I wzór

$$(1+x+x^2\ldots)^k = \frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^n$$

II wzór

$$(1+x^m+x^{2m}\ldots)^k = \frac{1}{(1-x^m)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^{m \cdot n}$$

III wzór

1) $$\sum_{n=0}^{\infty} (n+1)x^n = \left(\sum_{n=0}^{\infty} x^{n+1}\right)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$$

2) **Kombinatorycznie**

Wiemy, że n+1 to ilość rozwiązań równania x1 + x2 = n. Wynika to z tego, że jak już dobierzemy wartość dla x1 to wartość dla x2 jest z góry znana.

$$x_1 + x_2 = n \implies (1+x+x^2\ldots)^2 = \frac{1}{(1-x)^2}$$

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zad 1.

$$\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+2-1)x^n$$

$$= 2\sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^2} - \frac{1}{(1-x)} = \frac{1+x}{(1-x)^2}$$

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zad 2.

$$\sum_{n=4}^{\infty} (4n-16) \cdot x^{n-3} = x\sum_{n=4}^{\infty} 4(n-4) \cdot x^{n-4} \quad \begin{array}{l} n' = n-4 \\ \text{skoro } n \text{ jest od } 4 \\ \text{to } n' \text{ od } 0 \end{array}$$

$$= x\sum_{n'=0}^{\infty} 4n' \cdot x^{n'} = x\sum_{n'=0}^{\infty} (4(n'+1)-4) \cdot x^{n'} = x\left[\frac{4}{(1-x)^2} - \frac{4}{(1-x)}\right]$$

$$= 4x \cdot \frac{1+x}{(1-x)^2}$$

zad 3.

$$\sum_{n=0}^{\infty} n^2 \cdot x^n$$

Wiemy, że równanie $x_1 + x_2 + x_3 = n$ daje nam wynik: $\binom{n+3-1}{n} = \dfrac{(n+2)\cdot(n+1)}{2!} = \dfrac{1}{2}(n^2 + 3n + 2)$

$$= \sum_{n=0}^{\infty} 2 \cdot \frac{1}{2}(n^2 + 3n + 2 - 3n - 2)x^n = 2\sum_{n=0}^{\infty}\binom{n+3-1}{n}x^n - \sum_{n=0}^{\infty}(3n+2)\cdot x^n$$

$$= \frac{2}{(1-x)^3} - \sum_{n=0}^{\infty}(3(n+1)-1)x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{(1-x)}$$

zad 4.

$$\sum_{n=0}^{\infty}(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot x^{n+5}$$

$$= x^5 \cdot \sum_{n=0}^{\infty}\binom{n+5-1}{n}\cdot 4! \cdot x^n$$

$$= 24x^5 \cdot \frac{1}{(1-x^5)}$$

$(n+1)\ldots(n+4)$ przypomina nam $\binom{n+4}{n}$, tylko brakuje tu $4!$ w mianowniku

$$\binom{n+4}{n} = \binom{n+5-1}{n}$$

$$x_1 + x_2 \ldots + x_5 = n$$

$$\sum_{n=0}^{\infty}\binom{n+5-1}{n}x^n = \frac{1}{(1-x)^5}$$

zad 5

Jo Gorchńy ANALIZA

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

teraz albo możemy policzyć pochodną albo zgadnąć, że nasze wyrażenie to tak naprawdę całka z x^n

$1°\ f(x) = \dfrac{x^{n+1}}{n+1}$ dla ustalonego n, $f'(x) = x^n$

Skoro $\sum_{n=0}^{\infty}x^n = \left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)' \Rightarrow \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = \int_0^x\left(\sum_{n=0}^{\infty}t^n\right)dt = \int_0^x\frac{1}{1-t}\,dt$

$$\int_0^x\frac{1}{1-t}\,dt = \left[-\ln|1-t|\right]_0^x = -\ln(1-x)$$

bo szereg Maclaurina, wokół $x_0 = 0$

OPCJONALNE GÓWNO

$$a_n = \begin{cases} n+1, & \text{dla } n \text{ parz} \\ n-1, & \text{wpp} \end{cases} \qquad \text{dla } n \geqslant 0$$

