11.31

2023年11月29日 22:20

习题4.7(A)

2.(12) 原式 $= \lim\limits_{a\to4^+}\int_a^{20}\frac{dy}{y^2-16} \xlongequal{\text{设 } y=4\sec\theta} \lim\limits_{a\to0^+}\int_a^{\arccos\frac{1}{5}}\frac{4\sec\theta\tan\theta}{16\tan^2\theta}d\theta$

$= \frac{1}{4}\lim\limits_{a\to0^+}\int_a^{\arccos\frac{1}{5}}\csc\theta d\theta = \frac{1}{4}\lim\limits_{a\to0^+}\ln\left|\tan\frac{\theta}{2}\right|\Big|_a^{\arctan\frac{1}{5}}$

$= +\infty$

∴原积分发散.

(19) 先求 $I=\int e^{-ax}\sin bx\,dx$

$I = -\frac{1}{b}\int e^{-ax}d(\cos bx) = -\frac{\cos bx}{be^{ax}} - \frac{a}{b}\int e^{-ax}\cos bx\,dx$

$= -\frac{\cos bx}{be^{ax}} - \frac{a}{b^2}\int e^{-ax}d(\sin bx)$

$= -\frac{\cos bx}{be^{ax}} - \frac{a\sin bx}{b^2e^{ax}} - \frac{a^2}{b^2}I$

$\therefore I = -\frac{b\cos bx + a\sin bx}{(a^2+b^2)e^{ax}}$

原式 $= -\frac{b\cos bx+a\sin bx}{(a^2+b^2)e^{ax}}\Big|_0^{+\infty}$

$= -\left(0-\frac{b}{a^2+b^2}\right) = \frac{b}{a^2+b^2}$

(20) 原式 $= -\frac{1}{2}\int_0^{+\infty}\ln x\,d\left(\frac{1}{1+x^2}\right)$

$= -\frac{1}{2}\left[\frac{\ln x}{1+x^2}\Big|_0^{+\infty} - \int_0^{+\infty}\frac{dx}{x(1+x^2)}\right]$

$= -\frac{1}{2}\left[\frac{\ln x}{1+x^2}\Big|_0^{+\infty} - \int_0^{+\infty}\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx\right]$

$= -\frac{1}{2}\left[\frac{\ln x}{1+x^2} - \ln x + \frac{1}{2}\ln(1+x^2)\right]\Big|_0^{+\infty}$

记 $F(x) = \frac{\ln x}{1+x^2} - \ln x + \frac{1}{2}\ln(1+x^2)$

下面计算 $\lim\limits_{x\to+\infty}F(x)$ 和 $\lim\limits_{x\to0}F(x)$

$\lim\limits_{x\to+\infty}F(x) = \lim\limits_{x\to+\infty}\left(\frac{\ln x}{1+x^2}+\frac{1}{2}\ln\frac{1+x^2}{x^2}\right)$

$= \lim\limits_{x\to+\infty}\frac{\ln x}{1+x^2} + \frac{1}{2}\lim\limits_{x\to+\infty}\ln\left(1+\frac{1}{x^2}\right)$

$= 0$

$\lim\limits_{x\to0}F(x) = \lim\limits_{x\to0}\left[-\frac{x^2\ln x}{1+x^2}+\frac{1}{2}\ln(1+x^2)\right]$

$$\lim_{x\to0}F(x)=\lim_{x\to0}\left[-\frac{x^2\ln x}{1+x^2}+\frac{1}{2}\ln(1+x^2)\right]$$

$$=\left(\lim_{x\to0}-\frac{1}{1+x^2}\right)\cdot\left(\lim_{x\to0}x^2\ln x\right)+\frac{1}{2}\lim_{x\to0}\ln(1+x^2)$$

$$=-\lim_{x\to0}x^2\ln x=-\lim_{x\to0}\frac{\ln x}{x^{-2}}=-\lim_{x\to0}\frac{\frac{1}{x}}{-2x^{-3}}$$

$$=\lim_{x\to0}\frac{x^2}{2}=0$$

综上 原式 $=-\frac{1}{2}(0-0)=0$

(B) 2. 原式 $=\int_{1/2}^{1}\frac{dx}{\sqrt{x-x^2}}+\int_{1}^{3/2}\frac{dx}{\sqrt{x^2-x}}$

先计算 $I_1=\int_{1/2}^{1}\frac{dx}{\sqrt{x-x^2}}$:

$I_1=\frac{1}{2}\int_{0}^{1}\frac{dx}{\sqrt{x(1-x)}}$ $\because (1-x)+x=1$ $\therefore$ 设 $1-x=\sin^2\theta$

$$=\frac{1}{2}\int_{\pi/2}^{0}\frac{-2\sin\theta\cos\theta}{\sqrt{\cos^2\theta\sin^2\theta}}d\theta$$

$$=-\int_{\pi/2}^{0}d\theta=\frac{\pi}{2}$$

再计算 $I_2=\int_{1}^{3/2}\frac{dx}{\sqrt{x^2-x}}=\int_{1}^{3/2}\frac{dx}{\sqrt{x}\sqrt{x-1}}$ 设 $x=\sec^2\theta$

$$\int_{0}^{\cos\frac{\sqrt6}{3}}\frac{2\sec\theta\sec\theta\tan\theta}{\sec\theta\tan\theta}d\theta$$

$$=2\int_{0}^{\arccos\frac{\sqrt6}{3}}\sec\theta\, d\theta=2\ln\left|\sec\theta+\tan\theta\right|\Big|_{0}^{\arccos\frac{\sqrt{6}}{3}}.$$

$$=2\ln\left|\frac{3}{\sqrt6}+\frac{1}{\sqrt2}\right.=2\ln\frac{3+\sqrt3}{\sqrt6}=\ln\frac{12+6\sqrt3}{6}$$

$$=\ln(2+\sqrt3)$$

综上 原式 $=\frac{\pi}{2}+\ln(2+\sqrt3)$

4. 记 $\Gamma(n+1)=\int_0^{+\infty}x^ne^{-x}dx$

$$\Gamma(n+1)=\int_0^{+\infty}\frac{1}{n+1}e^{-x}d(x^{n+1})=\frac{x^{n+1}}{(n+1)e^x}\Big|_0^{+\infty}-\int_0^{+\infty}x^{n+1}\cdot\frac{-e^{-x}}{n+1}dx$$

$$=0+\frac{1}{n+1}\int_0^{+\infty}x^{n+1}e^{-x}dx=\frac{\Gamma(n+2)}{n+1}$$

$\therefore\ \Gamma(n+2)=(n+1)\Gamma(n+1)$

$$\Gamma(1)=\int_0^{+\infty}e^{-x}dx=-e^{-x}\Big|_0^{+\infty}=1$$

$$\therefore\int_0^{+\infty}x^ne^{-x}dx=\Gamma(n+1)=n\Gamma(n)=\cdots=n!\,\Gamma(1)=n!$$