21

Mass Matrix Templates: General Description

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§21.1. Templates: A Tool for Mass Matrix Customization

The present Chapter provides a general description of *template mass lumping*. This is a general approach through which *customized mass matrices* can be constructed for specific structural elements.The qualifier "customized" is defined more precisely later.

The standard procedures for constructing FEM mass matrices are well known. They are presented in Chapter 16–20. They lead to *consistent* and *diagonally-lumped* forms, respectively. Conventional forms of those models are denoted by $\mathbf{M}_C$ and $\mathbf{M}_L$, respectively, with additional subscripts or superscripts as necessary or convenient. Abbreviations CMM and DLMM, respectively, are also used. Collectively those two models take care of many engineering applications in structural dynamics. Occasionally, however, they fall short. The gap can be filled with a more general approach that relies on *templates*. These are algebraic forms that carry free parameters. The set of parameters is called the *template signature*. When given numerical values, the signature uniquely characterizes a mass matrix *instance*. Templates are described in this and the next 5 chapters.

The template approach has the virtue of generating a set of mass matrices that satisfy certain *a priori* constraints; for example symmetry, nonnegativity, invariance and linear momentum conservation. A mass matrix that satisfies those will be called *admissible*. In particular, the diagonally-lumped and consistent mass matrices should be obtained as instances. Thus those standard models are not excluded. Availability of free parameters, however, allows the mass matrix to be *customized* to special requirements.

Several customization scenarios are listed in Table 21.1, along with their acronyms. The last one: reduction of directional anisotropy in wave propagation, is not applicable to one-dimensional elements and therefore not treated in this paper.

The versatility of application will be evident from the examples. It will be also seen that optimizing templates for one scenario generally does not help with others, and in fact may make things worse. Thus, ability to adapt the mass matrix to particular needs as well as problem regions is an important virtue. Note that mesh and freedom configuration need not be modified in any way; only template signatures are adjusted.

An attractive feature of templates for FEM programming is that each "custom mass matrix" need not be coded and tested individually. It is sufficient to implement the template as a single element-level module, with free parameters as arguments. (Alternatively, useful instances may be identified by predefined mnemonic character strings, and converted to numerical signatures internally.) The signature is adjusted according to goals and needs. In particular the same module should be able to produce the conventional DLMM and CMM models as instances. This can provide valuable crosschecking with other programs while doing benchmarks.

In problems characterized by rapid transients, such as contact-impact and fragmentation, templates allow a flexible customization: reduced high-frequency pollution in elements in or near shock regions while maintaining low-frequency continuum fit away from such regions. In these scenarios, signatures may evolve in time.

§21.2. Is Customization Worth The Trouble?

The ability to customize a mass matrix is not free of development costs. The presence of free parameters makes template derivations considerably more complicated than those based on the

Table 21.1 Template Customization Scenarios

Acronym	*Customization*
LFCF	Low-frequency continuum fit: matching acoustic branch (AB) to continuum model
AMC	Angular momentum (= rotary inertia) conservation: useful for transverse motions.
RHFP	Reduced high-frequency pollution (spurious noise) in direct time integration (DTI)
MSTS	Maximum stable time step in conditionally stable direct time integration (DTI)
RDAW	Reduced directional anisotropy in wave propagation (not relevant to 1D meshes)

two standard procedures described in Chapter 16. Reason: everything must be carried along *symbolically*: geometry, material and fabrication properties, in addition to the free parameters. Consequence: hand computations rapidly become unfeasible, even for fairly simple 1D elements. Help from a computer algebra system (CAS) is needed to get timely results. A key issue is: when is this additional work justified? Two specific cases may be mentioned.

One is *high fidelity systems*. Dynamic analysis covers a wide range of applications. There is a subclass that calls for a level of simulation accuracy beyond that customary in engineering analysis. Examples are deployment of precision space structures, resonance analysis of machinery or equipment, adaptive active control systems, medical imaging, phononics (wave guidance at molecular level), vehicle signature detection, radiation loss in layered circuits, and molecular- and crystal-level simulations in micro- and nano-mechanics.

In static structural analysis an error of 20% or 30% in peak stresses is not cause for alarm — such discrepancies are usually covered adequately by safety factors. But a similar error in frequency analysis or impedance response of a high fidelity system can be disastrous. Achieving acceptable precision with a fine mesh, however, can be expensive. Model adaptivity comes to the rescue in statics. This approach is less effective in dynamics, however, on account of the time dimension and the fact that irregular meshes are prone to develop numerical pollution. Customized elements may provide a practical solution: achieving adequate accuracy with a *coarse regular* mesh.

