# 1

# Overview of Dynamical Systems

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## §**1.1. Scope**

This is a book about *dynamics*. Nowadays this term has acquired several meanings. It is used here in the traditional sense:

"The study of the relationship between motion and the forces affecting motion"

This is in fact meaning (**1a**) in the *American Heritage Dictionary of the English Language*. As such, it pertains to the science of Mechanics.[1]. The corresponding adjectives are *dynamic*[2] and its equivalent *dynamical*.[3]

But even the traditional definition is far too broad in two respects. First, Mechanics embodies a wide range of scales that span from cosmological through atomic and sub-atomic. Second, the study can focus on three aspects: theoretical, applied and computational. Our focus is restricted to a particular subset:

- *Classical Mechanics*, which obeys Newton's laws. This allows the use of continuum (field) models as well as certain "lumped" idealizations (point masses) that can be derived directly from such laws.
- *Computational Mechanics*, which relies on model-based simulation on digital computers. Of the various discretization methods, our focus will be on the Finite Element Method (FEM).

The main applications of this subset are to Solid and Structural Mechanics. Although on first sight this appears to be a bit limited in scope, many modeling and computational tools covered here are *application independent* in the sense discussed in the IFEM book [**?**, Chapter 7]. Hence the inclusion of qualifiers such as "Structural Mechanics" in the book title would be too confining.

## §**1.2. Dynamics versus Statics and Quasi-Statics**

Dynamic models in Classical Mechanics possess a common feature: the appearance of *inertial effects* modeled by Newton's second law, which states that inertia forces are proportional to mass times accelerations. Since accelerations are time derivatives of displacements, which characterize the motion, the formulation inevitably leads to differential equations in time, whose solutions exhibit *time dependence*. The converse is not true: time dependence does not necessarily require a dynamic model, as discussed below.

At the other extreme lies *statics*. This is the equilibrium mechanics of stationary bodies. The corresponding adjective is *static*, which means motionless, at rest, quiescent. Quantities associated with stationary bodies do not vary with time, whence modeling is greatly simplified.

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[1] The term *dynamics* has nowadays acquired a generalized meaning beyond Mechanics, as illustrated by definition (**2**) of that Dictionary: "The physical and moral forces that produce motion and changes in any field or system."

[2] Etymology: French *dynamique*, from Greek *dunamikos*: powerful, from *dunamis*, power, from *dunashai*, to be able. Its use in terms of active, energetic, vigorous, forceful, and the like, is comparatively recent: 1856 (from Emerson).

[3] The variant *Dynamical* tends to be often used in a more abstract sense. For example, MathWorld defines *dynamical system* as "a means of describing how a state evolves into another over the course of time." This is followed by "Technically, a dynamical system is a smooth action of the reals or integers into another object (usually a manifold)." This gobbledygook brings to mind one tongue-in-cheek comment about Lamb's *Hydrodynamics*: one can read the whole book without realizing that water is wet.

In between dynamics and statics lies the world of *quasi-static* scenarios, in which quantities vary with time but do so slowly that inertial and damping effects can be ignored.[4] For example one may imagine situations such as a roof progressively burdened by falling snow before collapse, the gradual filling of a dam over a decade, or the construction of a tunnel. Or foundation settlements: think of the Pisa tower before leaning was stopped.[5]

By contrast *dynamic analysis* is appropriate when the variation of displacements with time is so rapid that inertial effects cannot be ignored. There are numerous practical examples: earthquakes, rocket launches, vehicle crashes, explosive forming, air blasts, underground explosions, rotating machinery, airplane flutter, dancing robots. The structural accelerations, which are second derivatives with respect to time, must be kept in the governing equations.

Damping effects, which are usually associated with velocities (the first temporal derivatives of displacements), may be also part of a dynamic model. *Passive damping* effects are often neglected, however, since they tend to take energy out of a system and thus reduce the response amplitude. Hence ignoring such kind of damping may lead to conservative designs.

Developments in the AFEM [**?**], IFEM [**?**], MFEMS [**?**], and NFEM [**?**] books pertain to statics and quasi-statics. In this book, inertial effects will be *always* included, whereas damping effects are occasionally considered.

**Remark 1.1**. Quasi-static behavior should not be confused with *steady-state*. The latter describes the response to certain kinds of forced excitation (usually periodic) once effects of initial conditions disappear over the course of time. The opposite of steady-state is *transient*, as in "transient response." Those two qualifiers pertain only to dynamic systems.

