched functors $\mathcal{D} \to \mathcal{C}$. The operad $\mathrm{Diag}_{\mathcal{D}}$ is colored by the set of objects of $\mathcal{D}$. Its operations are defined as

$$\mathrm{Diag}_{\mathcal{D}}(X_1, \ldots, X_n, Y) = \begin{cases} \emptyset, & n \neq 1; \\ \mathrm{Map}_{\mathcal{D}}(X, Y), & n = 1. \end{cases}$$

Here $\mathrm{Map}_{\mathcal{D}}$ denotes the enriched hom object. The operadic composition and unit are induced by the composition and unit of $\mathcal{D}$. (The construction just described embeds enriched categories into nonsymmetric colored operads.)

A $\mathrm{Diag}_{\mathcal{D}}$-algebra in $\mathcal{C}$ consists of a collection of objects D_X in $\mathcal{C}$, for all $X \in \mathcal{D}$ together with morphisms $\mathrm{Mor}(X, Y) \otimes D_X \to D_Y$ that satisfy the obvious associativity and unitality conditions. This is precisely the data of a $\mathcal{V}$-enriched functor $\mathcal{D} \to \mathcal{C}$.

Theorem 5.10 now implies that the category of $\mathcal{D}$-diagrams admits a transferred model structure. At this point we remark that Theorem 6.6 likewise implies that cofibrations with cofibrant source are preserved by the forgetful functor if taking the pushout product with $1_C \to \mathrm{Map}_{\mathcal{D}}(X, X)$ and $\emptyset \to \mathrm{Map}_{\mathcal{D}}(X, Y)$ preserves (acyclic) cofibrations, which is true, for example, if individual hom objects are cofibrant and the unit maps are cofibrations.

Theorem 7.5 implies the desired rectification statement if $\mathcal{C}$ is flat. $\qquad\square$

9.6. **Monoidal diagrams.** Extending the results of the previous section, there is also a (symmetric) colored operad that encodes lax (symmetric) monoidal diagrams, i.e., lax (symmetric) monoidal $\mathcal{V}$-enriched functors $\mathcal{D} \to \mathcal{C}$, where $\mathcal{C}$ is now an algebra over the monoidal category $\mathcal{V}$ and $\mathcal{D}$ is a monoidal $\mathcal{V}$-enriched category. We therefore obtain a model structure on lax (symmetric) monoidal functors:

Proposition 9.6.1. *Assume that $\mathcal{C}$ is (symmetric) h-monoidal. For any $\mathcal{V}$-enriched symmetric monoidal small category $\mathcal{D}$, the category of lax (symmetric) monoidal $\mathcal{V}$-enriched functors $\mathcal{D} \to \mathcal{C}$ admits a transferred model structure. Furthermore, if $\mathcal{C}$ is (symmetric) flat over $\mathcal{V}$, then a weak equivalence $\mathcal{D} \to \mathcal{D}'$ induces a Quillen equivalence of the induced model categories.*

Proof. We consider the (symmetric) operad whose operations from a multisource $(s_1, \ldots, s_k)$ to a target t are given by the enriched morphism object from $s_1 \otimes \cdots \otimes s_k$ to t. The operadic composition and unit are induced by the monoidal category structure of $\mathcal{D}$.

An algebra in $\mathcal{C}$ over this operad consists of a collection of objects D_X in $\mathcal{C}$, for any $X \in \mathcal{D}$, together with morphisms $\mathrm{Mor}(X_1 \otimes \cdots \otimes X_k, Y) \otimes D_{X_1} \otimes \cdots \otimes D_{X_k} \to D_Y$ that satisfy the corresponding associativity and unitality conditions. This is precisely the data of a (symmetric) lax monoidal $\mathcal{V}$-enriched functor $\mathcal{D} \to \mathcal{C}$.

As before, Theorem 5.10 and Theorem 7.5 now imply the admissibility and rectification criteria as stated. $\square$

One could also ask for a model structure on lax functors whose fibrant objects are "weakly strong" monoidal functors, meaning that the canonical maps $A(X) \otimes A(Y) \to A(X \sqcup Y)$ and $1 \to A(\emptyset)$ are weak equivalences. This would be useful for factorization algebras, for example (see the next section). Such a model structure could be obtained by a left Bousfield localization with respect to the local objects defined above, however, it is not clear why such a left Bousfield localization should exist in this case.

9.7. **Prefactorization algebras.** As an application of the previous section we construct a model structure on prefactorization algebras. See §7.3 in Costello and Gwilliam's book [CG] for the relevant background. A *prefactorization algebra* on a $\mathcal{V}$-enriched monoidal site $(S, \sqcup, \emptyset)$ (it's useful to think of the monoidal structure as the disjoint union) is a symmetric lax monoidal $\mathcal{V}$-enriched functor from S to $\mathcal{C}$, where $\mathcal{C}$ is $\mathcal{V}$-enriched. A typical example of S is the category of smooth manifolds and their embeddings equipped with the Weiss topology, where morphism objects are either discrete or have the natural space structure. The previous section now immediately implies the following statement.

Proposition 9.7.1. *If $\mathcal{C}$ is symmetric h-monoidal, $\mathcal{V}$-enriched, and S is a $\mathcal{V}$-enriched site, then the category of prefactorization algebras over S with values in $\mathcal{C}$ admits a transferred model structure. Furthermore, if $\mathcal{C}$ is symmetric flat, then a functor of sites $S \to S'$ that induces the identity morphism on objects and is a componentwise weak equivalence on morphism gives a Quillen equivalence of the corresponding model categories.*

This raises the question whether the above model structure can be upgraded to factorization algebras. Fibrant objects in the resulting structure would be "weakly lax" functors defined in the previous section that satisfy the codescent condition with respect to the Grothendieck topology on S. As usual, one could try to enforce the codescent property using the obvious left Bousfield localization. However, the model category of prefactorization algebras constructed above is not left proper, so a special argument is needed to ensure that cobase changes of local acyclic cofibrations are local weak equivalences.

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Homotopy theory of symmetric powers

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Abstract. We introduce the symmetricity notions of symmetric h-monoidality, symmetroidality, and symmetric flatness. As shown in our paper arXiv:1410.5675, these properties lie at the heart of the homotopy theory of colored symmetric operads and their algebras. In particular, they allow one to equip categories of algebras over operads with model structures and to show that weak equivalences of operads induce Quillen equivalences of categories of algebras. We discuss these properties for elementary model categories such as simplicial sets, simplicial presheaves, and chain complexes. Moreover, we provide powerful tools to promote these properties from such basic model categories to more involved ones, such as the stable model structure on symmetric spectra.

Contents

1. Introduction

Model categories provide an important framework for homotopy-theoretic computations. Algebraic structures such as monoids, their modules, and more generally operads and their algebras provide means to concisely encode multiplication maps and their properties such as unitality, associativity, and commutativity. Homotopy coherent versions of such algebraic structures form the foundation of a variety of mathematical areas, such as algebraic topology, homological algebra, derived algebraic geometry, higher category theory, and derived differential geometry. This motivates the following question: what conditions on a monoidal model category $(\mathcal{C}, \otimes)$ are needed for a meaningful homotopy theory of monoids, modules, etc.? The first answer to this type of question was given by Schwede and Shipley's *monoid axiom*, which guarantees that for a monoid R in $\mathcal{C}$, the category $\mathbf{Mod}_R(\mathcal{C})$ of R-modules carries a model structure transferred from $\mathcal{C}$, see [SS00]. The monoid axiom asks that transfinite compositions of pushouts of maps of the form

$$Y \otimes s,$$

where s is an acyclic cofibration and Y is any object are again weak equivalences. Moreover, given two weakly equivalent monoids $R \xrightarrow{\sim} S$, the categories $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$ are Quillen equivalent if

$$Y \otimes X \to Y' \otimes X$$

is a weak equivalence for any weak equivalence $Y \to Y'$ and any cofibrant object X.

This paper is devoted to a thorough study of the homotopy-theoretic behavior of more general algebraic expressions in a model category, such as

$$(1.0.1) \qquad X_{\Sigma_n}^{\otimes n}, \qquad Y \otimes_{\Sigma_n} X^{\otimes n}, \qquad Z \otimes_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_e}} (X_1^{\otimes n_1} \otimes \cdots \otimes X_e^{\otimes n_e}),$$

where $X, Y, Z \in \mathcal{C}$, Y has an action of Σ_n, Z has an action of $\prod \Sigma_{n_i}$, and the subscripts denote coinvariants by the corresponding group actions. More specifically, we introduce *symmetricity properties* for a symmetric monoidal model category $\mathcal{C}$: *symmetric h-monoidality*, *symmetroidality*, and *symmetric flatness*.

Symmetric h-monoidality requires, in particular, that for any object Y as above and any acyclic cofibration s in $\mathcal{C}$, the map

$$(1.0.2) \qquad Y \otimes_{\Sigma_n} s^{\square n}$$

is a couniversal weak equivalence, i.e., a map whose cobase changes are weak equivalences. Here $s^{\square n}$ is the n-fold pushout product of s, which is a monoidal product on morphisms. Symmetric h-monoidality is a natural enhancement of h-monoidality introduced by Batanin and Berger in [BB13].

Symmetric flatness requires that for any Σ_n-equivariant map y whose underlying map in $\mathcal{C}$ is a weak equivalence and any cofibration $s \in \mathcal{C}$, the map

$$(1.0.3) \qquad y \,\square_{\Sigma_n} s^{\square n}$$

is a weak equivalence. This implies that $y \otimes_{\Sigma_n} X^{\otimes n}$ is a weak equivalence for any cofibrant object X. Among other things this means that the Σ_n-quotients in (1.0.1) are also homotopy quotients. See 4.2.7, 4.2.2 for the precise definitions.

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Expressions as in (1.0.1) are of paramount importance for handling monoids and, more generally, algebras over colored symmetric operads. Indeed, a free commutative monoid, more generally, a free algebra over a (colored) symmetric operad, involves such terms. In [PS14a], we show that symmetric h-monoidality ensures the existence of a transferred model structure on algebras over *any* symmetric colored operad, while symmetric flatness yields a Quillen equivalence of algebras over weakly equivalent operads. We also introduce *symmetroidality* in this paper, which can be used to govern the behavior of cofibrant algebras over operads.

Up to transfinite compositions present in the monoid axiom, which we treat separately, symmetric h-monoidality and symmetric flatness can be regarded as natural enhancements of the above conditions of Schwede and Shipley. However, it turns out to be hard to establish the symmetric h-monoidality, symmetroidality, and symmetric flatness for a given model category $\mathcal{C}$ directly. Therefore, in this paper, we also provide a powerful and convenient set of tools that enable us to quickly promote these properties through various constructions on model categories.

Theorem 1.0.4. *(See Theorem 4.3.8 for the precise statement.) To check that $\mathcal{C}$ is symmetric h-monoidal or symmetric flat it is enough to consider (1.0.2) and (1.0.3) for* generating *cofibrations s.*

Theorem 1.0.5. *(See Theorem 5.2.6 for the precise statement.) Given an adjunction of symmetric monoidal model categories,*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

which is sufficiently compatible with the monoidal products, such as $\mathcal{D} = \mathbf{Mod}_R(\mathcal{C})$, where R is a commutative monoid in $\mathcal{C}$, the symmetric h-monoidality and symmetric flatness of $\mathcal{C}$ imply the one of $\mathcal{D}$.

Theorem 1.0.6. *(See Theorem 6.2.2 for the precise statement.) Given a monoidal left Bousfield localization*

$$\mathcal{C} \rightleftarrows \mathcal{D} = \mathrm{L}_S^\otimes(\mathcal{C}),$$

the symmetric h-monoidality and symmetric flatness of $\mathcal{C}$ imply the one of $\mathcal{D}$.

As an illustration of these principles, consider the problem of establishing the symmetric h-monoidality, symmetroidality, and symmetric flatness for the monoidal model category of simplicial symmetric spectra. This allows one to establish the homotopy theory of operads and their algebras in spectra, such as commutative ring spectra or E_∞-ring spectra. First, by direct inspection (Subsection 7.1) one establishes these properties for the generating (acyclic) cofibrations of simplicial sets, i.e., $\partial\Delta^n \to \Delta^n$ and $\Lambda_k^n \to \Delta^n$. By Theorem 4.3.8, this shows that **sSet** is symmetric h-monoidal, symmetroidal, and flat. Next, again by direct inspection, one can show that positive cofibrations of symmetric sequences (i.e., cofibrations that are isomorphisms in degree 0) form a symmetric h-monoidal, symmetric flat class. Via Theorem 5.2.6 these properties can be transferred to modules over a (fixed) commutative monoid in symmetric sequences (specifically, the sphere spectrum), equipped with the positive unstable (i.e., transferred) model structure. Finally, by applying Theorem 6.2.2, one establishes them for the left Bousfield localization of the positive unstable model structure with respect to the stabilizing maps, which gives the positive stable model structure on simplicial symmetric spectra. These steps are carried out in detail for spectra in an abstract model category in [PS14b].

After recalling some basic notions pertaining to model categories in Section 2, we embark on a systematic study of the arrow category $\mathbf{Ar}(\mathcal{C})$ of a monoidal model category $\mathcal{C}$. Equipped with the pushout product of morphisms, we show that $\mathbf{Ar}(\mathcal{C})$ is again a monoidal model category (Subsection 3.1). We then recall the notion of h-monoidality due to Batanin and Berger [BB13], and the concept of flatness, which is well-known and has been independently studied by Hovey, for example, see [Hov14]. In Section 4, we define the above-mentioned symmetricity concepts. This extends the work of Lurie [Lur] and Gorchinskiy and Guletskiĭ [GG09]. An important technical key is Theorem 4.3.8, which shows the stability of these properties under weak saturation. This extends a similar statement of Gorchinskiy and Guletskiĭ [GG09, Theorem 5] about stability under weak saturation of a special case of symmetroidality (which we also prove in 4.3.8). Simplified expository accounts of this result were later given by White [Whi14a, Appendix A] and Pereira [Per14, §4.2]. Our proof uses similar ideas, but is shorter. The stability of the symmetricity and various other model-theoretic properties under transfers and left Bousfield localizations is shown in §5 and §6. Given that these two methods are the most commonly used tools to construct model structures, the main results of these sections (5.2.1, 5.2.6, 6.2.1, 6.2.2) should be useful to establish the symmetricity for many other model categories not considered in this paper. For example, the combination of h-monoidality and flatness allows to carry through the monoid axiom to a left Bousfield localization. This is illustrated in Section 7, where we discuss the symmetricity properties of model categories such as simplicial sets, simplicial modules, and simplicial (pre)sheaves, as well as topological spaces and chain complexes.

2. Model categories

In this section we recall parts of the language of model categories [Hov99], [Hir03], [MP12, Part 4] that is used throughout this paper. A *model category* is a complete and cocomplete category $\mathcal{C}$ equipped with a *model*

structure: a class W of morphisms (called weak equivalences) satisfying the 2-out-of-3 property together with a pair of weak factorization systems (C, AF) (cofibrations and acyclic fibrations) and (AC, F) (acyclic cofibrations and fibrations) such that $AC = C \cap W$ and $AF = F \cap W$.

An object X in a model category $\mathcal{C}$ is *cofibrant* if the canonical map $\emptyset \to X$ from the initial object to X is a cofibration. The class of cofibrant objects is denoted CO. Likewise, an object Y is *fibrant* if the canonical map $Y \to 1$ to the terminal object is a fibration. A model category is *pointed* if the unique map $\emptyset \to 1$ is an isomorphism.

Different model structures on the same category are distinguished using superscripts. The weakly saturated class generated by some class M of morphisms is denoted $\mathrm{cof}(M)$. The class of maps having the right lifting property with respect to all maps in M is denoted $\mathrm{inj}(M)$.

Definition 2.0.1. A model category is *cofibrantly generated* [Hir03, Definition 11.1.2] if its cofibrations and acyclic cofibrations are generated by sets (as opposed to proper classes) that permit the small object argument, *quasi-tractable* if its (acyclic) cofibrations are contained in the weak saturation of (acyclic) cofibrations with cofibrant source (and target), *combinatorial* [Lur09, Definition A.2.6.1] if it is locally presentable and cofibrantly generated, *tractable* [Bar10, Definition 1.21] if it is combinatorial and quasi-tractable.

Combinatoriality or alternatively *cellularity* [Hir03, Definition 12.1.1] is the key assumption used to guarantee the existence of Bousfield localizations.

Definition 2.0.2. A model category $\mathcal{C}$ is *pretty small* if there is a cofibrantly generated model category structure $\mathcal{C}'$ on the same category as $\mathcal{C}$ such that $W_{\mathcal{C}} = W_{\mathcal{C}'}$, $C_{\mathcal{C}} \supset C_{\mathcal{C}'}$ and the domains and codomains X of some set of generating cofibrations of $\mathcal{C}'$ are compact, i.e., $\mathrm{Mor}(X, -)$ preserves filtered colimits.

Pretty smallness is stable under transfer and localization (Propositions 5.1.2(v) and 6.1.3). Lemma 2.0.3 implies that weak equivalences are stable under colimits of chains in a pretty small model category. Pretty smallness is a fairly mild condition that is satisfied for all basic model categories in Section 7.

Lemma 2.0.3. *Let λ be an ordinal and $f \colon \lambda \to \mathbf{Ar}(\mathcal{C})$ a cocontinuous chain of morphisms in a model category, i.e., a sequence of commutative squares*

$$
\begin{array}{ccc}
X_i & \longrightarrow & X_{i+1} \\
\downarrow {\scriptstyle f_i} & & \downarrow {\scriptstyle f_{i+1}} \\
Y_i & \longrightarrow & Y_{i+1}
\end{array}
$$

indexed by $i \in \lambda$ such that $f_i = \mathrm{colim}_{j<i} f_j$ for all limit ordinals $i \in \lambda$. Set $f_\infty = \mathrm{colim}\, f_i$.

(i) *[CS02, Proposition I.2.6.3] If every f_i (equivalently, only f_0) and every map $X_{i+1} \sqcup_{X_i} Y_i \to Y_{i+1}$ is an (acyclic) cofibration, then so is f_∞.*

(ii) *If cofibrations in $\mathcal{C}$ are generated by cofibrations with compact domain and codomain and every f_i is an acyclic fibration, then so is f_∞.*

(iii) *If $\mathcal{C}$ is pretty small and every f_i is a weak equivalence, then so is f_∞. In particular, colimits of chains are homotopy colimits. The same is true for arbitrary filtered colimits.*

(iv) *If $\mathcal{C}$ is pretty small then weak equivalences are stable under transfinite compositions, i.e., for any cocontinuous chain $X \colon \lambda \to \mathcal{C}$ of weak equivalences the map $X_0 \to \mathrm{colim}\, X$ is also a weak equivalence.*

Proof. (ii): Following the proof of [Hov99, Corollary 7.4.2], consider the lifting diagram

$$
\begin{array}{ccc}
A & \to & X_s \\
\downarrow & & \downarrow \\
B & \to & Y_s,
\end{array}
$$

where $A \to B$ is a generating cofibration and $s = \infty$. The horizontal maps factor through some stage X_α, and Y_β. We can take $\alpha = \beta$, increasing them if necessary. By further increasing α we can make the above diagram commutative for $s = \alpha$. Since $X_\alpha \to Y_\alpha$ is an acyclic fibration, we have a lifting $B \to X_\alpha$, which gives a lifting of the original diagram after postcomposing with $X_\alpha \to X_\infty$.

(iii): We may assume that $\mathcal{C}$ is such that its generating cofibrations have compact (co)domains. Proposition 4.1 in Raptis and Rosický [RR14] now implies the desired result. As indicated there in the preceding remark, the condition of local presentability is not used in the proof, so our assumptions are sufficient to invoke their proposition. (iv) is a particular case of (iii). $\qquad\square$

The notion of h-cofibrations due to Batanin and Berger recalled below is the basis of (symmetric) h-monoidality (Definitions 3.2.1, 4.2.7), which a key condition in the admissibility results of a subsequent paper [PS14a, Theorem 5.10]. There is a similar concept of *i-cofibrations*. By definition, an i-cofibration is a map along which pushouts are homotopy pushouts. In a left proper model category, this is the same as being an h-cofibration. In a non-left proper model category i-cofibrations behave better than h-cofibrations. For example the left properness

assumptions in Theorem 6.2.2(ii) and Lemma 3.2.7 is unnecessary if one uses i-cofibrations instead. Moreover, acyclic i-cofibrations, i.e., maps that are i-cofibrations and weak equivalences, coincide with couniversal weak equivalences in any (not necessarily left proper) model category, as can be shown. However, our main supply of h-cofibrations (or i-cofibrations) comes from h-monoidal (or i-monoidal) categories, which are automatically left proper (Lemma 3.2.2), so the two concepts agree in this case. In particular, there is no difference between h-monoidality and i-monoidality (or their symmetric versions). Hence we do not pursue a separate study of i-cofibrations in this paper.

Definition 2.0.4. [BB13, Definition 1.1] A map $f\colon X \to X'$ in a model category $\mathcal{C}$ is an *h-cofibration* if for any pushout diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & A & \xrightarrow{\ g\ } & B \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & A' & \xrightarrow{\ g'\ } & B'
\end{array}
$$

with a weak equivalence g, g' is also a weak equivalence. An *acyclic h-cofibration* is a map that is both an h-cofibration and a weak equivalence.

Example 2.0.5. In the category **sSet**, equipped with its standard model structure, a map is an (acyclic) cofibration if and only if it is an (acyclic) h-cofibration. By 2.0.6(v), we only need to prove the if-part. Suppose a noninjective map $f\colon A \to B$ is an h-cofibration. Then A has two nondegenerate simplices $a, a' \in A_n$ with $f(a) = f(a')$. Since any cofibration is an h-cofibration and h-cofibrations are stable under composition by 2.0.6(ii), we may first replace A by the union of all faces of a and a' and then by $S^n \vee S^n$, using the pushout along the map $A \to S^n \vee S^n$ collapsing all proper faces of a and a' to the base point. The pushout of $B \sqcup_{S^n \vee S^n} S^n$ (using the obvious collapsing map) is isomorphic to B. If B was also the homotopy pushout, there was a homotopy fiber square of derived mapping spaces

$$
\begin{array}{ccc}
\underline{\mathrm{R}}\mathrm{Map}(S^n \vee S^n, K(\mathbf{Z}, n)) & \longleftarrow & \underline{\mathrm{R}}\mathrm{Map}(S^n, K(\mathbf{Z}, n)) \\
{\scriptstyle f^*}\uparrow & & \uparrow \\
\underline{\mathrm{R}}\mathrm{Map}(B, K(\mathbf{Z}, n)) & \xleftarrow[\ \mathrm{id}\ ] & \underline{\mathrm{R}}\mathrm{Map}(B, K(\mathbf{Z}, n)),
\end{array}
$$

contradicting the fact that the path components of these spaces are $\mathbf{Z} \oplus \mathbf{Z}$, $\mathbf{Z}$, and $\mathrm{H}^n(B, \mathbf{Z})$, respectively.

Usually, h-cofibrations form a strictly larger class than cofibrations, though. We don't know an effective criterion characterizing h-cofibrations.

Lemma 2.0.6. *Suppose $\mathcal{C}$ is a model category.*

(i) *If $\mathcal{C}$ is left proper, a map is an h-cofibration if and only if pushouts along it are homotopy pushouts.*
(ii) *(Acyclic) h-cofibrations in $\mathcal{C}$ are stable under composition, pushouts and retracts.*
(iii) *If weak equivalences are stable under colimits of chains (e.g., if $\mathcal{C}$ is pretty small, see Lemma 2.0.3(iii)), then so are (acyclic) h-cofibrations. In particular, they are closed under transfinite composition, so they form a weakly saturated class.*
(iv) *Couniversal weak equivalences are acyclic h-cofibrations. The converse is true if $\mathcal{C}$ is left proper.*
(v) *Any acyclic cofibration is an acyclic h-cofibration. If $\mathcal{C}$ is left proper, any cofibration is an h-cofibration.*

Proof. Parts (i), (ii), (iv) are due to Batanin and Berger [BB13, Proposition 1.5, Lemmas 1.3, 1.6].

(iii): We use the notation of Lemma 2.0.3. For an object S under X_∞, there is a functorial isomorphism $S \sqcup_{X_\infty} Y_\infty = \mathrm{colim}\, S \sqcup_{X_i} Y_i$. Therefore, the pushout of a weak equivalence $s\colon S \to S'$ under X_∞ along f_∞ is the filtered colimit of the pushouts of $s \sqcup_{X_i} Y_i$. Each of those is a weak equivalence since f_i is an h-cofibration. By assumption, their colimit is also a weak equivalence, so f_∞ is an h-cofibration. For acyclic h-cofibrations, use Lemma 2.0.3(iii) one more time.

(v): The acyclic part is immediate from (iv). The nonacyclic part is [BB13, Lemma 1.2]. $\qquad\square$

Lemma 2.0.7. *If $G\colon \mathcal{D} \to \mathcal{C}$ is a functor between model categories that creates weak equivalences (for example, if the model structure on $\mathcal{D}$ is transferred from $\mathcal{C}$) and preserves pushouts along a map $d \in \mathrm{Mor}(\mathcal{D})$ and $G(d)$ is an (acyclic) h-cofibration then d is an (acyclic) h-cofibration.*

Proof. Given a pushout f' in $\mathcal{D}$ of a weak equivalence f under $\mathrm{dom}(d)$, we apply G and get a pushout in $\mathcal{C}$. As $G(d)$ is an h-cofibration, $G(f')$ is a weak equivalence, hence f' is a weak equivalence and therefore d is an h-cofibration. The acyclic part is similar, using that G detects weak equivalences. $\qquad\square$

3. Monoidal model categories

In this section, we study certain properties of monoidal model categories. We first review the standard definitions of a monoidal model category and, more generally, a model category with a (left module) action of a monoidal category. In Subsection 3.2, we recall the concepts of h-monoidality (due to Batanin and Berger) and flatness (due to Hovey). In the case of a symmetric monoidal model category, these notions will be refined in Section 4.

Definition 3.0.1. [Hov99, Definitions 4.1.6, 4.2.6] A *(symmetric) monoidal category* $(\mathcal{C}, \otimes, 1)$ is a (commutative) 2-monoid in the (large) bicategory of categories, functors, and natural transformations. For a monoidal category $\mathcal{C}$, a *left $\mathcal{C}$-module* $\mathcal{C}'$ over $\mathcal{C}$ is a left module over $\mathcal{C}$ regarded as a 2-monoid. The functor

$$\otimes \colon \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$$

will be referred to as the *scalar product*. To simplify the notation, Mac Lane's coherence theorem for monoidal categories will implicitly be used.

A *(symmetric) monoidal model category* is a closed (symmetric) monoidal category $\mathcal{C}$ such that

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

is a left Quillen bifunctor, i.e.,

$$c \,\square\, d \colon C_1 \otimes D_2 \sqcup_{C_1 \otimes D_1} C_2 \otimes D_1 \to C_2 \otimes D_2$$

is a cofibration in $\mathcal{C}$ for any two cofibrations $c \colon C_1 \to C_2$ and $d \colon D_1 \to D_2$ in $\mathcal{C}$, which is moreover acyclic if c or d is acyclic. This is also referred to as the *pushout product axiom*.

If a left $\mathcal{C}$-module $\mathcal{C}'$ (but not necessarily $\mathcal{C}$ itself) carries a model structure, we call $\mathcal{C}'$ it a *left $\mathcal{C}$-module with a model structure*.

A left $\mathcal{C}$-module $\mathcal{C}'$ with a model structure satisfies the *monoid axiom* if the class $\mathrm{cof}(\mathcal{C} \otimes \mathrm{AC}_{\mathcal{C}'})$ consists of weak equivalences in $\mathcal{C}'$ [SS00, Definition 3.3].

In the definition of a monoidal model category, we do not require the unit axiom (which asks that $(\mathrm{Q}(1) \to 1) \otimes X$ is a weak equivalence, where X is any cofibrant object and the map is the cofibrant replacement of 1). It is a special case of flatness (Definition 3.2.3).

Suppose $\mathcal{V}$ is a symmetric monoidal model category. A *$\mathcal{V}$-enriched model category* [Bar10, Definition 1.27.4.1] is a $\mathcal{V}$-enriched category $\mathcal{C}$ that is tensored and cotensored over $\mathcal{V}$ and such that the tensor functor $\mathcal{V} \times \mathcal{C} \to \mathcal{C}$ is a left Quillen bifunctor. We also assume the unit axiom for the $\mathcal{V}$-module $\mathcal{C}$, i.e., that for some (equivalently, any) cofibrant replacement $\mathrm{Q}(1_{\mathcal{V}}) \to 1_{\mathcal{V}}$ of the monoidal unit, $\mathrm{Q}(1_{\mathcal{V}}) \otimes X \to X$ is a weak equivalence for all cofibrant objects X. (This requirement is used in Proposition 4.3.5.) Two important examples of enriching categories for us are the categories of simplicial sets **sSet**, which gives us simplicial model categories, and connective chain complexes of abelian groups **Ch**$_+$, which gives us differential graded model categories. Chain complexes of various kinds are not enriched over simplicial sets, which necessitates considering different enriching categories. In both cases, 1 is cofibrant, so the unit axiom is trivial.

To ensure that $\mathcal{V}$-enriched left Bousfield localizations exist, we require the enriching model category $\mathcal{V}$ to be tractable or at least quasi-tractable (see Proposition 6.1.3). Both of the above examples are tractable.

3.1. **The pushout product.** In this section, we define an endofunctor **Ar** on the bicategory of cocomplete monoidal categories, cocontinuous strong monoidal functors, and monoidal natural transformations. Roughly speaking, **Ar** sends a category $\mathcal{C}$ to its category of morphisms equipped with a new monoidal structure, the *pushout product*. The underlying category of $\mathbf{Ar}(\mathcal{C})$ is the category of functors $\mathbf{Fun}(2, \mathcal{C})$, where $2 := \{0 \to 1\}$ is the walking arrow category. Its objects are morphisms in $\mathcal{C}$ and its morphisms are commutative squares in $\mathcal{C}$. If $\mathcal{C}$ is (co)complete, then $\mathbf{Ar}(\mathcal{C})$ is also (co)complete, because (co)limits in categories of functors are computed componentwise. In this section we study the monoidal structure of $\mathbf{Ar}(\mathcal{C})$ given by the pushout product and the projective model structure on $\mathbf{Ar}(\mathcal{C})$.

Definition 3.1.1. Given a cocomplete monoidal category $\mathcal{C}$, its (cocomplete) category $\mathbf{Ar}(\mathcal{C})$ of morphisms can be endowed with a monoidal structure (the *pushout product*) as follows. Interpret an object in $\mathbf{Ar}(\mathcal{C})$ as a functor $2 \to \mathcal{C}$. A finite family $f \colon I \to \mathbf{Ar}(\mathcal{C})$ of objects in $\mathbf{Ar}(\mathcal{C})$ (i.e., morphisms $f_i \colon X_i \to Y_i$ in $\mathcal{C}$) gives a functor $2^I \to \mathcal{C}^I \to \mathcal{C}$, where $\mathcal{C}^I \to \mathcal{C}$ is the monoidal product on $\mathcal{C}$. We interpret this functor as a cocone on the category $2^I \setminus \{1^I\}$ (observe that 1^I is the terminal object of the category 2^I) and the monoidal product of f is defined to be the universal map $\square f_i \colon \square f_i \to \bigotimes_i Y_i$ associated to this cocone, interpreted as an object in $\mathbf{Ar}(\mathcal{C})$. This defines a monoidal structure on $\mathbf{Ar}(\mathcal{C})$.

For example, the pushout product of two morphisms f_1 and f_2 is

$$f_1 \,\square\, f_2 \colon f_1 \,\square\, f_2 = X_1 \otimes Y_2 \sqcup_{X_1 \otimes X_2} Y_1 \otimes X_2 \to Y_1 \otimes Y_2.$$

We obtain a bifunctor

(3.1.2) $$\square \colon \mathbf{Ar}(\mathcal{C}) \times \mathbf{Ar}(\mathcal{C}) \to \mathbf{Ar}(\mathcal{C}).$$

Remark 3.1.3. If $(\mathcal{C}, \otimes)$ is braided or symmetric, then so is $(\mathbf{Ar}(\mathcal{C}), \Box)$. Moreover, if $\otimes$ preserves colimits of a certain type (e.g., sifted colimits) in one or both variables, then so does $\Box$. For example, if $\mathcal{C}$ is a closed monoidal category, then so is $\mathbf{Ar}(\mathcal{C})$, with the internal hom $\mathrm{Hom}(f_1, f_2)$ (which one can call the *pullback hom* from f_1 to f_2) being the morphism $\mathrm{Hom}(Y_1, X_2) \to \mathrm{Hom}(Y_1, Y_2) \times_{\mathrm{Hom}(X_1, Y_2)} \mathrm{Hom}(X_1, X_2)$. For brevity of the exposition, we only spell out the nonsymmetric, nonclosed case in the sequel.

Proposition 3.1.4. *A cocontinuous strong monoidal functor $F \colon \mathcal{C} \to \mathcal{D}$ between cocomplete monoidal categories induces a cocontinuous strong monoidal functor $\mathbf{Ar}(F) \colon \mathbf{Ar}(\mathcal{C}) \to \mathbf{Ar}(\mathcal{D})$.*

Proof. The functor $\mathbf{Ar}(F)$ is cocontinuous because colimits of diagrams are computed componentwise. To prove strong monoidality, suppose $f \colon I \to \mathbf{Ar}(\mathcal{C})$ is a finite family of objects in $\mathbf{Ar}(\mathcal{C})$. The diagram

$$
\begin{array}{ccccc}
2^I & \xrightarrow{f} & \mathcal{C}^I & \xrightarrow{\otimes} & \mathcal{C} \\
\Big\downarrow{\scriptstyle\mathrm{id}} & & \Big\downarrow{\scriptstyle F^I} & & \Big\downarrow{\scriptstyle F} \\
2^I & \xrightarrow{F(f)} & \mathcal{D}^I & \xrightarrow{\otimes} & \mathcal{D}.
\end{array}
$$

is commutative, meaning the left square is strictly commutative and the right square is commutative up to the canonical natural isomorphism coming from the monoidal structure on the functor F. The pushout product $\Box f$ is the universal map associated to the cocone $2^I \xrightarrow{f} \mathcal{C}^I \xrightarrow{\otimes} \mathcal{C}$ with the apex $1^I \in 2^I$, and similarly for $\Box \mathbf{Ar}(F)(f)$. Since F is cocontinuous, it preserves universal maps associated to cocones. Thus the image of the universal morphism associated to the cocone $2^I \to \mathcal{C}^I \to \mathcal{C}$ is also the universal morphism associated to the cocone $2^I \to \mathcal{C}^I \to \mathcal{C} \to \mathcal{D}$. The latter cocone is canonically isomorphic to the cocone $2^I \to \mathcal{D}^I \to \mathcal{D}$, which is the cocone defining $\Box \mathbf{Ar}(F)(f)$. $\qquad\qquad\square$

Definition 3.1.5. A morphism in the category $\mathbf{Ar}(\mathcal{C})$ for some monoidal category $\mathcal{C}$ is a *pushout morphism* if the corresponding commutative square in $\mathcal{C}$ is cocartesian.

Proposition 3.1.6. *For any cocomplete closed monoidal category $\mathcal{C}$ pushout morphisms in $\mathbf{Ar}(\mathcal{C})$ are closed under the pushout product.*

Proof. A pushout morphism can be presented as a functor $2 \times 2 \to \mathcal{C}$, where the first 2 is responsible for the morphism direction in $\mathbf{Ar}(\mathcal{C})$ and the second 2 is responsible for the morphism direction in $\mathcal{C}$. Schematically, we denote this by the commutative diagram

$$
\begin{array}{ccc}
00 & \to & 10 \\
\downarrow & & \downarrow \\
01 & \to & 11.
\end{array}
$$

A finite family of pushout morphisms $f \colon I \to \mathrm{Mor}(\mathbf{Ar}(\mathcal{C}))$ gives a functor $(2 \times 2)^I \to \mathcal{C}^I$, which we compose with the monoidal product $\mathcal{C}^I \to \mathcal{C}$ to obtain a functor $F \colon (2 \times 2)^I \to \mathcal{C}$. Consider now the category DC of all full subcategories A of $(2 \times 2)^I$ that are *downward closed*: if $Y \in A$ and $X \to Y$ is a morphism in $(2 \times 2)^I$, then also $X \in A$. Morphisms in DC are inclusions of subcategories. Taking the colimit of the functor F restricted to the given full subcategory A yields a cocontinuous functor $Q \colon \mathrm{DC} \to \mathcal{C}$. In particular, the set of all inclusions $A \to B$ in DC that are mapped to isomorphisms by Q forms a subcategory of DC closed under cobase changes of the underlying sets.

Suppose that $B \in \mathrm{DC}$ is obtained from $A \in \mathrm{DC}$ by adding an element $W \times 11$ and taking the downward closure, where $W \in (2 \times 2)^{I \setminus i}$ for some $i \in I$ is such that $W \times \{00, 01, 10\} \subset A$. The resulting inclusion $A \to B$ gives an isomorphism after we apply Q because the commutative square $2 \times 2 \xrightarrow{\times W} (2 \times 2)^I \xrightarrow{F} \mathcal{C}$ is a cocartesian square because each f_i is a cocartesian square and the monoidal product with a fixed object preserves cocartesian squares. This uses the closedness of the monoidal product.

Consider the following commutative square in DC, whose right entries are obtained by taking the left entries, replacing 0 in the first components by 1, and downward closing:

$$
\begin{array}{ccc}
\{00, 01\}^I \setminus \{01\}^I & \to & \{00, 01, 10, 11\}^I \setminus \{01, 11\}^I \\
\downarrow & & \downarrow \\
\{00, 01\}^I & \to & \{00, 01, 10, 11\}^I.
\end{array}
$$

The pushout product $\Box f_i$ is obtained by applying Q to the following map:

$$
\{00, 01, 10, 11\}^I \setminus \{01, 11\}^I \sqcup_{\{00,01\}^I \setminus \{01\}^I} \{00, 01\}^I \to \{00, 01, 10, 11\}^I.
$$

We present this morphism in DC as a composition of pushouts of generating maps explained in the previous paragraph, which implies that the map itself is sent to an isomorphism by Q. Such a presentation can be obtained by using the rule explained above to add all elements of $\{01, 11\}^I \setminus \{01\}^I$ to the source by induction on the number of 11's. If there are no 11's, the element $\{01\}^I$ belongs to the bottom left corner, proving our claim.

By induction, assuming that all tuples with less than k elements equal to 11 have already been added, take any tuple with exactly k components equal to 11 and observe that by replacing this component with 00, 01, or 10 we obtain a tuple already present in our set. Thus we can also add the tuple under consideration to our set. $\square$

The elementary proof of the following lemma is left to the reader. Together with Proposition 3.1.6, it can be rephrased by saying that $x \square -$ preserves finite cellular maps.

