I wzór

$$(1+x+x^2\dots)^k = \frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^n$$

II wzór

$$(1+x^m+x^{2m}\dots)^k = \frac{1}{(1-x^m)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^{m\cdot n}$$

III wzór

1) $$\sum_{n=0}^{\infty} (n+1)x^n = \left(\sum_{n=0}^{\infty} x^{n+1}\right)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$$

2) Kombinatorycznie

Wiemy, że n+1 to ilość rozwiązań równania x1 + x2 = n. Wynika to z tego, że jak już dobierzemy wartość dla x1 to wartość dla x2 jest z góry znana.

$$x_1 + x_2 = n \Rightarrow (1+x+x^2\dots)^2 = \frac{1}{(1-x)^2}$$

---

zad 1.

$$\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+2-1)x^n$$

$$= 2\sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^2} - \frac{1}{(1-x)} = \frac{1+x}{(1-x)^2}$$

---

zad 2.

$$\sum_{n=4}^{\infty} (4n-16)\cdot x^{n-3} = x\sum_{n=4}^{\infty} 4(n-4)\cdot x^{n-4}$$

$n' = n-4$
starczy n jest od 4 to n' od 0

$$= x\sum_{n'=0}^{\infty} 4n'\cdot x^{n'} = x\sum_{n'=0}^{\infty} (4(n'+1)-4)\cdot x^{n'} = x\left[\frac{4}{(1-x)^2} - \frac{4}{(1-x)}\right]$$

$$= 4x \cdot \frac{1+x}{(1-x)^2}$$

zad 3.

$$\sum_{n=0}^{\infty} n^2 \cdot x^n$$

Wiemy, że równanie $x_1+x_2+x_3=n$ daje nam wynik: $\binom{n+3-1}{n} = \frac{(n+2)\cdot(n+1)}{2!} = \frac{1}{2}(n^2+3n+2)$

$$= \sum_{n=0}^{\infty} 2\cdot\frac{1}{2}(n^2+3n+2-3n-2)x^n = 2\sum_{n=0}^{\infty}\binom{n+3-1}{n}x^n - \sum_{n=0}^{\infty}(3n+2)\cdot x^n$$

$$= \frac{2}{(1-x)^3} - \sum_{n=0}^{\infty}(3(n+1)-1)x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{(1-x)}$$

---

zad 4.

$$\sum_{n=0}^{\infty}(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot x^{n+5}$$

$$= x^5\cdot\sum_{n=0}^{\infty}\binom{n+5-1}{n}\cdot 4!\cdot x^n$$

$$= 24x^5\cdot\frac{1}{(1-x)^5}$$

$(n+1)\ldots(n+4)$ przypomina nam $\binom{n+4}{n}$, tylko brakuje tu $4!$ w mianowniku

$$\binom{n+4}{n} = \binom{n+5-1}{n}$$

$x_1+x_2\ldots+x_5=n$

$$\sum_{n=0}^{\infty}\binom{n+5-1}{n}x^n = \frac{1}{(1-x)^5}$$

---

zad 5

Jo bardziej ANALIZA

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

teraz albo możemy policzyć pochodną albo zgadnąć, że nasze wyrażenie to tak naprawdę całka z x^n

1° $f(x) = \frac{x^{n+1}}{n+1}$ dla ustalonego n, $f'(x) = x^n$

Skoro $\sum_{n=0}^{\infty}x^n = \left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)' \Rightarrow \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \int_0^x\left(\sum_{n=0}^{\infty}t^n\right)dt = \int_0^x\frac{1}{1-t}\,dt$

bo szereg Maclaurina, wokół $x_0=0$

$$\int_0^x\frac{1}{1-t}\,dt = \left[-\ln|1-t|\right]_0^x = -\ln(1-x)$$

OPCJONALNE GÓWNO

Zad 5.
I sposób

$$a_n = \begin{cases} n+1, & \text{dla } n \text{ parz} \\ n-1, & \text{wpp} \end{cases} \quad \text{dla } n \geq 0$$

$$\sum_{k=0}^{\infty} (2k+1) x^{2k} + \sum_{n=0}^{\infty} (2k+1-1) x^{2k+1}$$

parz niepar

liczymy całkę

$$\int_0^x (2k+1) \cdot t^{2k} dt = \left[(2k+1) \cdot \frac{t^{2k+1}}{2k+1} \right]_0^x = x^{2k+1}$$

$$\sum_{k=0}^{\infty} x^{2k+1} = x \sum_{k=0}^{\infty} x^{2k} = \frac{x}{1-x^2} = g(x)$$

$$f(x) \xrightarrow{\int} g(x)$$

$$f(x) \xleftarrow{\frac{d}{dx}} g(x)$$

Zatem: $f(x) =$

$$= \sum_{k=0}^{\infty} (2k+1) \cdot x^{2k} = \left(\frac{x}{1-x^2} \right)' = \frac{1-x^2 - (-2x) \cdot x}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

$$\sum_{k=0}^{\infty} (2k+1-1) x^{2k+1} = x \sum_{k=0}^{\infty} (2k+1) x^{2k} - x \sum_{k=0}^{\infty} x^{2k}$$

$$= x \cdot \frac{x^2+1}{(1-x^2)^2} - \frac{x}{(1-x^2)}$$

$$= x \cdot \frac{2x^2}{(1-x^2)^2} = \frac{2x^3}{(1-x^2)^2}$$

zad 5.

