Bayes Rule & Independence

Lecture 2

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Reading

Read: [BT] Section 1.3, 1.4. [OCW] Lecture 2.

Explore: [WF] Chapter 5, Sections 1, 2, 3

[BT] Introduction to Probability, 2nd Edition by Dimitri P. Bertsekas and John N. Tsitsiklis.

[WF] An Introduction to Probability Theory and Its Applications, Volume 1 by William Feller.

[OCW] Probabilistic Systems Analysis and Applied Probability Online Resource. MIT OCW https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-041sc-probabilistic-systems-analysis-and-applied-probability-fall-2013/index.htm

Grading and Assignments

Relative but closer to Absolute.

Assignments

- Practice problems are given in each lecture.
- More solved problems available as recitations/problem sets in MIT OCW.
- 30 min (2-3 problems) test every Tutorial (Friday 2PM) based on above.
- Scores of the best 10/12 or 8/10 test for giving the 20% marks.

Doubt clearing and additional problem solving during tutorials (2-330PM Friday).

Previously

- Sample Spaces
  - Counting
  - Permutations Combinations
  - Balls and Bins
  - Continuous Sample Spaces
- Probability Law
  - Axioms
  - Examples

Review of probability models

- Sample space Ω
  - Mutually exclusive
    Collectively exhaustive
  - Right granularity
- Event: Subset of the sample space
- Allocation of probabilities to events

1. $\mathbf{P}(A) \geq 0$
2. $\mathbf{P}(\Omega) = 1$
3. If $A \cap B = \emptyset$,
   then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

3'. If $A_1, A_2, \ldots$ are disjoint events, then:
$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots$

- Problem solving:
  - Specify sample space
  - Define probability law
  - Identify event of interest
  - Calculate...

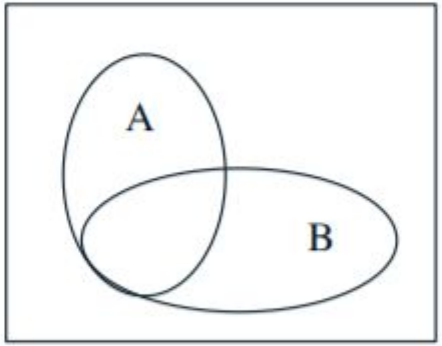
Problem 1

12 people wants to play badminton doubles tournament at IIIT. In how many ways can they form teams?

Problem 5

In a group of 5 people, every pair of them could be friends or non-friends. Assume that all possible relationships are equally likely. Find the probability of the event that there are at least three people who are mutually friends. (Try to write down the possibilities. Divide cleverly and conquer!)

Conditional probability



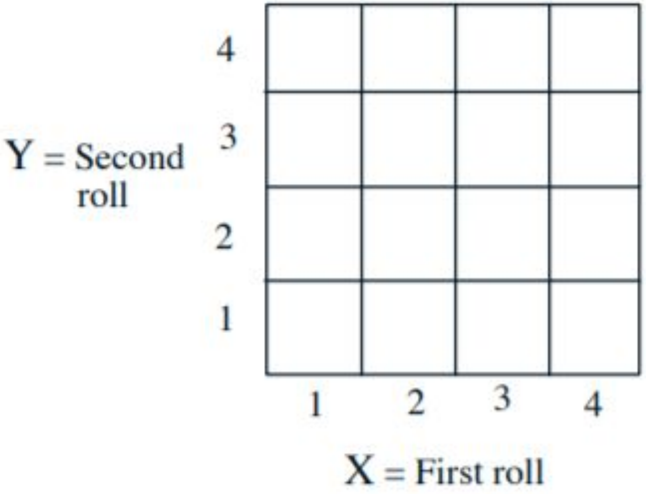
- $\mathbf{P}(A \mid B) =$ probability of A,
  given that B occurred
  - B is our new universe