Lemma 3.1.7. *Given two composable maps y and z, and another map x, $x \square (y \circ z)$ is the composition of the pushout of $x \square z$ along $x \boxdot z \to x \boxdot (y \circ z)$, followed by $x \square y$.*

We now extend the formation of arrow categories to monoidal model categories. A *strong monoidal left Quillen functor* between monoidal model categories is a left Quillen functor F that is also equipped with the structure of a strong monoidal functor, i.e., functorial isomorphisms $F(X \otimes Y) \cong F(X) \otimes F(Y)$ compatible with the unit and associativity of $\otimes$. Monoidal model categories, strong monoidal left Quillen functors, and monoidal natural transformations form a bicategory. (As in Remark 3.1.3, there are obvious variants for (symmetric) monoidal model categories, which we will not spell out explicitly.)

The following proposition was shown independently by Hovey under the additional assumption that $\mathcal{C}$ is cofibrantly generated [Hov14, Proposition 3.1].

Proposition 3.1.8. *The functor $\mathbf{Ar}$ described in Definition 3.1.1 and Proposition 3.1.4 descends to the bicategory of closed monoidal model categories, as described in the proof below.*

Proof. Given a closed monoidal model category $\mathcal{C}$, the monoidal category $\mathbf{Ar}(\mathcal{C})$ is complete and cocomplete. We equip $\mathbf{Ar}(\mathcal{C})$ with the projective model structure, which coincides with the Reedy model structure, where the nonidentity arrow $0 \to 1$ in 2 is declared to be positive. In particular, the projective model structure on $\mathbf{Ar}(\mathcal{C})$ exists. Fibrations and weak equivalences are defined componentwise. (Acyclic) cofibrations $f : g \to h$ are commutative squares

$$
\begin{array}{ccc}
W & \xrightarrow{\ p\ } & Y \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
X & \xrightarrow{\ q\ } & Z
\end{array}
$$

such that p and the universal map $Y \sqcup_W X \to Z$ are both (acyclic) cofibrations, hence q is also an (acyclic) cofibration. In particular, cofibrant objects in $\mathbf{Ar}(\mathcal{C})$ are morphisms $g : W \to X$ such that W is cofibrant and g is a cofibration in $\mathcal{C}$.

We now prove the pushout product axiom for $\mathbf{Ar}(\mathcal{C})$ from the one of $\mathcal{C}$ (Definition 3.0.1). Actually, we show that the pushout product of a finite nonempty family $f : I \to \mathrm{Mor}(\mathbf{Ar}(\mathcal{C}))$ of cofibrations in $\mathbf{Ar}(\mathcal{C})$ is a cofibration, and if one of the cofibrations is acyclic, then the resulting cofibration is also acyclic. The infrastructure of the following proof is the same as in the proof of Proposition 3.1.6. Just like there we get a functor $F : (2 \times 2)^I \to \mathcal{C}$ and a cocontinuous functor $Q : \mathrm{DC} \to \mathcal{C}$. Let

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a'} \\
B & \longrightarrow & B'
\end{array}
$$

be a cocartesian square in DC, i.e., $B' = A' \cup_A B$. If $Q(a)$ is a cofibration, then so is $Q(a')$. Suppose that for every $i \in I$ we select one of the morphisms $\{00\} \to \{00, 10\}$ or $\{00, 01, 10\} \to \{00, 01, 10, 11\}$ in $\mathrm{DC}(2 \times 2)$. Then the pushout product of these morphisms belongs to the above subcategory because of the pushout product axiom for $\mathcal{C}$. The first morphism above expresses the fact that the top arrow of a cofibration in $\mathbf{Ar}(\mathcal{C})$ is itself a cofibration and the second morphism corresponds to the canonical map from the pushout to the bottom right corner, which is also a cofibration. The pushout product mentioned above always has the form $A \setminus \{x\} \to A$, where the individual components of x are 10 respectively 11, according to the choice made above.

The pushout product of f is the functor Q applied to the commutative square

$$
\begin{array}{ccc}
\{00, 01, 10, 11\}^I \setminus \{10, 11\}^I \setminus \{01, 11\}^I & \to & \{00, 01, 10, 11\}^I \setminus \{01, 11\}^I \\
\downarrow & & \downarrow \\
\{00, 01, 10, 11\}^I \setminus \{10, 11\}^I & \to & \{00, 01, 10, 11\}^I.
\end{array}
$$

It remains to prove that Q applied to the top map and the map from the pushout of the left and top arrows (i.e., the union of all corners except for the bottom right corner) to the bottom right corner is a cofibration. We present the morphism in DC under consideration as a composition of pushouts of generating maps explained in the previous paragraph. This implies that the map itself is sent by Q to a cofibration.

For the top map, such a presentation can be obtained by using the rule explained above to add all elements of $\{10, 11\}^I \setminus \{11\}^I$ to the source by induction on the number of 11's. Assume that all tuples with less than k 11's have already been added and take any tuple with exactly k 11's. By applying the rule explained in the

previous paragraph to the family of maps that are either $\{00\} \to \{00, 10\}$ if the corresponding component is 10 or $\{00, 01, 10\} \to \{00, 01, 10, 11\}$ if the corresponding component is 11 we can conclude that the tuple under consideration can be added to our set.

For the map from the pushout of the top and left arrows to the bottom right corner observe that we only need to add the element $\{11\}^I$, which is possible because the conditions for the corresponding rule are satisfied.

For acyclic cofibrations observe that the rule in the previous paragraph now guarantees that the resulting map is an acyclic cofibration after we apply Q, precisely because the pushout product in $\mathcal{C}$ of a family of cofibrations, at least one of which is acyclic, is again an acyclic cofibration. The rest of the proof is exactly the same, because the category of acyclic cofibrations is also closed under pushouts.

Finally, **Ar** descends to strong monoidal left Quillen functors: if $F\colon \mathcal{C} \to \mathcal{D}$ is such a functor, then the induced functor $\mathbf{Ar}(F)\colon \mathbf{Ar}(\mathcal{C}) \to \mathbf{Ar}(\mathcal{D})$ is cocontinuous and strong monoidal (Proposition 3.1.4). It is a left Quillen functor because F preserves (acyclic) cofibrations and pushouts. $\square$

3.2. **H-monoidality and flatness.** In this section, we discuss the notion of h-monoidality and flatness of a left module $\mathcal{C}'$ with a model structure over a monoidal category $\mathcal{C}$.

H-monoidality was introduced by Batanin and Berger [BB13, Definition 1.7]. Essentially, h-monoidality ensures that category of modules over some monoid $R \in \mathcal{C}$ carries a model structure. This statement is referred to as the *admissibility* of the monoid R. The admissibility of monoids is also guaranteed by the monoid axiom [SS00, Theorem 4.1], which is a combination of two weak saturation properties, namely weak saturation by transfinite compositions and by pushouts. In this paper, we focus on admissibility conditions using pretty smallness and h-monoidality, which individually govern the homotopical behavior of transfinite compositions and of (certain) pushouts, respectively. Basic model categories are usually h-monoidal by Lemmas 3.2.4 and 3.2.5. On the other hand, h-monoidality is very robust since is stable under transfer and localization (5.2.5(i), 6.2.1(iii)). We don't know a similar statement for the monoid axiom (without the detour via pretty smallness and h-monoidality).

Definition 3.2.1. A class S of (acyclic) cofibrations in a left $\mathcal{C}$-module with a model structure (over a monoidal category $\mathcal{C}$) is *(acyclic) h-monoidal* if for any any object $C \in \mathcal{C}$ and any $s\colon S_1 \to S_2$ in S, the map

$$C \otimes s\colon C \otimes S_1 \to C \otimes S_2$$

is an (acyclic) h-cofibration (Definition 2.0.4). The category $\mathcal{C}'$ is *h-monoidal* if the classes of (acyclic) cofibrations are (acyclic) h-monoidal.

Lemma 3.2.2. [BB13, Lemma 1.8] *Any h-monoidal model category is left proper.*

We now define flatness, which is the main condition in rectification of modules over monoids. Its symmetric strengthening, symmetric flatness, plays the corresponding role for algebras over symmetric operads [PS14a, Theorem 7.5].

Definition 3.2.3. A class S of cofibrations in a left module $\mathcal{C}'$ over a model category $\mathcal{C}$ is *flat* if for all weak equivalences $y\colon Y_1 \to Y_2$ in $\mathcal{C}$ and all $s\colon S_1 \to S_2$ in S, the following map is a weak equivalence:

$$y \,\square\, s\colon Y_2 \otimes S_1 \sqcup_{Y_1 \otimes S_1} Y_1 \otimes S_2 \to Y_2 \otimes S_2$$

The category $\mathcal{C}'$ is *flat* if the class of all cofibrations is flat.

For example, if $\mathcal{C}'$ is flat then for any cofibrant object $X \in \mathcal{C}'$ and any weak equivalence $y \in \mathcal{C}$, the map $y \otimes X$ is a weak equivalence. In this slightly weaker form, flatness is independently due to Hovey [Hov14, Definition 2.4]. Actually, the notion appears already in [SS00, Theorem 4.3]. We use the above slightly stronger definition since it is stable under weak saturation of S (Theorem 3.2.8(ii)). This is useful to show the stability of flatness under transfer (Proposition 5.2.1(ii)) and localization (Proposition 6.2.1(i)).

In general, we avoid cofibrancy hypotheses where possible, in particular, we do not in general assume that the monoidal unit 1 is cofibrant. The combination of the following two lemmas is useful to establish h-monoidality and flatness in practice, though.

Lemma 3.2.4. *Let $\mathcal{C}$ be a model category in which all objects are cofibrant. Then $\mathcal{C}$ is left proper and quasi-tractable. Moreover, tractability follows from combinatoriality, while h-monoidality and flatness follow from monoidality.*

Proof. See [Hir03, Corollary 13.1.3] for left properness, [SS00, Remark 3.4] for flatness and [BB13, Lemma 1.8] for h-monoidality. $\square$

Lemma 3.2.5. *Assume that there are two model structures $\mathcal{C}$ and $\mathcal{C}_1$ on the same underlying category such that $\mathrm{W}_{\mathcal{C}} = \mathrm{W}_{\mathcal{C}_1}$ and $\mathrm{C}_{\mathcal{C}} \subset \mathrm{C}_{\mathcal{C}_1}$. Then the left properness of $\mathcal{C}_1$ implies the one of $\mathcal{C}$. If $\mathcal{C}$ is equipped with a monoidal structure, the same is true for monoidality, h-monoidality, and flatness.*

Proof. This follows from the definitions. For the h-monoidality, note that (acyclic) h-cofibrations only depend on weak equivalences. $\square$

Lemma 3.2.6. *(cf.* [BB13, Proposition 2.5]*) If $\mathcal{C}'$ is an h-monoidal left $\mathcal{C}$-module with a model structure (over a monoidal category $\mathcal{C}$) and its weak equivalences are stable under transfinite compositions (for example, $\mathcal{C}'$ is pretty small, see Lemma 2.0.3), then $\mathcal{C}'$ satisfies the monoid axiom.*

Proof. The monoid axiom says $\mathrm{cof}(\mathcal{C} \otimes \mathrm{AC}_{\mathcal{C}'}) \subset \mathrm{W}_{\mathcal{C}'}$, i.e., the weak saturation of any object of $\mathcal{C}$ with acyclic cofibrations, consists of weak equivalences. This is clear for retracts, and for colimits of chains by assumption. Finally, for $C \in \mathcal{C}$ and $f \in \mathrm{AC}_{\mathcal{C}'}$, $C \otimes f$ is an acyclic h-cofibration in $\mathcal{C}'$ by assumption. By Lemmas 2.0.6(iv) and 3.2.2, this is equivalent to being a couniversal weak equivalence. $\square$

We finish this section with two weak saturation properties. A slightly weaker statement than Theorem 3.2.8(ii) is independently due to Hovey [Hov14, Theorem A.2]. The following lemma is the basis of the interaction of h-monoidality and flatness, see for example the proof of 3.2.8(ii).

Lemma 3.2.7. *Let $\mathcal{C}'$ be a left proper model category that is a left module over a monoidal category $\mathcal{C}$. Let*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
A' & \longrightarrow & B'
\end{array}
$$

be a cocartesian square in $\mathcal{C}'$. Let $y\colon Y \to Y' \in \mathcal{C}$ be any morphism such that $y \square a$ is a weak equivalence in $\mathcal{C}'$, and both $Y \otimes a$ and $Y' \otimes a$ are h-cofibrations (Definition 2.0.4). Then $y \square b$ is a weak equivalence.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
Y \otimes A & \xrightarrow{y \otimes a} & Y' \otimes A \\
{\scriptstyle Y \otimes a}\downarrow & & \downarrow{\scriptstyle \alpha} \\
Y \otimes A' & \longrightarrow & y \boxdot a \xrightarrow{\ y\square a\ } Y' \otimes A'.
\end{array}
$$

As usual, $\boxdot$ denotes the domain of the pushout product $\square$. By assumption, $Y \otimes a$ is an h-cofibration, hence so is α by Lemma 2.0.6. Likewise, $Y' \otimes a$ is an h-cofibration. Hence the top square and the outer rectangle in the diagram below are homotopy pushouts (Lemma 2.0.6(i)). Hence so is the bottom square. By the claim in the proof of 2.0.6(iv), the map $y \square b$ is therefore also a weak equivalence:

$$
\begin{array}{ccc}
Y' \otimes A & \longrightarrow & Y' \otimes B \\
{\scriptstyle \alpha}\downarrow\ {\scriptstyle \text{h-cofib.}} & & \downarrow \\
y \boxdot a & \longrightarrow & y \boxdot b \\
{\scriptstyle \sim}\downarrow{\scriptstyle y\square a} & & \downarrow{\scriptstyle y\square b} \\
Y' \otimes A' & \longrightarrow & Y' \otimes B'.
\end{array}
$$

$\square$

Theorem 3.2.8. *Let $\mathcal{C}$ be a monoidal model category and let $\mathcal{C}'$ be a pretty small left $\mathcal{C}$-module with a model structure. We say some property of a class S of morphisms in $\mathcal{C}'$ is* stable under saturation *if it also holds for the weak saturation $\mathrm{cof}(S)$.*

(i) *If the scalar product $\otimes\colon \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'$ preserves all colimits in $\mathcal{C}'$, then the property of S of being (acyclic) h-monoidal is stable under saturation.*

(ii) *Suppose the scalar product $\otimes$ preserves filtered colimits in $\mathcal{C}'$. If S is h-monoidal then flatness of S is stable under saturation. In particular, if some class of generating cofibrations in $\mathcal{C}$ is flat and h-monoidal, then $\mathcal{C}$ is flat.*

Proof. (i): The stability of (acyclic) h-monoidality of S under weak saturation follows from Lemma 2.0.6(iii) and the preservation by $\mathcal{C}'$ of colimits in $\mathcal{C}'$.

(ii): For a weak equivalence $y\colon Y \to Y'$ in $\mathcal{C}$ and any $s \in S$, $y \square s$ is a weak equivalence by assumption. By h-monoidality of S, $Y \otimes s$ and $Y' \otimes s$ are h-cofibrations. Thus for any pushout s' of s, $y \square s'$ is a weak equivalence by Lemma 3.2.7. For a transfinite composition s_∞ of maps s_i, $y \square s_\infty$ is the transfinite composition of $y \square s_i$ by preservation of filtered colimits in the second variable. Therefore it is again a weak equivalence using pretty smallness (Lemma 2.0.3). As usual, retracts are clear. $\square$

4. Symmetricity properties

In this section we study three properties of a symmetric monoidal model category $\mathcal{C}$: symmetric h-monoidality, symmetroidality and symmetric flatness. As the name indicates, these involve the formation of pushout powers, i.e., expressions of the form

$$\underset{n}{\square} f := f^{\square n} := \underbrace{f \square \cdots \square f}_{n \text{ times}}.$$

After settling preliminaries on objects with a finite group action, these properties are defined in Subsection 4.2. The main result of Subsection 4.3 is Theorem 4.3.8 which shows the stability of these notions under weak saturation. This is a key step in showing that the properties also interact well with transfer and localization of model structures. Examples of model categories satisfying these properties are given in Section 7.

4.1. **Objects with a finite group action.** We first examine model-theoretic properties of objects with an action of a finite group, for example the permutation action of Σ_n on $f^{\square n}$. Given a finite group G, considered as a category with one object, and any category $\mathcal{C}$, define

$$(4.1.1) \qquad G\mathcal{C} := \mathbf{Fun}(G, \mathcal{C}).$$

This is the category of objects in $\mathcal{C}$ with a left G-action. It is symmetric monoidal if $\mathcal{C}$ is, by letting G act diagonally on the monoidal product. Given some $X \in G\mathcal{C}$ and any subgroup $H \subset G$, we write $X_H = \mathrm{colim}_H X$ for the coinvariants.

For any $X \in \mathcal{C}$ we define $G/H \cdot X := \coprod_{G/H} X \in G\mathcal{C}$ on which G-acts by the left G-action on G/H. More generally, given any $X \in H\mathcal{C}$ and any morphism of groups $H \to G$, we define

$$G \cdot_H X := (G \cdot X)_H,$$

where H acts on the right on G and on the left on X.

Lemma 4.1.2. *Suppose $\mathcal{C}$ is a cocomplete category and H is a subgroup of a finite group G. Any choice of a partition $G = \coprod_i H \cdot g_i$ of G into H-cosets induces a natural isomorphism*

$$\varphi(G \cdot_H -) \to (G/H) \cdot \varphi(-)$$

of functors $H\mathcal{C} \to \mathcal{C}$, where φ denotes the forgetful functor to $\mathcal{C}$.

Proof. The canonical projection $G \cdot \varphi X \to G/H \cdot \varphi X$ factors over $\varphi(G \cdot_H X)$. Conversely, given $g \in G$, the partition gives a unique $h \in H$ and i such that $g = hg_i$. Define $G/H \cdot \varphi X \to G \cdot_H \varphi X$ by $x_{gH} \mapsto (h^{-1}x)_{g_i}$. $\square$

Proposition 4.1.3. *Suppose $\mathcal{C}$ is a cofibrantly generated model category. The category $G\mathcal{C}$ carries the* projective model structure, *denoted $G^{\mathrm{pro}}\mathcal{C}$, whose weak equivalences and fibrations are precisely those maps in $G\mathcal{C}$ that are mapped to weak equivalences respectively fibrations in $\mathcal{C}$ by the forgetful functor $G\mathcal{C} \to \mathcal{C}$. The cofibrations of $G^{\mathrm{pro}}\mathcal{C}$ are generated by the maps of the form $G \cdot f$, where f runs over generating cofibrations of $\mathcal{C}$.*

Given a morphism of groups $H \to G$, there is a Quillen adjunction

$$(4.1.4) \qquad G \cdot_H - : H^{\mathrm{pro}}\mathcal{C} \leftrightarrows G^{\mathrm{pro}}\mathcal{C} : R,$$

where the right adjoint functor is the restriction.

Finally, suppose $\mathcal{C}$ is a symmetric monoidal model category. Given two groups G and H, the monoidal product on $\mathcal{C}$ induces a left Quillen bifunctor

$$(4.1.5) \qquad G^{\mathrm{pro}}\mathcal{C} \times H^{\mathrm{pro}}\mathcal{C} \to (G \times H)^{\mathrm{pro}}\mathcal{C}.$$

Proof. The existence of this model structure is standard, see, for example, Hirschhorn [Hir03, Theorem 11.6.1]. The adjunction (4.1.4) is seen to be a Quillen adjunction by looking at the right adjoint. The functor (4.1.5) is a left Quillen bifunctor because $(G \cdot I_\mathcal{C}) \square (H \cdot I_\mathcal{C}) = (G \times H) \cdot (I_\mathcal{C} \square I_\mathcal{C}) \subset (G \times H) \cdot \mathrm{C}_\mathcal{C}$, using the cocontinuity and monoidality of the functor $G \cdot -$ and the pushout product axiom for $\mathcal{C}$. $\square$

Proposition 4.1.6. *The functor $-^{\square n} \colon \mathbf{Ar}\mathcal{C} \to \Sigma_n \mathbf{Ar}\mathcal{C}$ preserves filtered colimits.*

Proof. The functor $-^{\square n}$ is the composition $\mathbf{Ar}(\mathcal{C}) \xrightarrow{\Delta} \Sigma_n \mathbf{Ar}(\mathcal{C})^n \xrightarrow{\square} \Sigma_n \mathbf{Ar}(\mathcal{C})$. The monoidal product $\mathbf{Ar}(\mathcal{C})^n \to \mathbf{Ar}(\mathcal{C})$ is separately cocontinuous because the monoidal structure is closed, so $-^{\square n}$ evaluated on $\mathrm{colim}\, D$ for some filtered diagram $D \colon I \to \mathbf{Ar}(\mathcal{C})$ can be computed as $\mathrm{colim}\, D^n$, where $D^n \colon I^n \to \mathbf{Ar}(\mathcal{C})$ is obtained by composing the nth cartesian power $I^n \to \mathbf{Ar}(\mathcal{C})^n$ of D with the monoidal product $\mathbf{Ar}(\mathcal{C})^n \to \mathbf{Ar}(\mathcal{C})$. For a filtered category I the diagonal $I \to I^n$ is a cofinal functor, thus the last colimit can be computed as $\mathrm{colim}_I \square^n D$. $\square$

Proposition 4.1.7. [Har09, Proposition 6.13] *Suppose $h \colon f \to g$ is a pushout morphism in $\mathbf{Ar}(\mathcal{C})$. Then $h^{\square n} \colon f^{\square n} \to g^{\square n}$ is a also a pushout morphism.*

Proof. This follows immediately from Proposition 3.1.6. $\square$

4.2. **Definitions.** We now define three properties of (morphisms in) a symmetric monoidal model category $\mathcal{C}$: symmetric flatness, symmetric h-monoidality and symmetroidality. They are appropriate strengthenings of flatness (Definition 3.2.3), h-monoidality (Definition 3.2.1) and the pushout product axiom. Symmetric flatness is the key condition required to obtain a rectification result for operadic algebras [PS14a, Theorem 7.1]. Approximately, it says that for any cofibrant object $X \in \mathcal{C}$, the map

$$y \otimes_{\Sigma_n} X^{\otimes n} \colon Y \otimes_{\Sigma_n} X^{\otimes n} \to Y' \otimes_{\Sigma_n} X^{\otimes n}$$

is a weak equivalence for any weak equivalence $y \colon Y \to Y'$ in $\Sigma_n \mathcal{C}$. Slightly more accurately, the definition is phrased in terms of more general cofibrations s using instead

$$y \,\square_{\Sigma_n}\, s^{\square n}.$$

For $s \colon \emptyset \to X$ this gives back the previous expressions. In order to ensure that the three symmetricity properties are stable under weak saturation (Theorem 4.3.8), we actually define them for a class of morphisms instead of a single morphism. In such cases, we use the following notational conventions.

Definition 4.2.1. Let $v := (v_1, \ldots, v_e)$ be a finite family of morphisms. For any sequence of nonnegative integers $n := (n_i)_{i \leq e}$, we write $\Sigma_n := \prod \Sigma_{n_i}$, $v^{\square n} := v_1^{\square n_1} \square \cdots \square v_e^{\square n_e}$, and $v^{\otimes n} := v_1^{\otimes n_1} \otimes \cdots \otimes v_e^{\otimes n_e}$. We write $m \leq n$ if $m_i \leq n_i$ for all i and $m < n$ if $m \leq n$ and $m \neq n$. Given a class S of morphisms, we write $v \subset S$ if all v_i are in S. Given another sequence of integers $(m_i)_{i=1}^e$, we write $mn := \sum m_i n_i$ and $\Sigma_m^n := \prod \Sigma_{m_i}^{n_i}$ and $\Sigma_n \rtimes \Sigma_m^n := \prod \Sigma_{n_i} \rtimes \Sigma_{m_i}^{n_i}$.

Definition 4.2.2. A class S of cofibrations in $\mathcal{C}$ is called *symmetric flat with respect to some class* $\mathcal{Y} = (\mathcal{Y}_n)$ of morphisms $\mathcal{Y}_n \subset \operatorname{Mor} \Sigma_n \mathcal{C}$ if

$$y \,\square_{\Sigma_n}\, s^{\square n} := (y \,\square\, s^{\square n})_{\Sigma_n}$$

is a weak equivalence in $\mathcal{C}$ for any $y \in \mathcal{Y}_n$, any finite multi-index $n \geq 1$ and any $s \in S$. We say S is *symmetric flat* if it is symmetrically flat with respect to the classes $\mathcal{Y}_n = (\mathrm{W}_{\Sigma_n^{\mathrm{pro}} \mathcal{C}})$ of projective weak equivalences (i.e., those maps in $\Sigma_n \mathcal{C}$ which are weak equivalences after forgetting the Σ_n-action). We say $\mathcal{C}$ is *symmetric flat* if the class of cofibrations is symmetric flat.

Example 4.2.3. A class S is symmetric flat (i.e., with respect to $\mathrm{W}_{\Sigma_n^{\mathrm{pro}} \mathcal{C}}$) if and only if $y \,\square_{\Sigma_n}\, s^{\square n}$ is a weak equivalence for a single map $s \in S$, i.e., no multi-indices are necessary in this case. The reader is encouraged to mainly think of this case.

The following definition is necessary to ensure that the small object argument can be applied to construct a model structure on operadic algebras [PS14a, Theorem 5.10]. Recall from [Hir03, Definition 10.4.1] or [Hov99, Definition 2.1.3] that an object $A \in \mathcal{C}$ is *small relative to some subcategory* $\mathcal{D} \subset \mathcal{C}$ if there is some cardinal λ such that for any λ-sequence $X_0 \to X_1 \to \cdots \to X_\beta \to \cdots$ $(\beta < \lambda)$ in $\mathcal{D}$, the canonical map of Hom-sets

$$\operatorname*{colim}_{\beta < \lambda} \operatorname{Hom}_{\mathcal{C}}(A, X_\beta) \to \operatorname{Hom}_{\mathcal{C}}(A, \operatorname*{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism. We will often apply this to $\mathcal{D} = \operatorname{cell}(I)$, the closure of a class I of maps under pushouts and transfinite composition. Also recall that, by definition, any object in a combinatorial model category is small with respect to all maps of $\mathcal{C}$, so is automatically admissibly generated in the sense below. Topological spaces are a non-combinatorial, but admissibly generated model category (Subsection 7.5).

Definition 4.2.4. A symmetric monoidal model category $\mathcal{C}$ is *admissibly generated* relative to a class S of morphisms in $\mathcal{C}$ if all cofibrant objects in $\mathcal{C}$ are small with respect to the subcategory

$$(4.2.5) \qquad\qquad\qquad \operatorname{cell}(Y \otimes_{\Sigma_n} s^{\square n})$$

for any finite family $s \subset S$, any multi-index $n > 0$, and any object $Y \in \Sigma_n \mathcal{C}$. We call $\mathcal{C}$ *admissibly generated* if it is cofibrantly generated and admissibly generated relative to the cofibrations $\mathrm{C}_{\mathcal{C}}$.

Lemma 4.2.6. [Hir03, Proposition 10.4.9] *For $\mathcal{C}$ to be admissibly generated relative to S it is enough that the (co)domains of some set of generating cofibrations are small with respect to (4.2.5).*

The notions of symmetric h-monoidal maps (respectively, symmetroidal maps) presented next are designed to ultimately address the (strong) admissibility of operads ([PS14a, Theorem 5.10]).

Definition 4.2.7. A class S of morphisms in a symmetric monoidal category $\mathcal{C}$ is called *(acyclic) symmetric h-monoidal* if for any finite family $s \subset S$ and any multi-index $n \neq 0$, and any object $Y \in \Sigma_n \mathcal{C}$ the morphism $Y \otimes_{\Sigma_n} s^{\square n}$ is an (acyclic) h-cofibration. We say $\mathcal{C}$ is *symmetric h-monoidal* if the class of (acyclic) cofibrations is (acyclic) symmetric h-monoidal.

The notion of power cofibrations presented next is due to Lurie [Lur, Definition 4.5.4.2] and Gorchinskiy and Guletskiĭ [GG09, Section 3], who also introduced symmetrizable maps.

Definition 4.2.8. Let $\mathcal{Y} = (\mathcal{Y}_n)_{n>0}$ be a collection of classes $\mathcal{Y}_n$ of morphisms in $\Sigma_n\mathcal{C}$, where $n > 0$ is any finite multi-index. We suppose that for $y \in \mathcal{Y}_n$, $y \,\square\, -$ preserves *injective* (acyclic) cofibrations in $\Sigma_n\mathcal{C}$, i.e., those maps which are (acyclic) cofibrations in $\mathcal{C}$.

A class S of morphisms in a symmetric monoidal category $\mathcal{C}$ is called *(acyclic) $\mathcal{Y}$-symmetroidal* if for all multi-indices $n > 0$ and all maps $y \in \mathcal{Y}_n$, the morphism

(4.2.9) $$y \,\square_{\Sigma_n}\, s^{\square n}$$

is an (acyclic) cofibration in $\mathcal{C}$ for all $s \in S$. If $\mathcal{Y}_n = \mathrm{C}_{\Sigma_n^{\mathrm{in}}\mathcal{C}}$, we say that S is *(acyclic) symmetroidal*. For $\mathcal{Y}_n = \{\emptyset \to 1_{\mathcal{C}}\}$, we say S is *(acyclic) symmetrizable*.

A map $f \in \mathcal{C}$ is called an *(acyclic) power cofibration* if the morphism $f^{\square n}$ is an (acyclic) cofibration in $\Sigma_n^{\mathrm{pro}}\mathcal{C}$ for all integers $n > 0$ (i.e., a projective cofibration with respect to the Σ_n-action).

The category $\mathcal{C}$ is called symmetric h-monoidal/$\mathcal{Y}$-symmetroidal/freely powered if the class of all (acyclic) cofibrations is (acyclic) symmetric h-cofibrant/(acyclic) $\mathcal{Y}$-symmetroidal/(acyclic) power cofibration.

Remark 4.2.10. In the definition of power cofibrations, no multi-indices are necessary: for power cofibrations s_i and any any multi-index $n = (n_i)$, $s^{\square n} := \square_i s^{\square n_i}$ is a $\Sigma_n := \prod \Sigma_{n_i}$ projective cofibration by the pushout product axiom.

Unlike the definition of power cofibrations in [Lur], we exclude the case $n = 0$, for this would require 1 to be cofibrant, which is not always satisfied. In fact, it is *never* satisfied for the positive model structures on symmetric spectra which is a main motivating example for us [PS14b].

We have the following implications (where symmetroidality is with respect to the classes $\mathcal{Y}_n$ of injective cofibrations in $\Sigma_n\mathcal{C}$):

(4.2.11)

power cofibration $\Longrightarrow$ symmetroidal map $\Longrightarrow$ cofibration

symmetric h-cofibration $\Longrightarrow$ h-cofibration.

The vertical implication holds if $\mathcal{C}$ is left proper. The dotted arrow is not an implication in the strict sense unless all objects in $\mathcal{C}$ are cofibrant. A symmetroidal map x is such that for all cofibrant objects $Y \in \Sigma_n^{\mathrm{in}}\mathcal{C}$, the map $Y \otimes_{\Sigma_n} x^{\square n}$ is a cofibration and therefore (again if $\mathcal{C}$ is left proper) an h-cofibration. Being a symmetric h-cofibration demands the latter for any object $Y \in \Sigma_n\mathcal{C}$. Every power cofibration is a symmetrizable cofibration since the coinvariants $\Sigma_n^{\mathrm{pro}}\mathcal{C} \to \mathcal{C}$ are a left Quillen functor. The implications in (4.2.11) are in general strict: in a monoidal model category $\mathcal{C}$ with cofibrant monoidal unit or, more generally, one satisfying the strong unit axiom, every object is h-cofibrant [BB13, Proposition 1.17], but of course not necessarily cofibrant. In the category **sSet** of simplicial sets every cofibration is a symmetrizable cofibration, but not a power cofibration (see Subsection 7.1).

The homotopy orbit $\mathrm{hocolim}_{\Sigma_n} X^{\otimes n}$ can be computed by applying the derived functor of the either of the following two left Quillen bifunctors to $(1_{\mathcal{V}}, X^{\otimes n})$ [Gam10, Theorem 3.2 and Theorem 3.3]:

(4.2.12) $$\Sigma_n^{\mathrm{op,in}}\mathcal{V} \times \Sigma_n^{\mathrm{pro}}\mathcal{C} \xrightarrow{\otimes} \mathcal{C},$$

(4.2.13) $$\Sigma_n^{\mathrm{op,pro}}\mathcal{V} \times \Sigma_n^{\mathrm{in}}\mathcal{C} \xrightarrow{\otimes} \mathcal{C}.$$

Here $\mathcal{V}$ denotes the symmetric monoidal model category used for the enrichment and the monoidal unit $1_{\mathcal{V}} \in \mathcal{V}$ is equipped with the trivial Σ_n-action. If $\mathcal{C}$ is freely powered, then for any cofibrant object $X \in \mathcal{C}$, $X^{\otimes n}$ is projectively cofibrant, i.e., cofibrant in $\Sigma_n^{\mathrm{pro}}\mathcal{C}$. Thus, the homotopy orbit is given by $(X^{\otimes n})_{\Sigma_n}$, provided that $1_{\mathcal{V}}$ is cofibrant [Lur, Lemma 4.5.4.11]. However, most model categories appearing in practice are not freely powered, so that $X^{\otimes n}$ needs to be *projectively* cofibrantly replaced to compute the homotopy colimit. This is usually a difficult task. On the other hand, when using (4.2.13), one needs to cofibrantly replace 1 in $\Sigma_n^{\mathrm{op,pro}}\mathcal{V}$, but no cofibrant replacement has to be applied to $X^{\otimes n}$, provided that X is cofibrant in $\mathcal{C}$. This makes the second approach to computing homotopy colimits much more easily applicable. This observation is used in Lemma 4.3.4 below, which in its turn is the key technical step in establishing the compatibility of symmetric h-monoidality and Bousfield localizations (Theorem 6.2.2(ii)).

4.3. **Basic properties and weak saturation.** In this section, we provide a few elementary facts concerning the symmetricity notions defined in Subsection 4.2. After this, we show the main theorem of this section (4.3.8), which asserts that the symmetricity notions behave well with respect to weak saturation.

The following two results have a similar spirit: we show that symmetric flatness can be reduced to (projective) acyclic fibrations, and that the class $\mathcal{Y}$ appearing in the definition of $\mathcal{Y}$-symmetroidality can be weakly saturated.

Lemma 4.3.1. *If S is symmetric flat with respect to $\mathcal{Y}$, it is also symmetric flat with respect to the class $\mathcal{Z}$, where $\mathcal{Z}_n$ consists of compositions $z = y \circ c$, with $y \in \mathcal{Y}_n$ and $c \in \mathrm{AC}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}}$, i.e., an acyclic projective cofibration. In particular, any class of cofibrations is symmetric flat with respect to $\mathrm{AC}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}}$. Moreover, being symmetric flat is equivalent to being symmetric flat with respect to the acyclic projective fibrations $\mathrm{AF}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}}$.*

Proof. Any acyclic projective cofibration c lies in the weak saturation of $\Sigma_n \cdot \mathrm{AC}_{\mathcal{D}}$. For $h \in \mathrm{AC}_{\mathcal{C}}$ and $c = \Sigma_n \cdot h$, we get an isomorphism in $\mathcal{C}$,

$$(\Sigma_n \cdot h) \,\square_{\Sigma_n}\, s^{\square n} \cong h \,\square\, s^{\square n}$$

which is an acyclic cofibration in $\mathcal{C}$ by the pushout product axiom, using that $s \in S$ is a cofibration. The assignment $c \mapsto c \,\square_{\Sigma_n}\, s^{\square n}$ is cocontinuous, so that the latter expression is an acyclic cofibration in $\mathcal{C}$ for all $c \in \mathrm{AC}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}}$. The pushout product $z \,\square_{\Sigma_n}\, s^{\square n}$ is the composition of a pushout of $c \,\square_{\Sigma_n}\, s^{\square n}$, which is an acyclic cofibration, followed by $y \,\square_{\Sigma_n}\, s^{\square n}$ which is a weak equivalence by assumption. $\square$

Lemma 4.3.2. *Let S, $\mathcal{Y}$, $\mathcal{C}$ be as in Definition 4.2.8. If S is $\mathcal{Y}$-symmetroidal, it is also $\mathrm{cof}(\mathcal{Y})$-symmetroidal.*

Proof. For a fixed $s \in S$, the functor $F_s \colon y \mapsto y \square_{\Sigma_n} s^{\square n}$ is cocontinuous. In particular $F_s(\mathrm{cof}(\mathcal{Y})) \subset \mathrm{cof}(F_s(\mathcal{Y})) \subset \mathrm{cof}(\mathrm{C})_{\mathcal{C}} = \mathrm{C}_{\mathcal{C}}$ and likewise for acyclic $\mathcal{Y}$-symmetroidal maps. $\square$

Definition 4.3.3. The cofibrant replacement of 1 in $\Sigma_n^{\mathrm{op,pro}}\mathcal{V}$ is denoted by $\mathrm{E}\Sigma_n$. (For $\mathcal{V} = \mathbf{sSet}$, this coincides with the usual definition of $\mathrm{E}\Sigma_n$ as a weakly contractible simplicial set with a free Σ_n-action.)

Proposition 4.3.5 is a key step in the proof of stability of symmetric h-monoidality and symmetroidality under left Bousfield localizations. It relies on the following technical lemma.

Lemma 4.3.4. *Suppose $\mathcal{C}$ is a symmetric monoidal, h-monoidal, flat model category, $y \in \Sigma_n\mathcal{C}$ is any map, s is a finite family of acyclic cofibrations with cofibrant domain that lies in some symmetric flat class S, and $y \square s^{\square n}$ is a weak equivalence in $\mathcal{C}$ for some multiindex $n > 0$. Then $y \square_{\Sigma_n} s^{\square n}$ is also a weak equivalence.*

Proof. Let

$$
\begin{array}{ccc}
A' & \xrightarrow{\ a\ }_{\sim} & A \\
{\scriptstyle y'}\downarrow & & \downarrow{\scriptstyle y} \\
B' & \xrightarrow[\ b\ ]{\sim} & B
\end{array}
$$

be the functorial cofibrant replacement of of $y \colon A \to B \in \mathbf{Ar}(\mathcal{C})$ (in the projective model structure, so that y' is a cofibration with a cofibrant domain). Functoriality and the fact that $y \in \mathbf{Ar}(\Sigma_n\mathcal{C})$ imply that $y' \in \mathbf{Ar}(\Sigma_n\mathcal{C})$. We claim that $y' \square s^{\square n}$ is a cofibrant replacement of $y \square s^{\square n}$ in $\mathbf{Ar}(\mathcal{C})$. Let $t := s^{\square n} \colon T \to S$. The map $b \otimes S$ is a weak equivalence by the flatness assumption. To see that $B' \otimes T \sqcup_{A' \otimes T} A' \otimes S \to B \otimes T \sqcup_{A \otimes T} A \otimes S$ is a weak equivalence we first note that these pushouts are homotopy pushouts by Lemma 2.0.6(i) since $A \otimes t$ is an h-cofibration. Thus it suffices that the three individual terms in the pushouts are weakly equivalent, which again follows from flatness. The claim is shown.

