Notes from reading "Introduction to Toric Varieties" by Fulton

TiGR participants

March 23, 2020

Chapter 1

Definitions and examples

1.1 Motivating Example

Let $X \subseteq \mathbb{A}^3_{\mathbb{C}}$ be given by $y^3 - xz = 0$. Denote its coordinate ring by $\mathbb{C}[X] = \mathbb{C}[x,y,z]/(y^3 - xz)$and consider the action of $(\mathbb{C}^*)^2$ on X given by:

$$(t_1, t_2) \cdot (x, y, z) = (t_1^{-2}t_2^{-3}x, t_1^{-1}t_2^{-1}y, t_1^{-1}z).$$

We have an induced action of $(\mathbb{C}^*)^2$ on the coordinate ring $\mathbb{C}[X]$ given by

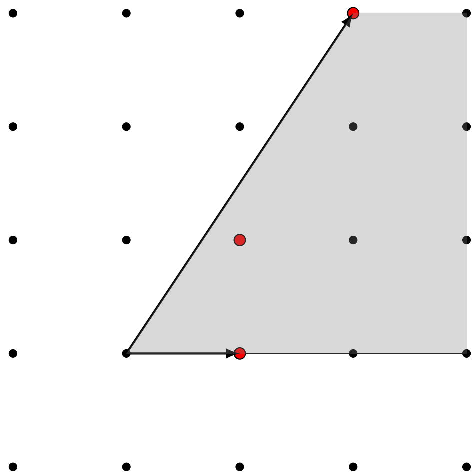
$$(t \cdot f)(p) = f(t^{-1} \cdot p).$$

Let $M = \{\chi \colon (\mathbb{C}^*)^2 \to \mathbb{C}^*\}$ be the lattice of characters of $(\mathbb{C}^*)^2$. Recall that this group is isomorphic with $\mathbb{Z}^2$, $u \in \mathbb{Z}^2$ corresponding to $\chi^u : t \mapsto t^u$. Given $\chi^u \in M$, define $\mathbb{C}[X]_u = \{f \in \mathbb{C}[X] | t \cdot f = \chi^u(t) \cdot f\}$. The class in $\mathbb{C}[X]$ of every monomial in $\mathbb{C}[x,y,z]$ can be written uniquely in the form $x^a y^b z^c$ with $b \in \{0,1,2\}$. We have $t_1^m t_2^n \cdot x^a y^b z^c = t_1^{m(2a+b+c)} t_2^{n(3a+b)} x^a y^b z^c$. Therefore, $x^a y^b z^c \in \mathbb{C}[X]_u$ if and only if $u = (2a+b+c, 3a+b)$. Let $u = (i,j)$. We search for a solution of the following system:

$$\begin{cases} 2a+b+c = i \\ 3a+b = j \\ b \in \{0,1,2\} \\ a \geq 0 \\ c \geq 0 \end{cases}$$

For every (i,j) there is at most one solution, so $\mathbb{C}[X] = \bigoplus_{u \in \mathbb{Z}^2} \mathbb{C}[X]_u$ decomposes $\mathbb{C}[X]$ into one or zero dimensional subrepresentations of $(\mathbb{C}^*)^2$. We will find those $u = (i,j)$ for which $\mathbb{C}[X]_u$ is non-zero. We need to have $j \geq 0$ and then a, b are uniquely determined. Therefore, if $j \geq 0$ then a solution exists if and only if $i - 2a - b \geq 0$. This is equivalent to $3i - 6a - 3b \geq 0$. We can rewrite this as $3i \geq 2j + b$ which is equivalent to $3i \geq 2j$ since $2j + b = 6a + 3b$ is divisible by 3 and $b \in \{0,1,2\}$.

Therefore, for every $u \in \mathbb{Z}^2$ we have $\dim_{\mathbb{C}} \mathbb{C}[X]_u \in \{0,1\}$ and those u for which $\dim_{\mathbb{C}} \mathbb{C}[X]_u = 1$ are the lattice points of the cone below (cones will be formally defined in the Section 1.2).



