Probability & Statistics

Problem set №7. Week starting April 22th

1. Independent random variables X, Y have distribution $U[0,1]$. Let x, y are values of r.vs. X, Y. The interval $[0,1]$ is therefore divided into three parts (it is possible that the length of one part equals 0). What is the probability that a triangle can be constructed from these three parts?

   [Problems 2–4] Let (X_1, X_2) be two-dimensional random variable with density function $f(x_1, x_2) = \dfrac{1}{\pi}$, when $0 < x_1^2 + x_2^2 < 1$.

2. Find marginal densities of variable (1-dimensional) X_1, X_2.
3. Prove that correlation coefficient of variables X_1, X_2 equals 0. Check that variables are dependent (not independent).
4. Let $X_1 = Y_1 \cos Y_2,\ \ X_2 = Y_1 \sin Y_2$ ($0 < Y_1 < 1,\ \ 0 \leqslant Y_2 \leqslant 2\pi$). Find density $g(y_1, y_2)$ of variable (Y_1, Y_2). Check if variables Y_1, Y_2 are independent.
5. Let n-dimensional random variable $\mathbf{X} = (X_1, \ldots, X_n)^T$ be given. Define $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$ as follow:
$$Y_1 = \bar{\mathbf{X}}, \quad Y_k = X_k - \bar{\mathbf{X}}, \quad \text{when } k = 2, \ldots, n.$$
   Find form (entries) of Jacobian determinant
$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \cdots & \dfrac{\partial x_1}{\partial y_n} \\[2ex] \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} & \cdots & \dfrac{\partial x_2}{\partial y_n} \\[2ex] \vdots & \vdots & \ddots & \vdots \\[2ex] \dfrac{\partial x_n}{\partial y_1} & \dfrac{\partial x_n}{\partial y_2} & \cdots & \dfrac{\partial x_n}{\partial y_n} \end{vmatrix}.$$
6. Given are random variables $X_1, \ldots, X_n$. Prove that
$$\sum_{k=1}^{n} (X_k - \mu)^2 = \sum_{k=1}^{n} \left(X_k - \bar{\mathbf{X}}\right)^2 + n\,(\bar{\mathbf{X}} - \mu)^2. \tag{1}$$

   [Problems 7–8] Assume that independent random variables X_k have the distribution $\mathrm{N}\left(\mu, \sigma^2\right)$.

7. Find (with proof) distribution of $M = \dfrac{n}{\sigma^2} \cdot (\bar{\mathbf{X}} - \mu)^2$
8. Let r.vs. $\bar{\mathbf{X}} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} X_k$ and $S^2 = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \left(X_k - \bar{\mathbf{X}}\right)^2$ be independent. Using equation (1) prove that $\dfrac{nS^2}{\sigma^2} \equiv \mathrm{Gamma}\left(\dfrac{1}{2}, \dfrac{n-1}{2}\right) \sim \chi^2(n-1)$

[Problems 9–10] Sides' length of rectangular are independent r.vs. X_1 i X_2 with distribution $U[1,2]$. $Y_1 = 2X_1 + 2X_2$ is a perimeter of the rectangular and $Y_2 = X_1 X_2$ denotes its area.

9. Find expected values and variances of Y_1, Y_2. (ANSWER: Y_1 – 6, $2/3$, Y_2 – $9/4$, $55/144$).
10. Find the correlation coefficient ρ of variables Y_1, Y_2. (ANSWER: $3\sqrt{330}/55$).

[Ex. 11–12] Random variable X is of normal distribution with parameters given below:

$$N \sim \left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 38 & -5 \\ -5 & 4 \end{bmatrix} \right).$$

11. Let $Y_1 = 3X_1 + X_2,\ Y_2 = -4X_1 + 2X_2$. Find distribution of Y.
12. Let $Y_1 = 2X_1 - 3X_2,\ Y_2 = 4X_1 + 2X_2$. Find value of the correlation coefficient ρ_{y_1,y_2}.

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