STAT 753: Stochastic Models and Simulations
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Homework 3

Theory

Consider the Markov chain with the transition matrix P $= \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix}$

1. Find the stationary distribution and the rate of convergence.

Solution

The stationary distribution can be found using the equations $\pi = \pi P$ and $\pi_1 + \pi_2 + \pi_3 = 1$, where P is the given transition matrix and $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$ are the corresponding probabilities for the 3 states.

$$\begin{aligned} \pi &= \pi P \\ \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} &= \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix} \\ &= \begin{Bmatrix} \pi_1 = & 0.3\pi_1 + 0.3\pi_2 \\ \pi_2 = & 0.7\pi_1 + 0.4\pi_2 + 0.7\pi_3 \\ \pi_3 = & 0.3\pi_2 + 0.3\pi_3 \end{Bmatrix} \end{aligned} \tag{1}$$

Solving this system of equations, we find that the stationary distribution is $\pi = [\frac{3}{13}, \frac{7}{13}, \frac{3}{13}]$. The rate of convergence is $|\lambda_i|^n$, where λ_i is the maximum absolute eigenvalue of the transition matrix that is less than 1. The eigenvalues can be found such that $det(P - \lambda I) =$.

$$\begin{aligned} 0 &= det(\begin{bmatrix} 0.3-\lambda & 0.7 & 0 \\ 0.3 & 0.4-\lambda & 0.3 \\ 0 & 0.7 & 0.3-\lambda \end{bmatrix}) \\ &= (0.3-\lambda)[(0.4-\lambda)(0.3-\lambda) - (0.3)(0.7)] - (0.7)[0.3(0.3-\lambda)] \\ &= -\lambda^3 + \lambda^2 + 0.09\lambda - 0.09 \\ & \begin{Bmatrix} \lambda_1 = 1 \\ \lambda_2 = 0.3 \\ \lambda_3 = -0.3 \end{Bmatrix} \end{aligned} \tag{2}$$

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Thus, the rate of convergence is $|\lambda_{2,3}|^n = |0.3|^n$.

2. Find the distribution at step 2 if the initial distribution is $\begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}$

Solution

The distribution for any step is given as $x(t+1) = x(t)P$. To find the distribution at step 2, we need to first find the distribution at step 1.

$$\begin{aligned} x(1) &= \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.3 & 0.55 & 0.15 \end{bmatrix} \\ x(2) &= \begin{bmatrix} 0.3 & 0.55 & 0.15 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.255 & 0.535 & 0.21 \end{bmatrix} \end{aligned} \tag{3}$$

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Alternatively, we can simply find the distribution at step 2 using the formula $x(n) = x(0)P^n$, such that

$$x(2) = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix}^2 . \tag{4}$$

This computation yields the same result - the distribution at step 2 is $\begin{bmatrix} 0.255 & 0.535 & 0.21 \end{bmatrix}$.

3. Find the distribution at step N if the initial position is $X_0 = 2$.

Solution

Since this Markov chain is irreducible, the distribution at step N should be equal to the stationary distribution found in question 1. We can show this by using the relationship $x(n) = c_1 v_1 \lambda_1^n + c_2 v_2 \lambda_2^n + c_3 v_3 \lambda_3^n$ where $[c_1, c_2, c_3]$ are constants, and v_1, v_2, v_3 are row eigenvectors corresponding to the eigenvalues $[\lambda_1 = 1, \lambda_2 = 0.3, \lambda_3 = -0.3]$ of the transition matrix P. The row eigenvectors can be found using the relationship $\lambda_i v_i = v_i P$. Doing this for the 3 eigenvalues yields

$$1 \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

$$\begin{aligned} a &= 0.3a + 0.3b \\ b &= 0.7a + 0.4b + 0.7c \\ c &= 0.3b + 0.3c \end{aligned} \tag{5}$$

$$v_1 = \begin{bmatrix} 3/7 & 1 & 3/7 \end{bmatrix}$$

$$0.3 \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

$$\begin{aligned} 0.3a &= 0.3a + 0.3b \\ 0.3b &= 0.7a + 0.4b + 0.7c \\ 0.3c &= 0.3b + 0.3c \end{aligned} \tag{6}$$

$$v_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

$$-0.3 \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

$$\begin{aligned} -0.3a &= 0.3a + 0.3b \\ -0.3b &= 0.7a + 0.4b + 0.7c \\ -0.3c &= 0.3b + 0.3c \end{aligned} \tag{7}$$

$$v_3 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

Plugging these eigenvectors into the above equation gives

$$x(n) = c_1 \begin{bmatrix} 3/7 & 1 & 3/7 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} 0.3^n + c_3 \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} (-0.3)^n. \tag{8}$$

