

FACULTY OF FUNDAMENTAL PROBLEMS OF TECHNOLOGY  
WROCLAW UNIVERSITY OF SCIENCE AND TECHNOLOGY

# EIGENMODES IN NEARLY INTEGRABLE QUANTUM CHAINS

JAKUB PAWŁOWSKI

INDEX NUMBER: 250193

Master thesis  
under supervision of  
prof. dr hab. Marcin Mierzejewski



Wrocław  
University  
of Science  
and Technology

WROCLAW 2023

This page is intentionally left blank.

*I would like to express my sincere gratitude to prof. dr hab.  
Marcin Mierzejewski for being my*

This page is intentionally left blank.

## Abstract

Apparent incompatibility of classical irreversible thermodynamics with

**Keywords:** *integrals of motion, ETH, integrability breaking, XXZ model*

This page is intentionally left blank.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Krylov subspace methods for quantum many-body systems</b>	<b>3</b>
2.1	Problems with Exact Diagonalization . . . . .	4
2.2	Calculation of extremal eigenvalues . . . . .	4
2.2.1	Arnoldi iteration . . . . .	4
2.2.2	Polynomial approximation and extremal eigenvalues . . . . .	7
2.2.3	Restriction to hermitian case: Lanczos iteration . . . . .	8
2.3	Time evolution via the Krylov propagator . . . . .	8
2.4	Physical interlude: Quantum Typicality . . . . .	8
2.5	Correlation functions and the search for integrals of motion . . . . .	8
<b>3</b>	<b>Spin transport in long range XXZ model</b>	<b>9</b>
<b>4</b>	<b>Relaxation eigenmodes in long range XXZ model</b>	<b>11</b>
<b>5</b>	<b>Summary</b>	<b>13</b>
	<b>Bibliography</b>	<b>15</b>
<b>A</b>	<b>Hilbert subspaces with fixed momentum</b>	<b>15</b>

This page is intentionally left blank.



# 1

## Introduction

The results concerning spin transport in the long range XXZ have already been published in Mierzejewski et al. [1]. Writing something about computational basis



# Krylov subspace methods for quantum many-body systems

One of the two purposes of this thesis is to develop and test a set of numerical tools based on the Krylov subspace methods, which is a family of iterative methods concerned with projecting high dimensional problems into smaller dimension subspaces and solving them therein. The titular Krylov subspace is introduced via

**Definition 2.1** *Let  $\mathbf{v} \in \mathbb{C}^m$  be a vector and  $A \in \mathbb{C}^{m \times m}$  be a matrix. A  $k$ -th **Krylov subspace** is defined as*

$$\mathcal{K}_k := \text{span}\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \dots, A^{k-1}\mathbf{v}\}$$

Maximal dimension of a Krylov subspace is bounded from above by  $\text{rank}(A) + 1$  [2].

This chapter serves as a pedagogical introduction to the core ideas of these methods, including some of the usually omitted mathematical details. For the initial part of this exposition we follow the excellent textbook of numerical linear algebra by Trefethen and Bau [3], whereas for further applications to quantum many-body physics we rely on the excellent treatments of the topic found in Sandvik [4] and PhD thesis by Crivelli [5].

**Reproduce Figure 32.1 about the difference between direct and iterative algorithms**

We start this chapter by quickly sketching the problems with "direct" algorithms such as Exact Diagonalization, and quickly follow with the fundamental iterative algorithm for sparse nonhermitian matrices, the Arnoldi iteration. Its outputs admits several possible interpretations, however we shall focus on the problem of locating extremal eigenvalues. Afterwards, we restrict our attention to the class of hermitian matrices, to which of course all typical tight-binding Hamiltonians belong to, and describe the Lanczos algorithm, which allows for efficient calculation of the ground state eigenvalue and eigenvector, and thus the ground state properties of a system. Yet in this work we are mainly interested in infinite temperature calculations, for which in principle sampling of the whole spectrum is required. To this end, in subsequent sections we develop a scheme for time evolution of arbitrary state, called the Krylov propagator [6], and combine it with the idea of Dynamical Quantum Typicality (DQT),



which states that a single pure state can have the same properties as an ensemble density matrix [7, 8, 9]. We finish this chapter with a proposal of employing this method to the identification of local integrals of motion in a given tight-binding system[10].