## §1.3. Overcoming Mustiness

Face it: Classical Mechanics smells musty. Its heyday was attained during the Victorian and Edwardian eras: the world of "Upstairs, Downstairs," just before relativity theory emerged. Reading Euler, Lagrange or Hamilton one can imagine horse carriages, powdered wigs, feather pens and blotting paper. But despite cowebs, it is not an obsolete subject. Far from it. But it needs to be spruced up with modern language and tools. Three "pick-me-ups" are used in this book.

*Systems Nomenclature*. General System Theory (GST) emerged as a discipline by 1968 [775]. Certainly not musty: GST is still a vigorous topic that has brought about a fresh and holistic approach to old questions. Lipstick on a pig? Perhaps, but terms like "open system" and "system environment" *are* shiny lipstick that connect well with related topics such as control.

*Linear Algebra*. Matrices are not exactly spring chickens. They were invented by Cayley and Sylvester in the mid XIX century. For over a century they were concealed behind suffocating tracts on determinants such as [501]. Linear and matrix algebra come to the expository forefront, however, once digital computers appeared in the early 1950s, and the baby boomer generation was

[4] Quasi-statics includes statics as limit when the motion (or the time) is frozen. It does *not* include dynamics as the opposite limit because inertial and damping effects cannot be recovered from quasi-static models. Thus the name is apt since there is no such a thing as quasi-dynamics.

[5] The quasi-static assumption can be done during design if dynamic effects can be accounted for through appropiate safety factors. For many types of structures (e.g., buildings, bridges, offshore towers) these are specified in building codes. This saves analysis time when dynamic effects are inherently nondeterministic, as in traffic, winds or water wave effects.

**Table 1.1. Some Terms Frequently Used in Dynamics**

| *Term* | *Definition* |
| --- | --- |
| **Body** | Any bounded aggregate of matter. |
| **Mass** | A measure of the resistance of a body to acceleration. |
| **Position** | A location in space. |
| **Point** | A geometric object devoid of any properties except position. |
| **Particle** | A body whose spatial extent and internal configuration are irrelevant in a specific context. |
| **Point-Mass** | A finite mass assigned to a position. Its spatial extent and internal configuration are irrelevant. |
| **Point-Mass Particle** | A punctiform particle with a finite mass. (Abbreviation: PMP) |

Space and time are not defined as they are considered primitives.

trained in their use. Linear Algebra is not only the natural language of numerical computation but that of the Finite Element Method. And it blends smoothly with the vector notation that is now the bread and butter of college Physics.

*Finite Elements*. The Finite Element Method (FEM), as used today, came out in 1956 [758] as an offspring of three interlaced developments: matrix structural analysis, energy methods, and digital computers [238]. Although FEM has reached middle age, it retains sufficient vigor to freshen up dynamics.

There is a surprising formal equivalence between certain widely-used dynamic FEM models and point-mass particle dynamics. This "one-to-one mapping" is discussed in Chapter 2. Suffices to say that decorating portions of the old musty stuff with the lipstick of modern tools makes the pig tolerably attractive.

## §1.4. Terminology and Notation

Some definitions of terms used frequently in this book are collected in Table 1.1 for further reference. Of those the definitions of mass[6] and particle[7] are the ones that have changed most over history.

Although particle and point-mass are often viewed as synonymous, they are not. The term *particle* is more versatile, especially in molecular and atomic physics. Thus saying "particle dynamics" means little until one specifies what kind of particles and effects we are talking about. On the other hand,

[6] From Latin *massa*, a large irregular lump of something. Newton spoke of "quantity of matter" and defined mass indirectly via proportions. The related term *inertia* is less specific, as it informally conveys inertness, inactivity, or sluggishness. In Mechanics inertia is the property of matter by which it retains its state of rest, or its velocity along a straight line, unless acted upon by an external force. This is actually Newton's First Law of Motion; cf. §3.2.1.

[7] From Latin *particula*, a small part. As noted by Truesdell in a critical essay on the axiomatics of classical mechanics [754, p. 512], the intended meaning has varied from author to author over the centuries. The roughly equivalent term *corpusculum*, which means "small body", is used by Newton and his contemporaries. In modern Physics, "particle" generally implies portions of matter considered at molecular or atomic scales.

