6-4(2)

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25. (1) $A=\begin{bmatrix}4&0&0\\0&-6&-2\\0&-2&-6\end{bmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix}4-\lambda&0&0\\0&-6-\lambda&-2\\0&-2&-6-\lambda\end{vmatrix}=(4-\lambda)\left[(6+\lambda)^2-4\right]$$

$$=(4-\lambda)(4+\lambda)(8+\lambda)=0$$

$\lambda=4、-4、-8$

1° 当 $\lambda=4$ 时 $A-\lambda I=\begin{bmatrix}0&0&0\\0&-10&-2\\0&-2&-10\end{bmatrix}$

$\Rightarrow \alpha_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \eta_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$

2° 当 $\lambda=-4$ 时 $A-\lambda I=\begin{bmatrix}8&0&0\\0&-2&-2\\0&-2&-2\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}$

$\Rightarrow \alpha_2=\begin{bmatrix}0\\-1\\1\end{bmatrix} \Rightarrow \eta_2=\begin{bmatrix}0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

3° 当 $\lambda=-8$ 时 $A-\lambda I=\begin{bmatrix}12&0&0\\0&2&-2\\0&-2&2\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}$

$\Rightarrow \alpha_3=\begin{bmatrix}0\\1\\1\end{bmatrix} \Rightarrow \eta_3=\begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$$\begin{bmatrix}x&y&z\end{bmatrix}\begin{bmatrix}1&0&0\\0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}x&\frac{-y+z}{\sqrt{2}}&\frac{y+z}{\sqrt{2}}\end{bmatrix}$$

令 $\begin{cases}x'=x\\y'=\frac{-y+z}{\sqrt{2}}\\z'=\frac{y+z}{\sqrt{2}}\end{cases}\Rightarrow\begin{cases}x=x'\\y=\frac{-y'+z'}{\sqrt{2}}\\z=\frac{y'+z'}{\sqrt{2}}\end{cases}\xrightarrow{\text{代入原式}} 4x'^2-3(-y'+z')^2-3(y'+z')^2$

$-4\cdot\frac{z'^2-y'^2}{2}-4(x'-\sqrt{2}z)-5=0$

$\Rightarrow 4(x'-\frac{1}{2})^2-4(y'-\frac{1}{2\sqrt{2}})^2-8z'^2-5=0$

$\therefore$ 双叶双曲面.

(2) $A=\begin{bmatrix}1&0&2\\0&-1&-2\\2&-2&0\end{bmatrix}$ $\det(A-\lambda I)=\begin{vmatrix}1-\lambda&0&2\\0&-1-\lambda&-2\\2&-2&-\lambda\end{vmatrix}$

$$\begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \qquad \begin{vmatrix} 2 & -2 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-1-\lambda)(-\lambda)-4\right]+2\cdot 2(1+\lambda)$$

$$= (1-\lambda)(\lambda^2+\lambda-4)+4(1+\lambda)$$

$$= \lambda^2+\lambda-4-\lambda^3-\lambda^2+4\lambda+4+4\lambda$$

$$= -\lambda^3+9\lambda = \lambda(3-\lambda)(3+\lambda)$$

$$\Rightarrow \lambda = 0,\ 3,\ -3$$

1° 当 $\lambda=0$ 时 $A-\lambda I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \alpha_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \eta_1 = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

2° 当 $\lambda=3$ 时 $A-\lambda I = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \alpha_2 = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} \Rightarrow \eta_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$

3° 当 $\lambda=-3$ 时 $A-\lambda I = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \alpha_3 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \eta_3 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{-2x+2y-z}{3} & \frac{-2x-y+2z}{3} & \frac{x+2y+2z}{3} \end{bmatrix}$$

令 $\begin{cases} x' = \frac{-2x+2y-z}{3} \\ y' = \frac{-2x-y+2z}{3} \\ z' = \frac{x+2y+2z}{3} \end{cases} \xRightarrow{\text{化入原式}} x''^2 - y''^2 = 1 \Rightarrow$ 双曲柱面.

(3) 同理 原式可化为 $x''^2 + y''^2 - 2z''^2 = 1 \Rightarrow$ 单叶双曲面.

(4) $2x''^2 + 4y''^2 + 4z''^2 = 1 \Rightarrow$ 椭球面.