Another possibility is that the stiffness matrix comes from a method that *avoids displacement shape functions*. For example, assumed-stress or strain elements. [Or, it could simply be an array of numbers provided by a black-box program, with no documentation explaining its source.] If this happens the concept of *consistent mass matrix,* in which velocity shape functions (VSF) are taken to coincide with displacement shape functions (DSF), loses the comfortable variational meaning outlined in §16.4.2. An expedient way out is to choose an element with similar geometry and freedom configuration derived with DSF and take those as VSF. But which element to pick? If time allows, constructing and customizing a template avoids uncritically rolling the dice.

§21.3. Mass Parametrization Techniques

There are several ways to parametrize mass matrices. Techniques found effective in practice are summarized below. Most of them are illustrated in the worked out examples of ensuing sections.

It is often advantageous to have several template expressions for the same element configuration. For example, to study the subset of diagonally lumped mass matrices (DLMM) it may be convenient to streamline the general form to one that produces only such matrices. Likewise for singular mass matrices. In that case we speak of template *variants*. These may overlap totally or partially: the

Table 21.2 Acronyms Used in Paper

Acronym	*Stands for*
AB	Acoustic branch in DDD: has physical meaning in continuum models
ABTS	AB Taylor series in DWN κ, centered at $\kappa = 0$
BLCD	Best linear combination (LFF sense) of the CMM and a selected DLMM
BLFD	Best possible DLMM (LFF sense); acronym also applies to MS pair with this mass
BLFM	Best possible FPMM (LFF sense); acronym also applies to MS pair with this mass
CMM	Consistent mass matrix: a special VDMM in which VSM and DSF coalesce
CMS	Component Mode Synthesis: model reduction framework for structural dynamics
CMT	Congruential (also spelled congruent) mass transformation
COB	Constant optical branch: OB frequency is independent of wavenumber
COF	Cutoff frequency: OB frequency at zero wavenumber (lowest one if multiple OB)
DDD	Dimensionless dispersion diagram: DCF Ω vs. DWN κ
DGVD	Dimensionless group velocity diagram: $\gamma_c = c/c_0$ vs. DWN κ
DIMM	Directionally invariant mass matrix: repeats with respect to any RCC frame
DOF	Degree(s) of freedom
DLMM	Diagonally lumped mass matrix; qualifier “diagonally” is often omitted
DSF	Displacement shape functions to interpolate displacements over element
DSM	Direct Stiffness Method: the most widely used FEM implementation
DTI	Direct time integration of EOM
DCF	Dimensionless circular frequency, always denoted by Ω
DWN	Dimensionless wavenumber, always denoted by κ
EOM	Equations of motion
FEM	Finite Element Method
FFB	Flexural frequency branch in Bernoulli-Euler or Timoshenko beam models
FPMM	Fully populated mass matrix (at element level); includes CMM as special case
HF	High frequency: short wavelength, small DWN, typically $\kappa > 1$
LCD	Mass matrix obtained as linear combination of the CMM and a selected DLMM
LF	Low frequency: long wavelength, small DWN, typically $\kappa < 1$
LFF	Low frequency fitting of AB to that of continuum
LLMM	Lobatto lumped mass matrix: a DLMM based on a Lobatto quadrature rule
MOF	Maximum overall frequency: largest frequency in DDD over Brillouin zone
MSA	Matrix Structural Analysis: invented by Duncan and Frazier at NPL (1934)
NCT	Non-continuum term: a term in the ABTS that is not present in the continuum
NND	Nonnegative definite; a qualifier reserved for symmetric real matrices
PD	Positive definite; a qualifier reserved for symmetric real matrices
PVP	Parametrized variational principle
OB	Optical branch (or branches) in DDD: no physical meaning in continuum models
OBTS	OB Taylor series in DWN κ, centered at $\kappa = 0$
RCC	Rectangular Cartesian Coordinate: qualifier to frame, system, axes, etc.
SDAV	Structural dynamics and vibration applications: low frequency range important
SF	Shape function
SFB	Shear frequency branch in the Timoshenko beam model
SLMM	Simpson lumped mass matrix: a LLMM based on Simpson’s 3-pt quadrature rule
SMS	Selective mass scaling: modifying a mass matrix by adding a scaled stiffness
VDMM	Variational derived mass matrix: Hessian of discretized kinetic energy
VP	Variational principle
VSF	Velocity shape functions to interpolate velocities and produce a VDMM

