Solution of task No. 8

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We have a function $f(x,y) = xy$ and area $[0,2] \times [0,1]$. Our task is to find distribution of $Z = X + Y$.

At first, let's change the variables: $(X,Y) \mapsto (Z,T)$. Let $Z = X + Y,\ T = Y$. Inverse transformation: $X = Z - T,\ Y = T$.

Determinant of Jacobian transformation: $J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial t} \\ \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ \\ 0 & 1 \end{vmatrix} = 1$

$g(z,t) = f(x(z,t), y(z,t)) \times |J| = (z-t)t$

$$\begin{cases} 0 < x < 2 \\ 0 < t < 1 \end{cases} \Rightarrow \begin{cases} 0 < z - t < 2 \\ 0 < t < 1 \end{cases} \Rightarrow \begin{cases} z - 2 < t < z & t \in [0,1] \\ 0 < t < 1 & z \in [0,3] \end{cases}$$

Integration interval for t is $[max\{0, z-2\},\ min\{1, z\}]$.
For $z \in [0,1] \to t \in [0,z]$, for $z \in [1,2] \to t \in [0,1]$, for $z \in [2,3] \to t \in [z-2,1]$.

$$\int g(z,t)\ dt = \int (z-t)t\ dt = \int zt - t^2\ dt = z\frac{t^2}{2} - \frac{t^3}{3} + C$$

$$g(z) = \begin{cases} z\frac{t^2}{2} - \frac{t^3}{3}\Big|_{t=0}^{z}, & z \in [0,1] \\ \\ z\frac{t^2}{2} - \frac{t^3}{3}\Big|_{t=0}^{1}, & z \in [1,2] \\ \\ z\frac{t^2}{2} - \frac{t^3}{3}\Big|_{t=z-2}^{1}, & z \in [2,3] \end{cases}$$