Modeling and Control of 2 DOF Robotic Arm

A Preprint

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1 Introduction: Motivation and Summary

- For two DOF robotic arm controller is implemented to control the θ_1 and θ_2 angle made by link 1 and link 2 respectively.

1. Perform the dynamic modeling of mechanical links and that of actuator and augment them into one state space matrix
2. Linearize the model about point of interest.
3. The closed loop system formation using LQR and LQG methods.
4. The closed loop should have following properties.
   - Closed loop stability
   - zero steady state error to step reference command
   - good low frequency reference command following
   - good low frequency disturbance attenuation
   - good high frequency noise attenuation
   - good stability robustness margins

2 Modeling

Consider two link planar arm in Fig 1, where the link frames have been illustrated. Both revolute joint are parallel. The DH parameters are specified in Table 1.[1]
a_1 is length of link 1, a_2 is length of link 2, θ_1 is angle swept by link 1 w.r.t x_o axis and θ_2 w.r.t axis made by length of link 1

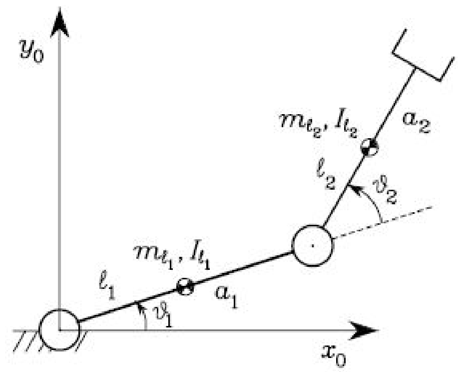

Figure 1: Two link planar arm

Index	a	α	θ	d
1	a_1	0	θ_1	0
2	a_2	0	θ_2	0

Table 1: DH parameters of 2 DOF robot

2.1 Forward Kinematic

With information of θ_1 and θ_2, the end effector location can be derived.
$T_2^0(\theta)$ represent transformation from base to end effector.

$$T_2^0(\theta) = \begin{pmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$c_{12} = cos(\theta_1 + \theta_2)$
$s_{12} = sin(\theta_1 + \theta_2)$

x_e and y_e denotes end effector position.
$x_e = a_1c_1 + a_2c_{12}$
$y_e = a_1s_1 + a_2s_{12}$

2.2 Model Parameters

a_1 = 1m –> Length of link 1
a_2 = 1m –> Length of link 2
L_1 = 0.5m –> Distance of CG on link 1 from joint 1
L_2 = 0.5m –> Distance of CG on link 2 from joint 2
mL_1 = 50kg –> Weight of link 1

mL_2 = 50kg –> Weight of link 2
IL_1 = 10 kg.m^2 –> Moment of inertia of link 1
IL_2 = 10 kg.m^2 –> Moment of inertia of link 2
kr_1 = 100 –> Gear ratio of motor 1
kr_2 = 100 –> Gear ratio of motor 2
mm_1 = 5kg –> Mass of motor 1
mm_2 = 5kg –> Mass of motor 2
Im_1 = 0.01 kg.m^2 –> Moment of inertia of motor 1
Im_2 = 0.01 kg.m^2 –> Moment of inertia of motor 2

2.3 Dynamic Modeling

Equation of motion representing the joint space model is represented as

M(q)$\ddot{q}$ + C(q,$\dot{q}$)$\dot{q}$ + G(q) = τ (Augmenting actuator (motor) dynamics into mechanical dynamics of link)

where :
M(q) is inertia matrix
C(q,$\dot{q}$) is centrifugal / Coriolis matrix
G(q) is gravity matrix
q is vector is joint variables
$\dot{q}$ is vector of joint velocity
$\ddot{q}$ is vector of joint acceleration
(Assuming no viscous friction torques, static friction torques, no force exerted by end effector on environment)

