STAT 753: Stochastic Models and Simulations
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Homework 9

Theory

1. For the Poisson process $N = (N(t), t \geq 0)$ with rate $\lambda = 1.2$. Assume τ_k is the time of the kth jump, find:

A. $\mathbb{P}(N(2) = 3)$
B. $\mathbb{P}(N(2) = 3, N(5) = 6, N(10) \geq 9)$
C. $\mathbb{E}[e^{2N(0.5)}]$
D. $\mathbb{E}[\tau_2]$
E. $\mathbb{P}(2 < \tau_2 < 3.2)$

Solution

A. $\mathbb{P}(N(2) = 3)$

The pdf for a Poisson distribution is given as

$$P(N(t) = k) = \frac{\lambda t^k}{k!} e^{-\lambda t}. \tag{1}$$

Here, $t = 2$, $k = 3$, and $\lambda = 1.2$. We can find the probability by plugging in values

$$\begin{aligned} P(N(t) = k) &= \frac{\lambda t^k}{k!} e^{-\lambda t} \\ P(N(2) = 3) &= \frac{(1.2(2))^3}{3!} e^{-1.2(2)} \\ &= 0.20901. \end{aligned} \tag{2}$$

B. $\mathbb{P}(N(2) = 3, N(5) = 6, N(10) \geq 9)$

A Poisson process has the properties $N(0) = 0$, $N(t) - N(s)$ is independent of $(N(u), 0 \leq u \leq s)$, and $N(t) - N(s) \sim Poi(\lambda(t - s))$ for any $0 \leq s < t$. For $\mathbb{P}(N(2) = 3, N(5) = 6, N(10) \geq 9)$, we can use these properties to rewrite the probability as

$$\begin{aligned}\mathbb{P}(N(2)=3,N(5)=6,N(10)\geq 9) = [&\mathbb{P}(N(10)\geq 9|N(5)=6,N(2)=3)\cdot \\ &\mathbb{P}(N(5)=6|N(2)=3)\cdot\mathbb{P}(N(2)=3)] \\ &= [\mathbb{P}(N(10)\geq 9|N(5)=6)\cdot\mathbb{P}(N(5)=6|N(2)=3) \\ &\cdot\mathbb{P}(N(2)=3)] \\ &= [\mathbb{P}(N(10)-N(5)\geq 3)\cdot\mathbb{P}(N(5)-N(2)=3) \\ &\cdot\mathbb{P}(N(2)=3)] \\ &= \mathbb{P}(N(5)\geq 3)\cdot\mathbb{P}(N(3)=3)\cdot\mathbb{P}(N(2)=3).\end{aligned} \tag{3}$$

The corresponding probabilities can be found by plugging in the given information:

$$\begin{aligned}\mathbb{P}(N(2)=3) &= \mathbb{P}(Poi(2\cdot 1.2)) \\ &= \frac{2.4^3}{6}e^{-2.4} \\ &= 0.20901 \\ \\ \mathbb{P}(N(3)=3) &= \mathbb{P}(Poi(3\cdot 1.2)) \\ &= \frac{3.6^3}{6}e^{-3.6} \\ &= 0.21246926 \\ \\ \mathbb{P}(N(5)\geq 3) &= 1-(\mathbb{P}(N(5)=0)+\mathbb{P}(N(5)=1)+\mathbb{P}N(5)=2) \\ &= 1-(e^{-6}+6e^{-6}+18e^{-6}) \\ &= 0.9380312\end{aligned} \tag{4}$$

The total probability is then $0.9380312\cdot 0.21246926\cdot 0.20901\approx 0.0416683$.

