Chapter 17

Pipes and Forbes Models

As the beginning of discussion on car-following models, this chapter introduces two simple models, i.e., Pipes model and Forbes model, both of which are derived from drivers' daily driving experiences.

17.1 Pipes Model

Pipes model [119] is based on a safe driving rule coined in California Motor Vehicle Code:

"*A good rule for following another vehicle at a safe distance is to allow yourself at least the length of a car between your vehicle and the vehicle ahead of you for every ten mile per hour of speed at which you are traveling.*"

Referring to Figure 18.1 and putting the safety rule in mathematical language results in the following:

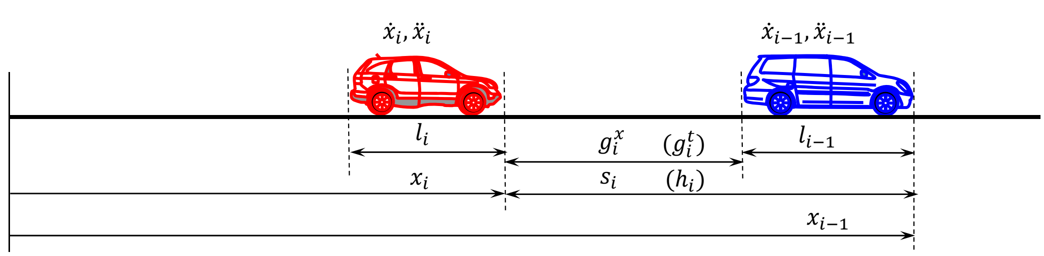

Figure 17.1: A car following scenario

$$g_i^x(t)_{min} = [(x_{i-1}(t) - x_i(t)) - l_{i-1}]_{min} = (s_i(t) - l_{i-1})_{min} = \frac{\dot{x}_i(t)}{0.447 \times 10} l_i \tag{17.1}$$

where $\dot{x}_i(t)$ is in m/s, and $g_i^x(t)$, $(x_{i-1}(t)$, and $x_i(t)$ is measured in meters. Assume a vehicle length of 6 meters, the model reduces to:

$$s_i(t)_{min} = 1.34\dot{x}_i(t) + 6 \tag{17.2}$$

or

$$h_i(t)_{min} = 1.34 + \frac{6}{\dot{x}_i(t)} \tag{17.3}$$

Equation 17.1, 17.2, or 17.3 constitutes the mathematical formulation of Pipes model.

17.1.1 Applications of Pipes Model

Pipes model can be applied in many ways, two foremost of which are automatic driving and computer simulation.

Automatic driving

Perhaps the simplest form of automatic driving is cruise control (CC). As an in-vehicle system, cruise control automatically controls the speed of a motor vehicle (by taking over the control of throttle) so that the vehicle maintains a constant speed set by its driver. Cruise control makes it easier to drive on long road trips and, hence, is a popular car feature. As vehicles keeps increasing on the road and the traffic becomes more and more crowded, the driver has to switch CC on and off so frequently that cruise control becomes less useful. In order to adapt to the dynamics of the vehicle in front, it is desirable that the cruise control system is able to adjust speed accordingly (rather than cruising at a constant speed) to maintain safe car following distance. Hence, adaptive or autonomous cruise control (ACC) system is developed. With the aid of distance sensors such as radar or laser, ACC allows the vehicle to slow down when approaching another vehicle and accelerate to the preset speed when traffic condition permits. To make this happen, the system requires an internal logic which relates vehicle speed to the distance between the vehicle and the one in front of it. Simple car-following models such as Pipes model can be employed as the basis of such an internal logic. More specifically, Equation 17.1 can be rearranged as follows:

$$\dot{x}_i(t) \leq g_i^x(t)\frac{0.447 \times 10}{l_i} = 0.745 g_i^x(t) \tag{17.4}$$

the constant coefficient is resulted if a vehicle length of 6 meters applies. Therefore, the ACC works as follows. At any moment t, the distance sensor measures the gap between the two vehicles $g_i^x(t)$. Then, the target speed that the vehicle needs to adapt to is set as $0.745 g_i^x(t)$ or less.