$$\sum_{k=0}^{\infty} (2k+1) x^{2k} + \sum_{n=0}^{\infty} (2k+1-1) x^{2k+1}$$

parz

niepan

liczymy całkę

$$\int_0^x (2k+1) \cdot t^{2k} \, dt$$

$$= \left[(2k+1) \cdot \frac{t^{2k+1}}{2k+1} \right]_0^x$$

$$= x^{2k+1}$$

$$\sum_{k=0}^{\infty} x^{2k+1} = x \sum_{k=0}^{\infty} x^{2k}$$

$$= \frac{x}{1-x^2} = g(x)$$

$$f(x) \xrightarrow{\int} g(x)$$

$$f(x) \xleftarrow{\frac{d}{dx}} g(x)$$

Zatem : $f(x) =$

$$= \sum_{k=0}^{\infty} (2k+1) \cdot x^{2k} = \left(\frac{x}{1-x^2} \right)'$$

$$= \frac{1-x^2 - (-2x) \cdot x}{(1-x^2)^2}$$

$$= \frac{1+x^2}{(1-x^2)^2}$$

$$\sum_{k=0}^{\infty} (2k+1-1) x^{2k+1} = x \sum_{k=0}^{\infty} (2k+1) x^{2k} - x \sum_{k=0}^{\infty} x^{2k}$$

$$= x \cdot \frac{x^2+1}{(1-x^2)^2} - \frac{x}{(1-x^2)}$$

$$= x \cdot \frac{2x^2}{(1-x^2)^2} = \frac{2x^3}{(1-x^2)^2}$$

zad 5.

II sposób

Mamy dany wzór:

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^{m \cdot n} = \frac{1}{(1-x^m)^k},$$ bo równanie $mx_1 + mx_2 \ldots + mx_k = n$

teraz mając sumę:

$$\sum_{k=0}^{\infty} (2k+1) \cdot x^{2k}$$ możemy postarać się uzyskać $(k+1)x^{2k} = \binom{k+2-1}{k} x^{2k}$

Zatem $\sum_{k=0}^{\infty} (2k+1)x^{2k} = \sum_{k=0}^{\infty} (2(k+1)-1) x^{2k} = 2\sum_{k=0}^{\infty} \binom{k+2-1}{k} x^{2k} - \sum_{k=0}^{\infty} x^{2k}$

$$= \frac{2}{(1-x^2)^2} - \frac{1}{(1-x^2)} = \frac{1+x^2}{(1-x^2)^2}$$ Dalej zadanie analogicznie

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zad 6.

$$\sum_{k=0}^{\infty} ((k+2)\cdot(k+3)+1) \cdot x^{6k+6} \quad \Big| \begin{array}{l} \text{podstawiamy} \\ k' = k+1 \Rightarrow 6k+6 = 6k' \end{array}$$

$$\sum_{k'=1}^{\infty} ((k'+1)\cdot(k'+2)+1) \cdot x^{6k'} = \sum_{k=1}^{\infty} (k'+1)(k'+2)\cdot x^{6k'} + \sum_{k'=1}^{\infty} x^{6k'}$$

$$x_1 + x_2 + x_3 = k \qquad \binom{k+3-1}{k} = \frac{(k+1)(k+2)}{2!}$$

$$\sum_{k'=0}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} = \sum_{k'=0}^{\infty} 2! \cdot \binom{k'+3-1}{k'} \cdot x^{6k'} = 2 \cdot \frac{1}{(1-x^6)^3}$$

ale nasza suma jest od $k'=1$!

zad 6 cd.

$$\sum_{k'=1}^{\infty} a_{k'} x^{6k'} = \sum_{k'=0}^{\infty} a_{k'} x^{6k'} - a_0 \cdot 1$$

u nas $a_0 = (0+1) \cdot (0+2) = 2$

Ostatecznie

$$\boxed{\frac{2}{(1-x^6)^3}} - 2 + \sum_{k'=1}^{\infty} x^{6k'} = \boxed{\frac{2}{(1-x^6)^3} - 2 + \frac{x^6}{(1-x^6)}}$$

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zad 7.

$$\sum_{n=4}^{\infty} (n+7) x^{n-2} \left| \begin{matrix} n' = n-2 \\ n' \in [4-2, \infty] \\ n = n'+2 \end{matrix} \right| \quad \sum_{n'=2}^{\infty} [(n'+2)+7] x^{n'} = x^2 \sum_{n'=2}^{\infty} (n'+9) \cdot x^{n'-2}$$

$$\left| \begin{matrix} m = n'-2 \\ m \in [2-2, \infty] \\ n' = m+2 \end{matrix} \right| \quad x^2 \sum_{m=0}^{\infty} [(m+2)+9] \cdot x^m = x^2 \sum_{m=0}^{\infty} (\underline{m+1}+10) x^m =$$

$$x^2 \left(\frac{1}{(1-x)^2} + \frac{10}{1-x} \right) = x^2 \cdot \left[\frac{11-10x}{(1-x)^2} \right]$$