Denote this cone by $\sigma^\vee$ and define the corresponding semigroup algebra $\mathbb{C}[\sigma^\vee \cap M] = \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C} \cdot \chi^u$ with multiplication induced by $\chi^u \cdot \chi^v = \chi^{u+v}$. Since $\sigma^\vee \cap M$ is generated as a semigroup by $(1,0),(1,1),(2,3)$, $\mathbb{C}[\sigma^\vee \cap M]$ is generated as a $\mathbb{C}$-algebra by $\chi^{(1,0)}$, $\chi^{(1,1)}$ and $\chi^{(2,3)}$. Moreover, the generators of the semigroup $\sigma^\vee \cap M$ satisfy a relation $3 \cdot (1,1) = (1,0) + (2,3)$, therefore we have an isomorphism $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[X]$ given by $\chi^{(1,0)} \mapsto x$, $\chi^{(1,1)} \mapsto y$ and $\chi^{(2,3)} \mapsto z$.

1.2 Convex polyhedral cones

Let N, M be dual lattices and let $V = N_\mathbb{R}$ and $V^* = M_\mathbb{R}$ be the corresponding dual vector spaces. A *convex polyhedral cone* is a set

$$\sigma = \{r_1 v_1 + ... + r_s v_s \in V : r_i \geq 0\}$$

generated by any finite set of vectors $v_1, ..., v_s \in V$. Such vectors, or sometimes the corresponding rays consisting of positive multiples of some v_i are called *generators* for the cone σ.

The *dimension* $\dim(\sigma)$ of σ is the dimension of the linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ. The *dual* $\sigma^\vee$ of any set σ is the set of equations of supporting hyperplanes, i.e.,

$$\sigma^\vee = \{u \in V^* : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Everything is based on the following fundamental fact from the theory of convex sets.

(*) If σ is a convex polyhedral cone and $v_0 \notin \sigma$, then there is some $u_0 \in \sigma^\vee$ with $\langle u_0, v_0 \rangle < 0$.

We list some consequences of (*). Since the proofs given in Fulton's text are easy to follow we do not replicate them here. A direct translation of (*) is the duality theorem:

(1) $(\sigma^\vee)^\vee = \sigma$.

A *face* τ, of σ is the intersection of σ with any supporting hyperplane: $\tau = \sigma \cap u^\perp = \{v \in \sigma : \langle u, v \rangle = 0\}$ for some u in $\sigma^\vee$. A cone is regarded as a face of itself, while others are called *proper* faces. Note that any linear subspace of a cone is contained in every face of the cone,

(2) Any face is also a convex polyhedral cone.

(3) Any intersection of faces is also a face.

(4) Any face of a face is a face.

A *facet* is a face of codimension one.

(5) Any proper face is contained in some facet.

(6) Any proper face is the intersection of all facets containing it.

(7) The topological boundary of a cone that spans V is the union of its proper faces (or facets).

When σ spans V and τ is a facet of σ, there is a $u \in \sigma^\vee$, unique up to multiplication by a positive scalar, with $\tau = \sigma \cap u^\perp$. Such a vector, which we denote by u_τ is an equation for the hyperplane spanned by τ.

(8) If σ spans V and $\sigma \neq V$, then σ is the intersection of the half-spaces $H_\tau = \{v \in V : \langle u_\tau, v \rangle \geq 0\}$, as τ ranges over the facets of σ.

From (8) we deduce the fact known as Farkas' Theorem:

(9) The dual of a convex polyhedral cone is a convex polyhedral cone.

This shows that polyhedral cones can also be given a dual definition as the intersection of half-spaces: for generators $u_1, ..., u_t$ of $\sigma^\vee$,

$$\sigma = \{v \in V : \langle u_1, v \rangle \geq 0, ..., \langle u_t, v \rangle \geq 0\}.$$

If we now suppose σ is *rational*, meaning that its generators can be taken from N, then $\sigma^\vee$ is also rational.

Proposition 1 (Gordan's Lemma). *If σ is a rational convex polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.*

It is often necessary to find a point in the *relative interior* of a cone σ, i.e., in the topological interior of σ in the space $\mathbb{R} \cdot \sigma$ spanned by σ. This is achieved by taking any positive combination of $\dim(\sigma)$ linearly independent vectors among the generators of σ. In particular, if σ is rational, we can find such points in the lattice.

(10) If τ is a face of σ, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$, with $\dim(\tau)+\dim(\sigma^\vee \cap \tau^\perp) = n = \dim(V)$. This sets up a one-to-one order-reversing correspondence between the faces of σ and the faces of $\sigma^\vee$. The smallest face of σ is $\sigma \cap (-\sigma)$.

(11) If $u \in \sigma^\vee$, and $\tau = \sigma \cap \tau^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot (-u)$.

