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Danish Term Structure Forecasting Under the HJM Framework

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Abstract

This paper explores the application of the Heath-Jarrow-Morton (HJM) framework to forecast the Danish term structure of interest rates. The primary goal is to develop a robust model for forecasting forward rates of various maturities, focusing on Danish forward rates. Utilizing a discrete version of the HJM model, this study incorporates the Pure Expectations Hypothesis (PEH) and assumes the equivalence of risk-neutral and real-world measures through the Radon-Nikodym derivative. The empirical analysis involves backtesting 23 sets of 3-month forecasts, demonstrating that the model performs reasonably well in a pre-2022 environment with forecasting errors consistently within ± 0.5 percentage points. Post-2022, the errors exhibit more significant variations, particularly for longer maturities, due to factors such as the ECB's aggressive rate hiking campaign and heightened market uncertainty. Compared to Sætherø (2018), this paper introduces key differences, including the use of a non-zero drift term, a 1-year rolling volatility estimation, and a specific focus on Danish forward rates. The study concludes with suggestions for future work, including the incorporation of time-varying risk premiums and macroeconomic indicators to enhance model accuracy, as well as forecast uses in estimating expected counterparty exposures.

Introduction

Forecasting interest rates accurately is crucial for financial institutions as it directly impacts decision-making processes related to risk-management. This seminar paper focuses on forecasting the Danish term structure of interest rates using the Heath-Jarrow-Morton (HJM) framework. The HJM model is particularly advantageous due to its ability to incorporate market observables directly and its requirement to specify only the volatility structure, with the drift derived from this estimation.

The primary objective of this paper is to develop and backtest a discrete HJM model to forecast forward rates of various maturities for Danish froward rates. The study relies on the Pure Expectations Hypothesis (PEH) and assumes the equivalence of risk-neutral and real-world measures via the Radon-Nikodym derivative. The methodology involves a detailed empirical analysis, including volatility estimation through Principal Component Analysis (PCA) and Monte Carlo simulations for backtesting the model.

This study introduces several key differences compared to Sætherø (2018). Firstly, it incorporates a non-zero drift term, enhancing the model's ability to capture expected future movements in interest rates more accurately. Secondly, it uses a 1-year rolling period for volatility estimation, as opposed to Sætherø (2018)s 2-year rolling volatility, allowing for a more responsive measure to recent market changes. Lastly, this paper focuses specifically on Danish forward rates, providing insights into a localized market that is highly relevant for Danish financial institutions.

The results demonstrate that the model performs reasonably well in a pre-2022 environment, with forecasting errors consistently within ± 0.5 percentage points. However, post-2022, the errors exhibit more significant variations, particularly for longer maturities, due to the ECB's aggressive rate hiking campaign and heightened market uncertainty. These findings highlight the need for continuous improvement and adaptation of the model to better reflect real-world complexities.

The paper concludes with suggestions for future work, including the incorporation of time-varying risk premiums and macroeconomic indicators to enhance model accuracy. Additionally, further research could expand upon this study by conducting an analysis similar to Sætherø (2018)'s backtesting of expected counterparty exposures for a portfolio of swaps, providing a comprehensive understanding of the credit risk implications in conjunction with interest rate forecasting.

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1 Interest Rates

In this seminar I have chosen to focus on forecasting Danish forward rates of different maturities. Therefore the main focus of this section will be to introduce the basic concepts when modeling these.

The following definitions and notation are from Brigo and Mercurio (2006).

Definition 1.0.1 (The Short Rate) *The Short rate can formally be defined as the instantaneous spot rate, which will be defined later.*

$$r(t) = \lim_{T \to t} R(t,T) \tag{1.0.1}$$

Definition 1.0.2 (Bank Account) *Let $B(t)$ be the bank account at time $t \geq 0$, where $B(0) = 1$, and the bank account evolves according to:*

$$dB(t) = r_t B(t) dt \tag{1.0.2}$$

Yielding:

$$B(t) = \exp\left\{ \int_0^t r_s ds \right\} \tag{1.0.3}$$

This expresses that an investment of 1 at time 0 will grow at the short rate r_t, and yield the amount in 1.0.3 at time t

Definition 1.0.3 (Discount Factors) *The discount factor $D(t,T)$ expresses the amount at time t which is equivalent to one unit payable at time T:*

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left\{ -\int_t^T r_s ds \right\} \tag{1.0.4}$$

The discount factor can either be a stochastic process or deterministic, depending on whether or not the short rate is chosen to be so.

Definition 1.0.4 (Continuously compounded spot rates) *The continuously compounded spot rate at time t with maturity T is the constant annual rate at which a loan grows until maturity:*

$$R(t,T) = -\frac{\ln D(t,T)}{T-t} \tag{1.0.5}$$

The term structure of interest rates correspondingly consists of multiple spot rates with different maturities T

Definition 1.0.5 (Forward Rates) *The forward rate is an agreed-upon rate at time t which is to begin accruing at time S until maturity T. We therefore have: $t \leq S \leq T$. Formally the continuously compounded forward rate is:*

$$F(t,S,T) = -\frac{\log D(t,T) - \log D(t,S)}{T-S} \tag{1.0.6}$$

Definition 1.0.6 (Instantaneous Forward Rates) *The instantaneous forward rate can then be denoted $f(t,T)$, and derived from the forward rate as:*

$$f(t,T) = \lim_{S \to T} F(t,T,S) = -\frac{\partial}{\partial T}(\log D(t,T)) \tag{1.0.7}$$

This essentially expresses the rate contracted at time t that accrues from time T. Note that the forward rate with expiry t is simply the short rate: $r(t) = f(t,t)$.

