Numerical Methods for Bayesian Inverse Problems

Lectures: Mo 9:00-11:00, Wed 9:00-11:00
Exercises: Mo 14:00-16:00
First class: November 2, 2020
All lectures and exercise classes take place online, the links will be published before the first class on the moodle page.

Instructors:
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Course description: This course discusses the mathematical foundations of Bayesian inversion and Bayesian inference as well as numerical methods for their efficient treatment. Specific focus will be on high-dimensional inverse problems for which the forward operator is given as the solution to a partial differential equation. This requires to formulate Bayes' theorem in a Banach space setting. Numerical methods covered will include Markov-Chain-Monte-Carlo methods (MCMC), quasi-Monte Carlo and sparse-grid quadrature, as well as modern variational inference techniques.

Prerequisites: This lecture is intended for MSc students or strong BSc students in mathematics or scientific computing. Students should have some background in probability theory, numerical analysis, PDEs and functional analysis. We'll try to make the course as self-contained as possible, and some of the necessary basics will shortly be recalled.

Tentative outline:

1. **Inverse problems:** introduction, Tikhonov regularization

2. **Review of important concepts in measure theory and functional analysis:** σ-algebras and measures, random variables, conditional and marginal distributions, distances between measures, Bochner integrals and L^p spaces

3. **Statistical inference in finite dimensions**: Bayes' Theorem and Bayesian modeling, prior and likelihood, noninformative priors, parameter estimation, MLE, MAP, posterior expectation

4. **The infinite dimensional setting**: inverse problems for elliptic and parabolic PDEs, random fields, Gaussian measures and Karhunen-Loève expansion, Bayes' theorem in infinite dimensions

5. **Monte Carlo methods**: Basic MCMC theory, variance reduction and importance sampling, Metropolis-Hastings, Gibbs sampling, Hamiltonian MCMC, Langevin dynamics, mixing times and curse of dimensionality

Further topics will be chosen from:

5. **Ratio estimators**: QMC, sparse-grid quadrature, multilevel methods, Laplace approximation

6. **Variational inference**: sampling via measure transport, tensor-train and sparse polynomial conditional sampling, Stein variational gradient descent, invertible neural networks, GANs

7. **Filtering**: particle filter, ensemble Kalman filter

The following list contains some general references on Bayesian inference and probability theory. Additional to these monographs, the course content will be based on various recent research papers, as attributed throughout the lecture.

References

[1] M. Dashti and A. Stuart, *The Bayesian Approach To Inverse Problems*, arXiv:1302:6989, 2013.

[2] J. Kaipio and E. Somersalo, *Statistical and Computational Inverse Problems*, Springer, 2004.

[3] C. P. Robert and G. Casella, *Monte Carlo Statistical Methods*, Springer, 2nd edition, 2004.

[4] L. Wasserman, *All of Statistics: A Concise Course in Statistical Inference*, Springer, 2004.

[5] A. Klenke, *Wahrscheinlichkeitstheorie*, 4. Ed., Springer, 2020.

[6] J. Liu, *Monte Carlo strategies in scientific computing*, Springer, 2001.

[7] J. Rosenthal, *A first look at rigorous probability theory*, World Scientific Publishing Co. Pte. Ltd., 2006.