Thus we have

$$\operatorname*{hocolim}_{\Sigma_n}(y \square s^{\square n}) = (\mathrm{E}\Sigma_n \otimes y' \square s^{\square n})_{\Sigma_n} \sim y \square_{\Sigma_n} s^{\square n}.$$

The last weak equivalence holds by symmetric flatness of S since $\mathrm{E}\Sigma_n \otimes y' \to y' \to y$ is a weak equivalence by the unit axiom for the $\mathcal{V}$-enrichment (note that the cofibrant replacement $\mathrm{E}\Sigma_n \to 1$ in $\Sigma_n^{\mathrm{pro}}\mathcal{V}$ is in particular a cofibrant replacement in $\mathcal{V}$). Finally, $y \square s^{\square n}$ is a weak equivalence in $\mathcal{C}$ by assumption. Therefore, the above homotopy colimit is a weak equivalence in $\mathcal{C}$. $\square$

Proposition 4.3.5. *The class of acyclic power cofibrations coincides with the intersection of W with the class of power cofibrations.*

A $\mathcal{Y}$-symmetroidal class S which consists of acyclic cofibrations with cofibrant source is acyclic $\mathcal{Y}$-symmetroidal, provided that $\mathcal{C}$ is h-monoidal and flat and S is symmetric flat in $\mathcal{C}$.

Proof. The first claim follows from the pushout product axiom.

For any $s \in S$ and any map $y \in \mathcal{Y}_n \subset \mathrm{Mor}(\Sigma_n\mathcal{C})$, $y \square s^{\square n}$ is a weak equivalence in $\Sigma_n\mathcal{C}$ by assumption on the class $\mathcal{Y}$ (see Definition 4.2.8). Now apply Lemma 4.3.4. $\square$

We now establish the compatibility of the three symmetricity properties with weak saturation. Parts (iv) and (v) of Theorem 4.3.8 are due to Gorchinskiy and Guletskiǐ [GG09, Theorem 5]. Part (ii) extends arguments in [GG11, Theorem 9], which shows a weak saturation property for symmetrically cofibrant objects in a stable model category. Of course, it also extends the analogous statement for nonsymmetric flatness (Theorem 3.2.8(ii)). Likewise, (iii) extends the weak saturation property of h-cofibrations (see Lemma 2.0.6). The proof of the closure under transfinite compositions in (iv) is reminiscent of §4 of Gorchinskiy and Guletsiǐ [GG09]. See also the expository accounts by White [Whi14a, Appendix A] and Pereira [Per14, §4.2]. In the proof of the theorem, we will need a combinatorial lemma that we establish first. Recall the conventions for multiindices in Definition 4.2.1.

Lemma 4.3.6. *Let $X_0^{(i)} \xrightarrow{v_0^{(i)}} X_1^{(i)} \xrightarrow{v_1^{(i)}} X_2^{(i)}$, $1 \le i \le e$ be a finite family of composable in a symmetric monoidal category. For a pair of multiindices $0 \le k \le n$ of length e, we set*

(4.3.7)
$$m_k := \Sigma_n \cdot_{\Sigma_{n-k} \times \Sigma_k} v_0^{\square n-k} \square v_1^{\square k}.$$

(i) *The map*

$$(v_1 v_0)^{\square n} \colon \overset{n}{\boxed{\cdot}}(v_1 v_0) \to X_2^{\otimes n}$$

is the composition of pushouts (with the attaching maps constructed in the proof) of the maps m_k ($0 \le k < n$), and the map $m_n = v_1^{\square n}$.

(ii) *The map*

$$\kappa \colon \overset{n}{\boxed{\cdot}}(v_1 v_0) \sqcup_{\boxed{\cdot}^n v_0} X_1^{\otimes n} \to X_2^{\otimes n}$$

is the composition of pushouts of m_k for $1 \le k < n$, and the map m_n. (Here 1 denotes the multiindex whose components are all equal to 1.)

Proof. We interpret the composable pair (v_0, v_1) as a functor $v \colon 3 = \{0 \to 1 \to 2\} \to \mathcal{C}^I$, where $I = \{1, \dots, e\}$. Let $\mathcal{E}$ be the category of posets C lying over $3^n = \prod_i 3^{n_i}$ and let $\Sigma_n \mathcal{E}$ be those posets with a Σ_n-action which is compatible with the Σ_n-action on 3^n. For all posets considered below, the map to 3^n will be obvious from the context. Consider the following functor:

$$Q \colon \Sigma_n \mathcal{E} \to \Sigma_n \mathcal{C}$$
$$C \to 3^n \mapsto \operatorname{colim}\left(C \longrightarrow 3^n \overset{v^n}{\longrightarrow} \mathcal{C}^n \overset{\otimes}{\longrightarrow} \mathcal{C} \right).$$

Being the composition of the two cocontinuous functors

$$\text{posets}/3^n \longrightarrow \text{posets}/\mathcal{C} \overset{\operatorname{colim}}{\longrightarrow} \mathcal{C},$$

Q is also cocontinuous. The map $(v_1 v_0)^{\square n}$ is obtained by applying Q to the map

$$\iota \colon \{0, 1, 2\}^n \backslash \{1, 2\}^n \to \{0, 1, 2\}^n$$

which adds all tuples containing only 1's and 2's. It is the composition of the maps

$$\iota_k \colon \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{<k}\} \to \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{\le k}\},$$

for $0 \le k \le n$, with $\prod_i (n_i + 1)$ maps in total. The superscript $*$ means that one adds as many elements as needed to get an n-multituple. For multiindices the above statements should be interpreted separately for each component. The map ι_k adds the Σ_n-orbit O consisting of tuples with k 2's and $n - k$ 1's, i.e., $\Sigma_n 1^{n-k} 2^k$. The cardinality of O is $\binom{n}{k}$. For $o \in O$, consider the downward closure D_o of o and $C_o := D_o \backslash \{o\}$.

There is a pushout diagram in $\Sigma_n \mathcal{E}$

$$
\begin{array}{ccc}
A := \coprod_{o \in O} C_o & \longrightarrow & \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{<k}\} \\
\downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle \iota_k} \\
B := \coprod_{o \in O} D_o & \longrightarrow & \{0, 1, 2\}^n \setminus \{1, 2\}^n \cup \{\Sigma_n 1^* 2^{\le k}\}.
\end{array}
$$

(For $k = n$ the top horizontal row is an identity, so $\iota_n = \mu_n$ in this case.) Any $o \in O$ determines a partition of $\coprod_i n_i$ into $\coprod_i \{1 \le j \le n_i \mid o_{i,j} = 1\}$ and $\coprod_i \{1 \le j \le n_i \mid o_{i,j} = 2\}$. Using this partition, we have $D_o = \Sigma_{n-k} 0^* 1^* \times \Sigma_k 0^* 1^* 2^*$ and $C_o = \Sigma_{n-k} 0^* 1^{<n-k} \times \Sigma_k 0^* 1^* 2^* \cup \Sigma_{n-k} 0^* 1^* \times \Sigma_k 0^* 1^* 2^{<k}$. Thus the map $Q(C_o \to D_o)$ is just $v_0^{\square n-k} \square v_1^{\square k}$. Using the cocontinuity of Q, this shows $Q(\mu_k) = m_k$.

The second part now follows immediately from the above once we observe that the codomain of ι_0 is precisely the domain of the map under consideration. $\qquad\square$

Theorem 4.3.8. *Let S be a class of morphisms in a symmetric monoidal model category $\mathcal{C}$. We say some property of S is* stable under saturation *if it also holds for the weak saturation $\operatorname{cof}(S)$.*

(i) *The property of being admissibly generated relative to S (Definition 4.2.4) is stable under saturation. Therefore, if $\mathcal{C}$ is cofibrantly generated and admissibly generated relative to some set of generating cofibrations, it is admissibly generated.*

(ii) *If S is symmetric h-monoidal then symmetric flatness of S relative to a class $\mathcal{Y} = (\mathcal{Y}_n)$ of weak equivalences in $\Sigma_n \mathcal{C}$ is stable under saturation. In particular, if some class of generating cofibrations in $\mathcal{C}$ is symmetric flat and symmetric h-monoidal, then $\mathcal{C}$ is symmetric flat.*

(iii) *The property of being (acyclic) symmetric h-monoidal is stable under saturation. In particular, if some class of generating (acyclic) cofibrations consists of (acyclic) symmetric h-cofibrations, then $\mathcal{C}$ is symmetric h-monoidal.*

(iv) *Being $\mathcal{Y}$-symmetroidal (Definition 4.2.8) is stable under saturation. In particular, if some class of generating (acyclic) cofibrations is (acyclic) $\mathcal{Y}$-symmetroidal, then $\mathcal{C}$ is $\mathcal{Y}$-symmetroidal.*

(v) *The same statement holds for power cofibrations.*

Proof. For a finite family of maps $v = (v^{(1)}, \ldots, v^{(e)})$ we use the multi-index notation of Definition 4.2.1. We prove the statements by cellular induction, indicating the necessary arguments for each statement individually in each step. The acyclic parts of (iii) and (iv) are the same as the nonacyclic parts, so they will be omitted. Fix an object $Y \in \Sigma_n \mathcal{C}$, respectively a map $y \in \mathcal{Y}_n \subset \mathrm{Mor}\,\Sigma_n \mathcal{C}$. For (ii) and (iv), respectively (i) and (iii), we write

$$g(v, n) := y \,\square_{\Sigma_n}\, v^{\square n}, \text{ respectively, } g(v, n) := Y \otimes_{\Sigma_n} v^{\square n}.$$

By Proposition 4.1.7, $g(-, n)$ preserves pushout morphisms $\varphi \colon v \to v'$ (in the sense that, say, $\varphi^{(1)}$ is a pushout morphism and all other $\varphi^{(j)}$'s are identities) and retracts. Thus, if $g(v, n)$ is an (acyclic) h-cofibration or (acyclic) cofibration, so is $g(v', n)$. This shows the stability of the properties of being symmetric h-monoidal and symmetroidal under cobase changes. For (ii), we additionally observe that $Y \otimes_{\Sigma_n} v^{\square n}$ is an h-cofibration and similarly with Y' since S is symmetric h-monoidal by assumption. By Lemma 3.2.7 (more precisely, replace $\otimes$ there by $\otimes_{\Sigma_n}$), applied to $a = v^{\square n}$ and $b = v'^{\square n}$, we see that $g(v', n)$ is a weak equivalence since $g(v, n)$ is one. For (i), we also use here and below that an object X is small relative to some class $\mathrm{cell}(T)$ if and only if it is small relative to its weak saturation [Hir03, Proposition 10.5.13].

We now show the stability of the three symmetricity properties and being admissibly generated relative to a class under transfinite composition: suppose $v^{(1)}$ is the transfinite composition

$$v^{(1)} \colon X_0^{(1)} \xrightarrow{v_0^{(1)}} \cdots \to X_i^{(1)} \xrightarrow{v_i^{(1)}} X_{i+1}^{(1)} \to \cdots \to X_\infty^{(1)} = \mathrm{colim}\, X_i^{(1)},$$

whose maps are obtained as pushouts

(4.3.9)
$$
\begin{array}{ccc}
A & \xrightarrow{\ s \in S\ } & A' \\
\downarrow & (*) & \downarrow \\
X := X_i^{(1)} & \xrightarrow{\ x := v_i^{(1)}\ } & X' := X_{i+1}^{(1)}.
\end{array}
$$

For the statements (ii), (iii), respectively (iv) we need to show that $g(v, n) = g((v^{(1)}, \ldots, v^{(e)}), n)$ is a weak equivalence, h-cofibration, or cofibration, respectively, provided that

$$\{v_i^{(1)}, i \leq \infty, v^{(2)}, \ldots, v^{(e)})\}$$

is a symmetric flat, symmetric h-monoidal, respectively symmetroidal class. Applying this argument e times gives the desired stability under transfinite compositions. We write $r_i^{(1)} \colon X_0^{(1)} \to X_i^{(1)}$ for the (finite) compositions of the $v_i^{(1)}$. Consider

(4.3.10)
$$\mathrm{id}_{(X_0^{(1)})^{\otimes n}} = (r_0^{(1)})^{\square n} \to (r_1^{(1)})^{\square n} \to \cdots \to (v^{(1)})^{\square n}.$$

As an object of $\Sigma_n \mathbf{Ar}(\mathcal{C})$,

(4.3.11)
$$g(v, n) = \underset{i}{\mathrm{colim}}\, g(\underbrace{(r_i^{(1)}, v^{(2)}, \ldots, v^{(e)})}_{=:v_i}, n) = \underset{i}{\mathrm{colim}}\, g(v_i, n),$$

since $-^{\square n}$ preserves filtered colimits (Proposition 4.1.6). We now show that v_i is a symmetric flat (respectively symmetric h-monoidal or symmetroidal) family, so that $g(v_i)$ is a weak equivalence (h-cofibration, cofibration, respectively). We consider the composition of two morphisms $r_0^{(1)}$ and $r_1^{(1)}$ only and leave the similar case of a finite composition of more than two maps to the reader. By Lemmas 3.1.7 and 4.3.6, $v_1^{\square n}$ is the (finite) composition of pushouts of $\Sigma_n \cdot_{\Sigma_m} w^{\square m}$, where $w = (r_0^{(1)}, r_1^{(1)}, v^{(2)}, \ldots, v^{(e)})$, and m runs through multi-indices of length $e + 1$ such that $0 \leq m^{(1)} \leq n^{(1)}$, $m^{(1)} + m^{(2)} = n^{(1)}$, and $m^{(k)} = n^{(k-1)}$ for $2 \leq k \leq e + 1$.

For (iii), each $g(w, m) = y \,\square_{\Sigma_m}\, w^{\square m}$ is an h-cofibration. Hence so is $g(v_1, n)$ since h-cofibrations are stable under pushouts and (finite) compositions by Lemma 2.0.6. By Lemma 2.0.6(iii), $g(v, n)$ is also an h-cofibration then.

Similarly, for (iv), each $g(w, m)$ is a cofibration, so that $g(v_1, n)$ is a cofibration. By Lemma 4.3.6, $(v_1^{(1)} \circ v_0^{(1)})^{\square n}$ is the composition of a pushout of $(v_0^{(1)})^{\square n}$ and the map

$$\boxed{\,\cdot\,}^{n^{(1)}}(v_1^{(1)} \circ v_0^{(1)}) \sqcup_{\boxed{\cdot}^{n^{(1)}}(v_0^{(1)})} (X_1^{(1)})^{\otimes n} \to (X_2^{(1)})^{\otimes n}.$$

Here, as usual, $\boxed{\,\cdot\,}^{n^{(1)}} -$ denotes the domain of the $-^{\square n^{(1)}}$. The latter map is the composition of pushouts of the maps $g(w, m)$, where w and m are as above, except that now $0 \leq m^{(1)} < n^{(1)}$. Again, these are cofibrations, so the above map is a cofibration. By Lemma 2.0.3(i), $g(v, n)$ is therefore a cofibration.

For (ii), each $g(w, m)$ is a weak equivalence. The map $g(v_1, n)$ is the composition of pushouts of $g(w, m)$ along $Y \otimes_{\Sigma_n} \Sigma_n \cdot_{\Sigma_m} w^{\square m} = Y \otimes_{\Sigma_m} w^{\square m}$. The latter map (and similarly for Y') instead of Y is an h-cofibration by the symmetric h-monoidality assumption. Thus the pushouts of $g(w, m)$, the compositions of which are $g(v_1, n)$, are weak equivalences by Lemma 3.2.7 (again, replace $\otimes$ by $\otimes_{\Sigma_n}$ there). We have shown that $g(v_1, n)$ is a weak equivalence. By Lemma 2.0.3(iii), $g(v, n)$ is then also weak equivalence.

For (i), we again use that $g(v_1, n)$ is in the weak saturation of maps $g(w, m)$ and the above-mentioned stability of smallness under weak saturation.

(v) can be shown using the same argument but considering $g(v) := v^{\square n} \in \Sigma_n \mathcal{C}$ instead. By Remark 4.2.10 it is unnecessary to use multi-indices in this proof. $\qquad\square$

5. Transfer of model structures

In this section, we fix an adjunction
$$(5.0.1) \qquad\qquad F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$
such that $\mathcal{C}$ is a model category and $\mathcal{D}$ is complete and cocomplete. One can ask whether it is possible to construct a model structure on $\mathcal{D}$ from this data. The following definitions turn out to be convenient in practice.

Definition 5.0.2. A model structure on $\mathcal{D}$ is *transferred along* G if the weak equivalences and fibrations in $\mathcal{D}$ are those morphisms which are mapped by G to weak equivalences and fibrations in $\mathcal{C}$, respectively.

If a transferred model structure on $\mathcal{D}$ exists, it is unique, so we also speak of *the* transferred model structure.

5.1. **Existence and basic properties.** The existence of the transferred model structure is addressed by the following proposition. Note that the condition that G maps $F(J)$-cellular maps (i.e., transfinite compositions of pushouts of maps in $F(J)$) to weak equivalences is necessary because F is a left Quillen functor, in particular it maps J to acyclic cofibrations in $\mathcal{D}$, which are closed under cobase changes and transfinite compositions.

Proposition 5.1.1. [Hir03, Theorem 11.3.2] *Suppose that $\mathcal{C}$ is a cofibrantly generated model category and $\mathcal{D}$ is a complete and cocomplete category. Fix some sets I and J of generating cofibrations and acyclic cofibrations in $\mathcal{C}$. Suppose that the functor G maps $F(J)$-cellular maps to weak equivalences in $\mathcal{C}$. The transferred model structure on $\mathcal{D}$ exists if $F(I)$ and $F(J)$ permit the small object argument [Hir03, Definition 10.5.15]. For example, the latter condition is satisfied if $\mathcal{D}$ is locally presentable, in which case $\mathcal{D}$ is a combinatorial model category.*

The next proposition describes basic properties of transferred model structures. Part (vi) can be applied to adjunctions of the form $\mathcal{C} \rightleftarrows \mathbf{Mod}_R$, where R is a commutative monoid which is cofibrant as an object of the underlying symmetric monoidal model category $\mathcal{C}$. It is a special case of much more general left properness results by Batanin and Berger [BB13].

Proposition 5.1.2. *The following properties hold for a transferred model structure on $\mathcal{D}$. We write I (respectively J) for a class of generating (acyclic) cofibrations of $\mathcal{C}$.*

(i) *Suppose that $\mathcal{V}$ is a symmetric monoidal model category and (F, G) is a $\mathcal{V}$-enriched adjunction of $\mathcal{V}$-enriched categories that are tensored and powered over $\mathcal{V}$. If $\mathcal{C}$ is a $\mathcal{V}$-enriched model category, then so is $\mathcal{D}$.*

(ii) *The class $F(I)$ (respectively, $F(J)$) generates (acyclic) cofibrations of $\mathcal{D}$.*

(iii) *If $\mathcal{C}$ is quasi-tractable, then so is $\mathcal{D}$.*

(iv) *If $\mathcal{C}$ is combinatorial or tractable, then so is $\mathcal{D}$, provided that $\mathcal{D}$ is locally presentable.*

(v) *Suppose that G preserves filtered colimits. If $\mathcal{C}$ is pretty small, then so is $\mathcal{D}$, provided that $\mathcal{D}$ is locally presentable, or, more generally, $F(I')$ and $F(J')$ permit the small object argument, where I' and J' come from pretty smallness.*

(vi) *Suppose that G preserves pushouts along maps in $F(I)$. Also suppose that G preserves filtered colimits. Finally suppose that (a) $G(F(I))$ consists of cofibrations or (b) $\mathcal{C}$ is pretty small and $G(F(I))$ consists of h-cofibrations. Then, if $\mathcal{C}$ is left proper, so is $\mathcal{D}$.*

(vii) *If G preserves filtered colimits and sends cobase changes of $F(I)$ (respectively cobase changes of $F(I)$ along maps with cofibrant targets) to cofibrations, then G preserves cofibrations (respectively, cofibrations with cofibrant source).*

Proof. (i): By [Hov99, Lemma 4.2.2] it suffices to check that for any cofibration $j : K \to L$ in $\mathcal{V}$ and any fibration $\pi : E \to B$ in $\mathcal{D}$ the natural map
$$\zeta : E^K \to E^L \times_{B^L} B^K$$
is a fibration in $\mathcal{D}$ that is acyclic if either j or π is. The map $G(\zeta)$ is an (acyclic) fibration because G preserves fiber products and $\mathcal{V}$-powers being a $\mathcal{V}$-enriched right adjoint.

(ii): By adjunction, a morphism f in $\mathcal{D}$ has a right lifting property with respect to $F(I)$ if and only if $G(f)$ has a right lifting property with respect to I, which is true if and only if $G(f)$ is an acyclic fibration in $\mathcal{C}$, equivalently f is an acyclic fibration in $\mathcal{D}$. Likewise for acyclic cofibrations.

(iii) The domains of $F(I)$ are cofibrant because F is a left Quillen functor and the domains of I are cofibrant.

(iv): The combinatoriality of $\mathcal{D}$ is immediate from (ii).

(v): By Definition 2.0.2, there is another model structure $\mathcal{C}'$ on the underlying category of $\mathcal{C}$ with the same weak equivalences and a smaller class of cofibrations that is generated by a set of morphisms with compact domains and codomains. By assumption $F(\mathrm{C}_{\mathcal{C}'})$ permits the small object argument and similarly for acyclic cofibrations. This verifies the condition for the existence of the transfer of the model structure $\mathcal{C}'$. Thus the model structure $\mathcal{C}'$ transfers to a model structure $\mathcal{D}'$ on the category underlying $\mathcal{D}$ and its cofibrations are a

subset of cofibrations of $\mathcal{D}$. The (co)domains of the generating set of cofibrations $F(I')$ are compact because G preserves filtered colimits and therefore F preserves compact objects.

(vi): We have to show that the pushout of any weak equivalence $f_0 \colon D_0 \to E_0$ along a $\mathcal{D}$-cofibration $D_0 \to D$ is a weak equivalence. Every cofibration $D_0 \to D$ is obtained as a retract of a transfinite composition $d \colon D_0 \to D_1 \to \cdots \to D_\infty = D$, where every map $d_i \colon D_i \to D_{i+1}$ is a cobase change of a map $F(c_i)$ for some generating cofibration $c_i \in I_\mathcal{C}$. Thus for each i we have the following diagram of cocartesian squares, where the objects E_i and the morphisms $E_i \to E_{i+1}$ and $D_i \to E_i$ are constructed inductively using pushouts and colimits:

$$
\begin{array}{ccccc}
C_i & \longrightarrow & D_i & \longrightarrow & E_i \\
\downarrow{\scriptstyle F(c_i)} & & \downarrow{\scriptstyle d_i} & & \downarrow \\
C_{i+1} & \longrightarrow & D_{i+1} & \longrightarrow & E_{i+1}
\end{array}
$$

All vertical maps are cofibrations in $\mathcal{D}$. Apply G to this diagram. The left square and the big rectangle in the resulting diagram are again cocartesian by assumption, hence the right square is also cocartesian.

If the morphism $G(F(c_i))$ is an (h-)cofibration in $\mathcal{C}$, then so is its cobase change $G(d_i)$ and therefore so is their transfinite composition $G(D_0) \to G(D_\infty)$: for h-cofibrations this is Lemma 2.0.6, using the assumption that $\mathcal{C}$ is pretty small. For cofibrations this is true because cofibrations in any model category are weakly saturated. Cofibrations in a left proper model category are h-cofibrations. Thus in both cases under consideration the morphism $G(D_0) \to G(D_\infty)$ is an h-cofibration. The latter morphism is isomorphic to $G(d)$, because G preserves filtered colimits. Pushouts along h-cofibrations are homotopy pushouts and therefore preserve weak equivalences. Thus $D_\infty \to E_\infty$ is a weak equivalence, being the cobase change of the weak equivalence $D_0 \to E_0$ along the h-cofibration $D_0 \to D_\infty$.

(vii): Cofibrations in $\mathcal{D}$ are retracts of transfinite compositions of cobase changes of elements in $F(I)$. All three operations are preserved by the functor G by assumption. Thus it is sufficient to observe that $G(F(I))$ consists of cofibrations in $\mathcal{C}$, which are weakly saturated, hence G preserves cofibrations. The preservation of cofibrations with cofibrant source is shown the same way. $\qquad\square$

5.2. **Transfer of monoidal and symmetricity properties.** We now transfer monoidal properties along an adjunction of monoidal categories. We restrict to monoidal categories, as opposed to left modules, merely for notational convenience.

Proposition 5.2.1. *Let*
$$
F \colon \mathcal{C} \leftrightarrows \mathcal{D} \colon G
$$
be an adjunction between (symmetric) monoidal model categories. Suppose that the model structure on $\mathcal{D}$ is transferred from $\mathcal{C}$, respectively, and that the left adjoint F is a strong (symmetric) monoidal functor between (symmetric) monoidal categories. If $\mathcal{C}$ is a (symmetric) monoidal model category, then so is $\mathcal{D}$.

Proof. By Proposition 5.1.2(ii), to prove the pushout product axiom it is enough to verify that $F(\mathrm{C}_\mathcal{C}) \square F(\mathrm{C}_\mathcal{C}) \subset \mathrm{C}_\mathcal{C}$ and similarly with acyclic cofibrations. This uses the preservation by $\otimes_\mathcal{D}$ of colimits in both variables. Since F is strong monoidal and cocontinuous, we have $F(\mathrm{C}_\mathcal{C}) \square F(\mathrm{C}_\mathcal{C}) = F(\mathrm{C}_\mathcal{C} \square \mathrm{C}_\mathcal{C}) = F(\mathrm{C}_\mathcal{C}) \subset \mathrm{C}_\mathcal{D}$. Likewise for acyclic cofibrations. $\qquad\square$

Definition 5.2.2. A *Hopf adjunction* is an adjunction between monoidal categories such that there is a functorial isomorphism for $C \in \mathcal{C}$, $D \in \mathcal{D}$,
$$
(5.2.3) \qquad\qquad G(F(C) \otimes D) \cong C \otimes G(D).
$$

Remark 5.2.4. If the monoidal products $\otimes_\mathcal{C}$ and $\otimes_\mathcal{D}$ are closed, this is equivalent to G being strong closed, i.e., internal homs are preserved up to a coherent isomorphism.

Proposition 5.2.5. *Suppose the model structure on monoidal model category $\mathcal{D}$ is transferred along a Hopf adjunction between monoidal model categories. Also suppose that G preserves pushouts along maps of the form $D \otimes F(s)$, where $D \in \mathcal{D}$ is any object and s is any morphism in S. Let S be a class of cofibrations in $\mathcal{C}'$. We say that a property of the class S* transfers, *if the same property holds for $F(S)$.*

  (i) *Suppose $\mathcal{C}$ and $\mathcal{D}$ are left proper. Then the (acyclic) h-monoidality of S transfers. The h-monoidality of $\mathcal{C}$ transfers to $\mathcal{D}$ if $\mathcal{D}$ is pretty small.*
  (ii) *The flatness of S transfers. The flatness of $\mathcal{C}$ transfers to $\mathcal{D}$ if $\mathcal{D}$ is pretty small and h-monoidal.*
  (iii) *If G also preserves filtered colimits then the monoid axiom transfers from $\mathcal{C}$ to $\mathcal{D}$.*

Proof. (i) and (ii) are shown exactly the same way as their symmetric counterparts, see Parts (ii) and (i) of Theorem 5.2.6, using Theorem 3.2.8 instead.

(iii): The preservation of colimits under $\otimes_\mathcal{D}$ and Proposition 5.1.2(ii), the assumption that G preserves the weak saturation, the Hopf adjunction property, and the monoid axiom for $\mathcal{C}$ give inclusions

$$
G(\mathrm{cof}(\mathcal{D} \otimes \mathrm{AC}_\mathcal{D})) \subset G(\mathrm{cof}(\mathcal{D} \otimes F(\mathrm{AC}_\mathcal{C}))) \subset \mathrm{cof}(G(\mathcal{D} \otimes F(\mathrm{AC}_\mathcal{C}))
$$
$$
= \mathrm{cof}(G(\mathcal{D}) \otimes \mathrm{AC}_C) \subset \mathrm{cof}(\mathcal{C} \otimes \mathrm{AC}_\mathcal{C}) \subset \mathrm{W}_\mathcal{C}.
$$

$\square$

The following theorem shows that the three symmetricity properties interact well with transfers. It is the symmetric counterpart of Proposition 5.2.5.

Theorem 5.2.6. *Let*

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

be a Quillen adjunction of symmetric monoidal model categories such that the model structure on $\mathcal{D}$ is transferred from $\mathcal{C}$. We assume F is strong monoidal and, for parts (i), (ii), and (v) we also assume that (a) the adjunction is a Hopf adjunction; (b) G preserves pushouts along maps of the form $D \otimes F(c)$, where $D \in \mathcal{D}$ is any object and c is any morphism in $\mathcal{C}$; and (c) that G commutes with the coinvariants functor $(-)_{\Sigma_n}$ for all n.

Let S be a class of cofibrations in $\mathcal{C}$. We say that a property of the class S transfers, *if the same property holds for $F(S)$.*

(i) Symmetric flatness of S transfers. Moreover, the symmetric flatness of $\mathcal{C}$ transfers to $\mathcal{D}$ if, in addition, $\mathcal{D}$ is pretty small and symmetric h-monoidal.

(ii) Suppose $\mathcal{C}$ and $\mathcal{D}$ are left proper. Then the (acyclic) symmetric h-monoidality of S transfers. The symmetric h-monoidality of $\mathcal{C}$ transfers if, in addition, $\mathcal{D}$ is pretty small.

(iii) For some class $\mathcal{Y}$ of morphisms as in Definition 4.2.8, the $\mathcal{Y}$-symmetroidality of S transfers in the sense that $\operatorname{cof}(F(S))$ is $F(\mathcal{Y})$-symmetroidal. In particular, if $\mathcal{C}$ is $\mathcal{Y}$-symmetroidal, then $\mathcal{D}$ is $\operatorname{cof}(F(\mathcal{Y}))$-symmetroidal.

(iv) Then the property of being freely powered transfers. In particular, if $\mathcal{C}$ is freely powered, then so is $\mathcal{D}$.

(v) Suppose G preserves filtered colimits. If $\mathcal{C}$ is admissibly generated, then so is $\mathcal{D}$.

Proof. For all properties, the transfer for the given class S is proven using a specific argument. The transfer of the property from $\mathcal{C}$ to $\mathcal{D}$ follows from the fact that $F(\mathrm{C}_{\mathcal{C}})$ generates the cofibrations of $\mathcal{D}$ (Proposition 5.1.2(ii)), and likewise for acyclic cofibrations. Then, a weak saturation property (indicated below) is used. Let $s \in S$ be any map.

(i): For any weak equivalence y in $\Sigma_n \mathcal{D}$ we have to show that $y \square_{\Sigma_n} F(s)^{\square n}$ is a weak equivalence. Indeed, $G(y \square_{\Sigma_n} F(s)^{\square n})$ is isomorphic to $G(y) \square_{\Sigma_n} F(s^{\square n})$ by the Hopf adjunction property, the cocontinuity of G, and the strong monoidality of F which ensures that F commutes with pushout products (Proposition 3.1.4). This is a weak equivalence since $\mathcal{C}$ is symmetric flat. The symmetric flatness of $\mathcal{C}$ transfers by Theorem 4.3.8(ii), using $S = I_{\mathcal{C}}$.

(ii): We need to show that $Y \otimes_{\Sigma_n} F(s)^{\square n} = Y \otimes_{\Sigma_n} F(s^{\square n})$ is an h-cofibration for all $Y \in \Sigma_n \mathcal{D}$. By Lemma 2.0.7, this is true since $G(Y \otimes_{\Sigma_n} F(s^{\square n})) = G(Y) \otimes_{\Sigma_n} s^{\square n}$ is an (acyclic) h-cofibration by the (acyclic) symmetric h-monoidality of S. The symmetric h-monoidality of $\mathcal{C}$ transfers to $\mathcal{D}$ by Theorem 4.3.8(iii).

(iii): As F is strongly monoidal and cocontinuous, $F(y) \square_{\Sigma_n} F(s^{\square n}) = F(y \square_{\Sigma_n} s^{\square n})$. This shows the $F(\mathcal{Y})$-symmetroidality since F preserves cofibrations and acyclic cofibrations. Then apply Lemma 4.3.2. The claim about the symmetroidality of $\mathcal{D}$ follows from Theorem 4.3.8(iv).

(iv): Replace $y \square_{\Sigma_n} s^{\square n}$ by $s^{\square n}$ in (iii) and use Theorem 4.3.8(v).

(v): The cofibrant generation transfers to $\mathcal{D}$ by Proposition 5.1.2(ii). By Lemma 4.2.6 and Theorem 4.3.8(i), we only have to show that $(\mathrm{co})\mathrm{dom}(F(I))$ are small with respect to $\mathrm{cell}(Y \otimes_{\Sigma_n} s^{\square n})$, where $s = F(t)$ are finite families of *generating* cofibrations, i.e., t are cofibrations in $\mathcal{C}$. By adjunction, this is equivalent to $(\mathrm{co})\mathrm{dom}(I)$ being small with respect to

$$G(\mathrm{cell}(Y \otimes_{\Sigma_n} F(t)^{\square n})) \subset \mathrm{cell}(G(Y \otimes_{\Sigma_n} F(t)^{\square n})) = \mathrm{cell}(G(Y) \otimes_{\Sigma_n} t^{\square n})$$

which holds by assumption. $\square$

Remark 5.2.7. If $\mathcal{C}$ is symmetroidal (i.e., symmetroidal with respect to the injective cofibrations in $\Sigma_n \mathcal{C}$), $\mathcal{D}$ need not be symmetroidal: for example, for $\mathcal{C} = \mathbf{sSet}$ and $\mathcal{D} = \mathbf{Mod}_R(\mathbf{sSet})$ with $R = \mathbf{Z}/4$, i.e., simplicial sets with an action of $\mathbf{Z}/4$. In this case, R has a $\mathbf{Z}/2$-action, so R is injectively cofibrant in $\Sigma_2 \mathbf{Mod}_R$, but $R \otimes_{R, \Sigma_2} R^{\otimes_R 2} = R/2$ is not cofibrant as an R-module.

5.3. **Modules over a commutative monoid.** In this section we apply the criteria developed above to the case of the category of modules over a commutative monoid R in a symmetric monoidal model category $\mathcal{C}$. An example of this situation occurs in the construction of unstable model structures on symmetric spectra, which are by definition modules over a commutative monoid in symmetric sequences [HSS00, Theorem 5.1.2].

As R is commutative, the category $\mathbf{Mod}_R$ of R-modules has a symmetric monoidal structure:

$$X \otimes_R Y := \operatorname{coeq}(X \otimes R \otimes Y \rightrightarrows X \otimes Y).$$

The free-forgetful adjunction

$$F = R \otimes - : \mathcal{C} \leftrightarrows \mathbf{Mod}_R : U$$

has the following properties: $R \otimes -$ is strong monoidal since $(R \otimes X) \otimes_R (R \otimes Y) \cong R \otimes (X \otimes Y)$. Moreover, it is a Hopf adjunction: $(R \otimes C) \otimes_R D \cong C \otimes D$. Finally, U also has a right adjoint, the internal hom functor $\operatorname{Hom}(R, -)$ (also known as the cofree R-module functor). In particular, U is cocontinuous.

The following theorem summarizes the properties of the transferred model structure on $\mathbf{Mod}_R$. The existence of the model structure is due to Schwede and Shipley [SS00, Theorem 4.1(2)]. As in Theorem 5.2.6, we say that some model-theoretic property *transfers* if it holds for $\mathbf{Mod}_R$, provided that it does for $\mathcal{C}$. The transfer of left properness to $\mathbf{Mod}_R$ (and much more general algebraic structures) was established by Batanin and Berger under the assumption that $\mathcal{C}$ is strongly h-monoidal [BB13, Theorems 2.11, 3.1b]. The transfer of symmetric flatness, symmetric h-monoidality and symmetroidality is new.

Theorem 5.3.1. *Suppose $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category that satisfies the monoid axiom and R is a commutative monoid in $\mathcal{C}$. The transferred model structure on $\mathbf{Mod}_R$ exists and is a cofibrantly generated symmetric monoidal model category.*

Combinatoriality, (quasi)tractability, admissible generation, pretty smallness, $\mathcal{V}$-enrichedness, and the property of being freely powered transfer from $\mathcal{C}$ to $\mathbf{Mod}_R$. Moreover, if $\mathcal{C}$ is symmetroidal with respect to some class $\mathcal{Y}$ (Definition 4.2.8), then $\mathbf{Mod}_R$ is symmetroidal with respect to $\mathrm{cof}(R \otimes \mathcal{Y})$, the weak saturation of maps of free R-module maps generated by all $y \in \mathcal{Y}$.

If either R is a cofibrant object in $\mathcal{C}$ or if $\mathcal{C}$ is pretty small and h-monoidal, then left properness transfers.

If $\mathcal{C}$ is pretty small and h-monoidal, then flatness, symmetric flatness, h-monoidality, symmetric h-monoidality, and the monoid axiom transfer from $\mathcal{C}$ to $\mathbf{Mod}_R$.

Proof. The existence of the transferred model structure follows from Proposition 5.1.1 after we observe that $F(J) = R \otimes J$ and the class of $F(J)$-cellular maps consists of weak equivalences by the monoid axiom. It is symmetric monoidal by Proposition 5.2.1. The transfer of combinatoriality, (quasi)tractability, pretty smallness, enrichedness, and left properness were established in Proposition 5.1.2. The transfer of flatness, h-monoidality, and the monoid axiom is shown in Proposition 5.2.5, while their symmetric counterparts are treated in Theorem 5.2.6. □

6. Left Bousfield localization

Left Bousfield localizations of various types (e.g., ordinary, enriched, monoidal) of model categories present reflective localizations of the corresponding locally presentable ∞-categories, i.e., they invert the reflective saturation of a given class of maps in a (homotopy) universal fashion. If the Bousfield localization of a given model category exists, it can be constructed as a model structure on the same underlying category, with a larger class of weak equivalences and the same class of cofibrations. Examples for left Bousfield localizations abound, e.g., local model structures on simplicial presheaves (see Section 7) and the stable model structure on symmetric spectra are left Bousfield localizations. (Right Bousfield localizations, which preserve fibrations and present coreflective localizations, are somewhat more rare.)

6.1. **Existence and basic properties.** Consider the following bicategories (specified by their objects, 1-morphisms, and 2-morphisms):

- model categories, left Quillen functors, and natural transformations;
- $\mathcal{V}$-enriched model categories, $\mathcal{V}$-enriched left Quillen functors, and $\mathcal{V}$-enriched natural transformations ($\mathcal{V}$ is a symmetric monoidal model category);
- (symmetric) monoidal model categories, strong (symmetric) monoidal left Quillen functors, and (symmetric) monoidal natural transformations;
- same as above, but $\mathcal{V}$-enriched.

There are obvious forgetful functors that discard enrichments or monoidal structures.

Definition 6.1.1. Fix one of the bicategories W defined above. Suppose $\mathcal{C} \in W$ and S is a class of morphisms in $\mathcal{C}$. A *left Bousfield localization of $\mathcal{C}$ with respect to S* is a 1-morphism $j\colon \mathcal{C} \to \mathrm{L}_S\mathcal{C}$ such that precomposition with j induces an equality between the category of morphisms $\mathrm{L}_S\mathcal{C} \to \mathcal{E}$ (note these are in particular left Quillen functors) and the category of morphisms $\mathcal{C} \to \mathcal{E}$ whose left derived functors send elements of S to weak equivalences in $\mathcal{E}$.