**Table 1.2. Terminology from General Systems Theory**

| *Term* | *Definition* |
|---|---|
| **GST System** | An entity that maintains its existence through the interaction of its parts. |
| **GST State** | The relevant properties, values or characteristics of a system component, or of the entire system. |
| **GST Event** | A change in the state of the system, or part of a system. |
| **GST Behavior** | A system event that initiates other events. |
| **GST Environment** | A set of entities and their relevant properties that are not part of the system, but such that a change on any of which can produce a change in the system. |
| **GST Open System** | A system that is influenced by entities outside the system. |
| **GST Closed System** | A self-contained system that is not affected by entities outside the system. |
| **GST Dynamic System** | A system whose state changes over time. It may be open or closed. |

By "maintains its existence" the definition of 'GST System emphasizes self-regulation processes.

"point-mass particle dynamics" or "PMP dynamics" is precise: it means Newtonian mechanics applied to particles of finite mass. Its essentials have not changed over the past 4 centuries.

As implied by the book title, matrix notation will be heavily used. Both matrices and vectors are identified by **bold** symbols. These symbols are lower case for vectors and upper case for matrices. Occasionally an arrow will be placed above a vector symbol to emphasize that it is a *field*.

Real time will be always denoted by $t$. Derivatives with respect to $t$ are abbreviated by superposed dots. For example, if $u(t)$ is a scalar motion, the associated *velocity* and *acceleration* are compactly written

$$\dot{u} \equiv \frac{du(t)}{dt}, \qquad \ddot{u} \equiv \frac{d^2 u}{dt^2}. \tag{1.1}$$

Sometimes we will denote velocity by $v$ and acceleration by $a$ when appropriate. Note that partial differentiation with respect to $t$ is not necessary when working with individual point-mass particles.

## §1.5. Systems Theory Terminology

The name General Systems Theory (GST), was introduced by von Bertalanffy in the late 1960 [775].[8] Some GST definitions are collected in Table 1.2, taken from [4,659]. The contrast with those in Table 1.1 is obvious. The GST definitions are nebulous: no images are conveyed. This is a consequence of ambitious goals: GST was originally intended to apply conjointly to areas as diverse as physical, climatological, biological, and even cultural systems, with initial emphasis on self-regulatory closed systems.[9]

This kind of generality has a silver lining: it illuminates features that connect different disciplines as long as they are not too discordant. For example: structural dynamics and control — or, to make

[8] GST development was helped by the success of operations research and control theory for servomechanisms during World War II and its aftermath.

[9] A contemporaneous but shorter-lived overreach: catastrophe theory, hampered by ridiculous claims from its founders, is mercifully gone. GST is still around, but has wisely evolved toward concrete applications.

**Table 1.3. Systems-Related Terminology for Physical Systems**

| *Term* | *Definition* |
|---|---|
| **System** | A functionally related set of components regarded as a physical entity. |
| **Configuration** | The relative disposition or arrangement of system components. |
| **State** | The condition of the system as regards its form, structure or constitution. |
| **State Variables** | A set of variables that uniquely characterizes the system state. |
| **Dynamic System** | A system whose state changes over time. It may be open or closed. |
| **Response** | The value of the state variables as a function of time. |
| **Event** | A change in the state variables produced by an agent. |
| **Behavior** | A pattern of events. |
| **Environment** | A set of entities that do not belong to the system, but can influence its behavior. |
| **Open System** | A system that is influenced by entities outside the system (its environment). |
| **Closed System** | A system that is not affected by entities outside the system. |
| **Interaction** | The mutual effect of a system component, or group of such components, on other components. |
| **Forces** | The action agents through which effects are transmitted between system components, or between environment entities and system components. |
| **Internal Forces** | Forces that act between system components. |
| **External Forces** | Forces that act between environment entities and system components. |
| **Input** (general) | The set of all actions that can influence the state of a system, or component. |
| **Input** (restricted) | The external forces acting on the system, or component, as function of time. |
| **Output** (general) | The set of all quantities that characterize the state of a system, or component. |
| **Output** (restricted) | The response of the system, or system component. |
| **Black Box** | A system, or system component, which may be viewed solely as a transformer from input to output without knowledge of its internal workings. |
| **White Box** | A system, or system component, which is described in sufficient level of detail to fully derive the transformation from input to output. |
| **Grey Box** | A system, or system component, which is described at a level of detail intermediate between that of a white box and a black box. |

