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习题4.6(A)

1. (1) 原式 $=\int \frac{(x-2)+(x+5)}{(x-2)(x+5)}dx$

$=\int \frac{1}{x-2}+\frac{1}{x+5}dx$

$=\ln|(x-2)(x+5)|+C$

(2) 原式 $=\int \frac{x\,dx}{(x-1)(x^2+x-2)}=\int \frac{x\,dx}{(x-1)^2(x+2)}$

$=\frac{1}{9}\left(\int \frac{2x+1}{(x-1)^2}dx-\int \frac{2}{x+2}dx\right)$

$=\frac{1}{9}\left(\int \frac{2x+1}{(x-1)^2}dx-2\ln|x+2|\right)$

下面求 $\int \frac{2x+1}{(x-1)^2}dx=\int \frac{2x+1}{x^2-2x+1}dx$

$=\int \frac{(2x-2)dx}{x^2-2x+1}+\int \frac{3}{(x-1)^2}dx$

$=\int \frac{d(x^2-2x+1)}{x^2-2x+1}+\int \frac{3}{(x-1)^2}d(x-1)$

$=2\ln|x-1|-\frac{3}{x-1}+C$

$\therefore$ 原式 $=\frac{1}{9}\left(2\ln\left|\frac{x-1}{x+2}\right|-\frac{3}{x-1}\right)+C$

(3) 原式 $=\frac{1}{2}\int\left(\frac{2}{x}-\frac{x+1}{x^2+1}-\frac{1}{x+1}\right)dx$

$=\frac{1}{2}\int\left(\frac{2}{x}-\frac{\frac{1}{2}\cdot 2x}{x^2+1}-\frac{1}{x^2+1}-\frac{1}{x+1}\right)dx$

$=\frac{1}{2}\left(2\ln|x|-\frac{1}{2}\ln|x^2+1|-\arctan x-\ln|x+1|\right)+C$

(4) 原式 $=\int 2x+\frac{x-2}{1+x^2}dx$

$=x^2+\frac{1}{2}\int \frac{2x-4}{1+x^2}dx$

$=x^2+\frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}-2\int \frac{dx}{1+x^2}$

$=x^2+\frac{1}{2}\ln(1+x^2)-2\arctan x+C$

2. (1) 原式 $=-\int \cos^4 x(1-\cos^2 x)\,d(\cos x)$

$\xlongequal{\text{记}u=\cos x}\int u^6-u^4\,du$

$=\frac{u^7}{7}-\frac{u^5}{5}+C=\frac{\cos^7 x}{7}-\frac{\cos^5 x}{5}+C$

(2) 原式 $= \int \frac{2\sin x\cos x}{2-\sin^2 x}dx = \int \frac{2\sin x}{2-\sin^2 x}d(\sin x)$

$= \int \frac{d(\sin^2 x)}{2-\sin^2 x} = -\int \frac{d(2-\sin^2 x)}{2-\sin^2 x}$

$= -\ln|2-\sin^2 x| + C$

(3) 原式 $= \int \frac{dx}{\sin 2x + 2\sin x} = \int \frac{dx}{2\sin x(\cos x+1)} = \int \frac{dx}{4\sin\frac{x}{2}\cos\frac{x}{2}\, 2\cos^2\frac{x}{2}}$

$= \int \frac{d(\frac{x}{2})}{4\sin\frac{x}{2}\cos^3\frac{x}{2}} = \frac{1}{4}\int \csc\frac{x}{2}\sec^3\frac{x}{2}\, d(\frac{x}{2})$

$= \frac{1}{4}\int \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\, d(\tan\frac{x}{2})$

$= \frac{1}{4}\int \tan\frac{x}{2} + \frac{1}{\tan\frac{x}{2}}\, d(\tan\frac{x}{2})$

$= \frac{1}{4}\left(\frac{\tan^2\frac{x}{2}}{2} + \ln\left|\tan\frac{x}{2}\right|\right) + C$

(4) 原式 $= \int \frac{dx}{2\sin^2\frac{x}{2} + 2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \int \frac{1}{\tan^2\frac{x}{2} + \tan\frac{x}{2} + 1}\sec^2\frac{x}{2}\, d(\frac{x}{2})$

$= \int \frac{d(\tan\frac{x}{2})}{\tan^2\frac{x}{2} + \tan\frac{x}{2} + 1} \xlongequal{设\tan\frac{x}{2}=t} \int \frac{dt}{t^2+t+1}$

$= \int \frac{d(t+\frac{1}{2})}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}} + C$

