2-4

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习题 2

15. (1) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 3 & 5 \end{pmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{13}{2} \end{bmatrix} \Rightarrow \text{rank} = 3$

考虑增广矩阵 $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{3} & 0 & 0 & \frac{1}{3} \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{13}{6} & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{5}{13} & \frac{1}{13} \\ 0 & 0 & 1 & 0 & -\frac{3}{13} & \frac{2}{13} \end{array}\right]$

$\therefore \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 3 & 5 \end{pmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{5}{13} & \frac{1}{13} \\ 0 & -\frac{3}{13} & \frac{2}{13} \end{bmatrix}$

16. 记原矩阵为 A.

则 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & \frac{1}{a_{n-1}} & 0 \end{bmatrix}$

17 记 $A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$, 则 $A^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & & & \\ & & & 1 & -1 & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$

$\therefore X = A^{-1} \cdot \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$

$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$

$$\begin{pmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$$

18. (2) 原方程可化为 $Ax=b$: $\begin{bmatrix} 2 & 4 & 3 \\ 2 & -3 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

考虑增广矩阵 $[A|b]$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & -1 \\ 1 & -\frac{3}{2} & 0 & 0 \\ 1 & 0 & 5 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & -1 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & 1 \\ 0 & -2 & \frac{7}{2} & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & -1 \\ 0 & -\frac{7}{2} & -\frac{3}{2} & 1 \\ 0 & 0 & \frac{61}{14} & \frac{38}{7} \end{array}\right]$$

$$\therefore x = \begin{bmatrix} -\frac{75}{61} \\ -\frac{50}{61} \\ \frac{76}{61} \end{bmatrix}$$

19. (2) 法一: 由 Gauss-Jordan 法可得:

$$A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -13 & 6 & -1 \\ -29 & 13 & -2 \end{bmatrix}$$

$$\therefore A^{-1}C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

法二: 注意到 $A_{c1} = C_{c2}$, $A_{r2} = C_{c3}$, $A_{r3} = C_{c1}$

$$\therefore A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = C$$

$$\therefore A^{-1}C = A^{-1}A\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

22. 记 I 为 identity matrix

则有 $D^{-1}D = \begin{bmatrix}I&0\\0&I\end{bmatrix}$ 即 $D^{-1}\begin{bmatrix}0&A\\B&C\end{bmatrix} = \begin{bmatrix}I&0\\0&I\end{bmatrix}$

设 $D^{-1} = \begin{bmatrix}M&N\\P&Q\end{bmatrix}$，则有：

$$\begin{bmatrix}M&N\\P&Q\end{bmatrix}\begin{bmatrix}0&A\\B&C\end{bmatrix} = \begin{bmatrix}I&0\\0&I\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}NB&MA+NC\\QB&PA+QC\end{bmatrix} = \begin{bmatrix}I&0\\0&I\end{bmatrix}$$

$$\because \begin{bmatrix}NB\\QB\end{bmatrix} = \begin{bmatrix}I\\0\end{bmatrix}$$

$$\therefore \quad N = B^{-1} \quad Q = 0$$

$$又\because \begin{bmatrix}MA+NC\\PA+QC\end{bmatrix} = \begin{bmatrix}MA+B^{-1}C\\PA\end{bmatrix} = \begin{bmatrix}0\\I\end{bmatrix}$$

$$\therefore \quad P = A^{-1} \quad M = -B^{-1}CA^{-1}$$

综上 $D^{-1} = \begin{bmatrix}-B^{-1}CA^{-1}&B^{-1}\\A^{-1}&0\end{bmatrix}$

23. 证明：$E^k - A^k = E^k - 0 = E^k$

$$\Rightarrow (E-A)\sum_{i=0}^{k-1}E^{k-1-i}A^i = E^k$$

$$\because E^k = E,\ k\in\mathbb{Z}$$

$$\therefore (E-A)\sum_{i=0}^{k-1}A^i = E$$

$$\therefore (E-A)^{-1} = \sum_{i=0}^{k-1}A^i = E + A + A^2 + \cdots + A^{k-1} \quad Q.E.D$$

24. 由课本内容可知 $AA^* = dE$

$\therefore$ 有 $|A||A^*| = |dE| = d^n|E| = d^n = |A|^n \quad \therefore |A^*| = |A|^{n-1} \quad Q.E.D$