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Complex Variables

Arithmetic
~~Addition~~ with complex numbers,

$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

- $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$
- $z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$
- $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
  $(\because i^2 = -1)$

(Imaginary part of $x_1 + iy_1$ is y_1 not iy_1).

- $z_1/z_2 = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$
  $= \dfrac{x_1x_2 + y_1y_2 + i(x_2y_1 - y\ldots}{x_2^2 + y_2^2}$

— × —

Ref ⇒ Complex Conjugate of $z_1 = x_1 + iy_1$
is $\bar{z}_1 = x_1 - iy_1$

$\boxed{z\bar{z} = x^2 + y^2}$

Ref ⇒ Modulus ⇒ $|z|^2 = z \cdot \bar{z}$
$\text{Mod}(z) = |z| = \sqrt{x^2 + y^2}$

— × —

Note: Depict $z_1 + z_2$ & $z_1 - z_2$ on complex planes.

— × —

Note; $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$ $\quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$
$\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

— × —

Q) $\begin{cases} z_1 = 4+3i \\ z_2 = 2-5i \end{cases}$

Find $\left(\dfrac{\overline{z_1}}{z_2}\right)$

— × —

*** Triangle Inequality;

$$\boxed{|z_1+z_2| \leq |z_1| + |z_2|}$$

Proof; $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$

$$= (z_1+z_2)\cdot(\overline{z_1}+\overline{z_2})$$

$$= z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_2}$$

$$= |z_1|^2 + |z_2|^2 + z_2\overline{z_1} + z_1\overline{z_2}$$

$$= |z_1|^2 + |z_2|^2 + 2\,Re(z_1\overline{z_2})$$

$\because z_1\overline{z_2} = \overline{(z_2\overline{z_1})}$

$$\Rightarrow |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1\overline{z_2})$$

But $2Re(z_1\overline{z_2}) \leq 2|z_1\overline{z_2}|$

$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2+|z_2|^2 + \cancel{2|z_1\bar{z_2}|} + 2|z_1\bar{z_2}|$$

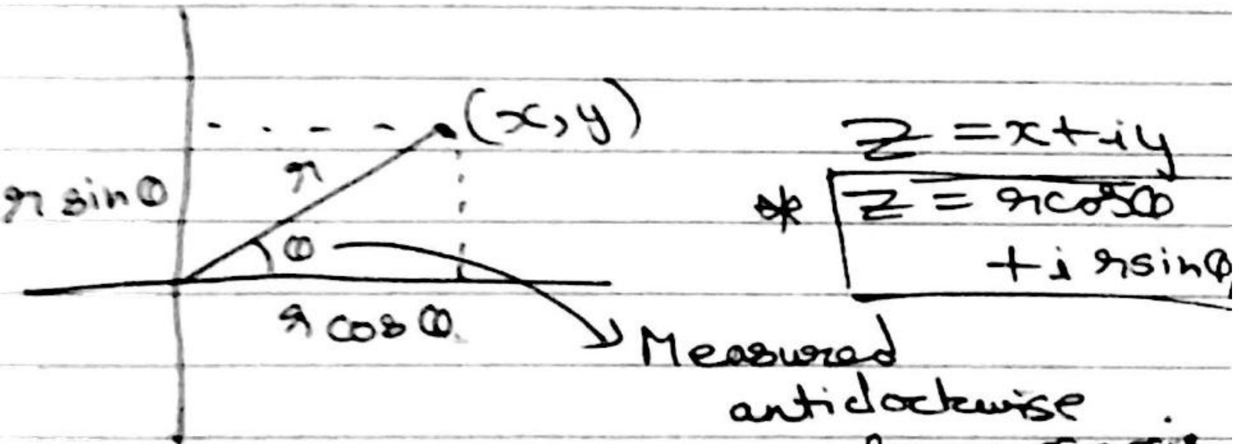
$$\le |z_1|^2+|z_2|^2 + 2|z_1||\bar{z_2}|$$

$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2+|z_2|^2 + 2|z_1||z_2|$$

$$\therefore \Rightarrow |z_1+z_2| \le |z_1|+|z_2|$$

— x —

* Polar Representation;



$z = x+iy$

$$* \boxed{z = r\cos\theta + i\,r\sin\theta}$$

- $r = \sqrt{x^2+y^2} = |z|$

- $\theta = \tan^{-1}(y/x)$

$$* \boxed{-\pi \le \theta < \pi}$$

— x —

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Q) Convert to complex polar coordinates

a) $1+i$

$r = \sqrt{1^2+1^2}$
$= \sqrt{2}$

$\theta = \tan^{-1}(1/1)$
$= \pi/4$

(Since x & y are +ve)
(1'st quadrant)

$\therefore 1+i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$

b) $-1-i$

$r = \sqrt{(-1)^2+(-1)^2}$
$= \sqrt{2}$

$\theta = \tan^{-1}(-1/-1)$
$= 5\pi/4$

Since $\pi + \frac{\pi}{4}$

(Since x & y are -ve)
(3^{rd} quadrant)

$\therefore -1-i = \sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

—— x ——

* **Multiplication in polar coordinates;**

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \cdot z_2 = r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

* $$\boxed{z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)}$$

—— x ——

Note; $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ from above

* Note: $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$
$\because (\arg(z_1) = \theta_1)$

— × —

* **Note:** Suppose $\frac{z_1}{z_2} = z$, $|z| = ?$ $\arg(z) = ?$

Proof: $z_1 = z z_2 \rightarrow$ ①

$$|z_1| = |z| \cdot |z_2|$$

$$\therefore |z| = \frac{|z_1|}{|z_2|}$$

From ①,

$$\arg(z_1) = \arg(z) + \arg(z_2)$$

$$\therefore \arg(z) = \arg(z_1) - \arg(z_2)$$

— × —

Euler's formula:

* $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\boxed{z = re^{i\theta}}$$

— × —

Q) Without binomial expansion show $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$ is real for any n

$\sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2 = r$

$\Rightarrow 2^n\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^n + 2^n\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^n$

$\rightarrow \cos\theta + i\sin\theta$, $\rightarrow \cos\theta_2 + i\sin\theta_2$

$\Rightarrow 2^n(\cos n\theta + i\sin n\theta) + 2^n(\cos n\theta_2 + i\sin n\theta_2)$

$\pi/6$ $\quad$ $-\pi/6$

— × —

** De Moivres Formula;

$\Rightarrow r^n(\cos\theta + i\sin\theta)^n = r^n(\cos(n\theta) + i\sin(n\theta))$

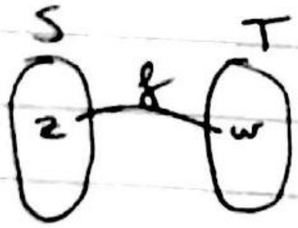
(Proof by induction)

<u>Note</u>; We can use this to prove formulas like ~~$\cos^2\theta - \sin^2\theta = \cos$~~

$\cos 2\theta = 2\cos^2\theta - 1$

— x —

* Complex Functions;



$z, w \in \mathbb{C} \rightarrow$ Complex numbers

• $w = f(z) = z^2$ (Suppose we pick this)

Complex valued function

$\Rightarrow z = x + iy$

$f(x+iy) = (x^2 - y^2) + i(2xy)$

$f(z) = u + iv$ (Here since u & v are different
f(z) is differentiable

— y —

Limit;

A function $f(z)$ is said to have a limit L as $z \to z_0$ written as

$$\lim_{z \to z_0} f(z) = L,$$

Any path from $z \neq z_0$ for this type of multivariable calculus (2D inputs)

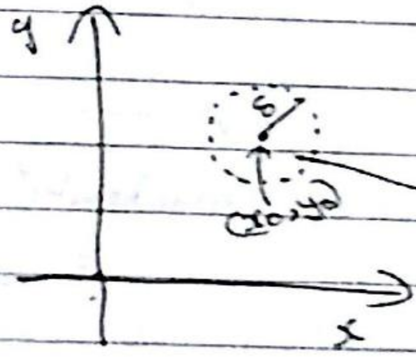
If f is defined in the neighbourhood of z_0 except possibly at z_0 and $f(z)$ is close to L for z close to z_0

~~i.e. we can find $\delta > 0 \in R$ such that $\forall z \neq z_0$~~

i.e. for every $\varepsilon > 0$, $\varepsilon \in R$, we can find a positive $\delta \in R$ such that $z \neq z_0$ in the disc $|z - z_0| < \delta$ we have $|f(z) - L| < \varepsilon$

(neighbourhood)

Here we don't have left hand & right hand limits (since we have many paths from z to z_0 unlike 1D calculus)

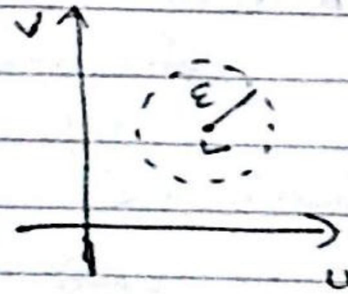


$z = x + iy$

$z_0 = x_0 + iy_0$

Open set (since $|z - z_0| \neq \delta$)

Now, $f(z) = u + iv$



(This should be true for all z, except maybe when $z = z_0$)

$\Rightarrow$ If this is true, then $f(z)$ is said to have a limit L as $z \to z_0$.

— x —

** * Continuity

A function $f(z)$ is said to be continuous at z_0 if $f(z_0)$ exists and if $\lim_{z \to z_0} f(z) = f(z_0)$

— x —

** * Differentiability;

The derivative of a complex fuction f at a point z_0 written

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\Delta z = z - z_0$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

— x —

* Q) $f(z) = z^3$ (No singularity) → Differentiable at all z

Find $f'(z)$

singularity → Point(s) where function doesn't exist

Note;(Finding derivatives from first principles means we use the definition of differentiability to get the derivative)

$$\rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^3 - z^3}{\Delta z}$$

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$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} + (\Delta z)^3 + 3z^2(\Delta z) + 3z(\Delta z)^2 - z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (\Delta z)^2 + 3z^2 + 3z(\Delta z)$$

$$= 0 + 3z^2 + 3z \times 0$$

$$= 3z^2 //$$

$$\therefore f'(z) = 3z^2$$

— x —

** Rules of Differentiability;

- $(cf)' = cf'$ (c is a constant)
- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $(f/g)' = \dfrac{gf' - fg'}{g^2}$ //

— x —

*** 1) $f(z) = \bar{z}$, This is not differentiable

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\bar{z}} + \overline{\Delta z} - \cancel{\bar{z}}}{\Delta z}$$

(Now, it gets ambiguous)

$$= \boxed{\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Does not exist

Now, we try pick 2 paths to z_0 from z.

$\Rightarrow$ Suppose we pick $\Delta x = 0$, (Come along y axis)

$$\Rightarrow \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

$\Rightarrow$ Suppose we pick $\Delta y = 0$ (Come along x axis)

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

(So Limits coming from 2 different paths don't match, so Limit doesnt exist)

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$\Rightarrow$ We can also pick along $y = x$ axis

$\Delta x = \Delta y$

$$\lim_{\Delta x \to 0} \frac{\Delta x - i\Delta x}{\Delta x + i\Delta x} = \frac{1-i}{1+i}$$ (This is again different)

$\therefore$ Since the Limit doesnt exist, $f(z) = \overline{z}$ is not differentiable.

— x —

Q) Show that $f(z) = \mathrm{Re}\, z = x$ is not differentiable at any z.

Q) Show that $f(z) = z^2$ is differentiable only at $z = 0$.

(Holomorphic functions)

Analytic Functions:

If D is entire domain, then f is entire function

- A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and must be differentiable at all points of D. (Open region domain)

—x—

- A function $f(z)$ is said to be ~~differentiable~~ analytic at $z = z_0$ if it is analytic in a neighbourhood of z_0.

—x—

Note: Polynomial functions are analytic

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n$$

$\forall z \in \mathbb{C} \rightarrow$ Complex numbers

Note: Rational functions

$$f(z) = \frac{h(z)}{g(z)}$$ is analytic if

h, g are analytic and $g(z) \neq 0$.

—x—

Cauchy Rieman equation:

Theorem: (Necessary condition) Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of point $z = x + iy$ and is differentiable at the point z. Then the point z, the first-order partial derivative of u and v exists and satisfy the C.R equations ~~$u_x = v_y$ and $u_y = -v_x$~~

$\boxed{u_x = V_y}$, $\boxed{u_y = -V_x}$

—— × ——

* (What is partial derivative)
* ex; $h(x,y) = x^2 + yx^3 + y^2$

$\Rightarrow$ a) Partial derivative wrt x,

$$\frac{\delta h}{\delta x} = h_x = 2x + y \times 3x^2 + 0$$

b) Partial derivative wrt y

$$\frac{\delta h}{\delta y} = h_y = x^3 + 2y$$

—— × ——

** : If $f(z)$ is analytic in a domain D

** then, partial derivatives (u_x, u_y, v_x, v_y) exist and satisfy C.R equations.

—— × ——

ex; $f(z) = \bar{z} = x - iy$

$\Rightarrow$ $u = x, v = -y$

$\Rightarrow u_x = 1$

$u_y = 0$

$v_x = 0$

$v_y = -1$

$\Rightarrow u_x \neq v_y$ so it fails.

(So $f(z)$ is not differentiable)

$\Rightarrow$ And not analytic. at any (x,y)

—— × ——

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Theorem 2: (Sufficient Condition) *** If two real valued functions $u(x,y)$ and $v(x,y)$ of two real variables x,y have continuous first partials (i.e. derivatives) and that satisfy C.R equations in some domain D, then $f(z)$ is analytic in D.

— x —

ex: $f(z) = z^3 = (x+iy)^3$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

//

$$\left.\begin{aligned} u_x &= 3x^2 - 3y^2 \\ u_y &= -6xy \\ v_x &= 6xy \\ v_y &= 3x^2 - 3y^2 \end{aligned}\right\}$$

u,v have first partials ~~that are~~ continuous

For continuous just check division by 0 and negative roots

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

$\therefore f(z)$ is analytic from theorem 2 of C-R equations

— x —

Note; Polynomial real valued functions are continuous and all their partial derivatives exist and are continuous.

— x —

C-R equations in polar form:

$x = r\cos\theta$
$y = r\sin\theta$
$z = x + iy = r\cos\theta + ir\sin\theta$

$f(z) = u(r,\theta) + iv(r,\theta)$
is analytic, then CR equation look like so,

$\Rightarrow u_r = \frac{1}{r} v_\theta$

$\Rightarrow v_r = -\frac{1}{r} \times u_\theta$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \times \frac{\partial \theta}{\partial x}$

$\Rightarrow u_x = u_r \frac{\partial r}{\partial x} + u_\theta \frac{\partial \theta}{\partial x} \longrightarrow (1)$

$\Rightarrow x = r\cos\theta$, $\therefore \frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial r}{\partial x} = \frac{1}{\cos\theta} = \sec\theta$

$\Rightarrow x = r\cos\theta$ $\therefore \frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial \theta}{\partial x} = \frac{-1}{r\sin\theta} = -\frac{1}{r}\operatorname{cosec}\theta$

Substituting in (1),

$u_x = u_r \sec\theta - \frac{u_\theta}{r}\operatorname{cosec}\theta$

$$\Rightarrow \underline{\underline{V_y}} = \frac{\partial v(r,\theta)}{\partial y} = \frac{\partial v}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\times\frac{\partial \theta}{\partial y} \quad \rightarrow ②$$

$\Rightarrow$ [redacted]

$\Rightarrow y = r\sin\theta$

$$\Rightarrow \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore V_y = V_r \times \operatorname{cosec}\theta + \frac{V_\theta}{r}\sec\theta$$

$\Rightarrow$ Now we know $u_x = V_y$ (From CR equations)

$\therefore \leftarrow$ [redacted]

$$\Rightarrow \underline{\underline{u_y}} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\times\frac{\partial \theta}{\partial y} \quad \rightarrow ③$$

$\Rightarrow y = r\sin\theta$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\Rightarrow u_y = u_r \operatorname{cosec}\theta + \frac{u_\theta}{r}\sec\theta$$

$$\Rightarrow V_x = \frac{\partial v}{\partial r}\times\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\times\frac{\partial \theta}{\partial x} \quad \rightarrow ④$$

$$V_x = V_r \times \sec\theta + V_\theta \times \left(\frac{-1}{r}\right)\operatorname{cosec}\theta$$

Now, using ① to ④,

$u_x = v_y$ & $u_y = -v_x$

$$u_r \sec\theta - \frac{u_\theta}{r} \operatorname{cosec}\theta = v_r \operatorname{cosec}\theta + \frac{v_\theta}{r} \sec\theta \rightarrow (1)$$

&

$$u_r \operatorname{cosec}\theta + \frac{u_\theta}{r} \sec\theta = -v_r \sec\theta + \frac{v_\theta}{r} \operatorname{cosec}\theta \rightarrow (2)$$

$\Rightarrow$ ~~(1) × cosec θ~~

~~(1) × sec θ ± (2) × cosec θ~~

$\Rightarrow \cancel{u_r \sec^2\theta - \frac{u_\theta}{r} \operatorname{cosec}\theta \sec\theta = v_r \operatorname{cosec}\theta \sec\theta +}$

$$\Rightarrow (1) \Rightarrow \sec\theta\left(u_r - \frac{v_\theta}{r}\right) = \operatorname{cosec}\theta\left(v_r + \frac{u_\theta}{r}\right)$$

$$\Rightarrow (2) \Rightarrow \sec\theta\left(\frac{u_\theta}{r} + v_r\right) = \operatorname{cosec}\theta\left(\frac{v_\theta}{r} - u_r\right)$$

Dividing;

$$\frac{u_r - \frac{v_\theta}{r}}{\frac{u_\theta}{r} + v_r} = \frac{v_r + \frac{u_\theta}{r}}{\frac{v_\theta}{r} - u_r}$$

$$\Rightarrow -\left(u_r - \frac{v_\theta}{r}\right)^2 = \left(v_r + \frac{u_\theta}{r}\right)^2$$

$$\Rightarrow \left(v_r + \frac{u_\theta}{r}\right)^2 + \left(u_r - \frac{v_\theta}{r}\right)^2 = 0$$

(Only when both are 0

$\therefore\ v_r = -\frac{u_\phi}{r}, \quad u_r = \frac{v_\phi}{r}$

—— x ——

D.I.Y

Q) Check for analyticity

1) $f(z) = 1/z^5$ & 2) $f(z) = z + \frac{1}{z}$

—— x ——

Laplace Equations;

$\Delta \to$ Laplace operator

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$$ → For 2 variables

→ Proof can be done with CR equations & first derivatives

$\Rightarrow$ For $u(x,y)$ → Some function that satisfies Laplace

$\Rightarrow u_{xx} + u_{yy} = 0$ or $\boxed{\Delta u = 0}$

—— x ——

Theorem 3; If $f(z) = u(x,y) + i\,v(x,y)$ is an analytic in domain D, then u & v satisfy laplace equations and has continuous second partial derivatives.

Proof ✓ via CR equations & derivatives

—— x ——

Defn: $\Rightarrow$ Solutions of Laplace equations having continuous II-order partial derivatives are called harmonic functions

Note: If u, v are harmonic, then they satisfy C-R equations.

