

I wzór

$$(1+x+x^2+\dots)^k = \left(\frac{1}{1-x}\right)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^n$$

II wzór

$$(1+x^m+x^{2m}+\dots)^k = \left(\frac{1}{1-x^m}\right)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^{m \cdot n}$$

III wzór

$$1) \sum_{n=0}^{\infty} (n+1) x^n = \left(\sum_{n=0}^{\infty} x^{n+1} \right)' = \left(\frac{x}{1-x} \right)' = \frac{1}{(1-x)^2}$$

2) Kombinatorycznik

Wiemy, że $n+1$ to ilość rozwiązań równania $x_1 + x_2 = n$. Wynika to z tego, że jak już dobierzemy wartość dla x_1 to wartość dla x_2 jest z góry znana.

$$x_1 + x_2 = n \Rightarrow (1+x+x^2+\dots)^2 = \frac{1}{(1-x)^2}$$

zad 1.

$$\sum_{n=0}^{\infty} (2n+1) x^n = \sum_{n=0}^{\infty} (2n+2-1) x^n$$

$$= 2 \sum_{n=0}^{\infty} (n+1) x^n - \sum_{n=0}^{\infty} x^n = \frac{2}{(1-x)^2} - \frac{1}{(1-x)} = \frac{1+x}{(1-x)^2}$$

zad 2.

$$\sum_{n=4}^{\infty} (4n-16) \cdot x^{n-3} = x \sum_{n=4}^{\infty} 4(n-4) \cdot x^{n-4} \quad \left| \begin{array}{l} n' = n-4 \\ \text{stano } n \text{ jest od } 4 \\ \text{to } n' \text{ od } 0 \end{array} \right.$$

$$= x \sum_{n'=0}^{\infty} 4n' \cdot x^{n'} = x \sum_{n'=0}^{\infty} (4(n'+1)-4) \cdot x^{n'} = x \left[\frac{4}{(1-x)^2} - \frac{4}{(1-x)} \right]$$

$$= 4x \cdot \frac{1+x}{(1-x)^2}$$

Zad 3.

$$\sum_{n=0}^{\infty} n^2 \cdot x^n$$

Wiemy, że równanie $x_1 + x_2 + x_3 = n$ daje nam wyznik: $\binom{n+3-1}{n} = \frac{(n+2) \cdot (n+1)}{2!} = \frac{1}{2}(n^2 + 3n + 2)$

$$\begin{aligned} &= \sum_{n=0}^{\infty} 2 \cdot \frac{1}{2}(n^2 + 3n + 2 - 3n - 2) x^n = 2 \sum_{n=0}^{\infty} \binom{n+3-1}{n} x^n - \sum_{n=0}^{\infty} (3n+2) \cdot x^n \\ &= \frac{2}{(1-x)^3} - \sum_{n=0}^{\infty} (3(n+1)-1) x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{(1-x)} \end{aligned}$$

Zad 4.

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4) \cdot x^{n+5} \\ &= x^5 \cdot \sum_{n=0}^{\infty} \binom{n+5-1}{n} \cdot 4! \cdot x^n \\ &= 24 x^5 \cdot \frac{1}{(1-x)^5} \end{aligned}$$

$(n+1) \dots (n+4)$ przypomina nam $\binom{n+4}{n}$, tylko brakuje tu $4!$ w mianowniku

$$\binom{n+4}{n} = \binom{n+5-1}{n}$$

$$x_1 + x_2 + \dots + x_5 = n$$

$$\sum_{n=0}^{\infty} \binom{n+5-1}{n} x^n = \frac{1}{(1-x)^5}$$

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Zad 5

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

teraz albo możemy policzyć pochodną albo zgadnąć, że nasze wyrażenie to tak naprawdę całka z x^n