Table 21.3 Template Related Nomenclature

Term or abbreviation	*Meaning*
Template	An algebraic expression for a FEM matrix that contains free parameters. So far used to construct stiffness and mass matrices of linear FEM models
Signature	The set of free parameters that uniquely defines a template
Instance	Matrix (or matrices) obtained by setting the signature to numeric values
Subset	Generic term for template specialization: includes families and variants
Family	A template subset in which some free parameters are linked by constraints
Variant	A template subset that introduces free parameters from scratch (the "subset" may be the original template if reparametrized)
Admissible	Qualifier applied to instances that satisfy predefined conditions such as positiveness, element mass conservation, and fabrication symmetries
MS template	Mass-stiffness pair template: both **M** and **K** have free parameters
FD template	Frequency-dependent template: free parameters may depend on frequency
FDM template	Frequency-dependent mass template
FDS template	Frequency-dependent stiffness template
FDMS template	Frequency-dependent mass-stiffness template
EW template	Entry weighted parametrization of a template; see §21.3.3
ML template	Multilevel parametrization of a template; see §21.3.4
MW template	Matrix weighted parametrization of a template; see §21.3.1
SP template	Spectral parametrization of a template; see §21.3.2

DLMM variant is plainly a subset of the general mass template. The key difference between a template subset and a variant is that the latter *redefines free parameters from scratch*.

For the reader's convenience, acronyms often used in this paper are listed in Table 21.2. A set of definitions and abbreviations pertaining to templates are collected in Table 21.3.

Notational conventions for mathematical expressions that appear in this paper are summarized in Table 21.4. Specific conventions used for free template parameters are given in Table 21.5.

§21.3.1. Matrix-Weighted Parametrization

A *matrix-weighted* (MW) mass template for element e is a linear combination of $(k+1)$ component mass matrices, $k \geq 1$ of which are weighted by parameters μ_i, $(i = 1, \ldots k)$:

$$\mathbf{M}^e \stackrel{\text{def}}{=} \mathbf{M}_0^e + \mu_1 \mathbf{M}_1^e + \ldots \mu_k \mathbf{M}_k^e. \tag{21.1}$$

Here $\mathbf{M}_0^e$ is the *baseline mass matrix*. This should be an admissible mass matrix on its own if $\mu_1 = \ldots \mu_k = 0$. The simplest instance of (21.1) is a linear combination of the consistent mass matrix (CMM) and a diagonally-lumped mass matrix (DLMM):

$$\mathbf{M}^e \stackrel{\text{def}}{=} (1-\mu)\mathbf{M}_C^e + \mu \mathbf{M}_L^e. \tag{21.2}$$

This can be reformatted as (21.1) by writing $\mathbf{M}^e = \mathbf{M}_C^e + \mu(\mathbf{M}_L^e - \mathbf{M}_C^e) = \mathbf{M}_0^e + \mu\, \mathbf{M}_1^e$. Here $k = 1$, the baseline is $\mathbf{M}_0^e \equiv \mathbf{M}_C^e$, $\mu \equiv \mu_1$ and $\mathbf{M}_1^e$ is the "mass deviator" $\mathbf{M}_L^e - \mathbf{M}_C^e$. The specialization (21.2) is often abbreviated to "linear combination of consistent and diagonally lumped masses," with acronym LCD; cf. Table 21.2. The rationale behind (21.2) is that the CMM typically overestimates

Table 21.4 General Notational Conventions For Mathematical Expressions

*Letter symbol**	*Used for*	*Examples*
UC bold	Matrices	$\mathbf{K}$, $\mathbf{M}$
LC bold	Vectors	$\mathbf{u}$, $\dot{\mathbf{u}}$
US roman	Scalar coefficients or functions	a, b, $\bar{Q}$, $u(x,t)$
SS LC roman	Subscripted variants of scalar coefficients	$\hat{c}_1$, $\hat{c}_2$
SS LC roman	Vector entries conforming with vector symbol	u_i: entries of $\mathbf{u}$
DS UC roman	Matrix entries conforming with matrix symbol	K_{ij}: entries of $\mathbf{K}$
Greek letters	Dimensionless quantities except as noted below†	θ, ψ, Ω
Superposed dot	Temporal differentiation	$\ddot{\mathbf{u}} \equiv d^2\mathbf{u}(t)/dt^2$
Prime	1D spatial differentiation, usually with respect to x	$v'(x) \equiv dv(x)/dx$

* UC:uppercase; LC:lowercase; US:unsubcripted; SS:single subscripted; DS:double subscripted
† Exemption made for well established symbols; e.g. ω: frequency or ρ: mass density

Table 21.5 Notational Conventions For Template Parameters

*Symbol**	*Used for*
α_i	Free parameters in basic stiffness matrix template (not used in this paper)
β_i	Free parameters in higher order stiffness matrix template
μ_i	Original free parameters in mass template. Additional letter subscripts may be appended as appropriate to distinguish template families or variants
ν_i, χ_i	Alternative notations for mass template parameters. Often derived from the original μ_i to streamline expressions, or to identify families or variants

* The subscript index is suppressed if only one parameter appears; e.g. β, μ.

natural frequencies while a DLMM usually underestimates them. Thus a linear combination has a good chance of improving low-frequency accuracy for some $\mu \in [0, 1]$.