$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right) + C$

3. (1) 设 $x = t^2+1$，则 $dx = 2t\,dt$

原式 $= \int \frac{t}{t^2+1}\cdot 2t\,dt = 2\int \frac{t^2}{t^2+1}dt$

$= 2\int 1 - \frac{1}{t^2+1}\,dt$

$= 2(t - \arctan t) + C$

$= 2(\sqrt{x-1} - \arctan\sqrt{x-1}) + C$

(2) 设 $x = t^3-2$，则 $dx = 3t^2dt$

原式 $= \int \frac{3t^2}{1+t}dt = 3\int \frac{t^2-1+1}{1+t}dt$

$= 3\int t-1+\frac{1}{1+t}\,dt$

$= 3\left(\frac{t^2}{2} - t + \ln|1+t|\right) + C$

$= 3\left(\frac{1}{2}(x+2)^{\frac{2}{3}} - \sqrt[3]{x+2} + \ln|1+\sqrt[3]{x+2}|\right) + C$

$(x^2+1)-(x^2-1)$

$$= 3\left(\frac{1}{2}(x+2)^{\frac{2}{3}} - \sqrt[3]{x+2} + \ln\left|1+\sqrt[3]{x+2}\right|\right)+C$$

(B)(1) 原式 $= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx$

$$= \frac{1}{2}\left(\int\frac{x^2+1}{x^4+1}dx - \int\frac{x^2-1}{x^4+1}dx\right)$$

$$= \frac{1}{2}\left(\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx\right)$$

$$= \frac{1}{2}\left(\int\frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2}dx - \int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}dx\right)$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{(x+\frac{1}{x})-\sqrt{2}}{(x+\frac{1}{x})+\sqrt{2}}\right|\right]+C$$

$$= \frac{\sqrt{2}}{4}\arctan\frac{x^2-1}{\sqrt{2}x} + \frac{\sqrt{2}}{8}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + C$$

(2) 原式 $= \int\frac{dx}{(x-1)(x+1)(x^2+1)} = \frac{1}{4}\int\left(\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}\right)dx$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\arctan x + C$$

(3) 原式 $= \frac{1}{2}\int\frac{(x^2+1)-(x^2-1)}{x^4+x^2+1}dx$

$$= \frac{1}{2}\left(\int\frac{x^2+1}{x^4+x^2+1}dx - \int\frac{x^2-1}{x^4+x^2+1}dx\right)$$

$$= \frac{1}{2}\left(\int\frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+3} - \int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{3}}\arctan\frac{x-\frac{1}{x}}{\sqrt{3}} - \frac{1}{2}\ln\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right|\right)$$

$$= \frac{\sqrt{3}}{6}\arctan\frac{x^2-1}{\sqrt{3}x} + \frac{1}{4}\ln\left|\frac{x^2+x+1}{x^2-x+1}\right| + C$$

(4) 原式 $= \int\tan x(\sec^2 x-1)dx = \int\tan x\sec^2 x - \int\tan x\,dx$

$$= \int\tan x\, d(\tan x) + \int\frac{1}{\cos x}d(\cos x)$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

(5) 原式 $= \int\csc^2 x\sec x\,dx = -\int\sec x\, d(\cot x)$

$$= -\sec x\cot x + \int\cot x\tan x\sec x\,dx$$

$$= -\sec x\cot x + \int\sec x\,dx.$$

下求 $\int\sec x\,dx = \int\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx$

$$= \int\frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx = \int\frac{d(\tan x+\sec x)}{\tan x+\sec x}$$

$$= \ln|\tan x+\sec x|+C$$

原式 $= -\csc x + \ln|\tan x + \sec x| + C = -\csc x + \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$

(6) 原式 $= \int \frac{dx}{4\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2}} = \int \frac{d(\tan\frac{x}{2})}{\tan^2\frac{x}{2}+2}$

$= \frac{1}{\sqrt{2}}\arctan\frac{\tan\frac{x}{2}}{\sqrt{2}} + C$

(7) 设 $x = t^6$，则 $dx = 6t^5 dt$

原式 $= \int \frac{6t^5}{(1+t^2)t^3}dt = 6\int \frac{t^5}{t^5+t^3}dt$

$= 6\int 1 - \frac{1}{t^2+1}dt = 6(t - \arctan t) + C$

$= 6\left[\sqrt[6]{x} - \arctan(\sqrt[6]{x})\right] + C$

(8) 原式 $= \int \frac{x+2-2\sqrt{x+1}}{x}dx = \int 1 + \frac{2}{x} - \frac{2\sqrt{x+1}}{x}dx$

$= x + 2\ln|x| - 2\int \frac{\sqrt{x+1}}{x}dx$ 设 $x = t^2 - 1$

$= x + 2\ln|x| - 2\int \frac{t}{t^2-1} 2t\,dt$

$= x + 2\ln|x| - 4\int 1 + \frac{1}{t^2-1}dt$

$= x + 2\ln|x| - 4\left(t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right) + C$

$= x + 2\ln|x| - 4\sqrt{x+1} - 2\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

$= x - 4\sqrt{x+1} + 4\ln(\sqrt{x+1}+1) + C$