$1^\circ f(x) = \frac{x^{n+1}}{n+1}$ dla ustalonego n , $f'(x) = x^n$

Skoro $\sum_{n=0}^{\infty} x^n = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \right)' \Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int_0^x \left(\sum_{n=0}^{\infty} t^n \right) dt = \int_0^x \frac{1}{1-t} dt$

$$\int_0^x \frac{1}{1-t} dt = [-\ln|1-t|]_0^x = -\ln(1-x)$$

to szereg Maclaurina, notat $x_0 = 0$

$$a_n = \begin{cases} n+1, & \text{dla } n \text{ parz} \\ n-1, & \text{wpp} \end{cases} \quad \text{dla } n \geq 0$$

$$\sum_{k=0}^{\infty} (2k+1) x^{2k} + \sum_{n=0}^{\infty} (2k+1-1) x^{2k+1}$$

parz *nieparz*

liczymy całkę

$$\int_0^x (2k+1) \cdot t^{2k} dt$$

$$= \left[(2k+1) \cdot \frac{t^{2k+1}}{2k+1} \right]_0^x$$

$$= x^{2k+1}$$

$$\sum_{k=0}^{\infty} x^{2k+1} = x \sum_{k=0}^{\infty} x^{2k}$$

$$= \frac{x}{1-x^2} = g(x)$$

$$f(x) \xrightarrow{\int} g(x)$$

$$f(x) \xleftarrow{\frac{d}{dx}} g(x)$$

Natem: $f(x) =$

$$\sum_{k=0}^{\infty} (2k+1) \cdot x^{2k} = \left(\frac{x}{1-x^2} \right)'$$

$$= \frac{1-x^2 - (-2x) \cdot x}{(1-x^2)^2}$$

$$= \frac{1+x^2}{(1-x^2)^2}$$

Zad 5.
I sposób

$$\sum_{k=0}^{\infty} (2k+1-1) x^{2k+1} = x \sum_{k=0}^{\infty} (2k+1) x^{2k} - x \sum_{k=0}^{\infty} x^{2k}$$

$$= x \cdot \frac{x^2+1}{(1-x^2)^2} - \frac{x}{(1-x^2)}$$

$$= x \cdot \frac{2x^2}{(1-x^2)^2} = \frac{2x^3}{(1-x^2)^2}$$

Zad 5.

II sposób

Mamy dany mór:

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^{m \cdot n} = \frac{1}{(1-x^m)^k}, \text{ bo równanie } mx_1 + mx_2 + \dots + mx_k = n$$

teraz musimy sumę:

$$\sum_{k=0}^{\infty} (2k+1) \cdot x^{2k} \text{ możemy postawić się wreszcie } (k+1)x^{2k} = \binom{k+2-1}{k} x^{2k}$$

$$\text{Zatem } \sum_{k=0}^{\infty} (2k+1)x^{2k} = \sum_{k=0}^{\infty} (2(k+1)-1)x^{2k} = 2 \sum_{k=0}^{\infty} \binom{k+2-1}{k} x^{2k} - \sum_{k=0}^{\infty} x^{2k}$$

$$= \frac{2}{(1-x^2)^2} - \frac{1}{(1-x^2)} = \frac{1+x^2}{(1-x^2)^2} \quad \text{Dalej zadanie analogicznie}$$

Zad 6.

$$\sum_{k=0}^{\infty} ((k+2) \cdot (k+3) + 1) \cdot x^{6k+6} \quad \left| \begin{array}{l} \text{podstawiamy} \\ k' = k+1 \Rightarrow 6k+6 = 6k' \end{array} \right|$$

$$\sum_{k'=1}^{\infty} ((k'+1) \cdot (k'+2) + 1) \cdot x^{6k'} = \sum_{k'=1}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} + \sum_{k'=1}^{\infty} x^{6k'}$$

$$x_1 + x_2 + x_3 = k \quad \binom{k+3-1}{k} = \frac{(k+1)(k+2)}{2!}$$

$$\sum_{k'=0}^{\infty} (k'+1)(k'+2) \cdot x^{6k'} = \sum_{k'=0}^{\infty} 2! \cdot \binom{k'+3-1}{k'} \cdot x^{6k'} = 2 \cdot \frac{1}{(1-x^6)^3}$$

ale nasza suma jest od $k'=1$!

zad 6 cd.

