21 juni 2024

Planlagt: 09:00-12:00

Eksamensnr: 13

Plads: EH-2153

Side 1 af 4

1a.
The arrow debreu state prices are the risk neutral probabilities at a given state in this economy. Say we are at time t in a two period binomial model, then the arrow debreu state prices would be $\psi_1 = q^2 \; and \; \psi_2 = (1-q)^2$, where $q = \frac{e^{rT}-d}{u-d}$, and relative changes to up and down nodes. In our binomial model of length T are state prices are: $\psi_1 = q^T \; and \; \psi_2 = (1-q)^T$

1b.
Barring arbitrage, for both the put and call the price at t=0 will be the discounted risk neutral expectation to the payoff at time T, and we will have to raise this entire expression to the Tth eksponent.

$$C_0 = e^{-rT}\left[q(S_0u-K)^+ - (1-q)(S_0d-K)^+\right]^T$$
$$P_0 = e^{-rT}\left[q(K-S_0u)^+ - (1-q)(K-S_0d)^+\right]^T$$

And given that maturity and strike T and K is the same for both derivatives, from the put-call parity it also follows that:

$$C_0 = S_0 - e^{-rT}K + P_0$$
$$P_0 = Ke^{-rT} - S_0 + C_0$$

1c.
The forward contract yields no positive payoff if it is out of the money, S < K. But the option portfolio, which is a buterfly portfolio, yields both positive payoff when the forward is out, and in the money. This is because the call yields a positive payoff when S > K, and the put yields a positive payoff when S < K. And it should also be noted that the option to not excercise in the option portfolio implies a premium of the cost to enter into the portfolio, as apposed to the forward portfolio where excercise is mandatory.

2a.
The Merton model describes a companies equity as value of assets less zero coupon bond debt, essentially:

$$E_t = V_t - p(t,T)N = V_t - N$$

At maturity T this becomes:

$$E_T = V_T - N$$

So considering this as a call option, the value of the assets are the underlying, and the notional amount in the ZCB is the strike of this option. Given that the value of the companies assets follow a geometric brownian motion – just as the stock, we then have that the value of equity at t=0 is:

$$E_0 = V_0\phi(d_1) - Ne^{-rT}\phi(d_2)$$

where:

$$d_1 = \frac{\ln\left(\frac{V_0}{N}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Then the value of the zero coupon bond debt at time 0 must be:

$$Ne^{-rT}\phi(d_2) = V_0\phi(d_1) - E_0$$

Or:

$$Ne^{-rT} = \frac{V_0\phi(d_1) - E_0}{\phi(d_2)}$$

$$NP(t,T) = \frac{V_0\phi(d_1) - E_0}{\phi(d_2)}$$

2b.
$\phi(d2)$ can be seen as cumulative standard normal probability of the company defaulting on its debt.

2c.
We know that the book value of the company can be written as equity less debt

$$B_0 = E_0 - D_0$$

Inserting equity and the fact that debt is a ZCB on a notional N:

$$B_0 = V_0\phi(d_1) - Ne^{-rT}\phi(d_2) - Ne^{-rT}$$
$$B_0 = V_0\phi(d_1) - Ne^{-rT}(\phi(d_2) + 1)$$

Inserting the implicitly defined value of the book value on the LHS and rearanging:

$$e^{-(r+s)T}N = V_0\phi(d_1) - Ne^{-rT}(\phi(d_2) + 1)$$
$$\frac{e^{rT}N}{e^{(r+s)T}N} = V_0\phi(d_1) - Ne^{-rT}(\phi(d_2) + 1)$$
$$e^{sT} = \frac{V_0}{Ne^{-rT}}\phi(d_1) - (\phi(d_2) + 1)$$
$$-e^{sT} = \phi(d_2) + 1 - \frac{V_0}{Ne^{-rT}}\phi(d_1)$$

Defining G and taking logs we get the result:

$$-sT = \ln\left[\phi(d_2) - G\phi(d_1)\right]$$
$$s = \frac{1}{T}\ln\left[\phi(d_2) + G\phi(-d_1)\right]$$

3a.
This means that the short term interest rate is mean reverting. We can see this from the (b-r). So when some constant b = r the drift term is removed. And we can see that this mean reversion happens with speed a. This is a fair assumption in relation to the real world, where it is known that equities do not in general mean revert, but interest rates do. Note also that the model allows for negative rates, where if the interest rate r is negative then the model implies that the short term interest rate explodes, as the drift becomes positive for r < 0.

3b.
The duration is defined as $D = -\frac{\frac{\partial P(t,T)}{\partial r}}{P(t,T)}$. In this model this will yield $B(t,T)$ for a ZCB. See the following:

$$\frac{\partial P(t,T)}{P(t,T)} = -B(t,T)A(T)e^{-B(t,T)r(t)}$$
$$= -B(t,T)P(t,T)$$

$$\Rightarrow D = B(t,T)$$

3c.
The purpose of the extra flexibility is to be able to model the entire yield curve throughout time, and not just short rates. The Vasicek model does not allow for modelling of spot rates as their times to maturity change after inital auction. In the real world, bonds with longer maturities do not develop the same as bonds with shorter maturities. Bonds with longer maturities are e.g. less sensitive to changes in interest rates, i.e. have a larger duration. So the Hull-whte model allows for this ”flexibility” that seems to be closer to how the real world develops.