STAT 753: Stochastic Models and Simulations
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University of Nevada, Reno - Spring 2024

Homework 12

Theory

1. Consider the switch SDE: $dX(t) = -c \cdot \text{sgn}(X(t) - m)dt + \sigma dW(t)$. Here, $c, \sigma > 0$, and sgn is the *signum function*:

$$\text{sgn}(y) := \begin{cases} +1, y > 0; \\ 0, y = 0; \\ -1, y < 0. \end{cases} \tag{1}$$

Show that the Laplace distribution with mean m and variance v is a stationary distribution, and find the variance v.

Solution For a SDE with density $\varphi(y)$, if it has a stationary distribution it will solve the equation

$$\int_{-\infty}^{\infty} Af(x)\varphi(x)dx = 0. \quad \text{where} \quad Af(x) = g(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x). \tag{2}$$

Integrating this equation gives the expression

$$-(g(x)\varphi(x))' + \frac{1}{2}(\sigma^2(x)\varphi(x))'' = 0. \tag{3}$$

Since $\varphi(y)$ represents a density function, Equation 3 has the boundary conditions $\varphi(\infty) = 0$ and $\varphi(-\infty) = 0$. For a Laplace distribution with mean $m = \mu$ and variance $v = 2b^2$, $\varphi(x)$ is equal to

$$\varphi(x) = \frac{1}{2b}\exp\left(\frac{-|x-\mu|}{b}\right). \tag{4}$$

This problem can then be broken up into the cases where (1): $X(t) - m > 0$ and (2): $X(t) - m < 0$. For (1), $g(x) = -c$ and $\sigma(x) = \sigma$. We can solve Equation 3 such that:

$$\begin{aligned} -(-c\varphi(x))' + \frac{\sigma^2}{2}(\varphi(x))'' &= 0 \\ \int_{-\infty}^{\infty} c\varphi'(x) + \frac{\sigma^2}{2}(\varphi(x))'' &= 0 \\ c\varphi(x) + \frac{\sigma^2}{2}(\varphi(x))' &= k_1. \end{aligned} \tag{5}$$

For the Laplace distribution where $|x-\mu|>0$,

$$\begin{aligned}\varphi(x) &= \frac{1}{2b}\exp\left(\frac{-(x-\mu)}{b}\right)\\ \varphi'(x) &= -\frac{1}{2b^2}\exp\left(\frac{-(x-\mu)}{b}\right).\end{aligned}\tag{6}$$

If we let $k_1=0$, we can then plug in $\varphi(x)$ and $\varphi'(x)$ and solve for $v=2b^2$:

$$\begin{aligned}c\varphi(x)+&\frac{\sigma^2}{2}(\varphi(x))'=k_1\\ c\frac{1}{2b}\exp\left(\frac{-(x-\mu)}{b}\right)+&\frac{\sigma^2}{2}(-\frac{1}{2b^2}\exp\left(\frac{-(x-\mu)}{b}\right)) &=0\\ \frac{c}{2b} &= \frac{\sigma^2}{4b^2}\\ b &= \frac{\sigma^2}{2c}\\ 2b^2 &= \frac{\sigma^4}{2c^2}.\end{aligned}\tag{7}$$

Thus, for case (1), the mean is μ and variance is $\frac{\sigma^4}{2c^2}$. Now, for case (2), $g(x)=c$ and $\sigma(x)=\sigma$, which gives the expression $-c\varphi(x)+\frac{\sigma^2}{2}(\varphi(x))'=k_1$ when integrating Equation 3. Since $|x-\mu|<0$, the density function can be rewritten as

$$\begin{aligned}\varphi(x) &= \frac{1}{2b}\exp\left(\frac{-(\mu-x)}{b}\right)\\ \varphi'(x) &= \frac{1}{2b^2}\exp\left(\frac{-(\mu-x)}{b}\right).\end{aligned}\tag{8}$$

