Social Networks & Recommendation Systems

VII. Probabilistic aspects of the complex networks.

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European Funds
Knowledge Education Development

Warsaw University of Technology

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Before classes

Reminder

From SNARS_5:

- Properties of the ER graphs.

From SNARS_6:

- Mean-field approach to the BA model.

From other courses:

- generating functions approach in the combinatorics.

To think about:

- What do you think, which of the graphs: Erdős-Rényi or Barábasi-Albert is more vulnerable for intetional attacks and random failures? Why?

Lecture

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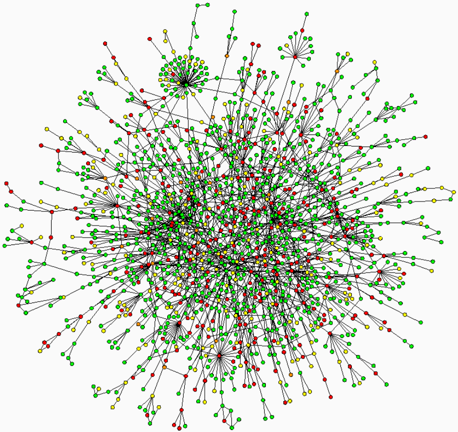
Conclusion:
I stronlgy recommend reading Durret to those who are dissatisfied!

R. Durret



https://services.math.duke.edu/~rtd/RGD/RGD.html

Let's have a look to chapter 4.1.

Master equation

The equation describing the changes in the probability distribution over time

$$\frac{d\mathcal{P}_i}{dt} = \sum_j \mathcal{P}_j T_{j\to i} - \sum_j \mathcal{P}_j T_{i\to j},$$

which in the discrete version takes the following form

$$\mathcal{P}_i(t+1) - \mathcal{P}_i(t) = \sum_j \mathcal{P}_j(t) T_{j\to i} - \sum_j \mathcal{P}_j(t) T_{i\to j},$$

and this is what we will focus on.

Master equation for BA networks

We will follow

- chapter 4.1 in Durret's book,
- S.N. Dorogovstev, J.F.F. Mendes i A.N Samukhin, *Structure of growing networks with preferential linking*, Phys. Rev. Letters. 85, 4633–4636 (2000).

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$$N_k(t+1) - N_k(t) = \frac{m(k-1)}{2mt} N_{k-1}(t) - \frac{mk}{2mt} N_k(t) + \delta_{km}$$

$$N_{m-1}(t) = 0.$$

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Question:

Can you justify the components of this equation?

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Question:

Is this solution mathematically exact?

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Why?

Precision vs. simplicity of the method?

i.e. mathematicians vs. physicists...

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$$N_{m-1}(t) = 0.$$

How to solve it?

`goto` Project;

Master equation for BA networks – solution

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Solution:

$$\mathcal{P}(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

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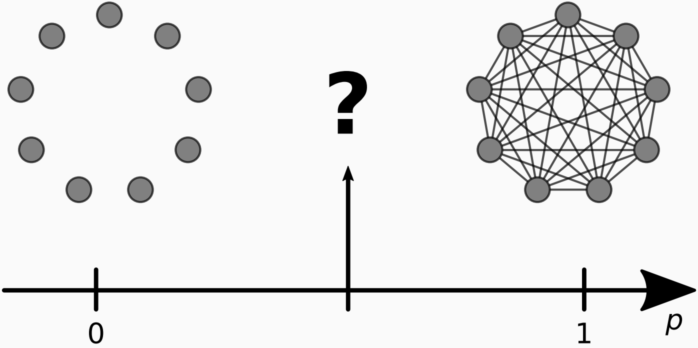
Question:

Argue that for $k \gg 1$ above results agrees with mean-field approach.

Percolation – ER graphs

Let us now return to the ER graphs



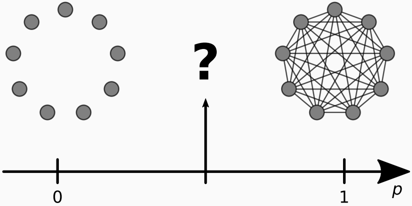
What is happening in the middle? We will follow sec. 4.3.2 in Fronczak and Fronczak book.

What is percolation?

wikipedia

Mathematical model of liquid leakage through porous material.

How does this relate to graphs?



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i.e. when percolation claster's size is equal to $N^* \propto N$.

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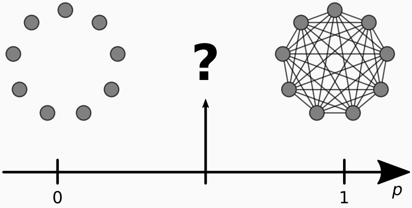

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Attention!

We will approach in the physicist way. For more detailed approaches (sic!) see chapter 2 in Durret's book.

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During Project we will prove that percolation threshold can be defined as

$$\sum_k k\mathcal{Q}(k) \geqslant 2,$$

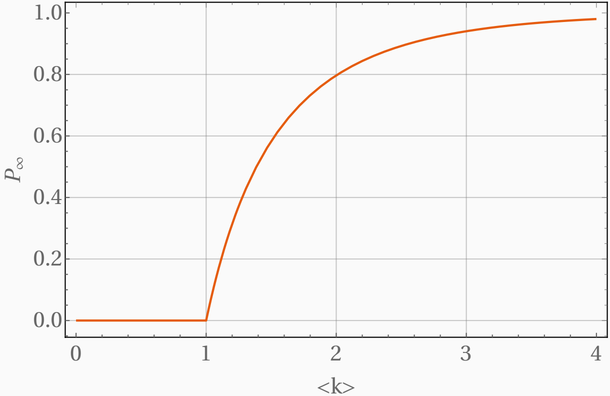
which is equivalent to $\langle k \rangle_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2$.

For ER graphs we obtain

$$\langle k \rangle = 1,$$

which means

$$p_c = \frac{1}{N}.$$



Thank you for your attention!

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