C. $\mathbb{E}[e^{2N(0.5)}]$

For a Poisson process, $N(t)-N(s)\sim Poi(\lambda(t-s))$. We are given $\lambda=1.2$ and $t=0.5$, so $N(0.5)-N(0)\sim Poi(1.2(0.5-0))$. This simplifies to $N(0.5)\sim Poi(0.6)$. The MGF of a Poisson distribution is

$$\mathbb{E}[e^{tX}] = e^{\lambda(e^t-1)}. \tag{5}$$

For this Poisson distribution, $\lambda=0.6$ and $t=2$. The expected value is then

$$\begin{aligned}\mathbb{E}[e^{2N(0.5)}] &= e^{0.6(e^2-1)} \\ &\approx 46.220973.\end{aligned} \tag{6}$$

D. $\mathbb{E}[\tau_2]$

For a Poisson process, the time steps follow distribution $\tau_k\sim\Gamma(k,\lambda)$. For a Gamma distribution, the expected value is k/λ. For this case, $k=2$ and $\lambda=1.2$, therefore

$\mathbb{E}[\tau_2] = 2/1.2 = 1.666.$

E. $\mathbb{P}(2 < \tau_2 < 3.2)$

For $\tau_2 \sim \Gamma(2, 1.2)$, we can find $\mathbb{P}(2 < \tau_2 < 3.2)$ by integrating the pdf of τ_2 from 2 to 3.2.

$$\begin{aligned}
\mathbb{P}(2 < \tau_2 < 3.2) &= \int_2^{3.2} \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x} dx \\
&= \int_2^{3.2} \frac{1.2^2}{(2-1)!} x^{2-1} e^{-1.2x} dx \\
&= \int_2^{3.2} 1.44 x e^{-1.2x} dx \\
&= \int_2^{3.2} 1.44 x e^{-1.2x} dx \\
&= 1.44\left(-\frac{xe^{-1.2x}}{1.2}\Big|_2^{3.2} - \frac{e^{-1.2x}}{1.2^2}\Big|_2^{3.2}\right) \\
&= 0.204412010673.
\end{aligned} \tag{7}$$

2. For the compound Poisson process $X = (X(t), t \geq 0)$ corresponding to the Poisson process from Theory 1, with jumps Z_k having exponential distribution with mean 2.5, find:

A. $\mathbb{E}[X(2)]$
B. $Var(X(2))$
C. $\mathbb{E}[e^{-3X(2)}]$

Solution

A. $\mathbb{E}[X(2)]$

For the compound Poisson process $X(t)$ with $\tau_k \sim \Gamma(k, \lambda = 1.2)$ and $Z_k \sim Exp(\lambda = 1/\mu) \sim Exp(0.4)$, $\mathbb{E}[X(t)] = \lambda\mu t$. Here, λ corresponds to the distribution of τ_k and μ corresponds to the distribution of Z_k. The expected value is then $\mathbb{E}[X(2)] = 1.2 \cdot 2.5 \cdot 2 = 6$.

B. $Var(X(2))$

For an IID random variable $Z_k \sim Exp(\lambda)$ with mean 2.5, the mean is equal to $1/\lambda$, therefore $\lambda = 0.4$. The variance of an exponential distribution is $1/\lambda^2 = 1/0.4^2 = 6.25$. Now, for the compound Poisson process $X(t)$ with $\tau_k \sim \Gamma(k, 1.2)$ and $Z_k \sim Exp(0.4)$, the variance at $t = 2$ can be computed as $Var(X(2)) = \lambda t(\mu^2 + \sigma^2) = 1.2 \cdot 2(2.5^2 + 6.25) = 30$.

C. $\mathbb{E}[e^{-3X(2)}]$

For the MGF of X(2), we derived in class

$$\begin{aligned} \mathbb{E}[e^{uX(t)}] &= e^{(M_z(u)-1)\lambda t} \\ M_z(u) &= \mathbb{E}[e^{Z_k u}]. \end{aligned} \tag{8}$$

In this problem, $Z_k \sim Exp(\lambda)$ where $\lambda = 1/\mu$. The corresponding MGF for Z_k is then $M_z(u) = \lambda/(\lambda + u)$ for $u < \lambda$. We can then solve for $M_z(u)$ and $\mathbb{E}[e^{-3X(2)}]$:

$$\begin{aligned} M_z(u) &= \frac{1/2.5}{1/2.5 - (-3)} \\ &\approx 0.1176471 \\ \mathbb{E}[e^{uX(t)}] &= e^{(M_z(u)-1)\lambda t} \\ &= e^{(0.1176471-1)1.2\cdot 2} \\ &\approx 0.1203144. \end{aligned} \tag{9}$$

Thus, $\mathbb{E}[e^{-3X(2)}] \approx 0.1203144$.