Solving, once again, gives

$$\begin{aligned}-c\varphi(x)+&\frac{\sigma^2}{2}(\varphi(x))'=k_1\\ -c\frac{1}{2b}\exp\left(\frac{-(\mu-x)}{b}\right)+&\frac{\sigma^2}{2}(\frac{1}{2b^2}\exp\left(\frac{-(\mu-x)}{b}\right)) &=0\\ \frac{c}{2b} &= \frac{\sigma^2}{4b^2}\\ b &= \frac{\sigma^2}{2c}\\ 2b^2 &= \frac{\sigma^4}{2c^2}.\end{aligned}\tag{9}$$

Thus, the variance and mean are the same as in case (1). The Laplace distribution is thus stationary with mean $\mu=m$ and variance $2b^2=\frac{\sigma^4}{2c^2}$.

For the first coding problem, we are asked to simulate this process for chosen parameters. By choosing $\sigma=.2$, $m=.5$, and $c=.5$, we can see that this process will have Laplace distribution with mean $m=0.5$ and variance $\frac{\sigma^4}{2c^2}=0.0032$.

2. Consider the Ornstein-Uhlenbeck process in the log scale modeling the Volatility Index VIX:

$$d\ln V(t) = (-0.1188\ln V(t) + 0.3482)dt + 0.1589dW(t), V(0) = 16. \tag{10}$$

Find the mean and variance of V in its stationary distribution.

Solution

For an OU process $dX(t) = c(m - X(t))dt + \sigma dW(t)$, the stationary distribution is normal with mean m and variance $\sigma^2/2c$. So, $ln(X(t)) \sim N(\mu = m, \varrho^2 = \sigma^2/2c)$ as $t \to \infty$. We can find the mean and variance of $ln(X(t))$ so that

$$\begin{aligned} \mu &= E[e^{tln(X(t)}]\Big|_{t=1} \\ \sigma^2 &= E[e^{tln(X(t)}]\Big|_{t=2} - E[e^{tln(X(t)}]^2\Big|_{t=1} \end{aligned} \tag{11}$$

The MGF for the normal distribution is

$$E[e^{tx}] = e^{\mu t + \sigma^2 t^2/2} \tag{12}$$

For this problem, $m = 0.3482/0.1188 \approx 2.931$ and variance $\sigma^2 = 0.1589^2/2(0.1188) \approx 0.1063$. We can plug in these values and solve:

$$\begin{aligned} \mu &= e^{\mu + \sigma^2/2} \\ &= e^{2.931 + 0.1063/2} \\ &\approx 19.769 \\ \\ v &= e^{\mu(2) + \sigma^2(4)/2} - (e^{\mu + \sigma^2/2})^2 \\ &= e^{2(2.931) + 0.1063(2)} - (e^{2.931 + 0.1063/2})^2 \\ &= 434.627 - 19.769^2 \\ &\approx 43.818. \end{aligned} \tag{13}$$

The stationary distribution for this process is $\sim N(19.769, 43.818)$ after the burn-in period.

Code

1. Q1

Solution

```python

import numpy as np
import seaborn as sns

def SDE1(mu, sigma,c, dt, T):
    simX=[0]
    N=int(T/dt)
    #X = np.random.laplace(mu, var/np.sqrt(2),N)
    noise = np.random.normal(0,np.sqrt(dt),N)

    for i in range(N):
        old = simX[-1]
        if (old-mu)>0:
            new= -c*dt + sigma*noise[i] + old
        elif (old-mu)<0:
            new= c*dt + sigma*noise[i] + old
        elif (old-mu)==0:
            sigma*noise[i] + old

        simX.append(new)
    return simX

sigma=.2
m=.5
c=.5
b= sigma**2/(2*c)
dt=0.01
T=20

sims=[]
for i in range(2000):
    sims.append(SDE1(m,sigma,c, dt, T))

print('Empirical mean:', np.mean(np.array(sims)[:,500]))
print('Empirical variance:', np.var(np.array(sims)[:,500]))

print('Theoretical mean:', m)
print('Theoretical variance:', 2*b**2)

>> Empirical mean: 0.5035253512152671
>> Empirical variance: 0.0034892051137381036
>> Theoretical mean: 0.5
>> Theoretical variance: 0.0032000000000000015

```

After a break-in period of 5 seconds, we can see that the empirical and theoretical means and variances of the stationary distribution are quite close.