Finally, let us end this section by also expressing the discount factor in 1.0.4 as a function of the instantaneous forward rate and the inverse of the continuously compounded spot rate in 1.0.4:

$$D(t,T) = \exp\left\{ -\int_t^T f(t,s)ds \right\} \tag{1.0.8}$$

$$D(t,T) = \exp\left\{ -R(t,T)(T-t) \right\} \tag{1.0.9}$$

2 Stochastic Interest Rate Modelling

2.1 One-Factor Models

A one-factor model seeks to capture the dynamics of the risk-free short rate $r(t)$. The stochastic nature of the rate is often described by a stochastic differential equation:

$$dr = \mu(r,t)dt + \sigma(r,t)dW(t), \tag{2.1.1}$$

where:

- dr: the change in the short rate over a small time interval dt,
- $\mu(r,t)$: the drift term representing the expected direction and magnitude of changes in r,
- $\sigma(r,t)$: the diffusion term, expressing the variability of changes in r,
- $dW(t)$: a standard Brownian motion term as defined in A.1

The drift term, $u(r,t)$, may include a market price of risk adjustment, denoted as $\lambda(t)$, when implemented under a real-world measure.

Multi-factor models stand in contrast to the one-factor models. These incorporate multiple sources of randomness to describe the behavior of various interest rate variables simultaneously, often focusing on rates of different maturities. This enables capturing several points on the yield curve, providing a more comprehensive understanding of interest rate dynamics. These models extend the one-factor framework, such that dr, $\mu(r,t)$, $\sigma(r,t)$, $dW(t)$ become vectors. The one I make use of in this paper is the Heath Jarrow Morton Model.

2.2 Heath Jarrow Morton Model

The notation used follows from Sætherø (2018).

2.2.1 Model Framework

The HJM model is utilized to describe the dynamics of the entire forward rate curve $f(t,T)$, where $0 \leq t \leq T \leq T^*$ and T^* is an assumed ultimate maturity. Here the forward rate comes into play. Recall the forward rate presented in equation 1.0.7, then assume it follows the dynamics in this SDE:

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)^\top dW(t), \tag{2.2.1}$$

Where the variables are vectors as mentioned in the previous section, effectively making this a multi-factor model.

2.2.2 Risk-Neutral Measure

In the risk-neutral framework, the evolution of the discount factors plays a critical role. By integrating equation 2.2.1 we have that: $df(t,T) = \frac{dD(t,T)}{D(t,T)}$. This leads to the following SDE for discount factor dynamics:

$$\frac{dD(t,T)}{D(t,T)} = r(t)dt + \nu(t,T)^\top dW(t), \tag{2.2.2}$$

Where $\nu(t,T)$ is the discount factor diffusion term. Applying Ito's Lemma as defined in A.2 to 2.2.1, the dynamics of $df(t,T)$ become:

$$d(\log D(t,T)) = \left(r(t) - \frac{1}{2}\nu(t,T)^\top \nu(t,T)\right) dt + \nu(t,T)^\top dW(t). \tag{2.2.3}$$

This can then be differentiated w.r.t T yielding the risk neutral form of the forward rate dynamics

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial T} \log D(t,T) \\ &= -\frac{\partial}{\partial T}\left[r(t) - \frac{1}{2}\nu(t,T)^\top \nu(t,T)\right] dt - \frac{\partial}{\partial T}\nu(t,T)^\top dW(t). \end{aligned} \tag{2.2.4}$$

Comparing this with the standard form of $df(t,T)$ in equation 2.2.1, and noting that $r(t)$ does not depend on T, the risk-neutral drift and volatility are then:

$$\begin{aligned} \sigma(t,T) &= -\frac{\partial}{\partial T}\nu(t,T) \\ \mu(t,T) &= \sigma(t,T)^\top \int_t^T \sigma(t,u)du. \end{aligned} \tag{2.2.5}$$

Substituting these back into 2.2.1, we obtain the final form for the evolution of forward rates under the risk-neutral measure:

$$df(t,T) = \left(\sigma(t,T)^\top \int_t^T \sigma(t,u)du\right) dt + \sigma(t,T)^\top dW(t), \tag{2.2.6}$$

This underscores that the drift in the risk-neutral setting is entirely governed by the volatility structure of the forward rates. Transitioning from the real-world measure P to the risk-neutral measure Q can then be done using the Radon-Nikodyn derivative in A.3. Then, making use of Girsanov's theorem in A.4, confirms that the volatility structures in the real-world and risk-neutral measures are equivalent, allowing for the estimation of risk-neutral volatilities directly from historical data. This alignment is central to the HJM framework, ensuring that the model's risk-neutral dynamics faithfully reflect observable market behaviors.