Several terms are adaptations to physical systems of corresponding ones in Table 1.2.

it a bit more general: multiphysics. Table 1.3 takes up several useful concepts suggested by GST, and redefines them concretely for the dynamics of *physical systems*.[10]

Several changes, as well as a panoply of new terms, may be noticed. As regards *system*, the GST phrase "maintains its existence" is gone: a physical system, or a component thereof, may cease to exist through catastrophic events such as collision. The definition of *state variables*, missing in Table 1.2, is important, since those will be the primary unknowns to be carried along and solved for to get the system *response*. The notion of *force*, also missing there, is indirectly introduced as an agent or carrier of change.[11]

The distinction between the system proper and its environment is an important contribution of

[10] Meaning systems considered in the physical sciences: mechanics, thermomechanics, optics, electromagnetics, and so on, at all observable scales.

[11] The question of what *is* force is omitted since, as Joda says, it has no obvious answer.

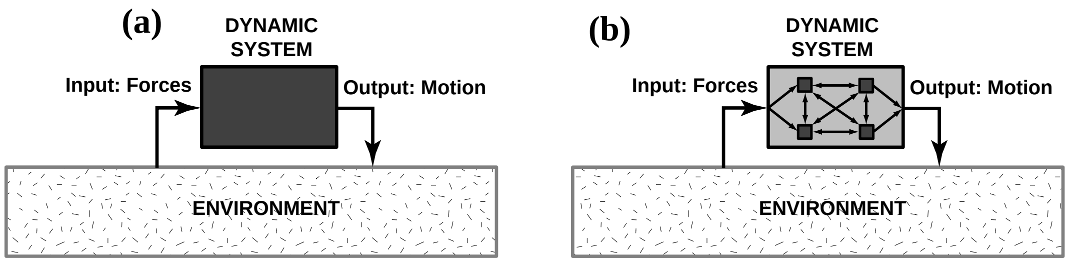


FIGURE 1.1. Open dynamic system that exchanges input and output with its environment: (a) black box view; (b) grey box view. Double arrow symbols in the latter stand for component interaction effects.

GST. It allows a clear distinction between *closed systems*, in which the dynamic model includes everything, and *open systems*, in which the environment is viewed as separate: environment entities influence the system response but not the other way around. The distinction is crucial in practical applications since it allows engineers and scientists to focus on important interactions while neglecting unimportant feedback.

The open vs. closed separation also offers an unambiguous way of classifying forces into *external* and *internal*. Many expositions still struggle with this particular decomposition.

Note that definitions of *input* and *output* span two levels: general and specific. In a general sense everything coming into the system from outside (its environment) is input, while everything going out is output. The specific sense restricts these to external forces and state variable response, respectively. This duality is necessary to accomodate the unsettled use of such terms in the literature.

Finally the definition of white, grey and black boxes responds primarily to te use of block diagrams to show system schematics. The distinction is important in verification and validation testing.

## §**1.6. Open Systems and Hierarchical Decomposition**

All dynamical systems considered in this book are *open*. The system proper is distinguished from its environment. An input-output relationship is established, as schematically shown in Figure 1.1. In this figure, (a) depicts the system as a *black box*, which conceals internal details. On the other hand, (b) gives some system details and thus may qualify as a grey box.

This view of open system can be continued hierarchically. Suppose the dynamic system is decomposed into first-level components. Each of these may be considered as an *open subsystem*, with the rest of the system as its environment. A first-level component may in turn be decomposed into second-level system components, which may also be viewed as *open sub-subsystems*, with the rest of the subsystem as environment. And so on. This process is continued into as many levels as necessary until further decomposition is deemed unnecessary.

This "divide and conquer", hierarchical multistage decomposition is natural for many engineering systems. Obviously it is not restricted to dynamics. It allows a problem to be decomposed into subproblems, subproblems into sub-subproblems, etc. Often the lowest component level is populated with black boxes, meaning that only their input-output transformer relationships need to be defined, while internal details of the transformer are of no relevance.

The Finite Element Method provides a systematic way to do multilevel decomposition using the concept of *superelements*. For static problems this technique is described in Chapter 10 of [**?**].