Assignment-2

(1) $F = \left\{ \overset{F_1}{\begin{pmatrix}1&0\\0&0\end{pmatrix}}, \overset{F_2}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}, \overset{F_3}{\begin{pmatrix}0&0\\1&0\end{pmatrix}}, \overset{F_4}{\begin{pmatrix}0&0\\0&1\end{pmatrix}} \right\}$ is the basis of $\mathbb{R}^{2\times 2}$.

$B = \begin{pmatrix}-2&1\\3&4\end{pmatrix}$, $\quad L: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$, $L(M) = BM$, $\forall M \in \mathbb{R}^{2\times2}$.

Matrix representation of L

$L\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}-2&1\\3&4\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}-2&0\\3&0\end{pmatrix}$, $\quad L\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}-2&1\\3&4\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&-2\\0&3\end{pmatrix}$

$L\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}-2&1\\3&4\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\4&0\end{pmatrix}$, $\quad L\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}-2&1\\3&4\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\0&4\end{pmatrix}$

Matrix representation of L w.r.t Standard basis F of $\mathbb{R}^{2\times2}$ is

$$\begin{bmatrix}-2&0&1&0\\0&-2&0&1\\3&0&4&0\\0&3&0&4\end{bmatrix}$$

Since
$L\begin{pmatrix}1&0\\0&0\end{pmatrix} = -2F_1 + 0F_2 + 3F_3 + 0F_4$

$L\begin{pmatrix}0&1\\0&0\end{pmatrix} = 0F_1 - 2F_2 + 0F_3 + 3F_4$

$L\begin{pmatrix}0&0\\1&0\end{pmatrix} = 1F_1 + 0F_2 + 4F_3 + 0F_4$

$L\begin{pmatrix}0&0\\0&1\end{pmatrix} = 0F_1 + 1F_2 + 0F_3 + 4F_4$.

(2) $L: \mathbb{R}^2 \to \mathbb{R}^2$, Matrix representation of L is $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = A$.

Rotation by angle θ: Consider the point $(a,b) = (r\cos\psi, r\sin\psi)$

$r\cos(\theta+\psi) = a' = r\cos\theta\cos\psi - r\sin\theta\sin\psi = a\cos\theta - b\sin\theta$.

$b' = r\sin(\theta+\psi) = r\sin\theta\cos\psi + r\sin\psi\cos\theta = a\sin\theta + b\cos\theta$.

$$\begin{bmatrix}a'\\b'\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = B\begin{bmatrix}a\\b\end{bmatrix}.$$

Wherever B is of the form $\begin{bmatrix}\cos\theta' & -\sin\theta'\\ \sin\theta' & \cos\theta'\end{bmatrix}$, geometrically it means transformation rotates the point by angle θ'.

$L^2: \mathbb{R}^2 \to \mathbb{R}^2$, Matrix representation is $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

$= \begin{pmatrix}\cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta\end{pmatrix} = \begin{pmatrix}\cos2\theta & -\sin2\theta\\ \sin2\theta & \cos2\theta\end{pmatrix}$, geometrically it means L^2 rotates point by angle 2θ.

Similarly, Matrix representation of any L^i is $\begin{pmatrix}\cos i\theta & -\sin i\theta\\ \sin i\theta & \cos i\theta\end{pmatrix}$ $\forall i = 3,4,\ldots n$.

(3) Let V be n-dimensional vector space over $\mathbb{R}$ & W be m-dimensional vector-space over $\mathbb{R}$ with $n>m$.

let $T: V \to W$ & $U: W \to V$ be 2 linear transformations.

Suppose that $UT: V \to V$ is invertible. Then UT is both one-one & onto.

i) UT is one-to-one implies $\text{Nullspace}(UT) = \{0\}$. let $v \in V$ such that $T(v)=0$ i.e $v \in \text{Nullspace}(T)$, then $UT(v) = U(T(v)) = 0$

$\Rightarrow \text{Nullspace}(T) \subset \text{Nullspace}(UT) = \{0\} \Rightarrow \text{Nullspace}(T) = \{0\}$

i.e T should be one-to one which is a contradiction because $\dim(V) > \dim(W)$

ii) UT is onto implies that for all $v \in V$, there exists $v' \in V$ such that $UT(v') = U(T(v')) = v$, This implies that for all $v \in V$, there exists $T(v') \in W$ such that $U(T(v')) = UT(v') = v$, $\therefore$ U should be onto which is a contradiction because $\dim(V) < \dim(W)$.

So, $UT: V \to V$ is not invertible.

