

I wariant

$$(1+x+x^2\dots)^k = \frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^n$$

II wariant

$$(1+x^m+x^{2m}\dots)^k = \frac{1}{(1-x^m)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^{mn}$$

III wariant

$$1) \sum_{n=0}^{\infty} (n+1)x^n = \left(\sum_{n=0}^{\infty} x^{n+1} \right)' = \left(\frac{x}{1-x} \right)' = \frac{1}{(1-x)^2}$$

2) Kombinatoryczne

Wiemy, że $n+1$ to ilość rozwiązań równania $x_1 + x_2 = n$. Wynika to z tego, że jak już dobierzemy wartość dla x_1 to wartość dla x_2 jest z góry znana.

$$x_1 + x_2 = n \Rightarrow (1+x+x^2\dots)^2 = \frac{1}{(1-x)^2}$$

zad 1.

$$\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+2-1)x^n$$

$$= 2 \sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^2} - \frac{1}{(1-x)} = \frac{1+x}{(1-x)^2}$$

$$\sum_{n=4}^{\infty} \binom{4}{n-4} \cdot x^{n-3} = x \sum_{n=4}^{\infty} \binom{4}{n-4} \cdot x^{n-4} \quad \begin{cases} n' = n-4 \\ \text{stąd } n \text{ jest od 4} \\ \text{to } n' \text{ od 0} \end{cases}$$

$$= x \sum_{n'=0}^{\infty} \binom{4}{n'} \cdot x^{n'} = x \sum_{n'=0}^{\infty} \binom{4(n'+1)-4}{n'} \cdot x^{n'} = x \left[\frac{4}{(1-x)^2} - \frac{4}{1-x} \right]$$

$$= 4x \cdot \frac{1+x}{(1-x)^2}$$

Zad 3.

$$\sum_{n=0}^{\infty} n^2 \cdot x^n$$

Wiemy, że równanie $x_1 + x_2 + x_3 = n$ daje nam mnożek: $\binom{n+3-1}{n} = \frac{(n+2) \cdot (n+1)}{2!} = \frac{1}{2}(n^2 + 3n + 2)$

$$\begin{aligned} &= \sum_{n=0}^{\infty} 2 \cdot \frac{1}{2}(n^2 + 3n + 2 - 3n - 2)x^n = 2 \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n - \sum_{n=0}^{\infty} (3n+2) \cdot x^n \\ &= \frac{2}{(1-x)^3} - \sum_{n=0}^{\infty} (3(n+1)-1) x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} \end{aligned}$$

Zad 4.

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4) \cdot x^{n+5} \\ &= x^5 \cdot \sum_{n=0}^{\infty} \binom{n+5-1}{n} \cdot 4! \cdot x^n \\ &= 24x^5 \cdot \frac{1}{(1-x)^5} \end{aligned}$$

$(n+1) \dots (n+4)$ przypomina nam $\binom{n+4}{n}$, tylko brakuje tu $4!$ w mianowniku

$$\binom{n+4}{n} = \binom{n+5-1}{n}$$

$$\sum_{n=0}^{\infty} \binom{n+5-1}{n} x^n = \frac{1}{(1-x)^5}$$

Zad 5

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$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

teraz albo możemy policzyć pochodną
albo zgadnąć, że nasze wyrażenie to
tak naprawdę całka z x^n

$$1^{\circ} f(x) = \frac{x^{n+1}}{n+1} \text{ dla ustalonego } n, f'(x) = x^n$$

$$\text{Skoro } \sum_{n=0}^{\infty} x^n = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right)' \Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int_0^x \left(\sum_{n=0}^{\infty} t^n \right) dt = \int_0^x \frac{1}{1-t} dt$$

do zera
Małorunnina,
wtedy $x_0 = 0$

$$\int_0^x \frac{1}{1-t} dt = [-\ln|1-t|]_0^x = -\ln(1-x)$$

Zad 5.
I sposób

$$a_n = \begin{cases} n+1, & \text{dla } n \text{ parz} \\ n-1, & \text{wpp} \end{cases}$$

