Probability & Statistics

Problem set №1. Week starting on March 2nd, 2020

1. **(2p.)** Check that
   (a) $\sum_{k=0}^{n} \binom{n}{k} p^k \, (1-p)^{n-k} = 1,$
   (b) $\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np.$
2. Prove that
   (a) $\sum_{k=0}^{\infty} e^{-\lambda} \, \frac{\lambda^k}{k!} = 1,$
   (b) $\sum_{k=0}^{\infty} k \cdot e^{-\lambda} \, \frac{\lambda^k}{k!} = \lambda.$
3. **Gamma function** is defined as the value of the integral
   $$\Gamma(p) = \int_0^{\infty} t^{p-1} \, \mathrm{e}^{-t} \, dt, \quad p > 0.$$
   Prove that $\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}$.
4. Let $f(x) = \lambda \exp(-\lambda x)$, where $\lambda > 0$. Find value of the integrals
   (a) $\int_0^{\infty} f(x) \, dx,$
   (b) $\int_0^{\infty} x \, f(x) \, dx.$
5. Check that $D_n = n$, where
   $$D_n = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}.$$
6. **(2p.)** Let $I = \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{2}\right\} dx$. Equation $I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2+y^2}{2}\right\} dy \, dx$ holds. Prove, by substitution $x = r\cos\theta$, $y = r\sin\theta$, that $I^2 = 2\pi$.
7. Symbol $\bar{s}$ denotes mean value of the sequence $s_1, \dots, s_n$. Prove that
   (a) $\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} x_k^2 - n \cdot \bar{x}^2,$
   (b) $\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}.$
8. **(2p.)** Given are vectors $\vec{\mu}, X \in \mathbb{R}^n$ and matrix $\Sigma \in \mathbb{R}^{n\times n}$. Let $S = (X - \vec{\mu})^T \Sigma^{-1} (X - \vec{\mu})$ and $Y = A \cdot X$, with invertible matrix A. Check that $S = (Y - A\vec{\mu})^T (A\Sigma A^T)^{-1} (Y - A\vec{\mu})$.

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