Inertia matrix M is a 2x2 matrix with following entries.
M(1,1) = $IL_1 + mL_1*L_1^2 + kr_1^2*Im_1 + IL_2 + mL_2*(a_1^2+L_2^2+2*a_1*L_2*\cos\theta_2 + Im_2 + mm_2*a_1^2$
M(1,2) = M(2,1) = $IL_2 + mL_2*(a_1^2+L_2^2+2*a_1*L_2*\cos\theta_2 + kr_2*Im_2$
M(2,2) = $IL_2 + mL_2*L_2^2 + kr_2^2*Im_2$

Centrifugal/Coriolis matrix C is a 2x2 matrix
$h = -mL_2*a_1*L_2*\sin\theta_2$;
C(1,1) = $h*\theta_2$;
C(1,2) = $h*(\dot{\theta}_1+\dot{\theta}_2)$;
C(2,1) = $-h*(\dot{\theta}_1)$;
C(2,2) = 0;

Gravity matrix G is 2x2 matrix
g = 9.81; m/s^2
G(1,1) = $mL_1*L_1 + mm_2*a_1 + mL_2*a_1*g*\cos\theta_1 + mL_2*L_2*g*\cos(\theta_1+\theta_2)$
G(1,2) = $mL_2*L_2*g*\cos(\theta_1+\theta_2)$

G(2,1) = G(1,2)
G(2,2) = G(1,2)

2.3.1 Controls

$u = [u1,u2]^T$
$u1 = \tau_1$ N-m –> Torque from motor 1
$u2 = \tau_2$ N-m –> Torque from motor 2

2.3.2 States

$x = [x_1,x_2,x_3,x_4]^T$
$x_1 = \theta_1$ (radians) –> Angle swept by Link 1
$x_2 = \theta_2$ (radians) –> Angle swept by Link 2
$x_3 = \dot{\theta}_1$ (rad/sec) –> Angular velocity of Link 1
$x_4 = \dot{\theta}_2$ (rad/sec) –> Angular velocity of Link 2

2.3.3 Outputs

$y = [y1,y2]^T$
$y_1 = \theta_1$ (radians) –> Angle swept by Link 1
$y_2 = \theta_2$ (radians) –> Angle swept by Link 2

2.4 Linearization Procedure

Starting with our dynamic equation
M(q)$\ddot{q}$ + C(q,$\dot{q}$)$\dot{q}$ + G(q) = τ
Isolating $\ddot{q}$ by moving all terms to RHS and taking left inverse of M(q) following steps from [2]
$\ddot{q} = M^{-1}*(\tau$ - C(q,$\dot{q}$)$\dot{q}$ - G(q))

Linearizing the nonlinear model about the equilibrium point $\theta_1 = 90^\circ$, $\theta_2 = 0^\circ$, (both links vertical), results in the following linear model.

Linearized inertia matrix
$M_{eq}(1,1) = IL_1 + mL_1*L_1^2 + kr_1^2*Im_1 + IL_2 + mL_2 *(a_1^2{+}L_2^2{+}2*a_1*L_2*\cos\theta_{2eq} + Im_2 + mm_2*a_1^2$
$M_{eq}(1,2) = M_{eq}(2,1) = IL_2 + mL_2 *(a_1^2{+}L_2^2{+}2*a_1*L_2*\cos\theta_{2eq} + kr_2*Im_2$
$M_{eq}(2,2) = IL_2 + mL_2*L_2^2 + kr_2^2*Im_2$

G(1,1) = $mL_1*L_1 + mm_2 * a_1 + mL_2*a_1*g*\cos\theta_1 + mL_2*L_2*g*\cos(\theta_1+\theta_2)$
G(1,2) =mL_2*L_2*g$*\cos(\theta_1+\theta_2)$
G(2,1) = G(1,2)

G(2,2) = G(1,2)

Linearized gravity matrix
$G_{eq}(1,1)$ = -(mL_1*L_1 + mm_2 * a_1 + mL_2*a_1*g*sin θ_{1eq}) - mL_2*L_2*g*sin$(\theta_{1eq}+\theta_{2eq})$
$G_{eq}(1,2)$ = -(mL_2*L_2*g*sin$(\theta_{1eq}+\theta_{2eq})$)
$G_{eq}(2,1) = G_{eq}(2,2) = G_{eq}(1,2)$