dla $n \geq 0$

$$\sum_{k=0}^{\infty} (2k+1)x^{2k} + \sum_{n=0}^{\infty} (2k+1-1)x^{2k+1}$$

parz
nieparz

Liczmy całkę

$$\int (2k+1) \cdot t^{2k} dt$$

$$= \left[(2k+1) \cdot \frac{t^{2k+1}}{2k+1} \right]_0$$

$$\sum_{k=0}^{\infty} x^{2k+1} = x \sum_{k=0}^{\infty} x^{2k}$$

$$= \frac{x}{1-x^2} = g(x)$$

$$f(x) \xrightarrow{\quad} g(x)$$

$$f(x) \xleftarrow{\frac{d}{dx}} g(x)$$

zatem: $f(x) =$

$$= \sum_{k=0}^{\infty} (2k+1) \cdot x^{2k} = \left(\frac{x}{1-x^2} \right)'$$

$$= \frac{1-x^2 - (-2x) \cdot x}{(1-x^2)^2}$$

$$= \frac{1+x^2}{(1-x^2)^2}$$

$$\sum_{k=0}^{\infty} (2k+1-1)x^{2k+1} = x \sum_{k=0}^{\infty} (2k+1)x^{2k} - x \sum_{k=0}^{\infty} x^{2k}$$

$$= x \cdot \frac{x^2+1}{(1-x^2)^2} - \frac{x}{(1-x^2)}$$

$$= x \cdot \frac{2x^2}{(1-x^2)^2} = \frac{2x^3}{(1-x^2)^2}$$

Zad 5.

II sposób

Uwaga dany mówiąc:

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^{m+n} = \frac{1}{(1-x)^k}, \text{ bo równanie } mx_1 + mx_2 + \dots + mx_k = n$$

teraz mając sumę:

$$\sum_{k=0}^{\infty} (2k+1) \cdot x^{2k} \quad \text{możemy postawić się wierszki } (k+1)x^{2k} = \binom{k+2-1}{k} x^{2k}$$

$$\begin{aligned} \text{Zatem } \sum_{k=0}^{\infty} (2k+1)x^{2k} &= \sum_{k=0}^{\infty} (2(k+1)-1)x^{2k} = 2 \sum_{k=0}^{\infty} \binom{k+2-1}{k} x^{2k} - \sum_{k=0}^{\infty} x^{2k} \\ &= \frac{2}{(1-x^2)} - \frac{1}{(1-x^2)} = \boxed{\frac{1+x^2}{(1-x^2)^2}} \end{aligned}$$

Dalej zadanie analogicznie

Zad 6.

$$\sum_{k=0}^{\infty} ((k+2) \cdot (k+3) + 1) \cdot x^{6k+6} \quad \left| \begin{array}{l} \text{podstawiamy} \\ k' = k+1 \Rightarrow 6k+6 = 6k' \end{array} \right.$$

$$\sum_{k'=1}^{\infty} ((k'+1) \cdot (k'+2) + 1) \cdot x^{6k'} = \sum_{k'=1}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} + \sum_{k'=1}^{\infty} x^{6k'}$$

$x_1 + x_2 + x_3 = k$ $\binom{k+3-1}{k} = \frac{(k+1)(k+2)}{2!}$

$$\sum_{k'=0}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} = \sum_{k'=0}^{\infty} 2! \cdot \binom{k'+3-1}{k'} \cdot x^{6k'} = \boxed{2 \cdot \frac{1}{(1-x^6)^3}}$$

ale nasza suma jest od $k'=1$!

zad 6 col.

$$\sum_{k=1}^{\infty} a_k \cdot x^{6k} = \sum_{k=0}^{\infty} a_k \cdot x^{6k} - a_0 \cdot 1$$

u mas $a_0 = (0+1) \cdot (0+2) = 2$

ostatcznie

$$\frac{2}{(1-x^6)^3} - 2 + \sum_{k=1}^{\infty} x^{6k} = \boxed{\frac{2}{(1-x^6)^3} - 2 + \frac{x^6}{(1-x^6)}}$$

zad 7.