Scanned by CamScanner

Question 4:

Given vector space $V = \{ \sum_{j=1}^{2} (a_j \cos jx + b_j \sin jx) : a_j, b_j \in \mathbb{R} \}$

$= \{ a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x : a_1, a_2, b_1, b_2 \in \mathbb{R} \}$

x is some indeterminate

1. Find basis F for V :

A set is said to be a basis for a vector space iff it is linearly independent and spans the vector space.

Consider the set $F = \{ \cos x, \sin x, \cos 2x, \sin 2x \}$

To check if the set F is linearly independent we need to see if $\exists$ no non-zero scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_1 \cos x + \alpha_2 \sin x + \alpha_3 \cos 2x + \alpha_4 \sin 2x = 0$$

Using Taylor's series expansion,

$$\alpha_1 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) + \alpha_2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

$$+ \alpha_3 \left(1 + \frac{4x^2}{2!} + \frac{16x^4}{4!} + \dots \right) + \alpha_4 \left(2x - \frac{8x^3}{3!} + \dots \right) = 0$$

$\Rightarrow \alpha_1 + \alpha_3 = 0 \rightarrow (1)$

$\alpha_1 + 4\alpha_3 = 0 \rightarrow (2)$

$\alpha_2 + 2\alpha_4 = 0 \rightarrow (3)$

$\alpha_2 + 8\alpha_4 = 0 \rightarrow (4)$

(2) - (1) $3\alpha_3 = 0$

$\Rightarrow \alpha_3 = 0$

From (1) $\Rightarrow \alpha_1 = 0$

(4) - (3) $6\alpha_4 = 0$

$\Rightarrow \alpha_4 = 0$

$\Rightarrow \alpha_2 = 0$ (From (3))

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

Hence the set F is linearly independent.

Now we need to prove that the set F spans V.

Let $v_1 \in V$ where $v_1 = a_1 \cos x + a_2 \sin x + a_3 \cos 2x + a_4 \sin 2x$

$(a_1, a_2, a_3, a_4 \in \mathbb{R})$

We see that $v_1 \in \text{span}(F)$

$\therefore \quad v_1 \subseteq \text{span}(F) \quad \rightarrow (5)$

Let $v_2 \in \text{span}(F)$ where $v_2 = b_1 \cos x + b_2 \sin x + b_3 \cos 2x + b_4 \sin 2x$

We see that $v_2 \in V$

$\therefore \quad \text{Span}(F) \subseteq V \quad \rightarrow (6)$

From (5) & (6), $\quad \text{span}(F) = V$

$\therefore$ F spans V

Hence F forms a basis of vector span V.

2. $$L\left(\sum_{j=1}^{2} (a_j \cos jx + b_j \sin jx)\right) = \sum_{j=1}^{2} (-j a_j \sin jx + j b_j \cos jx)$$

Now we need to find matrix representation A of L w.r.t. basis F

Let ~~$F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$~~ $F = \{\beta_1, \beta_2, \beta_3, \beta_4\}$

where $\beta_1 = \cos x$, $\beta_2 = \sin x$, $\beta_3 = \cos 2x$, $\beta_4 = \sin 2x$

$$A = [L]_F = \left[[L\beta_1]_F \quad [L\beta_2]_F \quad [L\beta_3]_F \quad [L\beta_4]_F \right]$$

$L\beta_1 = L(\cos x) = -\sin x \Rightarrow [L\beta_1]_F = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$L\beta_2 = L(\sin x) = \cos x \Rightarrow [L\beta_2]_F = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$L\beta_3 = L(\cos 2x) = -2 \sin 2x \Rightarrow [L\beta_3]_F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

$L\beta_4 = L(\sin 2x) = 2 \cos 2x \Rightarrow [L\beta_4]_F = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

$$\therefore A = [L]_F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

Question 5:

Given: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad B = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$

TP: Matrices A and B are similar over field of complex numbers

Proof: Two matrices say A and B are said to be similar iff $\exists$ an invertible matrix P such that $P^{-1}AP = B$

i.e. $AP = PB$

Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume A and B are similar

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

$$\begin{bmatrix} a\cos\theta - c\sin\theta & b\cos\theta - d\sin\theta \\ a\sin\theta + c\cos\theta & b\sin\theta + d\cos\theta \end{bmatrix} = \begin{bmatrix} ae^{i\theta} & be^{-i\theta} \\ ce^{i\theta} & de^{-i\theta} \end{bmatrix}$$

$\Rightarrow a\cos\theta - c\sin\theta = ae^{i\theta}$ $\qquad b\cos\theta - d\sin\theta = be^{-i\theta}$

$a\cancel{\cos\theta} - c\sin\theta = a\cancel{\cos\theta} + ai\sin\theta$ $\qquad b\cancel{\cos\theta} - d\sin\theta = b\cancel{\cos\theta} - bi\sin\theta$

