11.28

2023年11月29日 22:20

习题4.5 (B)

2. (1) 记 $I=\int e^x\cos x\,dx$

$$I=\int \cos x\,d(e^x) = e^x\cos x-\int(-\sin x)\,d(e^x)$$

$$= e^x\cos x+e^x\sin x-\int e^x\cos x\,dx$$

$$= e^x(\sin x+\cos x)-I$$

$$\therefore I=\frac{e^x}{2}(\sin x+\cos x)+C$$

(2) 记 $I=\int xe^x\sin x\,dx$

$$I=\int x\sin x\,d(e^x)$$

$$= xe^x\sin x-\int \sin x+x\cos x\,d(e^x)$$

$$= xe^x\sin x-\int e^x\sin x\,dx-\int x\cos x\,d(e^x)$$

$$= xe^x\sin x-\frac{e^x}{2}(\sin x-\cos x)-xe^x\cos x+\int \cos x-x\sin x\,d(e^x)$$

$$= xe^x\sin x-\frac{e^x}{2}(\sin x-\cos x)-xe^x\cos x+\int e^x\cos x\,dx-I$$

$$\therefore 2I= xe^x\sin x-\frac{e^x}{2}(\sin x-\cos x)-xe^x\cos x+\frac{e^x}{2}(\sin x+\cos x)+C$$

$$= xe^x(\sin x-\cos x)+e^x\cos x+C = e^x\left[x\sin x-(x-1)\cos x\right]+C$$

$$\therefore I=\frac{e^x}{2}\left[x\sin x-(x-1)\cos x\right]+C$$

(3) $$\int xe^x\cos x\,dx=\int xe^x\sin(x+\tfrac{\pi}{2})\,d(x+\tfrac{\pi}{2})$$

$$=\int(x+\tfrac{\pi}{2})e^x\sin(x+\tfrac{\pi}{2})\,d(x+\tfrac{\pi}{2})-\frac{\pi}{2}\int e^x\cos x\,dx$$

$$=\frac{\int(x+\frac{\pi}{2})e^{x+\frac{\pi}{2}}\sin(x+\frac{\pi}{2})\,d(x+\frac{\pi}{2})}{e^{\pi/2}}-\frac{\pi}{2}\int e^x\cos x\,dx$$

$$=\frac{\frac{e^{x+\frac{\pi}{2}}}{2}\left[(x+\frac{\pi}{2})\cos x+(x+\frac{\pi}{2}-1)\sin x\right]}{e^{\pi/2}}-\frac{\pi}{2}\frac{e^x}{2}(\sin x+\cos x)+C$$

$$=\frac{e^x}{2}\left[x\cos x+(x-1)\sin x\right]+\frac{e^x}{2}\frac{\pi}{2}(\cos x+\sin x)-\frac{\pi}{2}\frac{e^x}{2}(\sin x+\cos x)+C$$

$$=\frac{e^x}{2}\left[x\cos x+(x-1)\sin x\right]+C$$

总习题(4)

15. (1) 原式 $=\frac{1}{2}\int\frac{\ln x}{(1+x^2)^{3/2}}\,d(x^2) = -\frac{1}{2}\int \ln x\,d\left(\frac{2}{\sqrt{1+x^2}}\right)$

$$=-\frac{1}{2}\left(\frac{2\ln x}{\sqrt{1+x^2}}-\int\frac{2}{x\sqrt{1+x^2}}dx\right)$$

$$=-\frac{\ln x}{\sqrt{1+x^2}}+\int\frac{dx}{x\sqrt{1+x^2}}$$

下面求 $I=\int\frac{dx}{x\sqrt{1+x^2}}$，设 $x=\tan\theta$，则

下面求 $I=\int\frac{dx}{x\sqrt{1+x^2}}$，设 $x=\tan\theta$，则

$$I=\int\frac{\sec^2\theta}{\tan\theta\sec\theta}d\theta=\int\csc\theta d\theta$$
$$=\int\frac{\csc\theta(\cot\theta+\csc\theta)}{\cot\theta+\csc\theta}d\theta$$
$$=-\int\frac{-\csc^2\theta-\cot\theta\csc\theta}{\cot\theta+\csc\theta}d\theta$$
$$=-\int\frac{d(\cot\theta+\csc\theta)}{\cot\theta+\csc\theta}$$
$$=-\ln|\csc\theta+\cot\theta|+C$$
$$=\ln\left|\frac{\sqrt{x^2+1}-1}{x}\right|+C$$