Proposition 2. *Let σ be a rational convex polyhedral cone, and let u be in $S_\sigma = \sigma^\vee \cap M$. Then $\tau = \sigma \cap u^\perp$ is a rational convex polyhedral cone. All faces of σ have this form, and*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

Finally, we need the following strengthening of (*), known as a *Separation Lemma*, that separates convex sets by a hyperplane:

(12) If σ and σ' are convex polyhedral cones whose intersection τ is a face of each, then there is a u in $\sigma^\vee \cap (-\sigma')^\vee$ with

$$\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp.$$

Proposition 3. *If σ and σ' are rational convex polyhedral cones whose intersection τ is a face of each, then*

$$S_\tau = S_\sigma + S_{\sigma'}.$$

(13) For a convex polyhedral cone σ, the following conditions are equivalent:

(i) $\sigma \cap (-\sigma) = \{0\}$;

(ii) σ contains no nonzero linear subspace;

(iii) there is a u in $\sigma^\vee$ with $\sigma \cap u^\perp = \{0\}$;

(iv) $\sigma^\vee$ spans V^*.

A cone is called *strongly convex* if it satisfies the conditions of (13). Any cone is generated by some minimal set of generators. If the cone is strongly convex, then the rays generated by a minimal set of generators are exactly the one-dimensional faces of σ (as seen by applying (*) to any generator that is not in the cone generated by the others); in particular, these minimal generators are unique up to multiplication by positive scalars.

Since we are mainly concerned with these cones, we will often say "σ is a cone in N" to mean that σ is a strongly convex rational polyhedral cone in $N_\mathbb{R}$. We will sometimes write "$\tau \prec \sigma$" or "$\sigma \succ \tau$" to mean that τ is a face of σ. A cone is called *simplicial*, or a *simplex*, if it is generated by linearly independent generators.

1.3 Affine toric varieties

Let R be a ring. We work in the category of R-schemes so affine space, multiplicative group, products, etc. are all over R. When σ is a strongly convex rational polyhedral cone, we have seen that $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup. Any additive semigroup S determines a "group ring" $R[S]$, which is a commutative R-algebra. As an R-module it has a basis χ^u, as u varies over S, with multiplication determined by the addition in S:

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}.$$

The unit 1 is χ^0. Generators $\{u_i\}$ for the semigroup S determine generators $\{\chi^{u_i}\}$ for the R-algebra $R[S]$.

Any finitely generated commutative R-algebra A determines an affine scheme of finite type over R, which we denote by $\mathrm{Spec}(A)$. We review this construction and its related notation. If generators of A are chosen, this presents A as $R[X_1, ..., X_m]/I$, where I is an ideal; then $\mathrm{Spec}\, A$ can be identified with the subscheme $V(I)$ of affine space $\mathbb{A}^m$. In our applications, A will be a domain, so $\mathrm{Spec}(A)$ will be an integral scheme. Although $\mathrm{Spec}(A)$ officially includes all prime ideals of A, when we speak of a ***point*** of $\mathrm{Spec}(A)$ we will mean an R-point, i.e. a homomorphism of R-schemes $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$, unless we specify otherwise. Any homomorphism $A \to B$ of R-algebras determines a morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of R-schemes. In particular, R-points correspond to R-algebra homomorphisms from A to R. If $X = \mathrm{Spec}(A)$, for each nonzero element $f \in A$ the principal open subset

$$X_f = \mathrm{Spec}(A_f) \subset X = \mathrm{Spec}(A)$$

corresponds to the localization homomorphism $A \to A_f$.

For $A = R[S]$ constructed from a semigroup, the points are easy to describe: they correspond to homomorphisms of semigroups from S to R, where R is regarded as an abelian semigroup via multiplication:

$$(\mathrm{Spec}(R[S]))(R) = \mathrm{Hom}_{sg}(S, R).$$

Indeed, this follows from the adjunction $R[-] \dashv G$ where G is the forgetful functor from the category of R-algebras to the category of monoids mapping an R-algebra to the underlying multiplicative monoid.

