Quantitative Macroeconomics
Final Project

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1 Simple Variant of Krusell-Smith Algorithm.

1.1 Restate the steps of the proof.

According to the Proposition 3, equilibrium dynamics, for some initial capital stock k_0, are given by:

$$k_{t+1} = \frac{1}{(1+g)(1+\lambda)} s(\tau)(1-\tau)(1-\alpha)\zeta_t k_t^\alpha \tag{1}$$

The saving rate is given by:

$$s(\tau) \equiv \frac{\beta\phi(\tau)}{1+\beta\phi(\tau)} \leq \frac{\beta}{1+\beta} \tag{2}$$

where:

$$\phi(\tau) \equiv \mathbb{E}_t \left[\frac{1}{1 + \frac{1-\alpha}{\alpha(1+\lambda)e_{t+1}} \lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))} \right] \leq 1 \tag{3}$$

Proof: It is known that all households are exactly the same ex-ante and that assumptions 1 and 4 from the original paper must be satisfied. In the first step we guess that:

$$a_{2,t+1} = s(\tau 1-\tau)w_t = s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^\alpha \tag{4}$$

We also know that:

$$k_{t+1} = \frac{K_{t+1}}{\Upsilon_{t+1}(1+\lambda)} \tag{5}$$

After we insert for $K_{t+1} = a_{2,t+1}$ we receive:

$$k_{t+1} = \frac{s(1-\tau)(1-\alpha)\zeta_t k_t^\alpha}{(1+g)(1+\lambda)} \tag{6}$$

Hence we receive the same equation as in the proposition (equation (1)). Now we can verify what will happen with the consumption. We know that everything that is not consumed in period t is saved to period t+1 (equation (4)) and from budget constraints of the model (p.584 in the paper) we can derive consumption.

For young generation:

$$c_{1,t} = (1-s)(1-\tau)(1-\alpha)\Upsilon_t \zeta_t k_t^\alpha \tag{7}$$

And for old generation:

$$c_{i,2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t \zeta_t k_t^\alpha \zeta_{t+1} \alpha \varrho_{t+1} k_{t+1}^{\alpha-1} + \\ (1-\alpha)\Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))$$

When we apply the formula for capital from equation (6) into (8) we get:

$$c_{i,2,t+1} = (\alpha\varrho_{t+1}(1+\lambda) + (1-\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))) \times \Upsilon_{t+1}\zeta_{t+1}k_{t+1}^\alpha \tag{8}$$

From household's maximization problem (p.583) we get following first order condition:

$$\beta \mathbb{E}_t \left[\frac{c_{1,t}(1+r_{t+1})}{c_{i,2,t+1}} \right] - 1 = 0 \tag{9}$$

After we insert the formulas for consumption from (7) and (9) into the FOC (10) we get:

$$1 = \frac{(1-s)\beta}{s} \mathbb{E}_t \left[\frac{1}{1 + \frac{1-\alpha}{\alpha(1+\lambda)e_{t+1}} \lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))} \right]$$

$$1 = \frac{\beta(1-s)}{s}\phi$$

Where ϕ is the same as in the equation (3). Now we can easily derive the saving rate and we will get exactly equation (2). Which proves that the initial guess was correct. QED

1.2 Simulation

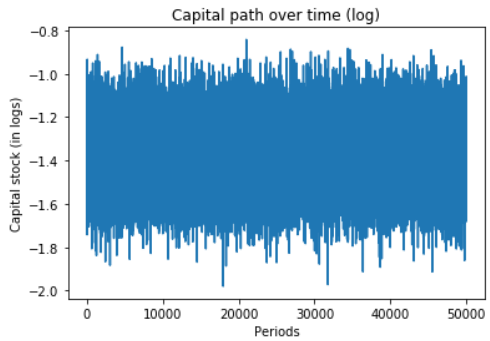
I've written the code in Python, the .py file corresponding to this part of the project has the name: *Exercise1_2.py*. Before we start the simulation we need to compute the capital in the steady state. Firstly we calculate ϕ and saving rate for the given parameters. Then we compute steady state capital as follows:

$$
\begin{aligned}
\ln(k^*) &= \ln(s(\tau)) + ln((1-\tau)) + ln(\frac{1-\alpha}{1-\lambda}) + ln(\zeta_t) + ln(k^{*\alpha}) \\
\ln(k^{*1-\alpha}) &= \ln(s(\tau)) + ln((1-\tau)) + ln(\frac{1-\alpha}{1-\lambda}) + ln(\zeta_t) \\
\ln(k^*) &= \frac{\ln(s(\tau)) + ln((1-\tau)) + ln(\frac{1-\alpha}{1-\lambda}) + ln(\zeta_t)}{1-\alpha}
\end{aligned}
$$

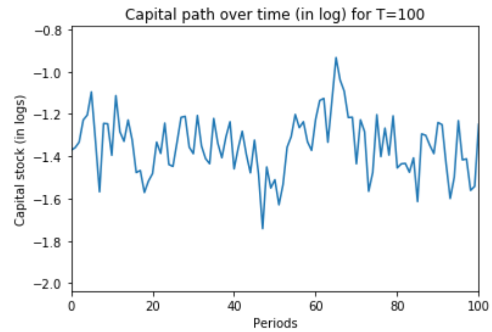
For our parameters we receive: $\phi = 0.5625$, $s = 0.2734$, $\ln(k^*) = -1.372$. Hence our capital at the beginning of the simulation at $t = 0$ is equal to $k^* = 0.254$.