$\therefore$ 原式 $=-\frac{\ln x}{\sqrt{x^2+1}}+\ln\left(\frac{\sqrt{x^2+1}-1}{x}\right)+C$

(2) 原式 $=\int\tan^2\frac{x}{2}dx=2\int\sec^2\frac{x}{2}-1\,d(\frac{x}{2})$

$$=2\left(\tan\frac{x}{2}-\frac{x}{2}\right)+C$$
$$=2\tan\frac{x}{2}-x+C$$

(3) 原式 $\xlongequal{设u=x^2}\frac{1}{2}\int u\sqrt{1+u}\,du$

$$\xlongequal{设t=\sqrt{u+1}}\frac{1}{2}\int(t^2-1)t\cdot 2t\,dt$$
$$=\int t^4-t^2\,dt=\frac{t^5}{5}-\frac{t^3}{3}+C$$
$$=\frac{(x^2+1)^{5/2}}{5}-\frac{(x^2+1)^{3/2}}{3}+C$$

(4) 原式 $=\frac{1}{18}\left[-\frac{6x}{1+x^3}-\ln(x^2-x+1)+2\ln|x+1|+2\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}}\right]+C$

Proof:

For the integrand $\frac{x^3}{(1+x^3)^2}$, use partial fractions:

$$I=\int\frac{x^3}{(1+x^3)^2}dx=\int\frac{x^3}{(x+1)^2(x^2-x+1)^2}$$
$$=\int\frac{3-x}{9(x^2-x+1)}+\frac{x-1}{3(x^2-x+1)^2}+\frac{1}{9(x+1)}-\frac{1}{9(x+1)^2}dx$$
$$=\frac{1}{9}\int\frac{3-x}{x^2-x+1}dx+\frac{1}{3}\int\frac{x-1}{(x^2-x+1)^2}+\frac{1}{9}\ln|x+1|+\frac{1}{9(x+1)}$$

First calculate $\int\frac{3-x}{x^2-x+1}dx$:

$$\int\frac{3-x}{x^2-x+1}dx=-\frac{1}{2}\int\frac{2x-6}{x^2-x+1}dx$$
$$=-\frac{1}{2}\left[\int\frac{d(x^2-x+1)}{x^2-x+1}-5\int\frac{dx}{x^2-x+1}\right]$$
$$=-\frac{1}{2}\ln(x^2-x+1)+\frac{5}{2}\int\frac{d(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}}$$
$$=-\frac{1}{2}\ln(x^2-x+1)+\frac{5}{2}\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C$$
$$=-\frac{1}{2}\ln(x^2-x+1)+\frac{5\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}}+C$$

$$= -\frac{1}{2}\ln(x^2 - x + 1) + \frac{5\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C$$

Then calculate $\int \frac{x-1}{(x^2-x+1)^2}\,dx$:

$$\int \frac{x-1}{(x^2-x+1)^2}\,dx = \frac{1}{2}\int \frac{2x-2}{(x^2-x+1)^2}\,dx$$
$$= \frac{1}{2}\Big[\int \frac{d(x^2-x+1)}{(x^2-x+1)^2} - \int \frac{dx}{(x^2-x+1)^2}\Big]$$
$$= \frac{1}{2}\Big[-\frac{1}{x^2-x+1} - \int \frac{dx}{(x^2-x+1)^2}\Big]$$

In which

$$\int \frac{dx}{(x^2-x+1)^2} = \int \frac{d(x-\frac{1}{2})}{[(x-\frac{1}{2})^2+\frac{3}{4}]^2}$$

Let $x-\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$

$$= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{9}{16}\sec^4\theta}\,\theta = \frac{8\sqrt{3}}{9}\int \cos^2\theta\,d\theta$$
$$= \frac{4\sqrt{3}}{9}\int(\cos 2\theta + 1)\,d\theta = \frac{2\sqrt{3}}{9}(\sin 2\theta + 2\theta) + C$$
$$= \frac{4\sqrt{3}}{9}\arctan\frac{2x-1}{\sqrt{3}} + \frac{2\sqrt{3}}{9}\frac{\sqrt{3}}{2}\frac{2x-1}{x^2-x+1} + C$$
$$= \frac{4\sqrt{3}}{9}\arctan\frac{2x-1}{\sqrt{3}} + \frac{2x-1}{3(x^2-x+1)} + C$$