$\therefore \boxed{-c = ai}$ $\qquad \boxed{d = bi}$

$a\sin\theta + c\cos\theta = ce^{i\theta}$ $\qquad b\sin\theta + d\cos\theta = de^{-i\theta}$

$a\sin\theta + c\cancel{\cos}\theta = c\cancel{\cos}\theta + ci\sin\theta$ $\qquad b\sin\theta + d\cancel{\cos\theta} = d\cancel{\cos\theta} - di\sin\theta$

$\boxed{a = ci}$ $\qquad \boxed{b = -di}$

$\therefore P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ci & b \\ c & bi \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ i & i \end{bmatrix}$

where $c = i$
$b = 1$
$a = ci = -1$
$d = bi = i$
& $a, b, c, d \in \mathbb{C}$

To check if P is invertible

$|P| = -i - i = -2i \neq 0$

$\therefore$ P is invertible

Since there exists an invertible matrix P such that $P^{-1}AP = B$, matrices A and B are similar over field of complex numbers.

Given:

Question 6: V ~~is~~ is a finite-dimensional vectorspace

T is a linear operator on V.

$$\text{Rank}(T^2) = \text{Rank}(T)$$

TP: Range and nullspace of T intersect trivially i.e. have only zero vector in common.

Proof: $\because$ $\text{Rank}(T^2) = \text{Rank}(T)$

$\Rightarrow \dim(\text{range}(T^2)) = \dim(\text{range}(T)) \longrightarrow ①$

By Rank Nullity Theorem,

$\dim(T) = \dim(\text{Range}(T)) + \dim(\text{Nullspace}(T))$

$\dim(T^2) = \dim(\text{Range}(T^2)) + \dim(\text{Nullspace}(T^2))$

From ①, $\dim(T) - \dim(T^2) = \dim(\text{Nullspace}(T)) - \dim(\text{Nullspace}(T^2))$

$\dim(T) - \dim(\text{Nullspace}(T)) = \dim(T^2) - \dim(\text{Nullspace}(T^2))$

Let T^2 be represented by 2 Transforms T_1 and T_2 such that $T^2(\alpha) = T_2(T_1(\alpha))$

$\Rightarrow \text{Rank}(T^2) = \text{rank}(T_2) = \text{rank}(T) \rightarrow ②$

$\text{Rank}(T_2) = \dim(T_2) - \overset{\dim}{\cancel{\dim}}(\text{Nullspace}(T_2))$

$\text{Rank}(T_2) = \overset{\text{Rank}}{\cancel{\dim}}(T) - \overset{\dim}{}(\text{nullspace}(T_2))$ ($\dim(T_1) = \text{Rank } T$ $\because$ T is linear operator on V)

From ②, $\dim(\text{nullspace}(T_2)) = 0$

$\Rightarrow \text{nullspace}(T_2) = \{0\}$

In range of T_2, $\exists$ only $\alpha = 0$ s.t. $T_2\alpha = 0$

$\therefore \text{range}(T_2) \cap \text{Nullspace}(T_2) = \{0\}$

~~But range (T) = range (T_2)~~

But $\text{range}(T) = \text{range}(T)$

$\therefore \text{Range}(T) \cap \text{Nullspace}(T) = \{0\}$

Q7

⇒ Dimension of V is mn & dimension of W is pn. Now we know that if $\{\alpha_1, \alpha_2 -- \alpha_{mn}\}$ is basis for V, then $\{T\alpha_1, T\alpha_2, ---, T\alpha_{mn}\}$ is a basis for W, that an invertible linear transformation must take a basis to a basis. Thus if there's an invertible linear transformation between V & W, it must be that both spaces have the same dimension. Thus if T is invertible then $pn = mn$, implying $p = m$.

The matrix B is then invertible because the $B \to BX$ is one-one & non invertible matrices have non-trivial solutions to $BX = 0$. Conversely if $p = n$ & B is invertible, then we can define the inverse transformation T^{-1} by $T^{-1}(A) = B^{-1}A$ & it follows that T is invertible.

Q8

→

Using Rank Nullity Theorem

$\dim(V) = \dim(\text{range space}(T)) + \dim(\text{null space}(T))$

According to question
Range space & Null space of T are identical

$\dim(V) = \text{Rank}(T) + \text{Nullity}(T)$

Since $\text{Rank}(T) = \text{Nullity}(T)$

$\dim(V) = 2\,\text{Rank}(T)$
& n is dimension of V
So, $n = 2\,\text{Rank}(T)$
& Hence n is an even no.