Computer simulation

Pipes model can also be used to simulate a platoon of vehicles moving on a one-lane highway. Before the simulation starts, the following variables need to be initialized, i.e. assign a value to each of them:

l_i length of vehicle i $\in$ {1,2,...,I}
τ_i perception-reaction time of driver i
v_i desired speed of driver i
ΔA_i maximum acceleration of vehicle i
ΔB_i maximum deceleration of vehicle i
Δt simulation time step

At time step j, the displacement x and speed v of each vehicle is updated:

```
FOR i = 1:I
  s(j,i) = x(j-1,i-1) - x(j-1,i);
  s_min(j,i) = l(i) * (v(j-1,i)/(0.447 * 10) + 1);
  IF s(j,i) < s_min(j,i)
     v(j,i) = MAX([0, v(j-1,i) - dB_i]);
  ELSE
     v(j,i) = MIN([v_i, v(j-1,i) + dA_i]);
  END
  x(j,i) = x(j-1,i) + v(j,i) * dt;
END
```

In the above code segment, the actual spacing between vehicle i and its leading vehicle, $s(j, i)$, is computed as the difference of their locations in previous time step. The minimum safe spacing, $s_{min}(j,i)$ is determined according to California Motor Vehicle Code. Then, $s(j,i)$ is compared against $s_{min}(j,i)$. If $s(j,i)$ is less than $s_{min}(j,i)$, reduce the speed of the vehicle by ΔB_i, but don't go beyond 0. Otherwise, increase the speed of the vehicle by ΔA_i without exceeding its desired speed v_i. Then, update the position of the vehicle, advance time by one step, and continue with the next vehicle.

Note that car-following models used for automatic control and computer simulation carry different objectives. The objective of automatic control is to guarantee safety yet achieving mobility (e.g., arriving at destination without delay). As such, automatic control calls for "an ideal (or the best) driver/model" that is able to operate the vehicle in the best way. In contrast, the purpose of computer simulation is to reproduce part of the real world as realistic as possible. Consequently, computer simulation necessities "a representative driver/model" that is able to mimic the behavior of day-to-day driving which is usually not perfect.

17.1.2 Properties of Pipes Model

In mathematical modeling, it is always interesting to understand how a system's microscopic behavior relates to its macroscopic behavior, or alternatively to interpret the microscopic basis of a macroscopic phenomenon. In traffic flow theory, microscopic car-following models are typically related to macroscopic speed-density relationships or equivalently fundamental diagram.

Typically, the linkage between microscopic and macroscopic models can be addressed in two ways. One approach is to run simulation based on the microscopic model. Such a microscopic simulation typically involves random variables such as perception-reaction time, desired speed, acceleration rate, etc. As a result, simulation results vary in different runs. Hence, the macroscopic behavior implied by the microscopic model can be obtained by a statistical analysis of these simulation results.

The other approach is analytical, i.e., one tries to aggregate or integrate the microscopic model (which typically involves ordinary differential equations) under some equilibrium or steady-state assumptions. If a system is in steady state, any property of the system is unchanging in time. More specifically, a traffic system in steady state would consists of homogeneous vehicles which exhibit unform behavior over time and space. Therefore, under steady state condition, vehicles lose their identities (e.g. $\tau_i \to \tau$ and $l_i \to l$), vehicles travel at uniform speed (i.e. $\dot{x}_i = \dot{x}_j \to v$ and $\ddot{x}_i \to 0$), drivers' desired speeds converge to free-flow speed (i.e. $v_i \to v_f$), and vehicle spacing $s_i(t)$ is replaced by the reciprocal of traffic density $\frac{1}{k}$. Hence, Pipes model reduces to:

$$\frac{1}{k} = 1.34v + 6 \text{ or } v = \frac{0.745}{k} - 4.47 \tag{17.5}$$

where k is measured in veh/m and v in m/s. The above speed-density relationship gives rise to the following flow-density and speed-flow relationships:

$$q = 0.745 - 4.47k \tag{17.6}$$

and

$$v = \frac{6q}{1 - 1.34q} \tag{17.7}$$

Equations 17.5, 17.6, and 17.7 constitute the mathematical representation of the fundamental diagram implied by Pipes model.