Lastly, we need to solve for c_1, c_2, and c_3 using the initial condition $X_0 = 2$, which corresponds to the distribution $x(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$. We can solve a system of equations by using the equation $x(0) = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} P$.

$$\begin{cases} 0 = \frac{3}{7}c_1 - c_2 + c_3 \\ 1 = c_1 - 2c_3 \\ 0 = \frac{3}{7}c_1 + c_2 + c_3 \end{cases} \quad => \begin{cases} c_1 = \frac{7}{13} \\ c_2 = \frac{-3}{13} \\ c_3 = 0 \end{cases} \tag{9}$$

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Since λ_2^n and λ_3^n are less than 1, those terms will go to zero as $n => \infty$. Thus, we are left with $x(n) = \frac{7}{13}\begin{bmatrix} 3/7 & 1 & 3/7 \end{bmatrix} = \begin{bmatrix} 3/13 & 7/13 & 3/13 \end{bmatrix}$, which is the stationary distribution found in question 1.

Coding

1. Simulate $N = 5000$ steps of this Markov chain if the initial distribution is $X_0 = 2$. Compute shares of times when this simulation is in state 1, state 2, and state 3. Compare with the stationary distribution in Theory 1.

Solution

```
import numpy as np
import matplotlib.pyplot as plt

#transition matrix
P = np.array([[0.3, 0.7, 0],
              [0.3, 0.4, 0.3],
              [0, 0.7, 0.3]])
x0 = 2 #initial state
N = 5000
sim = []
sim.append(x0)
np.random.seed(1)
for i in range(1, N):
        current_prob = P[sim[i-1]-1] #get the row associated with the current state
        new_state = np.random.choice(np.arange(1, 4), p=current_prob) #choose new value between 1-3
        sim.append(new_state)

# plot distribution of states and get estimated stationary distro
plt.hist(sim, bins=np.arange(1, P.shape[0] + 2)-0.5, color='grey', alpha=.5)
plt.title("Distribution of states over 5000 steps")
plt.xlabel("State")
plt.ylabel("Steps")
plt.xticks([1,2,3])
plt.show()

unique_states, counts = np.unique(sim, return_counts=True)
print('Distribution after 5000 steps:', counts/n)
>> Distribution after 5000 steps: [0.2332 0.5364 0.2304]
```