## 2.1 Problems with Exact Diagonalization

The most straightforward numerical method for studying discrete quantum many-body systems is without a doubt Exact Diagonalization (ED) [11]. It belongs to the family of the so-called direct algorithms and allows one to obtain numerically exact set of eigenvalues and eigenvectors and subsequently compute any desired properties of the system, be it thermal expectation values, time evolution, Green's functions etc. Unfortunately, the starting point of any ED calculation is the expression of the Hamiltonian as a dense matrix, in the Hilbert space basis of choice. Taking into account the fact that the dimension many-body Hilbert space grows exponentially with the size of the system, the memory cost quickly becomes prohibitive, even when exploiting conservation laws and related symmetries. For example, in the case of a spin chain of length  $L$ , with on-site basis dimension being 2, the full dimension of the Hilbert space would be  $\mathcal{D} = 2^L$ . Taking a modest length of 25 sites, that gives  $2^{25} = 33554432 \approx 3.36 \cdot 10^7$  basis states and a memory footprint of Hamiltonian matrix of around 9PB (using double-precision floating point numbers), which is 9000 times more than the typical consumer hard drive capacity of 1TB. Even assuming some kind of distributed memory platform allowing for handling such large matrices, the computational complexity of ED, requiring  $O(\mathcal{D}^3)$  operations, is the next major hurdle. Therefore, it is exceedingly difficult to probe the thermodynamic limit physics and ED calculations suffer from finite size effects.

Closer investigation of the Hamiltonian matrix, expressed in computational basis<sup>1</sup> quickly reveals the inefficiency of dense storage. Looking at Figure (Here figure with Hamiltonian, basis ordered by magnetization), we see that most of the matrix elements are zero. In fact only about  $\mu \propto \mathcal{D}$  out of  $\mathcal{D}^2$  matrix elements are non-zero. Hence, a numerical scheme leveraging this sparsity is highly desirable. This is exactly what the Krylov subspace algorithms do, by the virtue of requiring only a "black box" computation of matrix-vector product, which can be fairly easily implemented in a way requiring only  $O(\mu\mathcal{D})$  operations.

## 2.2 Calculation of extremal eigenvalues

Our goal in this section is to develop the Lanczos algorithm for ground state search of hermitian matrices, and along the way understand how and why it works.

### 2.2.1 Arnoldi iteration

The Lanczos algorithm is special case of a more general algorithm, called Arnoldi iteration, designed to transform a general, nonhermitian matrix  $A \in \mathbb{C}^{m \times m}$  via a orthogonal<sup>2</sup> similarity transformation to a Hessenberg form  $A = QHQ^\dagger$ . Such transformation always exist [12].

<sup>1</sup>For spin systems, it is the eigenbasis of  $S^z$  operator.

<sup>2</sup>Orthogonal in this context means that  $Q^\dagger Q = I_{m \times m}$

**Definition 2.2** A square,  $m \times m$  matrix  $H$  is said to be in upper **Hessenberg** form if  $\forall i, j \in \{1, \dots, n\} : i > j + 1 \implies (A)_{i,j} = 0$ . It is said to be in **lower Hessenberg form**, if its transpose is in upper Hessenberg form.

A Hessenberg matrix differs from a triangular one by one additional super- or subdiagonal. Such form is desirable, because many numerical algorithms in linear algebra experience considerable speedup from leveraging triangular structure of a matrix, and sometimes those benefits carry over to this almost-triangular case. A particularly important strength of the Arnoldi iteration is that it can be interrupted before completion (cf. fig 32.1), thus producing only an approximation of the Hessenberg form in situation where  $m$  is so large, that full computations are infeasible (eg. in quantum many-body physics).