2.2.3 Real World Measure

Now that we have a risk-neutral measure to price the swaps in this paper, we now need a model to simulate the path of the real-world interest rates. Despite knowing from Girsanov's theorem that the volatility can be shown to be equivalent under both measures, it is not the same for the drift. A simple assumption that we can make is that the drift is equivalent under both measures, as is done in e.g. Sætherø (2018). One major drawback from this assumption is that this also assumes a positive market price of risk, implying that the pure expectations hypothesis holds true:

Definition 2.2.1 (Pure Expectations Hypothesis) *The Pure Expectations Hypothesis posits that forward rates are unbiased estimators of future spot rates. Formally this looks like:*

$$R(t_0, t_n)(t_n - t_0) = \sum_{i=0}^{n-1} E[R(t_i, t_{i+1})](t_{i+1} - t_i).$$

Here, $R(t_0, t_n)(t_n - t_0)$ represents the total return on a safe investment between times t_0 and t_n, which is assumed to be equivalent to the expected total rolling return on sequentially maturing investments from t_0 to t_n. This has been rejected in e.g. Sarno et al. (2007). Despite this, I decide to continue with this assumption, as it simplifies the model. The effects on the empirical analysis will be discussed later on.

2.2.4 Discretization and Simulation

Simulating the continuous HJM model can be challenging due to the complex nature of the volatility structure. So in order to facilitate simulation without overly restricting the volatility's form, a discrete approximation is required.

Let $\hat{f}(t_i, t_j)$ represent the discrete forward rate for time $t = t_j$ at time t_i. Both t_i and t_j are discretized using a uniform grid: $0 = t_0 < t_1 < \ldots < t_M = T^*$. Under this grid, the discrete discount factors look like:

$$\hat{D}(t_i, t_j) = \exp\left(\sum_{l=i}^{j-1} \hat{f}(t_i, t_l)(t_{l+1} - t_l)\right). \tag{2.2.7}$$

Which are calculated using market observables as in equation 1.0.9. To align the discrete discount factors with the continuous model at the initial time $t_i = 0$, the following condition is used as in Sætherø (2018):

$$\begin{aligned} \int_0^{t_j} f(0,u)\,du &= \sum_{l=0}^{j-1} \hat{f}(0,t_l)(t_{l+1}-t_l) \\ \hat{f}(0,t_l) &= \frac{1}{t_{l+1}-t_l}\log\frac{D(0,t_l)}{D(0,t_{l+1})} \end{aligned} \tag{2.2.8}$$

for all $l = 1, 2, .., M-1$.

The discrete version of the instantaneous model with M factors as described in 2.2.6 is then expressed as:

$$\hat{f}(t_i,t_j) = \hat{f}(t_{i-1},t_j) + \hat{\mu}(t_{i-1},t_j)(t_j - t_{i-1}) + \sum_{k=1}^{M} \hat{\sigma}_k(t_{i-1},t_j)\sqrt{t_i - t_{i-1}}W_{ik}, \tag{2.2.9}$$

For all $i = 1, 2, .., M-1$ and $j = 1, 2, .., M$. W_{ik} are independent vectors of standard normal distributed variables, with length M. The drift term, $\hat{\mu}(t_{i-1}, t_j)$, is approximated by discretization of equation 2.2.5:

$$\hat{\mu}(t_{i-1},t_j)(t_{j+1}-t_j) = \sum_{k=1}^{M} \hat{\mu}_k(t_{i-1},t_j) \tag{2.2.10}$$

$\hat{\mu}(t_{i-1}, t_j)$ is given by:

$$\hat{\mu}(t_{i-1},t_j) = \frac{1}{(t_{j+1}-t_j)}\frac{1}{2}\left[\left(\sum_{l=i}^{j} \hat{\sigma}k(t_{i-1},t_l)(t_{l+1}-t_l)\right)^2 - \left(\sum_{l=i}^{j-1} \hat{\sigma}k(t_{i-1},t_l)(t_{l+1}-t_l)\right)^2\right].$$

This ensures that the drift terms account for the volatility structure in a manner consistent with the continuous model dynamics. As opposed to Sætherø (2018), I will make use of this drift term in one of the models, and then compare it to the simplified model Sætherø (2018) uses where the discrete drift is assumed to be zero. Lastly, to simulate the discrete forward rate equation 2.2.9, an initial forward rate curve $\hat{f}(0,t)$ for $0 \leq t \leq T$ is required. This is given by combining equations 2.2.8 and 1.0.9, and using time steps Δt, thereby transforming our spot rate curve points into forward rates.

$$\hat{f}(0,t) = \frac{1}{\Delta t}\Big(R(0,t+\Delta t)(1+\Delta t) - R(0,t)t\Big) \tag{2.2.11}$$

This method of discretization ensures that the model can be implemented in a computationally efficient manner while preserving the complex dynamics prescribed by the continuous HJM framework.