$\boxed{\text{II sposób}}$

Mamy dany wzór:

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^{m \cdot n} = \frac{1}{(1-x^m)^k}$$

, bo równanie $mx_1 + mx_2 \ldots + mx_k = n$

teraz mając sumę:

$\sum_{k=0}^{\infty} (2k+1) \cdot x^{2k}$ możemy postarać się uzyskać $(k+1)x^{2k} = \binom{k+2-1}{k} x^{2k}$

Zatem $$\sum_{k=0}^{\infty} (2k+1) x^{2k} = \underbrace{\sum_{k=0}^{\infty} (2(k+1)}-1) x^{2k} = 2\sum_{k=0}^{\infty} \binom{k+2-1}{k} x^{2k} - \sum_{k=0}^{\infty} x^{2k}$$

$$= \frac{2}{(1-x^2)^2} - \frac{1}{(1-x^2)} = \boxed{\frac{1+x^2}{(1-x^2)^2}}$$

Dalej zadanie analogicznie

---

zad 6.

$$\sum_{k=0}^{\infty} ((k+2) \cdot (k+3) + 1) \cdot x^{6k+6}$$ | podstawiamy $k' = k+1 \Rightarrow 6k+6 = 6k'$ |

$$\sum_{k'=1}^{\infty} ((k'+1) \cdot (k'+2) + 1) \cdot x^{6k'} = \sum_{k'=1}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} + \sum_{k'=1}^{\infty} x^{6k'}$$

$x_1 + x_2 + x_3 = k \quad \binom{k+3-1}{k} = \frac{(k+1)(k+2)}{2!}$

$$\sum_{k'=0}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} = \sum_{k'=0}^{\infty} 2! \cdot \binom{k'+3-1}{k'} \cdot x^{6k'} = 2 \cdot \frac{1}{(1-x^6)^3}$$

ale nasza suma jest od $k'=1$!

zad 6 cd.

$$\sum_{k'=1}^{\infty} a_{k'} x^{6k'} = \sum_{k'=0}^{\infty} a_{k'} x^{6k'} - a_0 \cdot 1$$

u nas $a_0 = (0+1)\cdot(0+2) = 2$

Ostatecznie

$$\frac{2}{(1-x^6)^3} - 2 + \sum_{k'=1}^{\infty} x^{6k'} = \frac{2}{(1-x^6)^3} - 2 + \frac{x^6}{(1-x^6)}$$

---

zad 7.

$$\sum_{n=4}^{\infty} (n+7) x^{n-2} \left| \begin{matrix} n' = n-2 \\ n' \in [4-2, \infty] \\ n = n'+2 \end{matrix} \right| \quad \sum_{n'=2}^{\infty} [(n'+2)+7] x^{n'} = x^2 \sum_{n'=2}^{\infty} (n'+9) \cdot x^{n'-2}$$

$$\left| \begin{matrix} m = n'-2 \\ m \in [2-2, \infty] \\ n' = m+2 \end{matrix} \right| \quad x^2 \sum_{m=0}^{\infty} [(m+2)+9] \cdot x^m = x^2 \sum_{m=0}^{\infty} (\underbrace{m+1}+10) x^m =$$

$$x^2 \left(\frac{1}{(1-x)^2} + \frac{10}{1-x} \right) = x^2 \cdot \left[\frac{11-10x}{(1-x)^2} \right]$$

---

zad 8.

$x_1 + 2x_2 + 4x_3 + 8x_4 = 8n+7 \ \wedge \ x_2, x_3$ dodatnie

$x_2 = y_2 + 1, \quad x_3 = y_3 + 1$

$x_1 + 2(y_2+1) + 4(y_3+1) + 8x_4 = 8n+7$

$x_1 + 2y_2 + 4y_3 + 8x_4 = 8n+1$, wnioskujemy x_1 nieparz

$x_1 = 2y_1 + 1$

$2y_1 + \cancel{1} + 2y_2 + 4y_3 + 8x_4 = 8n + \cancel{1}$

$y_1 + y_2 + 2y_3 + 4x_4 = 4n$

Oznaczmy $x_4 = i, \ i \in [0, n]$

$y_1 + y_2 + 2y_3 = 4n - 4i$

Oznaczmy: $y_3 = j, \ j \in [0, 2n-2i]$

zad 8 cd.

$y_1 + y_2 = 4n - 4i - 2j$, z komb. z powt: $\binom{(4n-4i-2j)+2-1}{4n-4i-2j} = (4n-4i-2j+1)$

$$\sum_{i=0}^{n}\sum_{j=0}^{2n-2i}(4n-4i-2j+1)$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{2n-2i}(-4i+4n-2j+1) = \sum_{i=0}^{n}\left[-\cancel{2}\cdot\frac{(2n-2i)(2n-2i+1)}{\cancel{2}} + \underbrace{(4n-4i+1)}\underbrace{(2n-2i+1)}\right]$$

$$= \sum_{i=0}^{n}\left[-(4n^2-8ni+2n-2i+4i^2)+(8n^2-16ni-6i+6n+1+8i^2)\right]$$

$$= \sum_{i=0}^{n}(4n^2-8ni+4n-4i+4i^2+1) = (n+1)\cdot\underbrace{(4n^2+4n+1)} + \sum_{i=0}^{n}(i(-8n-4)+4i^2)$$

$$= 4n^3+8n^2+5n+1 + \overset{(n^2+n)}{\overset{\|}{\frac{n\cdot(n+1)}{\cancel{2}}}}\cdot(-\overset{4}{\cancel{8}}n-\overset{2}{\cancel{4}}) + 4\sum_{i=0}^{n} i^2 \leftarrow$$

$$= (4n^3+8n^2+5n+1)+(-4n^3-6n^2-2n) + 4\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= (2n^2+2n^2+3n+1) + \frac{2}{3}(2n^3+3n^2+n)$$

$$= \frac{4}{3}n^3+4n^2+\frac{11}{3}n+1 = \frac{4n^3+12n^2+11n+3}{3} = a_n$$