I performed two simulations, one where I draw all 3 kinds of shocks from log-normal distributions and the second where I discretize the shocks following the instructions from the assignment.

In the 1st simulation the capital path looks like on the graphs below (the first one is a bit messy as we have 50k periods in our simulation so the second subfigure is limited to T=100). Even though 3 shocks are drawn, it turns out that only ζ matters as two other shocks show up only in the expectation in ϕ function and we know that $\mathbb{E}_{t+1}(\rho) = \mathbb{E}_{t+1}(\eta) = 1$ and they are independent. Hence ϕ and saving rate are constant over time in this case.



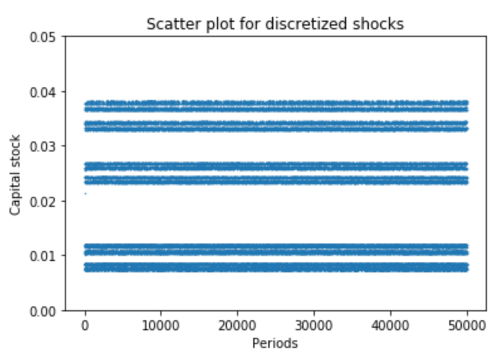
(a) 1st subfigure, T=50k



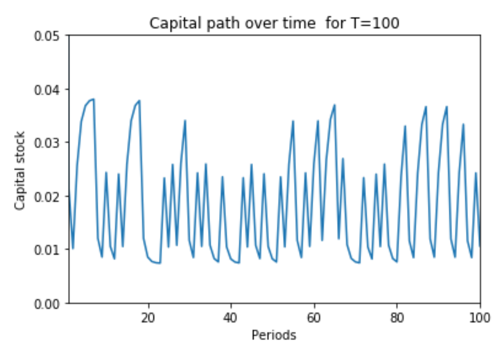
(b) 2nd subfigure, T=100

Figure 1: 1st simulation

In the 2nd simulation, that is with discrete shocks, fluctuations are even more visible, as we jump between booms and recessions. Firstly I draw the current state (boom or recession) with equal probabilities (0.5). Then I discretize ζ and ϱ with the values equal to +- standard deviation (as the expected values of $ln(\zeta)$ and $ln(\varrho)$ are = 0). More precisely $ln(\zeta)$ = +- 0.13 and $ln(\varrho)$ = +- 0.5. When we have $z = z^b$ (boom) the values have positive sign and if $z = z^r$ (recession) they are negative. After that $ln(\eta)$ is discretized into 11 nodes using quadrature take from QuantEcon package: `quantecon.quad.qnwlogn`. On the first figure I show the scatter plot for all of the simulated points, one can easily notice that we are now dealing with discretized shocks. And on the 2nd figure I show the capital path for the first 100 periods, as again it is more visible that way.



(a) Scatter plot, T=50k



(b) 2nd subfigure, T=100

Figure 2: 2nd simulation

1.3 Simple implementation of the Krusell-Smith algorithm

The Python code to this part of the project is in *Exercise1_3.py* file.

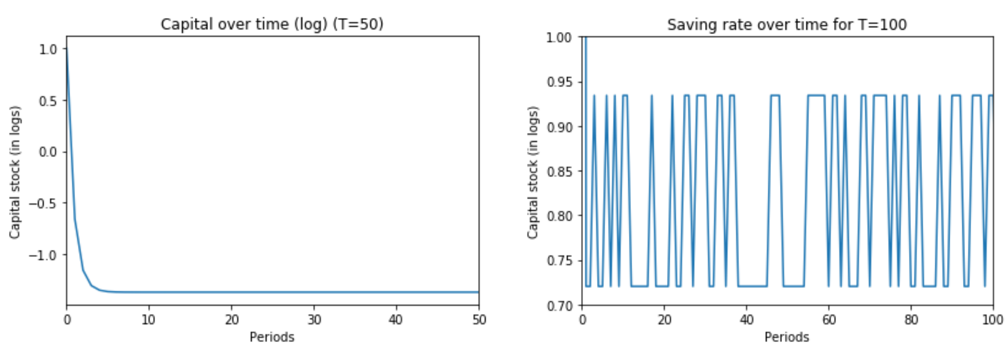
1.3.1 a)

Theoretical values are as given[1]:

$$\psi_0 \approx -0.96$$
$$\psi_1 = 0.3$$

1.3.2 b)

i) Convergence In the first step we randomly generate the state of the economy (recession or boom) and shocks the same way as in the exercise 1.2. Then we compute the capital path using Equation (2) from the assignment and theoretical values of ψ_i derived in subpoint (a). It converges precisely in 31 iterations, but the value is really close to be completely accurate after just a few iterations as it can be seen on the first graph below. Finally, using the equations (1)-(3) from HL paper we generate optimal saving rate for all periods and this is our solution. The optimal saving rate is different in our two states of the economy and that is why we see jumps from one value to the other at the 2nd subfigure.