2.3 Volatility

As can be seen from 2.2.9, the way I choose to model volatility plays a crucial role in the HJM model. There are many methods to choose from. Let us start by loosely defining volatility in this context as the standard deviation of specified increments of some time series: $A = \{a_1, a_2, ..., a_n\}$. As argued in Sætherø (2018), measuring the increments as the absolute difference between interest rates at different time points: $d_1 = \{a_2 - a_1, a_3 - a_2, ..., a_n - a_{n-1}\}$, can be optimal in a low-interest rate environment. The dataset I cover also shows a low interest rate environment, therefore I will also measure the increments as d_1, as apposed to $d_2 = \{\frac{a_2 - a_1}{a_1}, \frac{a_3 - a_2}{a_2}, ..., \frac{a_n - a_{n-1}}{a_{n-1}}\}$.

To effectively model volatility, I will utilize a Principal Component Analysis (PCA) as in Sætherø (2018). The PCA procedure involves calculating the covariance matrix Σ from the observed data, which is then decomposed into its eigenvalues and eigenvectors. The covariance matrix Σ can be expressed as $\Sigma = V\Lambda V^{-1}$, where Λ is a diagonal matrix containing the eigenvalues λ_i, and V is the matrix of corresponding eigenvectors.

Volatility factors are derived from the eigenvalues and eigenvectors, scaled to annual terms using the factor $\sqrt{265}$: $\hat{\sigma}_j(t_{i-1}, t_j) = \sqrt{\lambda_j 265} V_j$ where $j = 1, \ldots, M$ represents the different maturities. 265 is assumed as trading days per year.

To determine the number of principal components to include, I will calculate the total contribution of each component to the variance, denoted as TC_i, using the formula: $TC_i = \frac{\lambda_i}{\sum_{i=1}^{M} \lambda_i} 100\%$ Components will be included in descending order of their variance contribution until at least 95% of the total variability in the data is explained. The PCA will be performed on each set of training data. This means that all data up until each forecast point will be used.

By applying PCA, we can capture the essential patterns in the volatility structure, allowing for a robust and computationally efficient implementation of the HJM model. The first few components typically represent significant movements such as parallel shifts and twists in the yield curve, which are critical for accurate short-rate forecasting.

3 Backtest Methodology

3.1 Forecast length

The model presented in equation 2.2.9 specifies that the forward rate for a given maturity at time $t+1$ is a function of the value at time t. How we decide to specify the time increment $\Delta t = t + 1 - t$ could technically be any length. However, it would seem to make sense to use daily steps as I wish to forecast the daily developments over a given forecast range.

My choice of forecast range will be 3 months in advance, which is modelled as 63 days. this is chosen as it would make sense to further this paper, and use the forecasted developments of different maturities to price a range of fixed income derivatives, e.g swaps. As most DKK fixed income instruments as swaps are related to the 3 month Cibor, it would be very useful to know the value of the 3 month cibor 3 months in advance.

3.2 Backtest

The backtest will start from $t =$ 2018-01-01, and step 3 months in time until $t =$2024-01-01 is reached. For each forecast time step the given volatility measure specified later on will be calculated aswell as the drift for the M maturities from the principal component analysis will be calculated. Thereafter, a Monte Carlo simulation of N iterations will be run for $T = 63$ time steps and M maturities. For each time step the forward rate at time t, $f(t, M)$ is updated, by calculating the drift and volatility term in 2.2.9, this forward rate is then used as an input for the next time step. The result from this is a $NxTxM$ array we can call B_t. Averaging across the N iterations will then yield a matrix containing the forecasted daily forward rates for each maturity 63 days in advance, which we can call F_t, where subscript t still specifies the forecast start date.

4 Empirical Analysis

4.1 Data Description

My Dataset consists of daily spot rates from 2012-01-01 till 2024-01-01 on different maturities. The maturities break down as: CIBOR 1M - 6M, DKK FRAs 9M - 24M, DKK Swaps 3Y -30Y.