Therefore

$$\int \frac{x-1}{(x^2-x+1)^2}\,dx = \frac{1}{2}\Big[-\frac{1}{x^2-x+1} - \frac{4\sqrt{3}}{9}\arctan\frac{2x-1}{\sqrt{3}} - \frac{2x-1}{3(x^2-x+1)}\Big] + C$$
$$= \frac{1}{2}\Big[-\frac{4\sqrt{3}}{9}\arctan\frac{2x-1}{\sqrt{3}} - \frac{2x+2}{3(x^2-x+1)}\Big] + C$$
$$= -\frac{2\sqrt{3}}{9}\arctan\frac{2x-1}{\sqrt{3}} - \frac{x+1}{3(x^2-x+1)} + C$$

Above all

$$I = \frac{1}{9}\Big[\int \frac{3-x}{x^2-x+1}\,dx + 3\int \frac{x-1}{(x^2-x+1)^2} + \ln|x+1| + \frac{1}{x+1}\Big]$$
$$= \frac{1}{9}\Big[-\frac{1}{2}\ln(x^2-x+1) + \frac{5\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} +$$
$$3\Big(-\frac{2\sqrt{3}}{9}\arctan\frac{2x-1}{\sqrt{3}} - \frac{x+1}{3(x^2-x+1)}\Big) + \ln|x+1| + \frac{1}{x+1}\Big] + C$$

$$= \frac{1}{9}\Big[-\frac{1}{2}\ln(x^2-x+1) + \frac{5\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} - \frac{2\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}}$$
$$-\frac{x+1}{x^2-x+1} + \ln|x+1| + \frac{1}{x+1}\Big] + C$$
$$= \frac{1}{9}\Big[-\frac{1}{2}\ln(x^2-x+1) + \ln|x+1| + \sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} - \frac{3x}{x^3+1}\Big] + C$$
$$= \frac{1}{18}\Big[-\frac{6x}{1+x^3} - \ln(x^2-x+1) + 2\ln|x+1| + 2\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}}\Big] + C$$

$Q.E.D$

(5) 令 $x=-t$，则

$$\int_{-\pi/2}^{\pi/2}\frac{e^x}{1+e^x}\sin^4 x\,dx = \int_{\pi/2}^{-\pi/2}\frac{e^{-t}}{1+e^{-t}}\sin^4(-t)(-dt) = \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^t}\sin^4 t\,dt$$

∴有 $\int_{-\pi/2}^{\pi/2}\frac{e^x}{1+e^x}\sin^4 x\,dx = \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^x}\sin^4 x\,dx$

原式 $= \int_{-\pi/2}^{\pi/2}\left(1-\frac{1}{1+e^x}\right)\sin^4 x\,dx = \int_{-\pi/2}^{\pi/2}\sin^4 x\,dx - \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^x}\sin^4 x\,dx$

$$原式 = \int_{-\pi/2}^{\pi/2}\left(1-\frac{1}{1+e^x}\right)\sin^4 x\,dx = \int_{-\pi/2}^{\pi/2}\sin^4 x\,dx - \int_{-\pi/2}^{\pi/2}\frac{1}{1+e^x}\sin^4 x\,dx$$

$$\therefore 原式 = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\sin^4 x\,dx = \frac{1}{2}\cdot 2\int_0^{\pi/2}\sin^4 x\,dx$$

$$= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$$

(6) $\because d(\arcsin\sqrt{x}) = \frac{1}{\sqrt{1-x}}d\sqrt{x} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}dx.$

$$\therefore 2\sqrt{x}\,d(\arcsin\sqrt{x}) = \frac{dx}{\sqrt{1-x}}$$

$$\therefore 原式 = \int_0^{\frac{1}{2}} 2\sqrt{x}\arcsin\sqrt{x}\,d(\arcsin\sqrt{x})$$

$$\underline{\underline{设 u=\arcsin\sqrt{x}}}\ \int_0^{\pi/4} 2u\sin u\,du = -\int_0^{\pi/4} 2u\,d(\cos u)$$

$$= -\left(2u\cos u\Big|_0^{\pi/4} - \int_0^{\pi/4} 2\cos u\,du\right)$$

$$= \frac{4-\pi}{2\sqrt{2}}$$

(7) $$原式 = \int_{-\pi}^{5\pi}\left(\frac{\cos x+\cos 3x}{2}\cos 4x + \frac{\cos x-\cos 3x}{2}\sin 4x\right)dx$$

$$= \frac{1}{2}\int_{-\pi}^{5\pi}\left[\cos x(\cos 4x+\sin 4x) + \cos 3x(\cos 4x-\sin 4x)\right]dx$$

