Existence and Non-Existence of Solution of System of Linear Equations

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MA3.101 - Linear Algebra

Theorem . Consider a linear system $A\mathbf{x} = \mathbf{b}$, where A is a $m \times n$ matrix, and $\mathbf{x}$, $\mathbf{b}$ are vectors with orders $n \times 1$, and $m \times 1$, respectively. Suppose rank $(A) = r$ and $\text{rank}([A \;\; \mathbf{b}]) = r_a$. Then exactly one of the following statement holds:

1. If $r_a = r < n$, the set of solutions of the linear system is an infinite set and has the form
$$\{\mathbf{u}_0 + k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots \mathbf{u}_{n-r} \;:\; k_i \in \mathbb{R},\; 1 \leq i \leq n-r\}$$
where $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-r}$ are $n \times 1$ vectors satisfying $A\mathbf{u}_0 = \mathbf{b}$ and $A\mathbf{u}_i = \mathbf{0}$ for $1 \leq i \leq n-r$.

2. If $r_a = r = n$, the solution set of the linear system has a unique $n \times 1$ vector $\mathbf{x}_0$ satisfying $A\mathbf{x}_0 = \mathbf{0}$.

3. If $r < r_a$, the linear system has no solution.

Proof. Suppose $[C \;\; \mathbf{d}]$ is the row reduced echelon form of the augmented matrix $[A \;\; \mathbf{b}]$. Then by the solution set of the linear system $C\mathbf{x} = \mathbf{d}$ is same as the solution set of the linear system $A\mathbf{x} = \mathbf{b}$. So, the proof consists of understanding the solution set of the linear system $C\mathbf{x} = \mathbf{d}$.

1. Let $r = r_a < n$. Then $[C \;\; \mathbf{d}]$ has its first r rows as the non-zero rows. So, the matrix C has r leading columns. Let the leading columns be $1 \leq k_1 < k_2 < \cdots < k_r \leq n$. Then we observe the following.

   (a) The entries c_{ik_i} for $1 \leq i \leq r$ are leading terms. That is, for $1 \leq i \leq r$, all entries in the k_i^{th} column of C is zero, except the entry c_{ik_i}. The entry $c_{li_l} = 1$.

   (b) corresponding is each leading column, we have r BASIC VARIABLES, $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$.

   (c) The remaining $n-r$ columns correspond to the $n-r$ FREE VARIABLES, $x_{p_1}, x_{p_2}, \ldots, x_{p_{n-r}}$. So, the free variables correspond to the columns $1 \leq p_1 < p_2 < \cdots < p_{n-r} \leq n$.

For $1 \le i \le r$, consider the i^{th} row of $[C \;\; \mathbf{d}]$. The entry $c_{ik_i} = 1$ and is the leading term. Also, the first r rows of the augmented matrix $[C \;\; \mathbf{d}]$ give rise to following linear equations.

$$
\begin{aligned}
x_{k_i} + \sum_{j=1}^{n-r} c_{ip_j} x_{p_j} &= d_i, \quad \text{for} \quad 1 \le i \le r \\
\Rightarrow x_{k_i} &= d_i - \sum_{j=1}^{n-r} c_{ip_j} x_{p_j}, \quad \text{for} \quad 1 \le i \le r
\end{aligned}
$$

Let $\mathbf{y}^t = (x_{k_1}, \ldots, x_{k_r}, x_{p_1}, \ldots, x_{p_{n-r}})$. Then the set of solutions consists of

$$
\mathbf{y} = \begin{bmatrix} x_{k_1} \\ \vdots \\ x_{k_r} \\ x_{p_1} \\ \vdots \\ x_{p_{n-r}} \end{bmatrix} = \begin{bmatrix} d_1 - \sum\limits_{j=1}^{n-r} c_{1p_j} x_{p_j} \\ \vdots \\ d_r - \sum\limits_{j=1}^{n-r} c_{rp_j} x_{p_j} \\ x_{p_1} \\ \vdots \\ x_{p_{n-r}} \end{bmatrix}
$$

As x_{p_s} for $1 \le s \le n-r$ are free variables, let us assign arbitrary constants $a_s \in \mathbb{R}$ to x_{p_s}. That is, for $1 \le s \le n-r$, $\;x_{p_s} = a_s$. Then the set of solutions is given by

$$
\begin{aligned}
\mathbf{y} = \begin{bmatrix} x_{k_1} \\ \vdots \\ x_{k_r} \\ x_{p_1} \\ \vdots \\ x_{p_{n-r}} \end{bmatrix} &= \begin{bmatrix} d_1 - \sum\limits_{j=1}^{n-r} c_{1p_j} a_j \\ \vdots \\ d_r - \sum\limits_{j=1}^{n-r} c_{rp_j} a_j \\ x_{p_1} \\ \vdots \\ x_{p_{n-r}} \end{bmatrix} \\
&= \begin{bmatrix} d_1 \\ \vdots \\ d_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} -c_{1p_1} \\ \vdots \\ -c_{rp_1} \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -c_{1p_2} \\ \vdots \\ -c_{rp_2} \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_{n-r} \begin{bmatrix} -c_{1p_{n-r}} \\ \vdots \\ -c_{rp_{n-r}} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
&= \mathbf{v}_0 + a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_{n-r} \mathbf{v}_{n-r}
\end{aligned}
$$

Let us write ${\mathbf{v}_0}^t = (d_1, d_2, \ldots, d_r, 0, \ldots, 0)^T$. Also, for $1 \leq j \leq n-r$, let $\mathbf{v}_j$ be the vector associated with a_j in the above representation of the solution $\mathbf{y}$. Observe the following:

(a) If we assign $a_s = 0$, for $1 \leq s \leq n-r$, we get
$$C\mathbf{v}_0 = C\mathbf{y} = \mathbf{d}.$$

(b) If we assign $a_1 = 1$ and $a_s = 0$, for $2 \leq s \leq n-r$, we get
$$\mathbf{d} = C\mathbf{y} = C(\mathbf{v}_0 + \mathbf{v}_1).$$
So, we get $C\mathbf{v}_1 = \mathbf{0}$.

(c) In general, if we assign $a_t = 1$ and $a_s = 0$, for $1 \leq s \leq n-r; s \neq t$, we get $\mathbf{d} = C\mathbf{y} = C(\mathbf{v}_0 + \mathbf{v}_t)$. So, we get $C\mathbf{v}_t = \mathbf{0}$.

Note that a rearrangement of the entries of $\mathbf{y}$ will give us the solution vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Suppose that for $0 \leq j \leq n-r$, the vectors $\mathbf{u}_j$'s are obtained by applying the same rearrangement to the entries of $\mathbf{v}_j$'s which when applied to $\mathbf{y}$ gave $\mathbf{x}$. Therefore, we have $C\mathbf{u}_0 = \mathbf{d}$ and $C\mathbf{u}_j = \mathbf{0}$, $1 \leq j \leq n-r$. Now, using equivalence of the linear system $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ gives $A\mathbf{u}_0 = \mathbf{b}$ and $A\mathbf{u}_j = \mathbf{0}$, $1 \leq j \leq n-r$. Thus, we have obtained the desired result for the case $r = r_1 < n$.

2. $r = r_a = n$, $m \geq n$.

Here the first n rows of the row reduced echelon matrix $[C \;\; \mathbf{d}]$ are the non-zero rows. Also, the number of columns in C equals $n =$ rank $(A) =$ rank (C). So, all the columns of C are leading columns and all the variables $x_1, x_2, \ldots, x_n$ are basic variables. Thus, the row reduced echelon form $[C \;\; \mathbf{d}]$ of $[A \;\; \mathbf{b}]$ is given by
$$[C \;\; \mathbf{d}] = \begin{bmatrix} I_n & \tilde{\mathbf{d}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$
Therefore, the solution set of the linear system $C\mathbf{x} = \mathbf{d}$ is obtained using the equation $I_n\mathbf{x} = \tilde{\mathbf{d}}$. This gives us, a solution as $\mathbf{x}_0 = \tilde{\mathbf{d}}$. That is, the solution set consists of a single vector $\tilde{\mathbf{d}}$.

3. $r < r_a$.

As C has n columns, the row reduced echelon matrix $[C \;\; \mathbf{d}]$ has $n+1$ columns. The condition, $r < r_a$ implies that $r_a = r+1$. We now observe the following:

as rank$(C) = r$, the $(r+1)$ th row of C consists of only zeros. Whereas the condition $r_a = r+1$ implies that the $(r+1)^{\text{th}}$ row of the matrix $[C \;\; \mathbf{d}]$ is non-zero. Thus, the $(r+1)^{\text{th}}$ row of $[C \;\; \mathbf{d}]$ is of the form $(0, \ldots, 0, 1)$. Or in other words, $\mathbf{d}_{r+1} = 1$. Thus, for the equivalent linear system $C\mathbf{x} = \mathbf{d}$, the $(r+1)^{\text{th}}$ equation is

$0\,x_1 + 0\,x_2 + \cdots + 0\,x_n = 1$. This linear equation has no solution. Hence, in this case, the linear system $C\mathbf{x} = \mathbf{d}$ has no solution. Therefore, the linear system $A\mathbf{x} = \mathbf{b}$ has no solution.

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