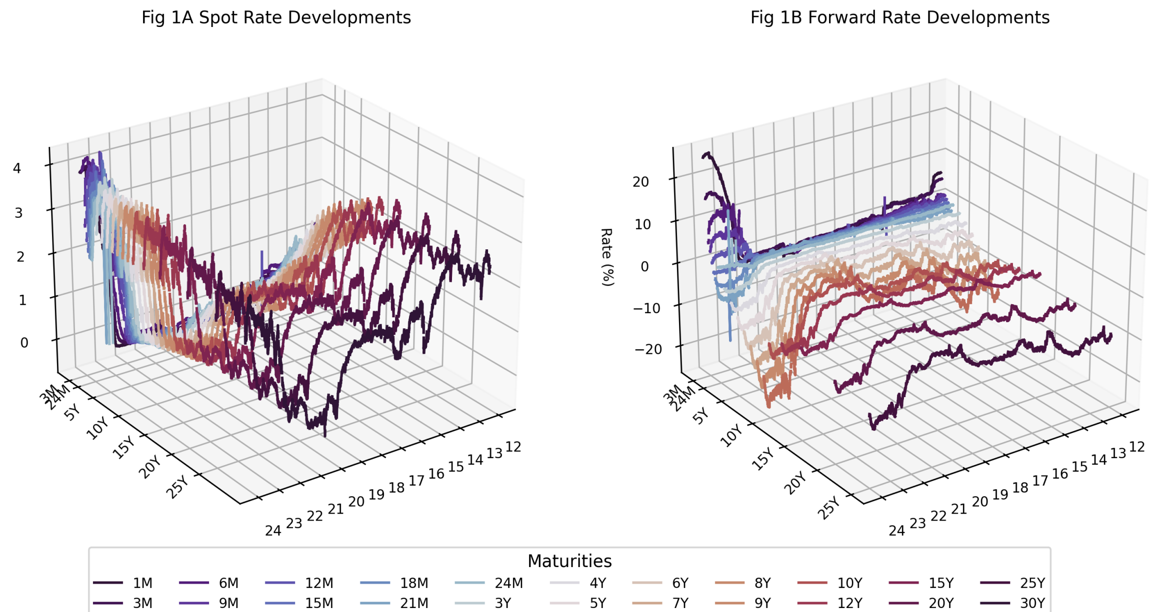

Figure 1: Spot and Forward Rate Developments.

The development of the selected maturities for spot and forward rates is illustrated in Figure 1. The spot rates exhibit noticeable fluctuations over the observed period, with certain periods showing higher volatility than others. For example, the period from 2014 to 2016 displays relatively stable rates, particularly for shorter maturities like CIBOR 1M and 3M. However, from 2018 onwards, the rates show increased volatility, particularly during the years 2020 and 2021, due to central back intervention under and post Covid.

Forward rates also demonstrate substantial volatility, especially for longer maturities such as 10Y and 20Y. The sharp movements in forward rates around 2020 again reflect the market's reaction to monetary policies and economic measures taken to counteract the impacts of the pandemic. This will naturally affect the 3 month forecasts I will perform, given that I do not account for any central bank deposit rate hike announcements.

4.2 Volatility Estimation

The estimated volatility used as input for the drift term, and thereby forward rate development as specified in 2.2.9, will be the 1Y rolling annualized volatility of the absolute increments 265 days before the forecasting cutoff. This is due to the highly volatile nature of the out of sample period.

We don't want to capture too much of the variability as these large spikes in rates, typically don't happen at the frequency they did during Covid. Keep in mind that we can't achieve a "completely valid" backtest due to this, as we would not have known how volatile the forecasting period would have been. This stands in contrast to the annualized 2y rolling volatility used in Sætherø (2018). For the first forecast beginning at time t = 2018-01-01, volatility for maturity j will be measured using the absolute increment series $d_{1t-1} \in [a_1 = t-1-265 : n = t-1]$.

4.2.1 Principal Component Analysis

The PCA is constructed such that we achieve the most valid backtest possible. This means that it is conducted on the subset of data from the start of the dataset up until the first forecast. The period: 2012-01-01 to 2017-31-12. As apposed to using the entire dataset as in Sætherø (2018), this does not make use of any data we would not have had at the beginning of the first forecast. The results of the PCA in 1 show that we optimally would make use of the 10 maturities listed.

Maturity	Eigenvalue	Explained cumulative variability
1M	10.287	0.49
3M	2.097	0.589
6M	1.915	0.68
9M	1.519	0.753
12M	1.319	0.816
15M	1.017	0.864
18M	0.912	0.907
21M	0.432	0.928
24M	0.407	0.947
3Y	0.359	0.964

Table 1: PCA Results: Eigenvalues and Explained Variability for data up until 1st forecast.

By comparison, Sætherø (2018) utilized 6 components derived from a larger dataset spanning 23 years. The difference in the number of components can be attributed to the length and variability of the datasets used. This approach, focuses on a shorter and more recent period and reflects the highly volatile nature of the dataset from 2018-2024 mentioned in section 4.1.

4.3 Simulation Convergence

Our choice of simulation iterations plays a pivotal role. We want to optimize the trade-off between time complexity and optimal simulation convergence. Figure 2 shows the 3 month out of sample forecast convergence for the first forecast using different choices of N iterations:

It seems that a choice of $N = 800$ iterrations would supply us with optimal convergence of forward rates.