$\Rightarrow$ Since u & v satisfy C-R equations they form the real part and imaginary part of an analytic complex function

$\Rightarrow f(z) = u + iv$

** $\Rightarrow$ v is said to be a <u>conjugate harmonic function</u> of u.

—x—

* Q) Let $u(x,y) = x^2 - y^2$
Verify u is harmonic, find conjugate harmonic. (!Note for finding conjugate harmonic, prove u to be harmonic first!)

$$\Rightarrow \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$\therefore \Delta u = 0$, u_{xx} & u_{yy} are continuous (Polynomial)

$\therefore$ u is harmonic

Now $u_x = v_y$ & $u_y = -v_x$

$\therefore v_y = 2x$ & $v_x = 2y$

$\therefore \ V = ?$ $(v_y = 2x$ & $v_x = 2y)$

$\Rightarrow \frac{\partial v}{\partial y} = 2x$, Integrate partially wrt y

$\Rightarrow V = 2xy + g(x) \rightarrow$ Constant of integration $\rightarrow (1)$

$\Rightarrow \frac{\partial V}{\partial x} = 2y$, Integrate partially wrt x

$\rightarrow V = 2xy +$ [illegible] $h(y) \rightarrow$ Constant of integration $\rightarrow (2)$

$\therefore$ Now, $\cancel{2xy} + g(x) = \cancel{2xy} + h(y)$

$\therefore g(x) = h(y)$

So $g(x) = h(y) = \underset{\text{Constant}}{c}$

$\therefore V = 2xy + c \rightarrow$ Conjugate harmonic

— x —

Q) $u(x,y) = x^2 - y^2 - y$

$\Rightarrow \Delta u = 0$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y - 1$

Now $u_x = v_y$ & $u_y = -v_x$

$\therefore V_y = 2x$ & $V_x = 2y + 1$

Integ wrt y	Integ wrt x
$V = 2xy + g(x)$	$V = 2xy + x + h(y)$

$\Rightarrow \cancel{2xy} + g(x) = \cancel{2xy} + x + h(y)$

$\Rightarrow g(x) = x + h(y)$

$\therefore$ we can see $g(x) = x + c$

$h(y) = c \rightarrow$ since no y terms

$\therefore v = 2xy + x + c$ } Conjugate harmonic

— x —

Q) $u(x,y) = x^3 - 3xy^2$

① Verify its harmonic
② Conjugate with C.R eqns

$u_x = 3x^2 - 3y^2$
$u_{xx} = 6x$
$u_y = -6xy$
$u_{yy} = -6x$

$\Delta u = 0$

$\Rightarrow v_y = 3x^2 - 3y^2, \quad v_x = 6xy$

$\Rightarrow \cancel{3x^2y} - y^3 + g(x) = \cancel{3x^2y} + h(y)$

$\Rightarrow h(y) = -y^3 + c$

$g(x) = c$

$\therefore V = 3x^2y - y^3 + c$

— x —

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DIY

* Q) Let $f(z)$ be an analytic in domain D
If $|f(z)| = K$ then $f(z)$ is a constant

—— x ——

Proof: Let $f(z) = u(x,y) + iv(x,y)$

$|f(z)|^2 = u^2 + v^2 = k^2$ (Given constant)

Case i) When $k = 0$

$\Rightarrow u = v = 0, \therefore f(z) \equiv 0$ } Constant

Case ii) When ~~k=0~~ $k \neq 0$

Since f is analytic,

$u_x = v_y$ & $u_y = -v_x$

$\Rightarrow$ Take $u^2 + v^2 = k^2$, Diff partially

$2uu_x + 2vv_x = 0$

$2uu_y + 2vv_y = 0$

$\Rightarrow 2uu_x - 2vu_y = 0$

$2uu_y + 2vu_x = 0$

$\Rightarrow uu_x = vu_y$ & $uu_y = -vu_x$

$\Rightarrow \frac{u}{-v} = \frac{v}{u} \Rightarrow u^2 = -v^2$

$\Rightarrow u^2 + v^2 = 0$

$\Rightarrow$ ~~Only when $u = 0$ & $v = 0$~~

$\therefore$ They are ?

—— x ——

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* Standard Derivatives;
(of Some functions)

1) $n \neq 0$, $\frac{d}{dz}(az^n) = anz^{n-1}$

2) $\frac{d}{dz}\sin z = \cos z$

3) $\frac{d}{dz}\cos z = -\sin z$

4) ~~de~~ $\tan z = \frac{\sin z}{\cos z}$,

5) $\frac{d}{dz}e^z = e^z$

— x —

*

Defn: Entire function;

*

- A function $f(z)$ is entire if it is analytic $\forall\, z \in \mathbb{C}$.

ex; e^z or Polynomial functions

— x —

Complex Integration;

(Also called Line Integrals;)

- $f(z)$ is integrated over a curve C in the complex plane.

⇒ If curve C is parametrized by $z(t) = x(t) + iy(t)$ → t is some parameter.
then direction of curve is direction of increasing t.

⇒ C has to be a smooth curve, i.e. C has continuous and non-zero derivatives ~~at every point~~ $\dot{z} = dz/dt$ at every point $z \in C$

⇒ Tangent is unique and is continuously turning along C.

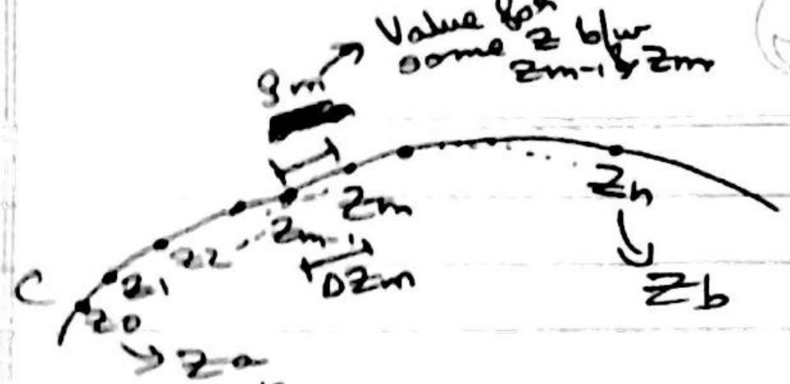
—x—

Note; $\int_C f(z)\,dz$ where C is an open curve

$\oint_C f(z)\,dz$ where C is a closed curve

—x—

⇒ Representing C parametrically
$z(t) = x(t) + iy(t)$
$a \le t \le b$



$$\Rightarrow S_n = \sum_{i=1}^{n} f(\rho_i)\cdot \Delta z_i$$

$$n = 2, 3, \ldots \infty$$

- Limit of sequence $\{S_n\}_{n=1}^{\infty}$ gives us the integral over C.

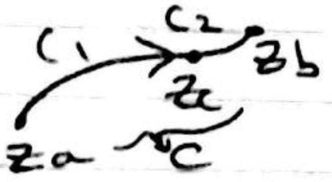
—— x ——

** Properties;

1) Linearity;

$$\int_C [A f(z) + B g(z)]\, dz = A\int_C f(z)\,dz + B\int_C g(z)\,dz$$

2) $$\int_{z_a}^{z_b} f(z)\,dz = -\int_{z_b}^{z_a} f(z)\,dz$$

3) 

} z_a to z_b is C

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

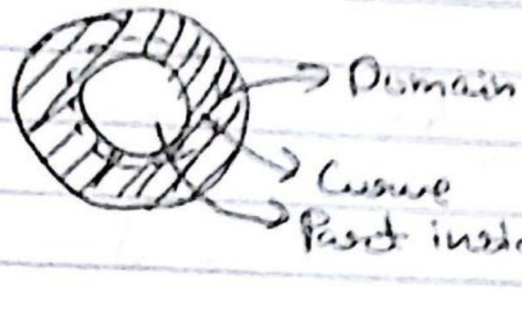
—— x ——

* Simply Connected Domain:

- If every simple closed curve (a curve which does not intersect itself) in D encloses only points in D

  & doesn't touch itself either

ex: of not simply connected domain is an annulus.



—— , ——

** (Weak Theorem) → Not Path Dependent

* **Theorem 1**: Let $f(z)$ be an analytic in a simple connected domain D, then there exists an indefinite integral of $f(z)$ in D.

i.e. there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D for all paths joining z_0 to z_1 in D.

We have,

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

—— , ——

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Note; D – simply connected
f – analytic
C – Simply closed curve in D

$$\oint_C f(z)\,dz = 0$$

— x —

**
*Q) Find $\int_C \frac{1}{z}\,dz$ where C is the simply closed curve unit circle centered at origin.

Let $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi} e^{-i\theta} \times ie^{i\theta}\,d\theta = i\theta\Big]_0^{2\pi} = 2\pi i$$

(z = 0 is excluded from our domain D, so our domain is not simply connected, so our integral's value is not 0) although C is a simply closed curve

— x —

Q) Evaluate $\displaystyle\int_0^{1-i} z^3\,dz = \frac{z^4}{4}\Big]_0^{1-i}$

$$= \frac{(1-i)^4}{4} = \frac{\sqrt{2}^4 e^{-i\pi/4 \times 4}}{4}$$

$$= -1$$

— x —

~~Or we can convert $z = re^{i\theta}$~~
$\cancel{\Rightarrow \theta = e^{i\theta}}$

$\cancel{\Rightarrow 1 - i = e^{-i\pi/4}}$

—— x ——

* **Note:** In a simply connected domain, integrals dont depend on path but only endpoints

—— x ——

** **Use of a path:**

1) Circle: $x^2 + y^2 = 1$
$\Rightarrow$ Parametrize it,

$\Rightarrow$ For increasing values of θ, we should traverse in anticlockwise
$(\cos\theta, \sin\theta)$

$\to$ Clockwise, (As $\theta\uparrow$, we traverse clockwise)
here ~~$(\cos\theta, \sin\theta)$~~
$(\cos(-\theta), \sin(-\theta))$

—— x ——

*** $\nearrow$ Lot stronger theorem. (Path Dependant) (Need not be analytic etc)

Theorem 2: Let C be a piecewise smooth curve represented by $z(t)$, $a \le t \le b$. Let $f(z)$ be continuous on C then
***

$\nearrow$ Draw curve without lift

$$\int_C f(z)\,dz = \int_a^b f(z(t)) \times \dot{z}(t)\,dt \qquad \Bigg| \qquad \dot{z}(t) = \frac{d}{dt} z(t)$$

Proof: $z = x + iy$

$dz = dx + i\,dy$

$z(t) = x(t) + iy(t)$

$\dot{z}(t) = \dfrac{d(x(t))}{dt} + i\dfrac{d(y(t))}{dt} = \dot{x}(t) + i\dot{y}(t)$

LHS

Let $f(z) = u + iv$

$\Rightarrow \int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$

$= \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$

RHS

$\int_a^b f(z(t)) \times \dot{z}(t)\,dt$

$= \int_C f(z(t))\,(\dot{x}(t) + i\dot{y}(t))\,dt$ [we try remove the parame

$\Rightarrow \dot{x}(t)dt = dx \quad , \dot{y}(t)dt = dy$

$\Rightarrow \int_C (u+iv)(dx + i\,dy)$

$= \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$

$\therefore$ LHS = RHS

Q) Evaluate $\int_C \bar{z} \cdot dz$ from $z=0$ to $z=4+2i$

along C given by

(Integrant $\bar{z}$ is not analytic so we cannot use them!)

a) $z = t^2 + it$ (Parabolic)

b) Line from $z=0$ to $z=2i$ and then line from $z=2i$ to $4+2i$.

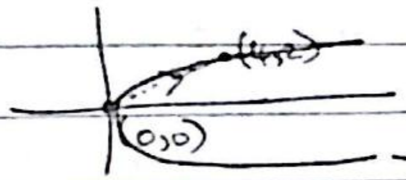
(Path Dependant)

C.R equation to check

a) $\Rightarrow z = t^2 + it$

$\Rightarrow x = t^2, \; y = t$

$x = y^2$



$\rightarrow z$ is a parabola

(Now we need to find limits in terms of t)

$\rightarrow t^2 = 0, \; t_1 = 0$

$t^2 = 4 \; \& \; t = 2, \; t_2 = 2$

$$\therefore \int_C \bar{z} \cdot dz = \int_0^2 (t^2 - it) \cdot (2t + i)\, dt$$

$$= \int_0^2 (2t^3 + it^2 - 2it^2 + t)\, dt$$

$$= \int_0^2 (2t^3 - it^2 + t)\, dt$$

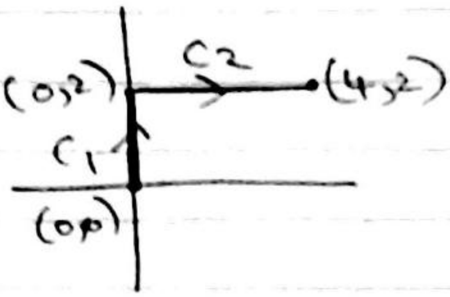
$$= \left. \frac{2t^4}{4} - \frac{it^3}{3} + \frac{t^2}{2} \right]_0^2$$

$$= 8 - 8i/3 + 2 = 10 - 8i/3$$

C_1

b) From 0 to $2i$, $(0, t)$ $\rightarrow z = 0 + it$

~~t=0 to t=~~ $t = 0$ to $t = 2$



$$\int_C f(z)\cdot dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

For C_2,

$2i$ to $4+2i$ $\quad (t, 2)$ $\rightarrow z = t + 2i$

$t = 0$ to $t = 4$

$$\therefore \int_{C_1} f(z)\,dz = \int_0^2 (-it) \times i \, dt$$

$$= \left. t^2/2 \right]_0^2 = 2$$

$$\int_{C_2} f(z)\,dz = \int_0^4 (t - 2i)\,dt = \left. \frac{t^2}{2} - 2ti \right]_0^4 = 8 - 8i$$

$$\therefore \int_C f(z)\,dz = 8 + 2 - 8i = 10 - 8i$$

$\therefore$ We can see that the path does matter.

—— x ——

DIY

Q) Prove that if $f(z)$ is integrable along a curve c having finite length L and if there exists a positive numbers M such that $|f(z)| \leq M$ on C $\left|\int_C f(z)\,dz\right| \leq M \cdot L$ //

~~Q) Sketch cu~~

DIY —— x ——

Q) Sketch curves

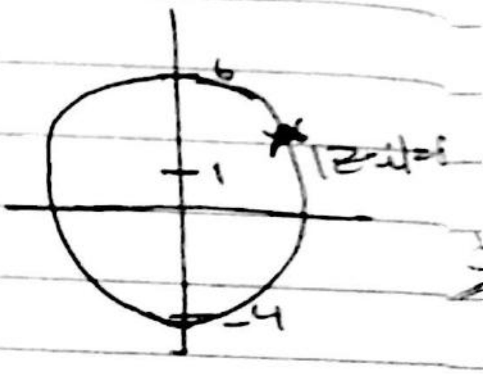
a) $t + 3t^4 i$ $\quad -1 \leq t \leq 1$

b) $t + \frac{4}{t} i$ $\quad 1 \leq t \leq 4$

Q) Evaluate integrals & state which theorem you use.

* DIY
** 1.) $\int_C \left[\frac{3}{z-i} - \frac{1}{(z-i)^2}\right] dz$ where C is the ~~unit~~ circle $|z-i|=5$

(D = Inside circle) Singularity at $z=i$, so the ~~contour~~ domain is not simply connected domain
(Integrand not analytic in entire D)
∴ Use theorm 2//



$|z-i|=5$

$x^2+(y-1)^2=25$, $z(\theta) = 5\cos\theta + i + 5i\sin$

$x(\theta) = 5\cos\theta$, $y(\theta) = 1+5\sin\theta$

$0 \le \theta \le 2\pi$ (Counter clockwise

(If you want ~~anti~~clock, then
$x(t) = 5(\cos(-\theta))$ & $y(t) = 1+5\sin$

— x —

Not analytic at $z=0$

(Independence of Path)

Thm: **Cauchy's Integral Theorem:**

If $f(z)$ is analytic in a simply connected domain D. Then for every simple closed path C in D.

$$\oint_C f(z)\,dz = 0$$

(Analyticity is a sufficient condition but not necessary) $(a \to b \;\&\; b \not\to a)$

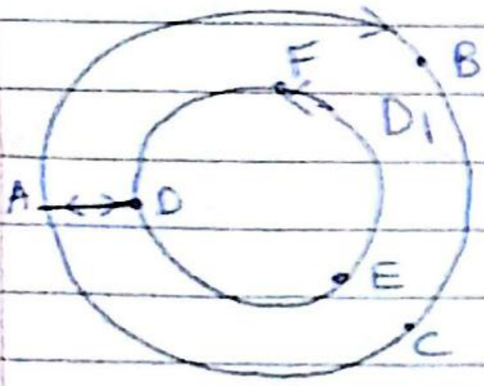
$(a \to b)$

— x —

Multiply Connected Domain:

~~simply connected~~

- Domains with parts removed come under this. ex: Annulus



- Let D_1 be the domain (Annulus).
- So now our curve ABCADEFDA is completely within our domain

$\therefore$ From Cauchy's Integral Theorem

$$\int_{ABCADEFDA} = 0$$

But $\displaystyle\int_{ABCADEFDA} = \int_{ABCA} + \cancel{\int_{AD}} + \int_{DEFD} + \cancel{\int_{DA}}$ $= 0$

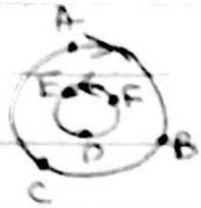
$$\therefore \oint_{ABCA} = -\oint_{DEFD}$$

(Integral of inner or outer loops is negative if opp direction)

---x---

Note;

$$\oint_{ABC} = \oint_{DFE} \quad , \quad \oint_{ABC} = -\oint_{EFD}$$



$$\oint_{ABC} = -\oint_{DFE}$$

---x---

DIY

- Get a relation b/w integrals C_1, C_2 & C_3

(f is analytic in D_1 and on curves C_1, C_2, C_3)

---x---

Cauchy's Integral Formula;

Let $f(z)$ be analytic in a simply connected domain D. Then for any point $z_0 \in D$ and any simple closed path C in D that encloses z_0.

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

Not analytic

singularity

↳ Counter clockwise.

---x---

ex; $\oint_C \frac{1}{z} dz$ (Let C be a unit circle centered at 0)

(C → counter clockwise direction)

Using Cauchy's Integral Formula

$f(z) = 1$

$z - z_0 = z, \therefore \underbrace{z_0 = 0}_{\text{Singularity}}$

$\therefore \int_C \frac{1}{z} dz = 2\pi i \times 1$

$= 2\pi i$ //

— x —

* * 6) Evaluate

$\oint_C \frac{z^2+1}{z^2-1} dz = \oint_C \frac{z^2+1}{(z+1)(z-1)} dz$

(singularities $z = 1$ & -1)

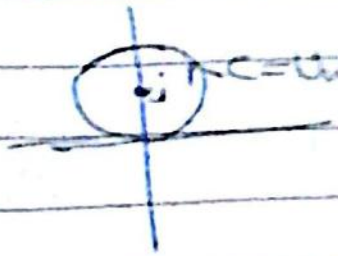
for ~~(counterclockwise)~~

a) $C \Rightarrow$ circle $|z-1| = 1$ (Only 1 singularity is there)

Using Cauchy's Integral Formula $\Rightarrow \oint_C \frac{(z^2+1)/(z+1)}{z-1} dz$

$= 2\pi i \times (2/2) = 2\pi i$ //

b)

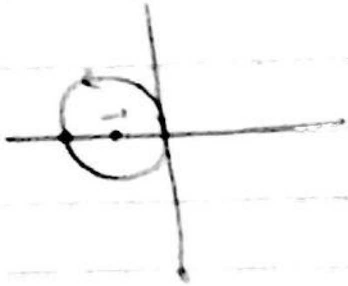


C = unit circle, center = i

$|z - i| = 1$

(No singularities within curve)

$\therefore$ Integral $= 0$ //

c) 

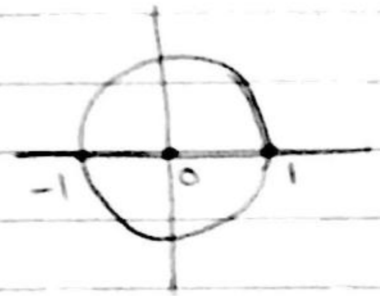
Here the singularity -1 is in our doma[in].

$$\therefore \int_C \frac{(z^2+1)/(z-1)}{z+1}$$

Cauchy's I.F

$$= 2\pi i \times (2/-2)$$

$$= -2\pi i$$

d) 

Here both singularities are present.

$$\therefore \frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

Partial Fractions

$$\therefore \int_C \frac{(z^2+1)}{2(z-1)} - \int_C \frac{z^2+1}{2(z+1)}$$

$\Rightarrow$ Using Cauchy's IF

$$\Rightarrow \int_C \frac{z^2+1}{z^2-1} = 2\pi i \times 1 - 2\pi i \times 1$$

$$= 0$$

— x —

Date
Page

** Note: Test for analyticity = CR equations
(Analytic integrations are path independent)

— x —

** Note; Derivatives of Analytic functions are also analytic.

— x —

** Derivatives of analytic functions;

Complex analytic functions have derivates of all orders.

** Thrm; If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, the value of these derivatives at z_0 in the domain D is given by,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \times \oint_c \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

n^{th} derivative

($n=0$, Cauchys integral formula)

— x —

** Thrm: Morera's Theorem; (Similar to ~~...~~ converse of cauchy's thrm with addition constraints)

If $f(z)$ is continuous in a simply connected domain D and if $\oint_c f(z)dz = 0$ for every ~~closed curve~~
closed curve/path in D then f is analytic.

— x —

* Q) $\oint_C \frac{1}{z^2} dz$, $C : |z|=1$

$\Rightarrow$ We can use the formula,

$$\oint_C \frac{1}{(z-0)^{2}} \longrightarrow f(n) \text{ is analytic}$$

$\therefore n=1$ $\qquad$ ~~$f(z)$~~ $\qquad f(n)=1$

$$\Rightarrow f'(z_0) \times \frac{2\pi i}{n!} = \oint_C \frac{1}{z^2}$$

$$\Rightarrow 2\pi i \times 0 = 0 //$$

(It does not contradict morera's theorem since $\frac{1}{z^2}$ is not analytic but '1' is analytic).

It's not continuous, so no contradiction.

---x---

* Q) Evaluate;

a) $\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)} dz$

Not analytic singularities exist

$|z|=3$ Curve.

b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$

$(\alpha = \pi z^2)$

a) $\oint_C \left(\frac{a}{(z-1)} + \frac{b}{(z-2)}\right)(\sin\alpha + \cos\alpha)\, dz$

$= \oint_C \left(\frac{1}{z-2} - \frac{1}{z-1}\right)(\sin\alpha + \cos\alpha)\, dz$

$= \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} - \oint \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}$ → Analytic (we can assert no need to prove)

$= 2\pi i \times 1 - 2\pi i \times (-1) = 4\pi i$ //

b) $\oint_C \frac{e^{2z}}{(z+1)^4}\, dz = \cancel{\frac{e^{2z}}{4.3!} \times 2\pi i}$ ~~Using the formula~~

$\cancel{= \frac{4\pi i}{3} \times e^{2z}}$

$= \frac{2\pi i}{3!} \times f^{(3)}(-1) = \frac{2\pi i}{6} \times 8 \times e^{-2}$

$e^{2z} \to$ Analytic

$\therefore f^{(3)} = 8e^{2z}$ //

$= \frac{8\pi i}{3e^2}$ //

— x —

* DIY * Let $f(z)$ be analytic in a region R bounded by two concentric circles C_1 & C_2, and on the boundary. Prove that if $z_0 \in R$, then

$$f(z_0) = \frac{1}{2\pi i} \times \oint_{C_1} \frac{f(z)}{z - z_0}\, dz - \oint_{C_2} \frac{1}{2\pi i} \times \frac{f(z)}{z - z_0}\, dz$$

(Use Cauchy's Integral Formula)

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \int_{DC} \frac{f(z)}{z-z_0}\,dz$$

$$= \oint + \left(+ \int_{C_2} \frac{f(z)}{z-z_0}\,dz \right) + \int_{CD} \frac{f(z)}{z-z_0}$$

$$f(z_0) = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \int_{C_2} \frac{f(z)}{z-z_0}\,dz$$

— x —

Page

* **Sequences;** (No summation)
(Numbers separated by commas)

$z \in \mathbb{C}$

$z_1, z_2, \ldots$ is a sequence $= \{z_n\}_{n=1}^{\infty}$

* • $\{z_n\}_{n=1}^{\infty}$ is convergent if $\lim_{n\to\infty} z_n = L$

• $\{z_n\}_{n=1}^{\infty}$ is divergent if limit does not exist.

—x—

ex; $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent to 0

$\{i^n\}_{n=1}^{\infty}$ is divergent $\Rightarrow \{i, -1, -i, 1, \ldots\}$

—x—

Thrm1: A sequence $\{z_n\}_{n=1}^{\infty} = \{x_n + iy_n\}_{n=1}^{\infty}$ converges to $L = a + ib$ iff sequence $\{x_n\}_{n=1}^{\infty}$ converges to a & $\{y_n\}_{n=1}^{\infty}$ converges to b.

—x—

* **Series;** (Generally infinite series)

$\sum_{n=1}^{\infty} z_n$, Convergence of series using sequence of partial sums

* Partial Sum; ~~(Infinite series generally)~~ (S_n)

$S_1 = z_1$

$S_2 = z_1 + z_2$

$\vdots$

$S_n = z_1 + z_2 + \cdots + z_n \rightarrow n^{th}$ partial sum

The series $\sum_{n=1}^{\infty} z_n$ converges if sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges.

* i.e. $\lim_{n\to\infty} S_n = s$ (exists) and $\sum_{n=1}^{\infty} z_n = s$.

—— x ——

Divergent Series;

- If series does not converge we say it diverges

—— x ——

* Note; $s = \sum_{n=1}^{k} z_n + R_k \rightarrow$ Remainder

$s = S_k + R_k$, $\therefore$ Error $R_k = s - S_k$

$\downarrow$

As k↑, exact

$\lim_{k\to\infty} R_k = s - s = 0$ //

—— x ——

Thm2 : Let $z_n = x_n + iy_n$
**

The series $\sum_{n=1}^{\infty} z_n$ converges (sum exists) to $u+iv$ iff $u = \sum_{n=1}^{\infty} x_n$ & $v = \sum_{n=1}^{\infty} y_n$

—— x ——

*
Thm3 : Test for divergence of series,
*

- If the series $\sum_{n=1}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$. (Not sufficient condition)

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ex: $\sum_{n=1}^{\infty} n$, $\lim_{n\to\infty} n \neq 0$,

$\therefore \sum_{n=1}^{\infty} n$ diverges

ex; $\sum_{n=1}^{\infty} \frac{2n+1}{n+1}$ diverges , $\lim_{n\to\infty} \frac{2n+1}{n+1} \neq 0$

—— x ——

**ex*, Counter example, (To Thrm 3)

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges , $\lim_{n\to\infty} \frac{1}{n} = 0$, (Harmonic Series)

- But it is a divergent series and not convergent.

—— x ——

Thrm4: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p \in R$ (p series test)

① For $p \leq 1$, series diverges.
② For $p > 1$, series converges.

—— x ——

Geometric Series;

$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots$ converges with sum $\frac{1}{1-q}$ if $|q| < 1$ and diverges if $|q| \geq 1$.

—— x ——

* ## Absolute Convergence:

* A series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges.

* • If $\sum_{n=1}^{\infty} z_n$ converges and $\sum_{n=1}^{\infty} |z_n|$ does not then we say $\sum_{n=1}^{\infty} z_n$ is conditionally convergent.

—x—

Note: Absolute convergence $\xRightarrow{\text{Implies}}$ Conditional convergence

—x—

Ex: The series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges (we know)

but $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (Theorem 4)

—x—

~~Alternating~~

—x—

*

Thm 5: Ratio Test

*

• If a series $\sum_{n=1}^{\infty} z_n$ $(z_n \neq 0, n = 1, 2, \ldots)$ is such that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$, then

a) The series converges absolutely (If $L < 1$) ($\sum_{n=1}^{\infty} |z_n|$ converges)

b) The series diverges (If $L > 1$)

c) Can't say anything with this test (If $L = 1$)

—x—

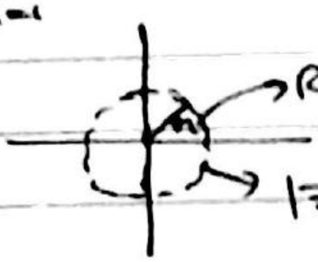
Note: **Geometric Series:**

$\sum_{n=1}^{\infty} z^n$, Use Ratio test

Consider, ~~$\lim_{n\to\infty} |z^{n+1}/z^n|$~~

$$\lim_{n\to\infty}\left|\frac{z^{n+1}}{z^n}\right| = |z| = L$$

* From the ratio test, if $L<1$ $(|z|<1)$ then $\sum_{n=1}^{\infty} z^n$ converges.



Radius of convergence $= 1$, $z_0 = 0$ ↓ Center of convergence

$|z|<1$

— x —

* Q) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$, Using ratio test,

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}\times(n!)}{(n+1)!\times z^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1}|z|$$

$$= 0$$

$\therefore$ The series is absolutely convergent.

- Radius of convergence $= \infty$.

— x —

~~Q3) Find the center and radius of~~

Q14 Find the center and radius of the following power series.

1) $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$

2) $\sum_{n=1}^{\infty} \frac{(n!)^2 i^n}{(2n)!}(z+1)^n$

3) $\sum_{n=1}^{\infty} \frac{n+5i}{(2n)!}(z-i)^n$

1) Center $= -i\sqrt{2}$, $\lim_{n\to\infty}\left|\frac{z+i\sqrt{2}}{n}\right| \times n+1 < 1$

$|z+i\sqrt{2}| < \lim_{n\to\infty} \frac{n}{n+1}$

$r = 1$

2) Center $= -1$, $\lim_{n\to\infty}\left|\frac{(z+1)(n+1)!^2 \times i^{n+1}}{(2n+2)! \times \frac{n!^2 \times \cancel{i^n}}{2n!}}\right| <$

$\Rightarrow \lim_{n\to\infty}\left|\frac{(z+1)(n+1)^2 \times i}{(2n+2)(2n+1)}\right| < 1$

$\Rightarrow |z+1| < \lim_{n\to\infty} \frac{(2\times(n+1)\times \ldots}{(n+1)(n+1} \ldots$

$\therefore r = 4$

$\cancel{\ldots} < 4$

3) Center $= i$, $\lim_{n\to\infty}\left|\frac{(z-i)\times(n+1+5i)}{(2n+2)(2n+1)(n+5i)}\right| < 1$

$\Rightarrow r = \infty$

Power Series;

$\sum_{n=1}^{\infty} a_n (z-z_0)^n$, here z_0 = Center of power series

Any function of n (complex)

If series converges for $|z-z_0|<R$ then R is radius of convergence.

—— x ——

On the boundary of disc. (Check for convergence)

ex; 1) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \frac{z \times n^2}{(n+1)^2}$

Power Series

$= \left|\frac{z n^2}{n^2+2n+1}\right|$

$= |z| \leq 1$

$\therefore |z-0|<1$ } Radius 1 disc.

We cant use ratio test for boundary ← Lets check if a point on the disk is convergent or not.

$|z|=1$,

$\Rightarrow$ ~~$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \leq \sum_{n=1}^{\infty} \left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|z|^n}{n^2}$~~

~~Here $|z|=1$, on boundary~~

~~$\therefore \sum_{n=1}^{\infty} \frac{z^n}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$~~

$\Rightarrow$ For any z on $|z|=1$, We know from absolute convergence if $\sum_{n=1}^{\infty} \left|\frac{z^n}{n^2}\right|$ converges then $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is convergent.

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$\Rightarrow$ On the boundary $|z|=1$

$$\Rightarrow \sum_{n=1}^{\infty}\left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$ which converg… from p-seri… te…

$\therefore$ Since absolute convergence is satisfied, $\sum_{n=1}^{\infty}\frac{z^n}{n^2}$ is convergent.

—— x ——

DIY: Test convergence on boundary

- $\sum_{n=1}^{\infty}\frac{z^n}{n!} \to e^z-1$ $\to$ No Boundary! (Using p-series test)

$\Rightarrow$ Ratio test $=1$, (Boundary)

$\lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 1$, ~~test = n+1~~ $\to$ No boundary

~~Absolute convergence~~ $\sum_{n=1}^{\infty}\left|\frac{z^n}{n!}\right| = \sum_{n=1}^{\infty}\frac{(n+1)^n}{n!}$

~~This~~ $\cancel{\lim_{n\to\infty}\frac{(n+1+1)^{n+1}}{n+1\times(n+1)^n}} = \cancel{\frac{(n+2)^{n+1}}{(n+1)^n}}$

$\Rightarrow$ We can see that this is ~~not~~ ~~absolutely~~ convergent,

$\therefore$ ~~Its not~~ absolutely ~~convergent on boundary, since on~~ ~~differentiation also its not converge~~

(standard form of power series) —— x ——

Note: $\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + \ldots$

At center $z=z_0$ & power series converges to a_0.

For this form of series (Power)

$$\boxed{R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$$ ~~for ratio test~~ (Using ratio test)

Radius of Convergence

—— x ——

Operations on Power Series;

- Termwise addition, subtraction & multiplication

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

— x —

* DIY: $f(z) \cdot g(z) = \sum_{n=0}^{\infty} [\ ?\ ] z^n$. Cauchy product

$$\sum_{r=0}^{n} a_r b_{n-r}$$

$$\therefore \boxed{f(z) \cdot g(z) = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} a_r b_{n-r} \right] \cdot z^n}$$

Cauchy Product

— x —

* **Differentiation**:

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$
$$= \sum_{n=1}^{\infty} n a_n z^{n-1}$$

* **Integration**;

$$\oint_C f(z)\, dz = a_0 z + \frac{a_1 z^2}{2} + \frac{a_2 z^3}{3} + \cdots\cdots$$
$$= \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

— x —

** Thrm 6: A power series with a non-zero radius of convergence R represents an analytic function at every interior point of the circle of convergence.
**

Power series of this type are analytic

ex; $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $R \neq 0$

Then by the thrm, for a function f which is a power series is analytic

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

Then $f'(z)$ also has a radius of convergence R.

— x —

* Q) Find the radius of convergence of,

$$f(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)(n!)^2} \times z^{n+1}$$ → Power series (So we can diff term by term)

*(D.I.Y → Solve with ratio test)

$$\Rightarrow f'(z) = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$$

$$\text{Radius } R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

(To solve it in standard form, we needed to differentiate).

$$= \lim_{n\to\infty} \frac{2n!}{n!\,n!} \cdot \frac{(n+1)!\cdot(n+1)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}$$

From thrm 6, we know radius of $f(z) = 1/4$

DIY : Find radius of convergence

$$f(z) = \sum_{n=0}^{\infty} \frac{2n}{n!} (z+i)^{2n-1}$$

~~(Integral will get $f(z)$ to be similar to standard~~

Use ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{2(n+1)}{(n+1)!} \times (z+i)^{2n+1}}{\frac{2n}{n!} \times (z+i)^{2n-1}} \right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{(z+i)^2}{n} < 1$$ This is 0, so always true, radius $= \infty$. ~~Radius of convergence~~

$\Rightarrow$ ~~$(z+i)^2 \le n$~~ , ~~$|z+i| \le \sqrt{n}$~~

Taylor Series ; (Only when $f(z)$ is analytic at $z = z_0$ & entire given domain)

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f(z) = a_0 + f'(z_0)(z-z_0) + \frac{1}{2!} f''(z_0)(z-z_0)^2 + \cdots$$

$\Rightarrow$ Using cauchy's integral formula of higher powers.

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{n!} \frac{n!}{2\pi i} \oint_C \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega$$

$$a_n = \frac{1}{2\pi i} \times \oint_C \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega$$

** Maclaurins Series;

- Taylor series when $z_0 = 0$

$$f(z) = a_0 + f'(z_0)z + \frac{1}{2!}f''(z_0)(z)^2 + \dots$$

— x —

* Taylor's Theorem;

Let $f(z)$ be analytic in domain D, Let $z = z_0 \in D$. Then there exists a unique taylor series with center $z = z_0$ that represents $f(z)$.

— x —

** Note; Basic Functions;

1) Geometric series: $\frac{1}{1-z} = 1 + z + z^2 + \dots$ $(|z| < 1)$

2) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ → Taylor series since e^z is an analytic function.

3) $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2n}}{(2n)!}$

4) $\sin z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$

} we can also use to prove these

5) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

6) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

Note; $\cosh z = \frac{e^z + e^{-z}}{2}$, $\sinh z = \frac{e^z - e^{-z}}{2i}$

we got by continuously differentiating

7) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

—x—

<u>Note</u>; Find radius of convergence of all the above functions.

—x—

Laurent Series:

(when there are singularities in our domain)

(Only 1 singularity in domain)

Laurent's Theorem:

- If $f(z)$ is an analytic function on two concentric circles C_1 & C_2 and in the annulus centered at z_0 and between them then $f(z)$ is,

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Positive part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part}}$$

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n} \longrightarrow (1)$$

, where $a_{-n} = b_n$ (Not a power series)

$$\text{where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* \rightarrow \text{Variable of integral.}$$

$b_n = a_{-n}$ $n = 0, \pm 1, \pm 2, \ldots$

where C is any closed path that lies in the annulus and encircles the inner circle.

- Series converges and represents $f(z)$ in the open annulus, obtained from the given annulus by continuously increasing the outer circle C_1 and decre-

C_2 until each of the two circles reach a point where $f(z)$ is singular..

** (Q) Find Laurent series about the indicated singularity,

1) $f(z) = \dfrac{e^z}{(z-1)^3}, z=1$

$$f(z) = \frac{e \cdot e^{z-1}}{(z-1)^3} = \frac{e}{(z-1)^3}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\dots\right]$$

$$= e\left[\frac{1}{(z-1)^3}+\frac{1}{(z-1)^2}+\frac{1}{2(z-1)}+\frac{1}{6} + \frac{(z-1)}{24}+\dots\right]$$

<u>Principle part</u> has 3 terms.