2. Q2

Solution

```
def SDE2(m,c, sigma, dt, T, X0):
    simX=[X0]
    N= int(T/dt)
    noise = np.random.normal(0,np.sqrt(dt),N)

    for i in range(N):
        old = simX[-1]
        new= (np.exp(c*m*dt)) * (old**(-c*dt)) * np.exp(sigma*noise[i]) *
        ↪  old
        simX.append(new)
    return simX

c = .1188
m = 0.3482/c
sigma=0.1589
dt= .01
T=30
X0=16

p2 = sigma*2 / 2*c

np.random.seed(1)
sims=[]
for i in range(4000):
    sims.append(SDE2(m,c,sigma, dt, T, X0))

p2 = sigma**2 / (2*c)
print('Empirical mean:', np.mean(np.array(sims)[:,1000:]))
print('Empirical variance:', np.var(np.array(sims)[:,1000:]))

print('Theoretical mean:', np.exp(m + (p2/2)))
print('Theoretical variance:', (np.exp(2*m + 4*p2/2) -  np.exp(m +
↪  p2/2)**2))

>> Empirical mean: 19.42318764448729
>> Empirical variance: 42.29081056308366
>> Theoretical mean: 19.76890582446917
>> Theoretical variance: 43.81740930293188
```

After a break-in period of 10 seconds, the theoretical and empirical solutions are similar.

Stat753_HW12_JaleesaHoule

April 30, 2024

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
```

1 Question 1

Consider the switch SDE: $dX(t) = -c \cdot \operatorname{sgn}(X(t) - m)dt + \sigma dW(t)$. Here, $c, \sigma > 0$, and sgn is the signum function:

$$\operatorname{sgn}(y) := \begin{cases} +1, y > 0; \\ 0, y = 0; \\ -1, y < 0. \end{cases} \tag{1}$$

Show that the Laplace distribution with mean m and variance v is a stationary distribution, and find the variance v.

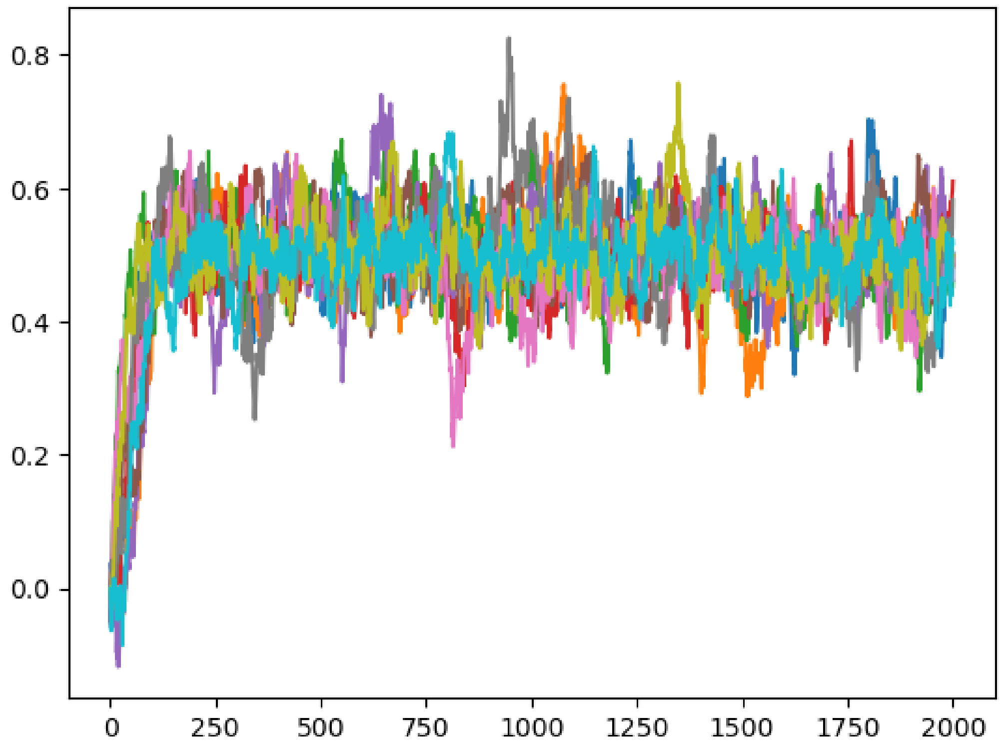
[2]:
```
def SDE1(mu, sigma,c, dt, T):
    simX=[0]
    N=int(T/dt)
    noise = np.random.normal(0,np.sqrt(dt),N)