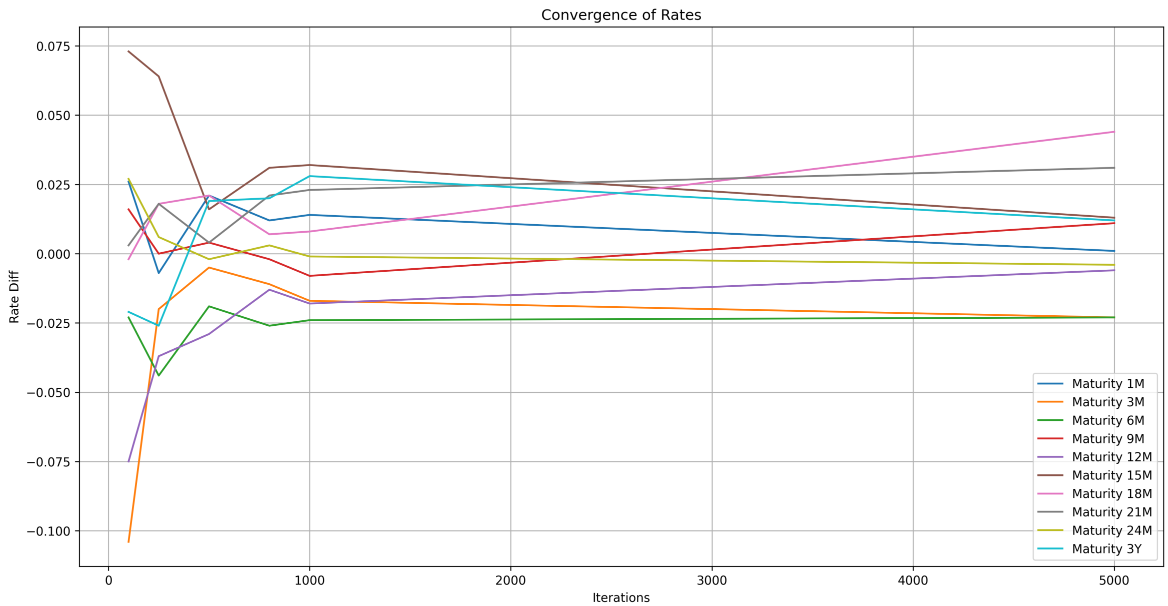

Figure 2: Rate Convergence.

4.4 Forecasting Results

4.4.1 Forecast Errors

Figure 3 Shows the results from the backtest of the 23 sets of 3 month forecasts. The results are presented pre and post-2022, due to the great variation in the last part of the dataset. First, looking at the forecasting errors for the entire forward curve, the error level is consistently between ± 0.5 %-points in the pre 2022 environment in figure 3B. For a swap agreement on a notional of 10 million DKK, this corresponds to an error in each party's expected floating leg payment of $\pm 50,000$ DKK. The error jumps a couple of times in the pre 2022 time series, which as mentioned in section 4.1, is due to central bank intervention during covid. But it is also due to the backtesting methodology of stepping 3 months at a time. A lot can happen in 3 months, therefore the simulation can easily miss key forecast-altering events. The post 2022 forecasting errors in figure 3D show the same story. They lay around ± 0.5 %-points except for small jumps mid 2022. The longer maturities show large forecasting errors in late 2023. This can most likely be attributed to the ECB rate hike regime that has fueled a market sentiment that interest rates and inflation will stay higher than has been seen post-financial crisis for a significant amount of time.

4.4.2 3M CIBOR

Looking specifically at the 3M CIBOR forecasts in figures 3A and 3C, we can see that the 3month forward forecasts perform acceptably. They can tolerated. The rate development consistently lies inside the 95% confidence bands, and errors don't seem to ever get unacceptably large, even when considering the mentioned backtesting concerns. The forecast could easily be improved upon by combining it with market know-how. Before an upcoming deposit rate hike one could run the forecast, adjust for any market expectations of the upcoming hike, compare this to past 3 month CIBOR developments post rate hikes, and come up with a revised expected CIBOR development.

