12.5

2023年12月9日 9:54

1. 已知 $\int_{-\infty}^{+\infty} e^{-x^2}dx=\sqrt{\pi}$，$n\geq 1$ 是自然数，计算积分：

(1) $\Gamma(\frac{1}{2})$ (2) $\Gamma(\frac{n+1}{2})$ (3) $\int_0^{+\infty} x^n e^{-x^2}dx$

(1) $\sqrt{\pi}=\int_{-\infty}^{+\infty} e^{-x^2}dx = 2\int_0^{+\infty} e^{-x^2}dx$

$\underline{\underline{设u=x^2}}$ $2\int_0^{+\infty}\frac{1}{2\sqrt{u}}e^{-u}du = \int_0^{+\infty}u^{-\frac{1}{2}}e^{-u}du$

$\Gamma(\frac{1}{2})=\int_0^{+\infty}x^{-\frac{1}{2}}e^{-x}dx=\sqrt{\pi}$

(2) 1° 当 $n=2k-1$，$k\in\mathbb{N}^*$ 时

$\Gamma(\frac{n+1}{2})=\Gamma(k)=(k-1)!$

2° 当 $n=2k$，$k\in\mathbb{N}^*$ 时

$\Gamma(\frac{n+1}{2})=\Gamma(k+\frac{1}{2})=(k-1+\frac{1}{2})(k-2+\frac{1}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$

$=(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{1}{2}\sqrt{\pi}$

(3) 记 $I_n=\int_0^{+\infty}x^n e^{-x^2}dx$

$I_n=\frac{1}{2}\int_0^{+\infty}x^{n-1}e^{-x^2}d(x^2)$

$=-\frac{1}{2}\int_0^{+\infty}x^{n-1}d(e^{-x^2})$

$=-\frac{1}{2}\left(x^{n-1}e^{-x^2}\Big|_0^{+\infty}-\int_0^{+\infty}\frac{x^{n-2}}{n-1}e^{-x^2}dx\right)$

$=-\frac{1}{2}\left(0-\frac{1}{n-1}I_{n-2}\right)$

$=\frac{1}{2(n-1)}I_{n-2}$

$I_0=\sqrt{\pi}$ $I_1=\int_0^{+\infty}xe^{-x^2}dx$

$=-\frac{1}{2}e^{-x^2}\Big|_0^{+\infty}=\frac{1}{2}$

综上 $I_n=\begin{cases}\dfrac{\sqrt{\pi}}{2^{n/2}(n-1)!!}, & n为偶数.\\ \dfrac{1}{2^{(n+1)/2}(n-1)!!}, & n为奇数.\end{cases}$

2. $\Gamma(m)\Gamma(n)=\int_0^{+\infty}u^{m-1}e^{-m}du\int_0^{+\infty}v^{n-1}e^{-n}dv$

设 $s=u+v$ $=\int_0^{+\infty}\int_0^{+\infty}u^{m-1}v^{n-1}e^{-m-n}du\,dv$

设 $s=u+v$ $\quad = \int_0^{+\infty}\int_0^{+\infty} u^{m-1} v^{n-1} e^{-m-n}\, du\, dv$

$= \int_0^{+\infty}\int_0^{s} u^{m+1}(s-u)^{n-1} e^{-s}\, du\, ds$

设 $t=\frac{u}{s}$ $\quad = \int_0^{+\infty}\int_0^{s} u^{m+1}\left(1-\frac{u}{s}\right)^{n-1} s^{n-1} e^{-s}\, du\, ds$

$= \int_0^{+\infty}\int_0^{1} (st)^{m-1}(1-t)^{n-1} s^{n-1} e^{-s}\cdot s\, dt\, ds$

$= \int_0^{+\infty}\int_0^{1} t^{m-1}(1-t)^{n-1} s^{m+n-1} e^{-s}\, dt\, ds$

$= \int_0^{+\infty} s^{m+n-1} e^{-s}\, ds \int_0^{1} t^{m-1}(1-t)^{n-1}\, dt$

$= \Gamma(m+n)B(m,n)$

综上 $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

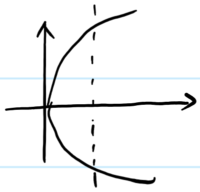
习题4.8 (A)

1. (1) 令 $x^2 = 2x+3 \Rightarrow x=-1$ 或 3

$S = \int_{-1}^{3} 2x+3-x^2\, dx$

$= x^2+3x-\frac{x^3}{3}\Big|_{-1}^{3} = \frac{32}{3}$

(3)



$S = 2\int_0^5 \sqrt{2x}\, dx$

$= 2\sqrt{2}\,\frac{2x^{\frac{3}{2}}}{3}\Big|_0^5$

$= \frac{4\sqrt{2}}{3}\, x^{\frac{3}{2}}\Big|_0^5$

$= \frac{20\sqrt{10}}{3}$

(5) $S = 4\int_0^1 \frac{\sqrt{1-x^2}}{3}\, dx = \frac{4}{3}\int_0^1 \sqrt{1-x^2}\, dx$

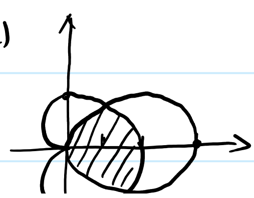
$\xlongequal{设\ x=\sin\theta} \frac{4}{3}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2} \cos 2\theta + 1\, d\theta$

$= \frac{2}{3}\left(\frac{1}{2}\sin 2\theta + \theta\right)\Big|_0^{\pi/2} = \frac{\pi}{3}$

2. (1) $S = 4\int_0^{\pi/2} \frac{1}{2}(2a\cos\theta)^2\, d\theta$

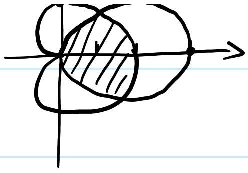
$= 8a^2\int_0^{\pi/2} \cos^2\theta\, d\theta = 2a^2\pi$

(2)



令 $3\cos\theta = 1+\cos\theta$

$\Rightarrow \theta = \frac{\pi}{3}$ 或 $\frac{5\pi}{3}$



$$\Rightarrow \theta = \frac{\pi}{3} \text{ 或 } \frac{5\pi}{3}$$

$$S = 2\left(\int_0^{\pi/3} \frac{1}{2}(1+\cos\theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 d\theta \right)$$

$$= \int_0^{\pi/3} 1+2\cos\theta + \cos^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} 9\cos^2\theta \, d\theta$$

$$= \left(\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + \frac{\theta}{2}\right)\Big|_0^{\pi/3} + 9\left(\frac{1}{4}\sin 2\theta + \frac{\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}$$

$$= \frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8} + 9\left(\frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2} + \frac{9\sqrt{3}}{8} + \frac{3\pi}{4} - \frac{9\sqrt{3}}{8} = \frac{5\pi}{4}$$