In the case when objects of W are monoidal, we use the notation $\mathrm{L}^\otimes$ instead of L to remind the reader of this fact. The above definition can be located in the ordinary case in [Bar10, Definition 4.2] or [Hir03, Theorem 3.3.19], in the enriched case in [Bar10, Definition 4.42] (which also implicitly contains the unenriched monoidal case because any symmetric monoidal model category is enriched over itself), and in the enriched monoidal case implicitly in [Bar10, Proposition 4.47]. Gorchinskiy and Guletskiĭ [GG09, Lemma 26] give an explicit formula for the underlying model category of a monoidal Bousfield localization. The term "monoidal Bousfield localization" is due to White [Whi14b], who also gives an exposition of the existence of monoidal Bousfield localizations.

Remark 6.1.2. The above definition talks about equality of categories to ensure that the underlying category of a left Bousfield localization does not change. One can replace equality with isomorphism or equivalence, which would yield an isomorphic or equivalent underlying category.

Proposition 6.1.3. *Fix one of the bicategories W defined above. Suppose $\mathcal{C} \in W$ and S is a set (as opposed to a proper class) of morphisms in $\mathcal{C}$. Suppose furthermore that $\mathcal{C}$ is left proper and combinatorial (or cellular). If objects of W are $\mathcal{V}$-enriched or monoidal, assume that $\mathcal{V}$ and $\mathcal{C}$ are quasi-tractable. Then the left Bousfield localization $\mathrm{L}_S\mathcal{C}$ exists and is left proper and combinatorial (or cellular).*

  (i) *If $\mathcal{C}$ is tractable or pretty small, then so is $\mathrm{L}_S\mathcal{C}$.*

  (ii) *If $U\colon W \to W'$ is the forgetful functor that discards $\mathcal{V}$-enrichments, then $U(\mathrm{L}_S\mathcal{C}) = \mathrm{L}_{S_\mathcal{V}}U(\mathcal{C})$, where $S_\mathcal{V}$ is the $\mathcal{V}$-enriched saturation of S, which consists of the derived tensor products of the elements of S and the objects of $\mathcal{V}$ (or some class of homotopy generators of $\mathcal{V}$, e.g., the set of domains and codomains of some set of generating cofibrations of $\mathcal{V}$).*

  (iii) *If $U\colon W \to W'$ is the forgetful functor that discards monoidal structures, then $U(\mathrm{L}_S\mathcal{C}) = \mathrm{L}_{S^\otimes}U(\mathcal{C})$, where $S^\otimes$ is the monoidal saturation of S, which consists of the derived monoidal products of the elements of S and the objects of $\mathcal{C}$ (or some class of homotopy generators of $\mathcal{C}$, e.g., the set of domains and codomains of some set of generating cofibrations of $\mathcal{C}$).*

Proof. The ordinary localization exists by [Bar10, Theorem 4.7] (combinatorial case) and [Hir03, Theorem 4.1.1] (cellular case). The original proof is due to Smith and tractability is due to Hovey [Hov04, Proposition 4.3]. In the enriched case, existence and the statement about the underlying model category is proved in [Bar10, Theorem 4.46]. This also covers the unenriched monoidal case, because every symmetric monoidal model category is enriched over itself. For the enriched monoidal case, see [Bar10, Proposition 4.47]. Barwick's proofs also work for the cellular case, under the assumption of quasi-tractability.

By the formulas for enriched and monoidal localizations, it is enough to show the pretty smallness statement for the ordinary localization $\mathcal{D} = \mathrm{L}_S\mathcal{C}$. Consider the localization $\mathcal{D}' := \mathrm{L}_S\mathcal{C}'$, where $\mathcal{C}'$ is the second model structure on $\mathcal{C}$ (Definition 2.0.2). We have $\mathrm{W}_{\mathcal{D}'} = \mathrm{W}_{\mathcal{D}}$ because both S-local objects and S-local weak equivalences only depend on S and weak equivalences. Thus $\mathcal{D}$ is pretty small. $\qquad\square$

Remark 6.1.4. Any left Bousfield localization of an **sSet**-enriched model category is automatically **sSet**-enriched [Hir03, Theorem 4.1.1(4)].

Remark 6.1.5. If $\mathcal{C}$ is $\mathcal{V}$-enriched and monoidal and both $\mathcal{C}$ and $\mathcal{V}$ are quasi-tractable, then monoidal localizations and $\mathcal{V}$-enriched monoidal localizations agree: to show this we may replace the maps in S by weakly equivalent maps that are cofibrations with cofibrant source. Then the maps in $S^\otimes = S \otimes (\mathrm{co})\mathrm{dom}(I_\mathcal{C})$ are weakly equivalent to $S \otimes (\mathrm{co})\mathrm{dom}(I_\mathcal{C}) \otimes \mathrm{Q}(1_\mathcal{V})$ by the unit axiom of the $\mathcal{V}$-enrichment. The latter class is contained in $S_\mathcal{V}^\otimes$. Vice versa, $S_\mathcal{V}^\otimes = S \otimes (\mathrm{co})\mathrm{dom}(I_\mathcal{V}) \otimes (\mathrm{co})\mathrm{dom}(I_\mathcal{C})$ is contained in $S \otimes (\mathrm{co})\mathrm{dom}(I_\mathcal{C})$ since $\otimes\colon \mathcal{V} \times \mathcal{C} \to \mathcal{V}$ is a left Quillen bifunctor.

The standard description of fibrant objects and adjunctions of Bousfield localizations admit the following variants for monoidal localizations.

Lemma 6.1.6. *If $\mathcal{D}$ is the monoidal left Bousfield localization $\mathrm{L}_{S^\otimes}\mathcal{C}$ of a monoidal model category $\mathcal{C}$, then fibrant objects in $\mathcal{D}$ are those fibrant objects W in $\mathcal{C}$ such that the derived internal Hom,*

$$\underline{\mathrm{R}}\mathrm{Hom}_\mathcal{C}(\xi, W)$$

is a weak equivalence in $\mathcal{C}$ for any $\xi \in S$.

Proof. By [Hir03, Proposition 3.4.1], fibrant objects in $\mathcal{D}$ are those fibrant objects of $\mathcal{C}$ such that the derived mapping space $\underline{\mathrm{R}}\mathrm{Map}_\mathcal{C}(\mathrm{CO}_\mathcal{C} \otimes^\mathrm{L} \xi, W)$ or, equivalently, $\underline{\mathrm{R}}\mathrm{Map}_\mathcal{C}(\mathrm{CO}_\mathcal{C}, \underline{\mathrm{R}}\mathrm{Hom}(\xi, W))$ is a weak equivalence for any $\xi \in S$. The objects $\mathrm{CO}_\mathcal{C}$ are homotopy generators of $\mathcal{C}$, so this is equivalent to $\underline{\mathrm{R}}\mathrm{Hom}(\xi, W)$ being a weak equivalence [Hov01, Proposition 3.2]. $\qquad\square$

Lemma 6.1.7. *If $F\colon \mathcal{C} \leftrightarrows \mathcal{C}' : G$ is a Quillen adjunction of monoidal model categories such that F is strong monoidal, then there is a Quillen adjunction*

$$F \colon \mathcal{D} := \mathrm{L}_{S^\otimes}\mathcal{C} \leftrightarrows \mathcal{D}' := \mathrm{L}_{\underline{\mathrm{L}}F(S)^\otimes}\mathcal{C}' : G,$$

(assuming the left Bousfield localizations exist), which is a Quillen equivalence if $\mathcal{C} \leftrightarrows \mathcal{C}'$ is one.

Proof. The class $F(\mathrm{CO}_\mathcal{C})$ is a class of homotopy generators of $\mathcal{C}'$. Hence $\mathcal{D}'$ can be computed as the (nonmonoidal) localization with respect to the class $F(\mathrm{CO}_\mathcal{C}) \otimes^\mathrm{L} \underline{\mathrm{L}}F(S) = F(\mathrm{CO}_\mathcal{C} \otimes^\mathrm{L} S)$. Thus, by [Hir03, Proposition 3.3.18, Theorem 3.3.20], the left Quillen functor $\mathcal{C} \to \mathcal{C}' \to \mathcal{D}'$ factors over a left Quillen functor $\mathcal{D} \to \mathcal{D}'$ since $\underline{\mathrm{L}}F(\mathrm{CO}_\mathcal{C} \otimes^\mathrm{L} S)$ consists of weak equivalences in $\mathcal{D}'$. Moreover, $\mathcal{D} \leftrightarrows \mathcal{D}'$ is a Quillen equivalence if $\mathcal{C} \leftrightarrows \mathcal{C}'$ is one. $\qquad\square$

6.2. **Localization of monoidal and symmetricity properties.** Here is a tool to transport h-monoidality and flatness along a Bousfield localization. An example application in the context of symmetric spectra is given in [PS14b, Subsection 3.3]. The idea of combining h-monoidality and flatness was is independently used by White [Whi14b].

Proposition 6.2.1. *Suppose $\mathcal{V}$ is a symmetric monoidal model category, $\mathcal{C}$ is a $\mathcal{V}$-enriched monoidal model category such that the monoidal left Bousfield localization $\mathcal{D} := \mathrm{L}_T^{\otimes}\mathcal{C}$ with respect to some class T exists. We say that a property of a class S of cofibrations in $\mathcal{C}$ localizes if it holds for S regarded as a class of cofibrations in $\mathcal{D}$. Likewise, we say that some property of $\mathcal{C}$ localizes, if it also holds for $\mathcal{D}$.*

(i) Flatness of S localizes. In particular, the flatness of $\mathcal{C}$ localizes.

(ii) If $\mathcal{C}$ and $\mathcal{D}$ are left proper, any (acyclic) h-cofibration f in $\mathcal{C}$ is also an (acyclic) h-cofibration in $\mathcal{D}$.

(iii) If $\mathcal{C}$ is left proper and $\mathcal{D}$ is left proper, quasi-tractable, pretty small, and flat, then the h-monoidality of S or of $\mathcal{C}$ localizes.

(iv) If $\mathcal{D}$ is pretty small and h-monoidal (which holds, for example, if $\mathcal{C}$ is left proper, pretty small, h-monoidal, and flat), then $\mathcal{D}$ also satisfies the monoid axiom.

Proof. (i): We have to show that $y \square s$ is a weak equivalence in $\mathcal{D}$ for all weak equivalences y in $\mathcal{D}$ and $s \in S$. By the pushout product axiom (of $\mathcal{D}$), we may assume y is a trivial fibration in $\mathcal{D}$ or, equivalently, one in $\mathcal{C}$. Now invoke the flatness of S in $\mathcal{C}$ and use $\mathrm{W}_{\mathcal{C}} \subset \mathrm{W}_{\mathcal{D}}$.

(ii): The acyclic part follows from the nonacyclic one and the inclusion $\mathrm{W}_{\mathcal{C}} \subset \mathrm{W}_{\mathcal{D}}$. Given a diagram $A \leftarrow B \xrightarrow{f} C$, where f is an h-cofibration in $\mathcal{C}$, we have to show by Lemma 2.0.6(i) that $C \sqcup_B A$ is a homotopy pushout in $\mathcal{D}$. The identity functor $\mathbf{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C}) \rightarrow \mathbf{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{D})$ is a left Quillen functor if we equip both functor categories with the projective model structure. Since it also preserves all weak equivalences, it preserves homotopy colimits, i.e., sends the homotopy pushout $C \sqcup_B A \sim C \sqcup_B^{h,\mathcal{C}} A$ to a homotopy pushout in $\mathcal{D}$.

(iii): As the cofibrations in $\mathcal{C}$ and $\mathcal{D}$ are the same, the nonacyclic part of the h-monoidality of $\mathcal{D}$ follows from (ii). Acyclic h-cofibrations are weakly saturated by Lemma 2.0.6(iii). Therefore, it is enough to show $f \otimes X \in \mathrm{W}_{\mathcal{D}}$ for any $f \colon Y \to Z \in J_{\mathcal{D}}$ and any object X. The quasi-tractability of $\mathcal{D}$ (Proposition 6.1.3) allows us to assume that Y (hence Z) is cofibrant. Writing $\mathrm{Q}(-)$ for the cofibrant replacement (equivalently in $\mathcal{C}$ or $\mathcal{D}$) we see that $X \otimes f$ is a weak equivalence since $\mathrm{Q}(X) \otimes f$ is one (by the pushout product axiom for $\mathcal{D}$) and $q \otimes Y$ and $q \otimes Z$ are weak equivalences in $\mathcal{D}$ (by flatness).

(iv): Apply Lemma 3.2.6 to $\mathcal{D}$. $\qquad\square$

The following proposition provides a method to transport the symmetricity notions to a Bousfield localization. It is the symmetric counterpart of Proposition 6.1.3.

Theorem 6.2.2. *Suppose $\mathcal{V}$ is a symmetric monoidal model category, $\mathcal{C}$ is a $\mathcal{V}$-enriched symmetric monoidal model category such that the $\mathcal{V}$-enriched symmetric monoidal left Bousfield localization $\mathcal{D} := \mathrm{L}_T^{\otimes}\mathcal{C}$ with respect to some class T of morphisms exists.*

We say that a property of a class S of cofibrations in $\mathcal{C}$ localizes if it holds for S regarded as a class of cofibrations in $\mathcal{D}$. Likewise, we say that some property of $\mathcal{C}$ localizes, if it also holds for $\mathcal{D}$.

(i) Let $\mathcal{Y} = (\mathcal{Y}_n)$ be some classes of morphisms in $\Sigma_n\mathcal{C}$. The property of S of being symmetric flat with respect to $\mathcal{Y}$ localizes. In particular, the symmetric flatness of S and of $\mathcal{C}$ localizes.

(ii) If $\mathcal{C}$ is left proper and $\mathcal{D}$ is left proper, quasi-tractable, pretty small and symmetric flat, then the symmetric h-monoidality of S or of $\mathcal{C}$ localizes.

(iii) The property of S of being (acyclic) $\mathcal{Y}$-symmetroidal localizes provided that $\mathcal{D}$ is flat and h-monoidal and provided that S consists of cofibrations with cofibrant source and is symmetric flat in $\mathcal{D}$. In particular if $\mathcal{D}$ is h-monoidal and symmetric flat and $\mathcal{C}$ is $\mathcal{Y}$-symmetroidal then $\mathcal{D}$ is also $\mathcal{Y}$-symmetroidal.

(iv) The property of being freely powered localizes.

(v) Suppose $\mathcal{D}$ is quasi-tractable. Then the property of being admissibly generated localizes.

Proof. (i): The $\mathcal{Y}$-symmetric flatness of S states that $y \square_{\Sigma_n} s^{\square n}$ is a weak equivalence in $\mathcal{C}$ for all $y \in \mathcal{Y}_n$ and $s \in S$. Since weak equivalences of $\mathcal{C}$ are contained in the ones of $\mathcal{D}$ this property obviously localizes. The additional claims concern the symmetric flatness of S (or the class of all cofibrations on $\mathcal{C}$) with respect to $\mathrm{W}_{\Sigma_n^{\mathrm{pro}}\mathcal{D}}$. By Lemma 4.3.1, this is equivalent to symmetric flatness with respect to $\mathrm{AF}_{\Sigma_n^{\mathrm{pro}}\mathcal{D}} = \mathrm{AF}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}}$ which holds since S is symmetric flat with respect to $\mathrm{W}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}}$ by assumption.

(ii): As (acyclic) h-cofibrations of $\mathcal{C}$ are contained in the ones of $\mathcal{D}$ (Proposition 6.2.1(ii)), a class S which is (acyclic) symmetric h-monoidal in $\mathcal{C}$ is also (acyclic) symmetric h-monoidal in $\mathcal{D}$.

Now suppose that $\mathcal{C}$ is symmetric h-monoidal. We want to show that (acyclic) $\mathcal{D}$-cofibrations form an (acyclic) symmetric h-monoidal class (in $\mathcal{D}$). Again using the above fact, it is enough to show the acyclic part. Once again, we may restrict to generating acyclic cofibrations (4.3.8(iii)). Thus, let s be a finite family of generating acyclic cofibrations in $\mathcal{D}$. By quasi-tractability, we may assume they have cofibrant domains. Setting $y \colon \emptyset \to Y$, the pushout product $y \square s^{\square n}$ is just $Y \otimes s^{\square n}$, which is a weak equivalence by the h-monoidality of $\mathcal{D}$ ensured by Proposition 6.2.1(iii). Using the flatness and h-monoidality of $\mathcal{D}$ (Proposition 6.2.1(i), (iii)), Lemma 4.3.4 applies to s and y and shows that $Y \otimes_{\Sigma_n} s^{\square n}$ is a weak equivalence.

(iii): The stability of the nonacyclic part of $\mathcal{Y}$-symmetroidality is obvious. The acyclic part follows from Proposition 4.3.5, using the cofibrancy assumption and the symmetric flatness of S in $\mathcal{D}$. Similarly, by 4.3.8(iv),

the symmetroidality of $\mathcal{D}$ follows by using a set S of generating acyclic cofibrations (of $\mathcal{D}$) with cofibrant domain, which is possible thanks to the tractability of $\mathcal{D}$.

(iv): This follows from Proposition 4.3.5.

(v): This is clear since $\mathrm{C}_{\mathcal{C}} = \mathrm{C}_{\mathcal{D}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

7. Examples of model categories

We discuss the model-theoretic properties of Section 2, Subsection 3.2, and Section 4 for simplicial sets, simplicial presheaves, simplicial modules, topological spaces, chain complexes, and symmetric spectra.

7.1. **Simplicial sets.** The most basic example of a monoidal model category is the category **sSet** of simplicial sets equipped with the cartesian monoidal structure $A \otimes B = A \times B$ and the Quillen model structure, see, e.g., [GJ99, Theorem I.11.3]. All objects are cofibrant, so **sSet** is left proper, flat, and h-monoidal by Lemma 3.2.4.

Simplicial sets are symmetroidal: given any monomorphism $y \in \Sigma_n\mathbf{sSet}$ and a finite family of monomorphisms $v \in \mathbf{sSet}$, $y\square_{\Sigma_n} v^{\square n}$ is a monomorphism. Indeed, $y\square v^{\square n}$ is a Σ_n-equivariant monomorphism and passing to Σ_n-orbits preserves monomorphisms. By Theorem 4.3.8(iv), the acyclic part of symmetroidality follows if $y\square_{\Sigma_n} v^{\square n}$ is a weak equivalence for any y as above and any finite family of horn inclusions $v \colon \Lambda_k^m \to \Delta^m$ (where m and k are multiindices). To this end we first construct a homotopy $h \colon \Lambda \times \Delta^m \to \Delta^m$ from the identity map $\Delta^m \to \Delta^m$ to the composition $\Delta^m \longrightarrow \Delta^0 \overset{k}{\longrightarrow} \Delta^m$ such that $\Lambda_k^m \subset \Delta^m$ is preserved by the homotopy. Here Λ is the 2-horn, which can be depicted as $0 \to 1 \leftarrow 2$. We parametrize h by Λ and not by the usual Δ^1 since Δ^m is not fibrant. The map h is uniquely specified by its value on vertices, i.e., $\{0,1,2\} \times \{0,\ldots,m\} \to \{0,\ldots,m\}$. We have $(0,i) \mapsto i$, $(1,i) \mapsto \max(k,i)$, $(2,i) \mapsto k$. Thus we have constructed a simplicial deformation retraction $\Lambda \times (\Lambda_k^m \to \Delta^m) \to (\Lambda_k^m \to \Delta^m)$ that contracts the inclusion $\Lambda_k^m \to \Delta^m$ to the identity map $\Delta^0 \to \Delta^0$. (Morphisms of maps are commutative squares, as usual.) The map h gives rise to a simplicial deformation retraction

$$\Lambda \times (y\,\square_{\Sigma_n}\, v^{\square n}) \overset{\Delta}{\to} (\Lambda^{\times n} \times (y\,\square\, v^{\square n}))_{\Sigma_n} \cong y\,\square_{\Sigma_n}\, (\Lambda \times v)^{\square n} \overset{h}{\to} y\,\square_{\Sigma_n}\, v^{\square n}$$

using the fact that the diagonal $\Delta \colon \Lambda \to \Lambda^{\times n}$ is Σ_n-equivariant. It contracts the map $y\,\square_{\Sigma_n}\, v^{\square n}$ to the map $y\,\square_{\Sigma_n}\, (\mathrm{id}_{\Delta_0})^{\square n}$. For $n > 0$ the latter map is the identity map on the codomain of y, in particular, a weak equivalence, hence so is $y\,\square_{\Sigma_n}\, v^{\square n}$.

Symmetroidality and cofibrancy of all objects implies that **sSet** is symmetric h-monoidal.

The category **sSet** is far from freely powered: the map $(\partial\Delta^1 \to \Delta^1)^{\square 2}$ is not a Σ_2-projective cofibration, since Σ_2 does not act freely on the complement of the image.

Simplicial sets are not symmetric flat: $\mathrm{E}\Sigma_n \to *$ is Σ_n-equivariant and a weak equivalence of the underlying simplicial sets, but $\mathrm{B}\Sigma_n := (\mathrm{E}\Sigma_n)_{\Sigma_n} \to *$ is not a weak equivalence: recall that $\mathrm{B}\Sigma_2$ is weakly equivalent to $\mathbf{RP}^\infty$, the infinite real projective space.

Similar statements hold for pointed simplicial sets equipped with the smash product.

The category **sSet** also carries the Joyal model structure [Lur09, Theorem 2.2.5.1]. It is an interesting question whether it is symmetric h-monoidal.

7.2. **Simplicial presheaves.** A more general example than simplicial sets is the category

$$\mathbf{sPSh}(S) = \mathbf{Fun}(S^{\mathrm{op}}, \mathbf{sSet})$$

of simplicial presheaves on some site S. The *projective model structure* on this category is transferred from the Quillen model structure on **sSet** along

$$(7.2.1) \qquad\qquad\qquad\qquad \prod_{X \in S} \mathbf{sSet} \leftrightarrows \mathbf{sPSh}(S).$$

It is pretty small by 5.1.2(v) and left proper by 5.1.2(vi). The monoid axiom, h-monoidality, flatness, and symmetric h-monoidality follow from the corresponding properties of the injective model structure by Lemma 3.2.5. Alternatively, even though (7.2.1) is not a Hopf adjunction, the arguments of Proposition 5.2.5 can be generalized to (7.2.1). The projective model structure is not in general symmetroidal (for $X \in S$, $(X^n)_{\Sigma_n}$ is in general not projectively cofibrant).

In the *injective model structure* on $\mathbf{sPSh}(S)$, weak equivalences and cofibrations are checked pointwise. It is combinatorial [Lur09, Proposition A.2.8.2] and therefore tractable. It is pretty small (as the second model structure in Definition 2.0.2, take the projective structure), left proper, h-monoidal and flat (Lemma 3.2.4). The symmetric monoidality, symmetric h-monoidality and symmetroidality (with respect to injective cofibrations $\mathcal{Y}_n = \mathrm{C}_{\Sigma_n^{\mathrm{in}}\mathbf{sPSh}(S)}$) follows from the one of **sSet**.

There are various *intermediate model structures* on $\mathbf{sPSh}(S)$, such as Isaksen's flasque model structure [Isa05]. They also have pointwise weak equivalences but other choices of cofibrations which lie between projective and injective cofibrations. For such intermediate model structures, monoidality, h-monoidality, symmetric h-monoidality, symmetroidality, the monoid axiom, and flatness follow from the injective case and pretty smallness follows from the projective case.

The properties mentioned above are stable under *Bousfield localization*. For example, given some Grothendieck topology τ on the site S, the τ-local projective model structure is the left Bousfield localization of the projective model structure with respect to τ-hypercovers [DHI04, Theorem 6.2]. Since hypercovers are stable under product with any $X \in S$ by [DHI04, Proposition 3.1], this is a monoidal localization. It is also **sSet**-enriched by Remark 6.1.4. By Proposition 6.2.1, the localized model structure is again left proper, tractable, monoidal and h-monoidal, pretty small, flat, and satisfies the monoid axiom. It is symmetric h-monoidal at least if τ has enough points, for in this case local weak equivalences are maps which are stalkwise weak equivalences [Jar87, page 39].

7.3. **Simplicial modules.** Let R be a commutative simplicial ring and consider the transferred model structure on simplicial R-modules via the free-forgetful adjunction

$$(7.3.1) \qquad\qquad R[-] : \mathbf{sSet} \rightleftarrows \mathbf{sMod}_R : U.$$

The model category $\mathbf{sMod}_R$ is pretty small by Proposition 5.1.2. As for chain complexes, $\mathbf{sMod}_R$ is flat, but not symmetric flat (unless R is a rational algebra).

Simplicial R-modules are symmetric h-monoidal. The nonacyclic part follows from the fact that monomorphisms, i.e., *injective cofibrations*, of simplicial R-modules are h-cofibrations.

We reduce the acyclic part of symmetric h-monoidality of $\mathbf{sMod}_R$ to the one of $\mathbf{sSet}$ using the cocontinuous strong monoidal functor $R[-] : (\mathbf{sSet}, \times) \to (\mathbf{sMod}_R, \otimes)$, which preserves weak equivalences. Given any object $Y \in \Sigma_n \mathbf{sMod}_R$ and any finite family w of generating cofibrations of $\mathbf{sMod}_R$, i.e., $w = R[v]$, we have a deformation retraction

$$R[\Lambda] \otimes (Y \otimes_{\Sigma_n} R[v]^{\square n}) \xrightarrow{R[\Delta]} (R[\Lambda]^{\otimes n} \otimes Y \otimes_{\Sigma_n} R[v]^{\square n})_{\Sigma_n} \cong Y \otimes_{\Sigma_n} (R[\Lambda \times v])^{\square n} \xrightarrow{R[h]} Y \otimes_{\Sigma_n} R[v]^{\square n}$$

of $Y \otimes_{\Sigma_n} w^{\square n}$ to a weak equivalence, which shows that the former is also a weak equivalence.

Simplicial R-modules are symmetroidal with respect to the class $\mathcal{Y} = (\mathcal{Y}_n) = (R[\mathrm{C}_{\Sigma_n^{\mathrm{in}} \mathbf{sSet}}])$, which follows immediately from the symmetroidality of simplicial sets and cocontinuity and strong monoidality of $R[-]$. Note that $\mathbf{sMod}_R$ is not symmetroidal, as can be shown as in Remark 5.2.7.

7.4. **Chain complexes.** The category $\mathbf{Ch}(\mathbf{Mod}_R)$ of unbounded chain complexes of R-modules, for some commutative ring R, carries the *projective model structure* whose weak equivalences are the quasiisomorphisms and fibrations are the degreewise epimorphisms. It is enriched over $\mathbf{Ch}(\mathbf{Mod_Z})$ (equipped with the projective model structure). The generating (acyclic) cofibrations are given by all shifts of the canonical inclusion $[0 \to R] \to [R \xrightarrow{\mathrm{id}} R]$ ($[0 \to 0] \to [R \xrightarrow{\mathrm{id}} R]$, respectively) [Hov99, Definition 2.3.3, Theorem 2.3.11]. In particular, the model structure is tractable and pretty small. It is flat, as can be seen using Theorem 3.2.8(ii). The category is h-monoidal by [BB13, Corollary 1.14].

It is *not* symmetric flat, for the same reason as $\mathbf{sSet}$ above. Moreover, it is neither symmetric h-monoidal nor symmetroidal: for the chain complex $A = [\mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z}]$ in degrees 1 and 0, we have

$$A^{\otimes 2} = [\mathbf{Z} \xrightarrow{(1,-1)} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{+} \mathbf{Z}],$$

where from left to right we have the sign representation, the regular and the trivial representation of Σ_2. However, $(A^{\otimes 2})_{\Sigma_2} = [\mathbf{Z}/2 \longrightarrow \mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z}]$ is not exact nor cofibrant.

By the Dold-Kan correspondence $N : (\mathbf{sMod}_R, \times) \rightleftarrows (\mathbf{Ch}_R^+, \otimes)$ between simplicial R-modules and connective chain complexes of R-modules, the projective model structures correspond to each other. However, N fails to be a strong symmetric monoidal functor. Instead, $\times$ corresponds to the *shuffle* tensor product $\widetilde{\otimes}$ of chain complexes, which is much bigger than the usual tensor product. According to Subsection 7.3, $(\mathbf{Ch}_R^+, \widetilde{\otimes})$ is symmetric h-monoidal. The reason why a similar argument fails for $\otimes$ is that the (smaller) ordinary tensor product fails to allow for a Σ_n-equivariant diagonal map for an interval object.

If R contains $\mathbf{Q}$, the picture changes drastically: every R-module M with a Σ_n-action is projective as an R-module if and only if it is projective as an $R[\Sigma_n]$-module (Maschke's theorem). Thus, the projective and injective model structure (with respect to the Σ_n-action) on $\Sigma_n \mathbf{Ch}(\mathbf{Mod}_R)$ agree. Therefore, $\mathbf{Ch}(\mathbf{Mod}_R)$ is symmetric flat and freely powered (and therefore symmetroidal and symmetric h-monoidal).

With appropriate additional assumptions, the statements above can be generalized to chain complexes in a Grothendieck abelian category $\mathcal{A}$. For example, flatness and h-monoidality of $\mathbf{Ch}(\mathcal{A})$ require that projective objects $P \in \mathcal{A}$ are flat, i.e., $P \otimes -$ is an exact functor.

7.5. **Topological spaces.** The category $\mathbf{Top}$ of compactly generated weakly Hausdorff topological spaces carries the Quillen model structure which is transferred from $\mathbf{sSet}$ via the singular simplicial set functor. Thus left properness, pretty smallness, symmetroidality, and symmetric h-monoidality of $\mathbf{sSet}$ transfers to $\mathbf{Top}$ by Theorem 5.2.6(iii). Moreover, $\mathbf{Top}$ is monoidal [Hov99, Corollary 4.2.12], h-monoidal by [BB13, Example 1.15], and flat [EKMM97, Theorem III.3.8]. It is cellular [Hir03, Propositions 4.1.4], though not locally presentable and therefore not combinatorial. However, it is admissibly generated. This follows from Theorem 4.3.8(i) and

the following facts about topological spaces: the maps in (4.2.5) are inclusions of topological spaces, since products with arbitrary spaces and coinvariants by finite group actions preserve those. Inclusions are stable under pushouts and transfinite compositions. Finally, any topological space is small relative to inclusions by [Hov99, Lemma 2.4.1]. Alternatively, one can use Smith's Δ-generated topological spaces, which are combinatorial and pretty small.

7.6. **Symmetric spectra.** The positive stable model structure on symmetric spectra with values in an abstract model category $\mathcal{C}$ is both symmetric flat and symmetric h-monoidal. With a careful choice of the model structure on symmetric sequences, it is also symmetroidal. As a special case, this shows that any model category is Quillen equivalent to one which is symmetric flat and symmetroidal. For this, only mild conditions on $\mathcal{C}$ are necessary (such as flatness and h-monoidality, but not their symmetric counterparts). See [PS14b, Theorem 3.3.4] for the precise statement.

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SYMMETRIC OPERADS IN ABSTRACT SYMMETRIC SPECTRA

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ABSTRACT. We show that all colored symmetric operads in symmetric spectra valued in a symmetric monoidal model category are admissible, i.e., algebras over such operads carry a model structure. For example, this applies to commutative ring spectra and E_∞-ring spectra in simplicial sets or motivic spaces. Moreover, any weak equivalence of operads in spectra gives rise to a Quillen equivalence of their categories of algebras. For example, any E_∞-ring spectrum of simplicial sets or motivic spaces can be strictified to a commutative ring spectrum. We apply this to construct a strictly commutative ring spectrum representing Deligne cohomology. We also discuss applications to Toën-Vezzosi homotopical algebraic contexts and Goerss-Hopkins obstruction theory.

CONTENTS

1. INTRODUCTION

Ever since Brown's representability theorem, spectra occupy a central place in a variety of areas. They are the objects representing cohomology theories, i.e., for some cohomology theory $H^*(-)$, one can find a spectrum E such that the cohomology of all spaces X is given by morphisms of spectra (up to homotopy) from the infinite suspension of X to a suspension of the spectrum:

$$H^n(X) = [\Sigma^\infty X, \Sigma^n E].$$

Most cohomology theories in algebraic topology, algebraic geometry, and beyond carry a commutative and associative product

$$H^m(X) \otimes H^n(X) \to H^{m+n}(X).$$

This makes it desirable to refine the multiplicative structure on the cohomology to one on the representing spectrum. Ideally, one would like a strictly commutative and associative product

$$E \wedge E \to E$$

1

that gives back the above product. In this case E is called a commutative ring spectrum. The following theorem is the basis of the homotopy theory of commutative ring spectra and spectra with a much more general multiplicative structure, namely algebras over symmetric colored operads:

Theorem 1.0.1. *(See Theorem 3.4.1.) Suppose $\mathcal{C}$ is a symmetric monoidal model category satisfying some mild additional assumptions (see Assumption 3.2.1 for the precise list), R is a commutative monoid in symmetric sequences in $\mathcal{C}$, O is a (symmetric colored) operad in symmetric R-spectra (i.e., R-modules in symmetric sequences in $\mathcal{C}$). Then the stable positive model structure on R-spectra exists and gives rise to a model structure on O-algebras in R-spectra.*

For example, this applies to $\mathcal{C} = \mathbf{sSet}_\bullet$, $R_n = (S^1)^{\wedge n}$, and O being the commutative operad (i.e., $O_n = S^0$), in which case R-modules are simplicial symmetric spectra and O-algebras are simplicial commutative symmetric ring spectra. If O is the Barratt-Eccles operad (i.e., $O_n = \mathrm{E}\Sigma_n$), then O-algebras are simplicial symmetric E_∞-ring spectra.

Another example is the category $\mathcal{C}$ of pointed simplicial presheaves $\mathbf{sPSh}_\bullet(\mathbf{Sm}/S)$ on the site of smooth varieties over a scheme S, equipped with the projective, flasque, or injective model structure, or any localization thereof (such as the Nisnevich $\mathbf{A}^1$-localization), and $R_n = (\mathbf{P}^1)^{\wedge n}$. In this case R-modules are known as motivic symmetric $\mathbf{P}^1$-spectra and commutative monoids are (strictly) commutative motivic symmetric ring spectra.

We also give a supplementary condition that guarantees, for example, that the underlying spectrum of a cofibrant commutative ring spectrum is nonpositively cofibrant (see Theorem 3.4.3 for the precise statement).

In practice, it is often hard to construct *strictly* commutative ring spectra. Often it is the case that we instead can construct an algebra over an operad weakly equivalent to the commutative operad Comm, for example, the Barratt-Eccles operad E_∞. Essentially, this means that instead of defining a single product, there is a whole space of binary products and more generally n-ary products. A bigger space of n-ary products gives us more freedom to construct examples. The following theorem says in particular that a multiplication whose space of n-ary operations is contractible, can be strictified to a strictly commutative and associative product.

Theorem 1.0.2. *(See Theorem 3.4.4.) With $\mathcal{C}$ and R as above, any morphism $f\colon O \to P$ of operads in R-spectra induces a Quillen adjunction between O-algebras and P-algebras, which is a Quillen equivalence if f is a weak equivalence.*

We also study operadic algebras in spectra with values in Quillen equivalent categories (Theorem 3.4.9). As a special case we obtain the following Quillen invariance:

Theorem 1.0.3. *(See Corollary 3.4.10.) For a weak equivalence $\varphi : R \xrightarrow{\sim} S$ of commutative monoids in $\Sigma\mathcal{C}$, and any levelwise fibrant operad P in S-spectra and any levelwise cofibrant operad O in R-spectra, there are Quillen equivalences*

$$\varphi_* : \mathbf{Alg}_O^{\mathrm{s},+}(\mathbf{Mod}_R) \quad \leftrightarrows \quad \mathbf{Alg}_{S\otimes_R O}^{\mathrm{s},+}(\mathbf{Mod}_S) : \varphi^*$$

$$\varphi_* : \mathbf{Alg}_{\varphi^* P}^{\mathrm{s},+}(\mathbf{Mod}_R) \quad \leftrightarrows \quad \mathbf{Alg}_P^{\mathrm{s},+}(\mathbf{Mod}_S) : \varphi^*.$$

After a few recollections on model categories in Section 2, we define the notion of a *(strongly) admissible model structure* on symmetric sequences ($\Sigma\mathcal{C}$) in Section 3.1. The admissibility of the model structure on $\Sigma\mathcal{C}$ will ultimately give rise to the admissibility of all symmetric operads. If the model structure is strongly admissible, it has the extra property that positive cofibrations c, i.e., those that are trivial in level 0, are symmetric cofibrations (Theorem 3.1.6(5)), i.e., $(c^{\Box n})_{\Sigma_n}$ is a cofibration. Using a general transfer technique developed in [PS, Section 5], we transfer these model-theoretic properties to the unstable model category of symmetric spectra (Section 3.2).

More conceptually speaking, we look at the category of R-modules, where R is any commutative monoid in $\Sigma\mathcal{C}$. We refer to this category as R-spectra. We then perform the usual stabilization (Section 3.3) using the technique of Bousfield localization. In Theorem 3.4.1, we show the existence of a model structure on algebras over operads in R-spectra, which means that every operad in R-spectra is admissible. The key argument is that for a positive acyclic cofibration f, i.e., one whose level 0 is an isomorphism, the n-fold pushout product $f^{\Box n}$ has very good properties. For example, for any spectrum X with a Σ_n-action, $X \otimes_{\Sigma_n} f^{\Box n}$ is a couniversal weak equivalence. This is weaker than being an acyclic cofibration, but enough to obtain the admissibility of all operads.

The model categories discussed above are connected by the following chain of Quillen adjunctions. The middle adjunction is a Bousfield localization, while the other two adjunctions serve to transfer the model structure on the left to the right. The superscripts indicate the precise choice of model structure: "+" and "s, +" refer to the positive and stable positive structures. Underneath we indicate the place where the model structure in question is defined.

$$\Sigma^+ \mathcal{C} \quad \leftrightarrows \quad \mathbf{Mod}_R^+ \quad \leftrightarrows \quad \mathbf{Mod}_R^{\mathrm{s},+} \quad \leftrightarrows \quad \mathbf{Alg}_O(\mathbf{Mod}_R^{\mathrm{s},+}) \tag{1.0.4}$$
$$\text{3.1.6} \qquad\quad \text{3.2.5} \qquad\quad \text{3.3.4} \qquad\qquad \text{3.4.1}$$

Along the way we prove the *monoid axiom* for the stable model structures on R-modules, which was previously unknown.

We go on to proving the operadic rectification result cited above (see Theorem 3.4.4) using the notion of symmetric flatness which again holds for the stable *positive* model structure on R-modules.