- $f(z)$ converges for all $z \neq 1$

<u>DIY</u>: Find the laurent series about $z=-2$ for ~~f(z)~~ $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$

$$f(z) = (z+2-5)\left(\sum_{n=0}^{\infty}(-1)^n \frac{1}{(z+2)^{2n+1}} \times \frac{1}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{1}{(z+2)^{2n}} \times \frac{1}{(2n+1)!} + \sum_{n=0}^{\infty}(-1)^n \times \frac{(-5)}{(2n+1)!} \times \frac{1}{(z+2)^{2n+1}}$$

Laurent series expansion about −2 is,

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \times \left(\frac{1}{(2n+1)!}\right)\left(\frac{1}{(z+2)^{2n}} - \frac{5}{(z+2)^{2n+1}}\right)$$

— x —

* Q) Expand $\frac{1}{1-z}$ in.

a) Non negative powers of z.
b) Negative powers of z.

a) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots\dots , |z|<1$

(Non negative powers)

Expansion about $z=0$

b) When $|z|>1$,

$$\frac{1}{1-z} = -\frac{1}{z} \times \frac{1}{\left(1-\frac{1}{z}\right)}$$

$$= \frac{-1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \dots\right.$$

$$\frac{1}{1-z} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots\dots$$

Principle part is ∞,
(Negative powers)

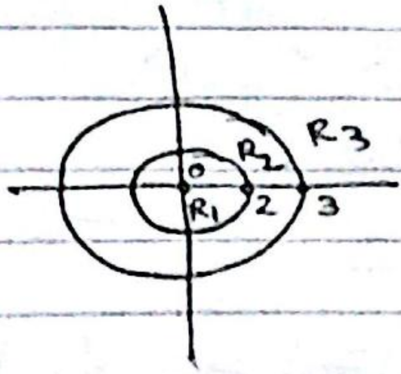
— x —

Note; Expansions are different in different regions, depending on singulari

— x —

Q) Find Laurent series of $f(z)$ about $z=0$

$$f(z) = \frac{-2z+3}{z^2-5z+6} = \frac{-2z+3}{(z-3)(z-2)}$$



- The expansions on R_1, R_2, R_3 need not be the same,

$\Rightarrow z=2, z=3$ are singularities.

$$\Rightarrow f(z) = \frac{-2z+3}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$\Rightarrow -2z+3 = Az - 3A + Bz - 2B$$

$$A+B=-2 \quad , A = -2-B$$

$$-3A-2B=3$$

$$\rightarrow 6+3B-2B=3 \quad , B=-3$$

$$A=1$$

$$\therefore f(z) = \frac{1}{z-2} - \frac{3}{z-3}$$

a) Region R_1, $|z|<2$

$$\frac{1}{z-2} = \frac{-1}{2-z} = \frac{-1}{2(1-z/2)} \qquad |z|<2 \quad \therefore |z/2|<1$$

$$\therefore \frac{1}{z-2} = -\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right)$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \longrightarrow (A)$$

classmate

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$$\frac{-3}{z-3} = \frac{3}{3-z} = \frac{3}{3(1-z/3)} = \frac{1}{1-z/3}$$

$|z| < 2$, $\therefore |z/3| < 2/3$

$$\therefore \frac{-3}{z-3} = \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n$$

$$\therefore f(z) = \sum_{n=0}^{\infty}\left(\left(\frac{z}{3}\right)^n - \left(\frac{z}{2}\right)^n \times \frac{1}{2}\right)$$

b) Region R2: $|z| > 2$ & $|z| < 3$

$$\cancel{\frac{1}{z-2} = \frac{1}{z(1-2/z)} = \frac{-1}{2}\times\left(\frac{-1}{z/2}\right)\times\left(\frac{1}{1-\frac{2}{z}}\right)}$$

$$\Rightarrow \frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right)$$

$$= \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$\Rightarrow \frac{-3}{z-3} = \frac{1}{1-z/3}, \; |z| < 3, \text{ so } |z/3| < 1$$

$$= \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$\therefore f(z) = \sum_{n=0}^{\infty}\left(\left(\frac{z}{3}\right)^n + \left(\frac{2}{z}\right)^n \times \frac{1}{z}\right)$$

c) Region R3: $|z| > 3$,

$$\cdot \frac{1}{z-2} = \frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{}$$

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$$\Rightarrow \frac{-3}{z-3} = \frac{-3}{z\left(1-\frac{3}{z}\right)} = \frac{-3}{z}\left(\frac{1}{1-\frac{3}{z}}\right)$$

$$|z| > 3, \quad \left|\frac{3}{z}\right| < 1$$

$$\therefore \frac{-3}{z-3} = \frac{-3}{z}\left(\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n\right)$$

$$= -\left(\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^{n+1}\right)$$

$$\therefore f(z) = \sum_{n=0}^{\infty}\left(\left(\frac{2}{z}\right)^n \times \frac{1}{z} - \left(\frac{3}{z}\right)^{n+1}\right)$$

- We can see that the function $f(z)$ is different in each region. (Laurent series expansion about $z=0$).

Singularities;

- $z = z_0$ is a singularity if $f(z)$ is not analytic at $z = z_0$.

** Types of Singularities;

i) Removable singularity
ii) Isolated singularity → Poles / Isolated essential singularity.
iii) Non-isolated singularity

* Isolated Singularity:

- $z = z_0$ is an isolated singularity of $f(z)$ if $z = z_0$ has a neighbourhood with no further singularities.

—— x ——

ex: $f(z) = \tan z$, $z = \pm \pi/2, \pm 3\pi/2, \ldots$

$\Rightarrow$ These are all isolated singularity (Since we can find a neighbourhood with no further singularities for each singularity).

** ex: * $g(z) = \tan\left(\frac{1}{z}\right)$, $z = 0$ is a singularity

$z = \pm 2/\pi, \pm 2/3\pi, \ldots$

$z = \pm \frac{2}{(2n+1)\pi}$

- Now for large values of n, we can find another singularity in the neighbourhood of $z = 0$. (Whatever we choose our neighbourhood we can always find an n, such that $z = \pm \frac{2}{(2n+1)\pi}$ is in the neighbourhood).

$\therefore z = 0$ } Non isolated singularities
$z = \pm \frac{2}{(2n+1)\pi}$ } Isolated singularity.

—— x ——

Consider, Region $0 < |z - z_0| < R$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\substack{\text{Analytic} \\ \text{at } z = z_0}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part}} \longrightarrow (1)$$

~~Defn~~

Poles:

—— x ——

* Defn: If the principal part has only finitely many terms i.e. $\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots \frac{b_m}{(z-z_0)^m}$

then we call $z = z_0$ → where z_0 is a singularity

a pole of order m.

* ⇒ If $m=1$, then z_0 is a simple pole

—— x ——

* Defn: If the principal part of (1) has infinitely many terms, we say $f(z)$ has an isolated essential singularity at $z = z_0$.

—— x ——

** ex; $\frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)} = f(z)$ Center $= 0$ (Use Laurent Series) (There are singularities)

(Analyze singularities) using Laurent Series

i) $|z| < 1$, $\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots \longrightarrow (1)$

$z = 0$ & $z = 1$ are singularities

- Here $z = 0$ lies in our domain $|z| < 1$ so we can analyze it.

(Expansion about $z_0 = 0$).

⇒ $z_0 = 0$ is a pole of order 3.

* ii) Now for $z_0 = 1$, we need to expand about $z_0 = 1$.

(This expansion will be different from above)

$f(z) = \frac{1}{z^3(1-z)}$ [illegible]

[illegible]

$\frac{-1}{z-1}\left[\sum_{n=0}^{\infty} \ldots\right]$ [illegible]

$\Rightarrow$ Residue Here we only have principal term

$z_0 = 1$ is a pole of [illegible]

—— x ——

*** Test Theorem for poles;

$f(z)$ has a pole at $z = z_0$ of order m iff $\lim_{z \to z_0} (z - z_0)^m f(z) = D$ where

D is a finite non-zero constant.

—— x ——

ex: $f(z) = \frac{1}{z^3 - z^4}$, At $z_0 = 0$

$\Rightarrow \lim_{z \to 0} z^3 \times \left(\frac{1}{z^3 - z^4}\right) = \frac{1}{1-z} = 1$

$\therefore$ Pole of order 3.

At $z_0 = 1$, $\lim_{z \to 1} (z-1) \frac{1}{z^3(z-1)} = 1 \neq 0$

$\therefore$ 1 is a simple pole

ex; Isolated essential singularity

$f(z) = e^{1/z}$, $z=0$ is a singularity

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\times\frac{1}{z^2} + \dots$$

$\therefore$ $z=0$ is an isolated essential singularity (Infinite principal part)

— x —

* Zeros;

- $z=z_0$ is a zero of $f(z)$ if $f(z_0)=0$ in domain.
- $z=z_0$ ~~of order~~ is a zero of order n if $f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(n-1)}(z_0) = 0$ and $f^{n}(z_0) \neq 0$.

$\Rightarrow$ If $n=1$, then $z=z_0$ is a simple zero.

— x —

ex; $f(z) = z^3$, $z_0 = 0$ is a zero

$f(0)=0$, $f'(0)=0$, $f''(0)=0$,

$f'''(0) = 6 \neq 0$

$\therefore$ Order is 3.

$\therefore$ $z=0$ is a zero of order 3

— x —

DIY: Find the zeros and their orders

1. $f(z) = 1+z^2$, $z = \pm i$
2. $f(z) = (1-z^4)^2$, $z = \pm 1, \pm i$
3. $g(z) = 1 - z^4$, $z = \pm 1, \pm i$

1. $f'(z) = 2z$, at $z = \pm i$ $f'(z) \neq 0$
   ∴ ~~Ste~~ zero of order 1
   (simple zeros)

2. $f'(z) = 2(1-z^4) \times (-4z^3)$
   $= -8z^3 + 8z^7$
   $f''(z) = -24z^2 + 56z^6$
   At $z = 1, -1, i, -i$ $f''(z) \neq 0$
   ∴ They are all zero's of order 2

3. $g'(z) = -4z^3$
   $g''(z) = -12z^2$
   $g'''(z) = -24z$
   $g''''(z) = -24$

   Here $z = \pm 1, \pm i$
   $g'(z) \neq 0$ for all those values
   ∴ They are all simple zero's
   (order = 1)

— x —

*Thm: If $f(z)$ is an analytic function then the zeros of $f(z)$ are isolated
i.e. There is a neighbourhood in which there are no further zeros.

— x —

*Thm: Poles & Zeros;

- Let $f(z)$ be analytic at $z = z_0$, and have a zero of n^{th} order at $z = z_0$, then $1/f(z)$ has a pole of order n at $z = z_0$.

— x —

ex; $f(z) = z\sin z$, $\frac{1}{f(z)} = \frac{1}{z\sin z} = g(z)$

- Zeros of $f(z)$ are, $z = 0$, $z = k\pi$ where $k = \pm 1, \pm 2, \ldots$

$f(z) = z\sin z$
$f'(z) = \sin z + z\cos z$
$f''(z) = 2\cos z - z\sin z$
$f'''(z) = -3\sin z - z\cos z$

$\Rightarrow$ For $z_0 = 0$, $f(0) = 0$
$f'(0) = 0$
$f''(0) = 2 \neq 0$
, $\therefore z_0 = 0$ is a zero of order 2.

For $z = k\pi$, $k \neq 0$ $\Rightarrow$ Zeros of order 1 } Simple zero

$f(z) = 0$, $f'(z) = k\pi\cos k\pi + \sin k\pi \neq 0$, $\therefore$ Order of zero is 1

• Poles of $g(z)$ are zeros of $f(z)$ from th[e]

$\therefore\ g(z) = \dfrac{1}{z\sin z}$ → We can't use th[e] theorem (so use Laurent) [for] this th[at]... → Trig function

Singularities are $z=0$, $z=k\pi$ where $k=\pm1,\pm$[...]

From theorem,

$z=0$ is a pole of order 2 and

$z=k\pi$ is a pole of order 1 (simple pole)

— x —

*** Residues; (when asked to find via Laurent series expansion, always binomial expansion and then compare term[s])

• Aim to evaluate integral of form, $\oint f(z)\,dz$. If $f(z)$ is analytic, then integral is 0 else we have troubles, so we use residues.

Contour clockwise.

⟹ Laurent series expansion of $f(z)$ is,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2}$$

$$0 < |z-z_0| < R$$

• $f(z)$ converges $\forall$ points z near z_0 but not at z_0 → singularity.

$\Rightarrow b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz$ } From Laurent expansion

- We obtain laurent series expansion by using binomial ~~theorem~~ theorem etc,

$\Rightarrow$ Comparing coefficients we get,

$\underline{\underline{b_1}} = \frac{1}{2\pi i} \oint_C f(z)\,dz$ } The residue of $f(z)$ at $z = z_0$

$\Rightarrow \oint_C f(z)\,dz = b_1 \times 2\pi i$

(Use residues with expansion)

*** Q) Integrate $z^{-4} \sin z$ counterclockwise around $C: |z| = 1$. $\rightarrow$ Domain

$\oint_C f(z)\,dz = \oint_C \frac{1}{z^4} \sin z\,dz$, $z = 0$ is a singularity within our domain

(So the residue from laurent = ?)

$z_0 = 0$, We expand upon z_0 (Singularity)

$$\Rightarrow f(z) = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$b_1 = -1/6$ } Residue

$$\therefore \oint_C f(z)\,dz = b_1 \times 2\pi i = -\frac{\pi i}{3}$$

— x —

(Residue theorem → [illegible])

DIY: Find $\oint_C f(z)\,dz$ where $f(z) = \frac{1}{z^3 \ldots}$

$C \rightarrow$ clockwise around $|z| = 1$

$\Rightarrow$ Use residue.

$(z_0 = 0)$ is a singular

$$f(z) = \frac{1}{z^3(1-z)}$$

$\Rightarrow$ So we expand about singularity

$$f(z) = \frac{1}{z^3}\left(1 + z + z^2 + \ldots\right)$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \ldots$$

So residue $= b_1 = 1$

$\therefore$ ~~$\oint_C f(z)\,dz = 2\pi i \times b_1 = 2\pi i$~~

• ~~Now expand about $(z_0 = 1)$, other~~ singular

From prev expansion (Binomial expansion for -ve powers)

~~then find residue b_1'~~ $\rightarrow$ Out of domain

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i(b_1 + \cancel{b_1'})$$

From residue theorem

—x—

Calculating residues (Laurent expansion compare)

1) Formula for residue if z_0 is a simple pole;

$$f(z) = \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad (b_1 \neq 0)$$

$$\Rightarrow (z-z_0) f(z) = b_1 + (z-z_0) \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Take Limit as $z \to z_0$, ($f(z_0)$ is not analytic at $z=z_0$)

$$\Rightarrow \boxed{b_1 = \lim_{z \to z_0} (z-z_0) f(z)} \to \text{If } z_0 \text{ is a simple pole}$$

Residue

2) Formula for residue at a pole of any order;

- Let z_0 be a pole of order m.

Laurent expansion

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$0 < |z-z_0| < r$$ The series converges (In this disc)

$\Rightarrow$ Multiply by $(z-z_0)^m$ on both sides

$$\Rightarrow \underbrace{(z-z_0)^m f(z)}_{\text{Let this be } g(z)} = b_m + (z-z_0)b_{m-1} + \dots + (z-z_0)^{m-1} b_1 + (z-z_0)^m \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Residue here (of $f(z)$ at z_0) is coeff. of $(z-z_0)^{m-1}$!

$\Rightarrow g(z)$ is analytic throughout, so we can use taylor series (ie. treat RHS as taylor series).

$$b_1 = \frac{1}{(m-1)!} \times g^{(m-1)}(z) \rightarrow \text{At } z = z_0$$

$$\ast\ \ast \quad \text{Residue of } f(z) \text{ at } z=z_0 = \boxed{\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{(m-1)}}{dz^{(m-1)}} \left[(z-z_0)^m f(z) \right]}$$

— x —

Note; $\Rightarrow$ If z_0 is a double pole, then

$$b_1 = \underset{\text{at } z=z_0}{\text{Residue}} f(z) = \lim_{z \to z_0} \frac{d}{dz} \left[(z-z_0)^2 f(z) \right]$$

— x —

Note; From this residue we can get the integral; of $f(z)$.

— x —

Note; While getting residues, always show what the orders of poles are (singularities).

— x —

$$\Rightarrow \underbrace{(z-z_0)^m f(z)}_{\text{Let this, } g(z)} = b_m + (z-z_0)b_{m-1} + \cdots + (z-z_0)^{m-1}b_1 + (z-z_0)^m \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Residue here (of $f(z)$ at z_0) is coefficient of $(z-z_0)^{m-1}$!

$\Rightarrow g(z)$ is analytic throughout, so we can use taylor series (ie. treat RHS as taylor series).

$$b_1 = \frac{1}{(m-1)!} \times g^{(m-1)}(z) \rightarrow \text{At } z = z_0$$

* Residue of $f(z)$ at $z=z_0$ $= \boxed{\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{(m-1)}}{dz^{(m-1)}} \left[(z-z_0)^m f(z) \right]}$

— x —

Note; $\Rightarrow$ If z_0 is a double pole, then

$$b_1 = \underset{\text{at } z=z_0}{\text{Residue}} f(z) = \lim_{z \to z_0} \frac{d}{dz} \left[(z-z_0)^2 f(z) \right]$$

— x —

Note; From this residue we can get the integral; of $f(z)$.

— x —

*Note; While getting residues, always show what the orders of poles are (singularities)

*Q) Find residues of the singularities

*
$$f(z) = \frac{z}{(z-1)(z+1)^2}$$

a) $z=1$, ~~Its~~ a simple pole.

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1) f(z) = \lim_{z\to 1} \frac{z}{(z+1)^2} = \frac{1}{4}$$

b) $z=-1$ is a double pole,

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} \frac{d}{dz}\left[(z+1)^2 \cdot f(z)\right]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= \lim_{z \to -1} \frac{(z-1) - z}{(z-1)^2} = \lim_{z \to -1} \frac{-1}{(z-1)^2}$$

$$= -1/4$$

— x —

Many Singularities in curve (closed); (contour integration)

can't use simple

** Residue Theorem; laurent.

• Let f be an analytic inside a simple closed curve C and in C, except for finitely many singularities $z_1, z_2, \ldots, z_k$ inside C. Then the integral of $f(z)$

taken counterclockwise around C,

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \underset{z=z_j}{\text{Res}}\, f(z)}$$

• Proved using cauchy's theorem ↑

———— ∝ ————

DIY

Q) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 \cdot (z^2+4)}$, Find $\oint_C f(z)\,dz$.
(Contour integral

Poles → $-1, 2i, -2i$ } Lie within our domain $|z| \leq 3$

At $z_0 = -1$, $\lim_{z \to -1} (z+1)^2 \times f(z) = 0$
$\neq 0$

$\therefore$ -1 is a pole of order 2

At $z_0 = 2i$ $\lim_{z \to 2i} (z-2i) f(z) = 0 \neq 0$

& similarly for $z_0 = -2i$

$\therefore$ $2i$ & $-2i$ are poles of order 1.