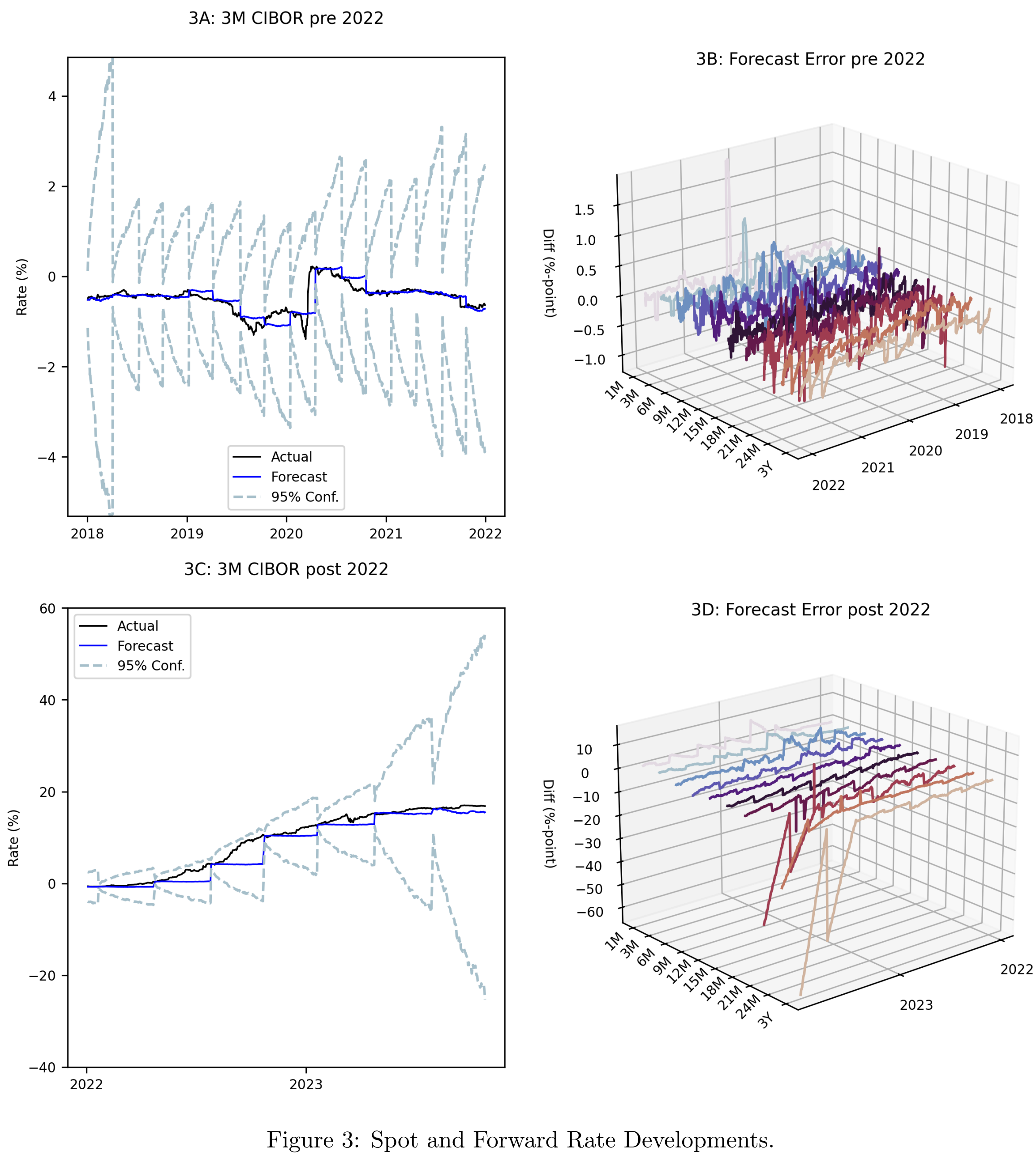

Figure 3: Spot and Forward Rate Developments.

5 Discussion

In constructing the discrete HJM model, I relied on the Pure Expectations Hypothesis (PEH). The PEH posits that forward rates are unbiased estimators of future spot rates. This simplifies the model by assuming that the expected future spot rate equals the forward rate, which is derived under the assumption of no arbitrage in a risk-neutral framework. However, the PEH has been empirically challenged, as noted in Sarno et al. (2007), suggesting that forward rates may not always be perfect predictors of future spot rates due to risk premiums and other market frictions.

My model operates under the risk-neutral measure Q, assuming equivalence to the real-world measure P by way of the Radon-Nikodym derivative. This assumption, while simplifying the mathematical complexity of the model, may introduce discrepancies when translating risk-neutral valuations to real-world predictions. The Radon-Nikodym derivative allows for the transformation between these measures, enabling the use of historical data to estimate risk-neutral volatilities directly.

Using Girsanov's theorem, which states that the volatility structures in the real-world and risk-neutral measures are equivalent, I assumed that the drift under both measures is the same. This simplifies the model but also implies a positive market price of risk. In reality, this assumption may not hold perfectly, particularly in times of market stress or significant central bank interventions, such as during and post Covid.

These modeling choices reflect a trade-off between mathematical tractability and empirical accuracy. While the risk-neutral framework provides a robust method for pricing derivatives and forecasting interest rates, the real-world deviations observed, particularly in longer maturities, highlight the need for incorporating additional factors, such as time-varying risk premiums and macroeconomic indicators, into the model. Future work could focus on enhancing the model by integrating these elements to better capture the nuances of real-world interest rate dynamics.

Overall, the forecasting model performs reasonably well, but it is evident that incorporating more real-time data and market insights could further improve accuracy, especially for longer maturities during periods of economic turbulence. Despite these limitations, the framework provides valuable insights and a solid foundation for further research and model refinement.

6 Conclusion

This seminar paper has explored using the Heath-Jarrow-Morton (HJM) framework to forecast interest rates, focusing on Danish spot rates of various maturities and instruments. The study utilized a discrete version of the HJM model, incorporating the Pure Expectations Hypothesis (PEH) and the equivalence of risk-neutral and real-world measures via the Radon-Nikodym derivative, and Girsanovs theorem.

The empirical analysis, which involved backtesting 23 sets of 3-month forecasts, demonstrated that the model performs reasonably well in a pre-2022 environment, with forecasting errors consistently within ± 0.5 percentage points. These errors translate to a deviation of $\pm 50,000$ DKK on a notional amount of 10 million DKK for swap agreements. The stability of the forecast errors in the pre-2022 period can be attributed to a relatively stable economic environment and effective central bank interventions during the Covid.

Post-2022, the forecasting errors exhibit more significant variations, particularly for longer maturities. This can be attributed to the ECB's aggressive rate-hiking campaign at the end of 2023 and heightened market uncertainty regarding long-term economic conditions and inflation expectations. These factors contribute to larger discrepancies between forecasted and actual rates, highlighting the limitations of the model in capturing real-world complexities.

Compared to Sætherø (2018), this paper introduces several key differences. Firstly, the model used in this study incorporates a non-zero drift term, enhancing its ability to capture expected future movements in interest rates more accurately. Secondly, the volatility estimation is based on a 1-year rolling period, as opposed to Sætherø (2018)s 2-year rolling volatility, allowing for a more responsive measure to recent market changes. Lastly, this paper specifically analyzes Danish forward rates, providing insights into a localized market that is highly relevant for Danish financial institutions.