We finish our paper with the following applications (Section 4): we show that $\mathbf{Mod}_R^{+,\mathrm{s}}$ is an homotopical algebra context in the sense of Toën and Vezzosi [TV08]. This allows to do derived algebraic geometry over ring spectra. We also show that the Goerss-Hopkins axioms [GH04] and [GH] for a convenient category of spectra are satisfied by this model category, which allows one to run the Goerss-Hopkins obstruction machine in settings other than ordinary spaces. In Section 4.3 we show how to use the rectification result to construct commutative ring spectra. In Section 4.4, we finish with an application to Deligne cohomology:

Theorem 1.0.5. *(See Theorem 4.4.8.) There is a strictly commutative motivic* $\mathbf{P}^1$*-spectrum representing Deligne cohomology with integral coefficients, including the product structure and all higher product operations such as Massey products.*

It is a pleasure to acknowledge the wealth of ideas that have helped to shape this paper. For us, a starting point was an observation by Lurie that guarantees both the existence of a model structure on commutative monoids in a model category $\mathcal{C}$ and a rectification result [Lur, Section 4.4.4]. It requires that $f^{\square n}$ is a Σ_n-projective acyclic cofibration for all acyclic cofibrations $f \in \mathcal{C}$. Roughly, this means that Σ_n acts freely on the complement of the image of this iterated pushout product. This is a harder condition than just asking that $f^{\square n}/\Sigma_n$ is an acyclic cofibration. In fact, Lurie's condition is rarely satisfied in practice. It holds for chain complexes over a field of characteristic zero, but fails for the categories of simplicial sets or symmetric spectra in simplicial sets (even when endowed with the positive model structure).

The positive model structure on spectra is due to Smith. It was studied in the context of topological spaces by Mandell, May, Schwede, and Shipley, who showed the existence of model structures on commutative ring spectra and noted the rectification of E_∞-ring spectra in topological spaces [MMSS01, Theorem 15.1, Remark 0.14]. The positive model structure on symmetric spectra with values in an arbitrary model category has been studied by Gorchinskiy and Guletskiǐ [GG11]. They showed the homotopy orbits property (under a strong assumption related to Lurie's condition mentioned above). This property is a key step in the operadic rectification. Harper also proved a rectification result as in Theorem 1.0.2 [Har09, Theorem 1.4] for $\mathcal{C} = \mathbf{sSet}_\bullet$, which was generalized to $\mathcal{C}$ being the category of simplicial presheaves with the injective model structure by Hornbostel [Hor13, Theorem 3.6]. These two model categories possess special features that substantially simplify the proof, one of them being the fact that all objects are cofibrant.

In another direction, Harper showed the existence of a model structure on algebras over operads [Har10, Theorem 1.4] under the assumption that every symmetric sequence is projectively cofibrant. Again, this is a strong assumption, which applies to such special categories as chain complexes over a field of characteristic zero. In this case, rectification goes back to Hinich [Hin97]. A recent application was the construction of motives (with rational coefficients) over general bases by Cisinski and Déglise [CD09, Theorem 4.1.8]. In fact, our paper grew out from the desire to construct a convenient (i.e., fibrant) ring spectrum representing (higher) algebraic cobordism groups. We plan to present such applications in a separate paper.

2. Model-categorical preliminaries

This paper uses the language of model categories. Very briefly, we recall the less standard notions developed in [PS, Sections 2, 4]. A *pretty small* model category $\mathcal{C}$ has, by definition, another model structure on the same underlying category which has the same weak equivalences, but fewer cofibrations which are

required to be generated by a set of maps whose (co)domain is compact. A class S of morphisms in a symmetric monoidal model category $\mathcal{C}$ is *(acyclic) symmetric i-monoidal*, if for any finite multi-index $n = (n_1, \ldots, n_e)$, $n_i \geq 1$, and any object $Y \in \Sigma_n \mathcal{C} := \prod_i \Sigma_{n_i} \mathcal{C}$, and any finite family of maps $s = (s_i)$ in S, the map

$$Y \otimes_{\Sigma_n} s^{\square n} := (Y \otimes s_1^{\square n_1} \square \cdots \square s_e^{\square n_e})_{\Sigma_n}$$

is an (acyclic) i-cofibration, which means that pushouts along this map are homotopy pushouts (and that it is moreover a weak equivalence in the acyclic case). The category $\mathcal{C}$ is symmetric i-monoidal if this condition holds for the class of (acyclic) cofibrations. A related condition is called *(acyclic) $\mathcal{Y}$-symmetroidality* of S: it requires that for any map y in a fixed class of morphisms $\mathcal{Y}_n \subset \operatorname{Mor} \Sigma_n \mathcal{C}$ (for example all injective cofibrations), and any finite family s of maps in S, the map

$$y \square_{\Sigma_n} s^{\square n}$$

is an (acyclic) cofibration. Finally, a class S of morphisms in $\mathcal{C}$ is called *symmetric flat*, if for any weak equivalence $y \in \Sigma_n \mathcal{C}$ (i.e., a Σ_n-equivariant map which is a weak equivalence in $\mathcal{C}$) and any finite family of maps (s_i) in S, the map

$$y \square_{\Sigma_n} s^{\square n}$$

is a weak equivalence. While symmetric i-monoidality is satisfied relatively often, symmetroidality and in particular symmetric flatness are more rare. For example, simplicial sets are symmetric i-monoidal and symmetroidal, but not symmetric flat. Simplicial presheaves with the projective model structure are symmetric i-monoidal. For a commutative ring R the category of chain complexes of R-modules is symmetric i-monoidal, symmetroidal, and symmetric flat precisely if R contains $\mathbf{Q}$, but none of these properties hold otherwise. These and further basic examples are discussed in [PS, Section 7]. A more sophisticated example is the positive stable model structure on R-modules in symmetric sequences, i.e., *symmetric R-spectra* with values in an abstract model category $\mathcal{C}$ (subject to some mild conditions). This category is symmetric i-monoidal, symmetroidal, and symmetric flat. See Theorem 3.3.4 for the precise statement.

A *monoidal left Bousfield localization* $\mathrm{L}_S^{\otimes} \mathcal{C}$ of a symmetric monoidal model category $\mathcal{C}$ with respect to a class S is the left Bousfield localization in the bicategory of symmetric monoidal model categories. Its underlying model category can be computed as $\mathrm{L}_{S^{\otimes}} \mathcal{C}$, where $S^{\otimes}$ denotes the monoidal saturation of S in $\mathcal{C}$. If $\mathcal{C}$ is tractable, $S^{\otimes}$ can be taken to be $S \otimes (\mathrm{co})\mathrm{dom}(I)$, where I is some set of generating cofibrations with cofibrant source. See [PS, Section 6.1] for further details.

For a finite group G and a subgroup H and some object X with a left H-action, we write $G \cdot_H X := \operatorname{colim}_H (\coprod_G X)$. It carries a natural left G-action.

3. Model structures on symmetric spectra

3.1. **Symmetric sequences.** In this section, let $\mathcal{C}$ be a tractable, pretty small, left proper, symmetric monoidal model category. We study model structures on the category of *symmetric sequences*, which is the functor category

$$\Sigma \mathcal{C} := \mathbf{Fun}(\Sigma, \mathcal{C})$$

from the category Σ of finite sets and bijections or, equivalently, its skeleton. There is an obvious adjunction

$$G_n : \Sigma_n \mathcal{C} \leftrightarrows \Sigma \mathcal{C} : \mathrm{ev}_n, \tag{3.1.1}$$

where ev_n is the evaluation on n and $G_n(X)(m)$ is X for $m = n$ and the initial object of $\mathcal{C}$ else. For some fixed $k \geq 0$, these assemble to an adjunction

$$G_{\geq k} : \prod_{n \geq k} \Sigma_n \mathcal{C} \leftrightarrows \Sigma \mathcal{C} : \mathrm{ev}, \tag{3.1.2}$$

For $k = 0$ this is an equivalence of categories, but we will mostly be interested in $k = 1$ in the sequel.

The category $\Sigma \mathcal{C}$ is equipped with the monoidal structure (denoted $\otimes$) coming from the disjoint union of finite sets [HSS00, Definition 2.1.3]. It satisfies

$$G_n(X) \otimes G_{n'}(X') = G_{n+n'}(\Sigma_{n+n'} \cdot_{\Sigma_n \times \Sigma_{n'}} X \otimes X'). \tag{3.1.3}$$

Depending on the model category $\mathcal{C}$, there are typically many different model structures on $\Sigma \mathcal{C}$, so we isolate a short list of axioms that we rely upon in the sequel. (Strongly) admissible model structures on $\Sigma \mathcal{C}$

will ultimately guarantee that all operads in the stable positive model structure on symmetric R-spectra are (almost strongly) admissible (see Theorems 3.4.1, 3.4.3). Recall the notation $G \cdot_H -$ from Section 2.

Definition 3.1.4. A model structure on $\Sigma\mathcal{C}$ is called *level-k admissible* (or just admissible) for some fixed $k \geq 0$ if it is transferred along the adjunction (3.1.2) such that the model structures on the categories $\Sigma_m\mathcal{C}$ (denoted $\Sigma_m^{\mathrm{ad}}\mathcal{C}$) satisfy the following properties:

(1) Each $\Sigma_m^{\mathrm{ad}}\mathcal{C}$ is a tractable model category.
(2) The weak equivalences are given by $\mathrm{W}_{\Sigma_m^{\mathrm{ad}}\mathcal{C}} = \varphi^{-1}(\mathrm{W}_{\mathcal{C}})$, where φ denotes the functor that forgets any action of a finite group on some object in $\mathcal{C}$.
(3) The following identity functors are left Quillen functors between the projective, the admissible, and the injective model structures:
$$\Sigma_m^{\mathrm{pro}}\mathcal{C} \to \Sigma_m^{\mathrm{ad}}\mathcal{C} \to \Sigma_m^{\mathrm{in}}\mathcal{C}.$$
(4) For any decomposition $m = m' + m''$, $m', m'' \geq 0$, the following is a left Quillen bifunctor:
$$\Sigma_{m'}^{\mathrm{ad}}\mathcal{C} \times \Sigma_{m''}^{\mathrm{ad}}\mathcal{C} \xrightarrow{\otimes} (\Sigma_{m'} \times \Sigma_{m''})\mathcal{C} \xrightarrow{\Sigma_m \cdot_{\Sigma_{m'} \times \Sigma_{m''}} -} \Sigma_m^{\mathrm{ad}}\mathcal{C}. \qquad (3.1.5)$$

For any multi-index n, let $\mathcal{Y}_{n,u}$ be a class of morphisms in $\Sigma_n\Sigma_u\mathcal{C}$. We suppose that for any $y \in \mathcal{Y}_{n,u}$, $G_u(y) \square -$ preserves (acyclic) cofibrations in $\Sigma_n^{\mathrm{in}}\Sigma^{\mathrm{ad}}\mathcal{C}$, i.e., those Σ_n-equivariant maps which are (acyclic) cofibrations in $\Sigma^{\mathrm{ad}}\mathcal{C}$. For example, this condition is satisfied if $\mathcal{Y}_{n,u}$ is empty for $n \neq u$ and consists just of the single map $\emptyset \to 1_{\Sigma\mathcal{C}}$ for $n = u$. Another example is the class $\mathcal{Y}_{n,u}$ of injective cofibrations in $\Sigma_n^{\mathrm{in}}\Sigma^{\mathrm{ad}}\mathcal{C}$, i.e., maps which are cofibrations in $\Sigma^{\mathrm{ad}}\mathcal{C}$. The model structure is called *strongly $\mathcal{Y}$-admissible* if, in addition, the following condition holds:

(5) For any multi-index $n \geq 1$, any multi-index (of the same size) $t \geq 1$, any $y \in \mathcal{Y}_{n,u}$, any finite family of generating (acyclic) cofibrations $h \in \Sigma_t\mathcal{C}$ (i.e., $h_i \in \Sigma_{t_i}\mathcal{C}$), the expression
$$\Sigma_{tn+u} \cdot_{\Sigma_n \ltimes (\Sigma_u \times \Sigma_t^n)} y \square h^{\square n}$$
is an (acyclic) cofibration in $\Sigma_{tn+u}\mathcal{C}$.

For $k = 0$, we will drop the prefix "k-". For $k = 1$, we replace this prefix by "positive", e.g., the *positive admissible model structure*. We denote these model structures by $\Sigma^{\geq k}\mathcal{C}$, $\Sigma\mathcal{C}$, $\Sigma^+\mathcal{C}$. In order to emphasize the admissibility of the model structure, or for particular choices of admissible model structures, we write $\Sigma^{+,\mathrm{ad}}$ or $\Sigma^{+,\mathrm{pro}}$ etc.

We now study the model-theoretic properties of symmetric sequences. The abstract techniques of transfer and localizations of model structures established in [PS, Sections 5, 6] will then readily imply similar properties for the stable model structure on symmetric spectra. For example, the symmetric i-monoidality statement in Part (5) will give rise to the admissibility of symmetric operads, while the symmetric flatness is responsible for the rectification of algebras over operads. Recall our conventions on $\mathcal{C}$ (p. 4).

Theorem 3.1.6. *Any level-k admissible model structure has the following properties:*

(1) It is tractable. Its generating (acyclic) cofibrations are the maps $G_n(\phi)$, where $n \geq k$ and ϕ is a generating (acyclic) cofibration in $\Sigma_n\mathcal{C}$. The cofibrations are those maps which are cofibrations in degrees $n \geq k$ (in $\Sigma_n\mathcal{C}$) and isomorphisms in degrees $n < k$.

(2) The weak equivalences (fibrations) are those maps which are weak equivalences (fibrations) in $\Sigma_n\mathcal{C}$ for each $n \geq k$ (and arbitrary in degrees $n < k$).

(3) It is pretty small and left proper.

(4) For any $k \geq 0$, $\Sigma^{\geq k}\mathcal{C}$ is a symmetric monoidal model category. If the monoid axiom holds for $\mathcal{C}$, the same is true for $\Sigma^{\geq k}\mathcal{C}$. If $\mathcal{C}$ is i-monoidal, then so is $\Sigma^{\geq k}\mathcal{C}$. If $\mathcal{C}$ is i-monoidal and flat [PS, Definitions 3.2.1, 3.2.2], then $\Sigma^{\geq 0}\mathcal{C}$ is flat.

(5) Let $k > 0$. If $\mathcal{C}$ is i-monoidal, the (acyclic) cofibrations of the model structure $\Sigma^{\geq k}\mathcal{C}$ form an (acyclic) symmetric i-monoidal class in $\Sigma^{\geq 0}\mathcal{C}$. If $\mathcal{C}$ is i-monoidal and flat, then they form a symmetric flat class in $\Sigma^{\geq 0}\mathcal{C}$. Finally, if the model structure on $\Sigma\mathcal{C}$ is strongly $\mathcal{Y}$-admissible for some $\mathcal{Y} = (\mathcal{Y}_{n,u})$ as in Definition 3.1.4, then the (acyclic) cofibrations in $\Sigma^{\geq k}\mathcal{C}$ $(k > 0)$ form a class that is (acyclic) $\mathcal{Y}$-symmetroidal in the model structure in $\Sigma^{\geq 0}\mathcal{C}$, i.e., (acyclic) symmetroidal with respect to the class of maps $y \in \Sigma_n\mathcal{C}$ whose components $\mathrm{ev}_u y$ lie in $\mathcal{Y}_{n,u}$. In particular, $\Sigma^{\geq k}\mathcal{C}$ is $\mathcal{Y}$-symmetroidal in this case.

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Proof. The description of fibrations and weak equivalences is the definition of a transferred model structure. The description of generating (acyclic) cofibrations, as well as transfer of left properness, pretty smallness and tractability are basic properties of transfer [PS, Proposition 5.1.2]. For left properness note that $\mathcal{C}$ is left proper, hence so is $\Sigma_m^{\mathrm{in}}\mathcal{C}$, and therefore $\Sigma_m^{\mathrm{ad}}\mathcal{C}$, by the admissibility condition (3). Similarly, $\mathcal{C}$ is pretty small, hence so is $\Sigma_m^{\mathrm{pro}}\mathcal{C}$, and therefore $\Sigma_m^{\mathrm{ad}}\mathcal{C}$.

(4): For the pushout product axiom, it is enough to check $I \square I \subset \mathrm{C}(\Sigma\mathcal{C})$ and $I \square J \cup J \square I \subset \mathrm{AC}(\Sigma\mathcal{C})$. Here I (J) are the generating (acyclic) cofibrations of $\Sigma\mathcal{C}$. They are of the form $G_n(f)$, where $n \geq k$ and $f \in \Sigma_n\mathcal{C}$ is a generating (acyclic) cofibration (with respect to the chosen admissible structure). Using (3.1.3), we obtain our claim by Definition 3.1.4(4).

The monoid axiom requires the weak saturation $\mathrm{cof}(J \otimes \Sigma\mathcal{C})$ to be contained in $\mathrm{W}_{\Sigma\mathcal{C}}$. Equivalently, we need to check that

$$\varphi\mathrm{ev}_n(\mathrm{cof}(J \otimes \Sigma\mathcal{C})) \subset \mathrm{W}_{\mathcal{C}},$$

where $\Sigma\mathcal{C} \xrightarrow{\mathrm{ev}_n} \Sigma_n\mathcal{C} \xrightarrow{\varphi} \mathcal{C}$, for each $n \geq k$. Pick some generating acyclic cofibration $f = G_r(g)$, $g \in J_{\Sigma_r\mathcal{C}}$ ($r \geq k$) and any symmetric sequence X. By [PS, Lemma 4.1.2], there is a noncanonical isomorphism

$$
\begin{aligned}
\varphi\mathrm{ev}_n(f \otimes Y) &= \varphi(\Sigma_n \cdot_{\Sigma_r \times \Sigma_{n-r}} f_r \otimes Y_{n-r}) &\qquad (3.1.7)\\
&\cong \Sigma_n/(\Sigma_r \times \Sigma_{n-r}) \cdot \varphi(f_r) \otimes \varphi(Y_{n-r})\\
&= (\Sigma_n/(\Sigma_r \times \Sigma_{n-r}) \cdot \varphi(g)) \otimes \varphi(Y_{n-r})
\end{aligned}
$$

We now use that $\varphi(g) \in \mathrm{AC}_{\mathcal{C}}$ by 3.1.4(3) and likewise with a (finite) coproduct of copies of this. Therefore, the previous expression is contained in $\mathrm{AC}_{\mathcal{C}} \otimes \mathcal{C}$. Invoking the cocontinuity of $\varphi\mathrm{ev}_n$ and the monoid axiom in $\mathcal{C}$, we obtain our claim

$$\varphi\mathrm{ev}_n(\mathrm{cof}(J \otimes \Sigma\mathcal{C}))) \subset \mathrm{cof}(\varphi\mathrm{ev}_n(J \otimes \Sigma\mathcal{C})) \subset \mathrm{cof}(\mathrm{AC}_{\mathcal{C}} \otimes \mathcal{C}) \subset \mathrm{W}_{\mathcal{C}}. \qquad (3.1.8)$$

Using the i-monoidality of $\Sigma^{\geq 0}\mathcal{C}$ it is enough to check flatness for generating cofibrations. Thus we need to show $y \square G_n(c)$ is a weak equivalence for any weak equivalence $y \in \Sigma^{\geq 0}\mathcal{C}$ and any cofibration c in $\Sigma_n\mathcal{C}$, $n \geq 0$. We have $y \square G_n(c) = \coprod_{r \geq 0} G_{n+r}(\Sigma_{n+r} \cdot_{\Sigma_r \times \Sigma_n} y_r \square c)$. It is enough to see that $\Sigma_{n+r} \cdot_{\Sigma_r \times \Sigma_n} y_r \square c$ is a weak equivalence. Again by [PS, Lemma 4.1.2], it is isomorphic, in $\mathcal{C}$, to a finite coproduct of copies of $y_r \square c$ which is a weak equivalence in $\mathcal{C}$ by the flatness of $\mathcal{C}$. Moreover, by the i-monoidality, $(\mathrm{co})\mathrm{dom}(y_r) \otimes c$ is an i-cofibration, so that $y_r \square c$ is a couniversal weak equivalence by [PS, Lemma 3.2.6]. These are stable under finite coproducts in any model category.

(5): By [PS, Theorem 4.3.9], symmetric i-monoidality, symmetroidality and symmetric flatness only have to be checked on generating (acyclic) cofibrations. The acyclic parts of the three statements are proven by replacing the words "cofibration" and "i-cofibration" by their acyclic analogues, so that proof is omitted. Let $v = (v_1, \ldots, v_e)$ be a finite family of generating cofibrations of $\Sigma^{\geq k}\mathcal{C}$. They are given by $v_i = G_{t_i}(h_i)$ for some generating cofibrations $h_i \in \Sigma_{t_i}\mathcal{C}$ and $t_i \geq k > 0$. Let $n = (n_i)$ be a multi-index with $n_i \geq 1$.

For an object $Y = G_u(Z)$ in $\Sigma_n\Sigma\mathcal{C}$, we have

$$
\begin{aligned}
Y \otimes_{\Sigma_n} v^{\square n} &= \left(G_{tn+u}(\Sigma_{tn+u} \cdot_{\Sigma_u \times \Sigma_t^n} Z \otimes h^{\square n}) \right)_{\Sigma_n} \\
&= G_{tn+u}(\Sigma_{tn+u} \cdot_{\Sigma_n \ltimes (\Sigma_u \times \Sigma_t^n)} Z \otimes h^{\square n}) &\qquad (3.1.9)
\end{aligned}
$$

The group Σ_n acts trivially on Σ_u and Σ_u acts trivially on $h^{\square n}$. In $\mathcal{C}$ (as opposed to $\Sigma_{tn+u}\mathcal{C}$), there is an isomorphism

$$\varphi(\Sigma_{tn+u} \cdot_{\Sigma_n \ltimes (\Sigma_u \times \Sigma_t^n)} Z \otimes h^{\square n}) = \frac{\Sigma_{tn+u}}{\Sigma_n \ltimes (\Sigma_u \times \Sigma_t^n)} \cdot \varphi\left(Z \otimes h^{\square n}\right), \qquad (3.1.10)$$

by [PS, Lemma 4.1.2]. This uses the positivity of the t_i which implies that $\Sigma_n \ltimes (\Sigma_u \times \Sigma_t^n)$ is a subgroup of Σ_{tn+u}.

For symmetric i-monoidality, we have to show that $Y \otimes_{\Sigma_n} v^{\square n}$ is an i-cofibration in $\Sigma^{\geq 0}\mathcal{C}$ for all $Y \in \Sigma_n\Sigma\mathcal{C}$. We may assume $Y = G_u(Z)$, where $u \geq 0$ and $Z \in \Sigma_m\Sigma_u\mathcal{C}$ is arbitrary. Here we use that (acyclic) i-cofibrations in an i-monoidal model category are stable under finite coproducts [BB13, Lemma 1.3] and therefore, using the pretty smallness and [PS, Lemma 2.0.2], under countable coproducts. We show the stronger statement that the above map is an i-cofibration in $\mathcal{C}$ in all degrees. Finally, the h_i are cofibrations, so that $h^{\square n}$ is also a cofibration (in $\mathcal{C}$, by the pushout product axiom). Hence, $Z \otimes h^{\square n}$ and therefore the right hand side of (3.1.10) are i-cofibrations in $\mathcal{C}$, using the i-monoidality of $\mathcal{C}$.

The symmetric flatness of the $\geq k$-cofibrations (for $k > 0$) in $\Sigma^{\geq 0}\mathcal{C}$ is proven similarly: replace $Y \otimes_{\Sigma_n} -$ by $y \square_{\Sigma_n} -$ for any weak equivalence $y \in \Sigma_n \Sigma^{\geq 0}\mathcal{C}$. Again, the reduction from a general weak equivalence y to $y = G_u(z)$, $u \geq 0$, z a weak equivalence in $\Sigma_n \Sigma_u \mathcal{C}$, is possible by pretty smallness. Now, note that $z \square h^{\square n}$ is a weak equivalence in $\mathcal{C}$ since $\mathcal{C}$ is flat.

Finally, for $\mathcal{Y}$-symmetroidality, we again reduce the claim that $y \square_{\Sigma_n} v^{\square n}$ is a cofibration in $\Sigma^{\geq 0}\mathcal{C}$ to the case $y = G_u(z)$ for $z \in \mathcal{Y}_{n,u}$. This is true provided that

$$\Sigma_{tn+u} \cdot_{\Sigma_n \rtimes (\Sigma_u \times \Sigma_t^n)} z \square h^{\square n}$$

is a cofibration in $\Sigma_{tn+u}\mathcal{C}$ which is exactly the strong admissibility condition 3.1.4(5). The $\mathcal{Y}$-symmetroidality of $\Sigma^{\geq k}\mathcal{C}$ also follows from this, noting that $tn + u \geq u \geq k$ in this case, so the previous expression is a cofibration in $\Sigma^{\geq k}\mathcal{C}$.

The i-monoidality of $\Sigma^{\geq k}\mathcal{C}$ for $k \geq 0$ also follows from these arguments: in (3.1.10), put $n = 1$, consider only single maps (as opposed to finite families) and use that $\Sigma_t \times \Sigma_u$ is a subgroup of Σ_{t+u}, even for $t = 0$. $\square$

Remark 3.1.11. The $\mathcal{Y}$-symmetroidality of $\Sigma^{\geq k}\mathcal{C}$ would hold for $k = 0$ provided that $\mathcal{C}$ itself is $\mathcal{Y}$-symmetroidal. This excludes the projective model structure on chain complexes of abelian groups, for example. (See [PS, Section 7] for a discussion of concrete model categories (not) satisfying symmetric i-monoidality, symmetroidality and symmetric flatness.) The positive structure does not require such an assumption. Likewise, the (nonsymmetric) flatness promotes to *symmetric flatness* of $\Sigma^{\geq k}\mathcal{C}$ for $k > 0$.

The strong admissibility (as opposed to mere admissibility) is necessary to ensure the symmetroidality of the positive model structure. For example, the argument above fails for the projective structure on $\Sigma\mathcal{C}$, for example for $\mathcal{C} = \mathbf{sSet}$: for $t = 1$, $v = G_1(h)$ where h is some cofibration (=monomorphism) in $\mathbf{sSet}$. However, Σ_n does not usually act freely on the complement of the image $h^{\square n}$, so this map is not a cofibration in $\Sigma^{\mathrm{pro}}\mathbf{sSet}$.

The model category $\Sigma^{\geq k}\mathcal{C}$ is *not* flat for $k > 0$: for *any* map $y \in \mathcal{C}$, $G_0(y)$ is a weak equivalence in $\Sigma^{\geq k}\mathcal{C}$, but $y \square G_k(c)$ is not.

We now give examples of strong admissible model structures. Lemma 3.1.12 shows that the injective model structure $\Sigma^{\mathrm{in}}\mathcal{C}$ is strongly admissible, except, possibly, for the tractability. Because of that, it suffices to check the nonacyclic parts of the requirements in 3.1.4(4) and (5). In other words, these requirements only depend on the cofibrations of $\Sigma^{\mathrm{ad}}\mathcal{C}$. The tractability requirement 3.1.4(1) (as opposed to, say, combinatoriality) is primarily of technical importance. It will be used to carry through monoidal properties to the stabilization of R-modules, which is helpful to prove the monoid axiom for the stable structure on R-modules (3.3.4(2)). Ignoring this necessity, the injective model structure $\Sigma^{\mathrm{in}}\mathcal{C}$ can be used in the sequel. However, fibrancy is very difficult to check in this model structure. A strongly admissible structure with controlled cofibrations (and therefore, acyclic fibrations) is provided by Theorem 3.1.18.

Lemma 3.1.12. *Let $\mathcal{C}$ be a combinatorial, symmetric monoidal model category. Then the injective model structure $\Sigma^{\mathrm{in}}\mathcal{C}$ is strongly admissible with tractability weakened to combinatoriality.*

Proof. The injective structure is combinatorial [Lur09, Proposition A.2.8.2.]. The first bifunctor in (3.1.5) is left Quillen since the pushout product commutes with φ and $\mathcal{C}$ is monoidal. The functor $\Sigma_m \cdot_{\Sigma_{m'} \times \Sigma_{m''}} -$ in (3.1.5) is a left Quillen functor by [PS, Lemma 4.1.2]. Using the notation of 3.1.4(5), h is an (acyclic) cofibration in $\mathcal{C}$, hence so is $h^{\square n}$ by the pushout product axiom and therefore $z \square h^{\square n}$ is again a cofibration in $\mathcal{C}$ by the assumption on $\mathcal{Y}$. Again [PS, Lemma 4.1.2], applied to the subgroup $\Sigma_n \rtimes (\Sigma_u \times \Sigma_t^n) \subset \Sigma_{tn+u}$, shows the strong admissibility. $\square$

Remark 3.1.13. The tractability of $\Sigma^{\mathrm{in}}\mathcal{C}$ holds if every object of $\mathcal{C}$ is cofibrant. This applies, for example, for simplicial sets or for simplicial presheaves with the *injective* model structure.

Lemma 3.1.14. *Let $\mathcal{C}$ be a tractable model category. Then the projective model structure $\Sigma^{\mathrm{pro}}\mathcal{C}$ is admissible.*
Set $\mathcal{Y}_{n,u}$ to be the projective cofibrations in $\Sigma_n \Sigma_u \mathcal{C}$. If every cofibration c in $\mathcal{C}$ is a power cofibration (i.e., $c^{\square n}$ is a projective cofibration, see [Lur, Section 4.4.4] *or* [PS, Definition 4.2.5]*), then the projective model structure $\Sigma^{\mathrm{pro}}\mathcal{C}$ is strongly $\mathcal{Y}$-admissible.*

Proof. The admissibility is standard, see for example [PS, Proposition 4.1.3]. As for strong admissibility, the generating cofibrations of $\Sigma_t^{\mathrm{pro}}\mathcal{C}$ are given by $\Sigma_t \cdot I_\mathcal{C}$. The following chain of inclusion shows our claim for

generating projective cofibrations in $\Sigma_n \Sigma_u \mathcal{C}$. The general case follows from this using [PS, Lemma 4.3.2].

$$
\begin{aligned}
\Sigma_u \cdot z \,\square\, (\Sigma_t \cdot I_\mathcal{C})^{\square n} &= \Sigma_u \cdot \Sigma_t^n \cdot z \,\square\, (I_\mathcal{C})^{\square n} \\
&\subset \Sigma_u \cdot \Sigma_t^n \cdot z \,\square\, \mathrm{C}_{\Sigma_n^{\mathrm{pro}}\mathcal{C}} \qquad\qquad (3.1.15) \\
&= \Sigma_u \cdot \Sigma_t^n \cdot z \,\square\, \mathrm{cof}(\Sigma_n \cdot I_\mathcal{C}) \\
&\subset \mathrm{cof}(\Sigma_u \cdot (\Sigma_n \rtimes \Sigma_t^n) \cdot z \,\square\, I_\mathcal{C}) \qquad (3.1.16) \\
&\subset \mathrm{C}_{\Sigma_u \times \Sigma_n \rtimes (\Sigma_t^n)^{\mathrm{pro}}\mathcal{C}}.
\end{aligned}
$$

The inclusion (3.1.15) holds by assumption. For (3.1.16), observe that Σ_n acts on Σ_t^n by permutation. $\qquad\square$

Remark 3.1.17. Under a mild condition on $\mathcal{C}$, namely that $\mathcal{C}$ has cellular fixed points [Gui], one can construct the so-called *mixed model structure* $G^{\mathrm{mix}}\mathcal{C}$. Its generating cofibrations (called equivariant cofibrations) are of the form $G/H \cdot I$, where $H \subset G$ is any subgroup. The weak equivalences of $G^{\mathrm{mix}}\mathcal{C}$ are the underlying weak equivalences. The mixed model structure is admissible, as can be easily shown.

The mixed model structure was introduced by Shipley for $\mathcal{C} = \mathbf{sSet}$ [Shi04, Proposition 1.3]. It turns out that for $\mathcal{C} = \mathbf{sSet}$ the mixed model structure $G^{\mathrm{mix}}\mathcal{C}$ agrees with the injective model structure $G^{\mathrm{in}}\mathcal{C}$ and therefore gives a strongly admissible model structure $\Sigma^{\mathrm{mix}}\mathcal{C}$. However, for a general model category such as $\mathcal{C} = \Sigma_2^{\mathrm{mix}}\mathbf{sSet}$ (i.e., the mixed model structure), the $G^{\mathrm{mix}}\mathcal{C}$ and $G^{\mathrm{in}}\mathcal{C}$ are distinct. For $G := \Sigma_2$, one checks that the projection

$$
Y := \mathrm{E}\Sigma_2 \sqcup \mathrm{E}\Sigma_2 \to X := * \sqcup *
$$

is an acyclic mixed (or equivariant) fibration, where G (Σ_2) acts on $Y \to X$ by permutation (by permutation and the natural Σ_2-action on $\mathrm{E}\Sigma_2$, respectively). The map does not have a section, so X is not cofibrant in $G^{\mathrm{mix}}\mathcal{C}$ (but in $G^{\mathrm{in}}\mathcal{C}$). See also [Har07] for more about this.

Theorem 3.1.18. *Suppose that $\mathcal{Y}$ is a set (as opposed to a class) of morphisms. Then $\mathcal{C}$ admits a strongly $\mathcal{Y}$-admissible model structure. We call it the* canonical strongly $\mathcal{Y}$-admissible model structure.

Proof. We use [Lur09, Proposition A.2.6.13] to construct a combinatorial model structure on each $\Sigma_m\mathcal{C}$. The weak equivalences will always be $W := \varphi^{-1}(W_\mathcal{C})$, as required by 3.1.4(2). This is a perfect class (in the sense of loc. cit.) since $\mathcal{C}$ is pretty small [PS, Lemma 2.0.2]. In addition we need to define a set I_m of maps in $\Sigma_m\mathcal{C}$. These will be the generating cofibrations of a model structure on $\Sigma_m\mathcal{C}$ provided that two conditions are met. (1) Any $f \in I_m$ is an i-cofibration in $\Sigma_m\mathcal{C}$. This will be satisfied as soon as I_m consists of injective cofibrations. (2) The class $\mathrm{inj}(I_m)$ is contained in W. This will be satisfied provided that I_m contains $I_{\Sigma_m^{\mathrm{pro}}\mathcal{C}}$ since all maps in $\mathrm{inj}(I_{\Sigma_m^{\mathrm{pro}}\mathcal{C}}) = \mathrm{AF}_{\Sigma_m^{\mathrm{pro}}\mathcal{C}}$ are in particular weak equivalences in $\mathcal{C}$.

We inductively construct I_m as follows. For $m = 0, 1$, we put $I_m = I_\mathcal{C}$. For $m > 1$, we define

$$
I_m^0 := \Sigma_m \cdot I_\mathcal{C} \cup \bigcup_{m=m'+m''} (\Sigma_m \cdot_{\Sigma_{m'} \times \Sigma_{m''}} I_{m'} \,\square\, I_{m''}) \cup \bigcup_{m=tn+u,\,y} \Sigma_m \cdot_{\Sigma_u \times (\Sigma_n \rtimes \Sigma_t^n)} y \,\square\, I_t^{\square n}. \qquad (3.1.19)
$$

The first union runs over partitions of m into *positive* parts. The second union runs over all multi-indices (of the same size) $t \geq 1$, $n \geq 1$ where at least one entry $n_i > 1$, all $u \geq 0$, and all $y \in \mathcal{Y}_{n,u}$ (which is a set by assumption). As usual, we have abbreviated $I_t^{\square n} := I_{t_1}^{\square n_1} \,\square\, \cdots \,\square\, I_{t_e}^{\square n_e}$. Note that m', m'', and the t_i are all strictly less than m. Therefore, $I_{m'}$ etc. is defined. Finally, we inductively define

$$
I_m := \cup_{j \geq 0} I_m^j, \quad I_{j+1} := I_j \,\square\, I_\mathcal{C}. \qquad (3.1.20)
$$

By Lemma 3.1.12, I_m consists of injective cofibrations. Moreover, $I_m \supset I_m^0 \supset \Sigma_m \cdot I_\mathcal{C} = I_{\Sigma_m^{\mathrm{pro}}\mathcal{C}}$, as requested above. Hence, W and I define a combinatorial model structure on $\Sigma_m\mathcal{C}$. By design, the functor in 3.1.4(3.1.5) is a left Quillen bifunctor. In fact, for partitions $m = m' + m''$ into positive parts, this is already true for I_m^0. For the partition $m = m + 0$, this holds by the construction in (3.1.20). Again by design, the strong admissibility requirement (5) is met for those multi-indices n where at least one n_i is at least 2. If all $n_i = 1$, then the expression in (5) reduces to $\Sigma_{t+u} \cdot_{\Sigma_u \times \Sigma_t} y \,\square\, h$ (where $t + u := \sum t_i + u$), which is the $(t+u)$-th level of $G_u(y) \,\square\, G_t(h)$. The latter map is a cofibration in $\Sigma\mathcal{C}$ by the assumption on $\mathcal{Y}$ made in Definition 3.1.4.

This also shows that the tractability of $\Sigma_{m'}\mathcal{C}$ etc. carries over to the one of the newly minted model structure on $\Sigma_m\mathcal{C}$. $\qquad\square$

Remark 3.1.21. It follows from the construction above that the canonical admissible model structure is minimal among strongly admissible ones in the sense that the identity is a left Quillen functor $\Sigma^{\mathrm{can}}\mathcal{C} \to \Sigma^{\mathrm{strongly\ ad}}\mathcal{C}$.

In our main application of strongly admissible model structures, Theorem 3.1.6(5), we actually only need 3.1.4(4) to hold for m', $m'' \geq 1$. For this purpose, one can use the model structures defined by W as above and I_m^0 (as opposed to I_m).

3.2. **Unstable model structures on spectra.** In Sections 3.2, 3.3 and 3.4, we will use the following convention:

Assumption 3.2.1. $\mathcal{C}$ is a tractable, pretty small, left proper, i-monoidal, flat, symmetric monoidal model category. We fix an admissible model structure on $\Sigma\mathcal{C}$, for example the projective model structure (Definition 3.1.4).

In practice, these assumptions are both mild and robust. They are satisfied for simplicial sets, simplicial presheaves, and chain complexes of abelian groups, for example. Moreover, if $\mathcal{C}$ has these properties, then so does any monoidal left Bousfield localization $\mathrm{L}_S^\otimes\mathcal{C}$, as well as any model structure that is transferred from $\mathcal{C}$ to $\mathcal{D}$, provided that the adjunction has good monoidal properties. The reader is referred to [PS, Sections 5, 6, 7] for further examples and precise statements of the above claims.

Suppose R is a commutative monoid in $\Sigma\mathcal{C}$. We denote the category of R-modules in $\Sigma\mathcal{C}$ by $\mathbf{Mod}_R$ and refer to it as the *category of R-spectra*. See [HSS00, Section 2.2] for more details. R-spectra form a symmetric monoidal category with the tensor product of R-modules M and N being

$$M \otimes_R N = \mathrm{coeq}(M \otimes R \otimes N \rightrightarrows M \otimes N),$$

where the tensor products on the right are computed in $\Sigma\mathcal{C}$. In this section we transfer any admissible model structure on symmetric sequences to R-spectra by means of the adjunction

$$R \otimes - : \Sigma\mathcal{C} \leftrightarrows \mathbf{Mod}_R : U. \tag{3.2.2}$$

Example 3.2.3. In many applications, R is the free commutative monoid on $G_1(A)$ for some object $A \in \mathcal{C}$, i.e., $R_n = A^{\otimes n}$ with Σ_n acting by permutations. In Proposition 3.3.9 we discuss the case $A = 1_\mathcal{C}$, the monoidal unit. More specifically, for $\mathcal{C} = \mathbf{sSet}_\bullet$ (pointed simplicial sets) and the pointed circle $A = S^1$, $\mathbf{Mod}_R$ is the category of simplicial symmetric S^1-spectra.