2.4.1 State Space Equation

$\dot{x} = Ax + Bu$
$y = Cx$
$u = [u1, u2]^T$
$x = [x_1, x_2, x_3, x_4]^T$
$y = [y1, y2]^T$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4.0421 & 0.6419 & 0 & 0 \\ 0.4017 & 1.7479 & 0 & 0 \end{pmatrix} \tag{2}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.0043 & -0.0017 \\ -0.0017 & 0.0088 \end{pmatrix} \tag{3}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{5}$$

2.4.2 Scaling

Convert angle to degrees from radians
Keep Torque as N-m
r2d = 180/Π
sx = diag([r2d, r2d, r2d, r2d])
sy = diag([r2d, r2d]);
A=sx*A*inv(sx)
B=sx*B

C=sy*C*inv(sx)

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4.0421 & 0.6419 & 0 & 0 \\ 0.4017 & 1.7479 & 0 & 0 \end{pmatrix} \tag{6}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.2482 & -0.0983 \\ -0.0983 & 0.5066 \end{pmatrix} \tag{7}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

Two-input two-output (TITO) linear model, the control inputs τ_1 and τ_2 are measured in N-m, angles θ_1 and θ_2 in deg, angular rates $\dot{\theta}_1$ and $\dot{\theta}_2$ is measured in deg/sec.

3 Analysis of System

3.1 Modal Analysis

3.1.1 Finding and Exciting the modes

V = Eigenvectors matrix
D = Eigevalues matrix

$$V = \begin{pmatrix} -0.4346 & 0.1589 & 0.4346 & -0.1589 \\ -0.0727 & -0.5945 & 0.0727 & 0.5945 \\ 0.8854 & -0.2035 & 0.8854 & -0.2035 \\ 0.1481 & 0.7615 & 0.1481 & 0.7615 \end{pmatrix} \tag{10}$$

$$D = \begin{pmatrix} -2.0370 & 0 & 0 & 0 \\ 0 & -1.2808 & 0 & 0 \\ 0 & 0 & 2.0370 & 0 \\ 0 & 0 & 0 & 1.2808 \end{pmatrix} \tag{11}$$

1. The right half plane pole at s = 2.0370 is a standard inverted pendulum instability. It is associated with $\dot{\theta}_1$ (Angular rate of Link 1). Both link have same weight of 50kg each but link 2 is supported by link 1, hence more net weight acting on link 1, so it also makes sense that link 1 is associated with high unstable pole.(Figure 2)

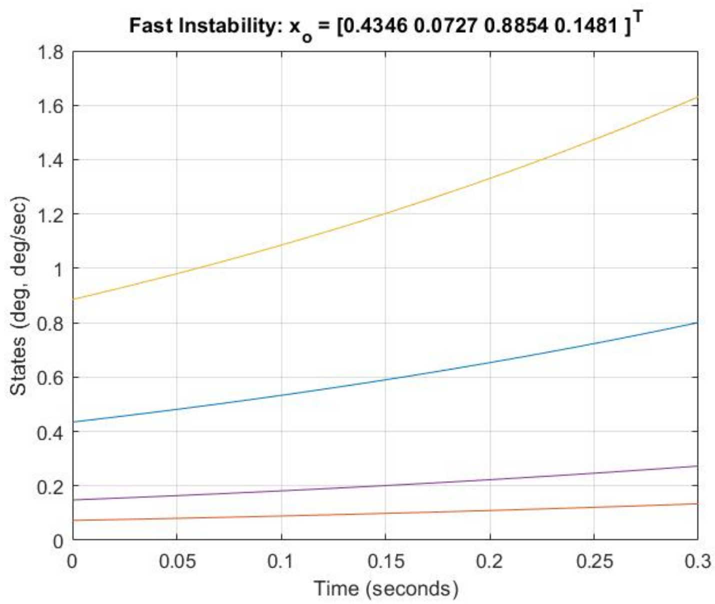

Figure 2: Exciting mode of RHP pole at s=2.0370

- θ_1 is blue, θ_2 is orange, $\dot{\theta}_1$ is yellow, $\dot{\theta}_2$ is purple
- Figure 2 shows natural response of mode 2.0370, given initial condition x_0 and zero input. $\dot{\theta}_1$ is growing at positive rate instead of decaying which makes it a mode unstable mode associated mainly to $\dot{\theta}_1$
- This system has characteristic of a real pendulum and hence the angle would not grow, but it will oscillate. Looking at figure 2 we conclude that linear model has limitation.