Assume now that we are able to only compute the first  $n < m$  columns of the equation  $AQ = QH$ . Let  $Q_n$  be the restriction of  $Q$  to  $n$  columns and let them be denoted by  $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n \in \mathbb{C}^m$ . Denoting by  $\tilde{H}_n$  the  $(n+1) \times n$  upper left section of  $H$ , which is also a Hessenberg matrix, we can write down the following  $n$ -step approximation to the full decomposition

$$AQ_n = Q_{n+1}\tilde{H}_n \quad (2.1)$$

From this equation we can deduce an  $n+1$  term recurrence relation for the column  $\mathbf{q}_{n+1}$ , however it is perhaps best illustrated with a simple example in the first place.

**Example 2.1** Let  $A \in \mathbb{C}^{3 \times 3}$ ,  $AQ = QH$  be the Hessenberg decomposition and corresponding matrix elements be denoted by lowercase letters. We consider the approximation for  $n = 2$ , i.e.  $AQ_2 = Q_3\tilde{H}_2$ . On the right hand side

$$\begin{aligned} AQ_2 &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix} = \begin{bmatrix} a_{11}q_{11} + a_{12}q_{21} + a_{13}q_{31} & a_{11}q_{12} + a_{12}q_{22} + a_{13}q_{32} \\ a_{21}q_{11} + a_{22}q_{21} + a_{23}q_{31} & a_{21}q_{12} + a_{22}q_{22} + a_{23}q_{32} \\ a_{31}q_{11} + a_{32}q_{21} + a_{33}q_{31} & a_{31}q_{12} + a_{32}q_{22} + a_{33}q_{32} \end{bmatrix} \\ &= \begin{bmatrix} (A\mathbf{q}_1)_1 & (A\mathbf{q}_2)_1 \\ (A\mathbf{q}_1)_2 & (A\mathbf{q}_2)_2 \\ (A\mathbf{q}_1)_3 & (A\mathbf{q}_2)_3 \end{bmatrix} \end{aligned}$$

On the left hand side

$$\begin{aligned} Q_3H_2 &= \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ 0 & h_{32} \end{bmatrix} = \begin{bmatrix} q_{11}h_{11} + q_{12}h_{21} & q_{11}h_{12} + q_{12}h_{22} + q_{13}h_{32} \\ q_{21}h_{11} + q_{22}h_{21} & q_{21}h_{12} + q_{22}h_{22} + q_{23}h_{32} \\ q_{31}h_{11} + q_{32}h_{21} & q_{31}h_{12} + q_{32}h_{22} + q_{33}h_{32} \end{bmatrix} \\ &= \begin{bmatrix} h_{11}(\mathbf{q}_1)_1 + h_{21}(\mathbf{q}_2)_1 & h_{12}(\mathbf{q}_1)_1 + h_{22}(\mathbf{q}_2)_1 + h_{32}(\mathbf{q}_3)_1 \\ h_{11}(\mathbf{q}_1)_2 + h_{21}(\mathbf{q}_2)_2 & h_{12}(\mathbf{q}_1)_2 + h_{22}(\mathbf{q}_2)_2 + h_{32}(\mathbf{q}_3)_2 \\ h_{11}(\mathbf{q}_1)_3 + h_{21}(\mathbf{q}_2)_3 & h_{12}(\mathbf{q}_1)_3 + h_{22}(\mathbf{q}_2)_3 + h_{32}(\mathbf{q}_3)_3 \end{bmatrix} \end{aligned}$$

From the above calculation and 2.1 we can read off two identities

$$\begin{aligned} A\mathbf{q}_1 &= h_{11}\mathbf{q}_1 + h_{21}\mathbf{q}_2 \\ A\mathbf{q}_2 &= h_{21}\mathbf{q}_1 + h_{22}\mathbf{q}_2 + h_{32}\mathbf{q}_3 \end{aligned}$$



Therefore we get, assuming  $\mathbf{q}_1$  is known,

$$\begin{aligned}\mathbf{q}_2 &= \frac{A\mathbf{q}_1 - h_{11}\mathbf{q}_1}{h_{21}} \\ \mathbf{q}_3 &= \frac{A\mathbf{q}_2 - h_{21}\mathbf{q}_1 - h_{22}\mathbf{q}_2}{h_{32}}\end{aligned}$$

Generalizing the above example, we arrive at the desired  $n + 1$  term recurrence relation for  $\mathbf{q}_{n+1}$

$$\mathbf{q}_{n+1} = \frac{A\mathbf{q}_n - \sum_{m=1}^n h_{mn}\mathbf{q}_m}{h_{n+1,n}} \quad (2.2)$$