A MW mass template represents a tradeoff. It cuts down on the number of free parameters. Such a reduction is essential for 2D and 3D elements. It makes it easier to satisfy conservation and nonnegativity conditions through appropriate choice of the $\mathbf{M}_i^e$. On the minus side it generally spans only a subspace of admissible mass matrices.

§21.3.2. Spectral Parametrization

A *spectrally parametrized* (SP) mass template has the form

$$\mathbf{M}^e \stackrel{\text{def}}{=} \mathbf{H}^T\,\mathbf{D}_\mu\,\mathbf{H}, \qquad \mathbf{D}_\mu = \mathbf{diag}\,[\,c_0\mu_0 \;\; c_1\mu_1 \;\; \ldots \;\; c_k\mu_k\,]. \tag{21.3}$$

in which $\mathbf{H}$ is a generally full matrix. Parameters $\mu_0 \ldots \mu_k$ appear as entries of the diagonal matrix $\mathbf{D}_\mu$. Scaling coefficients c_i may be introduced for convenience so the μ_i are dimensionless. Often the values of μ_0 and/or μ_1 are preset from conservation conditions.

Configuration (21.3) occurs naturally when $\mathbf{M}^e$ is constructed first in generalized coordinates, followed by congruential transformation to physical coordinates via $\mathbf{H}$. If the generalized mass is derived using mass-orthogonal functions (for example, Legendre polynomials in 1D elements), the unparametrized generalized mass $\mathbf{D} = \mathbf{diag}\,[\,c_0 \;\; c_1 \;\; \ldots c_k\,]$ is diagonal. Parametrization is effected by scaling its entries. As noted, some entries may be left fixed to satisfy *a priori* constraints.

Expanding (21.3) and collecting matrices that multiply each μ_i leads to a matrix weighted combination form (21.1) in which each $\mathbf{M}_i^e$ is a rank-one matrix. The analogy with the spectral representation theorem of symmetric matrices is obvious. But in practice it is usually better to work directly with the congruent representation (21.3).

As remarked later in §21.3.6, SP is especially convenient for constructing *singular* mass matrices under customization scenario RHFP of Table 21.1.

§21.3.3. Entry-Weighted Parametrization

An *entry-weighted* (EW) mass template applies free parameters directly to each entry of the mass matrix, except for *a priori* constraints on symmetry, invariance and conservation. As an example, for a one-dimensional (1D) element with three translational DOF we may start from

$$\mathbf{M}^e \stackrel{\text{def}}{=} m^e \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{12} & \mu_{22} & \mu_{23} \\ \mu_{13} & \mu_{23} & \mu_{33} \end{bmatrix}, \tag{21.4}$$

in which m^e is the total element mass, and the sum of all row sums is one. EW is often applied to entries of a "deviator matrix" that measures the change from a baseline matrix such as $\mathbf{M}_C$. For example, see the three-node bar template (23.2).

Because of its generality, EP can be expected to lead to optimal customized instances. But it is restricted to simple (usually 1D) elements because the number of parameters grows quadratically in the matrix size, whereas for the foregoing two schemes either it grows linearly, or stays constant.

§21.3.4. Multilevel Parametrization

A hierarchical combination of parametrization schemes can be used to advantage if the kinetic energy can be naturally decomposed from physical considerations. For example, the Timoshenko beam element covered in §24.2 uses a two-matrix-split template combined by a weighted form similar to (21.2) as top level. (The energy split is between translational and rotational inertia, respectively.) The two components are constructed by spectral and entry-weighted parametrization, respectively. Such combinations fall under the scope of *multilevel* (ML) parametrization.