Future work should focus on refining the model by incorporating additional factors such as time-varying risk premiums and macroeconomic indicators. Enhancing the model with real-time data and market insights could improve its accuracy, especially for longer maturities. Additionally, further research could expand upon this study by conducting an analysis similar to Sætherø (2018)s backtest of expected counterparty exposures for a portfolio of swaps. This would provide a comprehensive understanding of the credit risk implications in conjunction with interest rate forecasting.

In conclusion, the discrete HJM model offers a robust method for forecasting interest rates and pricing derivatives under normal market conditions, due to its simplicity and direct calibration from market observables. The model's primary strength lies in its use of market inputs and the need to specify only the volatility structure, with the drift being derived from this volatility estimation. However, the observed deviations during periods of economic turbulence underscore the need for continuous improvement and adaptation of the model to better reflect real-world complexities. Despite these limitations, the current framework provides valuable insights and establishes a solid foundation for yield curve forecasting.

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Appendices

A Important Concepts

The following theorems follow from Björk (2009) and Shreve et al. (2004)

A.1 Brownian Motion

A Brownian motion, or Wiener process, is a stochastic process $\{X(t), t \geq 0\}$ satisfying the following conditions:

1. $X(0) = 0$ almost surely.
2. The process has independent increments: for any $0 \leq s < t$, the increment $X(t) - X(s)$ is independent of the past values.
3. The process has Gaussian increments: for any $0 \leq s < t$, the increment $X(t) - X(s)$ is normally distributed with mean zero and variance $t - s$. That is, $X(t) - X(s) \sim N(0, t - s)$.
4. The process is continuous: the path $t \mapsto X(t)$ is continuous with probability one.

This formal definition underpins many financial models and stochastic differential equations.

A.2 Ito's Lemma

Ito's Lemma is a key result used to determine the differential of a function applied to a stochastic process. Let $X(t)$ be a stochastic process satisfying the stochastic differential equation:

$$dX(t) = \mu(X, t)dt + \sigma(X, t)dW(t), \tag{A.2.1}$$

where: - $\mu(X,t)$ is the drift term, - $\sigma(X,t)$ is the diffusion term, - $dW(t)$ represents increments of a standard Brownian motion.

If $f(X,t)$ is a twice continuously differentiable function of X and t, then the differential of f is given by:

$$df(X,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}\sigma^2(X,t)\frac{\partial^2 f}{\partial X^2}dt. \tag{A.2.2}$$

This formula allows the computation of the dynamics of functions of stochastic processes, essential in many financial models.

A.3 The Radon-Nikodym Derivative

The Radon-Nikodym derivative is a mathematical concept that facilitates the transformation between two probability measures. It plays an essential role in financial mathematics, especially when switching between different pricing measures.

Let P and Q be two equivalent probability measures (i.e., measures that agree on which events have zero probability). The Radon-Nikodym theorem states that there exists a function Z such that:

$$Z = \frac{dQ}{dP}, \tag{A.3.1}$$

where the derivative $\frac{dQ}{dP}$ is called the Radon-Nikodym derivative. This function Z allows us to convert expectations with respect to the measure P into expectations with respect to the measure Q and vice versa.

In financial modeling, this is particularly useful when comparing contingent claims under different equivalent martingale measures. For example, if $X(T)$ is the value of a contingent claim at time T, then the price of the claim at time t under measure Q can be derived from its price under measure P using the Radon-Nikodym derivative:

$$E_Q\left[X(T) \mid \mathcal{F}_t\right] = E_P\left[X(T) \cdot \frac{dQ}{dP} \mid \mathcal{F}_t\right]. \tag{A.3.2}$$

In practice, the Radon-Nikodym derivative can also be seen as a "change of numeraire," which allows the same security to be valued differently depending on the asset used as a benchmark for comparison.

A.4 Girsanov's Theorem

Girsanov's theorem describes the effect of changing the probability measure on the underlying stochastic processes.

Theorem (Girsanov's Theorem): Let $W(t)$ be a Brownian motion under the probability measure P. Suppose there exists an adapted process $f(t)$ that satisfies:

$$E_P\left[\int_0^t f^2(\tau)d\tau\right] < \infty, \tag{A.4.1}$$

for any $t \geq 0$. Define the Radon-Nikodym derivative by:

$$\frac{dP^*}{dP} = g(t), \tag{A.4.2}$$

where

$$g(t) = \exp\left(\int_0^t f(\tau)dW(\tau) - \frac{1}{2}\int_0^t f^2(\tau)d\tau\right). \tag{A.4.3}$$

This ensures that $g(t)$ is a martingale and that the new measure P^* is equivalent to P. Under the measure P^*, the process $W^*(t)$ defined by:

$$W^*(t) = W(t) - \int_0^t f(\tau)d\tau, \tag{A.4.4}$$

is a Brownian motion.

In simpler terms, Girsanov's theorem allows us to change the probability measure, which effectively shifts the drift of a Brownian motion by an adapted process $f(t)$. This theorem is pivotal in pricing derivatives and managing risks because it enables us to move between measures that price various financial instruments appropriately.