$$\sum_{n=4}^{\infty} (n+7) x^{n-2} \left| \begin{array}{l} n' = n-2 \\ n' \in [4-2, \infty] \\ n = n'+2 \end{array} \right| \quad \sum_{n'=2}^{\infty} [(n'+2)+7] x^{n'} = x^2 \sum_{n'=2}^{\infty} (n'+9) \cdot x^{n'-2}$$

$$\left| \begin{array}{l} m = n'-2 \\ m \in [2-2, \infty] \\ m = m+2 \end{array} \right| \quad x^2 \sum_{m=0}^{\infty} [(m+2)+9] \cdot x^m = x^2 \sum_{m=0}^{\infty} (m+1+10) x^m =$$

$$x^2 \left(\frac{1}{(1-x)^2} + \frac{10}{1-x} \right) = x^2 \cdot \left[\frac{11-10x}{(1-x)^2} \right]$$

zad 8.

$$x_1 + 2x_2 + 4x_3 + 8x_4 = 8n+7 \quad \wedge \quad x_2, x_3 \text{ dodatnie}$$

$$x_2 = y_2 + 1, \quad x_3 = y_3 + 1$$

$$x_1 + 2(y_2 + 1) + 4(y_3 + 1) + 8x_4 = 8n+7$$

$$x_1 + 2y_2 + 4y_3 + 8x_4 = 8n+1, \quad \text{maksymalny } x_1, \text{ np. } 1$$

$$2y_1 + \cancel{1} + 2y_2 + 4y_3 + 8x_4 = 8n+1$$

$$y_1 + y_2 + 2y_3 + 4x_4 = 4n$$

Oznaczamy $x_4 = i, i \in [0, n]$

$$y_1 + y_2 + 2y_3 = 4n - 4i$$

Oznaczamy: $y_3 = j, j \in [0, 2n-2i]$

zuvol 8 (d).

$$\begin{aligned} & \gamma_1 + \gamma_2 = h_n - h_{i-2j}, \text{ z. komb. z. point: } \binom{(h_n - h_{i-2j}) + 2 - 1}{h_n - h_{i-2j}} \\ & \sum_{i=0}^n \sum_{j=0}^{2n-2i} (h_n - h_{i-2j} + 1) \\ & = \sum_{i=0}^n \sum_{j=0}^{2n-2i} (-h_i + h_{n-2j+1}) = \sum_{i=0}^n \left[-\cancel{2} \cdot \frac{\cancel{(2n-2i)}(2n-2i+1)}{\cancel{2}} + (h_n - h_i + 1)(2n-2i+1) \right] \\ & = \sum_{i=0}^n \left[-(h_n^2 - 8ni + 2n-2i+h_i^2) + (8n^2 - 16ni - 6i + 6n+1 + 8i^2) \right] \\ & = \sum_{i=0}^n (h_n^2 - 8ni + h_n - h_i + h_i^2 + 1) = (n+1) \cdot (h_n^2 + 4n+1) + \sum_{i=0}^n (i(-8n-4) + 4i^2) \\ & = h_n^3 + 6n^2 + 5n + 1 + \frac{n \cdot (n+1)}{2} \cdot (-8n-4) + 4 \sum_{i=0}^n i^2 \quad \leftarrow \sum_{k=1}^n k^2 = 0^2 + 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} \\ & = (h_n^3 + 8n^2 + 5n + 1) + (-h_n^3 - 6n^2 - 2n) + h_n \cdot \frac{n(n+1)(2n+1)}{6} \\ & = (2n^3 + 2n^2 + 3n + 1) + \frac{2}{3}(2n^3 + 3n^2 + n) \\ & = \frac{4}{3}n^3 + \frac{4}{3}n^2 + \frac{11}{3}n + 1 = \frac{4n^3 + 12n^2 + 11n + 3}{3} = a_n \end{aligned}$$