$$\sum_{k'=1}^{\infty} a_{k'} x^{6k'} = \sum_{k'=0}^{\infty} a_{k'} x^{6k'} - a_0 \cdot 1$$

u nas $a_0 = (0+1) \cdot (0+2) = 2$

Ostatecznie

$$\frac{2}{(1-x^6)^3} - 2 + \sum_{k'=1}^{\infty} x^{6k'} = \frac{2}{(1-x^6)^3} - 2 + \frac{x^6}{(1-x^6)}$$

zad 7.

$$\sum_{n=4}^{\infty} (n+7) x^{n-2} \left| \begin{array}{l} n' = n-2 \\ n' \in [4-2, \infty] \\ n = n'+2 \end{array} \right| \sum_{n'=2}^{\infty} [(n'+2)+7] x^{n'} = x^2 \sum_{n'=2}^{\infty} (n'+9) \cdot x^{n'-2}$$

$$\left| \begin{array}{l} m = n'-2 \\ m \in [2-2, \infty] \\ n' = m+2 \end{array} \right| x^2 \sum_{m=0}^{\infty} [(m+2)+9] \cdot x^m = x^2 \sum_{m=0}^{\infty} (m+11) x^m =$$

$$x^2 \left(\frac{1}{(1-x)^2} + \frac{10}{1-x} \right) = x^2 \cdot \left[\frac{11-10x}{(1-x)^2} \right]$$

zad 8.

$$x_1 + 2x_2 + 4x_3 + 8x_4 = 8n+7 \wedge x_2, x_3 \text{ dodatnie}$$

$$x_2 = y_2 + 1, x_3 = y_3 + 1$$

$$x_1 + 2(y_2 + 1) + 4(y_3 + 1) + 8x_4 = 8n+7$$

$$x_1 + 2y_2 + 4y_3 + 8x_4 = 8n+1, \text{ wiemy, że } x_1 \text{ nieparzyste}$$

$$2y_1 + 2y_2 + 4y_3 + 8x_4 = 8n+1$$

$$y_1 + y_2 + 2y_3 + 4x_4 = 4n$$

$$\text{Oznaczmy } x_4 = i, i \in [0, n]$$

$$y_1 + y_2 + 2y_3 = 4n - 4i$$

$$\text{Oznaczmy: } y_3 = j, j \in [0, 2n-2i]$$

zuol 8 cd.

$$y_1 + y_2 = 4n - 4i - 2j, \text{ z komb. z powt: } \binom{(4n-4i-2j)+2-1}{4n-4i-2j} = (4n-4i-2j+1)$$

$$\sum_{i=0}^n \sum_{j=0}^{2n-2i} (4n-4i-2j+1)$$

$$= \sum_{i=0}^n \sum_{j=0}^{2n-2i} (-4i + 4n - 2j + 1) = \sum_{i=0}^n \left[-4 \cdot \frac{(2n-2i)(2n-2i+1)}{2} + (4n-4i+1)(2n-2i+1) \right]$$

$$= \sum_{i=0}^n \left[-(4n^2 - 8ni + 2n - 2i + 4i^2) + (8n^2 - 16ni - 6i + 6n + 1 + 8i^2) \right]$$

$$= \sum_{i=0}^n (4n^2 - 8ni + 4n - 4i + 4i^2 + 1) = (n+1) \cdot (4n^2 + 4n + 1) + \sum_{i=0}^n (i(-8n-4) + 4i^2)$$

$$= 4n^3 + 8n^2 + 5n + 1 + \frac{n \cdot (n+1)}{2} \cdot (-8n-4) + 4 \sum_{i=0}^n i^2 \leftarrow \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= (4n^3 + 8n^2 + 5n + 1) + (-4n^3 - 6n^2 - 2n) + 4 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= (2n^2 + 2n^2 + 3n + 1) + \frac{2}{3}(2n^3 + 3n^2 + n)$$

$$= \frac{4}{3}n^3 + 4n^2 + \frac{11}{3}n + 1 = \frac{4n^3 + 12n^2 + 11n + 3}{3} = a_n$$