We can now easily cast the above recurrence into a pseudocode algorithm: Step 9 of the

---

**Algorithm 1** Arnoldi iteration

---

**Input:**  $v \in \mathbb{C}^m$ ,  $A \in \mathbb{C}^{m \times m}$ , number of steps  $n$

**Output:** columns of  $Q_n$

```

1:  $\mathbf{q}_1 = \mathbf{v}/\|\mathbf{v}\|$ 
2: for  $i = 1 : n - 1$  do
3:    $\mathbf{q} = A\mathbf{q}_i$ 
4:   for  $j = 1 : i$  do
5:      $h_{ji} = \text{cdot}(\mathbf{q}_j, \mathbf{q})$   $\triangleright$  cdot is the complex dot product on  $\mathbb{C}^m$ .
6:      $\mathbf{q} = \mathbf{q} - h_{ji}\mathbf{q}_j$   $\triangleright$  In exact arithmetic, this ensures orthogonality.
7:   end for
8:    $h_{i+1,i} = \|\mathbf{q}\|$ 
9:    $\mathbf{q}_{i+1} = \mathbf{q}/h_{i+1,i}$ 
10: end for
```

---

Algorithm 1 may be questionable, as we are dividing by a norm of a vector, which after all can be equal to zero. However, in practical applications of Arnoldi iteration it usually means that our calculations have converged and the iterations may be stopped.

Examining closely the Arnoldi iteration algorithm, we notice that it is essentially the Gram-Schmidt procedure applied to the vectors  $\{\mathbf{v}, A\mathbf{v}, \dots, A^{n-1}\mathbf{v}\}$  and hence the vectors  $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$  form an orthonormal basis of the Krylov subspace  $\mathcal{K}_n$ . The orthonormality condition is concisely expressed by the fact that  $Q_n^\dagger Q_{n+1}$  is the  $n \times (n + 1)$  identity matrix. Multiplying the left-hand side of equation (2.1) by  $Q_n^\dagger$  we get

$$Q_n^\dagger A Q_n = \underbrace{Q_n^\dagger Q_{n+1}}_{\text{Id}_{n \times (n+1)}} \tilde{H}_n = H_n \in \mathbb{C}^{n \times n} \quad (2.3)$$

where  $H_n$  is the Hessenberg matrix  $\tilde{H}_n$  with its last row removed.

To understand the meaning of matrix  $H_n$  from the point of view of linear algebra, consider the following reasoning. Imagine we are given an endomorphism of the space  $\mathbb{C}^m$ , represented in the standard basis by a matrix  $A$ . We would like to restrict it to an endomorphism of the Krylov subspace  $\mathcal{K}_n$ ,  $n < m$ . Of course, as  $\mathbf{q} \in \mathcal{K}_n \implies \mathbf{q} \in \mathbb{C}^m$ , we can calculate the action of  $A$  on a vector from Krylov subspace in a straightforward way. However, the resulting vector  $A\mathbf{q}$  is not guaranteed to be an element of  $\mathcal{K}_n$ . We need to orthogonally project it back to the subspace. Such projection is realized by  $Q_n Q_n^\dagger \in \mathbb{C}^{m \times m}$  and hence, with respect to the

standard basis on  $\mathbb{C}^m$ , the desired restriction can be written as  $Q_n Q_n^\dagger A$ . Transforming it to the basis given by columns of  $Q_n$  we get  $Q_n^{-1} (Q_n Q_n^\dagger A) Q_n = Q_n^\dagger A Q_n$ . Thus, matrix  $H_n$  is the orthogonal projection of  $A$  to the subspace  $\mathcal{K}_n$ , represented in the basis  $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ .

$H_n$  is once again a square matrix, so we can talk about its eigenvalues in the usual fashion. These numbers are called the *Arnoldi eigenvalues estimates at step  $n$* , or the *Ritz values with respect to  $\mathcal{K}_n$* . Given the interpretation above, we may suspect that they would be related to the eigenvalues of the original matrix  $A$ . Indeed, as we shall see in a moment, some of the Ritz values are extremally good approximations of some of the original eigenvalues.