(a) Scatter plot, T=50k (b) 2nd subfigure, T=100

Figure 3: 2nd simulation

[1]For updated version of the exercise, in the first version there was $ln(1-\alpha)$ in the (1) equation and in 2nd version we had $ln(\frac{1-\alpha}{1-\lambda})$)

ii) Simulation Now we follow the same equation as in (1). When we simulate our economy for saving rates computed in (i) and we obtain the following capital path:

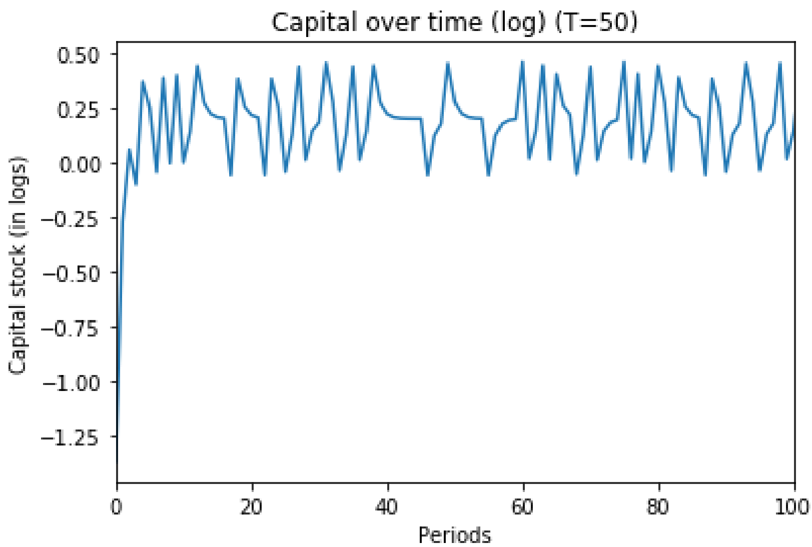


Figure 4: Capital path, T=100

iii) Regression At the beginning the first 500 periods are discarded. And after that I perform regression using sklearn.linear_model package for Python. The slope of the regression is almost equal to zero. Plot depicts the first 100 hundred to make it visible:

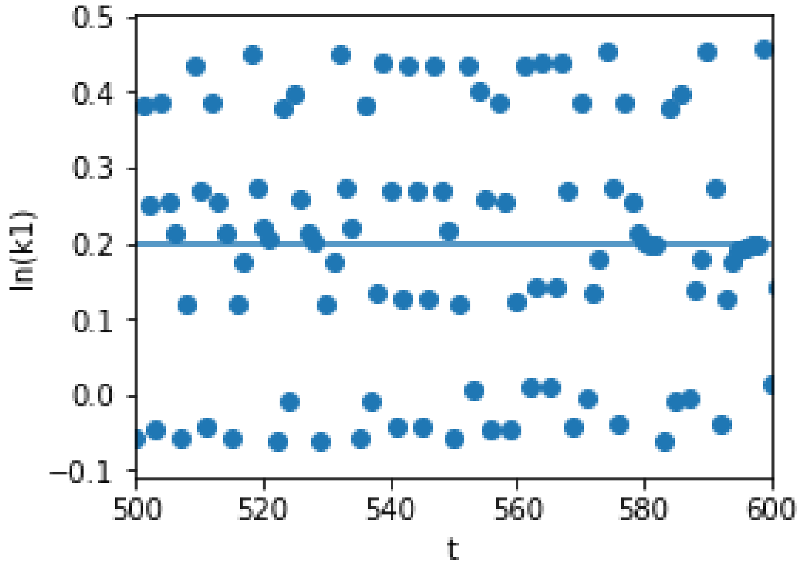


Figure 5: Regression

1.3.3 c) Comparison

When compared, the solution from $Exercise1_2.py$ (directly from analytical) is the same as the one in $Exercise1_3.py$ (numerical). In steady state:

$$\ln(k^*) = -1.372$$
$$k^* = 0.254$$

1.3.4 d)

For $\tau = 0.1$ new theoretical values are:

$$\psi_0 \approx -0.683$$
$$\psi_1 = 0.3$$

And the new steady state is given by:

$$\begin{aligned}\ln(k^*) &= -0.975\\ k^* &= 0.376\\ s^* &= 0.4\end{aligned}$$

2 Complex Variant of Krusell-Smith Algorithm

Unfortunately I ran out off time to finish this one, as it required switching to MatLab.