2. The right half plane pole at s = 1.2808 is a standard inverted pendulum instability. It is associated with $\dot{\theta}_2$ and θ_2 (Angular rate of Link 2 and angle swept by Link 2).(Figure 3)

   - θ_1 is blue, θ_2 is orange, $\dot{\theta}_1$ is yellow, $\dot{\theta}_2$ is purple
   - Figure 3 we can observe that θ_2 and $\dot{\theta}_2$ is rising exponentially where as θ_1 and $\dot{\theta}_1$ is decaying
   - Slow instability as compared to pole at 2.0370.

3. The left half plane pole at s = -2.0370 is a standard inverted pendulum damping mode. It is associated with $\dot{\theta}_1$ (Angular rate of Link 1).(Figure 4)

   - θ_1 is blue, θ_2 is orange, $\dot{\theta}_1$ is yellow, $\dot{\theta}_2$ is purple
   - Quick damping mode, as all the states are decaying to 0.

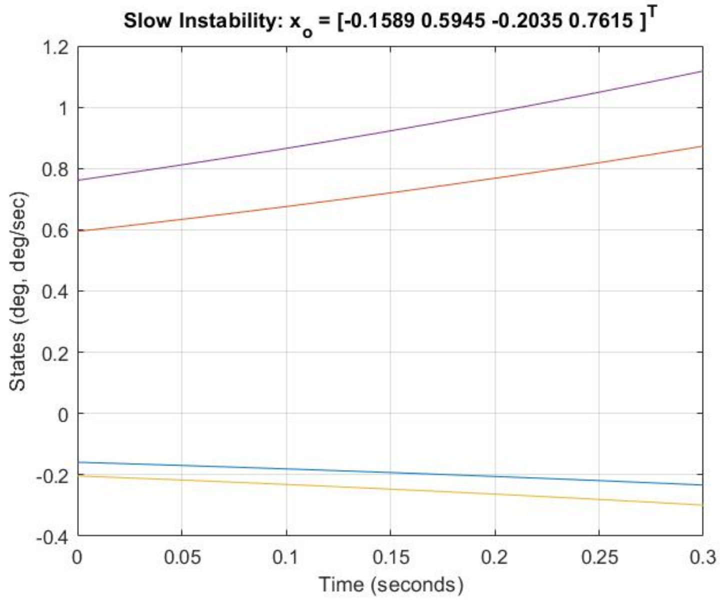

Figure 3: Exciting mode of RHP pole at s=1.2080

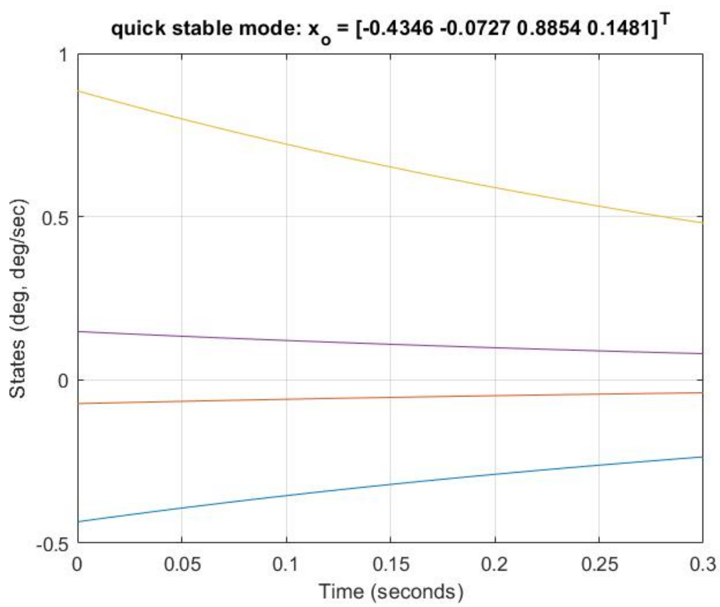

Figure 4: Exciting mode of LHP pole at s= -2.0370

4. The left half plane pole at s = -1.2808 is a standard inverted pendulum damping mode. It is associated with $\dot{\theta}_2$. (Angular rate of Link 2).(Figure 5)

