Probability & Statistics

Problem set №4. Week starting March 30th

1. Given is function $f(x,y) = C(x+y)\exp\{-(x+y)\}$, where $x>0$, $y>0$.
   (a) Compute the value of C such that $f(x,y)$ is the density of 2-dimensional r.v. (X,Y).
   (b) Check if variables X, Y are independent.
   (c) Find moments m_{10}, m_{01}.

   (a) $C = 1/2$,
   (b) $f_1(x) = \frac{1}{2}\cdot \mathrm{e}^{-x}(x+1)$, $\quad f_2(y) = \frac{1}{2}\cdot \mathrm{e}^{-y}(y+1)$ so variables X, Y are **dependent**.
   (c) $m_{10} = \mathrm{E}X = \int_0^\infty \frac{1}{2}\cdot x\,\mathrm{e}^{-x}\,(x+1)\,dx = 3 = \mathrm{E}Y = m_{01}$.

In exercises 2–11 we assume continuous random variables are considered. Symbols $f_X(x)$ and $F_X(x)$ mean – respectively – density and cdf of random variable X.

2. Is it possible to find C such that function $f_{XY}(x,y) = Cxy + x + y$, where $0 \leqslant x \leqslant 3$, $1 \leqslant y \leqslant 2$, would be density of 2-dimensional random variable?

   First we check what is the value of C such that $\int_0^3 \int_1^2 f_{XY}(x,y)\,dy\,dx = 1$. After some integration we find that $C = -\frac{32}{27}$. We have then

$$f_{XY} = x + y - \frac{32}{27}\cdot xy, \quad f_{XY}(3,2) = 5 - \frac{32}{27}\cdot 6 < 0.$$

About exercises 3–4. Given is function $f_{XY}(x,y) = -xy + x$, where $0 \leqslant x \leqslant 2$, $\quad 0 \leqslant y \leqslant 1$.

3. Check if X and Y are independent.

$$f(x,y) = x\cdot(1-y) = K_1\,x\cdot K_2\,(1-y) = f_X(x)\,f_Y(y), \quad \text{where} \quad K_1K_2 = 1.$$

   In addition – support[1] of density is a rectangle, – so X and Y are independent.

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4. Find probability: $P(1 \leqslant X \leqslant 3,\; 0 \leqslant Y \leqslant 0.5)$.

   $P(1 \leqslant X \leqslant 3,\; 0 \leqslant Y \leqslant 0.5) = \int_1^3 \int_0^{1/2} f(x,y)\,dy\,dx = \int_1^2 \int_0^{1/2} (x - xy)\,dy\,dx = \frac{9}{16}$.

Notation: Symbol $X \sim U[a,b]$ means that random variable X has uniform distribution on the interval $[a,b]$. In other words: $f_X(x) = \frac{1}{b-a}$, where $x \in [a,b]$.

5. Suppose that $X \sim U[0,1]$ and let $Y = X^n$. Prove that $f_Y(y) = \frac{y^{1/n-1}}{n}$, where $0 \leqslant y \leqslant 1$.

   $f_X(x) = 1$, $\;x \in [0,1]$. As for cdf we have $F_Y(t) = P(Y < y) = P(X < \sqrt[n]{t})$. After differentiation we have $f_Y(t) = \frac{1}{n} : t^{\frac{1}{n}-1}, ;\; t \in [0,1]..$

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6. Let $Y = X^2$ (in addition X is defined on $\mathbb{R}$). Prove that

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, \quad \text{where } y > 0.$$

[1] Area with non-zero density.

$F_Y(t) = P(X^2 < t) = P(-\sqrt{t} < X < \sqrt{t} = F_X(t) - F_X(-t)$. Hence $f_Y(t) = \dfrac{f_X(\sqrt{t}) + f_X(-\sqrt{t})}{2\sqrt{t}}$.
Note that $(-F_X(-s))' = f_X(s) \cdot s'$ and $s = \sqrt{t}$.

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7. Let $X \sim U[-1;1]$. Find density of $Y = |X|$.

   First we notice that Y has non-zero density on $[0,1]$. As for **cdf**[2] we have

$$F_Y(t) = P(Y < t) = P(-t < X < t) = F_X(t) - F_X(-t),$$

   so, after differentiation

$$f_Y(t) = f_X(t) + f_X(-t) = 1/2 + 1/2 = 1, \text{ and } t \in [0,1].$$

   Answer: $Y \sim U[0,1]$, intuitively – $[-1,1]$ collapses to $[0,1]$.

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8. Let X be (continuous) r.v. and let $Y = F_X(X)$. Prove that $Y \sim U[0;1]$.

   X is continuous random variable so $F_X(t)$ is invertible (at least on some interval).

$$F_Y(t) = P(Y < t) = P(F_X(X) < t) = P(X < F_X^{-1}(t)) = F_X(F_X^{-1}(t)) = t, \quad t \in [0,1].$$

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9. Density of random variable X is given by the formula $f_X(x) = xe^{-x}$, where $x \geqslant 0$. Find density of random variable $Y = X^2$.

$$F_Y(t) = P(Y < t) = P(X < \sqrt{t}) = F_X(\sqrt{t}).$$

   After differentiation

$$f_Y(t) = f_X(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \sqrt{t}\, \mathrm{e}^{-\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} = \frac{\mathrm{e}^{-\sqrt{t}}}{2}, \quad t \in [0.\infty).$$

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10. Let $X \sim U[a;b]$. Find value of variance $\mathrm{V}(X)$

    $\mathrm{E}X = \displaystyle\int_a^b x \cdot f(x)\, dx = \int_a^b \frac{x}{b-a}\, dx = \frac{a+b}{2}$. Next step: $\mathrm{E}(X^2) = \displaystyle\int_a^b x^2 f(x)\, dx = \int_a^b \frac{x^2}{b-a}\, dx = \frac{(b-a)^3}{3\,(b-a)}$.
    Since $\mathrm{V}X = E(X^2) - (EX)^2$ we obtain $\mathrm{V}X = \dfrac{b^2+ab+a^2}{3} - \dfrac{b^2+2ab+a^2}{4} = \dfrac{(b-a)^2}{12}$

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11. Random variable X has (standard) Cauchy distribution, i.e. $f_X(x) = \dfrac{1}{\pi(1+x^2)}$, where $x \in \mathbb{R}$.
    Prove that $Y = \dfrac{1}{X}$ has – also – (standard) Cauchy distribution.

    First we notice that $f_X(t)$ is an even function i.e. $f_X(t) = f_X(-t)$. This means that $F_X(t) = 1 - F_X(-t)$, where $t \in [0,1]$. We can consider only positive values of argument t.

$$F_Y(t) = P(Y < t) = P(1/X < t) = P(X > 1/t) = 1 - F_X(1/t).$$

    After differentiation

$$f_Y(t) = -f_X(1/t) \cdot \left(-\frac{1}{t^2}\right) = \frac{1}{\pi\,(1 + 1/t^2)} \cdot \frac{1}{t^2}, \text{ and } t > 0.$$

    Answer: Y has also standard Cauchy distribution. This is some exceptional property. Because of this property air-flying is possible (here should be some emoticon).

Witold Karczewski

[2] cdf ≡ dystrybuanta