- **Definition:** Assuming $\mathbf{P}(B) \neq 0$,

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$$

$\mathbf{P}(A \mid B)$ undefined if $\mathbf{P}(B) = 0$

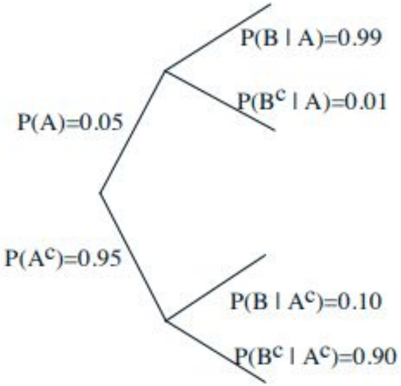
Die roll example



- Let B be the event: $\min(X, Y) = 2$
- Let $M = \max(X, Y)$
- $\mathbf{P}(M = 1 \mid B) =$
- $\mathbf{P}(M = 2 \mid B) =$

Models based on conditional probabilities

- Event A: Airplane is flying above
  Event B: Something registers on radar screen



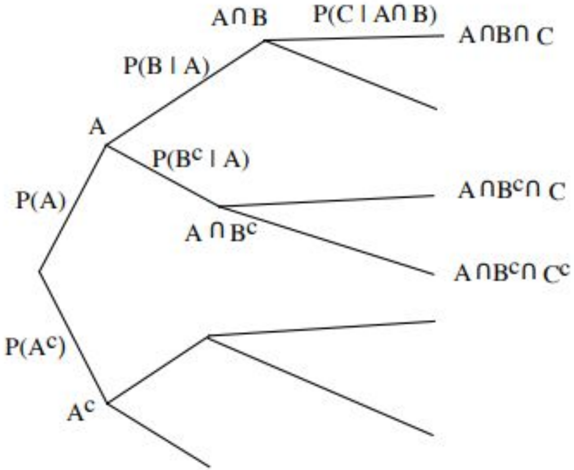
$\mathbf{P}(A \cap B) =$

$\mathbf{P}(B) =$

$\mathbf{P}(A \mid B) =$

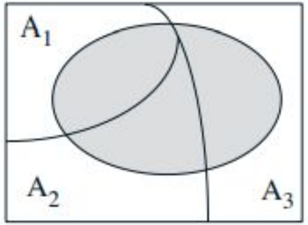
Multiplication rule

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A) \cdot \mathbf{P}(C \mid A \cap B)$$



Total probability theorem

- Divide and conquer
- Partition of sample space into A_1, A_2, A_3
- Have $\mathbf{P}(B \mid A_i)$, for every i

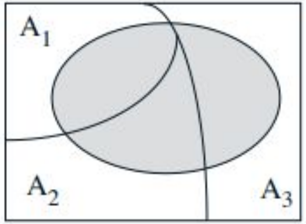


- One way of computing $\mathbf{P}(B)$:

$$\begin{aligned}\mathbf{P}(B) = \quad & \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) \\ + \; & \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) \\ + \; & \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)\end{aligned}$$

Bayes' rule

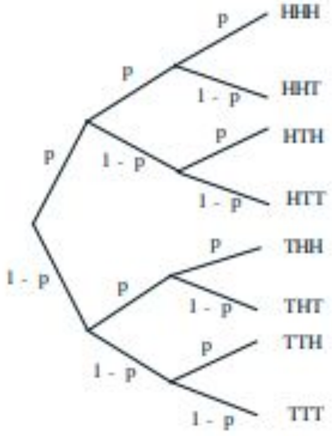
- "Prior" probabilities $\mathbf{P}(A_i)$
  - initial "beliefs"
- We know $\mathbf{P}(B \mid A_i)$ for each i
- Wish to compute $\mathbf{P}(A_i \mid B)$
  - revise "beliefs", given that B occurred



$$\begin{aligned}\mathbf{P}(A_i \mid B) &= \frac{\mathbf{P}(A_i \cap B)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\sum_j \mathbf{P}(A_j)\mathbf{P}(B \mid A_j)}\end{aligned}$$

Models based on conditional probabilities

- 3 tosses of a biased coin:
  $\mathbf{P}(H) = p$, $\mathbf{P}(T) = 1 - p$



$\mathbf{P}(THT) =$

$\mathbf{P}(1 \text{ head}) =$

$\mathbf{P}(\text{first toss is H} \mid 1 \text{ head}) =$