§21.3.5. Selective Mass Scaling

Selective Mass Scaling, or SMS, is a method proposed recently (references given in §H.4) in which the mass matrix is modified by a scaled version of the stiffness matrix. Thus $\mathbf{M}$ becomes

$$\mathbf{M}_K = \mathbf{M} + \frac{\mu_K}{\omega_{ref}^2}\,\mathbf{K}. \tag{21.5}$$

Here $\mu_K \geq 0$ is a dimensionless scaling factor whereas ω_{ref}^2 is a "reference" frequency used to homogenize physical dimensions. The modification (21.5) may be done at the element or system level. The objective is to "filter down" high frequencies in explicit DTI for applications such as contact-impact; e.g., vehicle crash simulation. Filtering aims to reduce spurious noise as well as increasing the stable timestep. It thus follows under customization scenarios RHFP and MSTS of Table 21.1. The basic idea can be explained as follows. Let ω_i and $\mathbf{v}_i$ denote the natural frequencies and associated orthonormalized eigenvectors, respectively, whereas $\hat{\omega}_i$ and $\hat{\mathbf{v}}_i$ are their counterparts for the modified eigenproblem $(\mathbf{M}_K + \hat{\omega}_i^2\,\mathbf{K})\,\hat{\mathbf{v}}_i = \mathbf{0}$. By inspection the eigenvectors are preserved: $\hat{\mathbf{v}}_i = \mathbf{v}_i$. Taking the Rayleigh quotient shows that the modified frequencies are

$$\hat{\omega}_i^2 = \frac{\omega_i^2}{R_i}, \quad \text{in which} \quad R_i = 1 + \mu_K \frac{\omega_i^2}{\omega_{ref}^2}. \tag{21.6}$$

Choosing $\mu_K > 0$ cuts down each frequency by $R_i > 0$. For low frequencies the modification is negligible if μ_K and ω_{ref}^2 are appropriately selected so that $R_i \approx 1$. For nonphysical high frequencies (mesh modes) the reduction can be significant In fact note that if $\omega_i^2 >> \mu_K/\omega_{ref}^2$, $\hat{\omega}_i^2$ cannot exceed the fixed bound $\omega_{max}^2 = \omega_{ref}^2/\mu_K$. The downside is that low frequency accuracy may suffer significantly, as illustrated later.

Although SMS may be presented as a variant of the MW parametrization technique of §21.3.1, it deserved to be considered on its own for the reasons stated in §H.4.

§21.3.6. Singular Mass Matrices

A thread linked to SMS but independently developed is that of *singular* mass matrices. This has been primarily advocated for multibody dynamics, as well as dynamical systems leading to differential-difference EOM that occur in active control with time lags. References are provided in §H.5. The objective is roughly similar to SMS: reduce high frequency noise pollution triggered by rapid transients and/or time lags. But now this is done by raising the optical branch (or branches) so as to widen the *acoustoptical gap* pictured described in §23.1.1 and illustrated in Figure 23.2. Noisy frequencies that fall in the gap decay exponentially.

There are several ways to produce such matrices. Under the template framework, the use of spectral parametrization (SP) is particularly convenient, as observed in §21.3.2. Other approaches include reduced numerical integration or injection of a convenient null space using mass matrix projection.

§21.3.7. Constant Optical Branch Variant

Instead of rising the optical branch (or branches) by making $\mathbf{M}^e$ singular, one may try to make the OB frequency independent. Templates that accomplish that feat are tagged as having a *Constant Optical Branch*, or COB for short. They form subsets collectively identified as the *COB variant*. The group velocity pertaining to a COB vanishes, so associated waveforms with that particular frequency do not propagate. COB templates were discovered during the course of this work, and are briefly studied in §23.1.13 for the three-node bar element.

§21.3.8. Mass-Stiffness Template Pairs

The concept of template was first developed for element stiffness matrices, as a natural generalization of its decomposition into basic and higher order parts. A brief historical account is provided in §H.7. Normally the stiffness template is optimized by imposing superconvergence conditions dealing with higher order patch tests while element aspect ratios are kept arbitrary. That optimal instance, if found, is kept fixed while a mass matrix template is subsequently investigated.

Maximum customization for dynamics can be expected if *both* stiffness and mass matrix templates can be simultanously adjusted. This is known as a *mass-stiffness* (MS) template. These may be of interest when improving dynamic behavior is paramount. Presently there is relatively little experience with this more ambitious approach. A note of caution: highly optimized MS templates may be abnormally sensitive to geometric or material perturbations away from a regular mesh.

§21.3.9. Frequency Dependent Templates

One final generalization should be mentioned: allowing free parameters to be function of the frequency. If this is done for the mass matrix, we speak of a *frequency dependent mass* (FDM) template. If this is done for both the mass and stiffness matrices, we call the combination a *frequency dependent mass-stiffness* (FDMS) template. Both cases are illustrated in §22.1.11–§22.1.13 for the two-node bar element.

Although this ultimate complication is largely a curiosity, it might be occasionally useful in problems that profit from transformation to the frequency domain. For example: a linear dynamic system driven by a harmonic excitation of slowly varying frequency, if only the long term (steady-state) response is considered. Such systems may arise in parametric stability and active control.