At $z_0 = -1$, $b_1 = \lim_{z \to -1} \dfrac{d}{dz}\left[\dfrac{z^2 - 2z}{z^2+4}\right]$

$= \lim_{z \to -1} \dfrac{(z^2+4)(2z-2) - (z^2-2z)(2z}{(z^2+4)^2}$

$$= \frac{1+4(-4)-(+3)(-2)}{25} = \frac{-9}{25}$$

- At $z_0 = 2i$, $b_1' = \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2 \times (z+2i)} = \frac{-4-4i}{(-3+4i)(4i)} \to ①$
- At $z_0 = -2i$, $b_1'' = \lim_{z \to -2i} \frac{z^2 - 2z}{(z+1)^2 (z-2i)} = \frac{-4+4i}{(-3-4i)(4i)} \to ②$

$$\oint f(z)\,dz = 2\pi i \left(b - \tfrac{9}{25} + ① + ②\right)$$

Q) $f(z) = e^z \operatorname{cosec}^2 z$, Find contour integral

$= e^z / \sin^2 z \longrightarrow$ Only 1 pole.

$\Rightarrow$ Pole at $z=0$, Order of pole $= ?$
Flip function is $(1/f(z))$, then find order of zeros at $z=0$, this is the order of pole of $f(z)$

$$\Rightarrow \frac{\sin^2 z}{e^z} = f(z)$$

$$\Rightarrow f'(z) = \frac{\cancel{e^z} \times 2\sin z \cos z - \sin^2 z\, \cancel{e^z}}{e^{z\cancel{z}}}$$

$$= \frac{\sin z(2\cos z - \sin z)}{e^z}$$

At $z=0$, $f'(z) = 0$

$$\Rightarrow f''(z) = \frac{\cancel{e^z}\left(2(-\sin^2 z + \cos^2 z) - 2\sin z \cos z\right) - \sin z(2\cos z - \sin z)\cancel{e^z}}{e^{z\cancel{z}}}$$

At $z=0$, $f''(z) = 2/1 = 2 \neq 0$

$\therefore$ Order of pole at $z=0$ is 2.

Now we use, $b_1 = \lim_{z \to z_0} \frac{d}{dz}\left((z-z_0)^2 f(z)\right)$

$\therefore b_1 = \lim_{z \to 0} \frac{d}{dz}\left(\frac{e^z \times z^2}{\sin^2 z}\right)$

(Once we have b_1, $\oint_C f(z)\,dz = b_1 \times 2\pi i$)

— x —

Q) $f(z) = \frac{\cot z \times \coth z}{z^3}$, find contour integral $|z|<1$

↳ Its easier to use laurent expansion here. (Generally with complex trig, Laurent expansion is easier)

How to do without laurent expansion of cot & coth?

$\Rightarrow f(z) = \frac{\cos z \times \cosh z}{\sin z \ \sinh z \times z^3}$

$= \frac{\cos z}{\sin z}\cancel{\left(\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\right)\times\frac{1}{z^3}}$

$\cancel{= \frac{\cos z}{\sin z}\left(\frac{e^{2iz}+1}{e^{2iz}-1}\right)\times\frac{1}{z^3}}$

$= \frac{\cos z}{\sin z}\cancel{\left(\frac{\cos 2z + i\sin 2z + 1}{\cos 2z + i\sin 2z - 1}\right)\times\frac{1}{z^3}}$

$\cancel{= \sum_{n=0}^{\infty} z}$

$\cancel{\Rightarrow z = 0 \text{ is a pole}}$

$\cancel{g(z) = 1/f(z) = z^3 \times \tan z \tanh z}$

$\cancel{g'(z) = z^3 \times \tan z(1-\tanh^2 z) + \tanh z(z^3 \sec^2 z + 3z^2 \tan z)}$

At $z=0$, $g'(z) = 0$

!

$\Rightarrow$ Let us solve it using Laurent expansion

$f(z) = \frac{\left(\frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \cdots\right)\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} \cdots\right)}{z^3}$

$\cancel{b_1 \text{ here} = -\frac{1}{9} + \frac{-1}{45} + \frac{-1}{45} = \ldots}$

$\cot z = i\coth(iz)$
$\Rightarrow \cot z \times \coth z \longrightarrow \frac{1}{z}$

$$\oint_C \frac{\cot z \times \coth z}{z^7} dz = 2\pi i \times b_1 = \cancel{\frac{-14\pi i}{45}}$$

$$b_1 = -\frac{1}{9} - \frac{1}{19} - \frac{1}{45} = \frac{-7}{49}$$

$$\therefore \oint \dots = 2\pi i \times -\frac{7}{49} = \frac{-14\pi i}{49}$$

** (3) Evaluate $\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz$, $C: |z|=3$

All singularities lie in this range.

Use residue theorem,

poles are, Double pole at $z=0$
Simple poles at $z^2+2z+2=0$
$\Rightarrow z = -1 \pm i$

1) Residue at $z=0$,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{d}{dz}(z-0)^2 f(z)$$

$$= \lim_{z\to 0} \frac{d}{dz} \frac{e^{zt}}{z^2+2z+2}$$

$$= \frac{(z^2+2z+2) \times t \times e^{zt} - e^{zt}(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{2t-2}{4} = \frac{t-1}{2}$$

2) $$\operatorname*{Residue}_{z=-1+i} f(z) = \lim_{z\to(-1+i)} \frac{(z-(-1+i))e^{zt}}{z^2(z^2+2z+2)}$$

$$= \lim_{z\to(-1+i)} \frac{e^{zt}}{z^2(z+1+i)} = \frac{1}{4}e^{(-1+}$$

3) $\underset{z=-1-i}{\text{Res}} f(z) = \lim_{z \to z_0} \dfrac{e^{zt}}{z^2(z+1-i)} = \frac{1}{4} e^{(-1-i)t}$

$$\therefore \oint_C f(z) = \frac{t-1}{2} + \frac{1}{4}\left[e^{-t}(e^{it}+e^{-it})\right]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{2}\cos t$$

—— x ——

(Q) Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a>|b|$

Let $z = e^{i\theta}$, $|z|=1$

$dz = ie^{i\theta}d\theta$

$dz = iz\,d\theta$

$\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-\frac{1}{z}}{2i} = \dfrac{z^2-1}{2iz}$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_C \frac{1}{iz}\,\frac{dz}{\left(a+b\left(\frac{z^2-1}{2iz}\right)\right)}$$

$C: |z|=1$ → we can choose any circle centred at origin

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_C \frac{2\,dz}{bz^2+2aiz-b}$$

→ Singularities are, simple poles at

$$z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b}$$

$$= \frac{-ai \pm \sqrt{b^2-a^2}}{b}, \quad a>|b|$$

classmate
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$\therefore \; z = \dfrac{-ai \pm i\sqrt{a^2-b^2}}{b} \qquad \left(\frac{a}{b} > 1\right)$

$\Rightarrow -\dfrac{ai}{b} - \dfrac{i\sqrt{a^2-b^2}}{b} \Big\} < -1$, So its out of our domain

$\therefore$ We only need to worry about 1 pole $= \dfrac{-ai + i\sqrt{a^2-b^2}}{b}$

$\Rightarrow$ Residue at pole $= \lim_{z \to z_0} (z - z_0) f(z)$

$= \lim_{z \to z_0} \left[\left(z + \dfrac{ai - i\sqrt{a^2-b^2}}{b} \right) \left(\dfrac{2}{bz^2 + 2aiz - b} \right) \right]$

$= \lim_{z \to z_0} \left[\dfrac{2}{\cancel{b}} \left(\dfrac{\cancel{b}}{zb - (ai - i\sqrt{a^2-b^2})} \right) \right]$

$= 2 \left[\dfrac{1}{-ai + i\sqrt{a^2-b^2} + ai + i\sqrt{a^2-b^2}} \right]$

$= \dfrac{1}{i\sqrt{a^2-b^2}}, \quad \oint = \dfrac{2\pi i}{i\sqrt{a^2-b^2}} = \dfrac{2\pi}{\sqrt{a^2-b^2}}$

— x —

DIY

C — Counter clockwise

Evaluate;

1) $\oint_C \dfrac{z^2 \cdot \sin z}{4z^2 - 1} \, dz, \quad C: |z| = 2$

2) $\oint_C \dfrac{z+1}{z^4 - 2z^3} \, dz, \quad C: |z| = \frac{1}{2}$ $\rightarrow$ Ans $= -\frac{3}{4}\pi i$

3) $\oint_C \dfrac{e^{-z^2}}{\sin 4z} \, dz, \quad C: |z| = 1$

4) $\int_0^{2\pi} \dfrac{1 + \sin\theta}{3 + \cos\theta} \, d\theta$ $\rightarrow$ Do not use school methods $\Rightarrow$ Ans $= \pi/\sqrt{2}$

5) $\int_0^{2\pi} \dfrac{\cos\theta}{13 - 12\cos 2\theta} \, d\theta \rightarrow$ Ans $= 0$.

6) $\int_0^{2\pi} \dfrac{\cos 3\theta}{5 - 4\cos\theta} \, d\theta$ $\downarrow$ Ans $= \dfrac{\pi}{60}$

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Probability Theory

* $S \rightarrow$ Sample Space
  - Set of all outcomes

→ Event x is a subset of sample space
  $A \subseteq$ $x \subseteq S$.

—— x ——

ex, Rolling a die,
$S = \{1, 2, 3, 4, 5, 6\}$
$A \rightarrow$ Getting a number < 4, → Event A
$A = \{1, 2, 3\} \subseteq S \rightarrow$ subset

—— x ——

* <u>Mutually exclusive events;</u>

- $A \cap B = \phi$, then A & B are mutually exclusive

ex. A = Getting even number
  B = Getting odd number } On rolling a die.
  $A \cap B = \phi$

—— x ——

<u>Note;</u> $A^c \rightarrow$ Complement of A.

$A^c = S - A$, $A \cup A^c = S$

—— x ——

Defn: If the sample space S of an experiment consists of finitely many outcomes that are equally likely then the probability

$$P(A) = \frac{n(A)}{n(S)} \rightarrow \text{Cardinalities}$$

$$0 \leq P(A) \leq 1 \;,\; P(S) = 1$$

— x —

ex: Rolling a pair of dice, what is the probability of obtaining a sum of 5.

$A \rightarrow$ Sum of 5 on pair of dice,

$n(S) = 36.$

$A = \{(1,4), (2,3), (4,1), (3,2)\}$

$n(A) = 4$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

— x —

Note: Axioms of probability; 1) $0 \leq P(A) \leq 1$

2) $P(S) = 1$

* 3) If A & B are mutually exclusive events, ie $A \cap B = \phi$, then

$P(A \cup B) = P(A) + P(B)$

can be extended to n terms.

— x —

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Thm 2: $P(A^c) = 1 - P(A)$ ← (Proved)

Let $n(A) = k$, then $n(A^c) = n(S) - k$

Thm 3: If A & B are two events in S then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Ex: Find prob of getting odd number or a number less than 4 on rolling a die.

A — odd number

B — number less than 4.

$A = \{1, 3, 5\}$

$B = \{1, 2, 3\}$

$P(A) = \frac{3}{6}$, $P(B) = \frac{3}{6}$, $A \cap B = \{1, 3\}$

$P(A \cap B) = 2/6$, $P(A \cup B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6}$

$P(A \cup B) = 4/6 = 2/3$

** Baye's Theorem; (Conditional probability)

⇒ $P(B|A) = \frac{P(A \cap B)}{P(A)}$ } We reduce sample space to A.

Probability that B occurs only when A occurs.

$P(A|B) = P(A \cap B) / P(B)$

⇒ If A & B are independent events,

$P(A \cap B) = P(A) \cdot P(B)$

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DIY

Q) Sample space of a random experiment is given by $S = \{a, b, c, d\}$ with probabilities
$P(a) = 0.2$, $P(b) = 0.3$, $P(c) = 0.4$, $P(d) = 0.1$

$A = \{a, b\}$
$B = \{b, c, d\}$

i) $P(A) = 0.5$
ii) $P(B) = 0.8$
iii) $P(A^c) = 0.5$ $(1 - P(A))$
iv) $P(A \cup B) = 0.5 + 0.8 - 0.3 = 1$
v) $P(A \cap B) = P(b) = 0.3$ //

— x —

DIY Inspect envelopes by drawing 4 of them without replacement, from 100 envelopes, what is the prob of getting clean envelopes when 3% contain spots.

$$P = \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98} \times \frac{94}{97}$$

— x —

* Random Variables; (R.V)

- In an experiment the unknown quantity or observation is called a random variable or stochastic variable.

— x —

Just for notation

* Defn: * A random variable (R.V) $X(\mathcal{E})$ is a function defined on a sample space S of an experiment. Its range is a set of real numbers. For every number $a \in R$, the probability $P(X=a)$ is defined.

- If the range of X $(R(X))$ is finite or countable then X is a <u>discrete random variable</u>.

- If we measure X, then X is a <u>continuous random variable</u>.

—— x ——

I) Discrete Random Variables;

1) ** Cummulative distribution function; (cdf) $(F(x))$

$$\Rightarrow \underbrace{F}_{cdf}(\overset{\text{value of random variable.}}{x}) = P(\underbrace{X \le x}_{\text{Events whose random variable} \le x.})$$

—— x ——

* ex; * In the experiment of tossing a fair coin 3 times, if X is the R.V that denotes the number of heads obtained, Find a) $P(X=2)$
b) $P(X<2)$

sample space

$S = \{HHT, HTH, THH, HTT, THT, TTH, HHH, TTT\}$

a) $P(X=2) = 3/8$

b) $P(X<2) = P(X=0 \text{ or } X=1) = \frac{4}{8} = 1/2$

— x —

<u>Note:</u> $\Rightarrow \boxed{P(a<X\leq b) = F(b)-F(a)}$

$X \leq a$ & $a<X\leq b$ are independent events.

$P(X\leq a \cup a<X\leq b)$

$= P(X\leq a) + P(a<X\leq b)$

— x —

$\Rightarrow$ <u>Properties of $F(x)$</u>; Cummulative distribution function

1. $0 \leq F(x) \leq 1$
2. $F(x_1) \leq F(x_2)$ if $x_1 < x_2$
3. $\lim\limits_{x\to\infty} F(x) = 1$
4. $\lim\limits_{x\to -\infty} F(x) = 0$

— x —

2) <u>Probability function</u>; ($f(x)$ of X)

$$f(x) = \begin{cases} p_i, & x = x_i \quad (i = 1, 2, \ldots) \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \text{Cdf} \rightarrow F(x) = \sum_{x_i \leq x} f(x_i) = \sum_{x_i\leq x} p_i$$

Step function. (For discrete random variables)

— x —

** ex: In an experiment where you roll the dice, what is the probability of getting atleast 4 & atmost 8.

- We define our random variable, say

X — sum of number on dice.

$P(3 < X \leq 8) = F(8) - F(3)$

$F(3) = P(X \leq 3)$

(1,0), (1,1), (2,1) → 5/36

, (0,1), (1,2), (..)

$F(8) = P(X \leq 8) = 26/36$

$\therefore P(3 < X \leq 8) = 26/36 - 3/36 = 23/36$

—— x ——

ex: Experiment — rolling a fair die.

*

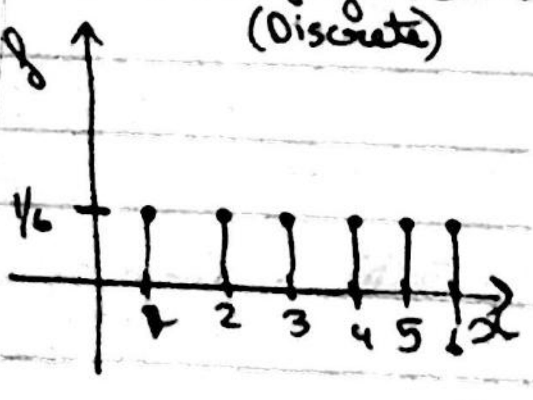
R.V X = Number on the die.

$R(X) = \{1, 2, 3, 4, 5, 6\}$

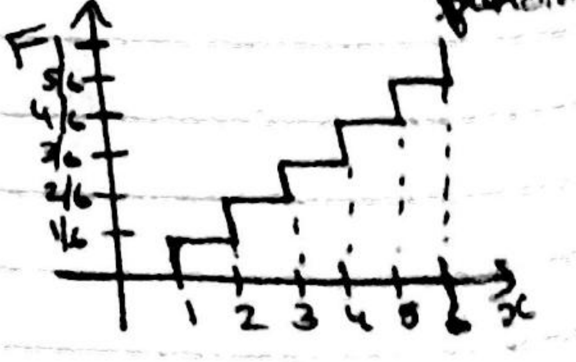
$x \in R(X)$

$P(X = x) = 1/6$

Probability function (Discrete)



Cummulative dist function



—— x ——

Ex) Experiment → Rolling a pair of dice.
X – sum of numbers on two dice.

Plot $f(x)$ & $F(x)$ $(2 \le X \le 12)$

—x—

II) Continuous Random Variables;

* 1) Cumulative Distribution function,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

* 2) $f(x)$ = Probability density function.
  - f — non repetitive
    - continuous except possibly at finite number of points
    - $f(x) = F'(x)$ where f is continuous.

Probability ⇒ $\boxed{P(a < X \le b) = \int_a^b f(x)\,dx}$

$\downarrow$

$F(b) - F(a)$

—x—

Note; $P(a < X \le b) = P(a \le X < b) = P(a \le X \le b)$
$= P(a < X < b)$

⇒ The reason why these are the same is since on integration of with or without boundary is the same, unlike in discrete random variab.

—— x ——

**

<u>DIY</u>: Show that (In discrete R.V)

** 1) $P(a < X \leq b) = F(b) - F(a)$

Exam 2) $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$

3) $P(a \leq X < b) = P(\alpha = a) + F(b) - F(a) - P$

4) $P(a < X < b) = F(b) - F(a) - P(X = b)$

(Note: Pick disjoint sets & consider unions) to prove

—— x ——

Suppose a discrete R.V has the following pdf (pf), (Discrete)

$p(1) = 1/2$, $p(4) = 1/8$

$p(2) = 1/4$

$p(3) = 1/8$ this is the discrete R.V

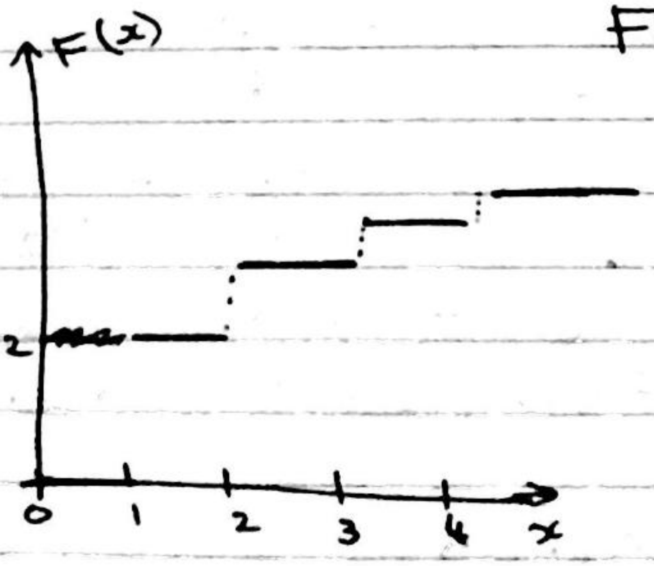
a) Find & sketch the cdf $F_X(x)$ ← Random variable X

b) i) $P(X \leq 1)$

ii) $P(1 < X \leq 3)$

iii) $P(1 \leq X \leq 3)$

a)



$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{3}{4}, & 2 \leq x < 3 \\ \frac{7}{8}, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

b) i) $P(X \leq 1) = 1/2$, $P(1 < X \leq 3) = 3/8$

$P(1 \leq X \leq 3) = \underline{7/8}$ // x ——

* Continuous R.V ;

- cdf , $F(x) = \int_{-\infty}^{x} f(v)\,dv$

$f \rightarrow$ Probability density function (pdf)

$f \rightarrow$ continuous except possibly at finite number of points.

