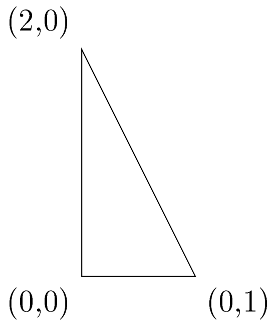
Task No. 9

April 22, 2020

From task we have $f(x,y) = \frac{15}{2}x^2y$, $T = X/Y$ and the area shown below:



We want to change variables: $(X,Y) \mapsto (U,T)$.
$T = X/Y$. Let $U = X$.
Inverse transformation: $X = U$, $Y = U/T$.

To make substitution of variables we need to compute the Jacobian:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{t} & \frac{-u}{t^2} \end{vmatrix} = -\frac{u}{t^2}$$

As $x \in [0,1]$, $y \in [0, -2x+2]$ and $T = X/Y$ then $t \in [0, \infty]$. Let's find the range of U.

$$\begin{cases} 0 < x < 1 \\ 0 < y < -2x+2 \end{cases} \Rightarrow \begin{cases} 0 < u < 1 \\ 0 < \frac{u}{t} < -2u+2 \end{cases} \Rightarrow \begin{cases} 0 < u < 1 \\ 0 < u < \frac{2t}{1+2t} \end{cases} \Rightarrow \begin{cases} 0 < u < 1 \\ 0 < u < 1 - \frac{1}{1+2t} \end{cases}$$

From equations above and $t \in [0, \infty]$ we get $u \in (0,\ 1 - \dfrac{1}{1+2t})$.

Now we can change variables:
$g(u,t) = f(x(u,t),\ y(u,t)) \cdot |J| = \dfrac{15}{2}u^2 \cdot \dfrac{u}{t} \cdot \left|-\dfrac{u}{t^2}\right| = \dfrac{15}{2} \cdot \dfrac{u^4}{t^3}$
Last equation holds because both t and u are non-negative.
We see that density of T is one of marginal densities of $g(u,t)$ so all we need to do is to compute it:
$g_2(t) = \displaystyle\int_0^{1-\frac{1}{1+2t}} g(u,t)\,du = \int_0^{1-\frac{1}{1+2t}} \frac{15}{2} \cdot \frac{u^4}{t^3} = \frac{3}{2} \cdot \frac{u^5}{t^3}\bigg|_0^{1-\frac{1}{1+2t}} = \frac{3}{2} \cdot \frac{(1-\frac{1}{1+2t})^5}{t^3}$