3. Using the Central Limit Theorem, for the compound Poisson process from Theory 2, find the approximate value of u such that $X(1000) < u$ with probability 99%. This u is the 99% quantile of $X(1000)$

Solution

The CLT states that for large values of t

$$\frac{X(t) - \mathbb{E}(X(t))}{\sqrt{Var(X(t))}} \Longrightarrow N(0,1). \tag{10}$$

For $X(1000)$, $\mathbb{E}[X(1000)] = 1.2 \cdot 2.5 \cdot 1000 = 3000$ and $Var(X(1000)) = 1.2 \cdot 1000(2.5^2 + 6.25) = 15000$. From the Normal distribution table, the quantile corresponding to 99% is $x_{99} = 2.326$. We can use this information to solve for $X(1000)$:

$$\begin{aligned}\frac{X(1000) - \mathbb{E}(X(1000))}{\sqrt{Var(X(1000))}} &\approx N(0,1)\\ \frac{X(1000) - 3000}{\sqrt{1500}} &\leq x_{99}\\ X(1000) \leq (2.326)(50\sqrt{6}) + 3000\\ X(1000) &\leq 3284.876.\end{aligned} \tag{11}$$

The value of u such that it is the 99% quantile of $X(1000)$ is 3284.876.

Code

1. For the Poisson process $N = (N(t), t \geq 0)$ with rate $\lambda = 1.2$. Assume τ_k is the time of the kth jump, find:

A. $\mathbb{P}(N(2) = 3)$

Solution

```
import numpy as np
import pandas as pd
import scipy

def simulate_poisson_process(l,t,mu):
    jumpTimes= np.append(np.zeros(1),np.cumsum(np.random.exponential(scale=1/l, size=t)))
    values= np.arange(0, t+1)
    compound_values=  np.append(np.zeros(1), np.cumsum(np.random.exponential(scale=mu, size=t)))
    return values, compound_values, jumpTimes

def find_value_at_T(values,jumptimes, T):
    idx = np.where(jumptimes<T)[-1][-1]
    return values[idx]

N = 10000
l = 1.2
mu=2.5
timesteps=1500

events=[]
compound_events = []
taus = []
np.random.seed(12345)
for i in range(N):
    a,b,c = simulate_poisson_process(l,timesteps,mu)
    events.append(a)
    compound_events.append(b)
    taus.append(c)

values_at_t2 = []

for i in range(N):
    values_at_t2.append(find_value_at_T(events[i], taus[i], 2))

P1= values_at_t2.count(3)/N
print('P(N(2)=3)=', P1)
```

```
>>
>> P(N(2)=3)= 0.2096
```

The empirical value for $\mathbb{P}(N(2) = 3)$ was found to be 0.2096, whereas the theoretical value was 0.20901. These solutions are in strong agreement.

B. $\mathbb{P}(N(2) = 3, N(5) = 6, N(10) \geq 9)$

Solution

```
P2_count = 0

values_at_t2 = []
values_at_t5 = []
values_at_t10 = []

for i in range(N):
    values_at_t2.append(find_value_at_T(events[i], taus[i], 2))
    values_at_t5.append(find_value_at_T(events[i], taus[i], 5))
    values_at_t10.append(find_value_at_T(events[i], taus[i], 10))

    if (values_at_t2[i] ==3) and (values_at_t5[i] ==6) and (values_at_t10[i] >=9):
        P2_count+=1

P2=P2_count/N
print('P((N(2)=3, N(5)=6, N(10)>=9)= ', P2)
>>
>> P((N(2)=3, N(5)=6, N(10)>=9)=  0.0414
```

The empirical value for $\mathbb{P}(N(2) = 3, N(5) = 6, N(10) \geq 9)$ was found to be 0.0414, whereas the theoretical value was 0.0416683. Again, these solutions are in strong agreement.