$$= \frac{1}{2}\int_{-\pi}^{5\pi}\left(\frac{\cos 3x+\cos 5x}{2} + \frac{\sin 3x+\sin 5x}{2} + \frac{\cos x+\cos 7x}{2} + \frac{\sin x+\sin 7x}{2}\right)dx$$

$$= \frac{1}{4}\int_{-\pi}^{5\pi}\left[\sin x+\cos x+\sin 3x+\cos 3x+\sin 5x+\cos 5x+\sin 7x+\cos 7x\right]dx$$

$$= 0$$

(8) $$原式 = \frac{1}{1+2^{\frac{1}{x}}}\Bigg|_{-1}^{0^-} + \frac{1}{1+2^{\frac{1}{x}}}\Bigg|_{0^+}^{1}$$

$$= 0-\frac{2}{3}+\frac{1}{3}-0 = -\frac{1}{3}$$

(9) $$原式 = \int_{-\pi/2}^{\pi/2}\frac{\cos x}{\sin x}dx = \int_{-\pi/2}^{\pi/2}\frac{d(\sin x)}{\sin x}$$

$$= \ln|\sin x|\Big|_{-\pi/2}^{0^-} + \ln|\sin x|\Big|_{0^+}^{\pi/2}$$

$$= 2\ln|\sin x|\Big|_0^{\pi/2} = \infty$$

$\therefore$ 原积分发散.

(10) $$原式 = \int_0^1\frac{dx}{\sqrt{1-x^2}} + \int_1^2\frac{dx}{\sqrt{x^2-1}}$$

$$= \arcsin x\Big|_0^1 + \int_0^{\pi/3}\frac{\sec\theta\tan\theta}{\tan\theta}d\theta$$

$$= \frac{\pi}{2} + \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/3}$$

$$= \frac{\pi}{2}+\ln(2+\sqrt{3})$$

19. (1) $$原式 = \lim_{n\to\infty}\frac{\sum_{i=1}^{n} i^p}{n^{p+1}} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p = \int_0^1 x^p dx = \frac{1}{1+p}$$

(2) $$原式 = \lim_{x\to 0}\frac{-\frac{1}{2}\int_0^x f(x^2-t^2)\,d(x^2-t^2)}{x^4}$$

$$\underline{\underline{设 u=x^2-t^2}}\ \lim_{x\to 0}\frac{-\frac{1}{2}\int_{x^2}^{0} f(u)\,du}{x^4}$$

$\underline{\underline{设\ u=x^2-t^2}}\ \lim_{x\to 0}\frac{-\frac{1}{2}\int_{x^2}^{0}f(u)du}{x^4}$

$=\lim_{x\to 0}\frac{\int_0^{x^2}f(u)du}{2x^4}=\lim_{x\to 0}\frac{f(x^2)\cdot 2x}{8x^3}=\lim_{x\to 0}\frac{f(x^2)}{4x^2}$

$=\lim_{x\to 0}\frac{f'(x^2)\cdot 2x}{8x}=\frac{1}{4}$

22.(1) $A\int_0^a g(x)dx=\int_0^a Ag(x)dx=\int_0^a(f(x)+f(-x))g(x)dx$

$=\int_0^a f(x)g(x)dx+\int_0^a f(-x)g(x)dx$

$=\int_0^a f(x)g(x)dx+\int_a^0 f(-x)g(-x)d(-x)$

$=\int_0^a f(x)g(x)dx+\int_{-a}^0 f(x)g(x)dx=\int_{-a}^a f(x)g(x)dx$

(2) $\because\ \arctan e^x+\arctan e^{-x}=\frac{\pi}{2}$ 且 $|\sin x|$ 为偶函数.

$\therefore$ 原式 $=\frac{\pi}{2}\int_0^{\pi/2}|\sin x|dx=\frac{\pi}{2}\left(-\cos x\Big|_0^{\pi/2}\right)=\frac{\pi}{2}$

23. $S=\int_0^{\pi/2}|\sin(x-a)|dx\qquad a\in(0,\frac{\pi}{2})$

$=\int_0^a -\sin(x-a)dx+\int_a^{\pi/2}\sin(x-a)dx.$

$=-\int_0^a\sin(x-a)d(x-a)+\int_a^{\pi/2}\sin(x-a)d(x-a)$

$=\cos(x-a)\Big|_0^a-\cos(x-a)\Big|_a^{\pi/2}$

$=1-\cos a-(\sin a-1)=2-\sin a-\cos a=2-\sqrt{2}\sin(a+\frac{\pi}{4})$

$\therefore$ 当 $a=\frac{\pi}{4}$ 时 S 取最小值 $S_{min}=2-\sqrt{2}$