When S_σ arises from a strongly convex rational polyhedral cone, we set $A_\sigma = R[S_\sigma]$, and $U_\sigma = \mathrm{Spec}(R[S_\sigma]) = \mathrm{Spec}(A_\sigma)$, the corresponding ***affine toric scheme***. All of these semigroups will be sub-semigroups of the group $M = S_{\{0\}}$. If $e_1, ..., e_n$ is a basis for N, and $e_1^*, ..., e_n^*$ is the dual basis of M, write

$$X_i = \chi^{e_i^*} \in R[M].$$

As a semigroup, M has generators $\pm e_1^*, ..., \pm e_n^*$, so

$$R[M] = R[X_1, X_1^{-1}, X_2, X_2^{-1}, ..., X_n, X_n^{-1}] = R[X_1, ..., X_n]_{X_1 \cdot ... \cdot X_n},$$

which is the ring of *Laurent polynomials* in n variables. So

$$U_{\{0\}} = \mathrm{Spec}(R[M]) \cong \mathbb{G}_m^n$$

is an affine algebraic torus. All of our semigroups S will be sub-semigroups of a lattice M, so $R[S]$ will be a subalgebra of $R[M]$; in particular, $R[S]$ will be a domain if R is an integral domain. When a basis for M is chosen as above, we usually write elements of $R[S]$ as Laurent polynomials in the corresponding variables X_i. Note that all of these algebras are generated by *monomials* in the variables X_i.

The torus $T = T_N$ corresponding to M or N can be written intrinsically:

$$T_N(R) = (\mathrm{Spec}(R[M]))(R) = \mathrm{Hom}_{group}(M, R^*) = N \otimes_{\mathbb{Z}} R^*.$$

The above uses the adjunction $R[-] \dashv [-]^*$ where $[-]^*$ is the functor from the category of R-algebras to the category of groups mapping underlying ring to its group of invertible elements.

For a basic example, let σ be the cone with generators $e_1, ..., e_k$ for some k, $1 \leq k \leq n$. Then

$$S_\sigma = \mathbb{Z}_{\geq 0} \cdot e_1^* + \mathbb{Z}_{\geq 0} \cdot e_2^* + ... + \mathbb{Z}_{\geq 0} \cdot e_k^* + \mathbb{Z} \cdot e_{k+1}^* + ... + \mathbb{Z} \cdot e_n^*.$$

Hence $A_\sigma = R[X_1, X_2, ..., X_k, X_{k+1}, X_{k+1}^{-1}, ..., X_n, X_n^{-1}]$, and

$$U_\sigma = \mathbb{A}^k \times \mathbb{G}_m^{n-k}.$$

It follows from this example that if σ is generated by k elements that can be completed to a basis for N, then U_σ is a product of affine k-space and an algebraic torus of rank $(n-k)$. In particular, such affine toric schemes are smooth over R.

Next we look at a singular example. Let N be a lattice of rank 3, and let σ be the cone generated by four vectors v_1, v_2, v_3, and v_4 that generate N and satisfy $v_1 + v_3 = v_2 + v_4$. The scheme U_σ is a "cone over a quadric surface", a scheme met frequently when singularities are studied. If we take $N = \mathbb{Z}^3$ and $v_i = e_i$ for $i = 1, 2, 3$, so $v_4 = e_1 + e_3 - e_2$, then S_σ is generated by $e_1^*, e_3^*, e_1^* + e_2^*$, and $e_2^* + e_3^*$, so

$$A_\sigma = R[X_1, X_3, X_1X_2, X_2X_3] = R[W, X, Y, Z]/(WZ - XY).$$

A homomorphism of semigroups $S \to S'$ determines a homomorphism $R[S] \to R[S']$ of algebras, hence a morphism $\mathrm{Spec}(R[S']) \to \mathrm{Spec}(R[S])$ of affine R-schemes. In particular, if τ is contained in σ, then S_σ is a sub-semigroup of S_τ, corresponding to a morphism $U_\tau \to U_\sigma$. For example, the torus $T_N = U_{\{0\}}$ maps to all of the affine toric schemes U_σ that come from cones σ in N.

Lemma 4. *If τ is a face of σ, then the map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

Proof. By Proposition 2 in §1.2, there is a $u \in S_\sigma$ with $\tau = \sigma \cap u^\perp$ and

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

This implies immediately that each basis element for $R[S_\tau]$ can be written in the form $\chi^{w-pu} = \chi^w/(\chi^u)^p$ for $w \in S_\sigma$. Hence

$$A_\tau = (A_\sigma)_{\chi^u},$$

which is the algebraic version of the required assertion. □

More generally, if $\varphi\colon N' \to N$ is a homomorphism of lattices such that $\varphi_{\mathbb{R}}$ maps a (rational strongly convex polyhedral) cone σ' in N' into a cone σ in N, then the dual $\varphi^\vee\colon M \to M'$ maps S_σ to $S_{\sigma'}$, determining a homomorphism $A_\sigma \to A_{\sigma'}$, and hence a morphism $U_{\sigma'} \to U_\sigma$.