The model category used in motivic homotopy theory is $\mathcal{C} = \mathbf{sPSh}_\bullet(\mathbf{Sm}/S)$ (pointed simplicial presheaves on the site of smooth schemes over some base scheme S), for which we take the pointed projective line $A = (\mathbf{P}_S^1, \infty)$ or, alternatively, $A = \mathbf{A}^1/(\mathbf{A}^1 \setminus \{0\})$ [Jar00]. The category $\mathbf{Mod}_R$ is known as the category of motivic $\mathbf{P}^1$-spectra. In the projective model structure on *pointed* simplicial presheaves (or any localization thereof), $(\mathbf{P}_S^1, \infty)$ is *not* cofibrant. This is why we avoid imposing any cofibrancy hypotheses on R, unlike Hovey [Hov01, Section 8]. The flatness of $\mathcal{C}$ ensures that the category of R-spectra is replaced by a Quillen equivalent category if R is replaced by a weakly equivalent commutative monoid, see [SS00, Theorem 4.3]. This is used in Section 4.4 to construct a strictly commutative $\mathbf{P}^1$-spectrum representing Deligne cohomology.

Definition 3.2.4. Suppose that $\Sigma\mathcal{C}$ is equipped with a level-k admissible model structure denoted $\Sigma^{\geq k}\mathcal{C}$. The *level-$k$ admissible model structure* $\mathbf{Mod}_R^{\geq k}$ on $\mathbf{Mod}_R$ is the model structure transferred from $\Sigma^{\geq k}\mathcal{C}$ along (3.2.2). As in Theorem 3.1.6, $\mathbf{Mod}_R^{\geq 0}$ and $\mathbf{Mod}_R^{\geq 1}$ are called the *admissible* and *positive admissible model structure* and are denoted by $\mathbf{Mod}_R$ and $\mathbf{Mod}_R^+$ respectively.

We now study this transferred model structure on $\mathbf{Mod}_R$. The existence of this model structure is a consequence of the monoid axiom of Schwede and Shipley [SS00, Theorem 4.1(2)], but can also be derived from i-monoidality. Note that under mild auxiliary assumptions, i-monoidality implies the monoid axiom [PS, Lemma 3.2.5]. For symmetric spectra in simplicial sets, the transferred injective (equivalently, mixed) model structure is called the level S-model structure [Shi04, Proposition 2.2]. For symmetric spectra in an abstract model category, the transferred projective model structure was studied by Hovey [Hov01, Theorem 8.2]. The positive model structure studied in [GG11, Proposition 1] is also based on the projective model structure. The projective and mixed model structures are admissible, but (in a general model category $\mathcal{C}$) not strongly admissible. The strong admissibility of the model structure on $\Sigma\mathcal{C}$ will (almost) guarantee the strong admissibility of operads (Theorem 3.4.3). The stability of left properness under passing to a category of R-modules (and much more general algebraic structures) was established by Batanin and Berger [BB13, Theorem 2.11].

The symmetric i-monoidality, symmetric flatness and symmetroidality are, to the best of our knowledge, new. They are the key input in establishing the existence of a model structure on commutative ring spectra and algebras over more general operads.

Theorem 3.2.5. *Let $\mathcal{C}$ be a model category satisfying Assumption 3.2.1. Suppose that $\Sigma\mathcal{C}$ is a equipped with a level-k admissible model structure. Let R be a commutative monoid in $\Sigma\mathcal{C}$.*

The level-k admissible model structure $\mathbf{Mod}_R^{\geq k}$ exists and is tractable. Its generating (acyclic) cofibrations are $R \otimes f$, where f runs through the generating (acyclic) cofibrations of $\Sigma\mathcal{C}$. The weak equivalences and (acyclic) fibrations in $\mathbf{Mod}_R^{\geq k}$ are transferred from $\Sigma^{\geq k}\mathcal{C}$.

For any $k \geq 0$, $\mathbf{Mod}_R^{\geq k}$ has the following properties: it is symmetric monoidal, i-monoidal, left proper. If $\mathcal{C}$ satisfies the monoid axiom, then so does $\mathbf{Mod}_R^{\geq k}$. For $k = 0$ (!), $\mathbf{Mod}_R^{\geq 0}$ is also flat.

For any $k > 0$, the following holds: the (acyclic) cofibrations of $\mathbf{Mod}_R^{\geq k}$ form an (acyclic) symmetric i-monoidal and symmetric flat class in $\mathbf{Mod}_R^{\geq 0}$. Moreover, if the admissible model structure on $\Sigma\mathcal{C}$ is strongly $\mathcal{Y}$-admissible in the sense of Definition 3.1.4(4), then the (acyclic) cofibrations of $\mathbf{Mod}_R^{\geq k}$ are (acyclic) $\mathrm{cof}(R \otimes \mathcal{Y})$-symmetroidal in $\mathbf{Mod}_R^{\geq 0}$, i.e., (acyclic) symmetroidal with respect to the weak saturation of the class of maps $R \otimes y$, $y \in \mathcal{Y}$. [1]

For a map $\varphi : R \to S$ of commutative monoids in $\Sigma\mathcal{C}$, there is a Quillen adjunction

$$\varphi_* = S \otimes_R - : \mathbf{Mod}_R^{\geq k} \leftrightarrows \mathbf{Mod}_S^{\geq k} : \varphi^*,$$

which is a Quillen equivalence if φ is a weak equivalence (in $\Sigma\mathcal{C}$).

Proof. By [PS, Theorem 8.2.5], the tractability, i-monoidality, left properness, monoid axiom transfers from $\Sigma^{\geq k}\mathcal{C}$ to $\mathbf{Mod}_R^{\geq k}$. Similarly, the properties of the cofibrations of $\Sigma^{\geq k}\mathcal{C}$ ($k > 0$) of being symmetric i-monoidal, symmetric flat or symmetroidal transfer from symmetric sequences to R-modules by [PS, Proposition 5.2.5, Proposition 5.2.6], using that (3.2.2) is a Hopf adjunction with a strong monoidal left adjoint.

The Quillen adjunction between R- and S-spectra follows since both model structures are transferred from $\Sigma^{\geq k}\mathcal{C}$. If φ is a weak equivalence, φ_* is a Quillen equivalence by the flatness of $\mathbf{Mod}_R^{\geq k}$ and [SS00, Theorem 4.3]. $\square$

3.3. **Stable model structures on spectra.** In this section we localize the unstable model structure on R-modules to obtain the stable model structure. Consider the Quillen adjunction

$$F_n : \mathcal{C} \leftrightarrows \mathbf{Mod}_R : \mathrm{Ev}_n \tag{3.3.1}$$

obtained by composing the adjunctions $\Sigma_n \cdot - : \mathcal{C} \leftrightarrows \Sigma_n\mathcal{C}$, (3.1.1) and (3.2.2). The right adjoint evaluates at the nth level (after forgetting the R-module structure and the Σ_n-action). The left adjoint is given by $F_n(X) = G_n(\Sigma_n \cdot X) \otimes R$.

Definition 3.3.2. Suppose $k \geq 0$. Consider the symmetric monoidal left Bousfield localization, i.e., the localization in the bicategory of $\mathcal{V}$-enriched symmetric monoidal model categories, of the level-k admissible model structure $\mathbf{Mod}_R^{\geq k}$ on R-modules with respect to the set

$$\xi^R := \{\xi_n^R := \xi_n : F_n(\mathrm{Q}R_n) \to R, \ n \geq 0\}.$$

Here Q is the cofibrant replacement functor in $\mathcal{C}$. This model structure is called the *stable level-k admissible model structure*. It is denoted $\mathbf{Mod}_R^{\mathrm{s}, \geq k, \mathrm{ad}}$ or $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$. As usual, we drop the prefix k- for $k = 0$ (denoted $\mathbf{Mod}_R^{\mathrm{s}}$) and speak of the *stable positive model structure* in the case $k = 1$ (denoted $\mathbf{Mod}_R^{\mathrm{s}, +}$).

Remark 3.3.3. For $n \geq k$, the map ξ_n above is the homotopy adjoint of the identity map $R_n := \mathrm{Ev}_n(R) \to \mathrm{Ev}_n(R) \in \mathcal{C}$ with respect to the adjunction (3.3.1). See [PS, Section 2], for example, for a general discussion of homotopy adjoints.

If $\mathcal{C}$ is $\mathcal{V}$-enriched, then $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ is the $\mathcal{V}$-enriched monoidal localization by [PS, Remark 6.1.5]. The name "stable model structure" for this model structure is standard, even though this model structure is not stable for all R, for example for $R_n = 1_\mathcal{C}$ (see the discussion following Proposition 3.3.9). See, however, Theorem 3.3.4(5).

---

[1] In the case of symmetric spectra in $\mathcal{C} = \mathbf{sSet}_\bullet$ and $R_n = S^n$, the n-sphere, claims have been made that every cofibration in $\Sigma^{+, \mathrm{in}}\mathcal{C}$ (positive injective structure) is in fact a power cofibration. This is a stronger statement than symmetroidality. However, there is a counterexample as follows: the object $(R \otimes G_1(*_+))^{\otimes_R 2} = R \otimes G_2(\Sigma_2 \cdot *_+)$ is not cofibrant in $\Sigma_2^{\mathrm{pro}}\mathbf{Mod}_R$ because its evaluation in degree 2 is $(* \sqcup *)_+$ on which both copies of Σ_2 act by permutation. This object is not cofibrant in $\Sigma_2^{\mathrm{pro}}\Sigma_2^{\mathrm{in}}\mathbf{sSet}_\bullet$, see Remark 3.1.17.

Suppose R is the free commutative monoid on $G_1(R_1)$, i.e., $R_n = R_1^{\otimes n}$. Suppose further that R_1 is either cofibrant in $\mathcal{C}$ or monoidally cofibrant, i.e., there is a cofibration $1 \to R_1$. Then the above localization agrees with the one with respect to ξ_1 only, since $F_1(QR_1)^{\otimes n} = F_n((QR_1)^{\otimes n})$ and $(QR_1)^{\otimes n} \sim Q(R_1^{\otimes n})$ by [PS, Lemma 9.4.5].

In the case $k = 0$, the projective structure on $\Sigma\mathcal{C}$ and $R = \mathrm{Sym}(G_1(R_1))$ with a cofibrant object $R_1 \in \mathcal{C}$, the stable model structure has been defined by Hovey in [Hov01, Definition 8.7] as the localization (in the bicategory of mere model categories, i.e., disregarding the monoidality and $\mathcal{V}$-enrichment of $\mathbf{Mod}_R$) with respect to the set of maps

$$\zeta_n(C) : F_{n+1}(C \otimes R_1) \to F_n(C)$$

adjoint to the map $C \otimes R_1 \to \mathrm{Ev}_{n+1}F_n(C) = \Sigma_{n+1} \cdot C \otimes R_1$ given by the identity element of Σ_{n+1}. Here $n \geq 0$ and C runs through the (co)domains of generating cofibrations of $\mathcal{C}$. Hovey's definition agrees with the one above. Indeed, by [PS, Proposition 6.1.3], the monoidal localization with respect to $\xi_1 = \zeta_0(1)$ is the (ordinary) localization with respect to the set $F_n(C) \otimes_R Q\zeta_0(1)$, which is equivalent to the one by $F_n(C) \otimes_R \zeta_0(1)$ by the flatness of $\mathbf{Mod}_R$. One checks that this map is just $\zeta_n(C)$. The objects $F_n(\mathcal{C})$ are precisely the (co)domains of generating cofibrations of the (projective, nonpositive) structure $\mathbf{Mod}_R^{\geq 0, \mathrm{pro}}$.

For the same type of commutative monoid Gorchinskiy and Guletskiĭ define the stable positive structure to be the localization with respect to Hovey's class, but for $n \geq 1$. Both their definition and Definition 3.3.2 have the property that positive stable weak equivalences agree with nonpositive stable equivalences [GG11, Theorem 9], Theorem 3.3.4(3), so that the model structure in loc. cit. is Quillen equivalent to the one defined above.

We now study the stable model structures, especially the stable positive one. Its most striking properties are symmetric i-monoidality, symmetroidality and symmetric flatness. In the generality stated below, these properties are new. However, various aspects of this description are well-known. For example, parts (1) and (2) are proved in [MMSS01, Theorem 14.2] in the case of symmetric spectra in simplicial sets. With a slightly different definition, see Remark 3.3.3, Part (3) is due to Gorchinskiy and Guletskiĭ [GG11, Theorem 9]. In a general model category, the question whether the monoid axiom holds in the stable model structure was unknown (see remarks at the end of Section 7 in [Hov01]). Part (2) settles this question for a broad class of model categories. If $\mathcal{C}$ consists of the Nisnevich $\mathbf{A}^1$-localization of simplicial presheaves with the injective model structure, the existence of the stable positive model structure has been shown by Hornbostel [Hor13, Theorem 3.4] in the case where the chosen model structure is the mixed model structure. A special case of symmetric flatness (namely the case where the weak equivalence $y \in \Sigma_n\mathbf{Mod}_R$ is given by the projective cofibrant replacement of $1_{\mathbf{Mod}_R} = R$, $\mathrm{E}\Sigma_n \to R$) is due to Gorchinskiy and Guletskiĭ [GG11, Theorem 11]. They prove this statement under the assumption that every cofibration in $\mathbf{Mod}_R^{+, \mathrm{pro}}$ (i.e., the transfer of the positive projective structure on $\Sigma\mathcal{C}$ to R-modules) is a power cofibration. As was explained in Lemma 3.1.14, this condition ensures that the projective structure is strongly admissible (which only holds in very special cases). The more general symmetric flatness will be used to show the operadic rectification (Theorem 3.4.4).

Theorem 3.3.4. *Again, let $\mathcal{C}$ be a model category satisfying Assumption 3.2.1, equip $\Sigma\mathcal{C}$ with an admissible model structure, and let R be a commutative monoid in $\Sigma\mathcal{C}$.*

(1) *The model category $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ exists. It is a left proper, tractable model category. Its fibrant objects are those objects W which are fibrant in $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ and such that the derived internal Hom in $\mathbf{Mod}_R^{\geq k}$,*

$$\mathrm{R}\,\mathrm{Hom}(\xi_n, W)$$

   is a weak equivalence for all $n \geq 0$.
(2) *For any $k \geq 0$, $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ is a symmetric monoidal, i-monoidal and flat model category. It also satisfies the monoid axiom if $\mathcal{C}$ does.*
(3) *The class of stable level-k weak equivalences $\mathrm{W}_{\mathrm{s}, \geq k} := \mathrm{W}_{\mathbf{Mod}_R^{\mathrm{s}, \geq k}}$, is independent of k. In particular, the categories $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ are Quillen equivalent for all $k \geq 0$.*
(4) *The model structure $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ is independent of the choice of the admissible model structure in the sense that for any two choices of admissible model structures on $\Sigma\mathcal{C}$, the resulting stable level-k model structures are Quillen equivalent.*

(5) *Suppose that $\mathcal{C}$ is pointed. Let us write $S^1 \in \mathcal{C}$ for some cofibrant representative of the suspension of the monoidal unit $1_{\mathcal{C}}$, i.e., the homotopy pushout $* \sqcup_{1_{\mathcal{C}}}^h *$. Suppose that R is such that R_1 is weakly equivalent to $S^1 \otimes B$ for some cofibrant object $B \in \mathcal{C}$. Then the model structure $\mathbf{Mod}_R^{s, \geq k}$ is stable in the sense that it is pointed and the suspension and loop functors are inverse Quillen equivalences on $\mathbf{Mod}_R^{s, \geq k}$ for $k \geq 0$ [SS03b, Definition 2.1.1].*

(6) *For any $k > 0$, $\mathbf{Mod}_R^{s, \geq k}$ is symmetric flat and symmetric i-monoidal. If, moreover, the admissible model structure on $\Sigma\mathcal{C}$ is strongly $\mathcal{Y}$-admissible, then the (acyclic) cofibrations of $\mathbf{Mod}_R^{s, \geq k}$ form an (acyclic) $\mathrm{cof}(R \otimes \mathcal{Y})$-symmetroidal class in $\mathbf{Mod}_R^{s, \geq 0}$. In particular $\mathbf{Mod}_R^{s, \geq k}$ is $\mathrm{cof}(R \otimes \mathcal{Y})$-symmetroidal in this case.*

(7) *For a weak equivalence $\varphi : R \to S$ of commutative monoids in $\Sigma\mathcal{C}$. Suppose that there is a weak equivalence $\underline{\mathrm{L}}\varphi_*(R) \sim S$ in $\mathbf{Mod}_S$, where $\underline{\mathrm{L}}\varphi_*$ denotes the left derived functor of $\varphi_* : \mathbf{Mod}_R \to \mathbf{Mod}_S$. For example, this condition is satisfied if $1_{\mathcal{C}}$ is cofibrant or if the map φ is a cofibration in $\mathbf{Mod}_R$. Then there is a Quillen equivalence for any $k \geq 0$,*

$$\varphi_* = S \otimes_R - : \mathbf{Mod}_R^{s, \geq k} \leftrightarrows \mathbf{Mod}_S^{s, \geq k} : \varphi^*. \tag{3.3.5}$$

Proof. The existence and the properties claimed in Part (1) follow from [PS, Proposition 6.1.3], since the corresponding unstable model structure on $\mathbf{Mod}_R^{\geq k}$ has these properties by Theorem 3.2.5. The description of fibrant objects is an application of [PS, Lemma 6.1.6].

(2): This follows from the corresponding properties of the unstable model structure established in Theorem 3.2.5 and the stability of these properties under monoidal left Bousfield localizations established in [PS, Theorem 6.2.2].

We now show (3), essentially reproducing the proof of [GG11, Theorem 9]. In the proof of this part, we will not explicitly mention that a model structure on $\mathbf{Mod}_R$ is level-0 or unstable, but will always indicate level-k (for $k > 0$) and/or stability where necessary. Moreover, a superscript indicates a certain model-categorical operation related to the model category structure in question. For example Q is the cofibrant replacement functor in $\mathbf{Mod}_R$, $\mathrm{Q}^{\geq k}$ the one of $\mathbf{Mod}_R^{\geq k}$. Similarly, $\underline{\mathrm{R}}\mathrm{Map}^{s, \geq k}$ is the derived mapping space of $\mathbf{Mod}_R^{s, \geq k}$. By definition, there is a Quillen adjunction, where Hom denotes the internal Hom:

$$F_k\mathrm{Q}(R_k) \otimes_R - : \mathbf{Mod}_R \leftrightarrows \mathbf{Mod}_R^{\geq k} : \Theta_k := \mathrm{Hom}(F_k\mathrm{Q}R_k, -). \tag{3.3.6}$$

It localizes to a Quillen adjunction

$$F_k\mathrm{Q}(R_k) \otimes_R - : \mathbf{Mod}_R^{s} \leftrightarrows \mathbf{Mod}_R^{s, \geq k} : \Theta_k. \tag{3.3.7}$$

In fact, $F_k\mathrm{Q}(R_k) \otimes_R^{\mathrm{L}} \xi^R$ is weakly equivalent to $F_k\mathrm{Q}(R_k) \otimes_R \xi^R$ by the flatness of $\mathbf{Mod}_R$. The latter set is contained in the monoidal saturation of ξ^R with respect to the model structure $\mathbf{Mod}_R^{\geq k}$ since $F_k\mathrm{Q}(R_k)$ is cofibrant in $\mathbf{Mod}_R^{\geq k}$. Therefore the derived functor of the left adjoint sends ξ^R to weak equivalences in $\mathbf{Mod}_R^{s, \geq k}$ which shows that (3.3.7) is a Quillen adjunction.

We first prove two preliminary claims. The first claim is that any $f \in \mathrm{W}_{\geq k}$ is a stable (nonpositive) weak equivalence. Both W_s and $\mathrm{W}_{\geq k}$ are preserved by (unstable nonpositive) fibrant replacement, so that we may assume that f is a map between nonpositively, a fortiori level-k fibrant objects. By Brown's lemma (applied to (3.3.6)), $\Theta_k(f) \in \mathrm{W} \subset \mathrm{W}_s$. Let $f \xrightarrow{\sim} f'$ be the fibrant replacement of f in the stable structure. In the following commutative diagram, $\sim$ indicates a stable equivalence.

$$
\begin{array}{ccccc}
f = \mathrm{Hom}(F_0 1, f) & \longrightarrow & \mathrm{Hom}(F_0 \mathrm{Q}1, f) & \longrightarrow & \Theta_k(f) \\
\downarrow {\scriptstyle \sim} & & \downarrow & & \downarrow \\
f' = \mathrm{Hom}(F_0 1, f') & \xrightarrow[\sim]{*} & \mathrm{Hom}(F_0 \mathrm{Q}1, f') & \xrightarrow[\sim]{**} & \Theta_k(f').
\end{array}
$$

The map $*$ is a stable weak equivalence since $F_0(\mathrm{Q}1) \otimes_R Y \to F_0(1) \otimes_R Y$ is a weak equivalence in $\mathbf{Mod}_R$ (and therefore $\mathbf{Mod}_R^{s}$) for any cofibrant object $Y \in \mathbf{Mod}_R$ by the flatness of $\mathbf{Mod}_R$ (Theorem 3.2.5). The map $**$ is a stable weak equivalence by the very definition of this model structure. Consequently, in the homotopy category $\mathbf{Ho}(\mathbf{Mod}_R^{s})$, f is a retract of the isomorphism $\Theta_k(f)$, so that f is also a stable weak equivalence. This finishes the first claim.

The second claim is that for any fibrant object $Z \in \mathbf{Mod}_R^{s, \geq k}$, $\mathrm{Hom}(\xi_k, Z) : Z \to \Theta_k(Z)$ is an (unstable) weak equivalence in $\mathbf{Mod}_R^{\geq k}$. Indeed, for any $n \geq k$ and any cofibrant object $T \in \Sigma_n \mathcal{C}$,

$$\begin{aligned}
\underline{\mathrm{R}}\mathrm{Map}^{\geq k}(F_n(T), Z) & \xrightarrow{\xi_k} & \underline{\mathrm{R}}\mathrm{Map}^{\geq k}(F_n(T) \otimes_R F_k(\mathrm{Q}R_k), Z) \\
& \sim & \underline{\mathrm{R}}\mathrm{Map}^{\geq k}(F_n(T), \Theta_k(Z))
\end{aligned}$$

are weak equivalences, the first by the definition of the *monoidal* Bousfield localization, the second by (homotopy) adjunction. Since the objects $F_n(T)$ are homotopy generators of $\mathbf{Mod}_R^{\geq k}$, we are done with the second claim.

The first claim implies that there is a Quillen adjunction

$$\mathrm{id} : \mathbf{Mod}_R^{s, \geq k} \leftrightarrows \mathbf{Mod}_R^s : \mathrm{id}. \tag{3.3.8}$$

Indeed, $\mathrm{Q}^{\geq k}(\xi_n^R)$ is level-k and therefore (by the first claim) stably weakly equivalent to ξ_n^R. Therefore, any fibrant object $T \in \mathbf{Mod}_R^s$, is also fibrant in $\mathbf{Mod}_R^{s, \geq k}$. For any $X \in \mathbf{Mod}_R$, the natural map of derived mapping spaces (in $\mathbf{Mod}_R^s$ and $\mathbf{Mod}_R^{s, \geq k}$, respectively) induced by the transformation of cofibrant replacement functors $\mathrm{Q}^{\geq k} X \to \mathrm{Q}X$,

$$\underline{\mathrm{R}}\mathrm{Map}^s(X, T) \to \underline{\mathrm{R}}\mathrm{Map}^{s, \geq k}(X, T)$$

is a weak equivalence. Indeed, $\mathrm{Q}^{\geq k} X \to \mathrm{Q}X$, is a positive weak equivalence and therefore a stable (nonpositive) equivalence by the first claim.

We finally prove the proper statement. For a morphism f and an object $Z \in \mathbf{Mod}_R$, we consider the commutative diagram whose horizontal maps stem from the Quillen adjunction (3.3.8):

$$\begin{array}{ccc}
\underline{\mathrm{R}}\mathrm{Map}^s(f, Z) & \longrightarrow & \underline{\mathrm{R}}\mathrm{Map}^{s, \geq k}(f, Z) \\
\downarrow & & \downarrow \\
\underline{\mathrm{R}}\mathrm{Map}^s(f, \Theta_k Z) & \longrightarrow & \underline{\mathrm{R}}\mathrm{Map}^{s, \geq k}(f, \Theta_k Z).
\end{array}$$

Suppose f is in $\mathrm{W}_{s, \geq k}$, so that $\underline{\mathrm{R}}\mathrm{Map}^{s, \geq k}(f, -)$ is a weak equivalence. For any fibrant object $Z \in \mathbf{Mod}_R^s$, the top horizontal map is a weak equivalence (of arrows, i.e., a weak equivalence of source and target) by the above consequence of the first claim. Thus $\underline{\mathrm{R}}\mathrm{Map}^s(f, Z)$ is a weak equivalence, i.e., f is in W_s.

Conversely, suppose $f \in \mathrm{W}_s$ so that $\underline{\mathrm{R}}\mathrm{Map}^s(f, -)$ is a weak equivalence. For any fibrant object $Z \in \mathbf{Mod}_R^{s, \geq k}$, $\Theta_k(Z)$ is fibrant in $\mathbf{Mod}_R^s$ by (3.3.7). Hence, by the consequence of the first claim, the bottom horizontal map is a weak equivalence. By the second claim $Z \to \Theta_k(Z)$ is in $\mathrm{W}_{\geq k} \subset \mathrm{W}_{s, \geq k}$, hence the right hand vertical map is a weak equivalence. We conclude that $\underline{\mathrm{R}}\mathrm{Map}^{s, \geq k}(f, Z)$ is a weak equivalence so that f is a weak equivalence in $\mathbf{Mod}_R^{s, \geq k}$.

(4): By Definition 3.1.4, weak equivalences in $\Sigma^{\mathrm{pro}} \mathcal{C}$ and $\Sigma^{\mathrm{ad}} \mathcal{C}$ are the same, so the same is true for $\mathbf{Mod}_R^{\geq k, \mathrm{pro}}$ and $\mathbf{Mod}_R^{\geq k, \mathrm{ad}}$ which are therefore Quillen equivalent. This localizes to a Quillen equivalence $\mathbf{Mod}_R^{s, \geq k, \mathrm{pro}} \sim \mathbf{Mod}_R^{s, \geq k, \mathrm{ad}}$ since they are monoidal localization with respect to the same set ξ^R of morphisms.

(5): By (3), we may assume $k = 0$. For a cofibrant object $X \in \mathbf{Mod}_R^{s, \geq 0}$, the suspension ΣX is weakly equivalent to $X \otimes S^1 = X \otimes F_0(S^1)$, where F_n is defined in (3.3.1). As F_1 is a left Quillen functor, $F_0(S^1) \otimes F_1(B) = F_1(S^1 \otimes B)$ is weakly equivalent to $F_1(\mathrm{Q}(R_1)) = R \otimes G_1(\mathrm{Q}(R_1))$, where Q is the cofibrant replacement functor. By definition of the stable model structure, this is stably weakly equivalent to $F_0(1_{\mathcal{C}}) = R$ which is the monoidal unit in $\mathbf{Mod}_R$. Thus the suspension functor is a Quillen equivalence on $\mathbf{Mod}_R^{s, \geq 0}$.

(6): Let $k > 0$. By Theorem 3.2.5, the cofibrations of $\mathbf{Mod}_R^{\geq k}$ are symmetric flat and symmetric i-monoidal in $\mathbf{Mod}_R^{\geq 0}$. By [PS, Theorem 6.2.2], they are also symmetric flat and symmetric i-monoidal in $\mathbf{Mod}_R^{s, \geq 0}$. Since (acyclic) i-cofibrations only depend on the weak equivalences, the symmetric i-monoidality and symmetric flatness of a class of morphisms also only depends on the weak equivalences. By Part (3), we therefore conclude that the stable level-k model structure is symmetric flat and symmetric i-monoidal for $k > 0$.

The nonacyclic part of $\mathrm{cof}(R \otimes \mathcal{Y})$-symmetroidality of $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ follows immediately from the one of $\mathbf{Mod}_R^{\geq k}$. The acyclic part follows from a variant of [PS, Theorem 6.2.2(iii)], as follows: by [PS, Theorem 4.3.9(iii)], it is enough to show that the generating acyclic cofibrations of $\mathbf{Mod}_R^{\mathrm{s}, \geq k}$ are acyclic $\mathrm{cof}(R \otimes \mathcal{Y})$-symmetroidal in $\mathbf{Mod}_R^{\mathrm{s}, \geq 0}$. By tractability, we may assume they have cofibrant source. Thus they are acyclic $\mathcal{Y}$-symmetroidal in $\mathbf{Mod}_R^{\mathrm{s}, \geq 0}$ by [PS, Proposition 4.3.5].

(7): If $1_{\mathcal{C}}$ is cofibrant, then so is $R \in \mathbf{Mod}_R$, so that $\underline{\mathrm{L}}\varphi_*(R) \sim \varphi_*(R) = S$. If there is a cofibration $R \to S$ in $\mathbf{Mod}_R$ then $S \otimes_R - : \mathbf{Mod}_R \to \mathbf{Mod}_R$ preserves weak equivalences by [PS, Lemma 9.4.8], so that again $\underline{\mathrm{L}}\varphi_*(R) = S \otimes_R \mathrm{Q}R \sim S \otimes_R R = S$.

To prove the proper statement, we may assume $k = 0$ by (3) and the 2-out-of-3-property of Quillen equivalences. By [PS, Lemma 6.1.7], $\mathbf{Mod}_R^{\mathrm{s}}$ is Quillen equivalent to the monoidal localization $\mathrm{L}^{\otimes}_{\underline{\mathrm{L}}\varphi_*(\mathrm{Q}\xi^R)}\mathbf{Mod}_S$. The map $\underline{\mathrm{L}}\varphi_*(\xi_n^R)$ is weakly equivalent to $G_n(\Sigma_n \cdot \mathrm{Q}R_n) \otimes S \to \underline{\mathrm{L}}\varphi_*(R)$. The target is, by assumption, weakly equivalent to S. The map $G_n(\Sigma_n \cdot \mathrm{Q}R_n) \otimes S \to S$ is the composition of $G_n(\Sigma_n \cdot \mathrm{Q}\varphi_n) \otimes S$, which is an unstable weak equivalence by Brown's lemma, followed by ξ_n^S which is a stable equivalence of S-modules. Hence $\mathrm{L}^{\otimes}_{\underline{\mathrm{L}}\varphi_*(\mathrm{Q}\xi^R)}\mathbf{Mod}_S$ is Quillen equivalent to $\mathrm{L}^{\otimes}_{\xi^S}\mathbf{Mod}_S = \mathbf{Mod}_S^{\mathrm{s}}$. $\qquad\square$

We finish this section by examining the special case $R = E$, where E is the free commutative monoid in $\Sigma\mathcal{C}$ on the monoidal unit. Its levels are given by $E_n = 1_{\mathcal{C}}$, the monoidal unit (with the trivial Σ_n-action). In this case, E-modules coincide with I-spaces, as defined by Sagave and Schlichtkrull [SS12]. By definition, these are functors from the category I of finite sets and injections to $\mathcal{C}$. Indeed, an E-module X is the same as a sequence of objects $X_n \in \Sigma_n\mathcal{C}$ with a Σ_n-equivariant bonding map $X_n \cong X_n \otimes 1 \to X_{n+1}$. This datum is equivalent to specifying an I-space whose value on objects and isomorphisms $\sigma \in \Sigma_n$ is given by the X_n and whose value on injections is given by compositions of bonding maps. What is more, the stable model structure on I-spaces defined in loc. cit. agrees with the stable model structure on $\mathbf{Mod}_E$:

Proposition 3.3.9. *Let $\mathcal{C}$ be a model category satisfying Assumption 3.2.1. We equip $\Sigma\mathcal{C}$ with the projective model structure and consider the resulting unstable and stable level-k projective model structures on E-modules. The unstable and stable level-k projective structures on $\mathbf{Mod}_E$ and the category $I\mathcal{C}$ of I-spaces coincide, i.e., all 5 classes of maps are preserved under the above equivalence.*

Proof. The unstable level-k projective model structures on E-modules and I-spaces coincide since they are both transferred from $\prod_{n \geq k}\mathcal{C}$.

For the stable structure it is enough to prove that stable weak equivalence of I-spaces correspond to stable weak equivalences of E-modules. Both model structures are left Bousfield localizations, so it is sufficient to establish that the stably fibrant E-modules are exactly the stably fibrant I-spaces. By 3.3.4(1), stably level-k fibrant E-modules are precisely those E-modules X that are unstably level-k fibrant and such that $\underline{\mathrm{R}}\mathrm{Hom}(F_n(\mathrm{Q}1) \to E, X)$ is a weak equivalence in $\mathbf{Mod}_E^{\geq k}$ for all $n \geq 0$, or, equivalently, the r-th level $(r \geq k)$ of this is a weak equivalence. As $\mathbf{Mod}_E^{\geq k}$ is flat and X is fibrant, the derived internal Hom is weakly equivalent to the underived one. One easily checks there is an isomorphism in $\mathcal{C}$,

$$\mathrm{Hom}_{\mathbf{Mod}_E}(F_n(\mathrm{Q}1), X)_r = \mathrm{Hom}_{\mathcal{C}}(\mathrm{Q}1, X_{r+n}) \sim \mathrm{Hom}_{\mathcal{C}}(1, X_{r+n}) = X_{r+n},$$

where we have used the flatness of $\mathcal{C}$ (actually, only the unit axiom [Hov99, Lemma 4.2.7(b)]). In other words stably level-k fibrant E-modules are those unstably fibrant E-modules such that $X_r \to X_{r+n}$ is a weak equivalence for all $n \geq 0$ and all $r \geq k$. These are exactly the stably level-k fibrant I-spaces [SS12, Section 3.1]. $\qquad\square$

By [Hov01, Theorem 9.1], $\mathbf{Mod}_E^{\mathrm{s}, \geq 0}$ and therefore $\mathbf{Mod}_E^{\mathrm{s}, +}$ is Quillen equivalent to $\mathcal{C}$. Thus, even if $\mathcal{C}$ is, say, not symmetric flat (such as $\mathcal{C} = \mathbf{sSet}$), it is Quillen equivalent to E-modules (or I-spaces), which is, by the theorems in Section 3.4, much better behaved. This point of view goes back to Jeff Smith.

3.4. **Algebras over colored symmetric operads in symmetric spectra.** We now exploit the excellent model-theoretic properties of the stable positive model structure $\mathbf{Mod}_R^{\mathrm{s}, +}$ on symmetric R-spectra to study algebras over operads in this category. A symmetric single-colored operad O in $\mathbf{Mod}_R$ consists of an R-module O_n with a Σ_n-action for each $n \geq 0$. It can be thought of as the space of n-ary operations. For different n, they are connected by $\Sigma_{r_1} \times \cdots \times \Sigma_{r_n}$-equivariant maps

$$O_n \otimes_R O_{r_1} \otimes_R \ldots \otimes_R O_{r_n} \to O_{r_1 + \cdots + r_n}.$$

Specifying an O-algebra structure on some $M \in \mathbf{Mod}_R$ amounts to specifying maps

$$O_n \otimes_{R,\Sigma_n} M^{\otimes n} \to M$$

which are again compatible in a suitable sense. For example, the commutative operad Comm is such that $\mathrm{Comm}_n = 1_{\mathbf{Mod}_R} = R$, so a Comm-algebra is exactly a strictly commutative ring spectrum.

Since there are no essential additional difficulties, we actually work with W-*colored symmetric operads* or just operads for short. The set W (called set of *colors*) is fixed. Instead of the indexing by $n \in \mathbf{N}$ in single-colored operads, W-colored operads are indexed by tuples (s, w) consisting of a map of sets $s : I \to W$ (the *multisource*), where I is a finite set and $w \in W$ (the *target*). Such tuples form a category $\mathbf{sSeq}_W$. This category is a groupoid and the automorphism group of (s, w) is given by $\Sigma_s := \prod_{r \in W} \Sigma_{s^{-1}(r)}$. The category $\mathbf{sColl}_W(\mathbf{Mod}_R) := \mathbf{Fun}(\mathbf{sSeq}_W, \mathbf{Mod}_R)$ of *symmetric collections* is equipped with the *substitution product*, denoted $\circ$, which turns this into a monoidal category. Its monoidal unit $R[1]$ is such that $R[1]_{s,w} = \emptyset$ except for $s : I = \{*\} \to W$, $s(*) = w$, in which case it is R, the monoidal unit of $\mathbf{Mod}_R$.

A symmetric W-colored operad is, by definition, a monoid in $(\mathbf{sColl}_W(\mathbf{Mod}_R), \circ)$. They form a category denoted $\mathbf{sOper}_W(\mathbf{Mod}_R)$. The multiplication $O \circ O \to O$ amounts to giving maps

$$O_{s,w} \otimes \bigotimes_{i \in I} O_{t_i, s(i)} \to O_{\cup_{i \in I} t_i, w}.$$

An O-algebra consists of $M_w \in \mathbf{Mod}_R$, for every $w \in W$, together with maps

$$O_{s,w} \otimes \bigotimes_{i \in I} M_{s(i)} \to M_w.$$

Of course, these are subject to appropriate associativity and unitality constraints. For a slightly less short summary of operads and their algebras, the reader may consult [PS, Section 9].

We now turn to the model-theoretic properties of algebras over operads in R-spectra. We show the admissibility of all operads (3.4.1), give a criterion for (almost) strong admissibility of levelwise cofibrant operads (3.4.3), rectification of algebras over weakly equivalent operads (3.4.4), and Quillen equivalences of algebras over operads in different categories of spectra (3.4.9) and finally the special case of R-spectra and S-spectra, where $R \sim S$ are weakly equivalent (3.4.10).

The admissibility of operads in symmetric spectra is due to Elmendorf and Mandell for $\mathcal{C} = \mathbf{Top}$ [EM06, Theorem 1.3], and Harper for $\mathcal{C} = \mathbf{sSet}_\bullet$ [Har09, Theorem 1.1]. It was generalized by Hornbostel to the category $\mathcal{C}$ of simplicial presheaves with the injective model structure and the injective model structure on $\Sigma\mathcal{C}$ [Hor13, Theorem 1.3]. In the latter two cases, all objects are cofibrant. This considerably simplifies the situation because all i-monoidality questions are trivial. The assumption that every object in $\mathcal{C}$ is cofibrant excludes the projective model structures on presheaves, which is a main motivating example for us. In fact, this paper grew out from an attempt to construct an algebraic cobordism spectrum, as a *fibrant* commutative ring spectrum. The fibrancy is necessary to actually compute the homotopy groups of this spectrum (i.e., the higher algebraic cobordism groups). For the injective model structure on presheaves the fibrancy condition is practically impossible to check.

