11.23

2023年11月24日 20:06

习题4.4 (B)

3. (1) $\int_0^{\pi/2} f(\sin x)dx \xlongequal{设x=\frac{\pi}{2}-t} \int_{\pi/2}^{0} f[\sin(\frac{\pi}{2}-t)]d(\frac{\pi}{2}-t)$

$\int_{\pi/2}^{0} f(\cos t)(-1)dt = \int_0^{\pi/2} f(\cos t)dt$ Q.E.D

(2) 原式 $= \int_0^{\pi} \frac{\sin x}{2-\sin^2 x}dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{2-\sin^2 x}dx$

$= -\frac{\pi}{2}\int_0^{\pi}\frac{d(\cos x)}{1+\cos^2 x}$

$= -\frac{\pi}{2}\left(\arctan(\cos x)\Big|_0^{\pi}\right)$

$= \frac{\pi^2}{4}$

6. $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$

$= \int_0^1 f(x)dx + \int_0^1 f(x+1)d(x+1)$

$= \int_0^1 f(x)dx + \int_0^1 f(x+1)dx$

$= \int_0^1 [f(x)+f(x+1)]dx$ Q.E.D

习题4.5 (A)

3. (1) 原式 $= \frac{1}{\sqrt{5}}\int 5te^{5t}d(5t)$

$= \frac{1}{\sqrt{5}}\int xe^x dx$

$= \frac{1}{25}(x-1)e^x + C = \frac{1}{25}(5t-1)e^{5t}+C$

(3) 原式 $= \frac{1}{2}\int x^2\ln x\, d(x^2)$

$= \frac{1}{4}\int u\ln u\, du$

$= \frac{1}{4}(u\ln u - u) + C = \frac{x^2}{4}(2\ln x - 1) + C$

(5) 原式 $= \frac{1}{9}\int (3z+3)e^{3z}d(3z)$

$= \frac{1}{9}\int (x+3)e^x dx$

$= \frac{1}{9}(x+2)e^x + C = \frac{1}{9}(3z+2)e^{3z}+C$

(7) 原式 $= \int x^5(\ln 5 + \ln x)dx$

(7) 原式 $= \int x^5(\ln 5+\ln x)\,dx$

$= \frac{x^6\ln 5}{6} + \int x^5 \ln x\,dx$

$= \frac{x^6\ln 5}{6} + \frac{1}{25}\int u\ln u\,du$

$= \frac{x^6\ln 5}{6} + \frac{x^5}{25}(5\ln x-1)+C$

(9) 原式 $= \frac{1}{7}\int \arctan u\,du = \frac{1}{7}\left(u\arctan u - \int\frac{u}{1+u^2}du\right)$

$= \frac{1}{7}\left(u\arctan u - \frac{1}{2}\int\frac{d(u^2+1)}{1+u^2}\right)$

$= \frac{1}{7}\left(u\arctan u - \frac{1}{2}\ln|1+u^2|\right)+C$

$= x\arctan(7x) - \frac{1}{14}\ln|1+49x^2|+C$

6. (1) 原式 $= (t\ln t - t)\Big|_1^5 = 5\ln 5-4$

(3) 原式 $= \int_3^5 x\,d(\sin x) = x\sin x\Big|_3^5 - \int_3^5 \sin x\,dx$

$= (x\sin x+\cos x)\Big|_3^5 = 5\sin 5+\cos 5-3\sin 3-\cos 3$

(5) 原式 $= \int_1^6 \ln u\,du = (u\ln u-u)\Big|_1^6$

$= 6\ln 6-5$

(7) 原式 $= \frac{1}{2}\int_0^1 \arctan(x^2)\,d(x^2) = \frac{1}{2}\int_0^1 \arctan u\,du$

$= \frac{1}{2}\left(u\arctan u - \frac{1}{2}\ln|1+u^2|\right)\Big|_0^1$

$= \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\ln 2\right) = \frac{\pi-2\ln 2}{8}$

(9) 原式 $= -\int_{\pi/4}^{\pi/2} x\,d(\cot x)$

$= -x\cot x\Big|_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}\cot x\,dx$

$= \frac{\pi}{4} + \int_{\pi/4}^{\pi/2}\frac{d(\sin x)}{\sin x}$

$= \frac{\pi}{4} + \ln|\sin x|\Big|_{\pi/4}^{\pi/2} = \frac{\pi+2\ln 2}{4}$

(11) 先求 $I = \int \sin(\ln x)\,dx$:

$I = \int \frac{x\sin(\ln x)}{x}dx = \int e^{\ln x}\sin(\ln x)\,d(\ln x)$

$\underline{\underline{记u=\ln x}}\ \int e^u\sin u\,du = \int \sin u\,d(e^u)$

$\underline{\underline{\text{设}u=\ln x}}$ $\int e^u \sin u\,du = \int \sin u\,d(e^u)$

$= e^u \sin u - \int e^u \cos u\,du = e^u \sin u - \int \cos u\,d(e^u)$

$= e^u(\sin u - \cos u) - I$

$\therefore I = \frac{1}{2}e^u(\sin u - \cos u) + C$

$= \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$

$\therefore$ 原式 $= \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right]\Big|_1^e$

$= \frac{e}{2}(\sin 1 - \cos 1) - \frac{1}{2}(0-1)$

$= \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$

(B)

3. (1) 原式 $= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}dx$

$= x\ln(1+x^2) - \int 2 - \frac{2}{1+x^2}dx$

$= x\ln(1+x^2) - 2x + 2\arctan x + C$

(3) 原式 $= \int \frac{x+\sin x}{2\cos^2\frac{x}{2}}dx = \frac{1}{2}\int \frac{x}{\cos^2\frac{x}{2}}dx + \frac{1}{2}\int \frac{\sin x}{\cos^2\frac{x}{2}}dx$

$= \int x\,d(\tan\frac{x}{2}) + \int \tan\frac{x}{2}dx$

$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}dx + \int \tan\frac{x}{2}dx$

$= x\tan\frac{x}{2} + C$

(5) 原式 $= -\int \frac{1}{e^x}d\left(\frac{1}{1+e^x}\right)$

$= -\frac{1}{e^x(1+e^x)} + \int \frac{1}{1+e^x}\left(-\frac{1}{e^x}\right)dx$

$= -\frac{1}{e^x} + \frac{1}{1+e^x} - \int \frac{1}{e^x} - \frac{1}{1+e^x}dx$

$= \frac{1}{1+e^x} + \int \frac{dx}{1+e^x}$

$= \frac{1}{1+e^x} + \int 1 - \frac{e^x}{1+e^x}dx$

$= \frac{1}{1+e^x} + x - \int \frac{d(1+e^x)}{1+e^x}$

$= \frac{1}{1+e^x} - \ln(1+e^x) + x - C$

(7) 原式 $= 2\int x\,d(\sqrt{e^x-1})$

$$\xlongequal{\text{记 } u=\sqrt{e^x-1}} 2\int \ln(u^2+1)\,du$$

$$= 2\left[u\ln(u^2+1) - \int \frac{2u^2}{u^2+1}du\right]$$

$$= 2\left[u\ln(u^2+1) - 2\int 1-\frac{1}{u^2+1}du\right]$$

$$= 2\left[u\ln(u^2+1) - 2u + 2\arctan u\right] + C$$

$$= 2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C$$

(9) 原式 $= \int_0^1 \ln(1+x)\,d\left(\frac{1}{2-x}\right)$

$$= \left.\frac{\ln(1+x)}{2-x}\right|_0^1 - \int_0^1 \frac{dx}{(2-x)(1+x)}$$

$$= \ln 2 - \int_0^1 \frac{dx}{(2-x)(x+1)}$$

$$= \ln 2 - \frac{1}{3}\int_0^1 \frac{1}{2-x} + \frac{1}{x+1}\,dx$$

$$= \ln 2 + \frac{1}{3}\int_0^1 \frac{d(x-2)}{x-2} - \frac{1}{3}\int_0^1 \frac{d(x+1)}{x+1}$$

$$= \ln 2 + \frac{1}{3}\ln|x-2|\Big|_0^1 - \frac{1}{3}\ln|x+1|\Big|_0^1$$

$$= \ln 2 - \frac{\ln 2}{3} - \frac{\ln 2}{3} = \frac{\ln 2}{3}$$

(11) 记 $I = \int_0^1 \frac{x}{e^x+e^{1-x}}dx$

$$I = \int_0^1 \frac{x}{e^x+e^{1-x}}dx \xlongequal{\text{设 } x=1-t} \int_1^0 \frac{t-1}{e^t+e^{1-t}}dt$$

$$= -I + \int_0^1 \frac{dt}{e^t+e^{1-t}}$$

$$\Rightarrow I = \frac{1}{2}\int_0^1 \frac{dt}{e^t+e^{1-t}}$$

$$= \frac{1}{2}\int_0^1 \frac{d(e^t)}{e+(e^t)^2} = \frac{1}{2\sqrt{e}}\int_0^1 \frac{d(e^{t-\frac{1}{2}})}{1+(e^{t-\frac{1}{2}})^2}$$

$$= \frac{1}{2\sqrt{e}}\arctan(e^{t-\frac{1}{2}})\Big|_0^1$$

$$= \frac{1}{2\sqrt{e}}\left(\arctan\sqrt{e} - \arctan\frac{1}{\sqrt{e}}\right)$$

(13) 设 $\arcsin\sqrt{\frac{x}{1+x}} = t$，则有：$\sqrt{\frac{x}{1+x}} = \sin t$，$x = \frac{\sin^2 t}{1-\sin^2 t} = \tan^2 t$.

$$\therefore \text{原式} = \int_0^{\pi/3} t\,d(\tan^2 t) = t\tan^2 t\Big|_0^{\pi/3} - \int_0^{\pi/3}\tan^2 t\,dt$$

$$= \pi - \int_0^{\pi/3} \sec^2 t - 1\,dt$$

$$= \pi - \int_0^{\pi/3} \sec^2 t - 1 \, dt$$

$$= \pi - \tan t \Big|_0^{\pi/3} + t \Big|_0^{\pi/3} = \frac{4\pi}{3} - \sqrt{3}$$

(15) 设$\sqrt{5-x} = t$，则有：$x = 5-t^2$，$dx = -2t\,dt$

$$\therefore 原式 = \int \frac{5-t^2}{t}(-2t)\,dt = 2\int t^2-5\,dt$$

$$= 2\left(\frac{t^3}{3} - 5t\right) + C$$

$$= \frac{2}{3}(5-x)^{\frac{3}{2}} - 10\sqrt{5-x} + C$$

(17) 设$\sqrt{5-t} = x$．则有：$t = 5-x^2$，$dt = -2x\,dx$

$$\therefore 原式 = \int \frac{12-x^2}{x}(-2x)\,dx$$

$$= 2\int x^2-12\,dx$$

$$= 2\left(\frac{x^3}{3} - 12x\right) + C$$

$$= \frac{2}{3}(5-t)^{\frac{3}{2}} - 24\sqrt{5-t} + C$$

4. (1) $原式 = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n} \frac{1}{n+i}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \frac{1}{1+\frac{i}{n}}$$

$$= \int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2$$

(2) $原式 = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n-1} \frac{i}{n^2}$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \frac{i}{n}$$

$$= \int_0^1 x\,dx = \frac{1}{2}$$

(3) $原式 = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n-1} \sin\frac{i\pi}{n}$

$$= \int_0^1 \sin(\pi x)\,dx = \frac{2}{\pi}$$