The semigroups S_σ arising from cones are special in several respects. First, it follows from the definition that S_σ is *saturated*, i.e., if $p \cdot u$ is in S_σ for some positive integer p, then u is in S_σ. In addition, the fact that σ is strongly convex implies that $\sigma^\vee$ spans $M_{\mathbb{R}}$, so S_σ generates M as a group, i.e.,

$$M = S_\sigma + (-S_\sigma).$$

If σ is a cone in N, the torus T_N acts on U_σ,

$$T_N \times U_\sigma \to U_\sigma,$$

as follows. A point $t \in T_N(R)$ can be identified with a map $M \to R^*$ of groups, and a point $x \in U_\sigma(R)$ with a map $S_\sigma \to R$ of semigroups; the product $t \cdot x$ is the map of semigroups $S_\sigma \to R$ given by

$$u \mapsto t(u)x(u).$$

The dual map on algebras, $R[S_\sigma] \to R[S_\sigma] \otimes R[M]$, is given by mapping χ^u to $\chi^u \otimes \chi^u$ for $u \in S_\sigma$. When $\sigma = \{0\}$, this is the usual product in the algebraic group T_N. These maps are compatible with inclusions of open subsets corresponding to faces of σ. In particular, they extend the action of T_N on itself.

To say it differently, the multiplication in $\mathbb{G}_m^n$ is known to correspond to the homomorphism $m^\#\colon R[M] \to R[M] \otimes R[M]$ given by $\chi^u \mapsto \chi^u \otimes \chi^u$. Therefore, if we define $\mu^\#\colon R[S_\sigma] \to R[M] \otimes R[S_\sigma]$ by $\chi^u \mapsto \chi^u \otimes \chi^u$ the following diagram is commutative:

$$\begin{tikzcd}
R[S_\sigma] \arrow[r, "\mu^\#"] \arrow[d, hook] & R[M] \otimes R[S_\sigma] \arrow[d, hook] \\
R[M] \arrow[r, "m^\#"] & R[M] \otimes R[M].
\end{tikzcd}$$

so that $\mu\colon T_N \times U_\sigma \to U_\sigma$ extends $m\colon T_N \times T_N \to T_N$. In order to check that this is an action we should check that the following diagrams are commutative:

$$\begin{tikzcd}
T_N \times T_N \times U_\sigma \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "m\times\mathrm{id}"] & T_N \times U_\sigma \arrow[d, "\mu"] \\
T_N \times U_\sigma \arrow[r, "\mu"] & U_\sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Spec} R \times U_\sigma \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\cong"'] & T_N \times U_\sigma \arrow[d, "\mu"] \\
 & U_\sigma
\end{tikzcd}$$

This can be verified directly on algebras using the fact that $e : \operatorname{Spec} R \to T_N$ corresponds to $R[M] \to R$ given by $\chi^u \mapsto 1$ for all $u \in M$ and the isomorphism in the right diagram corresponds to the standard isomorphism $R[S_\sigma] \to R \otimes_R R[S_\sigma]$.

Chapter 2

Singularities and compactness

2.1 Local properties of toric varieties

For any cone σ in a lattice N, the corresponding affine scheme U_σ has a distinguished point, which we denote by x_σ. This point in U_σ is given by a map of semigroups

$$S_\sigma = \sigma^\vee \cap M \to R,$$

defined by the rule

$$u \mapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is well defined since $\sigma^\perp$ is a face of $\sigma^\vee$, which implies that the sum of two elements in $\sigma^\vee$ cannot be in $\sigma^\perp$ unless both are in $\sigma^\perp$.

