Lecture Twenty

Some Things are Like Other Things

18th November 2025

CS260 Algorithms

Last time

We introduced SAT, the Boolean satisfiability problem: Can a given Boolean formula (in CNF) be satisfied?

SAT is in NP - we can **verify** a possible solution in polynomial time.

Theorem (Cook-Levin, ca. 1971). SAT is complete for NP.

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k-SAT is when we restrict the number of literals in each CNF clause to (at most) k.

1-SAT is O(n).

2-SAT is in P.

Transitivity of $\leq_P$

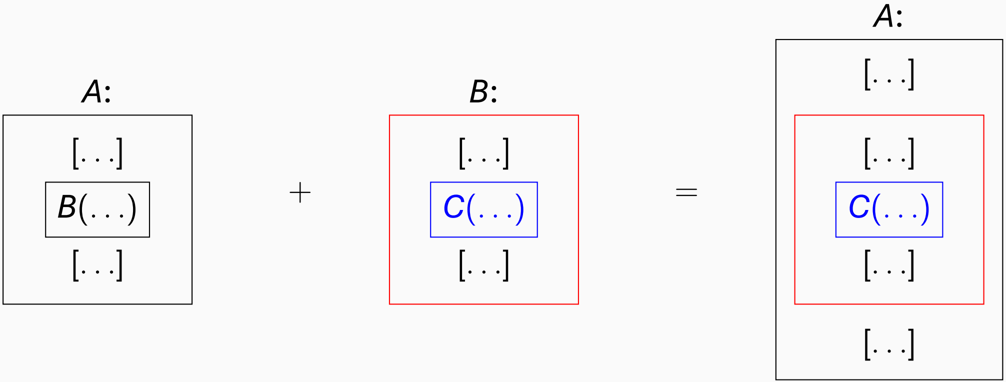
Suppose we have problems A, B, and C where $A \leq_P B$ and $B \leq_P C$.

What can we say about A and C?

Our reduction from A to B makes use of an oracle for B.

Our reduction from B to C makes use of an oracle for C.

Transitivity of $\leq_P$



No occurrences of B remain—this is a valid reduction from A to C.

Transitivity of $\leq_P$

Suppose (wlog.) our reduction from A to B runs in time $O(n^k)$,and our reduction from B to C runs in time $O(n^\ell)$.

Then the reduction from A to C runs in time $O((n^\ell)^k) = O(n^{k\cdot\ell})$, which is polynomial.

So we have $A \leq_P C$.

So $\leq_P$ is transitive!

We can create *chains* of polynomial-time reductions.

3-SAT

Theorem (Karp, 1972). 3-SAT is NP-complete.

Proof (sketch). We start by proving NP membership: a satisfying assignment is a valid witness, and can be checked in polynomial time. (The same argument as for SAT)

For NP hardness, Karp gives a reduction from SAT to 3-SAT. *(The idea is to break apart clauses of longer than 3 literals into smaller clauses.)*

Since $Y \leq_P$ SAT for all $Y \in$ NP, and we now know SAT $\leq_P$ 3-SAT, by transitivity of poly-reductions, we have $Y \leq_P$ 3-SAT as well. $\square$

Independent-Set

Theorem. Independent-Set is NP-complete.

Proof. *(demo)*

SET-COVER

SET-COVER

Consider

$$\mathcal{S} = \{\{4\}, \{1, 4\}, \{1, 2\}, \{5\}, \{1, 3, 5\}\}.$$

Can we choose 3 sets from $\mathcal{S}$ whose union is $\{1, 2, 3, 4, 5\}$?

Can we choose 2?

SET-COVER

SET-COVER: for a given

- universal set U,
- set $\mathcal{S}$ of *subsets* of U, and
- integer k,

is there a set $\mathcal{S}' \subseteq \mathcal{S}$ of size k such that $\bigcup \mathcal{S}' = U$?

The previous question was a SET-COVER instance with

- $U = \{1, 2, 3, 4, 5\}$,
- $\mathcal{S} = \{\{4\}, \{1, 4\}, \{1, 2\}, \{5\}, \{1, 3, 5\}\}$, and
- $k = 3$ (and then $k = 2$).

SET-COVER

Theorem. SET-COVER is complete for NP.

We can prove NP hardness by reduction from VERTEX-COVER to SET-COVER.

Each edge becomes an element in the universe U.
Each vertex v becomes a subset of U containing the edges incident to v.

Exercise. Write the complete proof!

Hint:
What would a selection of k sets correspond to in the original graph?

Next time

We look at some more NP-complete problems.