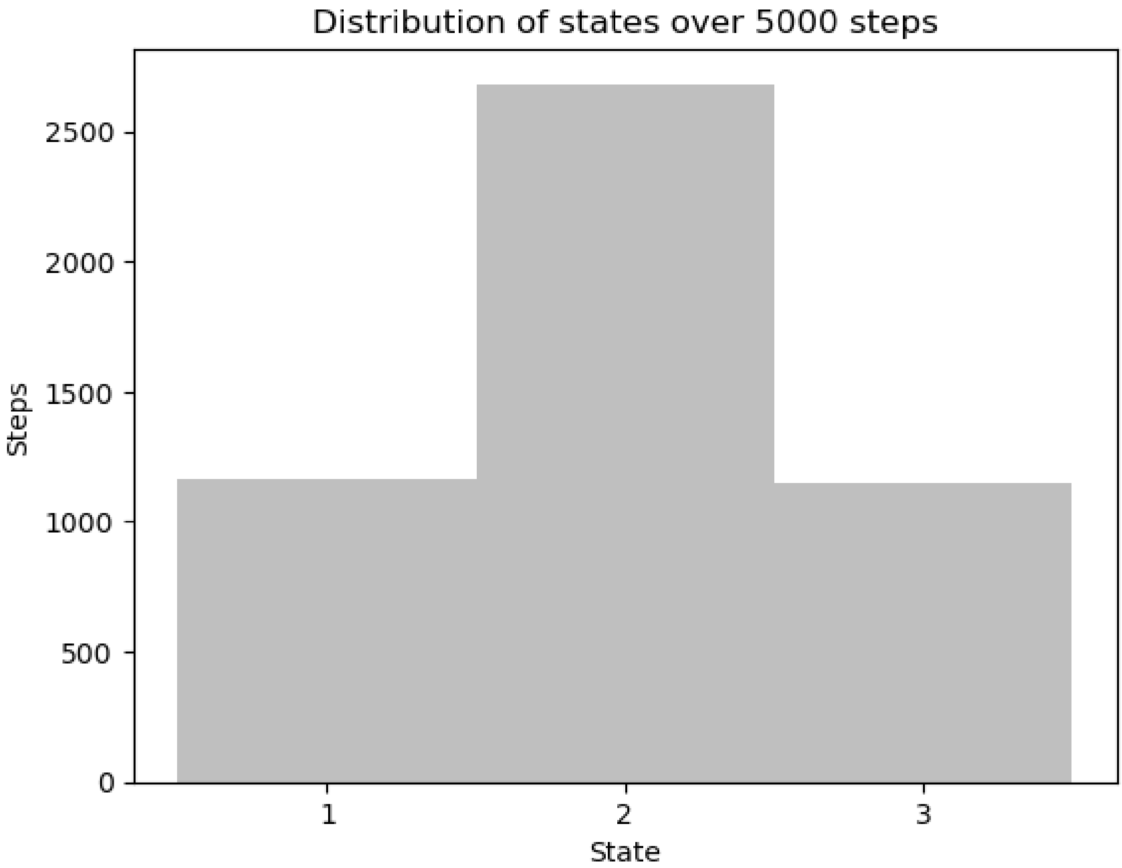

Figure 1: Distribution of states over 5000 steps.

The estimated stationary distribution from the above 5000 step simulation is $\begin{bmatrix} 0.2332 & 0.5364 & 0.2304 \end{bmatrix}$, whereas the theoretical stationary distribution is $\begin{bmatrix} 3/13 & 7/13 & 3/13 \end{bmatrix}$, which approximates to $\begin{bmatrix} 0.2307 & 0.5384 & 0.2307 \end{bmatrix}$. The corresponding errors between these probabilities are $1.05\%, 0.38\%$ and 0.16%, respectively. It is clear that the simulated Markov chain was sufficiently long enough to closely converge with the theoretical stationary distribution.

2. Simulate the two steps of this Markov chain for 1000 times if the initial distribution is $\begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix}$ and compute empirical distribution at time step 2. Compare with the Theory 2.

Solution

```
np.random.seed(1)
N = 1000
distro_step1 = []
distro_step2 = []
x0_prob = np.array([0.5, 0.5, 0])
for i in range(0,N):
    step0_state = np.random.choice(np.arange(1,4), p =x0_prob) #returns a 1 or 2

    x1_prob = P[int(step0_state)-1] #get the row associated with the current state
    step1_state = np.random.choice(np.arange(1, 4), p=x1_prob) #choose new value between 1-3
    distro_step1.append(step1_state)


    x2_prob = P[int(step1_state)-1] #get the row associated with the current state
    step2_state = np.random.choice(np.arange(1, 4), p=x2_prob) #choose new value between 1-3
    distro_step2.append(step2_state)

unique_states, counts = np.unique(distro_step2, return_counts=True)
print('Step 2 distribution after 1000 simulations:', counts/N)
>> Step 2 distribution after 1000 simulations: [0.258 0.539 0.203]
```

The theoretical distribution after 2 steps was found to be $\begin{bmatrix} 0.255 & 0.535 & 0.21 \end{bmatrix}$ whereas the distribution from 1000 simulations was $\begin{bmatrix} 0.258 & 0.539 & 0.203 \end{bmatrix}$. The corresponding errors between these probabilities are $1.17\%, 0.75\%$ and 3.3%, respectively. These errors are small, demonstrating fairly good agreement between the theoretical distribution and the average over 1000 simulations.