17.2 Forbes Model

Rather than ensuring safety distance between vehicles as Pipes model does, Forbes [38, 37] stipulates that

"*To ensure safety, the time gap between a vehicle and the vehicle in front of it should be always greater than or equal to reaction time.*"

This safety rule can be formulated as:

$$g_i^t(t) = h_i(t) - \frac{l_i}{\dot{x}_i} \geq \tau_i \tag{17.8}$$

Assume a reaction time of 1.5 seconds and vehicle length 6 m, the model becomes:

$$h_i(t) \geq 1.5 + \frac{6}{\dot{x}_i} \tag{17.9}$$

or

$$s_i(t) \geq 1.5\dot{x}_i + 6 \tag{17.10}$$

This is very similar to Pipes model except for a slight difference in the coefficient of the speed term which is interpreted as perception-reaction time τ_i. Therefore, Pipes model and Forbes model are essentially equivalent and can be generically expressed as:

$$s_i(t) \geq \tau_i \dot{x}_i + l_i \tag{17.11}$$

where τ_i and vehicle length l_i are model parameters. Note that applications and properties of Pipes model discussed above apply to Forbes model. In addition, the fundamental diagram implied by Pipes and Forbes models can be generically expressed as:

$$v = \frac{1}{\tau k} - \frac{l}{\tau} \tag{17.12}$$

$$q = \frac{1}{\tau} - \frac{l}{\tau}k \tag{17.13}$$

$$v = \frac{q1}{1 - \tau q} \tag{17.14}$$

where τ is average perception-reaction time and l is average vehicle length.

17.3 Benchmarking

Since Pipes and Forbes models are essentially equivalent, the following discussion addresses only Pipes model with the understanding that the result applies to Forbes model as well. Microscopic benchmarking refers to the scenario presented in 16.3.1 and macroscopic benchmarking refers to the scenario presented in 16.3.2.

17.3.1 Microscopic Benchmarking

For convenience, Pipes model is reproduced below:

$$\dot{x}_i(t + \Delta t) = \frac{s_i(t) - l_i}{\alpha} \tag{17.15}$$

where Δt is simulation time step and α is a constant resulted from unit conversion ($\alpha = 1.34$ if speed is in m/s and $l_i = 6$ m).

First, The model has problem with vehicle acceleration. Referring to the microscopic benchmarking scenario presented in 16.3.1, suppose that initially the leading vehicle is located at $x_{i-1}(0) = 5000$ m and the subject vehicle is at $x_i(0) = -102$ m and both vehicles are standing still. When the simulation begins, vehicle i starts to move according to Pipes model. A spacing of $s_i(0) = 5102$ m results in a speed of about 3803 m/s at the next time step (assuming $\Delta t = 1$ s), which requires an acceleration of 3803 m/s^2. It follows that an infinity speed and acceleration would be resulted if there is no leading vehicle in front. Therefore, the following external logic has to be imposed on Pipes model in order to limit its maximum acceleration:

$$\ddot{x}_i(t) = \frac{\dot{x}_i(t + \Delta t) - \dot{x}_i(t)}{\Delta t} \leq A_i \tag{17.16}$$

where A_i is the maximum acceleration of vehicle i, e.g. $A_i = 4$ m/s^2. With this addition, Pipes model loses its mathematical elegance which favors one-equation-for-all formulation. Even though an external logic is added, Pipes model still has problem with maximum speed. For example, it is true that the acceleration no longer exceeds A_i, but the vehicle can still reach unrealistic high speeds, e.g. $\dot{x}_i = 196$ m/s when $s_i = 590$ m. Therefore, another external logic has to be imposed to limit speed:

$$\dot{x}_i \leq v_i \tag{17.17}$$

where v_i is driver i's desired speed. The third problem is unrealistic deceleration. For example, at time $t = 424$, vehicle i is located at about $x_i = 8734$ m moving at speed $\dot{x}_i = 30$ m/s, while vehicle $i - 1$ stops at $x_{i-1} = 8762$ m. According to Pipes model, vehicle i's speed at the next step would be $\dot{x}_i \approx 16.42$ m/s. As such, the deceleration rate is $\ddot{x}_i = -13.58$ m/s^2. Hence, a third external logic has to be imposed to limit maximum deceleration B_i (e.g. -6 m/s^2):

$$\ddot{x}_i(t) = \frac{\dot{x}_i(t + \Delta t) - \dot{x}_i(t)}{\Delta t} \geq B_i \tag{17.18}$$

However, this addition introduces a new problem. For example, vehicle i's speed at the next step becomes $\dot{x}_i = 30 - 6 = 24$ m/s^2 and its location is $x_i = 8758$ m. This would leave a spacing of $s_i = 4$ m which is less than a vehicle length $l_{i-1} = 6$ m, i.e., vehicle i has collided into vehicle $i - 1$. Unfortunately, there is no easy remedy to the problem except for accepting the unrealistic deceleration behavior.

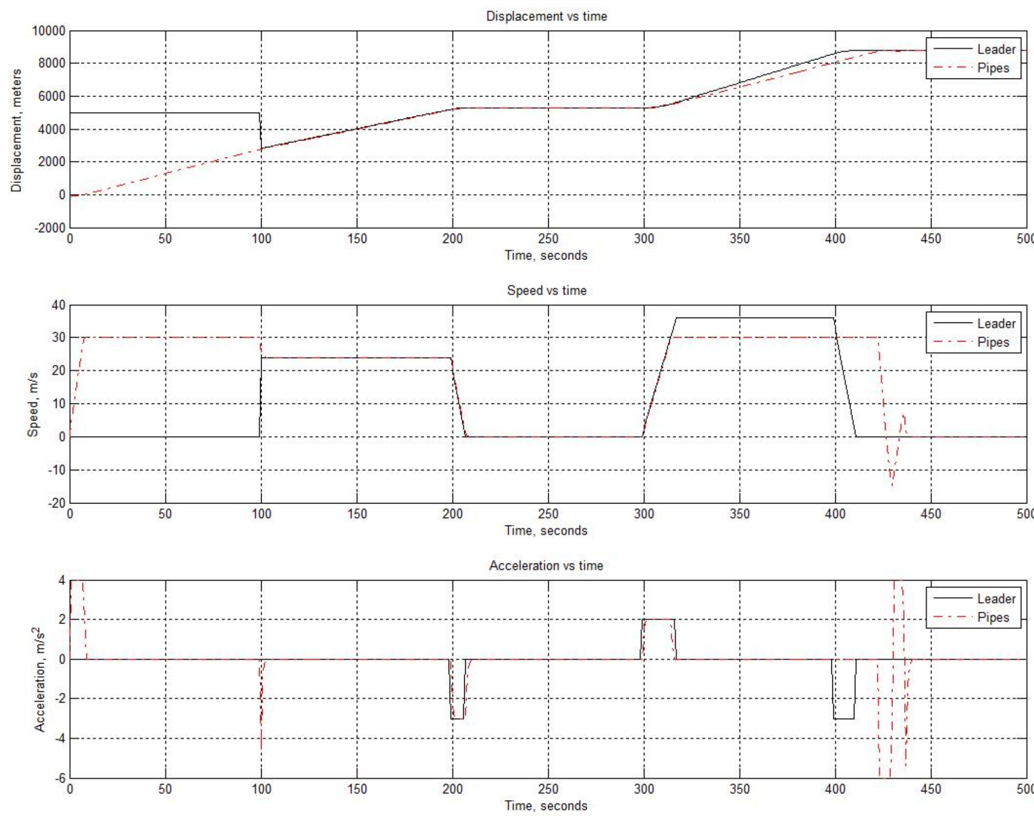

Figure 17.2: Microscopic benchmarking of Pipes model

Benchmarking result of Pipes model with constraints 17.16, 17.17, and 17.18 is plotted in Figure 17.2. The performance of the constrained Pipes model is summarized as follows and the discussion is based on the benchmarking scenario:

- Start-up: the model is able to start the vehicle up from stand-still. See the figure when $t > 0$ s.
- Speed-up: the model is able to speed up the vehicle. However, its acceleration profile (i.e. acceleration as a function of speed) is unrealistic because the vehicle is able to retain maximum acceleration at

high speeds. Normally, maximum acceleration is only available when a vehicle starts up. As the vehicle speeds up, acceleration decreases and eventually varnishes when the vehicle achieves its desired / cruising speed. See the figure when $0 < t < 100$ s.

- Free-flow: an external logic has to be imposed to limit maximum under free-flow condition. See the figure when $0 < t < 100$ s
- Cut-off: the model retains control and responds reasonably when a vehicle cuts in front. See the figure around $t = 100$ s.
- Following: the model is able to adopt the leader's speed and follow the leader by a reasonable distance. See the figure when $100 < t < 200$ s.
- Stop-and-go: the model is able to stop the vehicle safely behind its leader and start moving when the leader departs. See the figure when $200 \geq t \leq 300$ s.
- Trailing: the model is able to speed up normally without being tempted by its speeding leader. See the figure when $300 < t < 400$ s.
- Approaching: the model is unable to decelerate properly when approaching a stationary vehicle at a distance. The vehicle might collide into its leader when maximum deceleration is imposed. See the figure when $400 \geq t < 420$ s.
- Stopping: this portion is invalid since approaching fails. See the figure when $t \geq 420$ s.