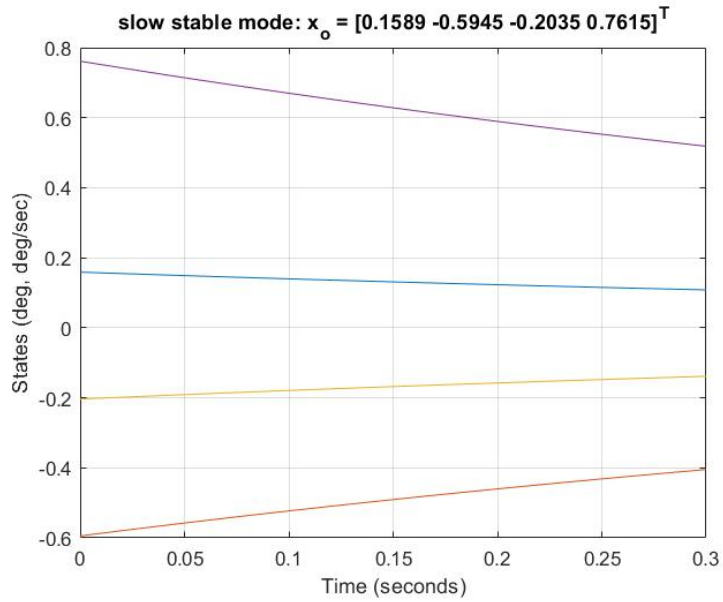

Figure 5: Exciting mode of LHP pole at s=-1.2808

- θ_1 is blue, θ_2 is orange, $\dot{\theta}_1$ is yellow, $\dot{\theta}_2$ is purple
- slow damping mode, all states are decaying to 0 but at slower rate.

3.1.2 Mode's time domain analysis

Index	Pole	Magnitude	Damping ζ	Frequency (rad/sec)	Time Constant (sec)
1	1.28e+00	1.28e+00	-1.00e+00	4.95e-01	-2.02e+00
2	2.04e+00	2.04e+00	-1.00e+00	1.42e+00	-7.03e-01
3	-1.28e+00	1.28e+00	-7.85e-02	6.30e+00	-2.02e+00
4	-2.04e+00	2.04e+00	-2.21e-01	6.44e+00	-7.03e-01

Table 2: Time domain characteristic of modes

3.2 Transmission Zeros

System has no finite transmission zeros

3.3 Transfer Function Matrix

From input 1 to output...

