Problem 4°

April 6, 2020

Proof that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\, dt = \left| t = \frac{x^2}{2}, dt = x\, dx \right| = \int_0^\infty \frac{\sqrt{2}}{x} e^{-x^2/2} x\, dx =$$
$$\sqrt{2} \int_0^\infty e^{-x^2/2}\, dx = \sqrt{2} \cdot \frac{1}{2} \cdot \sqrt{2\pi} = \sqrt{\pi}$$

Equation $\int_0^\infty e^{-x^2/2}\, dx = \frac{1}{2} \cdot \sqrt{2\pi}$ holds as from 1.6 we know $\int_{-\infty}^\infty e^{-x^2/2}\, dx = \sqrt{2\pi}$ and from evenness of $e^{-x^2/2}$ we have $\int_{-\infty}^\infty e^{-x^2/2}\, dx = 2 \int_0^\infty e^{-x^2/2}\, dx$.