$\Rightarrow F'(x) = f(x)$

- $P(a < X \leq b) = F(b) - F(a)$.

—— x ——

* Q) Let X in mm be the thickness of
* washers a machine turns out (mdas) assume X has the density function

$$f(x) = \begin{cases} kx, & 0.9 < x < 1.1 \\ 0, & \text{otherwise}. \end{cases}$$

(For continuous R.V , including or excluding ends does not matter)

a) Find k.

b) What is the prob that the washer will have thickness b/w 0.95mm & 1.05mm

Ans a) $\int_{-\infty}^{\infty} f(v)\,dv = 1$,

$$\therefore \int_{-\infty}^{\infty} f(v)\,dv = \int_{-\infty}^{0.9} f(v)\,dv + \int_{0.9}^{1.1} f(v)\,dv + \int_{1.1}^{\infty} f(v)$$

$$1 = 0 + \int_{0.9}^{1.1} kv\,dv + 0$$

$$1 = \frac{kv^2}{2}\Big]_{0.9}^{1.1} \Rightarrow \frac{2}{k} = (1.1)^2 - (0.9)^2$$
$$= 2 \times 0.2$$

$$\therefore\ k = 1/0.2 = 5$$

Ans b) $P(0.95 < X < 1.05) = F(1.05) - F(0.95)$

$$= \int_{0.95}^{1.05} 5 \times x\,dx = \frac{5x^2}{2}\Big]_{0.95}^{1.05}$$

$$= \frac{5 \times 2 \times 0.1}{2}$$

$$= 0.5$$

DIY: i) Consider the experiment of throwing a dart onto a circular plate with unit radius. Let X be the R.V representing the distance of the point where the dart lands from the origin of the plate. Assume that the dart always lands on the plate and the dart is equally likely to land anywhere on the plate.

a) What is the range of X $\quad (f(x) = k)$ $\because$ Equal prob.

$$0 \leq X \leq 1 \quad , \quad \underbrace{\int_0^1 k\,dx = kx = k}_{\therefore k = 1}$$

b) Find

i) $P(X < a) = \int_{-\infty}^{a} dx = \int_0^a dx = a$

(ending
not
matters)

(u)/

ii) $P(a < x < b) = \int_a^b dx = b - a$

—x—

Q) Consider the experiment of tossing a fair coin repeatedly.
Let R.V X denote the number of tosses required until the first head appears.

a) Find & sketch the cdf $F(x)$ of X.

b) Find i) $P(1 < X \leq 4)$

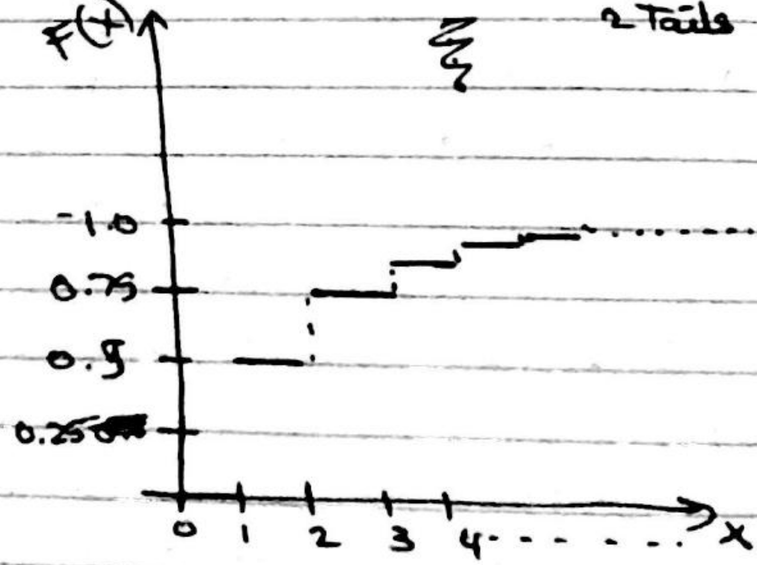
ii) $P(X > 4)$

Discrete R.V, $p(1) = \frac{1}{2}$ → head

$P(2) = \frac{1}{2} \times \frac{1}{2}$ (→ tails, → head)

$P(3) = \underbrace{\frac{1}{2} \times \frac{1}{2}}_{\text{2 Tails}} \times \underbrace{\frac{1}{2}}_{\text{1 head}}$

⋮



- $P(1 < X \leq 4) = F(4) - F(1) = 7/16$
- $P(X > 4) = F(\infty) - F(4) = 1 - 15/16$

$= 1/16$

—x—

took singularity on boundary.

*Q) Find the prob that none of the three
* bulbs in a traffic signal will have to be replaced during the first 1500 hours of operation if the lifetime X of a bulb is a R.V with density function

$$f(x) = \begin{cases} 6[0.25 - (x-1.5)^2] , & 1 \le x \le 2 \\ 0 , & \text{otherwise} \end{cases}$$

Measured in multiple of 1000 hours

$$? \quad P(X > 1.5) = 1 - P(X \le 1.5)$$

$$= 1 - F(1.5)$$

$$= 1 - \int_{-\infty}^{1.5} f(x)\,dx$$

$$= 1 - \int_{1}^{1.5} 1.5 - 6(x-1.5)^2 \, dx$$

$$= 1 - \Big[1.5x - 2(x-1.5)^3 \Big]_{1}^{1.5}$$

$$= 1 - \big[1.5 \times 1.5 - 1.5 - 0.25 \big]$$

$$= 1 - 1.5 \times (1.25)$$

$$P(x > 1.5) = 0.5$$

$$\therefore \text{Ans} = (0.5)^3 \rightarrow 3 \text{ bulbs}$$

— x —

After Mid2

* Expectation of a R.V X ; (Similar to mean)

- Discrete R.V : $E(X) = \sum_{i=1}^{n} x_i \cdot p(x_i)$
- Continuous R.V : $E(X) = \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{pdf}} \, dx$

— x —

Q) Find the expected value $E(X)$ where X is the outcome if we roll a dice.

$$E(X) = \sum_{i=1}^{6} i \times \frac{1}{6} = \frac{1+2+3+4+5+6}{6}$$

$$= 21/6 = 3.5$$

(Units of E.V same as R.V)

~~doesnt~~ need to be on the dice

— x —

* Q) Suppose you are expecting a message at some time past 5:00 pm. From experience we know that X, number of hours after 5 pm until message arrives, is a R.V with p.d.f $f(x) = \begin{cases} 2/3, & 0 < x < 3/2 \\ 0, & \text{otherwise} \end{cases}$. *

Find expected amount of time after 5pm until message arrives.

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{3/2} x f(x) \, dx$$

$$= \int^{3/2} \frac{2}{3} x \, dx = \frac{x^2}{3} \Big]$$

** Note:

$X \to$ R.V. If $g(X)$ is non constant and continuous for all X, then $g(X)$ is a R.V

$$E(g(X)) \underset{D.R.V}{=\!=} \sum_i g(x_i) \cdot p(x_i)$$

$$E(g(X)) \underset{C.R.V}{=\!=} \int_{-\infty}^{\infty} g(x) \cdot f(x) \cdot dx \to \text{We can prove this. (Considering } f(g(X)) \text{ etc)}$$

—— x ——

* Q) The time in hours it takes to locate and repair an electrical breakdown in a certain factory is a R.V X, where p.d.f

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

If the cost involved in a breakdown of duration x is x^3, find expected cost of such a breakdown.

$X \to$ R.V, X^3 is also a R.V

$$E(X^3) = \int_{-\infty}^{\infty} x^3 f(x) dx = \int_0^1 x^3 dx = \frac{x^4}{4} = \frac{1}{4}$$

—— x ——

Thrm: If a R.V X has a mean ($E(X)$) μ then the R.V $X' = a + bX$ ($a, b \in \mathbb{R}$) has a mean $a + bE(X)$. (DIY prove)

$\Rightarrow X' = a + bX$

$$E(X') = \sum_{i=1}^{n} (a + bx_i) \cdot p(x_i)$$

$$= a + b\sum_{i=1}^{n} x_i \cdot p(x_i)$$

$$\therefore E(x') = a + b\left(\sum_{i=1}^{n} x_i \cdot p(x_i)\right)$$
$$= a + bE(x)$$

—x—

** Variance ;

If X is a R.V with mean (E.V) $\mu = E(x)$ then the variance of X is ,

$$\sigma^2 = Var(X) = E\left[(x-\mu)^2\right]$$

(σ = Standard deviation)

—x—

Note ;

- Consider the three R.V X, Y, Z with pdf

$f_x = 0 \; , P(0) = 1 \; , \mu_x = 0$

$$f_Y = \begin{cases} -1, & P(-1) = 1/2 \\ 1, & P(1) = 1/2 \end{cases} \; , \mu_y = 0$$

$$f_Z = \begin{cases} -100, & P(-100) = 1/2 \\ 100, & P(100) = 1/2 \end{cases} \; , \mu_z = 0$$

$\Rightarrow$ We can see means are the same, but spread of data is different, so we use variance to denote spread of data.

—x—

*

Thm : $Var(X) = E(X^2) - [E(X)]^2$

*

$X \to R.V \; , \mu = E(x)$

$Var(x) =$ Variance

$$Var(X) = E\left[(X-\mu)^2\right] = E\left[X^2 - 2\mu X + \mu^2\right]$$

$$= E[X^2] - 2\mu E[X] + E[\mu^2]$$

DIY

$$= E[X^2] - 2(E[X])^2 + \mu^2$$

$$= E[X^2] - (E[X])^2$$

—— x ——

DIY

Thm: Show $Var(aX+b) = \cancel{a^2 E(X^2)}\ a^2 Var(X)$

$$Var(aX+b) = E\left[(aX+b)^2\right] - \left(E[aX+b]\right)^2$$

$$\cancel{E[aX+b] = aE(X)+b}$$

$$\cancel{= E[a^2X^2+b^2+2baX] - (E(aX+b))^2}$$

$$\cancel{= a^2E[X^2] + b^2 + 2baE[X] + b^2 - (E[X]a+b)^2}$$

$$\therefore E(a^2X^2+b^2+2abX) - (aE(X)+b)^2$$

$$\cancel{= a^2E[X^2] + 2baE[X] + b^2 - E[X]^2a^2 - b^2 - 2ab}$$

$$Var(aX+b) = E\left[\left((aX+b) - (aE(X)+b)\right)^2\right]$$

$$= E\left[(aX+b-aE(X)-b)^2\right]$$

$$= E\left[a^2(X-E(X))^2\right]$$

$$= a^2 \times E\left[(X-E(X))^2\right]$$

$$= a^2 \times Var(X)$$

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K Q) a) If the diameter X (in cm) of certain bolt K ** has the p.d.f $f(x) = \begin{cases} K(x-0.9)(1.1-x) & ; 0.9 < x < 1.1 \\ 0 ; & \text{otherwise} \end{cases}$

⇒ What are K, E(x) and Var(X) = σ^2

$$\int_{0.9}^{1.1} K(x-0.9)(1.1-x)\,dx = 1$$

Solving we get K = 750

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0.9}^{1.1} 750 \times x \times (x-0.9) \times (1.1-x)\,dx$$

(On solving) (t = x−1) substituting (to make it simpler)

$$= \int_{-0.1}^{0.1} 750 \times (t+1) \cdot (t+0.1) \cdot (0.1-t)\,dx$$

$$= 1$$

⇒ $Var(X) = E(X^2) - (E(X))^2$

$= 0.002$

— x —

b) If in (a) a defective bolt is one that deviates from 1.00 cm by more than 0.06 cm What percentage of defective should we export.

Probability that $|X-1| > 0.06$

(DIY)

$P(X-1 > 0.06) + P(1-X > 0.06)$

$\Rightarrow P(X > 1.06) + P(X < 0.94)$

$$= \int_{1.06}^{1.1} f(x)\,dx + \int_{0.9}^{0.94} f(x)\,dx$$

---

* * K^{th} moment; $(E(X^k))$

$E(X^k)$ is defined to be the k^{th} moment.

---

* Moment generating functions;

Defn: $\boxed{\phi(t) = E(e^{tX})} = \begin{cases} \sum e^{tx} p(x), & X \text{ is discrete R.V} \\ \int_{-\infty}^{\infty} e^{tx} \cdot f(x) \cdot dx, & X \text{ is continuous R.V} \end{cases}$

$\phi'(t) = E\left(\frac{d}{dt} e^{tX}\right)$

$\phi'(t) = E(Xe^{tX})$

* $\phi'(0) = E(X)$ = mean expected value

DIY $\phi''(0) = E(X^2) = 2^{nd}$ moment of X

$\phi^{k}(0) = E(X^k) = k^{th}$ moment of X.

** Markov's inequality;

If X is a R.V that takes only non negative values then for any value $a > 0$,

$$\boxed{P(X \geq a) \leq \frac{E(X)}{a}}$$

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Proof: X is a continuous R.V with pdf f.

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx$$

$$\Rightarrow E(x) \geq \int_{a}^{\infty} x f(x)\,dx$$

$$\Rightarrow E(x) \geq \int_{a}^{\infty} a f(x)\,dx$$ ($\because$ a is the least possible value of x)

$$\Rightarrow \frac{E(x)}{a} \geq \int_{a}^{\infty} f(x)\,dx$$

$$\therefore P(X \geq a) \leq \frac{E(x)}{a}$$

x

** Chebyshev's Inequality:

- If X is a R.V with mean μ and variance σ^2, then for any value $K>0$,

$$\boxed{P(|X-\mu| > K) \leq \frac{\sigma^2}{K^2}}$$

Proof: $(X-\mu)^2$ is a non negative R.V. Apply markov's inequality, Let $a = K^2$ for some K.

$$\Rightarrow P((X-\mu)^2 > K^2) \leq \frac{E\left((X-\mu)^2\right)}{K^2}$$

$\Rightarrow$ As $(x-\mu)^2 \geq K^2$, $|X-\mu| \geq K$

$$P(|X-\mu| > K) \leq \frac{\sigma^2}{K^2}$$

ex: Suppose it is known that the number of items used in a factory during a week is a R.V with mean 50, What can be said about the probability that this week's production exceeds 75.

$$P(X \geq 75) \leq \frac{E(X)}{75} = \frac{50}{75} = \frac{2}{3}$$

$$P(X \geq 75) \leq 2/3 \quad \text{(Markov's inequality)}$$

• If the variance of a week's production is 25, then what can be said about the probability of this week's production will be between 40 & 60.

$$P(|X-50| > 10) \leq 25/100$$

$$\Rightarrow 1 - P(|X-50| > 10) \geq 75/100 \quad \text{(Chebyshev's inequality)}$$

—x—

* Bernoulli R.V; (Or Binomial R.V)

• A R.V X is said to be a bernoulli R.V if its pdf is given by $P(X=1) = p$
$P(X=0) = 1-p$

$$E(X) = p$$

—x—

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Binomial/Bernoulli's

* ~~Bernoulli's~~ distribution;

- Suppose there are n independent ~~trials~~ trials each of which results in a success with probability of p, then X is said to take a binomial R.V with parameters (n,p)

Proof: $f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x}$

$$\Rightarrow \boxed{f(x) = \frac{n!}{(n-x)!\,x!} \cdot p^x q^{n-x}} \rightarrow \text{P.D.F of Bernoulli random variable.}$$

DIY $\Rightarrow \sum_{k=0}^{n} f(k) = 1$ $\therefore$ P

$$\Rightarrow \sum_{k=0}^{n} = {}^nC_0 \times p^0 \times q^n + {}^nC_1 \times p^1 \times q^{n-1} + \cdots$$

$$= (p+q)^n = 1^n = 1$$

—x—

* Mean of Binomial ~~Bernoulli~~ distribution;

$$E(X) = \sum_{k=0}^{n} k f(k) = \sum_{k=0}^{n} k \cdot \frac{n!}{(n-k)!\,k!} \cdot p^k \cdot q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} \cdot p^k \cdot q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n \times (n-1)!}{(n-k)!(k-1)!} \times p \times p^{k-1} \times q^{n-k}$$

$$= np \sum \frac{(n-1)!}{(n-k)!(k-1)!} \cdot p^{k-1} \cdot q^{n-k}$$

$= np(p+q)^{n-1} \longrightarrow$ DIY (From binomial expansion)

$\therefore E(X) = np(p+q)^{n-1}$

But $p+q=1$

$\therefore \boxed{E(X) = np}$

DIY: Variance of ~~Bernoulli~~ Binomial distribution

** $\boxed{Var(X) = npq}$ $\quad (Var(X) = E(X^2) - (E(X))^2)$

$E(X^2) = \sum_{k=0}^{n} k^2 f(k) = \sum_{k=0}^{n} k^2 \times \frac{n!}{(n-k)!k!} \times p^k \times q^{n-k}$

$= n \sum_{k=0} \frac{k \times (n-1)!}{(n-k)!(k-1)!} \times p^k \times q^{n-k}$

$= n\left(\sum_{k=0} \frac{(n-1)!}{(n-k)!(k-2)!} \times p^k \times q^{n-k} + \sum_{k=0} \frac{(n-1)!}{(n-k)!(k-1)!} p^k q^{n-k}\right)$

$= n(n-1) \cdot p^2 + n \cdot p$

$= n^2p^2 - np^2 + np$

$\therefore Var(X) = n^2p^2 - np^2 + np - n^2p^2 = np - np^2$

$= np(1-p)$

$= npq$

—x—x—

** Poisson R.V ; ($n \to \infty$ & $p \to 0$ in binomial distribution)

Probability function (pdf)

$\Rightarrow P(x=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

$k = 0, 1, 2, \ldots$, λ = Expectation

$\lambda > 0$, parameter.

