Probability & Statistics

Problem set №4. Week starting March 30th

1. Given is function $f(x, y) = C(x+y)\exp\{-(x+y)\}$, where $x > 0$, $y > 0$.
   (a) Compute the value of C such that $f(x, y)$ is the density of 2-dimensional r.v. (X, Y).
   (b) Check if variables X, Y are independent.
   (c) Find moments m_{10}, m_{01}.

   In exercises 2–11 we assume continuous random variables are considered. Symbols $f_X(x)$ and $F_X(x)$ mean – respectively – density and cdf of random variable X.

2. Is it possible to find C such that function $f_{XY}(x, y) = Cxy + x + y$, where $0 \leqslant x \leqslant 3$, $1 \leqslant y \leqslant 2$, would be density of 2-dimensional random variable?

   About exercises 3–4. Given is function $f_{XY}(x, y) = -xy + x$, where $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$.

3. Check if X and Y are independent.
4. Find probability: $P(1 \leqslant X \leqslant 3,\ \ 0 \leqslant Y \leqslant 0.5)$.

   Notation: Symbol $X \sim U[a, b]$ means that random variable X has uniform distribution on the interval $[a, b]$. In other words: $f_X(x) = \dfrac{1}{b-a}$, where $x \in [a, b]$.

5. Suppose that $X \sim U[0, 1]$ and let $Y = X^n$. Prove that $f_Y(y) = \dfrac{y^{1/n-1}}{n}$, where $0 \leqslant y \leqslant 1$.
6. Let $Y = X^2$ (in addition X is defined on $\mathbb{R}$). Prove that

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, \quad \text{where } y > 0.$$

7. Let $X \sim U[-1; 1]$. Find density of $Y = |X|$.
8. Let X be (continuous) r.v. and let $Y = F_X(X)$. Prove that $Y \sim U[0; 1]$.
9. Density of random variable X is given by the formula $f_X(x) = xe^{-x}$, where $x \geqslant 0$. Find density of random variable $Y = X^2$.
10. Let $X \sim U[a; b]$. Find value of variance $\mathrm{V}(X)$
11. Random variable X has (standard) Cauchy distribution, i.e. $f_X(x) = \dfrac{1}{\pi(1+x^2)}$, where $x \in \mathbb{R}$. Prove that $Y = \dfrac{1}{X}$ has – also – (standard) Cauchy distribution.

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