Ex:- $T(x, y) = (x, 0)$

$\text{Nullity}(T) = 1 \quad ((0,0))$
$\text{Rank}(T) = 1 \quad (x(1,0))$
$\dim(V) = 2 \quad ((1,0), (0,1))$

Sol- 9(i).

We know that,

$$\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$$

here, $W = \mathbb{F}^1$, i.e one-dimentional vector space over $\mathbb{F}$.

$$\therefore \dim \mathcal{L}(V, \mathbb{F}) = \dim(V) \cdot \dim(\mathbb{F})$$
$$= \dim(V).$$

Sol- 9.2.

$V = \mathbb{F}^n$, $V^* := \mathcal{L}(\mathbb{F}^n; \mathbb{F})$

$T_i : (x_1, \dots x_n) \to x_i$

→ First we will prove $\{T_1 T_2 \dots T_n\}$ are lin. indp.

Suppose $c_1T_1 + c_2T_2 + \dots + c_nT_n = 0$ is a zero mapping to $\mathbb{F}$.

$c_i \in \mathbb{F}$

Let $\alpha \in V$

$$(c_1T_1 + \dots + c_nT_n)(\alpha) = 0 \quad \text{—— (1)}$$

$$c_1 T_1(\alpha) + \dots + c_n T_n(\alpha) = 0$$

where, $\alpha = (a_1, \dots a_n)$

$$\therefore c_1a_1 + \dots c_na_n = 0$$

eqn (1) ~~the~~ has to be true for all $\alpha \in V$

Let $\alpha_i \in V$ and $\alpha_i = (0, \dots, 1, \dots 0)$ —— (2)

$\uparrow$ i_{th} position

Putting α_i in eqn (1), we get

$$c_1 \cdot 0 + \dots + c_i \cdot 1 + \dots c_n \cdot 0 = 0$$

$$\Rightarrow c_i = 0,$$ ~~This is true for case~~

Similary, we can get $c_1 = c_2 = \dots = c_n = 0$ if we put $1 \le i \le n$ in eqn (2).

$\therefore T_1, T_2, \dots T_n$ are linearly independent. —— (3)

Now, we will prove that $\{T_1 \dots T_n\}$ spans V^*.

From 9(a) we know that,

$$\dim L(V,F) = \dim V$$

$$\therefore \dim V^* = \dim V = n.$$

~~because~~ As $T_1 \dots T_n$ are lin. indp. ~~and~~

and the dimension of V^* is also 'n',

Hence $\{T_1 \dots T_n\}$ spans V^*. — (4)

from (3) & (4)

$\{T_1 \dots T_n\}$ ~~spans~~ forms basis of V^*.

Q10

$\Rightarrow$ Since W_i, $i = 1 \ldots r$ are T-invariant subspaces
$V = \bigoplus_{i=1}^{r} W_i$;

W_i's are the disjoint subspaces.
When T acts on a vector in W_i, it won't affect other Ws.

Let's say it is $A_1 v \in W$, where $v \in W_1$

Here the linear transformation is matrix A_1, taking vectors from W_1 to W_1. So if we want to extend A_1 to whole vector space V, A_1 elements will be 0 that will act on the basis of other W's.

$$\left(\begin{bmatrix} [b_1^1] \\ 0 \end{bmatrix} \quad \begin{bmatrix} [b_1^2] \\ 0 \end{bmatrix} \quad - - - - - \quad \begin{bmatrix} [b_1^{n_1}] \\ 0 \end{bmatrix} \right)$$

1, n_1

$\sum_{i=1} (n_i) \times 1$

$\uparrow$

Basis of extended A_1 that will act on whole V.

In the same way, we can write other basis,

Extended A_2

$\downarrow$

$$\left(\begin{bmatrix} 0 \\ [b_2^1] \end{bmatrix} \quad \begin{bmatrix} 0 \\ [b_2^2] \end{bmatrix} \quad - - - - \quad \begin{bmatrix} 0 \\ [b_2^{n_2}] \end{bmatrix} \right)$$

n_1, n_2

|

|

$$\left(\begin{array}{c} \\ \sum_{i=1}^{r-1} n_i \\ \sum_{i=1}^{r} n_i \end{array} \begin{bmatrix} 0 \\ [b_r^1] \end{bmatrix} \begin{bmatrix} 0 \\ [b_r^2] \end{bmatrix} - - - - - - \begin{bmatrix} 0 \\ [b_r^{n_r}] \end{bmatrix} \right)$$

Extended A_r

so

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 & - - \\ 0 & A_2 & 0 & - - & - - \\ | & & & & \\ | & & & & \\ 0 & - - & - - & - - & A_r \end{bmatrix}$$