### 2.2.2 Polynomial approximation and extremal eigenvalues

By carrying out the Arnoldi iterations for successive steps, and at each step  $n$  (or at just some of the steps) calculating the eigenvalues of the Hessenberg matrix  $H_n$ , we are left with sequences of Ritz values. Some of them often converge rapidly to, what we reasonably assume, eigenvalues of the original matrix  $A$ . However in practice, the maximal accessible  $n$  is much smaller than  $m$ , so we cannot expect to find all eigenvalues. As it turns out, Arnoldi iteration typically finds extremal eigenvalues, which fortunately are those that we are interested in.

Before we will understand the details, let us introduce different, seemingly unrelated problem of *polynomial approximation*. We can take any  $\mathbf{q} \in \mathcal{K}_n$  and using the defining basis of Krylov subspace  $\mathcal{K}_n$  (Definition 2.1), expand it as

$$\begin{aligned} \mathbf{q} &= a_0 \mathbf{v} + a_1 A \mathbf{v} + a_2 A^2 \mathbf{v} + \dots + a_{n-1} A^{n-1} \mathbf{v} \\ &= (a_0 + a_1 A + a_2 A^2 + \dots + a_{n-1} A^{n-1}) \mathbf{v} \end{aligned}$$

Utilizing the special structure of vectors from  $\mathcal{K}_n$ , we can define a polynomial  $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}$ , and concisely write our vector as  $\mathbf{q} = p(A) \mathbf{v}$ . As the vector  $\mathbf{q}$  was arbitrary, we have established an isomorphism between the  $n$ -th Krylov subspace and the space of complex polynomials of maximal degree  $n - 1$ . We are now ready to state the problem:

#### Arnoldi Approximation Problem

Given a matrix  $A \in \mathbb{C}^{m \times m}$  and a vector  $\mathbf{v} \in \mathbb{C}^m$ , find  
 $p \in P^n := \{a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + z^n \mid a_0, a_1, \dots, a_{n-1} \in \mathbb{C}\}$   
such that  $\|p(A) \mathbf{v}\|_2$  is minimized.

Remarkably, the Arnoldi approximation is the exact solution to this problem. This fact is interesting enough that we state it here as a theorem and, following Trefethen and Bau [3], provide a complete proof.

**Theorem 2.1** *If  $\dim(\mathcal{K}_n) = n$ , i.e. matrix having columns  $\mathbf{v}, A\mathbf{v}, \dots, A^{n-1}\mathbf{v}$  is of rank  $n$ , then the Arnoldi Approximation Problem has a unique solution  $p^n \in P^n$ , given by the characteristic polynomial of the matrix  $H_n$ , defined by (2.3).*

**Proof.** We start with an observation, that given a polynomial  $p \in P^n$ , the vector  $p(A) \mathbf{v}$  can be written as  $p(A) \mathbf{v} = A^n \mathbf{v} - Q_n \mathbf{r}$  for some  $\mathbf{r} \in \mathbb{C}^n$ . ■



### 2.2.3 Restriction to hermitian case: Lanczos iteration

From this point onwards, let us switch to the bracket notation favored by physicists, namely  $|v\rangle \equiv \mathbf{v}$ .

## 2.3 Time evolution via the Krylov propagator

## 2.4 Physical interlude: Quantum Typicality

## 2.5 Correlation functions and the search for integrals of motion



## Spin transport in long range XXZ model

The results concerning spin transport in the long range XXZ have already been published in Mierzejewski et al. [\[1\]](#).



# 4

## Relaxation eigenmodes in long range XXZ model

The results concerning spin transport in the long range XXZ have already been published in Mierzejewski et al. [\[1\]](#).



