Probability & Statistics

Problem set №9. May, 2020

1. Let r.vs. $X_1, X_2, \ldots, X_n$ be independent and have the same distribution $\mathrm{Exp}(\lambda)$. Let $Y_i = X_1 + \ldots + X_i$, dla $i = 1, \ldots, n$. Prove that density of variable $(Y_1, \ldots, Y_n)$ equals $f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \lambda^n \exp\left(-\lambda y_n\right)$, with $0 < y_1 < y_2 < \ldots < y_n$.
2. Given density from the previous exercise $f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$, prove that marginal density with respect to Y_n equals $f_{Y_n}(y_n) = \lambda^n \dfrac{y_n^{n-1}}{(n-1)!} \exp\left(-\lambda y_n\right)$, with $0 < y_n$.
3. Given is value of parameter a. Find MLE estimator of parameter θ of uniform distribution on interval $[\theta - a; \theta + a]$.
4. Find MLE estimator of parameter θ of uniform distribution on interval $[\theta - a; \theta + a]$ in case when value of parameter a is unknown.
5. Let $X_1, \ldots, X_5$ be independent and identically distributed continuous random variables. Symbol p denotes probability $P\left(X_1 < X_2 > X_3 < X_4 > X_5\right)$. Prove that p does not depend on density $f(x)$ of variables X_k. Find value of p.
6. X, Y, Z are independent variables of $U[0, 1]$ distribution. Find $P(X \geqslant YZ)$.
7. X_1, X_2, X_3 are independent random variables of $\mathrm{Exp}(\lambda)$ distribution. Find (3-dimensional) density of variable $(Y_1, Y_2, Y_3) = (X_1 + X_2, X_1 + X_3, X_2 + X_3)$.
8. (2p.) $X_1, \ldots, X_n$ are independent and of identical continuous distribution random variables. At time j record occurs ($j \leqslant n$), if $Xj \geqslant Xi$ for $1 \leqslant i \leqslant j$. Let r.v. Z equals number of record values in sequence $\{X_k\}$. Prove that $\mathrm{E}\left(Z\right) = \sum_{i=1}^{n} \dfrac{1}{j}$.

   [Ex. 9–10] Density of r.v. (X, Y) is given by:

   $$f(x, y) = 1, \quad 0 < x, y < 1.$$

9. Find density of r.v. $Z = {}^{X}\!/_{Y}$.
10. Find probability that first significant digit of Z equals 1.

Witold Karczewski