    for i in range(N):
        old = simX[-1]
        if (old-mu)>0:
            new= -c*dt + sigma*noise[i] + old
        elif (old-mu)<0:
            new= c*dt + sigma*noise[i] + old
        elif (old-mu)==0:
            sigma*noise[i] + old

        simX.append(new)
    return simX
```

[3]:
```
sigma=.2
m=.5
c=.5
b= sigma**2/(2*c)
dt=0.01
```

```
T=20
```

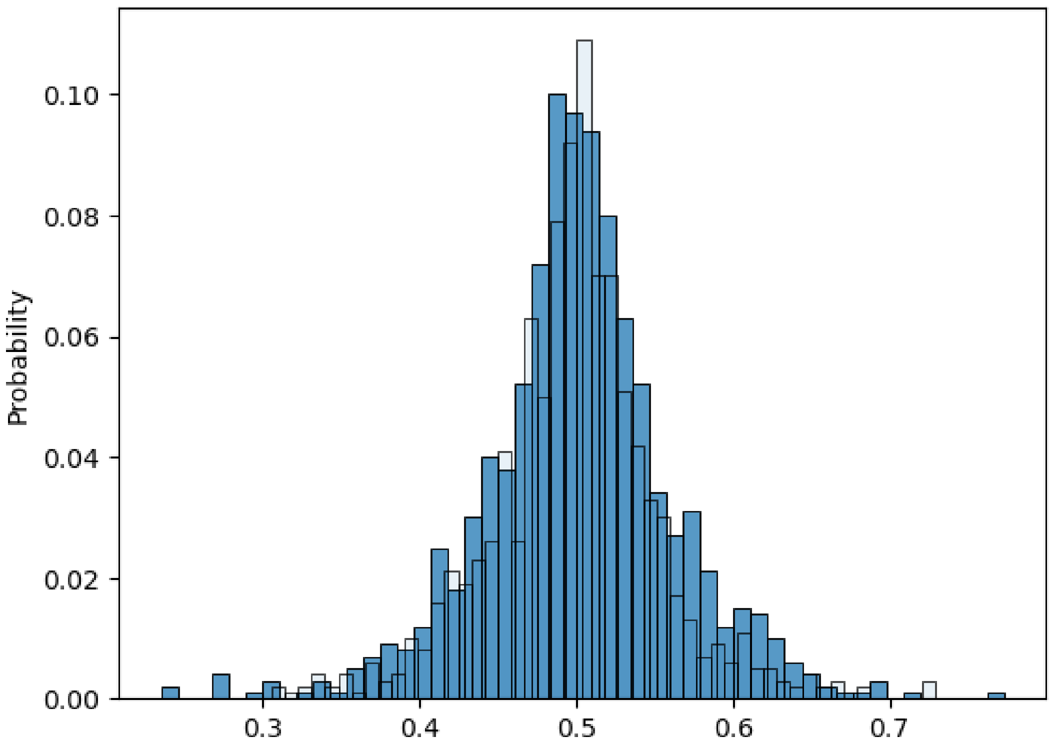
[4]:
```
sims=[]
np.random.seed(1)
for i in range(1000):
    sims.append(SDE1(m,sigma,c, dt, T))
```

[5]:
```
for i in range(10):
    plt.plot(sims[i])
```



[6]:
```
X = np.random.laplace(loc=m, scale=b , size=1000)
sns.histplot(np.array(sims)[:,1000],stat='probability', bins=50)
sns.histplot(X, bins=50,stat='probability', alpha=.1)
```

[6]:
```
<AxesSubplot:ylabel='Probability'>
```



[7]:
```
print('Empirical mean:', np.mean(np.array(sims)[:,500]))
print('Empirical variance:', np.var(np.array(sims)[:,500]))