The above benchmarking is based on the set of parameters in Table 17.1 and the outcome may vary under different set of parameters.

Table 17.1: Microscopic benchmarking parameters of Pipes model

l_i	v_i	τ_i	α	-
6 m	30 m/s	1.0 s	1.34	-
A_i	B_i	$x_i(0)$	$\dot{x}_i(0)$	$\ddot{x}_i(0)$
4.0 m/s^2	6.0 m/s^2	-120 m	0 m/s	0 m/s^2

17.3.2 Macroscopic Benchmarking

The fundamental diagram implied by Pipes model is plotted in Figure 17.3 against empirical observations. The "cloud" contains 5-minute observations of flow, speed, and density, the circles are empirical observations aggregated with respect to density, and the curves are the equilibrium relationships implied by Pipes model.

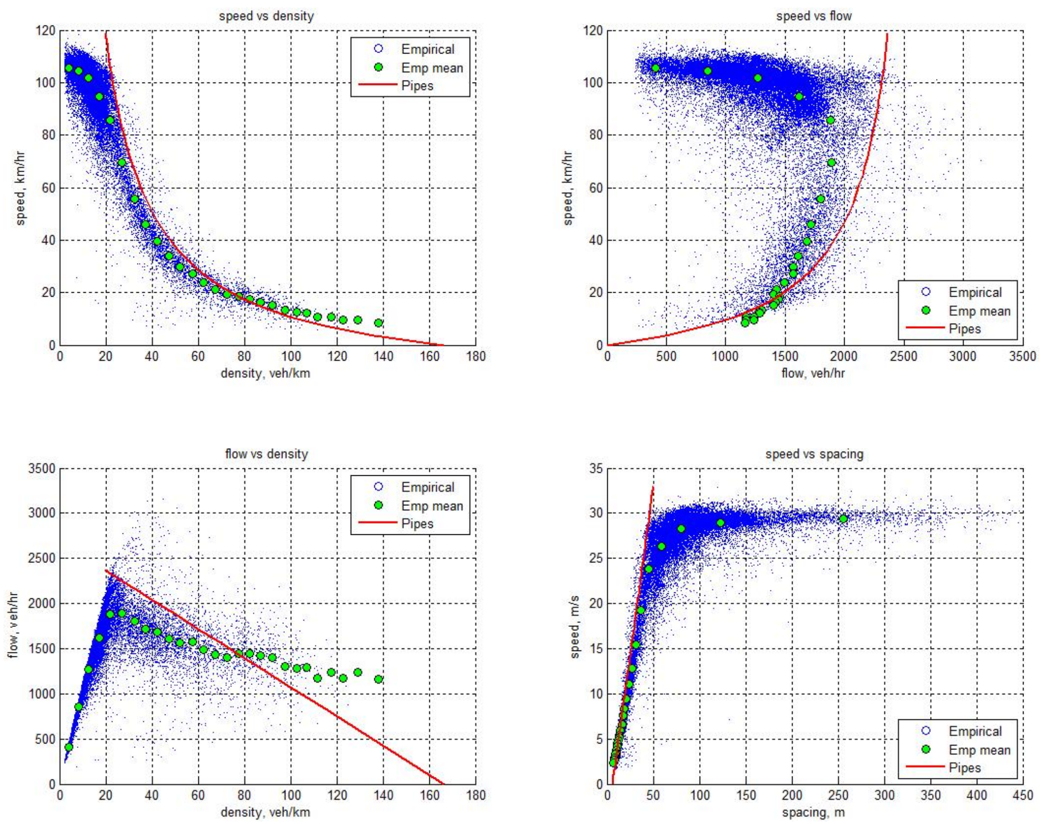

Figure 17.3: Fundamental diagram implied by Pipes model

These curves roughly fit the empirical data in the middle to upper range of density (e.g., $k > 20$ veh/km), but do not apply to the low density range (e.g., $k < 20$ veh/km). It appears that Pipes model is designed to literally describe car-following behavior. Cases when the leading vehicle is absent has to be handled by an external logic. In addition, Pipes model predicts that traffic speed would increase to infinity as density approaches zero.

The above benchmarking is based on the set of parameters in Table 17.2 and the outcome may vary under different set of parameters.

Table 17.2: Macroscopic benchmarking parameters of Pipes model

α (τ)	l
1.34 s	6 m

Since Forbes model is essentially the same as Pipes model, the above benchmarking results apply to Forbes model as well.