Proposition 5. *An affine toric scheme U_σ is smooth over* $\operatorname{Spec}(R)$ *if and only if σ is generated by part of a basis for the lattice N, in which case*

$$U_\sigma \cong \mathbb{A}^k \times \mathbb{G}_m^{n-k},\ k = \dim(\sigma).$$

Proof. It is enough to show that if U_σ is smooth over $\operatorname{Spec}(R)$ then σ is generated by part of a basis for the lattice N. Suppose to start that σ spans $N_\mathbb{R}$, so $\sigma^\perp = \{0\}$. Assume that U_σ is smooth over R and let $\mathfrak{n}$ be a maximal ideal of R with $K = R/\mathfrak{n}$. Take the base change:

$$\begin{tikzcd} U_\sigma \times_R K \arrow[r] \arrow[d, "\pi'"] & U_\sigma \arrow[d, "\pi"] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} R. \end{tikzcd}$$

Note that $U_\sigma \times_R K$ is the affine toric variety associated with the cone σ if we work in the category of K-schemes. Moreover, since π is smooth, so is π' by [1, Prop. 6.15]. Therefore, we may assume from the start that U_σ is a variety over a field K, that is we suppose that $R = K$ is a field. Then π being smooth implies that U_σ is a regular scheme by [1, Lem. 6.26]. Consider the maximal ideal $\mathfrak{m} = (\chi^u : u \in S_\sigma \setminus \{0\})$ of $A_\sigma = K[\sigma^\vee \cap M]$. The square $\mathfrak{m}^2$ is generated by all χ^u for those u that are sums of two elements of $S_\sigma \setminus \{0\}$. The cotangent space $\mathfrak{m}/\mathfrak{m}^2$ therefore has a basis of images of elements χ^u for those u in $S_\sigma \setminus \{0\}$

that are not the sums of two such vectors. For example, the first elements in M lying along the edges of $\sigma^\vee$ are vectors of this kind. Since U_σ is regular, the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is n-dimensional, since $\dim(U_\sigma) = \dim(T_N) = n$. This implies in particular that $\sigma^\vee$ cannot have more than n edges, and that the minimal generators along these edges must generate S_σ. Since S_σ generates M as a group, the minimal generators for S_σ must be a basis for M. The dual σ must therefore be generated by a basis for N. Hence U_σ is isomorphic to affine space $\mathbb{A}^n_K$.

Consider now a general case when σ has smaller dimension k. Let

$$N_\sigma = \sigma \cap N + (-\sigma \cap N)$$

be the sublattice of N generated (as a subgroup) by $\sigma \cap N$. Since σ is saturated, N_σ is also saturated, so the quotient group $N(\sigma) = N/N_\sigma$ is also a lattice. We may choose a splitting and write

$$N = N_\sigma \oplus N'',\ \sigma = \sigma' \oplus \{0\},$$

where σ' is a cone in N_σ. Decomposing $M = M' \oplus M''$ dually, we have $S_\sigma = ((\sigma')^\vee \cap M') \oplus M''$, so

$$U_\sigma \cong U_{\sigma'} \times T_{N''} \cong U_{\sigma'} \times \mathbb{G}_m^{n-k}.$$

As before, by taking a base change, we may assume that we work over a field K. Since U_σ is smooth over K and $U_{\sigma'}$,$T_{N''}$ are affine schemes of finite type over K, they must be both smooth. Indeed, $\dim U_\sigma = \dim U_{\sigma'} + \dim T_{N''}$ by [1, Prop. 5.37]. We may choose a presentation of $U_{\sigma'}$, $T_{N''}$ as subschemes of affine spaces and then the Jacobian corresponding to U_σ is a block matrix with blocks corresponsing to Jacobians of $U_{\sigma'}$ and $T_{N''}$. Therefore, the preceding discussion applies: σ' must be generated by a basis for N_σ. $\square$

We therefore call a cone *nonsingular* if it is generated by part of a basis for the lattice, and we call a fan *nonsingular* (fan is defined in §1.4) if all of its cones are nonsingular, i.e., if the corresponding toric scheme is smooth. Although a toric scheme may be singular, every toric scheme is normal if the base ring is an integrally closed domain:

Proposition 6. *If the base ring R is integrally closed, then each ring A_σ is integrally closed.*

Proof. If σ is generated by $v_1, ..., v_r$, then $\sigma^\vee = \cap \tau_i^\vee$, where τ_i is the ray generated by $v_i \in \sigma$, so $A_\sigma = \cap A_{\tau_i}$. We have seen that each A_{τ_i} is isomorphic to $R[X_1, X_2, X_2^{-1}, ..., X_n, X_n^{-1}]$, which is integrally closed (polynomial ring over a normal domain is a normal domain and localization of a normal domain is a normal domain), and the proposition follows from the fact that the intersection of integrally closed domains with common field of fractions is integrally closed. $\square$

Bibliography

[1] U. Görtz and T. Wedhorn. *Algebraic Geometry: Part I: Schemes. With Examples and Exercises.* Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 2010.