Theorem 3.4.1. *Any (symmetric W-colored) operad O in $\mathbf{Mod}_R$ is admissible, i.e., the category of O-algebras carries a model structure that is transferred along the adjunction*

$$O \circ - : \mathbf{Mod}_R^{\mathrm{s},+} \leftrightarrows \mathbf{Alg}_O(\mathbf{Mod}_R) : U.$$

We refer to it as the stable positive model structure *and denote it by* $\mathbf{Alg}_O^{\mathrm{s},+}(\mathbf{Mod}_R)$.

For example, for $O = \mathrm{Comm}$, this gives a model structure on strictly commutative ring spectra. For the operad $\mathbf{sOper}_W$ of W-colored operads, this gives a model structure on W-colored symmetric operads in spectra.

Proof. This follows from [PS, Theorem 9.2.11] whose assumptions are satisfied by Theorem 3.3.4. □

Example 3.4.2. For $\mathcal{C} = \mathbf{sSet}_\bullet$ and R given by $R_n = (S^1)^{\wedge n}$, $\mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}_R)$ is known as the category of commutative ring spectra (in simplicial sets). Another example is the case $\mathcal{C} = \mathbf{sPSh}_\bullet(\mathbf{Sm}/S)$ of pointed simplicial presheaves on the site of smooth schemes over some base scheme S and the monoid given by $R_n = (\mathbf{P}_S^1, \infty)^{\wedge n}$. Any of the standard model structures, for example the projective model structure or any monoidal localization, such as the Nisnevich localization or the Nisnevich-$\mathbf{A}^1$ localization satisfies the

Assumption 3.2.1. In this case $\mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}_R)$ is the category of (strictly) commutative motivic ring $\mathbf{P}^1$-spectra.

The next result addresses the strong admissibility of operads, i.e., the behavior of cofibrant algebras under the forgetful functor $\mathbf{Alg}_O^{\mathrm{s},+}(\mathbf{Mod}_R) \to \mathbf{Mod}_R$ [PS, Definition 9.2.1]. The main abstract result [PS, Theorem 9.2.19] works for operads whose levels $O_{s,w}$ are of the form $R \otimes$ some *positively* cofibrant object, which excludes the commutative operad, for example. The following variant does include this example. For $\mathcal{C} = \mathbf{sSet}_\bullet$ and the injective model structure on $\Sigma\mathbf{sSet}_\bullet$ and the commutative operad, the statement is due to Shipley [Shi04, Proposition 4.1]. By Lemma 3.1.12, the injective structure on $\Sigma\mathbf{sSet}_\bullet$ is strongly admissible, so our result generalizes Shipley's. Recall the notion of a strongly $\mathcal{Y}$-admissible model structure from Definition 3.1.4 and also the construction of such model structures from Theorem 3.1.18.

Theorem 3.4.3. *Suppose the admissible model structure on $\Sigma\mathcal{C}$ is strongly $\mathcal{Y}$-admissible with respect to some class $\mathcal{Y} = (\mathcal{Y}_n \subset \mathrm{Mor}_{\Sigma_n \Sigma\mathcal{C}})$. Suppose moreover that for all $(s : I \to W, w \in W) \in \mathbf{sSeq}_W$,*

$$(\eta_O)_{s,w} \in \mathrm{cof}(R \otimes \mathcal{Y}_n)(\subset \mathrm{Mor}_{\Sigma_n \mathbf{Mod}_R}),$$

where $\eta_O : R[1] \to O$ is the unit map of O and n is the finite multi-index given by $n_r = \sharp s^{-1}(r)$ for $r \in W$. (Note that only finitely many r appear since I is finite.) For example, if $\mathcal{Y}_n$ consists of $\{\emptyset \to 1\}$, this condition is satisfied for the commutative operad Comm.

Then the forgetful functor

$$\mathbf{Mod}_R^{\mathrm{s}} \leftarrow \mathbf{Alg}_O(\mathbf{Mod}_R^{\mathrm{s},+}) : U$$

preserves cofibrant objects and cofibrations between them. (Note that the "+" is missing at the left hand model structure.)

Proof. By [PS, Lemma 9.2.16], it is enough to notice that for any finite multi-index $n = (n_r)$, $n_r \geq 1$, any multi-source s as in the statement, any $w \in W$, and any finite family $x = (x_r)$ of generating cofibrations of $\mathbf{Mod}_R^{\mathrm{s},+}$

$$(\eta_O)_{s,w} \,\square_{\Sigma_n}\, x^{\square n} := (\eta_O)_{s,w} \,\square_{\prod_r \Sigma_{n_r}} \,\boxed{}_r\, x_r^{\square n_r}$$

is a cofibration in $\mathbf{Mod}_R^{\mathrm{s},\geq 0}$ by Theorem 3.3.4(6). $\qquad\square$

The following is a rectification result for algebras over weakly equivalent operads in spectra. For $\mathcal{C}$ being the category of compactly generated topological spaces, it is due to Goerss and Hopkins [GH, Theorem 1.2.4]. For R-spectra in spaces, where R is the free commutative monoid on the monoidal unit 1 in degree 1, this is due to Sagave and Schlichtkrull [SS12, Proposition 9.12], see also Proposition 3.3.9.

Theorem 3.4.4. *Let $\psi : P \to Q$ be a map of operads in $\mathbf{Mod}_R$. Then there is a Quillen adjunction*

$$Q \circ_P - : \mathbf{Alg}_P^{\mathrm{s},+}(\mathbf{Mod}_R) \rightleftarrows \mathbf{Alg}_Q^{\mathrm{s},+}(\mathbf{Mod}_R) : U$$

If ψ is a weak equivalence, i.e., if $P_{s,w} \to Q_{s,w}$ is a weak equivalence in $\mathbf{Mod}_R^{+,\mathrm{s}}$ for all $(s,w) \in \mathbf{sSeq}_W$, this is is a Quillen equivalence.

Example 3.4.5. For example, there is a Quillen equivalence of algebras over the Barratt-Eccles operad (i.e., E_∞-ring spectra) and commutative monoids in $\mathbf{Mod}_R$ (i.e., commutative ring spectra).

Another obvious application is that A_∞-ring spectra can be rectified to strictly associative ring spectra. See, e.g., [PS, Section 10.3] for a definition of A_∞.

Proof. Again, this follows from [PS, Theorem 9.2.11] and [PS, Theorem 9.3.1] whose assumptions are satisfied by Theorem 3.3.4. $\qquad\square$

We finally give two transport results that describe the category of operadic algebras in different categories of spectra. The first result is about a general weak monoidal Quillen adjunction. In the special case of algebras in R-spectra and S-spectra, where $R \sim S$ are weakly equivalent commutative monoids in $\Sigma\mathcal{C}$, we get a stronger result.

Let $\mathcal{D}$ be another symmetric monoidal model category satisfying Assumption 3.2.1. Let

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G \tag{3.4.6}$$

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be a Quillen adjunction. We suppose that G is symmetric lax monoidal. We pick commutative monoids $S \in \Sigma\mathcal{D}$ and $R \in \Sigma\mathcal{C}$ and a map of commutative monoids $\varphi : R \to G(S)$. Note that G preserves commutative monoids since it is symmetric lax monoidal. There are adjunctions

$$F^{\mathbf{Mod}} : \mathbf{Mod}_R^{\mathcal{C}} \quad \leftrightarrows \quad \mathbf{Mod}_S^{\mathcal{D}} : G, \tag{3.4.7}$$

$$(F^{\mathbf{Mod}})^{\mathbf{sOper}} : \mathbf{sOper}_W(\mathbf{Mod}_R^{\mathcal{C}}) \quad \leftrightarrows \quad \mathbf{sOper}(\mathbf{Mod}_S^{\mathcal{D}}) : G.$$

where G is in both cases the obvious functor and $F^{\mathbf{Mod}}$ and $(F^{\mathbf{Mod}})^{\mathbf{sOper}}$ are left adjoints whose existence is guaranteed since $\mathcal{C}$ and $\mathcal{D}$ and hence all categories in sight are locally presentable. See, e.g., [SS03a, Section 3] for the first and [PS, Section 9.4] for the second.

We equip $\Sigma\mathcal{C}$ and $\Sigma\mathcal{D}$ with some admissible model structures and we consider the condition that this datum induces a weak monoidal Quillen adjunction [SS03a, Definition 3.6]

$$F^{\mathbf{Mod}} : \mathbf{Mod}_R^{\mathrm{s},+,\mathcal{C}} \leftrightarrows \mathbf{Mod}_S^{\mathrm{s},+,\mathcal{D}} : G, \tag{3.4.8}$$

that is

$$F^{\mathbf{Mod}}(QR) \quad \to \quad S,$$
$$F^{\mathbf{Mod}}(C \otimes_R C') \quad \to \quad F^{\mathbf{Mod}}(C) \otimes_S F^{\mathbf{Mod}}(C').$$

are weak equivalences for all cofibrant objects $C, C' \in \mathbf{Mod}_R^{\mathrm{s},+,\mathcal{C}}$. Using the Quillen equivalences $\mathbf{Mod}_R^{\mathrm{s},+,\mathcal{C}} \sim \mathbf{Mod}_R^{\mathrm{s},\geq 0,\mathcal{C}}$ (Theorem 3.3.4(3)), this condition is equivalent for the nonpositive or the positive stable model structures. Since $F^{\mathbf{Mod}}(R \otimes -) = S \otimes F(-)$, the first condition holds if $1 \in \mathcal{C}$ is cofibrant. Using pretty smallness (via [PS, Lemma 2.0.2]), the second condition can be reduced to free R-modules C and C', so that it holds provided that the original adjunction (3.4.6) is weakly monoidal and that $\Sigma\mathcal{C}$ and $\Sigma\mathcal{D}$ both carry the projective model structure.

Theorem 3.4.9. *Suppose that (3.4.8) is a weak monoidal Quillen adjunction. Then, for any operad O in $\mathbf{Mod}_R$ and $P \in \mathbf{Mod}_S$, there are Quillen adjunctions*

$$F^{\mathbf{Alg}} : \mathbf{Alg}_O^{\mathrm{s},+}(\mathbf{Mod}_R^{\mathcal{C}}) \quad \rightleftarrows \quad \mathbf{Alg}_{F^{\mathbf{sOper}}(O)}^{\mathrm{s},+}(\mathbf{Mod}_S^{\mathcal{D}}) : G,$$

$$F_{\mathbf{Alg}} : \mathbf{Alg}_{G(P)}^{\mathrm{s},+}(\mathbf{Mod}_R^{\mathcal{C}}) \quad \rightleftarrows \quad \mathbf{Alg}_P^{\mathrm{s},+}(\mathbf{Mod}_S^{\mathcal{D}}) : G$$

They are Quillen equivalences if $(F^{\mathbf{Mod}}, G)$ is a weak monoidal Quillen equivalence and O is cofibrant and P is fibrant.

Proof. This is an immediate application of [PS, Theorem 9.4.10] whose assumptions are satisfied by Theorem 3.3.4. $\square$

In the special case $\mathcal{C} = \mathcal{D}$ and a weak equivalence $\varphi : R \to S$ in $\Sigma\mathcal{C}$, the transport of algebras applied to more general operads:

Corollary 3.4.10. *Suppose that there are Quillen equivalences*

$$\varphi_* : \mathbf{Mod}_R^{\mathrm{s},+} \leftrightarrows \mathbf{Mod}_S^{\mathrm{s},+} : \varphi^*. \tag{3.4.11}$$

(See Theorem 3.3.4(7) for sufficient criteria.) Then there are Quillen equivalences

$$\varphi_* : \mathbf{Alg}_O^{\mathrm{s},+}(\mathbf{Mod}_R) \quad \leftrightarrows \quad \mathbf{Alg}_{S \otimes_R O}^{\mathrm{s},+}(\mathbf{Mod}_S) : \varphi^*, \tag{3.4.12}$$

$$\varphi_* : \mathbf{Alg}_{\varphi^* P}^{\mathrm{s},+}(\mathbf{Mod}_R) \quad \leftrightarrows \quad \mathbf{Alg}_P^{\mathrm{s},+}(\mathbf{Mod}_S) : \varphi^* \tag{3.4.13}$$

for any operad O in $\mathbf{Mod}_R$ whose levels $O_{s,w}$ are cofibrant in $\mathbf{Mod}_R^{\mathrm{s},\geq 0}$ and any operad P in $\mathbf{Mod}_S$ whose levels $P_{s,w}$ are fibrant in $\mathbf{Mod}_S^{\mathrm{s},+}$.

Example 3.4.14. If $1_{\mathcal{C}}$ is cofibrant, $R = 1_{\mathbf{Mod}_R}$ is cofibrant in $\mathbf{Mod}_R^{\mathrm{s},\geq 0}$. The levels of the commutative operad $O = \mathrm{Comm}$ are given by $O_n = 1_{\mathbf{Mod}_R} = R$. We get $S \otimes_R O = \mathrm{Comm}$ and therefore a Quillen equivalence of commutative ring spectra.

Proof. The left adjoint in (3.4.11) is strong symmetric monoidal, the right adjoint is lax monoidal. It therefore gives an adjunction whose left adjoint is again strong monoidal.

$$\varphi_* : (\mathbf{sColl}(\mathbf{Mod}_R), \circ) \leftrightarrows (\mathbf{sColl}(\mathbf{Mod}_S), \circ) : \varphi^*$$

Therefore, by [AM10, Proposition I.3.91], both right and left adjoints preserve commutative monoids, i.e., operads, and the induced functors on operadic algebras form an adjoint pair. In other words, on the underlying spectra, $(\varphi_*)^{\mathbf{sOper}}$ is just φ_*. In the same vein, $(\varphi_*)^{\mathbf{Alg}}$ and $(\varphi_*)_{\mathbf{Alg}}$ are also just given by φ_* on the underlying level.

Applying Theorem 3.4.9 to $Q^{\mathbf{sOper}}(O)$ (the cofibrant replacement of O, using the model structure on operads in $\mathbf{Mod}_R^{s,+}$ established in Theorem 3.4.1) and RP (the fibrant replacement) and rectification (Theorem 3.4.4) we have to show weak equivalences of operads

$$
\begin{array}{rcl}
\varphi_*(Q^{\mathbf{sOper}}(O)) & \sim & \varphi_*(O) \\
\varphi^*(P) & \sim & \varphi^*(RP).
\end{array}
$$

The latter holds since $RP \to P$ is a weak equivalence of operads whose levels $(RP)_{s,w} \to P_{s,w}$ are a weak equivalence between fibrant objects in $\mathbf{Mod}_S^{s,+}$ (the latter by assumption). Being a right Quillen functor, φ^* preserves this weak equivalence by Brown's lemma.

The former weak equivalence is shown as follows: the weak equivalence $Q^{\mathbf{sOper}}(O) \to O$ gives a weak equivalence of the levels $Q^{\mathbf{sOper}}(O)_{s,w} \sim O_{s,w}$. For any $(s,w) \neq (w,w)$, $Q^{\mathbf{sOper}}(O)_{s,w}$ is cofibrant in $\mathbf{Mod}_R^{s,+}$ and a fortiori in $\mathbf{Mod}_R^{s,\geq 0}$. For $(s,w) = (w,w)$, the unit map $R = 1_{\mathbf{Mod}_R} \to Q^{\mathbf{sOper}}(O)_{w,w}$ is a cofibration [PS, Lemma 9.2.14(i)]. By [PS, Lemma 9.4.8], using the flatness of $\mathbf{Mod}_R^{s,\geq 0}$, $Q^{\mathbf{sOper}}(O)_{s,w} \otimes_R -$ preserves stable weak equivalences in both cases. Similarly for $O_{s,w}$, using the cofibrancy assumption on $O_{s,w}$. Hence we get a chain of weak equivalences in $\mathbf{Mod}_R^{s,\geq 0}$ or equivalently in $\mathbf{Mod}_R^{s,+}$:

$$
Q^{\mathbf{sOper}}(O)_{s,w} \otimes_R S \sim Q^{\mathbf{sOper}}(O)_{s,w} \sim O_{s,w} \sim O_{s,w} \otimes_R S.
$$

$\square$

4. Applications

We finish our paper by the following applications: we show that R-spectra form a suitable framework for derived algebraic geometry in the sense that they satisfy the axioms of Toën and Vezzosi. Moreover, we show that the axioms of Goerss and Hopkins used in their work on moduli problems of ring spectra are also satisfied for R-spectra. In Theorem 4.3.16, we use the rectification result (Theorem 3.4.4) to construct a strictly commutative ring spectrum (in simplicial presheaves) from a commutative differential graded algebra. As an example, we apply this to Deligne cohomology (Theorem 4.4.8).

4.1. **Toën-Vezzosi axioms.** In this section we prove that symmetric spectra in a symmetric monoidal model category form a *homotopical algebraic context* in the sense of Toën and Vezzosi [TV08], so that one can do derived algebraic geometry over ring spectra.

Definition 4.1.1. A *homotopical algebraic context* is a model category $\mathcal{D}$ such that:
  (i) $\mathcal{D}$ is a proper, pointed, combinatorial symmetric monoidal model category. The canonical morphism from the homotopy coproduct to the homotopy product of any finite family of objects is a weak equivalence. The homotopy category of $\mathcal{D}$ is additive.
  (ii) For any commutative monoid P in $\mathcal{D}$ the transferred model structure on $\mathbf{Mod}_P(\mathcal{D})$ exists and is a proper, flat, combinatorial symmetric monoidal model category.
  (iii) The transferred structure on commutative P-algebras and commutative nonunital P-algebras exists and is a proper combinatorial model category.
  (iv) Given a weak equivalence $f : E \to F$ in $\mathbf{Mod}_P(\mathcal{D})$ and a cofibrant commutative P-algebra Q, $Q \otimes_P f$ is a weak equivalence in $\mathbf{Mod}_Q(\mathcal{D})$.

Theorem 4.1.2. *Suppose $\mathcal{C}$ is a pointed symmetric monoidal model category satisfying Assumption 3.2.1. We fix an admissible model structure on $\Sigma\mathcal{C}$ and consider a commutative monoid $R \in \Sigma\mathcal{C}$ which is such that R_1 is weakly equivalent to $S^1 \otimes B$, where S^1 is a cofibrant representative of $* \sqcup_{1_{\mathcal{C}}}^h *$, the suspension of the monoidal unit, and $B \in \mathcal{C}$ is any cofibrant object. Then the stable positive model structure $\mathcal{D} := \mathbf{Mod}_R^{s,+}$ on the category of symmetric R-spectra defined in Theorem 3.3.4 is a homotopical algebraic context, except possibly for the properness of the model categories mentioned above.*

Proof. (i): This is a restatement of Theorem 3.3.4. The last statement follows from the stability of $\mathcal{D}$, which holds by the assumption on R_1.

(ii): Let $P \in \mathrm{Comm}(\mathcal{D})$, i.e., P is a commutative ring spectrum. The model structure on $\mathcal{D}$ transfers to a combinatorial, left proper, symmetric monoidal model structure on $\mathbf{Mod}_P$ by [PS, Theorem 8.2.5], using that $\mathcal{D}$ satisfies the monoid axiom by 3.3.4(2). Likewise, the flatness of $\mathcal{D}$ transfers to $\mathbf{Mod}_P$ by [PS, Proposition 5.2.5(ii)].

(iii): The categories of (nonunital) commutative P-algebras are algebras over the operad Comm and Comm^+ (which is given by $\mathrm{Comm}_n^+ = \emptyset$ for $n = 0$ and the monoidal unit 1 for $n > 0$), with values in $\mathbf{Mod}_P$. Again by [PS, Theorem 8.2.5], $\mathbf{Mod}_P$ is symmetric i-monoidal, so that any operad in $\mathbf{Mod}_P$, in particular Comm and Comm^+ are admissible, so the transferred model structure on (nonunital) commutative P-algebras exists [PS, Theorem 9.2.11].

(iv): As usual, we prove this by cellular induction. The first case is when $Q = \mathrm{Sym}(P \otimes X)$, where X is the (co)domain of a generating cofibration of $\mathcal{D}$ and Sym denotes the symmetric algebra on the P-module $P \otimes X$. As above, we have a canonical isomorphism in $\mathcal{C}$:

$$Q \otimes_P f = \coprod_{t \geq 0} ((P \otimes X)^{\otimes_P n})_{\Sigma_t} \otimes_P f = \coprod_{t \geq 0} f \otimes_{\Sigma_t} X^{\otimes t},$$

where Σ_t acts trivially on f. This is a weak equivalence in $\mathcal{D}$ since $\mathcal{D}$ is symmetric flat. As $\mathcal{D}$ is i-monoidal, weak equivalences are closed under finite coproducts [BB13, Proposition 1.15] and therefore, using the pretty smallness of $\mathcal{D}$, closed under countable coproducts.

Next, consider a cocartesian square in $\mathbf{Alg}_P$, where $i : X \to X'$ is a generating cofibration in $\mathcal{C}$,

$$\begin{array}{ccc} \mathrm{Sym}(P \otimes X) & \longrightarrow & \mathrm{Sym}(P \otimes X') \\ \downarrow & & \downarrow \\ Q & \longrightarrow & Q', \end{array} \qquad (4.1.3)$$

we want to show that our claim is true for Q', provided that it holds for Q. We again use the filtration that already appeared in the proof of [PS, Theorem 9.2.11]. In the case considered here, $O = \mathrm{Comm}$, so that $\mathrm{Env}(O, Q)_t = Q$ (with the trivial Σ_t-action). This description of the enveloping operad can be read off its explicit description in [Har09, Proposition 7.6] (in loc. cit., $\mathrm{Env}(O, Q)_t$ is denoted $O_Q[t]$, and the formula for $O_Q[t]$ simplifies to $O_Q[t] = \mathrm{colim}(O \circ A \Leftarrow O \circ (O \circ A))$ for $O = \mathrm{Comm}$). As in [PS, Theorem 9.2.11], we get a cocartesian square in $\mathbf{Mod}_P$,

$$\begin{array}{ccc} Q \otimes (\square^t i)_{\Sigma_t} = Q \otimes_P (\square_P^t (P \otimes i))_{\Sigma_t} & \longrightarrow & Q \otimes (X'^{\otimes t})_{\Sigma_t} = Q \otimes_P ((P \otimes X')^{\otimes_P t})_{\Sigma_t} \\ \downarrow & & \downarrow \\ Q_t & \longrightarrow & Q_{t+1}. \end{array}$$

We apply $f \otimes_P -$ to this square and get a cube whose front and back face are cocartesian (in $\mathbf{Mod}_Q$, or in $\mathcal{C}$):

$$\begin{array}{ccc} (F \otimes_P Q) \otimes_{\Sigma_t} \square^t i & \longrightarrow & (F \otimes_P Q) \otimes_{\Sigma_t} X'^t \\ & & \\ (E \otimes_P Q) \otimes_{\Sigma_t} \square^t i & \longrightarrow & (E \otimes_P Q) \otimes_{\Sigma_t} X'^t \\ & & \\ & F \otimes_P Q_t \longrightarrow F \otimes_P Q_{t+1} & \\ & & \\ & E \otimes_P Q_t \longrightarrow E \otimes_P Q_{t+1} & \end{array}$$

The top horizontal arrows of this cube are i-cofibrations (in $\mathcal{C}$, say), since i is a symmetric i-cofibration. Consequently the front and back face are homotopy pushout squares. The three arrows labeled with $\sim$ are

weak equivalences by induction and the case of free commutative P-algebras considered above. Therefore, the map $f \otimes_P Q_{t+1}$ is also a weak equivalence.

Now, any cofibrant P-algebra is a retract of transfinite compositions (in $\mathbf{Alg}_P$) of maps as in (4.1.3). The forgetful functor $\mathbf{Alg}_P \to \mathbf{Mod}_P$ commutes with sifted colimits, therefore with transfinite compositions (and retracts). Weak equivalences in $\mathcal{C}$ are stable under filtered colimits by [PS, Lemma 2.0.2]. This finishes the proof of (iv). □

4.2. **Goerss-Hopkins axioms.** In [GH04] and [GH], Goerss and Hopkins formulated a number of axioms that a category of spectra should satisfy in order to admit a good obstruction theory for lifting commutative monoid objects in the homotopy category of spectra to E_∞-spectra. They pointed out that the stable positive model structure on topological spectra satisfies these properties and raised the question whether the same property is true for spectra in a general model category. This was shown for spectra with values in simplicial presheaves by Hornbostel [Hor13, Section 3.3]. In this section, we answer this question in the positive for spectra in a very broad class of model categories, namely the ones satisfying Assumption 3.2.1.

We summarize the axioms of loc. cit. in the following definition:

Definition 4.2.1. A *Goerss-Hopkins context* is a symmetric monoidal tractable stable V-enriched model category $\mathcal{C}$ (V is a tractable symmetric monoidal model category) such that every operad O in $\mathcal{C}$ is admissible with the resulting model structure on O-algebras being tractable and V-enriched and every weak equivalence of operads induces a Quillen equivalence between their categories of algebras.

Theorem 4.2.2. *Suppose $\mathcal{C}$ is a pointed, symmetric monoidal, $\mathcal{V}$-enriched model category satisfying Assumption 3.2.1, $\Sigma\mathcal{C}$ is endowed with an admissible model structure, and R is a commutative monoid in $\Sigma\mathcal{C}$ such that R_1 is weakly equivalent to $S^1 \otimes B$, where S^1 is a cofibrant representative of the suspension of the monoidal unit and B is any cofibrant object. The category of R-spectra, equipped with the stable positive model structure established in Theorem 3.3.4, is a Goerss-Hopkins context.*

Proof. The model structure $\mathbf{Mod}_R^{s,+}$ is stable, symmetric monoidal and tractable by Theorem 3.3.4. Every operad O in $\mathbf{Mod}_R$ is admissible by Theorem 3.4.1, and weak equivalences of operads induce Quillen equivalent categories of algebras by Theorem 3.4.4. □

Definition 4.2.1 is slightly different from the list of properties mentioned in [GH04, Sections 1.1, 1.4] and [GH, Theorems 1.2.1, 1.2.3]: we omit the requirement that the homotopy category of $\mathcal{C}$ is equivalent to the homotopy category of Bousfield-Friedlander spectra, i.e., nonsymmetric spectra. The Quillen equivalence of symmetric and nonsymmetric spectra with values in an abstract model category is addressed by [Hov01, Corollary 10.4]. We have replaced the requirement of cellularity of the model structures for $\mathbf{Alg}_O\mathbf{Mod}_R$ by combinatoriality. The relation of these two properties is discussed in [PS, Section 7].

[GH, Axiom 1.2.3.5] can be rephrased by requiring that the forgetful functor $\mathbf{Alg}_O(\mathbf{Mod}_R) \to \mathbf{Mod}_R$ preserves cofibrations. In op. cit. this is only used in Theorem 1.3.4.2, which in its turn is only used in Theorem 1.4.9 to establish cellularity, which can be replaced by combinatoriality. Moreover, this property may fail for internal operads if, say, $O(1) \in \mathbf{Mod}_R$ is not cofibrant, so it is omitted in Definition 4.2.1. A positive result in this direction, for a general model category $\mathcal{C}$, is given by Theorem 3.4.3.

[GH, Axiom 1.2.3.6] states that for any $n \geq 0$, and any cofibrant object $X \in \mathbf{Mod}_R$, the functor $\Sigma_n^{\mathrm{in}}\mathcal{V} \to \mathbf{Mod}_R$, $K \mapsto K \otimes_{\Sigma_n} X^{\otimes n}$ preserves weak equivalences and cofibrations. This condition is again not present in Definition 4.2.1. It is used only in [GH, Theorem 1.2.4] (rectification for operads in R-spectra). Our proof of this statement is based on the symmetric flatness of the stable positive model structure on $\mathbf{Mod}_R$, which is a generalization of the preservation of weak equivalences by the above functor.

4.3. **Construction of commutative ring spectra.** In this section, we apply the results of Section 3 to the construction of strictly symmetric ring spectra.

We recall two technical tools: first, we study nonsymmetric lax monoidal right adjoints, such as the Dold-Kan functor $\Gamma : \mathbf{Ch}_+ \to \mathbf{sAb}$, and the endomorphism operad associated to such a functor. This is due to Richter [Ric03, Definition 3.1] (also see [AM10, Section 4.3.2]). Second, in order to capture the maximal information from the ring spectra constructed in Theorem 4.3.16, we will not only consider mapping spaces, but convolution algebras, which encode the multiplication on mapping spaces (see for example [AM10, Section 3.4.5]).

Definition 4.3.1. Let $G : \mathcal{D} \to \mathcal{C}$ be a lax monoidal (but not necessarily symmetric lax monoidal) functor between two symmetric monoidal categories, where $\mathcal{C}$ is enriched over a symmetric monoidal category $\mathcal{V}$. The *endomorphism operad* of G is the operad in V defined by

$$O_G(n) = \mathrm{Hom}_{\mathbf{Fun}(\mathcal{D}^n, \mathcal{C})}(G(-) \otimes \cdots \otimes G(-), G(- \otimes \cdots \otimes -)).$$

We say that G is O-lax monoidal for some operad O in V if there is a natural map $O \to O_G$.

For example, a symmetric lax monoidal functor G is just the same as a Comm-lax monoidal functor [AM10, Table 4.2].

Lemma 4.3.2. *Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G \tag{4.3.3}$$

be an adjunction of symmetric monoidal categories, where G is O-monoidal for some operad O. Also suppose that $\mathcal{C}$ and $\mathcal{D}$ are accessible.

(1) *There is an adjunction*

$$F^{\mathbf{Alg}} : \mathbf{Alg}_O \mathcal{C} \rightleftarrows \mathbf{Alg}_{\mathrm{Comm}} \mathcal{D} : G, \tag{4.3.4}$$

where G sends a commutative algebra $D \in \mathcal{D}$ to $G(D)$ with the O-algebra structure defined by

$$O(n) \otimes G(D)^{\otimes n} \to O_G(n) \otimes G(D)^{\otimes n} \to G(D^{\otimes n}) \to G(D).$$

(2) *[AM10, Proposition 3.91] If G is symmetric monoidal (so that $O = \mathrm{Comm}$) and F is strong symmetric monoidal, then $F^{\mathbf{Alg}}$ sends a commutative algebra $C \in \mathcal{C}$ to $F(C)$ with the commutative algebra structure*

$$F(C) \otimes F(C) \xrightarrow{\cong} F(C \otimes C) \to F(C),$$

where the first map is the isomorphism that is part of the strong symmetric monoidal functor.

Proof. The functor G preserves limits and filtered colimits of algebras, since these are created by the functor forgetting the algebra structure [PS, Section 8]. Since G is a functor between locally presentable categories, it therefore has a left adjoint $F^{\mathbf{Alg}}$. $\square$

Definition 4.3.5. Suppose that $\mathcal{C}$ is a closed symmetric monoidal category. The internal Hom functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ is symmetric monoidal. The induced functor

$$\mathrm{Hom}^{\mathbf{Alg}} : \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}^{\mathrm{op}}) \times \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}) = \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \to \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C})$$

is called the *convolution algebra*. More generally, given an operad O in $\mathcal{C}$, the *convolution O-algebra* is the functor

$$\mathrm{Conv} : \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}^{\mathrm{op}}) \times \mathbf{Alg}_O(\mathcal{C}) \to \mathbf{Alg}_O(\mathcal{C}). \tag{4.3.6}$$

which sends (X, Y) to the internal $\mathrm{Hom}(X, Y) \in \mathcal{C}$ equipped with the O-algebra structure induced by the comultiplication on X and the O-algebra structure on Y. Explicitly, it is defined by

$$\begin{aligned} O(n) \otimes \mathrm{Hom}(X, Y)^{\otimes n} &\to O(n) \otimes \mathrm{Hom}(X^{\otimes n}, Y^{\otimes n}) \\ &\to \mathrm{Hom}(X^{\otimes n}, O(n) \otimes Y^{\otimes n}) \\ &\to \mathrm{Hom}(X, Y). \end{aligned}$$

Lemma 4.3.7. *(cf. [AM10, 3.83]) In the situation of Lemma 4.3.2, let $\mathcal{C}' \subset \mathcal{C}$ be a full subcategory such that $F' := F|_{\mathcal{C}'}$ is symmetric oplax monoidal (so that F' preserves commutative coalgebras). The natural transformation*

$$\mathrm{Conv}_{\mathcal{C}}(-, G(-)) \to G(\mathrm{Conv}_{\mathcal{D}}(F'(-), D)) \tag{4.3.8}$$

is a morphism of functors $\mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}'^{\mathrm{op}}) \times \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{D}) \to \mathbf{Alg}_{O_G}(\mathcal{C})$. It is an isomorphism if the oplax structural map

$$F(T \otimes X) \to F(T) \otimes F(X), \tag{4.3.9}$$

is an isomorphism for any $T \in \mathcal{C}$ and any $X \in \mathcal{C}'$.

Proof. The underlying internal Hom's are given by the compositions

$$\Phi = \mathrm{Hom}(-, G-) \quad : \quad \mathcal{C}'^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{id} \times G} \mathcal{C}'^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}} \mathcal{C},$$

$$\Psi = G\,\mathrm{Hom}(F'-, -) \quad : \quad \mathcal{C}'^{\mathrm{op}} \times \mathcal{D} \xrightarrow{F' \times \mathrm{id}} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}} \mathcal{D} \xrightarrow{G} \mathcal{C}.$$

The functors G and $\mathrm{id} \times G$ are O_G-monoidal, and all other functors are symmetric lax monoidal, i.e., Comm-monoidal. Thus, their composition is $O_G \boxtimes \mathrm{Comm} = O_G$-monoidal. Here $- \boxtimes -$ denotes the Hadamard product of operads [AM10, Theorem 4.28]. The natural transformation $\Phi \to \Psi$ induces the transformation in (4.3.8) which is therefore a map of O_G-algebras. For the second claim, $\Phi \to \Psi$ is an isomorphism in this case, hence so is the transformation in (4.3.8). $\square$

We now consider the interaction of Conv and model structures. Suppose $\mathcal{C}$ is a symmetric monoidal model category. Then the convolution algebra (4.3.6) is a functor between categories with weak equivalences. To get homotopically meaningful information, we therefore have to derive it. A natural strategy to compute this (right) derived functor would be to endow the category of commutative coalgebras in $\mathcal{C}$ (=commutative algebras in $\mathcal{C}^{\mathrm{op}}$) with a model structure. The standard choice of such a model structure is the transferred structure along the forgetful functor

$$\mathcal{C}^{\mathrm{op}} \leftarrow \mathbf{Alg}_{\mathrm{Comm}}(\mathcal{C}^{\mathrm{op}}).$$

However, this is a notoriously difficult task (see, e.g., [BHK$^+$14]), which we will not undertake in this paper. Instead we use the following fact:

Lemma 4.3.10. *Let $\mathcal{C}$ be a symmetric monoidal model category. Let $X \in \mathcal{C}$ be a cofibrant object which is also endowed with a commutative coalgebra structure. The functor*

$$\mathrm{Conv}(X, -) : \mathbf{Alg}_O(\mathcal{C}) \to \mathbf{Alg}_O(\mathcal{C}).$$

is a right Quillen functor. Its derived functor will be denoted by $\underline{\mathrm{R}}\mathrm{Conv}(X, -)$.

Proof. We have to check $\mathrm{Conv}(X, -)$ preserves (acyclic) fibrations. These are created by the forgetful functor to $\mathcal{C}$. Forgetting the O-algebra structure, $\mathrm{Conv}(X, -)$ is just the internal $\mathrm{Hom}(X, -)$, which is a right Quillen functor since X is cofibrant and $\mathcal{C}$ is a monoidal model category. $\square$

We now upgrade Lemma 4.3.7 to model categories. We use the notation of Lemma 4.3.2 and Lemma 4.3.7.

Proposition 4.3.11. *Suppose that (4.3.3) is a Quillen adjunction between combinatorial model categories and the transferred model structures on the categories of algebras in (4.3.4) exist. Also suppose X is an object of $\mathcal{C}'$, which is cofibrant in $\mathcal{C}$ and such that the lax monoidal structural map (4.3.9) is a weak equivalence for all cofibrant objects $T \in \mathcal{C}$.*

(1) The adjunction (4.3.4), which exists by Lemma 4.3.2, is a Quillen adjunction. The map

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathcal{C}}(X, \underline{\mathrm{R}}GD) \xrightarrow{\sim} \underline{\mathrm{R}}G(\underline{\mathrm{R}}\mathrm{Conv}_{\mathcal{D}}(F(X), D)) \qquad (4.3.12)$$

is a weak equivalence in $\mathbf{Alg}_O\mathcal{C}$.

(2) In the situation of Lemma 4.3.2(2), suppose that (4.3.3) and (4.3.4) are Quillen equivalences. Then, for any object $C \in \mathbf{Alg}_{\mathrm{Comm}}\mathcal{C}$ there is a weak equivalence in $\mathbf{Alg}_{\mathrm{Comm}}\mathcal{C}$

$$\underline{\mathrm{L}}F^{\mathbf{Alg}}\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathcal{C}}(X, C) \xrightarrow{\sim} \underline{\mathrm{R}}\mathrm{Conv}_{\mathcal{D}}(F^{\mathbf{Alg}}X, \underline{\mathrm{L}}F^{\mathbf{Alg}}C).$$

Proof. (1): (4.3.4) is a Quillen adjunction since (acyclic) fibrations are created by the functors forgetting the respective operadic algebra structures.

By Lemma 4.3.7, (4.3.12) is a map of O-algebras. It is therefore enough to show that (4.3.12) is a weak equivalence in $\mathcal{C}$, i.e., after forgetting the O-algebra structure. This is an easy consequence of the assumption that (4.3.9) is a weak equivalence.

(2): In (4.3.12), put $D = \underline{\mathrm{L}}F^{\mathbf{Alg}}(C)$. As (4.3.4) is a Quillen equivalence, there is a weak equivalence $C \to \underline{\mathrm{R}}G(\underline{\mathrm{L}}F^{\mathbf{Alg}}(C))$. Hence we get a weak equivalence $\underline{\mathrm{R}}\mathrm{Conv}_{\mathcal{C}}(X, C) \xrightarrow{\sim} \underline{\mathrm{R}}G\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathcal{D}}(FX, \underline{\mathrm{L}}F^{\mathbf{Alg}}C)$ which implies our claim again using the Quillen equivalence (4.3.4). $\square$

We now prepare for Theorem 4.3.16 by fixing some notation related to the Dold-Kan equivalence. Let $\mathcal{A}$ be a symmetric monoidal Grothendieck abelian category. We fix a model structure on the category $\mathbf{s}\mathcal{A}$ of simplicial objects. We assume that this model structure transfers, via the Dold-Kan equivalence,

$$N : \mathbf{s}\mathcal{A} \rightleftarrows \mathbf{Ch}_+\mathcal{A} : \Gamma \qquad (4.3.13)$$

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to a model structure on connective chain complexes. Finally, we pick a model structure on chain complexes, such that the adjunction of the good truncation functor and the inclusion of chain complexes in nonnegative degrees,

$$\iota : \mathbf{Ch}_+ \mathcal{A} \rightleftarrows \mathbf{Ch}\mathcal{A} : \tau \qquad (4.3.14)$$

(with $\tau A_* := [\ldots \to A_1 \to \ker d_0]$) is a Quillen adjunction. We assume that these three model categories satisfy Assumption 3.2.1, i.e., they are pretty small, left proper, tractable, flat, i-monoidal, symmetric monoidal model categories. We also assume that the monoidal unit is cofibrant in $\mathbf{Ch}_+ \mathcal{A}$. (This is needed to apply Corollary 3.4.10.)

