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2023年11月12日 10:16

25. (1) 先求 $\begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & 1 & 1 \\ 1-\lambda & \lambda-1 & 0 \\ 0 & 1-\lambda & \lambda-1 \end{vmatrix}$

$$= \begin{vmatrix} \lambda & 2 & 1 \\ 1-\lambda & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda+2 & 2 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda+2)(\lambda-1)^2$$

∴ 1° 当 $\lambda \neq -2$ 且 $\lambda \neq 1$ 时 方程组有唯一解

考虑增广矩阵:

$$\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1-\lambda & \lambda-1 & 0 & 0 \\ 0 & 1-\lambda & \lambda-1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\begin{cases} x_1 = x_2 = x_3 \\ \lambda x_1 + x_2 + x_3 = 1 \end{cases} \Leftarrow \left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

所以 $x = \begin{bmatrix} 1/(\lambda+2) \\ 1/(\lambda+2) \\ 1/(\lambda+2) \end{bmatrix}$

2° 当 $\lambda=1$ 时 原方程可化为 $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

即方程组有无穷多解

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \ (a, b \in \mathbb{R})$$

3° 当 $\lambda=-2$ 时 原方程可化为 $\left[\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

由 row1 可知方程组无解.

(2) 由(1)可知 $\begin{vmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{vmatrix} = (\lambda+3)\lambda^2$

1° 当 $\lambda \neq -3$ 且 $\lambda \neq 0$ 时，考虑增广矩阵:

$$\left[\begin{array}{ccc|c} 1+\lambda & 1 & 1 & 1 \\ 1 & 1+\lambda & 1 & \lambda \\ 1 & 1 & 1+\lambda & \lambda^2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1+\lambda & 1 & 1 & 1 \\ -\lambda & \lambda & 0 & \lambda-1 \\ 0 & -\lambda & \lambda & \lambda^2-\lambda \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1+\lambda & 2 & 0 & 2-\lambda \\ -1 & 1 & 0 & \frac{\lambda-1}{\lambda} \\ 0 & -1 & 1 & \lambda-1 \end{array}\right] \leftarrow \left[\begin{array}{ccc|c} 1+\lambda & 1 & 1 & 1 \\ -1 & 1 & 0 & \frac{\lambda-1}{\lambda} \\ 0 & -1 & 1 & \lambda-1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 3+\lambda & 0 & 0 & \frac{2}{\lambda}-\lambda \\ -1 & 1 & 0 & \frac{\lambda-1}{\lambda} \\ 0 & -1 & 1 & \lambda-1 \end{array}\right] \Rightarrow x = \frac{1}{\lambda^2+3\lambda}\begin{bmatrix} 2-\lambda^2 \\ 2\lambda-1 \\ \lambda^3+2\lambda^2-\lambda-1 \end{bmatrix}$$

2° 当 $\lambda=0$ 时，原方程组可化为 $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

所以方程组无解

3° 当 $\lambda=-3$ 时，原方程组可化为 $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 7/3 \\ -1 & 1 & 0 & 4/3 \\ 0 & -1 & 1 & -4 \end{array}\right]$

所以方程组无解

(3) 先计算 $\begin{vmatrix} 3-2\lambda & 2-\lambda & 1 \\ 2-\lambda & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2-\lambda & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$

$= (1-\lambda)\left[(2-\lambda)^2-1\right] = 0$

1° 当 $\lambda \neq 1$ 且 $\lambda \neq 3$ 时 原方程有唯一個解

考虑 $\left[\begin{array}{ccc|c} 3-2\lambda & 2-\lambda & 1 & \lambda \\ 2-\lambda & 2-\lambda & 1 & 1 \\ 1 & 1 & 2-\lambda & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1-\lambda & 0 & 0 & \lambda-1 \\ 2-\lambda & 2-\lambda & 1 & 1 \\ 1 & 1 & 2-\lambda & 1 \end{array}\right]$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2-\lambda & 2 \\ 0 & 0 & (\lambda-1)(-\lambda+3) & \lambda-1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2-\lambda & 1 & 3-\lambda \\ 0 & 1 & 2-\lambda & 2 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} -1 \\ (4-\lambda)/(3-\lambda) \\ 1/(3-\lambda) \end{bmatrix}$$

2° 当 $\lambda=1$ 时，原方程组可化为 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

即原方程组有无穷多解

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

3° 当 $\lambda=3$ 时，原方程组可化为 $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

由 row3 可知原方程组无解