Probability and Counting
Lecture 1

Girish Varma

Grading

- Weekly Assignments: 20%
  - 8 in total, 2 per quarter.
  - Deadlines at the begining of tutorials.
  - Questions based on assignments will be asked to random student during tutorials. So make sure you understand your solutions.
- Quiz 1, Quiz 2: 10% each
  - Weight: 10% each
- MidSem: 25%
- EndSem: 35%
- Bonus Marks: 5%
  - For questions based on Bonus Topics

Textbook and References

Introduction to Probability, 2nd Edition by Dimitri P. Bertsekas and John N. Tsitsiklis.

An Introduction to Probability Theory and Its Applications, Volume 1 by William Feller.

Introduction to Probability and Statistics for Engineers and Scientists by Sheldon M. Ross.

Probabilistic Systems Analysis and Applied Probability Online Resource. MIT OCW
https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-041sc-probabilistic-systems-analysis-and-applied-probability-fall-2013/index.htm

Website: https://girishvarma.in/teaching/prob-stat/

How to Ace the course with min effort?

1. *Attend* all the lectures (90 mins each).
2. *Read* the corresponding topics in the textbook on the same day (1 hr after each lecture).
3. *Solve* the problems for the previous lecture before the next lecture (2 hr after each lecture).
4. *Ask* doubts and clear them during tutorials (60 mins).
5. *Explore* bonus topics for fun and get bonus points.

So weekly there is 4 hrs of classroom time and 6 hrs of homework.

If you are lagging behind, use the online course material to catch up. There are also 1-2 buffer classes in each quarter, to help you. If you are already comfortable with the topics, the buffer classes gives you time to explore bonus topics.

Summary

Sample Space

Counting

Balls and Bins

Permutations and Combinations

Probabilistic models

- Axioms of probability
- Simple examples

Sample Space

Set of all possible outcomes

Coin Toss

A Pair of Dies?

Balls and Bins

3 balls in 3 bins

#	Bin 1	Bin 2	Bin 3
1.	*abc*	–	–
2.	–	*abc*	–
3.	–	–	*abc*
4.	*ab*	*c*	–
5.	*a c*	*b*	–
6.	*bc*	*a*	–
7.	*ab*	–	*c*
8.	*a c*	–	*b*
9.	*bc*	–	*a*
10.	*a*	*bc*	–
11.	*b*	*a c*	–
12.	*c*	*ab*	–
13.	*a*	–	*bc*
14.	*b*	–	*a c*
15.	*c*	–	*ab*
16.	–	*ab*	*c*
17.	–	*a c*	*b*
18.	–	*bc*	*a*
19.	–	*a*	*bc*
20.	–	*b*	*a c*
21.	–	*c*	*ab*
22.	*a*	*b*	*c*
23.	*a*	*c*	*b*
24.	*b*	*a*	*c*
25.	*b*	*c*	*a*
26.	*c*	*a*	*b*
27.	*c*	*b*	*a*

A: One cell has multiple balls. B:First bin is not empty

C: Both A and B occur

Examples

Birthdays

10 throws of dies

Indistinguishable

$$1.\ \{\ast\ast\ast \mid - \mid - \}$$
$$2.\ \{ - \mid \ast\ast\ast \mid - \}$$
$$3.\ \{ - \mid - \mid \ast\ast\ast \}$$
$$4.\ \{\ast\ast \mid \ast \mid - \}$$
$$5.\ \{\ast\ast \mid - \mid \ast \}$$
$$6.\ \{ \ast \mid \ast\ast \mid - \}$$
$$7.\ \{ \ast \mid - \mid \ast\ast \}$$
$$8.\ \{ - \mid \ast\ast \mid \ast \}$$
$$9.\ \{ - \mid \ast \mid \ast\ast \}$$
$$10.\ \{ \ast \mid \ast \mid \ast \}.$$

Ordered Samples

Sampling without Replacements

Sampling with Replacements

Unordered Samples

Choose a subgroup

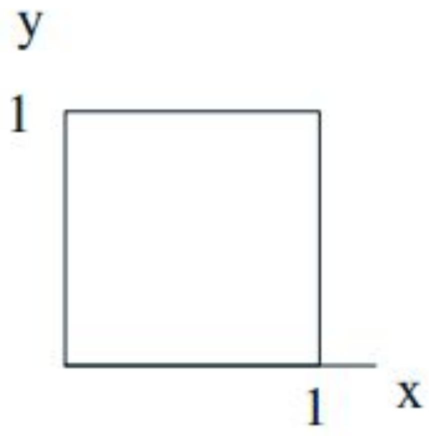
Choose 8 TAs from 50 TA applications

Partition into Subgroups

52 deck card given to 4 players

Sample space: Continuous example

$$\Omega = \{(x, y) \mid 0 \le x, y \le 1\}$$



Probability axioms

- **Event:** a subset of the sample space
- Probability is assigned to events

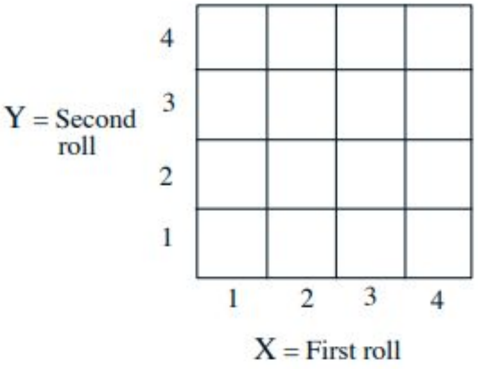
Axioms:

1. **Nonnegativity:** $\mathbf{P}(A) \geq 0$
2. **Normalization:** $\mathbf{P}(\Omega) = 1$
3. **Additivity:** If $A \cap B = \emptyset$, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$

- $$\begin{aligned} \mathbf{P}(\{s_1, s_2, \ldots, s_k\}) &= \mathbf{P}(\{s_1\}) + \cdots + \mathbf{P}(\{s_k\}) \\ &= \mathbf{P}(s_1) + \cdots + \mathbf{P}(s_k) \end{aligned}$$

- Axiom 3 needs strengthening
- Do weird sets have probabilities?

Probability law: Example with finite sample space



- Let every possible outcome have probability 1/16
  - $\mathbf{P}((X, Y)$ is (1,1) or (1,2)$) =$
  - $\mathbf{P}(\{X = 1\}) =$
  - $\mathbf{P}(X + Y$ is odd$) =$
  - $\mathbf{P}(\min(X, Y) = 2) =$

Discrete uniform law

- Let all outcomes be equally likely

- Then,

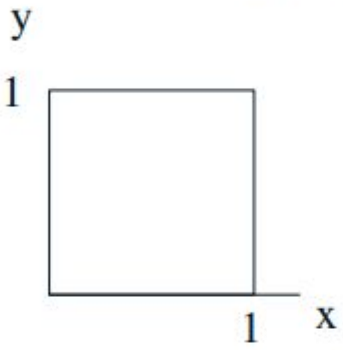
$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}}$$

- Computing probabilities $\equiv$ counting

- Defines fair coins, fair dice, well-shuffled card decks

Continuous uniform law

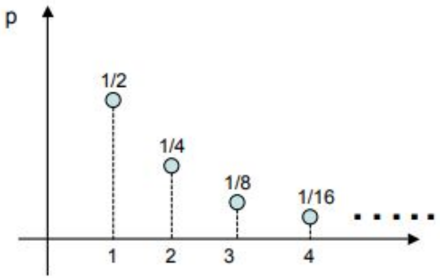
- Two "random" numbers in $[0, 1]$.



- **Uniform** law: Probability = Area
  - $\mathbf{P}(X + Y \leq 1/2) = ?$
  - $\mathbf{P}((X, Y) = (0.5, 0.3))$

Probability law: Ex. w/countably infinite sample space

- Sample space: $\{1, 2, \ldots\}$
  - We are given $\mathbf{P}(n) = 2^{-n}$, $n = 1, 2, \ldots$
  - Find $\mathbf{P}$(outcome is even)



$$\mathbf{P}(\{2, 4, 6, \ldots\}) \;=\; \mathbf{P}(2) + \mathbf{P}(4) + \cdots = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdot$$

- Countable additivity axiom (needed for this calculation):
  If $A_1, A_2, \ldots$ are disjoint events, then:

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots$$

Problem 1

12 people wants to play badminton doubles tournament at IIIT. In how many ways can they form teams?

Problem 2

A box contains 1 coupon labelled 1, 2 coupons labelled 2, and so on up to 10 coupons labelled 10. Two distinct coupons are drawn at random from the box. Write the sample space. Find the probability of the event that 2 coupons carry the same number.

Problem 3

Let A_1, A_2, A_3, A_4 be events. Let B be the event that at least two of A_is occur. Prove using the axioms that:

$P(B) = S_2 - 2S_3 + 3S_4$ where $S_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \mathbf{P}\{A_{i_1} \cap \ldots \cap A_{i_k}\}$ for $1 \le k \le n$.

and n=4

Problem 4

Problem 42. Place r_m distinguishable balls in m distinguishable bins. Let A_m be the event that at least one bin is empty[3].

(1) If $r_n = m^2$, show that $\mathbf{P}(A_m) \to 0$ as $m \to \infty$.

(2) If $r_m = Cm$ for some fixed constant C, show that $\mathbf{P}(A_m) \to 1$ as $n \to \infty$.

(3) Can you find an increasing function $f(\cdot)$ such that if $r_m = f(m)$, then $\mathbf{P}(A_m)$ does not converge to 0 or 1? [**Hint:** First try $r_m = m^\alpha$ for some α, not necessarily an integer].

Problem 5

In a group of 5 people, every pair of them could be friends or non-friends. Assume that all possible relationships are equally likely. Find the probability of the event that there are at least three people who are mutually friends. (Try to write down the possibilities. Divide cleverly and conquer!)

Problem 6

Problem 28. In a class with 108 people, one student gets a joke by e-mail. He/she forwards it to one randomly chosen classmate. The recipient does the same - chooses a classmate at random (could be the sender too) and forwards it to him/her. The process goes on like this for 20 steps and stops. What is the probability that the first person to get the mail does not get it again? What is the chance that no one gets the e-mail more than once?