9.28

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习题(A)

6. (1) 原式 $= \lim\limits_{x\to\infty}\left[\left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right]^{\frac{3}{2}}$

$= e^{\frac{3}{2}}$

(3) e

(5) 原式 $= \lim\limits_{x\to\infty}\left\{\left[1+\left(-\frac{1}{x}\right)\right]^{-x}\right\}^{-4}$

$= e^{-4}$

总习题(2)

4. (1) 原式 $= \lim\limits_{x\to4}\frac{(2x-8)(\sqrt{x-2}+\sqrt{2})}{(x-4)(\sqrt{2x+1}+3)}$

$= 2\lim\limits_{x\to4}\frac{\sqrt{x-2}+\sqrt{2}}{\sqrt{2x+1}+3}$

$= 2\times\frac{2\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$

(3) 原式 $= \lim\limits_{x\to\infty}\frac{\frac{x+1}{x}\cdot\frac{x^2+1}{x^2}\cdots\cdots\frac{x^n+1}{x^n}}{\left[(nx)^n+1\right]^{\frac{n+1}{2}}\Big/x^{\frac{n(n+1)}{2}}}$

$= \lim\limits_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)\cdots\cdots\left(1+\frac{1}{x^n}\right)}{\left[\frac{(nx)^n+1}{x^n}\right]^{\frac{n+1}{2}}}$

$= \lim\limits_{x\to\infty}\frac{1^n}{\left(n^n+\frac{1}{x^n}\right)^{\frac{n+1}{2}}} = n^{-\frac{n(n+1)}{2}}$

(5) 1° 当$x=0$时 原式$=0$

2° 当$x=\pm1$时 有 $1+\sin\pi x=1$

原式 $= \frac{x}{2}$

3° 当 $0<x<1$ 时 $1+\sin\pi x>1$

$$原式=\frac{x+\frac{\sin\pi x}{(1+\sin\pi x)^n}}{1+\frac{1}{(1+\sin\pi x)^n}}=x$$

4° 当 $-1<x<0$ 时 $0<1+\sin\pi x<1$

$$原式=\sin\pi x$$

6. 不妨设 $x<m<1$，则有 $\frac{2^x+3^x}{5}<\frac{2^m+3^m}{5}<1$

而显然 $0<\frac{2^x+3^x}{5}$

即有 $0<\left(\frac{2^x+3^x}{5}\right)^{\frac{1}{x}}<\left(\frac{2^m+3^m}{5}\right)^{\frac{1}{x}}$

又 $\because \lim\limits_{x\to 0}\left(\frac{2^m+3^m}{5}\right)^{\frac{1}{x}}=0$

由夹逼定理可得 $\lim\limits_{x\to 0}\left(\frac{2^x+3^x}{5}\right)^{\frac{1}{x}}=0$

8. 首先假设 x_n 收敛，且收敛于 A

那么当 $n\to\infty$ 时 有 $A=\frac{1}{A}+\frac{A}{2}\Rightarrow A=\sqrt{2}$

下证：$\lim\limits_{n\to\infty}x_n=\sqrt{2}\iff\forall\varepsilon>0,\exists N>0,\ni\forall n>N,$

都有 $|x_{n+1}-\sqrt{2}|<\varepsilon$

首先 $x_{n+1}=\frac{1}{x_n}+\frac{x_n}{2}\geqslant 2\sqrt{\frac{1}{2}}=\sqrt{2}$

$$\begin{aligned}\therefore |x_{n+1}-\sqrt{2}| &= x_{n+1}-\sqrt{2}\\ &=\frac{1}{x_n}+\frac{x_n}{2}-\sqrt{2}\\ &=\frac{(x_n-\sqrt{2})^2}{2x_n}\leqslant\frac{(x_n-\sqrt{2})^2}{2\sqrt{2}}\end{aligned}$$

即 $x_{n+1}-\sqrt{2}\leqslant\frac{\sqrt{2}}{4}(x_n-\sqrt{2})^2$

因此就有 $x_{n+1}-\sqrt{2}\leqslant\frac{\sqrt{2}}{4}(x_n-\sqrt{2})^2\leqslant\frac{\sqrt{2}}{4}\cdot\frac{1}{8}(x_{n-1}-\sqrt{2})^2$

$\vdots$

$\leqslant\frac{\sqrt{2}}{4}\cdot\frac{1}{8^{n-1}}\cdot(x_1-\sqrt{2})^2$

$$\leq \frac{\sqrt{2}}{4} \cdot \frac{1}{8^{n-1}} \cdot (x_1 - \sqrt{2})^2$$

$\because x_1 = 4 \quad \therefore x_{n+1} - \sqrt{2} \leq \frac{\sqrt{2}}{2} \cdot \frac{1}{8^{n-1}} \cdot (9 - 4\sqrt{2})$

$\therefore$ 想要 $|x_{n+1} - \sqrt{2}| < \varepsilon$ 成立，

只需要 $\frac{\sqrt{2}}{2} \cdot \frac{1}{8^{n-1}} \cdot (9-4\sqrt{2}) < \varepsilon$

$$\Leftrightarrow 8^{n-1} > \frac{9\sqrt{2}-8}{2\varepsilon}$$

$$\Leftrightarrow (n-1) \times 3 > \log_2 \frac{9\sqrt{2}-8}{\varepsilon} - 1$$

$$\Leftrightarrow n > \frac{\log_2 \frac{9\sqrt{2}-8}{\varepsilon} + 2}{3}$$

综上 取 $N = \left[\frac{\log_2 \frac{9\sqrt{2}-8}{\varepsilon} + 2}{3} \right] + 1$，

即可证明 $\lim_{n \to \infty} x_n = \sqrt{2}$