STAT 753: Stochastic Models and Simulations
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University of Nevada, Reno - Spring 2024

Homework 11

Theory

1. Take a standard Brownian motion W. Find the probability that its maximum over [0,9] is greater than 5, and $W(9) > 1$.

   Solution

   For a standard Brownian motion $W(t), t > 0$, with the corresponding distribution $M(t) = \max_{0 \leq s \leq t} W(s)$, the probability $P(W(9) > 5, W(9) > 1)$ can be found by observing two cases for $W(9) > 1$.

   First, there is the case where $(M(9) > 5, W(9) > 5)$. In this case, we note that if $W(9) > 5$ then $M(9) > 5$ must also be true. Therefore the probability in that case is $P(W(9) > 5)$.

   The second case occurs for $(M(9) > 5, 1 < W(9) < 5)$. In this instance, we can use the reflection principle where we reflect a Brownian motion $W(9) > 1$ over the line $y = 5$. By symmetry, the probability $P(M(9) > 5, 1 < W(9) < 5)$ is equal to $P(5 < W(9) < 9)$. By combining these two cases, we can transform to standard normal and solve:

$$\begin{aligned}
P(W(9) > 5, W(9) > 1) &= P(M(9) > 5, W(9) > 5) + P(M(9) > 5, 1 < W(9) < 5) \\
&= P(W(9) > 5) + P(5 < W(9) < 9) \\
&= P(Z > \frac{5}{3}) + P(\frac{5}{3} < Z < 3) \\
&= (1 - \Phi\left(\frac{5}{3}\right)) + (P(Z < 3) - P\left(Z < \frac{5}{3}\right)) \\
&= (1 - 0.9525) + (.9987 - .9525) \\
&\approx 0.0937.
\end{aligned} \tag{1}$$

2. Define a geometric Brownian motion $G(t) = \exp(\sigma W(t))$. Find the density of $G(t)$ given that $G(s) = g$. Assume that $0 \leq s < t$.

   Solution

   We can first define $X(t) = \sigma W(t)$ so that $G(t) = \exp(X(t))$. $X(t)$ is a Brownian motion with diffusion σ having independent increments $X(t) - X(s) \sim N(\mu(t - s), \sigma^2(t - s))$. In this case, $\mu = 0$, so the distribution for independent increments is then $X(t) - X(s) \sim N(0, \sigma^2(t-s))$. The independent increments of $G(t)$ can be described as

$$\frac{G(t)}{G(s)} = e^{X(t)-X(s)}, \text{ for } 0 \leq s < t. \tag{2}$$

This can be further manipulated so that

$$\begin{aligned} G(t) &= G(s)e^{X(t)-X(s)} \\ ln(G(t)) &= ln(G(s)) + X(t) - X(s) \\ &= ln(g) + X(t) - X(s). \end{aligned} \tag{3}$$

Since $X(t) - X(s) \sim N(0, \sigma^2(t-s))$ and $ln(g)$ is a constant, we can see $ln(G(t)) \sim N(ln(g), \sigma^2(t-s))$. Thus, $G(t) \sim lognorm(ln(g), \sigma^2(t-s))$.

For a log-normal distribution, the probability density is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(\frac{-(ln(x)-\mu)^2}{2\sigma^2}\right) \tag{4}$$

for $t \geq 0$. Using the known parameters for μ and σ^2, we can see that the density for $G(t)$ is

$$g(x) = \frac{1}{x\sigma\sqrt{(t-s)}\sqrt{2\pi}} \exp\left(\frac{-(ln(x/g))^2}{2\sigma^2(t-s)}\right). \tag{5}$$

Code

1. Simulate the standard Brownian motion from Theory 1, and compute the empirical probability in question.

Solution

```
import numpy as np
import matplotlib.pyplot as plt

def brownian_motion_sim( dt, T, mu=0, sigma=1, x0=None):
    N = int(T/dt)
    BM = np.append(np.zeros(1),np.cumsum(np.random.normal(mu*dt,
    ↪  sigma*np.sqrt(dt), N)))
    if x0 is not None:
        BM= np.cumsum(np.append(x0, np.random.normal(mu*dt,
        ↪  sigma*np.sqrt(dt), N)))
    time = np.linspace(0,T, N+1)

    return BM, np.round(time,2)

def find_value_at_T(values,jumptimes, T):
    idx = np.where(jumptimes == T)[-1][-1]
    return values[idx]

dt = 0.01
N = 10000
T=9
sims=[]
np.random.seed(1234)
for i in range(N):
    a,b = brownian_motion_sim(dt, T)
    sims.append(a)
    time = b


countw_9 = 0

sims_idx=[]
for i in range(N):
    w_9 = find_value_at_T(sims[i], time, 9)
    if (np.max(sims[i][:900])>5) and (w_9>1):
        countw_9+=1

print('P(M(9)>5, W(9)>1)=', countw_9/N)
>>
>> P(M(9)>5, W(9)>1)= 0.0912
```

We can see that the empirical probability was found to be $P(M(9) > 5, W(9) > 1) = 0.0912$, while the theoretical probability was found to be 0.0937 in Theory 1. The percent error between the empirical and theoretical solutions are approximately 2.67%.

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[4]:
```python
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
```

0.1 Take a standard Brownian motion W. Find the probability that its maximum over [0,9] is greater than 5, and $W(9) > 1$.

[5]:
```python
def brownian_motion_sim( dt, T, mu=0, sigma=1, x0=None):
    N = int(T/dt)
    BM = np.append(np.zeros(1),np.cumsum(np.random.normal(mu*dt, sigma*np.
 ↪sqrt(dt), N)))
    if x0 is not None:
        BM= np.cumsum(np.append(x0, np.random.normal(mu*dt, sigma*np.sqrt(dt),␣
 ↪N)))
    time = np.linspace(0,T, N+1)

    return BM, np.round(time,2)

def find_value_at_T(values,jumptimes, T):
    idx = np.where(jumptimes == T)[-1][-1]
    return values[idx]
```

[6]:
```python
dt = 0.01
N = 10000
T=9
sims=[]
np.random.seed(1000)
for i in range(N):
    a,b = brownian_motion_sim(dt, T)
    sims.append(a)
    time = b
```

[9]:
```python
countw_9 = 0

sims_idx=[]
for i in range(N):
    w_9 = find_value_at_T(sims[i], time, 9)
```

```
    if (np.max(sims[i][:900])>5) and (w_9>1):
        countw_9+=1
```

[10]:
```
print('P(M(9)>5, W(9)>1)=', countw_9/N)
```

```
P(M(9)>5, W(9)>1)= 0.0912
```

[ ]: