10.20

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习题 3.2 (A)

6.(2) $y' = \cos x - x\sin x$

$y'' = -\sin x - (\sin x + x\cos x) = -2\sin x + x\cos x$

(4) $y' = ae^{ax}\sin x + e^{ax}\cos x = e^{ax}(a\sin x + \cos x)$

$y'' = ae^{ax}(a\sin x + \cos x) + e^{ax}(a\cos x - \sin x)$

$= e^{ax}(a^2\sin x + 2a\cos x - \sin x)$

(6) $y = \sqrt{x^2 + x^4}$

$y' = \dfrac{2x + 4x^3}{2x\sqrt{1+x^2}} = \dfrac{1+2x^2}{\sqrt{1+x^2}} \Rightarrow \ln y' = \ln(1+2x^2) - \frac{1}{2}\ln(1+x^2)$

$\dfrac{y''}{y'} = \dfrac{4x}{1+2x^2} - \dfrac{2x}{2(1+x^2)}$

$\Rightarrow y'' = \dfrac{4x}{\sqrt{1+x^2}} - \dfrac{x(1+2x^2)}{(1+x^2)\sqrt{1+x^2}} = \dfrac{2x^3+3x}{(1+x^2)^{\frac{3}{2}}}$

(8) $y' = \ln x + 1$

$y'' = \dfrac{1}{x}$

7.(2) $y' = -\dfrac{f'(\frac{1}{x})}{x^2}$

$y'' = -\dfrac{-\frac{f''(\frac{1}{x})}{x^2}\cdot x^2 - 2xf'(\frac{1}{x})}{x^4} = \dfrac{f''(\frac{1}{x}) + 2xf'(\frac{1}{x})}{x^4}$

$y''' = \dfrac{\left\{-\frac{f'''(\frac{1}{x})}{x^2} + 2f'(\frac{1}{x}) + 2x\left[\frac{f''(\frac{1}{x})}{x^2}\right]\right\}x^4 - 4x^3\left[f''(\frac{1}{x}) + 2xf'(\frac{1}{x})\right]}{x^8}$

$= \dfrac{x^2 f'''(\frac{1}{x}) + 2x^4 f'(\frac{1}{x}) - 2x^3 f''(\frac{1}{x}) - 4x^3 f''(\frac{1}{x}) - 8x^4 f'(\frac{1}{x})}{x^8}$

$= \dfrac{f'''(\frac{1}{x}) - 6xf''(\frac{1}{x}) - 6x^2 f'(\frac{1}{x})}{x^6}$

(4) $y' = \dfrac{f'(\ln x)}{x} \Rightarrow xy' = f'(\ln x)$

$y' + xy'' = \dfrac{f''(\ln x)}{x} \Rightarrow y'' = \dfrac{f''(\ln x) - f'(\ln x)}{x^2}$

$\Rightarrow x^2 y'' = f''(\ln x) - f'(\ln x)$

$$2xy'' + x^2y''' = \frac{f'''(\ln x) - f''(\ln x)}{x}$$

$$\Rightarrow y''' = \frac{f'''(\ln x) - 3f''(\ln x) + 2f'(\ln x)}{x^3}$$

(B) 3. $y' = 2\arcsin x \cdot \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\, y' = 2\arcsin x$

$$-\frac{x}{\sqrt{1-x^2}} y' + \sqrt{1-x^2} \cdot y'' = \frac{2}{\sqrt{1-x^2}}$$

即 $(1-x^2)y'' - xy' = 2$

4. (1) $y = \frac{1}{1-x^2} \Leftrightarrow (1-x^2)y = 1$

$$\sum_{k=0}^{n} C_n^k (1-x^2)^{(k)} y^{(n-k)} = 0$$

$$\Leftrightarrow (1-x^2)y^{(n)} - 2nxy^{(n-1)} - n(n-1)y^{(n-2)} = 0$$

?

(2) $y' = 2\sin x\cos x = \sin 2x$

$y'' = 2\cos 2x \quad y''' = -4\sin 2x \quad y^{(4)} = -8\cos 2x$

$y^{(5)} = 16\sin 2x \quad \cdots\cdots$

假设 $y^{(4k+1)} = 2^{4k} \cdot \sin 2x$, $(k \in \mathbb{N})$

显然当 $k=0$ 和5时 假设成立

$$y^{(4k+2)} = 2^{4k+1} \cdot \cos 2x$$

$$y^{(4k+3)} = -2^{4k+2} \cdot \sin 2x$$

$$y^{(4k+4)} = -2^{4k+3} \cdot \cos 2x$$

$$y^{(4k+5)} = 2^{4(k+1)} \cdot \sin 2x$$

由数学归纳法可知 假设成立

综上
$$\begin{cases} y^{(4k+1)} = 2^{4k} \cdot \sin 2x \\ y^{(4k+2)} = 2^{4k+1} \cdot \cos 2x \\ y^{(4k+3)} = -2^{4k+2} \cdot \sin 2x \\ y^{(4k+4)} = -2^{4k+3} \cdot \cos 2x \end{cases}, \quad k \in \mathbb{N}$$

4. (1) $y=\frac{1}{1-x^2}=\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)$

$\therefore y^{(n)}=\frac{1}{2}\left[\left(\frac{1}{1+x}\right)^{(n)}+\left(\frac{1}{1-x}\right)^{(n)}\right]$

$\left(\frac{1}{1+x}\right)^{(1)}=-\frac{1}{(1+x)^2}$ $\left(\frac{1}{1+x}\right)^{(2)}=\frac{2}{(1+x)^3}$ ····

假设 $\left(\frac{1}{1+x}\right)^{(n)}=\frac{(-1)^n\cdot(n-1)!}{(1+x)^{n+1}}$，则有：

$\left(\frac{1}{1+x}\right)^{(n+1)}=\frac{(-1)^{n+1}\cdot n!}{(1+x)^{n+2}}$

由数学归纳法可知 假设成立

同理 易证 $\left(\frac{1}{1-x}\right)^{(n)}=\frac{n!}{(1-x)^{n+1}}$

综上 $y^{(n)}=\frac{1}{2}\left[\frac{(-1)^n n!}{(1+x)^{n+1}}+\frac{n!}{(1-x)^{n+1}}\right]$