* E(X) of poisson distribution;

Consider,

- Moment generating fuction,

$$\phi(t) = E\left[e^{tX}\right] = \sum_{k=0}^{\infty} e^{tk} \cdot p(k)$$

$$= \sum_{k=0}^{\infty} \frac{e^{tk} \cdot e^{-\lambda} \cdot \lambda^k}{k!}$$

$$\Rightarrow \phi'(t) = \sum_{k=0}^{\infty} \frac{\lambda^k \cdot e^{-\lambda} \cdot k e^{+tk}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \cdot e^{+tk}}{(k-1)!}$$

$$= e^{-\lambda} \cdot \lambda e^{t} \sum_{k=1}^{\infty} \frac{\lambda^{k-1} \cdot e^{t(k-1)}}{(k-1)!}$$

$$= \lambda e^{t-\lambda} \sum_{k=1}^{\infty} \frac{(\lambda e^t)^{k-1}}{(k-1)!}$$

$$\phi'(t) = \lambda e^{t-\lambda} \cdot e^{\lambda e^t}$$

$$\phi'(0) = \lambda = \mu = E(X)$$

$$\therefore \boxed{E(X) = \lambda}$$

(Alternatively you can prove $E(X) = \sum_{k=0}^{n} \frac{k e^{\lambda}}{k}$

DIY $\phi''(0) = E(X^2)$, $\boxed{Var(X) = \lambda}$

$$\cancel{\phi''(t) = \lambda e^{t-\lambda} \times (\lambda e^t \times e^{\lambda e^t} \times \lambda + e^t}$$

$$\cancel{+ e^{\lambda e^t} \times \lambda(+e^{t-\lambda})}$$

$$\Rightarrow \cancel{\phi''(t) = e^{t-\lambda}\left(\lambda + e^{\lambda e^t}_{\times \lambda t}\right)}$$

$$\Rightarrow \cancel{\phi''(t) \text{ at } t=0,}$$

$$\cancel{\phi''(0) = \lambda e^{-\lambda} \times (\lambda \times e^{\lambda} \times 0) + e^{\lambda} \times \lambda}$$

$$\phi''(t) = \frac{\lambda}{e^{\lambda}}\left(e^{t} \times \lambda e^{t} \times e^{\lambda e^{t}} + e^{\lambda e^{t}} \times e^{t}\right)$$

$$\phi''(0) = \frac{\lambda}{e^{\lambda}}\left(\lambda \times e^{\lambda} + e^{\lambda}\right) = \lambda^2 + \lambda$$

$$Var(X) = \phi''(0) - \phi'(0)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

—— x ——

* Q) It is known that disks produced by a certain company will be defective with probability 0.01 independant of each other. The company sells disks in packages of 10 and offers a money back guarentee that at most 1 out of 10 disks is defective. What is the probability that the packages are returned. If someone buys 3 packages, what is the probability that he will return exactly one of them.

$X \rightarrow$ Number of defectives in a package.

$$\left.\begin{matrix} p = 0.01 \\ n = 10 \end{matrix}\right] X - \text{Binomial R.V}$$

$$P(X>1) = 1 - P(X \leq 1) = 1 - P(X=0) - P(X=1)$$

$$P(X>1) = 1 - {}^{n}C_0 \times (0.01)^0 \times (0.99)^{10} - {}^{n}C_1 \times (0.01) \times (0.99)^9$$

$$= 1 - 10(0.99)^9 \times (0.98)$$

$$= 1 - 9.8 \times (0.99)^9 = \delta \rightarrow (1)$$

- Now, lets consider R.V (X) = Number of packages returned. (Binomial distribution)

$$n = 3, \; p = \delta \quad \text{(From (1))}$$

$$\Rightarrow P(X=1) = {}^3C_1 \times (\delta)^1 \times (1-\delta)^2$$

—x—

Q) Flaws of a computer tape occurs at an average of one flaw per 1200 ft. If one assumes a poisson distribution, what is the pdf of the random variable X, the number of flaws in a 4800ft roll?

$$\lambda = 4$$

$$P(X=k) = \frac{e^{-\lambda} \times \lambda^k}{k!} = \frac{e^{-4} \times 4^k}{k!}$$

~~No. of flaws~~

- Find probability that you have atmost 2 flaws,

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-4} + e^{-4} \times 4 + \frac{e^{-4} \times 16}{2}$$

$$P(X \leq 2) = 13e^{-4}$$

—x—

* Normal distribution:

- A R.V is said to be normally distributed with mean μ and variance σ^2 (and we write X is $N(\mu, \sigma^2)$) if its pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

$$-\infty < \mu < \infty$$
$$0 < \sigma < \infty \quad , -\infty < x < \infty$$

a) $f \geq 0$, Show that f is pdf.

** b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{k^2}{2}} \cancel{\sigma}\, dk = I \qquad k = \frac{x-\mu}{\sigma}$$

$$\frac{dk}{dx} = \frac{1}{\sigma}$$

To solve this type of integral,

Consider, (~~put k = ...~~ ~~since we have substituted the limits~~)

$$I^2 = I \cdot I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx\, dy$$

Use $x = r\cos\theta, y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$ } we can compute these values (related to Jacobians)

$$\Rightarrow I^2 = \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} \cdot r\,dr\,d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} -\left[e^{-r^2/2}\right]_0^{\infty} d\theta$$

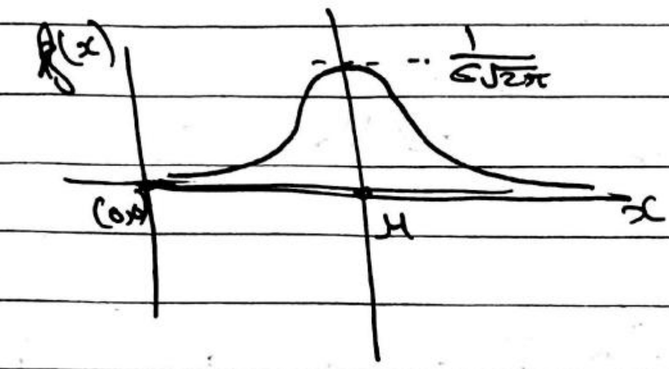
$$= \frac{1}{2\pi}\int_0^{2\pi} -[0-1]\,d\theta = \frac{1}{2\pi}\theta\Big]_0^{2\pi} = \frac{1}{2\pi}\cdot 2\pi = 1$$

$$\therefore I^2 = 1, \text{ So } I = 1$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

—x—

<u>Note:</u> Normal pdf is a bell shaped curve, symmetric about $x = \mu$ with a max value of $\frac{1}{\sigma\sqrt{2\pi}}$



—x—

<u>Note</u> $\Rightarrow$ If we shift the normal distibution curve to peak at $x=0$, we call it a standard distibution.

—x—

<u>Moment generating function</u>: (For Normal distribution)

$$\phi(t) = E(e^{tx}) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{tx} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2\sigma^2}((x-\mu)^2 - 2\sigma^2 tx)}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2\sigma^2}(x^2 - (2\mu + 2\sigma^2 t)x + \mu^2 \cdots}\,dx$$

$\Rightarrow$ The exponent can be rewritten as,

$$= \frac{-1}{2\sigma^2}\left((x-(\mu+\sigma^2 t))^2 + \mu^2 - (\mu+\sigma^2 t)^2\right)$$

$$\cancel{= \frac{-1}{2\sigma^2}\left(x^2 - (2\mu + 2\sigma^2 t)x + \mu^2\right)}$$

$$= \frac{-1}{2\sigma^2}\left[(x-(\mu+\sigma^2 t)\right]^2 + \frac{1}{2\sigma^2}\left[\sigma^4 t^2 + 2\mu\sigma^2 t\right]$$

• Putting this back,

$$\phi(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left[\frac{-1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2\right] * \exp\left[\frac{1}{2}(\sigma^2 t^2 + 2\mu t)\right] dx$$

$$\therefore \phi(t) = \exp\left(\frac{1}{2}(2\mu t + \sigma^2 t^2)\right) \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) \cdot dx$$

From previous $I^2 = 1$, $I = 1$ (Integration method)

$$\Rightarrow \text{But } \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right) dx = 1$$

$$* \therefore \boxed{\phi(t) = \exp\left(\frac{1}{2}(2\mu t + \sigma^2 t^2)\right)}$$

$\therefore$ Mean $= \phi'(0)$

$$\Rightarrow \phi'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times (\mu + \sigma^2 t)$$

$$\phi'(0) = e^0 \times (\mu) = \mu$$

$\therefore$ We get the mean $= \mu$

$$\text{Variance} = \phi''(0) - (\phi'(0))^2$$

$$\Rightarrow \phi''(t) = (\mu^2 + \sigma^2 t\mu) e^{\mu t + \frac{1}{2}\sigma^2 t^2} + \sigma^2 (t(\mu + \sigma^2 t) e^{\mu t + \frac{1}{2}\sigma^2 t^2} + e^{\mu t + \frac{1}{2}\sigma^2 t^2})$$

$$\therefore \phi''(0) = \mu^2 + \sigma^2$$

$$(\phi'(0))^2 = \mu^2$$

$$\therefore \text{Variance} = \sigma^2$$

—x—

* Standard or Unit normal distribution:

(Z is $N(0,1)$)

- X is $N(\mu, \sigma^2)$

then $Z = \frac{X-\mu}{\sigma}$ is normal R.V $N(0,$

$Z \rightarrow$ Standard or unit normal R.V

$\Rightarrow$ For a R.V $X \sim N(\mu, \sigma^2)$, Distribution function, $F(x) = P(X \leq x)$

$$\Rightarrow F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2\sigma^2}(\omega-\mu)^2\right] dt$$

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For a R.V $Z \sim N(0,1)$

$$F_Z(x) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2} \, dw}_{\text{Hard to get values, so approximate values will be given.}} = P(\cancel{Z} \leq x)$$

—x—

Q) Show ① $P(X<b) = F_Z\left(\frac{b-\mu}{\sigma}\right)$

② $P(a<X<b) = F_Z\left(\frac{b-\mu}{\sigma}\right) - F_Z\left(\frac{a-\mu}{\sigma}\right)$

$\rightarrow P(X<b) = P\left(\frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$

$= P\left(Z < \frac{b-\mu}{\sigma}\right)$

$= F_Z\left(\frac{b-\mu}{\sigma}\right)$

$\rightarrow P(a<X<b) = F_Z\left(\frac{b-\mu}{\sigma}\right) - F_Z\left(\frac{a-\mu}{\sigma}\right)$

—x—

Q) If X is $N(3,16)$ then find $P(4 \leq X \leq 8)$.

$\Rightarrow P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$

$= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$

$\Rightarrow \mu = 3, \sigma = 4,$

$\Rightarrow P\left(\frac{4-3}{4} \leq Z \leq \frac{8-3}{4}\right)$

$= P(0.25 \leq Z \leq 1.25)$

$= F_Z(1.25) - F_Z(0.25)$

From table with values of F_Z (standard normal distribution) $\{ = 0.89475 - 0.59871 = 0.29604$

—x—

For ~~as~~ a normal distribution,

(OIY)

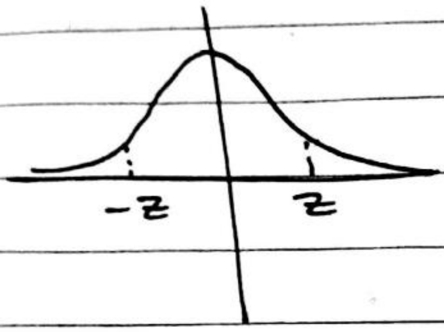
Note; $P(\mu-\sigma \le X \le \mu+\sigma) = 0.68$

** $P(\mu-2\sigma \le X \le \mu+2\sigma) = 0.955$

$P(\mu-3\sigma \le X \le \mu+3\sigma) = 0.997$

—— x ——

Note;



• For standard normal distribution

$$P(Z \ge z) = 1 - P(Z < z) = 1 - F_Z(z)$$

$$= -F_Z(z)$$

From symmetry

** $\therefore \boxed{P(Z \ge z) = -F_Z(z)}$

—— x ——

(OIY)

Q) ** If the resistance X of wires in an electrical network is normal with mean of 0.01 Ohm and standard deviation 0.001 Ohm. How many wires out of 1000, meet the specification that they have resistance between 0.001 & 0.011 Ohm.

$$P(0.001 \le X \le 0.011) = P\left(\frac{0.001-0.01}{0.001} \le Z \le \frac{0.011-0.01}{0.001}\right)$$

$$= F_Z\left(\frac{0.011-0.01}{0.001}\right) - F_Z\left(\frac{0.001-0.01}{0.001}\right)$$

$$=$$

Note: Always convert normal form to standard normal form and then use values of $F_Z(z)$ from table to evaluate the probabilities.

—x—

* Uniform Distribution;

- Let X be a R.V that denotes the outcome when a point is selected at random from an interval $[a,b]$, $a \leq b < \infty$. If the experiment is performed in a fair manner, then the probability that the point is selected from the interval $[a,x]$ $a \leq x < b$ is $\frac{x-a}{b-a}$.

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

$P(a \leq X \leq x)$

$X = U(a,b)$

$$* \boxed{f(x) = \frac{1}{b-a}}, \quad a < X < b \quad \text{if } X \text{ is continuous}$$

—x—

(Y) Show,

$$E(X) = \mu = \frac{a+b}{2}, \quad Var(X) = \sigma^2 = \frac{(b-a)^2}{12}$$

$$\phi(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$ for a uniform distribution.

(Moment generating function)

$$\phi(t) = \int_a^b \frac{1}{b-a} \times e^{tx} dx = \frac{1}{(b-a)\times t} e^{tx}\Big]_a^b$$

$$= \frac{1}{t(b-a)}\left(e^{tb} - e^{ta}\right)$$

$$\phi(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

? $$\phi'(t) = \frac{t(b-a)\left(be^{tb} - ae^{ta}\right) - \left(e^{tb} - e^{ta}\right)(b-a)}{t^2(b-a)^2}$$ (If $t \neq 0$) else 1.

$$E(x) = \int_a^b \frac{1}{b-a} \times x \times dx = \frac{x^2}{2(b-a)}\Big]_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{b+a}{2}$$

$Var(X) =$

—— x ——

Q) * Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10 minute period. Let X denote the time within the 10 minute period that the customer arrived. Find the probability that the customer arrives after 8 minutes.

$$P(X>8) = F(10) - F(8)$$

$$= 1 - \left(\frac{8-a}{b-a}\right) = 1 - \frac{8}{10}$$

$$= 0.2$$

—— x ——

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Page

Exponential Distribution;

Recap: Poisson: $f(k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$, $k = 0, 1, 2, \ldots$

Now consider, λ, λ is the mean number
Rate of changes that occur in unit interval.

* $\Rightarrow 1/\lambda$ is the mean waiting time for the first change. $1/\lambda = \theta \rightarrow E(k)$

* $\boxed{f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}}$, $0 \leq x < \infty$, $\theta > 0$

$X \rightarrow$ Continuous random variable.

- Let X denote waiting time until first change. CDF : $x > 0$,

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/\theta}, & x \geq 0 \end{cases}$$

Proof:
- For $x \geq 0$, $F(x) = P(X \leq x) = 1 - P(X > x)$

(No changes in $[0, x]$)

$$\Rightarrow F(x) = 1 - e^{-\lambda x}$$
$$= 1 - e^{-x/\theta}$$

$$\therefore F(x) = 1 - e^{-x/\theta} \quad (x \geq 0$$

$\boxed{f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}}$ (On differentiating $F(x)$ wrt x)

(DIY) Mean, $\mu = \theta$

Variance, $\sigma^2 = \theta^2$

$\Rightarrow \phi'(t) = \dfrac{\theta}{(1-\theta t)^2}$, $\phi'(0) = \theta = \mu$

$\rightarrow \phi''(t) = \dfrac{2\theta^2}{(1-\theta t)^3}$, $\phi''(0) = 2\theta^2$

$\text{Var}' = \phi''(t) - (\phi'(t))^2$ when $t = 0$

$\text{Var} = 2\theta^2 - \theta^2 = \theta^2$

—— x ——

(DIY) $M(t)$ = Moment generating function $= \phi(t) = \dfrac{1}{1-\theta t}, t$

$\phi(t) = \int_0^\infty e^{tk} \times p(k)\,dk = \int_0^\infty e^{tk} \times \frac{1}{\theta} \times e^{-k/\theta}\,dk$

Continuous R.V

$= \frac{1}{\theta}\int_0^\infty e^{(t-1/\theta)k}\,dk$

$= \frac{1}{\theta} \times \left(\frac{1}{t-1/\theta}\right) e^{(t-1/\theta)k}\Big]_0^\infty$

$= \dfrac{1/\theta}{t-1/\theta}(0-1)$

$\boxed{\phi(t) = \dfrac{1}{1-\theta t}} = \dfrac{\lambda}{\lambda - t}$

—— x ——

** Q) Customers arrive at a mean rate of 20 per hour according to an approximate poisson process. What is the probability that the shopkeeper will have to wait for more than 5 min, for the arrival of first customer.

$$\lambda = \frac{20}{60} = 1/3 \text{ per min}, \quad \theta = 3.$$

X is exponential distribution

$$P(X>5) = \int_5^\infty f(x)\,dx = \int_5^\infty \frac{1}{3} e^{-x/3}\,dx$$

$$= \frac{1}{3}\left[-3\left(0 - e^{-5/3}\right)\right] = e^{-5/3}$$

$$= 0.01389$$

—— x ——

(DIY)

Q) Let X be an exponential distribution with mean θ, show that

$$P(X > x+y \mid X > x) = P(X > y)$$

(Lowest memory property)

** Distribution with 2 or more R.V;

- Consider independant events. We say corrosponding R.V's are independent.

Let X_1 and X_2 be two R.V's,

X_1's space is R_1 $\quad X_1 : R_1 \to R$

X_2's space is R_2 $\quad X_2 : R_2 \to R$

Pick $A_1 \subset R_1$, $A_2 \subset R_2$

$P(x_1 \in A_1 \text{ and } x_2 \in A_2) = P(x_1 \in A_1) \cdot P(x_2 \in A_2)$

$\rightarrow$ ① (Independant events)

Joint probability density function

— x —

Note: * If ① is true for any $x_1 \in R_1$ & $x_2 \in R_2$ then we say RV's X_1 & X_2 are independent.

— x x —

* Q) Let X_1 be number obtained when a dice is rolled. $f_{X_1}(x_1) = 1/6$

$R_1 = \{1, 2, 3, 4, 5, 6\}$

Let X_2 be number of heads obtained on 4 independant tosses of a fair coin.

$R_2 = \{0, 1, 2, 3, 4\}$

- Use binomial R.V to model the coin to

Note:
Sampling distribution theory

$$pdf: f_{X_2}(x_2) = {}^4C_{x_2} \times \left(\frac{1}{2}\right)^{x_2} \times \left(\frac{1}{2}\right)^{4-x_2}$$

$$f_{X_2}(x_2) = {}^4C_{x_2} \times \frac{1}{16}$$

$\Rightarrow X_1$ & X_2 are independant R.V.

Compute:

a) $P(X_1 = 2 \text{ and } X_2 = 4) = \frac{1}{6} \times {}^4C_4 \times \frac{1}{16}$

$= P(X_1=2) \cdot P(X_2=4) = 1/96$

b) $P(X_1 = 1 \text{ or } 2 \text{ and } X_2 = 3 \text{ or } 4)$

$= P(X_1 = 1 \text{ or } 2) \cdot P(X_2 = 3 \text{ or } 4)$

$= \frac{2}{6} \times \left(\frac{1}{16} \times \left({}^4C_3 + {}^4C_4\right)\right)$

$= \frac{\cancel{2}}{6} \times \frac{1}{\cancel{16}\,8} \times (5) = \frac{5}{48}$

$\Rightarrow$ Consider $Y = X_1 + X_2$, $Y: R_3 \rightarrow R$

Space Y: $R_3 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Joint pdf of $X_1 + X_2 = Y = f(x_1, x_2)$ or $g(y)$

$g(1) = P(X_1 = 1 \text{ and } X_2 = 0)$

$g(2) = P((X_1 = 1 \text{ and } X_2 = 1) \text{ or } (X_1 = 2 \text{ and } X_2 = 0))$

$\vdots$

$\therefore \sum_y g(y) = 1$

—x—

Joint C.D.F : (For 2 independant R.V's)

- $F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2)$

- $P(a_1 < X_1 \le b_1, a_2 < X_2 \le b_2)$
  $= F(b_1, b_2) - F(a_1, a_2)$

— x —

Expected value of sum of independent R.V;

- $E(X_1 + X_2) = E(X_1) + E(X_2)$. $\Big\}$ Since its a linear operator.

$\Rightarrow$ If $X_1 + X_2$ are discrete R.V^s,

$$E(X_1 + X_2) = \sum_{x_2} \sum_{x_1} \left((x_1 + x_2) \cdot f_{X_1}(x_1) \cdot f_{X_2}(x_2)\right)$$

— x —

Theorem 1: Let $X_1, X_2, \ldots\ldots, X_n$ be n independent R.V's with joint pdf $f_1(x_1) \cdot f_2(x_2) \cdots f_n(x_n)$

Let the R.V $Y = \underbrace{u}_{\text{some Function}}(X_1, X_2, \ldots, X_n)$ have pdf $g(y)$, then in the discrete case

$$E(Y) = \sum_y y \cdot g(y).$$

$$E(Y) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} \left(u(x_1, x_2, \ldots, x_n) \cdot f_1(x_1) \cdot f_2(x_2) \cdots f_n(x_n)\right)$$

$\Rightarrow$ In the continuous case,

$$E(Y) = \int_{x_1=0}^{\infty} \cdots\cdots \int_{x_n=0}^{\infty} u(x_1, x_2, \ldots, x_n) \cdot f_1(x_1) \cdot \ldots \cdot f_n(x_n)\, dx_1 \cdot dx_2 \cdot \ldots \cdot dx_n$$

———x———

Theorem 2 : If $X_1, X_2, \ldots, X_n$ are independant R.V have pdf $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ respectively, and $E(u_i(x_i))$ exists $i = 1, 2, \ldots, n$

$$E(u_1(X_1) \cdot u_2(X_2) \cdot \ldots \cdot u_n(X_n)) = E(u_1(X_1)) \cdot E(u_2(X_2)) \cdot \ldots \cdot E(u_n(X_n))$$

Proof: Discrete case,

$$E(u_1(X_1) \cdot u_2(X_2) \cdot \ldots \cdot u_n(X_n))$$

$$= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} u_1(X_1) \cdot u_2(X_2) \cdot \ldots \cdot u_n(X_n) \cdot f_1(X_1) \cdot f_2(X_2) \cdot \ldots \cdot f_n(X_n)$$

$$= \sum_{x_1} (u(x_1) \cdot f_1(x_1)) \cdot \sum_{x_2} (u(x_2) f(x_2)) \cdot \ldots \cdot \sum_{x_n} (u(x_n) f(x_n))$$

$$= E(u_1(X_1)) \cdot E(u_2(X_2)) \cdot \ldots \cdot E(u_n(X_n))$$

Note: $E(g(x)) = \sum g(x) \cdot f(x)$, $f(x) \rightarrow$ pdf

———x———

Variance of 2 R.V's;

Note: Consider 2 R.V (Independant) (X_1 & X_2)

$$\Rightarrow Y = X_1 + X_2, \quad E(Y) = E(X_1) + E(X_2) = \mu_y$$

$$\Rightarrow \text{Variance} = \sigma_y^2 = E\left[(Y-\mu_y)^2\right]$$

$$= E\left[(X_1+X_2-\mu_{x_1}-\mu_{x_2})^2\right]$$

$$= E\left[\left[(X_1-\mu_{x_1}) + (X_2-\mu_{x_2})\right]^2\right]$$

$$= E\left[(X_1-\mu_{x_1})^2 + (X_2-\mu_{x_2})^2 + 2(X_1-\mu_{x_1})(X_2-\mu_{x_2})\right]$$

Since E is linear,

$$\sigma_y^2 = E\left[(X_1-\mu_{x_1})^2\right] + 2E\left[(X_1-\mu_{x_1})(X_2-\mu_{x_2})\right] + E\left[(X_2-\mu_{x_2})^2\right]$$

$$\Rightarrow \sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2E\left[(X_1-\mu_{x_1})(X_2-\mu_{x_2})\right]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2E(X_1-\mu_{x_1}) \cdot E(X_2-\mu_{x_2})$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2(E(X_1)-\mu_{x_1})(E(X_2)-\mu_{x_2})$$

$$\because E(X_1-\mu_{x_1}) = E(X_1) - E(\mu_{x_1}) = \mu_{x_1} - \mu_{x_1} = 0$$

$$\therefore \boxed{\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2}$$

Moment generating function; for 2 R.V's (Independant)

$Y = X_1 + X_2$

$$\Rightarrow M_y(t) = E(e^{tY})$$

$$= E\left(e^{t(X_1+X_2)}\right)$$

$$= E\left(e^{X_1 t} \cdot e^{X_2 t}\right)$$

$$= E(e^{X_1 t}) \cdot E(e^{X_2 t})$$

$$M_y(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

...rem 3: If $X_1, X_2, \dots, X_n$ are n independant R.V's with means $\mu_1, \mu_2, \dots, \mu_n$ and variance $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ then if $Y = \sum_{i=1}^{n} a_i X_i$, $a_i \in R$, $i = 1, 2, \dots, n$ (finite)

we have, $\mu_y = \sum_{i=1}^{n} a_i \mu_i \rightarrow (1)$

$$\sigma_y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2 \rightarrow (2)$$

$$② : \sigma_y^2 = E\left((Y-\mu_y)^2\right)$$

$$= E\left(\left(\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i \mu_i\right)^2\right)$$

$$= E\left[\left(\sum_{i=1}^{n} a_i (X_i - \mu_i)\right)^2\right]$$

$$\cancel{= \sum_{i=1}^{n} E\left(a_i (X_i - \mu_i)\right)}$$

$$= E\left[\sum_{i=1}^{n} a_i^2 (X_i - \mu_i)^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= E\left(\sum_{i=1}^{n} a_i^2 (X_i - \mu_i)^2\right) + 2E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= \sum_{i=1}^{n} E\left(a_i^2 (X_i - \mu_i)^2\right) + 0$$

∵ It 0 as we see it before

$$= \sum_{i=1}^{n} \sigma_i^2 \cdot a_i^2$$

—x—

Let independent R.V X_1, X_2 have means $\mu_1 = -4, \mu_2 = -3$, $\sigma_1^2 = 4, \sigma_2^2 = 9$.

μ & σ^2 of $Y = 3X_1 - 2X_2$

$$\mu_Y = \sum_i a_i \mu_i = -12 + 6 = -6$$

$$\sigma_y^2 = \sum_i a_i^2 \sigma_i^2 = 9(4) + 4(9)$$
$$= 9 \times 8 = 72$$

— x —

Sample Mean:

- Let $X_1, X_2, \ldots, X_n$ be n independant R.V's with same distribution with same mean μ & Variance σ^2. Define sample mean as
$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

Find $\mu_{\bar{X}}$ & $\sigma^2_{\bar{X}}$

Theorem 4: If $X_1, X_2, \ldots, X_n$ are independant R.V's with moment generating functions (MGF's) $M_{X_1}(t), M_{X_2}(t), \ldots, M_{X_n}(t)$ respectively.

a) MGF of $Y = \sum_{i=1}^{n} a_i X_i$

$$\boxed{M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)}, \quad a_i \in \mathbb{R}, \ i = 1, 2, \ldots$$

$$\Rightarrow M_Y(t) = E(e^{tY})$$

$$= E\left(e^{t(\sum_i a_i X_i)}\right)$$

$$= E\left(\prod_{i=1}^{n} e^{t(a_i X_i)}\right)$$

$$= \prod_{i=1}^{n} E\left(e^{t(a_i X_i)}\right)$$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

—— x ——

OIY

Q) If $X_1, X_2, \ldots, X_n$ are the outcomes of n Bernoulli trials, the MGF of X_i, $i = 1, 2, \ldots, n$ is,

All trials are equivalent

$$\boxed{M_{X_i}(t) = (1-p) + pe^t} \quad \text{— OIY}$$

Define $Y = \sum_{i=1}^{n} X_i$,

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \prod_{i=1}^{n} \left((1-p)+pe^{t}\right)$$

$$\boxed{M_Y(t) = \left((1-p)+pe^{t}\right)^{n}}$$

$\Rightarrow$ This is the MGF of a binomial distribution $B(n,p)$.

—x—

* MGF of Poisson Distribution:

* Poisson distribution,

$$f(K) = \frac{\lambda^{K} \cdot e^{-\lambda}}{K!} \quad , K = 0, 1, 2, \ldots,$$

$$M(t) = E(e^{tX}) = \boxed{e^{\lambda(e^{t}-1)} = M(t)}$$

$$M(t) = E(e^{tx}) = \sum_{k=0}^{\infty} \frac{e^{tk} \cdot \lambda^k \cdot e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \times e^{\lambda e^t}$$

$$M(t) = e^{\lambda(e^t - 1)}$$

—x—

Q) Let X_1, X_2, X_3 be independent RV's
* with poisson distribution having means
2, 1, 4 respectively.

a) Find MGF of $Y = X_1 + X_2 + X_3$
b) How is Y distributed

$\rightarrow$ $M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t)$

$$= e^{2(e^t-1)} \cdot e^{1(e^t-1)} \cdot e^{4(e^t-1)}$$

$$= e^{7(e^t-1)}$$

$\therefore$ Y is also a poisson distribution with $\lambda_Y = 7$.

DIV c) Compute $P(3 \leq Y \leq 5)$
- Write down new probability density function & use it.

Note: If $X_1, X_2, \ldots X_n$ are independent RV's with
** same mean μ & variance σ^2 then,
$Y = \sum_{i=1}^{n} X_i$ has the following properties.

1) $E(Y) = n\mu$
Y has a mean of $n\mu$
2) $\text{Variance}(Y) = n\sigma^2$
3) If $X_i = 1, 2, \ldots, n$ are normal R.V's,
then Y is a normal R.V.

—x—

** Central Limit Theorem;

*

- Let $X_1, X_2, \ldots, X_n$ be n independent RV's that have the same distribution function & they have same μ & variance σ^2.
- Let $Y_n = X_1 + X_2 + \ldots + X_n$.
- Then the R.V $W = \frac{Y_n - n\mu}{\sigma\sqrt{n}}$ is asymptotically normal, ie $\mu = 0$ & $\sigma^2 = 1$ ($N(0,1)$)

—x—

Note: ⇒ We know $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$ ⇒ $Y_n = n\bar{X}_n$

$$\Rightarrow W = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$$

—x—

(Asymptotic)
- So as $n \to \infty$, $\lim F_n(z) = F_Z(z)$

⇒ F_n is the cumulative density functio

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$$\Rightarrow \lim_{n\to\infty} P_n(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

(Using large sample sizes)

— x —

*Q) Let $\bar{X}$ denote sample mean of independent RV's
* X_i, $i = 1, 2, \ldots, 15$ of a certain distribution whose pdf is $f(x) = \frac{3}{2}x^2 - 1$, $-1 < x < 1$
Given $\mu = 0$, $\sigma^2 = 3/5$.

a) Find $P(0.03 < \bar{X} < 0.15)$ approximately.

$\Rightarrow$ We know $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow$ Standard Normal $(0,1)$ distribution by CLT

$$\Rightarrow P\left(\frac{0.03 - \mu}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{0.15 - \mu}{\sigma/\sqrt{n}}\right)$$

$$\Rightarrow P\left(\frac{0.03 \times \sqrt{15}}{\sqrt{3/5}} < Z < \frac{0.15 \times \sqrt{15}}{\sqrt{3/5}}\right)$$

$$\Rightarrow P(0.15 < Z < 0.75) = P(0.03 < \bar{X} < 0.15)$$

$\rightarrow N(0,1)$

$\therefore$ From tables,

$$F_Z(0.75) - F_Z(0.15) = \cancel{0.7422 - 0.8} = \cancel{0.1225}$$

$$= 0.7734 - 0.5596 = 0.2123$$

— x —

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$\Rightarrow Y = 50X_1 + 49X_2$, $\mu[X_1] = 0.5mm$
$\mu[X_2] = 0.05mm$

$\rightarrow E[Y] = 50E[X_1] + 49E[X_2]$
$= 25 + 2.45 = 27.45mm$

- Now $\sigma_y^2 = 50 \times 50 \, \sigma_{x_1}^2$
$+ 49 \times 49 \, \sigma_{x_2}^2$

$\sigma_y^2 = 2500 \times 25 \times 10^{-4}$
$+ 4 \times 10^{-4} \times 49 \times 49$

---x---

Marginal Distibution: (Bivariate distribution)

a) • Discrete distribution;

$\Rightarrow$ pdf $f(x,y)$, we want $P(X=x, Y=\text{arbitrary})$

pdf $f(x) = P(X=x, Y=\text{arbitary})$
$= \sum_y f(x,y)$

$\Rightarrow$ Marginal distribution of X is
$F_1(x) = P(X \leq x, Y=\text{arbitary})$

$$\therefore F_1(x) = \sum_{X^* \leq x} \cdot f_1(X^*)$$

- Marginal pdf,

$$f_2(y) = P(x = \text{arbitary}, Y = y)$$
$$= \sum_x f(x,y).$$

- Marginal distribution of Y is,

$$F_2(y) = P(X = \text{arbitary}, Y \leq y)$$

$$= \sum_{Y^* \leq y} f_2(Y^*)$$

b) <u>Continuous distribution:</u>

- X, Y are continuous R.V[s] with joint probability density function $f(x,y)$.

$\Rightarrow$ Then marginal distribution function

$$\cancel{F_1(x)} = F_1(x) = P(X \leq x, Y < \infty)$$

$$= \int_{-\infty}^{x} f_1(x^*) dx$$

Similarly $F_2(y)$

— x —

* $\Rightarrow$ Marginal density function

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$\Rightarrow \mu_x = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mu_y = \int_{-\infty}^{\infty} y f(y)\,dy$$

$\Rightarrow$ If x & y are independent $f(x,y) = f(x)\cdot f(y)$

* Discrete (X,Y)

Joint $P(x,y)$

Marginal density of X

$$P(X = x_i) = P_x(x_i) = \sum_y P(x_i, y)$$

Marginal density of Y

$$P(Y = y_i) = P_Y(y_i) = \sum_x P(x, y_i)$$

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* Covariance:

Defn: $\mathrm{Cov}(X,Y) = \sigma_{XY} = E[(X-\mu_x)(Y-\mu_y)]$
$= E(XY) - E(X)\cdot E(Y)$

$$\therefore \boxed{\sigma_{XY} = E[XY] - E[X]\cdot E[Y]}$$

- If $\mathrm{Cov}(X,Y) = 0$, then R.V's X & Y are uncorrelated.

Note: ** If two RV's are independent then they are uncorrelated but the reverse need not be true.

If (X,Y) is a bivariate random distribution with density $f(x,y)$.

a) $\Rightarrow$ Let X,Y be independent

$$\therefore f(x,y) = f_X(x) \cdot f_Y(y)$$

$$\Rightarrow E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx \cdot dy$$

$$= \int_{-\infty}^{\infty} y f_Y(y)\, dy \cdot \int_{-\infty}^{\infty} x f_X(x) \cdot dx$$

$$= E(X) \cdot E(Y)$$

$$\therefore Cov(XY) = 0$$

$\therefore$ They are uncorrelated.

~~b) If they are uncorrelated,~~

* $\Rightarrow$ Note converse is not true,

$\downarrow$

ex) Suppose the pdf of a bivariate R.V (X,Y) is given by

$$P(x_i, y_i) = \begin{cases} 1/3, & (1,0),(0,1),(2,1) \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ Want to check if $P(x, y) = P_x(x) \cdot P_y(y)$

$\Rightarrow P_X(x) = \sum_{y_i} p(x, y_i)$

$\therefore P_X(0) = \frac{1}{3}$, $P_X(1) = 1/3$, $P_X(2) = 1/3$

$\Rightarrow P_Y(y) = \sum_{x_j} p(x_j, y)$

$P_Y(0) = 1/3$, $P_Y(1) = 2/3$

$\Rightarrow$ At $(2,1)$, $P(2,1) = 1/3 \neq P_x(2) \cdot P_y(1)$

$\therefore$ (X, Y) are not independent

$\Rightarrow E(X, Y) = ?$

$E(X) = \sum_x x \, P_i(x) = 1/3 + 0 + 2/3$
$= 1$

$E(Y) = 2/3$

$\Rightarrow E(X,Y) = \sum_x \sum_y x \cdot y \, \cancel{P(X) \cdot P(Y)} \, P(x, y)$

$= 2 \times 1/3 = 2/3$

$\therefore$ We can see $Cov(X, Y) = 0$

$\therefore$ They are uncorrelated.

—— x ——

$\nearrow$ Different.

** Independent R.Vs $\rightarrow f(x,y) = f(x) \cdot f(y)$
Uncorrelated RVs $\rightarrow E(X,Y) = E(X) \cdot E(Y)$

—— x ——

DIY Q) Let (X, Y) be a bivariate R.V with joint
** PDF $f(x,y) = \frac{x^2+y^2}{4\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}$

$-\infty < x, y < \infty$

$\Rightarrow$ Show that they are not independent but are uncorrelated.

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———x———

** Correlation coefficient; $(\rho(x,y))$

$$\rho(x,y) = \boxed{\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}}$$

* Show $|\rho_{XY}| \leq 1$

Proof: $\sigma_{XY}^2 = \{E[(X-\mu_x)\cdot(Y-\mu_y)]\}^2$

now, we use cauchy - schwartz inequality,

———x———

** Cauchy Swchwartz Inequality;

- Bivariate R.V (X,Y)

$$\boxed{[E(XY)]^2 \leq E(X^2)\cdot E(Y^2)}$$

Proof: Start with $E[(X-\alpha Y)^2] \geq 0$

$$E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2] \geq 0$$

for any $\alpha \in R$, consider $\alpha = \frac{E[XY]}{E[Y^2]}$

$$\Rightarrow E[X^2] - \frac{2(E[XY])^2}{E[Y^2]} + \frac{(E[XY])^2}{E[Y^2]} \geq 0$$

$$\Rightarrow E[X^2] \geq \frac{(E[XY])^2}{E[Y^2]}, \therefore E[X^2]\cdot E[Y^2] \geq (E(XY))^2$$