print('Theoretical mean:', m)
print('Theoretical variance:', 2*b**2)
```

```
Empirical mean: 0.5038519384972718
Empirical variance: 0.0035413485192730544
Theoretical mean: 0.5
Theoretical variance: 0.0032000000000000015
```

2 Question 2

Consider the Ornstein-Uhlenbeck process in the log scale modeling the Volatility Index VIX:

$$d\ln V(t) = (-0.1188\ln V(t) + 0.3482)dt + 0.1589dW(t), V(0) = 16. \tag{2}$$

Find the mean and variance of V in its stationary distribution.

[8]:
```
def SDE2(m,c, sigma, dt, T, X0):
    simX=[X0]
    N= int(T/dt)
```

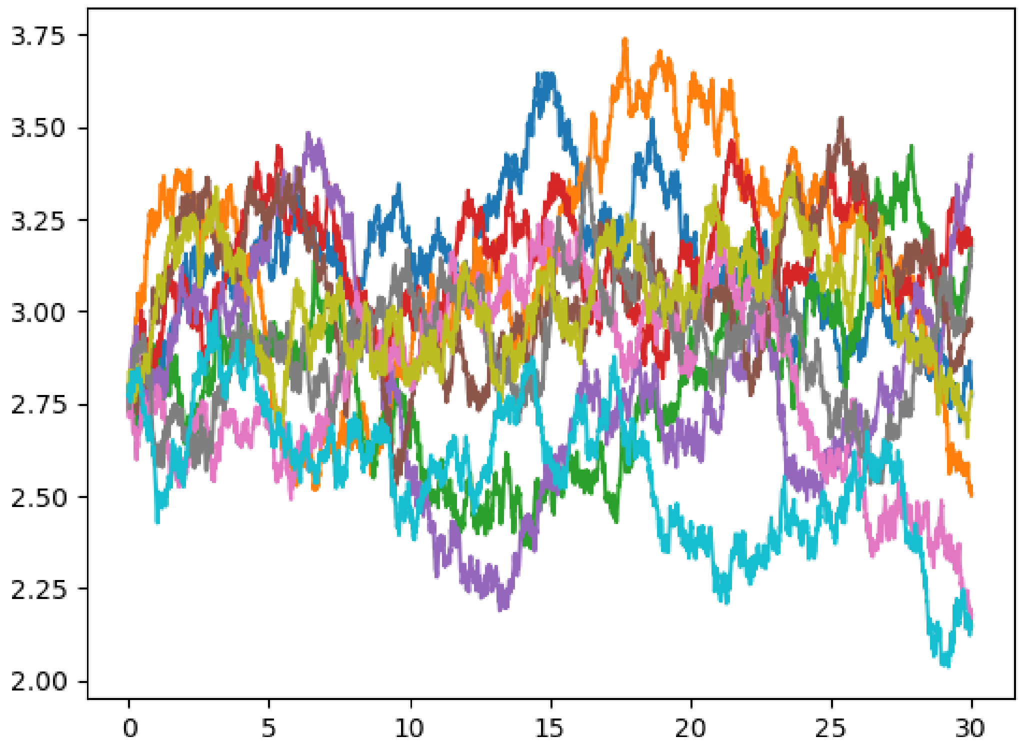
```python
    noise = np.random.normal(0,np.sqrt(dt),N)

    for i in range(N):
        old = simX[-1]
        new= (np.exp(c*m*dt)) * (old**(-c*dt)) * np.exp(sigma*noise[i]) * old
        simX.append(new)
    return simX
```

[9]:

```python
c = .1188
m = 0.3482/c
sigma=0.1589
dt= .01
T=30
X0=16

p2 = sigma*2 / 2*c

np.random.seed(1)
sims=[]
for i in range(1000):
    sims.append(SDE2(m,c,sigma, dt, T, X0))
```

[10]:

```python
time = np.linspace(0,T,int(T/dt)+1)
for i in range(10):
    plt.plot(time, np.log(sims[i]))
```

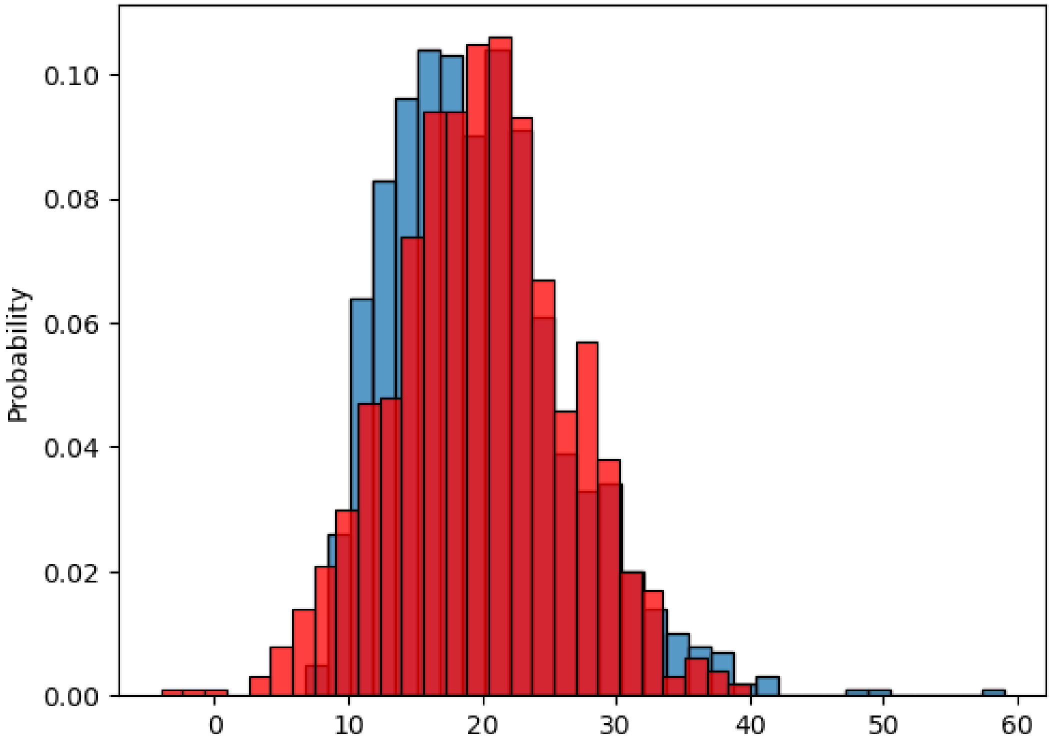


[11]:
```
p2 = sigma**2 / (2*c)
print('Empirical mean:', np.mean(np.array(sims)[:,1000:]))
print('Empirical variance:', np.var(np.array(sims)[:,1000:]))

print('Theoretical mean:', np.exp(m + (p2/2)))
print('Theoretical variance:', (np.exp(2*m + 4*p2/2) -  np.exp(m + p2/2)**2))
```

```
Empirical mean: 19.335080974581142
Empirical variance: 43.44341712167216
Theoretical mean: 19.76890582446917
Theoretical variance: 43.81740930293188
```

[17]:
```
X = np.random.normal(19.76890582446917, np.sqrt(43.81740930293188), size=1000)
sns.histplot(np.array(sims)[:,2500], stat='probability')
sns.histplot(X, stat='probability', color='red')
```

[17]: `<AxesSubplot:ylabel='Probability'>`



0.10
0.08
0.06
0.04
0.02
0.00
Probability
0
10
20
30
40
50
60


```
[ ]:
```