We fix the projective model structure on symmetric sequences with values in $\mathbf{s}\mathcal{A}$ etc. This is the model structure transferred from $\mathbf{s}\mathcal{A}$. Let $\tilde{R}_1 \in \mathbf{Ch}_+ \mathcal{A}$ and $R_1 \in \mathcal{A}$ be any objects. We regard R_1 as simplicially constant object in $\mathbf{s}\mathcal{A}$. We write R for the commutative monoid in $\Sigma\mathcal{A} \subset \Sigma\mathbf{s}\mathcal{A}$ whose n-th level is given by $R^{\otimes n}$ and likewise for $\tilde{R} \in \Sigma\mathbf{Ch}_+ \mathcal{A}$. We suppose there is a weak equivalence

$$\varphi : \tilde{R} \to N(R)$$

in $\Sigma\mathbf{s}\mathcal{A}$. The categories of R- and $\tilde{R}$-modules are equipped with their stable positive model structure (Theorem 3.3.4). To simplify the notation, we will write $\mathbf{Mod}_R^{\mathbf{s}} := \mathbf{Mod}_R^{\mathrm{s},+}(\Sigma\mathbf{s}\mathcal{A})$, $\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}+} := \mathbf{Mod}_{\tilde{R}}^{\mathrm{s},+}(\Sigma\mathbf{Ch}_+\mathcal{A})$, and similarly with $\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}$.

The normalization functor N in the Dold-Kan equivalence (4.3.13), applied to $\Sigma\mathcal{A}$ instead of $\mathcal{A}$, is symmetric lax monoidal and (nonsymmetric) oplax monoidal by means of the Alexander-Whitney and Eilenberg-Zilber maps (see for example [AM10, Section 5.4]). Therefore the right adjoint Γ is symmetric oplax monoidal and (nonsymmetric) lax monoidal. However, the lax monoidal structural map

$$\Gamma(A)^{\otimes n} \otimes \Gamma(B) \to \Gamma(A^{\otimes n} \otimes B)$$

is a Σ_n-equivariant isomorphism for any $B \in \mathbf{Ch}_+ \Sigma\mathcal{A}$ provided that A is a chain complex concentrated in degree 0. This can be checked using the explicit description of this map. Dually, there is a lax monoidal map for N,

$$N(A \otimes B) \to \mathrm{N}(A) \otimes \mathrm{N}(B),$$

which is an isomorphism if A is a constant simplicial object. Applying this to $A = R$, we obtain an adjunction

$$N : \mathbf{sMod}_R \rightleftarrows \mathbf{Mod}_{N(R)}^{\mathbf{Ch}+} : \Gamma. \qquad (4.3.15)$$

With these preparations, we can now state the construction of commutative ring spectra. Similar methods have been employed by Shipley to construct (noncommutative) ring spectra [Shi07, Theorem 1.1].

Theorem 4.3.16. *With the notation and assumptions fixed above, there is a functor*

$$\mathrm{H} : \mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}) \to \mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}_R^{\mathbf{s}})$$

defined by

$$\mathrm{H}(A) := \mathrm{Comm} \circ_O^{\mathbf{L}} (\underline{\mathrm{R}}\Gamma(R \otimes_{\tilde{R}}^{\mathbf{L}} \underline{\mathrm{R}}\tau A)).$$

The spectrum $\mathrm{H}(A)$ *represents the same cohomology as* A *in the sense that the following derived mapping spaces are weakly equivalent, where* X *is any object in* $\mathcal{A}$:

$$\underline{\mathrm{R}}\mathrm{Map}_{\mathbf{Mod}_R^{\mathbf{s}}}(R \otimes X, \mathrm{H}(A)) \sim \underline{\mathrm{R}}\Gamma\underline{\mathrm{R}}\tau \, \underline{\mathrm{R}}\mathrm{Map}_{\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}}(\iota(\tilde{R} \otimes N(X)), A).$$

Moreover, the multiplicative structure is preserved in the strongest possible sense: if $X \in \mathcal{A}(\subset \mathbf{s}\mathcal{A})$ *is cofibrant and in addition a commutative coalgebra, there is weak equivalence of convolution algebras*

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R \otimes X, \mathrm{H}(A)) \sim \mathrm{Comm} \circ_O^{\mathbf{L}} \underline{\mathrm{R}}\Gamma R \otimes_{\tilde{R}}^{\mathbf{L}} \underline{\mathrm{R}}\tau \, \underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}}(\iota(\tilde{R} \otimes N(X)), A).$$

Proof. We prove this using Proposition 4.3.11, a theorem of Richter [Ric03], and the rectification theorem 3.4.4.

The functor ι is strong monoidal and τ is symmetric lax monoidal (because of the Leibniz rule). Therefore, (4.3.14) induces a similar adjunction

$$\iota : \mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}+} \rightleftarrows \mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}} : \tau.$$

The unstable positive model structures on $\tilde{R}$-modules (Theorem 3.2.5) are transferred from (4.3.14) which is a Quillen adjunction by assumption. Therefore, by the universal property of the Bousfield localization, the

stable positive model structures are also related by a Quillen adjunction. Thus Proposition 4.3.11(1) yields a weak equivalence

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}}(\tilde{R}\otimes N(X),\underline{\mathrm{R}}\tau A) \sim \underline{\mathrm{R}}\tau\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}}(\iota(\tilde{R}\otimes N(X)),A). \tag{4.3.17}$$

The map $\varphi:\tilde{R}\to N(R)$ induces a Quillen adjunction

$$-\otimes_{\tilde{R}} N(R):\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}_+}\rightleftarrows\mathbf{Mod}_{N(R)}^{\mathbf{Ch}_+}:\text{restriction}.$$

The left adjoint is strong monoidal, the right adjoint is symmetric lax monoidal. Since φ is a weak equivalence by assumption, both this adjunction, as well as the induced adjunction of commutative algebra objects are Quillen equivalences (Corollary 3.4.10, using the cofibrancy of the unit in $\mathbf{Ch}_+(\mathcal{A})$). Proposition 4.3.11(2) gives a weak equivalence

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R(\mathbf{Ch}_+\Sigma\mathcal{A})}(N(R)\otimes X,R\otimes_{\tilde{R}}^{\mathrm{L}}\underline{\mathrm{R}}\tau A) \sim R\otimes_{\tilde{R}}^{\mathrm{L}}\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}}(\tilde{R}\otimes N(X),\underline{\mathrm{R}}\tau A). \tag{4.3.18}$$

The next step is the Dold-Kan equivalence. (4.3.13) is a Quillen adjunction by assumption. Therefore so is (4.3.15) (where both sides carry the stable positive model structures of Theorem 3.3.4). Let $O=O_\Gamma$ be the endomorphism operad of Γ. Using Proposition 4.3.11, we get a weak equivalence

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,\underline{\mathrm{R}}\Gamma(R\otimes_{\tilde{R}}^{\mathrm{L}}\underline{\mathrm{R}}\tau A)) \sim \underline{\mathrm{R}}\Gamma\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{N(R)}^{\mathbf{Ch}_+}}(N(R)\otimes X,N(R)\otimes_{\tilde{R}}^{\mathrm{L}}\underline{\mathrm{R}}\tau A). \tag{4.3.19}$$

Given a commutative monoid object $Z\in\mathbf{Mod}_R^{\mathbf{s}}$, it is easy to check that there is an isomorphism of O-algebras,

$$\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,UZ)\xrightarrow{\cong}U\,\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,Z).$$

Here U denotes the forgetful functors from commutative to O-algebras, by means of the unique map of operads $O\to\mathrm{Comm}$. This passes to a weak equivalence

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,\underline{\mathrm{R}}UZ)\xrightarrow{\sim}\underline{\mathrm{R}}U\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,Z). \tag{4.3.20}$$

Using that R is simplicially constant and therefore $N(R)$ is concentrated in degree 0, we can rewrite the adjunction (4.3.15) as the Dold-Kan equivalence applied to the abelian category $\mathbf{Mod}_R(\Sigma\mathcal{A})$:

$$N:\mathbf{sMod}_R(\Sigma\mathcal{A})\rightleftarrows\mathbf{Ch}_+\mathbf{Mod}_{N(R)}(\Sigma\mathcal{A}):\Gamma. \tag{4.3.21}$$

According to Richter's theorem [Ric03, Theorem 4.1], $O\to\mathrm{Comm}$ is a levelwise weak equivalence for the Dold-Kan equivalence on the abelian category $\mathbf{Ab}$. The proof of loc. cit. readily generalizes to a general abelian category such as $\mathbf{Mod}_R(\Sigma\mathcal{A})$. Thus, Theorem 3.4.4 establishes a Quillen equivalence

$$\mathrm{Comm}\circ_O -:\mathbf{Alg}_O(\mathbf{Mod}_R^{\mathbf{s}})\leftrightarrows\mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}_R^{\mathbf{s}}):U.$$

This Quillen equivalence and (4.3.20), applied to $Z=\mathrm{Comm}\circ_O^{\mathrm{L}}Y$ gives the following chain of weak equivalences of convolution algebras, i.e., commutative algebras in $\mathbf{Mod}_R$:

$$\begin{aligned}
\mathrm{Comm}\circ_O^{\mathrm{L}}\underline{\mathrm{R}}\mathrm{Conv}(X,Y) &\xrightarrow{\sim} \mathrm{Comm}\circ_O^{\mathrm{L}}\underline{\mathrm{R}}\mathrm{Conv}(X,\underline{\mathrm{R}}U_{O\to\mathrm{Comm}}\mathrm{Comm}\circ_O^{\mathrm{L}}Y)\\
&\xrightarrow{\sim} \mathrm{Comm}\circ_O^{\mathrm{L}}\underline{\mathrm{R}}U_{O\to\mathrm{Comm}}\underline{\mathrm{R}}\mathrm{Conv}(X,\mathrm{Comm}\circ_O^{\mathrm{L}}Y)\\
&\xrightarrow{\sim} \underline{\mathrm{R}}\mathrm{Conv}(X,\mathrm{Comm}\circ_O^{\mathrm{L}}Y).
\end{aligned} \tag{4.3.22}$$

Combining (4.3.17), (4.3.18), (4.3.19) and (4.3.22), we obtain the desired weak equivalence

$$\begin{aligned}
\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,\mathrm{H}(A)) &= \underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_R^{\mathbf{s}}}(R\otimes X,\mathrm{Comm}\circ_O^{\mathrm{L}}(\underline{\mathrm{R}}\Gamma(R\otimes_{\tilde{R}}^{\mathrm{L}}\underline{\mathrm{R}}\tau A)))\\
&\sim \mathrm{Comm}\circ_O^{\mathrm{L}}\underline{\mathrm{R}}\Gamma R\otimes_{\tilde{R}}^{\mathrm{L}}\underline{\mathrm{R}}\tau\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\tilde{R}}^{\mathbf{Ch}}}(\iota(\tilde{R}\otimes N(X)),A).
\end{aligned}$$

$\square$

4.4. **A commutative ring spectrum for Deligne cohomology.** In this section we construct a strictly commutative ring spectrum representing Deligne cohomology with integral coefficients. For a smooth projective variety $X/\mathbf{C}$, Deligne cohomology is defined as the hypercohomology group

$$\mathrm{H}^n_{\mathrm{D}}(X, \mathbf{Z}(p)) := \mathbf{H}^n(X^{\mathrm{an}}, \underbrace{[\mathbf{Z}(p) \to \Omega^0_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X]}_{=:\mathbf{Z}(p)_D}), \qquad (4.4.1)$$

where X^{an} is the smooth complex manifold associated to X, $\mathbf{Z}(p) := (2\pi i)^p \mathbf{Z}$ sits in degree 0 and Ω^*_X is the complex of holomorphic forms on X^{an}. Applications of Deligne cohomology range from arithmetic geometry, notably Beilinson's conjecture on special values of L-functions [Beĭ84] to higher Chern-Simons theory [Sch13, Section 5.5.8]. The product structure on Deligne cohomology is surprisingly subtle. It was defined by Beilinson by certain maps

$$- \cup_\alpha - : \mathbf{Z}(p)_D \otimes \mathbf{Z}(q)_D \to \mathbf{Z}(p+q)_D$$

that depend on a parameter $\alpha \in \mathbf{C}$ [EV88, Definition 3.2]. This parameter is used to show that the product on the complexes of sheaves is commutative and associative up to homotopy. In particular, $\bigoplus_{n,p} \mathrm{H}^n_{\mathrm{D}}(X, \mathbf{Z}(p))$ is a commutative ring. This was used by Holmstrom and the second author to construct a commutative ring spectrum representing Deligne cohomology [HS10]. This is the weakest possible requirement on the product operation on a spectrum: the multiplication is only commutative and associative up to homotopy. In a somewhat similar vein, Hopkins and Quick studied ring spectra that result from replacing the Betti cohomology part in Deligne cohomology by a different ring spectrum, such as complex cobordism [HQ12]. In this section, we provide a strictly commutative model for Deligne cohomology which is the strongest possible multiplicative structure on such a spectrum.

We emphasize that we are working with integral coefficients. For rational coefficients (i.e., with $\mathbf{Q}(p)$ instead of $\mathbf{Z}(p)$), it is possible to use Lurie's rectification result [Lur, Theorem 4.4.4.7] to obtain a strictly commutative ring spectrum. However, integral coefficients are interesting from many points of view. To refine the treatment of special L-values, which is up to rational factors in [Sch10], it will be necessary to have the integral structure available. One motivation for Hopkins' and Quick's work is to find new torsion algebraic cycles, which also requires integral coefficients. In yet another direction, one may speculate about the relation of modules over the Deligne cohomology spectrum and mixed Hodge modules by Saito [Sai91]. Again, for such considerations, it would be unnatural to throw away torsion.

Before discussing Deligne cohomology proper, we show how to turn a certain product structure on a fiber product of commutative differential graded algebras (cdga's) into a strictly commutative and associative one. As in Section 4.3, our complexes are regarded as chain complexes, i.e., $\deg d = -1$. Consider a diagram of cdga's, where we suppose that B takes values in $\mathbf{Q}$-vector spaces:

$$A \xrightarrow{a} B \xleftarrow{c} C.$$

Because of rational coefficients, a path object for B is given by $B \otimes Q$ [Beh02, Lemma 1.19], where Q is the chain complex of polynomial differential forms on Δ^1 familiar from rational homotopy theory. It is the complex in the left column, where the terms are in degrees 0 and -1, respectively. The complex R at the right is quasiisomorphic to Q:

$$
\begin{array}{ccc}
\mathbf{Q}[t] & \xrightarrow{\mathrm{ev}(0),\mathrm{ev}(1)} & \mathbf{Q} \oplus \mathbf{Q} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle (a,b)\mapsto a-b} \\
\mathbf{Q}[t]dt & \xrightarrow{\int_0^1 -} & \mathbf{Q}.
\end{array}
$$

We endow R with the multiplication $R \otimes R \to R$ given by the following matrix in terms of the standard basis $e_1, e_2 \in R_0, f \in R_{-1}$: $e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2, f \cdot e_2 = f, e_1 \cdot f = f$, and all other products of basis vectors are 0. This product is associative and left unital, but not commutative. Because of the latter defect, we consider the following diagram of associative left unital differential graded algebras

$$Q = Q \otimes \mathbf{Q}[0] \xrightarrow{\mathrm{id}\otimes 1} S := Q \otimes R \xleftarrow{(1,1)\otimes\mathrm{id}} \mathbf{Q}[0] \otimes R = R.$$

The horizontal maps are induced by the unit elements of Q and R, respectively. These maps are quasiisomorphisms. In addition, the augmentation maps $Q_0 = \mathbf{Q}[t] \to \mathbf{Q}^2$ and similarly for S and R commute with

these quasiisomorphisms. Therefore, there is a zigzag of quasiisomorphisms of associative (noncommutative, except for D) left unital differential graded algebras

$$D := A \times_B (B \otimes Q) \times_B C \xrightarrow{\sim} A \times_B (B \otimes S) \times_B C \xleftarrow{\sim} A \times_B (B \otimes R) \times_B C. \tag{4.4.2}$$

The right hand object is just $E := \mathrm{cone}(A \oplus C \xrightarrow{a-c} B)[1]$ or, equivalently, the homotopy pullback $A \times_C^h B$ in the model category of chain complexes, while D is the homotopy pullback in the (much more natural) model category of cdga's. The quasiisomorphisms in (4.4.2) are compatible with the respective product structures. In particular, the induced product on $\mathrm{H}^*(D)$ agrees with the one on $\mathrm{H}^*(E)$. Moreover, higher order multiplications, such as Massey products also agree.

In the sequel, we just write $\mathbf{PSh}_\bullet := \mathbf{PSh}(\mathbf{Sm/C}, \mathbf{Set}_\bullet)$ for the presheaves of pointed sets on the site of smooth schemes over $\mathbf{C}$. We write $\mathbf{sPShAb}$ for simplicial presheaves of abelian groups and $\mathbf{ChPSh}$ for chain complexes of presheaves of abelian groups and likewise $\mathbf{Ch_+PSh}$ for presheaves of chain complexes in degrees ≥ 0. We equip the categories $\mathbf{sPSh}_\bullet$, $\mathbf{sPShAb}$, $\mathbf{Ch_+PSh}$, $\mathbf{ChPSh}$ with the local projective model structure. They are the left Bousfield localizations of the projective model structures with respect to the covers

$$[(U \times_X V)_+ \rightrightarrows U_+ \sqcup V_+] \to X_+$$

(for $\mathbf{sPSh}_\bullet$, and likewise for the three other categories). Here $U \sqcup V \to X$ is a covering in the Zariski topology.

The corresponding categories of presheaves on the site $\mathbf{SmAn}$ of smooth complex manifolds are endowed with the local projective model structures with respect to the usual topology on $\mathbf{SmAn}$.

Remark 4.4.3. The results of this section hold unchanged if we replace the Zariski by the Nisnevich or etale topology on $\mathbf{Sm/C}$. We could also furthermore localize with respect to $\mathbf{A}^1$ on the algebraic side and with respect to the disk D^1 on the analytic side.

Lemma 4.4.4. *There is a chain of Quillen adjunctions of the model categories mentioned above*

$$\mathbf{sPSh}_\bullet \overset{\mathbf{Z}[-]}{\rightleftarrows} \mathbf{sPShAb} \overset{N}{\underset{\Gamma}{\rightleftarrows}} \mathbf{Ch_+PSh} \overset{\iota}{\underset{\tau}{\rightleftarrows}} \mathbf{ChPSh}.$$

The analogous categories for the site $\mathbf{SmAn}$ *are related to these categories by Quillen adjunctions, for example*

$$\mathbf{ChPSh} \overset{\mathrm{an}^*}{\underset{\mathrm{an}_*}{\rightleftarrows}} \mathbf{ChPSh}(\mathbf{SmAn}, \mathbf{Ab}).$$

All these model categories satisfy Assumption 3.2.1. Moreover, their monoidal units are cofibrant.

Proof. The Quillen adjunctions of these categories, equipped with the projective model structure, transfer from the standard Quillen adjunctions for simplicial sets etc. It passes to adjunctions of the local structures by the universal property of the Bousfield localization. The Quillen adjunctions to presheaves on $\mathbf{SmAn}$ hold since $\mathrm{an} : \mathbf{Sm} \to \mathbf{SmAn}$ sends Zariski covers to analytic covers. The properties required in Assumption 3.2.1 are discussed in [PS, Section 7.2]. Like any representable presheaf, the monoidal units, which are the representable presheaves associated to $\mathrm{Spec}\,\mathbf{C}$ (or $\mathrm{an}(\mathrm{Spec}\,\mathbf{C})$) are cofibrant. $\qquad\square$

We now turn towards the construction of our Deligne cohomology spectrum. The cdga corresponding to Betti cohomology is defined by $A = \bigoplus_{p \in \mathbf{Z}} \underline{\mathrm{R}}\mathrm{an}_* \mathbf{Z}(p)[-2p]$. Similarly, let $B := \bigoplus_p \underline{\mathrm{R}}\mathrm{an}_* \Omega^*[-2p]$, where Ω^* denotes the cdga of holomorphic differential forms. Finally, let

$$C : X \mapsto \bigoplus_p \left(\underset{\overline{X}}{\mathrm{colim}}\, F^p \Omega^*_{\overline{X}}(\log(\overline{X}\backslash X)) \right)[-2p],$$

be the Hodge filtration, i.e., the stupid truncation $\sigma^{\geq p}$ of the complex of meromorphic forms on $\overline{X}^{\mathrm{an}}$, which are holomorphic on X^{an} and have at most logarithmic poles at $\overline{X}\backslash X$. The colimit runs over all smooth compactifications $j : X \to \overline{X}$ such that $\overline{X}\backslash X$ is a strict normal crossings divisor.

We have obvious maps $A \xrightarrow{a} B \xleftarrow{c} C$ of cdga's of presheaves on $\mathbf{Sm/C}$ and consider the cdga D and the weakly equivalent dga $E = \mathrm{cone}(A \oplus C \xrightarrow{a-c} B)[1]$ defined above. On the other hand, we have the associative (but noncommutative) product on E which is the particular case $\alpha = 0$ of the classical product on the Deligne complexes [EV88, Definition 3.2]. The following result, which was already pointed out by Beilinson [Beĭ84, Remark 1.2.6], relates the two products:

Proposition 4.4.5. *The cdga D of presheaves on $\mathbf{Sm}/\mathbf{C}$ defined above represents Deligne cohomology with integral coefficients in the sense that there is a functorial isomorphism, for any $X \in \mathbf{Sm}/\mathbf{C}$:*

$$\bigoplus_{p \in \mathbf{Z}} \mathrm{H}_{\mathrm{D}}^{2p-n}(X, \mathbf{Z}(p)) = \mathrm{H}_n \, \underline{\mathrm{R}}\mathrm{Map}_{\mathbf{ChPSh}}(X, D).$$

Under this isomorphism, the product on the right hand term induced by the multiplication on D and the comultiplication given by the diagonal map $X \to X \times X$ agrees with the classical product on Deligne cohomology. Moreover, all classical higher order products induced by the multiplication on E, such as Massey products [Den95]*, agree with the corresponding higher order products on the cdga D, in the sense that the derived convolution algebras are weakly equivalent differential graded algebras of presheaves:*

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{ChPSh}}(X, D) \sim \underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{ChPSh}}(X, E).$$

Proof. The identification of Deligne cohomology with the right hand side is well-known, see for example [HS10, Lemma 3.2] for a very similar statement. Note that $X \in \mathbf{ChPSh}$ (i.e., the free abelian representable presheaf $\mathbf{Z}[X]$) is cofibrant in the projective model structure. Hence the derived convolution algebras are defined (Lemma 4.3.10). The extra information concerning the products follows immediately from the above discussion. □

In order to connect the cdga D of Proposition 4.4.5 to, say, algebraic K-theory, it is necessary to work with presheaves of simplicial sets. As is well-known to the experts (we learned it from Denis-Charles Cisinski), it is not possible to construct a strictly commutative simplicial abelian group representing Deligne cohomology or even Betti cohomology with integral coefficients. In fact, Steenrod operations preclude the existence of a strictly commutative simplicial abelian (pre)sheaf representing Betti cohomology with integral coefficients. This problem gives rise to an application of the operadic rectification for which we need to work in some category of symmetric spectra. Because of its interest from the viewpoint of motivic homotopy theory, we work in the category of symmetric $\mathbf{P}^1$-spectra.

The category of motivic symmetric $\mathbf{P}^1$-spectra is the categories of modules over the monoid $R \in \Sigma\mathbf{sPSh}_\bullet$ whose n-th level is $R_n = (\mathbf{P}^1, 1)^{\otimes n}$, i.e., the n-th smash power of $\mathbf{P}^1$, pointed by 1. Here and below we identify any scheme over $\mathbf{C}$ with its representable presheaf. Note that R_n is a constant simplicial presheaf. We will abbreviate $\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{s}} := \mathbf{Mod}_R(\Sigma\mathbf{sPSh}_\bullet)$. We have a similar category $\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{sAb}} := \mathbf{Mod}_{\mathbf{Z}[R]}(\Sigma\mathbf{sPShAb})$ of modules over the monoid $\mathbf{Z}[R]$ whose n-th level is $(\mathrm{coker}(\mathbf{Z} \xrightarrow{1} \mathbf{Z}[\mathbf{P}^1]))^{\otimes n}$.

Given the cdga $D = \bigoplus_p D_p$ of Proposition 4.4.5, we consider the symmetric sequence, again denoted by D, whose l-th level is given by $D(l) := \bigoplus_p D_{p+l}$, with a trivial Σ_l-action. Then D is a commutative monoid object in $\Sigma\mathbf{ChPSh}$. Turning D into a commutative monoid object in $\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{s}}$, i.e., a commutative symmetric $\mathbf{P}^1$-spectrum is equivalent to specifying a monoid map $R \to D$ in $\Sigma\mathbf{sPSh}_\bullet$, which is equivalent to specifying a pointed map $(\mathbf{P}^1, 1) \to D(1) = \bigoplus_p D_{p+1}$ in $\mathbf{sPSh}$ or, equivalently, a section on $\mathbf{P}^1$ of D whose restriction to the point $1 \in \mathbf{P}^1$ vanishes. Yet in other words, we need to specify of a line bundle with a flat connection on $\mathbf{P}^1$. As is well-known, a nontrivial line bundle (more precisely, a generator of $\mathrm{H}_{\mathrm{D}}^2(\mathbf{P}^1, \mathbf{Z}(1))$) is not representable by a global section, but has to be constructed by patching local data. In the parlance of homotopy theory, the nonfibrancy of D precludes the existence of the required map. We therefore replace $\mathbf{P}^1$ by a weakly equivalent model. This amounts to the standard idea of representing cohomology classes by Čech covers. Consider the object $\widetilde{\mathbf{P}^1} \in \mathbf{sPSh}$ defined as

$$\widetilde{\mathbf{P}^1} := [G \rightrightarrows \mathbf{P}^1\backslash 0 \sqcup \mathbf{P}^1\backslash\infty],$$

where the simplicial presheaf G is defined by the homotopy pullback diagram

$$
\begin{array}{ccc}
G & \longrightarrow & \underline{\mathrm{R}}\mathrm{an}_*\mathcal{U} \\
{\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle\sim} \\
\mathbf{G}_{\mathrm{m}} & \longrightarrow & \underline{\mathrm{R}}\mathrm{an}_*\mathrm{an}^*\mathbf{G}_{\mathrm{m}}
\end{array}
$$

where $\mathcal{U} = [U^\pm \rightrightarrows U^+ \sqcup U^-]$ is the simplicial scheme whose only nondegenerate simplices are in degrees 1 and 0, which is the Čech cover of $\mathbf{G}_{\mathrm{m}}^{\mathrm{an}}$ arising from the cover $\mathbf{G}_{\mathrm{m}}^{\mathrm{an}} = U^+ \cup U^-$, where $U^+ = \{z \in \mathbf{C}, +z \notin \mathbf{R}^{\geq 0}\}$ and similarly with U^-. The map $\mathcal{U} \to \mathrm{an}^*\mathbf{G}_{\mathrm{m}} = \mathbf{G}_{\mathrm{m}}^{\mathrm{an}}$ is a weak equivalence in the local model structure (with respect to the usual topology on $\mathbf{SmAn}$). Hence the map $G \to \mathbf{G}_{\mathrm{m}}$ is a weak equivalence. Likewise,

$[\mathbf{G}_{\mathrm{m}} \rightrightarrows \mathbf{P}^1\backslash 0 \sqcup \mathbf{P}^1\backslash\infty] \to \mathbf{P}^1$ is a weak equivalence in the (Zariski) local model structure. Therefore, the composition of these maps yields a weak equivalence $\widetilde{\mathbf{P}^1} \xrightarrow{\sim} \mathbf{P}^1$. It induces a Zariski-local weak equivalence

$$(\widetilde{\mathbf{P}^1})^{\otimes n} \xrightarrow{\sim} (\mathbf{P}^1)^{\otimes n}$$

in $\mathbf{sPSh}$: for this we may use the local injective model structure on $\mathbf{sPSh}_\bullet$. In this model structure, all objects are cofibrant, so weak equivalences are stable under tensor products.

Let $\widetilde{\mathbf{Z}\{1\}} := \mathrm{coker}(\mathbf{Z} \xrightarrow{1} N\mathbf{Z}[\widetilde{\mathbf{P}^1}]) \in \mathbf{Ch}_+\mathbf{PSh}$. As in Example 3.2.3, we consider the free commutative monoid $\tilde{R}$ on $\widetilde{\mathbf{Z}\{1\}}$, i.e., $\tilde{R}_n = (\widetilde{\mathbf{Z}\{1\}})^{\otimes n}$. By the above, the natural map

$$\tilde{R} \to N\mathbf{Z}[R]$$

is a Zariski-local weak equivalence in $\mathbf{Ch}_+\mathbf{PSh}$.

We now specify the $\tilde{R}$-module structure on the symmetric sequence D defined above. As for Betti cohomology, the map $N\mathbf{Z}[\widetilde{\mathbf{P}^1}] \to \mathrm{an}_*\mathbf{Z}(1)[-2] \subset A_1$ is given by the map $\mathrm{an}_* U^\pm = \mathrm{an}_*(\mathbf{H} \sqcup \mathbf{H}') \to \mathrm{an}_*\mathbf{Z}(1)$ which is given by the section $2\pi i$ on $\mathbf{H} := \{\Im z > 0\}$ and 0 on $\mathbf{H}' := \{\Im z < 0\}$. The map $N\mathbf{Z}[\widetilde{\mathbf{P}^1}] \to C_1$ is determined by the map $\mathbf{G}_{\mathrm{m}} \to F^1\Omega^1_{\mathbf{P}^1}(\log(\{0,\infty\}))$ given by the section $d\log z = dz/z \in \Omega^1_{\mathbf{P}^1}(\log(\{0,\infty\}))$.

Finally, the map $\widetilde{\mathbf{P}^1} \to \Omega^* \otimes Q[-2] \subset B_1$ is given by the following map of complexes (the leftmost term lies in degree 2):

$$
\begin{array}{ccccc}
\mathrm{an}_* U^\pm & \longrightarrow & \mathbf{G}_{\mathrm{m}} \sqcup \mathrm{an}_*(U^+ \sqcup U^-) & \longrightarrow & \mathrm{an}_*\mathrm{an}^*\mathbf{G}_{\mathrm{m}} \sqcup \mathbf{P}^1\backslash 0 \sqcup \mathbf{P}^1\backslash\infty \\
{\scriptsize\binom{2\pi i|_{\mathbf{H}}}{0|_{\mathbf{H}'}}\otimes(1-t)}\Big\downarrow & & {\scriptsize dz/z\otimes t+\binom{\log^+ z|_{U^+}}{\log^- z|_{U^-}}\otimes dt}\Big\downarrow & & \Big\downarrow 0 \\
\Omega^0 \otimes \mathbf{Q}[t] & \longrightarrow & \Omega^1 \otimes \mathbf{Q}[t] \oplus \Omega^0 \otimes \mathbf{Q}[t]dt & \longrightarrow & (\Omega^* \otimes Q)^2 \longrightarrow \cdots
\end{array}
$$

Here, $\log^+ z$ and $\log^- z$ are two branches of the complex logarithm (defined on U^+ and U^-, respectively) which agree on $\mathbf{H}'$ and satisfy $\log^+ z - \log^- z = 2\pi i$ on $\mathbf{H}$. One easily checks that this defines a map of complexes which yields a map

$$\widetilde{\mathbf{Z}\{1\}} \to D_1 = A_1 \times_{B_1} (B_1 \otimes Q) \times_{B_1} C_1. \tag{4.4.6}$$

This defines an $\tilde{R}$-module structure on the symmetric sequence $D = (D(l))_{l\geq 0}$ defined above. Therefore, we obtain a strictly commutative motivic $\widetilde{\mathbf{P}^1}$ ring spectrum, which we denote by $\widetilde{\mathrm{H}_{\mathrm{D}}}$. As above, we write $\mathbf{Mod}^{\mathbf{Ch}_+}_{\widetilde{\mathbf{P}^1}} := \mathbf{Mod}_{\tilde{R}}(\Sigma\mathbf{Ch}_+\mathbf{PSh})$ and likewise for $\mathbf{Mod}^{\mathbf{Ch}}_{\widetilde{\mathbf{P}^1}}$. The cohomology represented by $\widetilde{\mathrm{H}_{\mathrm{D}}}$ is Deligne cohomology, including all higher product operations:

Proposition 4.4.7. *The strictly commutative $\widetilde{\mathbf{P}^1}$ ring spectrum*

$$\widetilde{\mathrm{H}_{\mathrm{D}}} \in \mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}^{\mathbf{Ch}}_{\widetilde{\mathbf{P}^1}}),$$

defined above is such that, for any smooth scheme $X/\mathbf{C}$, there is a natural isomorphism of derived convolution algebras

$$\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}^{\mathbf{Ch}}_{\widetilde{\mathbf{P}^1}}}(\tilde{R} \otimes X, \widetilde{\mathrm{H}_{\mathrm{D}}}) \sim \underline{\mathrm{R}}U\,\underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{ChPSh}}(X, D),$$

where $U : \mathbf{Mod}^{\mathbf{Ch}}_{\widetilde{\mathbf{P}^1}} \xrightarrow{\mathrm{ev}_0} \Sigma\mathbf{ChPSh} \to \mathbf{ChPSh}$ is the forgetful functor and $U : \mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}^{\mathbf{Ch}}_{\widetilde{\mathbf{P}^1}}) \to \mathbf{Alg}_{\mathrm{Comm}}(\mathbf{ChPSh})$ is the induced functor (Lemma 4.3.2). In particular, by Proposition 4.4.5, all products and higher order operations such as Massey products are computed by $\widetilde{\mathrm{H}_{\mathrm{D}}}$.

Proof. Again, $X \in \mathbf{Ch}(\mathbf{PSh})$ is cofibrant, hence so is $\tilde{R} \otimes X$ as an $\tilde{R}$-module. Therefore the derived convolution algebras are well-defined. By Proposition 4.3.11(1), we have to check that $\underline{\mathrm{R}}U\widetilde{\mathrm{H}_{\mathrm{D}}} \to U\widetilde{\mathrm{H}_{\mathrm{D}}} = D$ is a weak equivalence. This is implied by the fibrancy of $\widetilde{\mathrm{H}_{\mathrm{D}}}$ which by Theorem 3.3.4(1) follows from the fact that the maps

$$D(l) \to \underline{\mathrm{R}}\mathrm{Hom}(\widetilde{\mathbf{Z}\{1\}}, D(l+1))$$

are weak equivalences. This can be checked by applying the derived mapping space $\mathrm{H}_n\,\underline{\mathrm{R}}\mathrm{Map}(X, -)$ for any $X \in \mathbf{Sm}/\mathbf{C}$ and any $n \in \mathbf{Z}$. By Proposition 4.4.5, we get

$$\oplus_p\mathrm{H}^{2p-n}_{\mathrm{D}}(X, \mathbf{Z}(p)) \longrightarrow \ker \bigoplus_p \left(\mathrm{H}^{2p+2-n}_{\mathrm{D}}(\mathbf{P}^1 \times X, \mathbf{Z}(p+1)) \to \mathrm{H}^{2p+2-n}_{\mathrm{D}}(X, \mathbf{Z}(p+1)) \right).$$

The map between them is the cup product with the element in $\zeta \in \mathrm{H}_{\mathrm{D}}^2(\mathbf{P}^1, \mathbf{Z}(1))$ represented by the map (4.4.6). The element ζ generates this cohomology group, since the forgetful map to Betti cohomology $\mathrm{H}_{\mathrm{D}}^2(\mathbf{P}^1, \mathbf{Z}(1)) \to \mathrm{H}^2(\mathbf{P}^1, \mathbf{Z}(1)) \cong \mathbf{Z}$ is an isomorphism which sends ζ to 1. By the projective bundle formula for Deligne cohomology [EV88, Proposition 8.5] the map above is an isomorphism. $\square$

Finally, we construct the strictly commutative Deligne cohomology spectrum:

Theorem 4.4.8. *There is a strictly commutative $\mathbf{P}^1$-spectrum with values in simplicial presheaves on $\mathbf{Sm}/\mathbf{C}$,*

$$\mathrm{H}_{\mathrm{D}} \in \mathbf{Alg}_{\mathrm{Comm}}(\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{s}})$$

defined by

$$\mathrm{H}_{\mathrm{D}} := \mathrm{Comm} \circ_O^{\underline{\mathrm{L}}} \underline{\mathrm{R}}\Gamma R \otimes_{\tilde{R}}^{\underline{\mathrm{L}}} \underline{\mathrm{R}}\tau \widetilde{\mathrm{H}_{\mathrm{D}}},$$

which represents Deligne cohomology with integral coefficients, i.e., for any smooth algebraic variety $X/\mathbf{C}$, there is an isomorphism

$$\pi_n \underline{\mathrm{R}}\mathrm{Map}_{\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{s}}}(R \otimes X, \mathrm{H}_{\mathrm{D}}) = \bigoplus_{p \in \mathbf{Z}} \mathrm{H}_{\mathrm{D}}^{2p-n}(X, \mathbf{Z}(p)).$$

The multiplication on the left induced by the ring spectrum structure on H_{D} agrees with the classical product on Deligne cohomology. Moreover, the convolution algebras are related by the following natural weak equivalence:

$$\mathrm{Comm} \circ_O^{\underline{\mathrm{L}}} \underline{\mathrm{R}}\Gamma R \otimes_{\tilde{R}}^{\underline{\mathrm{L}}} \underline{\mathrm{R}}\tau \, \underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{Ch}}}(\tilde{R} \otimes X, \mathrm{H}_{\mathrm{D}}) \sim \underline{\mathrm{R}}\mathrm{Conv}_{\mathbf{Mod}_{\mathbf{P}^1}^{\mathbf{s}}}(R \otimes X, \mathrm{H}_{\mathrm{D}}).$$

In particular, all higher order products on Deligne cohomology, such as Massey products, are represented by the commutative ring spectrum H_{D}.

Proof. This follows from Proposition 4.4.5, Proposition 4.4.7 and Theorem 4.3.16, applied to the Grothendieck abelian category $\mathcal{A} = \mathbf{PSh}(\mathbf{Sm}/\mathbf{C}, \mathbf{Ab})$ and the model structures mentioned in Lemma 4.4.4. $\square$

Remark 4.4.9. In the context of complex-analytic smooth manifolds, a variant of the Deligne complexes above is given by replacing the Hodge filtration as defined above by $F^p\Omega_X^*$. The resulting groups (called *analytic Deligne cohomology*) are the ones defined in (4.4.1) for all (including noncompact) manifolds. The above technique of rectifying this spectrum works essentially the same way. An even more basic case covered by the techniques above is a strictly commutative ring spectrum representing Betti cohomology with integral coefficients.

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