C. $\mathbb{E}[e^{2N(0.5)}]$

Solution

```
values_at_t_1half = []

for i in range(N):
    values_at_t_1half.append(find_value_at_T(events[i], taus[i],0.5))

print('E[e^2N(0.5)]=', np.mean(np.exp(2*np.array(values_at_t_1half))))
>>
>> E[e^2N(0.5)]= 44.21546876964519
```

The theoretical solution for $\mathbb{E}[e^{2N(0.5)}]$ was found to be 46.22097, while the empirical solution was approximately 44.21547. The percent error between these solutions is 4.34%. It should be noted that when running this code using different random seeds, the empirical solutions for this part varied between 25-50, while the other empirical solutions remained fairly consistent.

D. $\mathbb{E}[\tau_2]$

Solution

```
tau2 = []

for i in range(N):
    tau2.append(taus[i][2])

print('E[tau_2]=', np.mean(tau2))
>>
>> E[tau_2]= 1.670915291003944
```

The expected value for τ_2 was found to be 1.666 in Theory 1. The empirical value was approximately 1.6709, which is in strong agreement with Theory 1.

E. $\mathbb{P}(2 < \tau_2 < 3.2)$

Solution

```

tau2_series= pd.Series(tau2)
print('P(2 < tau_2 < 3.2)=', len(tau2_series[tau2_series.between(2,3.2,
↪  inclusive='neither')])/N)
>>
>> P(2 < tau_2 < 3.2)= 0.2051
```

In Theory 1, $\mathbb{P}(2 < \tau_2 < 3.2)$ was found to be approximately 0.20441, and the empirical solution was 0.2051, once again showing a strong agreement between solutions.

2. For the compound Poisson process $X = (X(t), t \geq 0)$ corresponding to the Poisson process from Theory 1, with jumps Z_k having exponential distribution with mean 2.5, find:

A. $\mathbb{E}[X(2)]$

Solution

```
values_at_X2 = []

for i in range(N):
    values_at_X2.append(find_value_at_T(compound_events[i], taus[i], 2))

print('E[X(2)]=', np.mean(values_at_X2))
>>
>> E[X(2)]= 5.983706190626745
```

The theoretical solution for $\mathbb{E}[X(2)]$ was found to be 6, and the estimated empirical solution was 5.9837. These solutions are in strong agreement.

B. $Var(X(2))$

Solution

```
print('Var(X(2))=', np.var(values_at_X2))
>>
>> Var(X(2))= 30.143141569183378
```

The theoretical variance of $X(2)$ was found to be 30, whereas the empirical solution was 30.143. The percent error of this solution is 0.47%, demonstrating that the simulation and theory are indeed in agreement.

C. $\mathbb{E}[e^{-3X(2)}]$

Solution

```
print('E[e^(-3X(2))]=',np.mean(np.exp(-3*np.array(values_at_X2))))
>>
>> E[e^(-3X(2))]= 0.1235090615340705
```

In Theory 2c, $\mathbb{E}[e^{-3X(2)}]$ was found to be 0.1203144. The approximate empirical solution was 0.123509. The percent error of the empirical solution is approximately 2.66%, demonstrating the theoretical and empirical solutions are reasonably close.

3. Using the Central Limit Theorem, for the compound Poisson process from Theory 2, find the approximate value of u such that $X(1000) < u$ with probability 99%. This u is the 99% quantile of $X(1000)$

Solution

```
values_at_t1000 = []

for i in range(N):
    values_at_t1000.append(find_value_at_T(compound_events[i], taus[i], 1000))
print('u=', np.percentile(values_at_t1000,99))
>>
>> u= 3288.081320425806
```

The theoretical solution for $u > X(1000)$ was found to be 3284.876. In the simulations run, u was found to be approximately 3288.08132. This solution demonstrates that $X(t)$ does approach a Normal distribution for sufficiently large t.

In [1]:
```python
import numpy as np
import pandas as pd
import scipy
```

In [2]:
```python
def simulate_poisson_process(l,t,mu):
    jumpTimes= np.append(np.zeros(1),np.cumsum(np.random.exponential(scale=1
    values= np.arange(0, t+1)
    compound_values=  np.append(np.zeros(1), np.cumsum(np.random.exponential
    return values, compound_values, jumpTimes

def find_value_at_T(values,jumptimes, T):
    idx = np.where(jumptimes<T)[-1][-1]
    return values[idx]
```

1. For the Poisson process $N = (N(t), t \geq 0)$ with rate $\lambda = 1.2$. Assume τ_k is the time of the kth jump, find:

In [3]:
```python
N = 10000
l = 1.2
mu=2.5
timesteps=1500
```

In [4]:
```python
events=[]
compound_events = []
taus = []
np.random.seed(12345)
for i in range(N):
    a,b,c = simulate_poisson_process(l,timesteps,mu)
    events.append(a)
    compound_events.append(b)
    taus.append(c)
```

In [5]:
```python
values_at_t2 = []

for i in range(N):
    values_at_t2.append(find_value_at_T(events[i], taus[i], 2))

P1= values_at_t2.count(3)/N
```

A. $\mathbb{P}(N(2) = 3)$

In [6]:
```python
print('P(N(2)=3)=', P1)
```

```
P(N(2)=3)= 0.2096
```

B. $\mathbb{P}(N(2) = 3, N(5) = 6, N(10) \geq 9)$

In [7]:
```python
P2_count = 0

values_at_t2 = []
values_at_t5 = []
values_at_t10 = []
```

```python
for i in range(N):
    values_at_t2.append(find_value_at_T(events[i], taus[i], 2))
    values_at_t5.append(find_value_at_T(events[i], taus[i], 5))
    values_at_t10.append(find_value_at_T(events[i], taus[i], 10))

    if (values_at_t2[i] ==3) and (values_at_t5[i] ==6) and (values_at_t10[i]
        P2_count+=1

P2=P2_count/N
```

In [8]:
```python
print('P((N(2)=3, N(5)=6, N(10)>=9)= ', P2)
```

```
P((N(2)=3, N(5)=6, N(10)>=9)=  0.0414
```

C. $\mathbb{E}[e^{2N(0.5)}]$

In [9]:
```python
values_at_t_1half = []

for i in range(N):
    values_at_t_1half.append(find_value_at_T(events[i], taus[i],0.5))
```

In [10]:
```python
print('E[e^2N(0.5)]=', np.mean(np.exp(2*np.array(values_at_t_1half))))
```

```
E[e^2N(0.5)]= 44.21546876964519
```

In [21]:
```python
np.min(np.array(values_at_t_1half))
```

Out[21]:
```
0
```

D. $\mathbb{E}[\tau_2]$

In [11]:
```python
tau2 = []

for i in range(N):
    tau2.append(taus[i][2])
```

In [12]:
```python
print('E[tau_2]=', np.mean(tau2))
```

```
E[tau_2]= 1.670915291003944
```

E. $\mathbb{P}(2 < \tau_2 < 3.2)$

In [13]:
```python
tau2_series= pd.Series(tau2)
print('P(2 < tau_2 < 3.2)=', len(tau2_series[tau2_series.between(2,3.2, incl
```

```
P(2 < tau_2 < 3.2)= 0.2051
```

2. For the compound Poisson process $X = (X(t), t \geq 0)$ corresponding to the Poisson process from Theory 1, with jumps Z_k having exponential distribution with mean 2.5, find:

A. $\mathbb{E}[X(2)]$

In [14]:
```
values_at_X2 = []

for i in range(N):
    values_at_X2.append(find_value_at_T(compound_events[i], taus[i], 2))
```

In [15]:
```
print('E[X(2)]=', np.mean(values_at_X2))
```

```
E[X(2)]= 5.983706190626745
```

B. $Var(X(2))$

In [16]:
```
print('Var(X(2))=', np.var(values_at_X2))
```

```
Var(X(2))= 30.143141569183378
```

D. $\mathbb{E}[e^{-3X(2)}]$

In [17]:
```
print('E[e^(-3X(2))]=',np.mean(np.exp(-3*np.array(values_at_X2))))
```

```
E[e^(-3X(2))]= 0.1235090615340705
```

3. Using the Central Limit Theorem, for the compound Poisson process from Theory 2, find the approximate value of u such that $X(1000) < u$ with probability 99\%. This u is the 99\% quantile of $X(1000)$

In [18]:
```
values_at_t1000 = []

for i in range(N):
    values_at_t1000.append(find_value_at_T(compound_events[i], taus[i], 1000
```

In [19]:
```
print('u=', np.percentile(values_at_t1000,99))
```

```
u= 3288.081320425806
```