# 5

## Summary

AAA



# Bibliography

- [1] M. Mierzejewski et al. “Quasiballistic transport in the long-range anisotropic Heisenberg model”. In: *Physical Review B* 107.4 (Jan. 2023), p. 045134. DOI: [10.1103/PhysRevB.107.045134](https://doi.org/10.1103/PhysRevB.107.045134). URL: <https://link.aps.org/doi/10.1103/PhysRevB.107.045134> (visited on 02/16/2023).
- [2] Valeria Simoncini and L. ELDÉN. “Krylov subspaces”. In: *The Princeton Companion to Applied Mathematics* (2015), pp. 113–114.
- [3] Lloyd N. Trefethen and David III Bau. *Numerical Linear Algebra*. Philadelphia, 1997. ISBN: 9780898713619.
- [4] Anders W. Sandvik. “Computational studies of quantum spin systems”. In: *AIP Conference Proceedings* 1297.2010 (Jan. 18, 2011), pp. 135–338. ISSN: 0094-243X. DOI: [10.1063/1.3518900](https://doi.org/10.1063/1.3518900). arXiv: [1101.3281](https://arxiv.org/abs/1101.3281).
- [5] Dawid Crivelli. “Particle and energy transport in strongly driven one-dimensional quantum systems”. PhD thesis. Uniwersytet Śląski w Katowicach, 2016.
- [6] Tae Jun Park and J. C. Light. “Unitary quantum time evolution by iterative Lanczos reduction”. In: *The Journal of Chemical Physics* 85.10 (Nov. 1986), pp. 5870–5876. ISSN: 0021-9606. DOI: [10.1063/1.451548](https://doi.org/10.1063/1.451548). URL: <https://aip.scitation.org/doi/10.1063/1.451548> (visited on 12/23/2022).
- [7] J. Gemmer and G. Mahler. “Distribution of local entropy in the Hilbert space of bi-partite quantum systems: origin of Jaynes’ principle”. en. In: *The European Physical Journal B - Condensed Matter and Complex Systems* 31.2 (Jan. 2003), pp. 249–257. ISSN: 1434-6036. DOI: [10.1140/epjb/e2003-00029-3](https://doi.org/10.1140/epjb/e2003-00029-3). URL: <https://doi.org/10.1140/epjb/e2003-00029-3> (visited on 12/23/2022).
- [8] Sheldon Goldstein et al. “Canonical Typicality”. In: *Physical Review Letters* 96.5 (Feb. 2006), p. 050403. DOI: [10.1103/PhysRevLett.96.050403](https://doi.org/10.1103/PhysRevLett.96.050403). URL: <https://link.aps.org/doi/10.1103/PhysRevLett.96.050403> (visited on 12/23/2022).
- [9] Sandu Popescu, Anthony J. Short, and Andreas Winter. “Entanglement and the foundations of statistical mechanics”. en. In: *Nature Physics* 2.11 (Nov. 2006), pp. 754–758. ISSN: 1745-2481. DOI: [10.1038/nphys444](https://doi.org/10.1038/nphys444). URL: <https://www.nature.com/articles/nphys444> (visited on 12/23/2022).
- [10] Marcin Mierzejewski, Peter Prelovšek, and Tomaž Prosen. “Identifying local and quasilo-cal conserved quantities in integrable systems”. In: *Physical Review Letters* 114.14 (Apr. 8, 2015), pp. 1–7. ISSN: 1079-7114. DOI: [10.1103/PhysRevLett.114.140601](https://doi.org/10.1103/PhysRevLett.114.140601). arXiv: [arXiv:1412.6974v2](https://arxiv.org/abs/1412.6974v2).

- [11] Alexander Weisse and Holger Fehske. “Exact Diagonalization Techniques”. en. In: ed. by H. Fehske, R. Schneider, and A. Weisse. *Lecture Notes in Physics*. Berlin, Heidelberg: Springer, 2008, pp. 529–544. ISBN: 9783540746867. DOI: [10.1007/978-3-540-74686-7\\_18](https://doi.org/10.1007/978-3-540-74686-7_18). URL: [https://doi.org/10.1007/978-3-540-74686-7\\_18](https://doi.org/10.1007/978-3-540-74686-7_18) (visited on 03/11/2023).
- [12] Stephan Ramon Garcia and Roger A. Horn. *A Second Course in Linear Algebra*. en. Google-Books-ID: nXOuDgAAQBAJ. Cambridge University Press, May 2017. ISBN: 9781107103818.





## Hilbert subspaces with fixed momentum