TF(1,1) =

$$\frac{0.24824(s-1.415)(s+1.415)}{(s+2.037)(s+1.281)(s-1.281)(s-2.037)}$$

TF(1,2) =

$$\frac{-0.098283(s-2.249)(s+2.249)}{(s+2.037)(s+1.281)(s-1.281)(s-2.037)}$$

TF(2,1) =

$$\frac{-0.098283(s-2.249)(s+2.249)}{(s+2.037)(s+1.281)(s-1.281)(s-2.037)}$$

TF(2,2) =

$$\frac{0.50663(s-2.03)(s+2.03)}{(s+2.037)(s+1.281)(s-1.281)(s-2.037)}$$

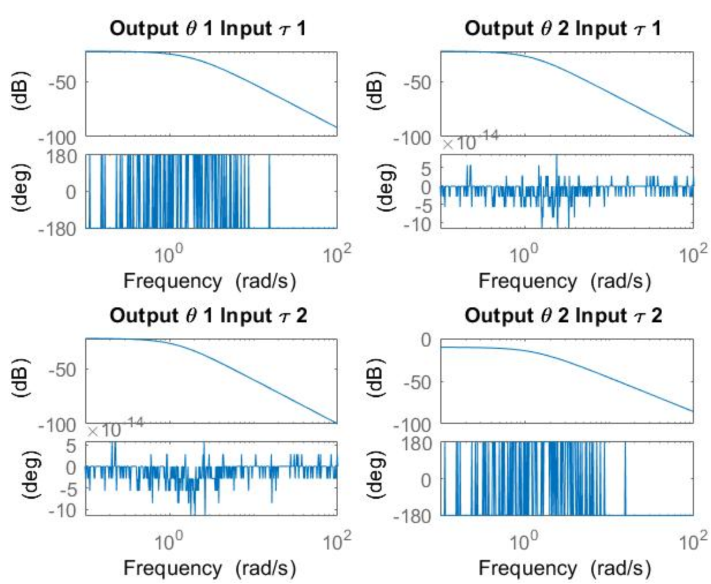

Figure 6: Bode plot for all four Transfer Function

3.4 Controllability

System is controllabl, rank of controllability matrix is 4

3.5 Observability

System is obervable, rank of observability matrix is 4

3.6 Plant frequency response at constant ω

3.6.1 Plant analysis at DC

To understand the system's input-output properties, we begin by examining the system at DC (under constant steady state conditions).
At DC manipulator transfer function will be as follows -
$DC = C * A^{-1} * B$

$$DC = \begin{pmatrix} -0.0730 & 0.0730 \\ 0.0730 & -0.3066 \end{pmatrix} \tag{12}$$

$DC = USV^H$

$$U = \begin{pmatrix} -0.2757 & 0.9612 \\ 0.9612 & 0.2757 \end{pmatrix} \tag{13}$$

$$S = \begin{pmatrix} 0.3276 & 0 \\ 0 & 0.0521 \end{pmatrix} \tag{14}$$

$$V = \begin{pmatrix} 0.2757 & -0.9612 \\ -0.9612 & -0.2757 \end{pmatrix} \tag{15}$$

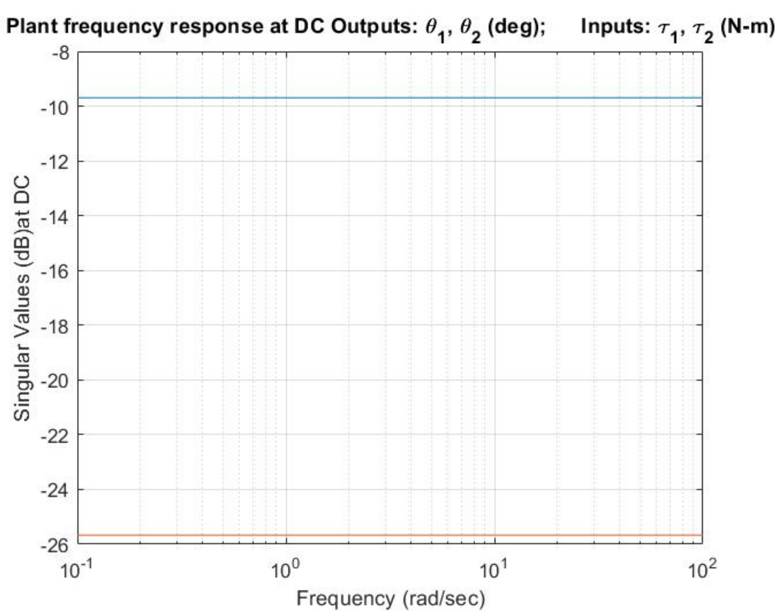

Figure 7: Plant frequency response at DC

System assessment from DC

1. Maximum Singular value

    - Maximum singular value of 0.3276 is associated with right singular (input) vector V(:,1) and left singular vector U(:,1).
    - Since the second component of V is much larger than its first component, it follows that maximum singular value is primarily associated with second control τ_2.
    - Since second component of U(:,1) is much larger than its first component, it follows that maximum singular value is primarily associated with the second output θ_2
    - The maximum singular value is therefore associated with τ_2 and θ_2 - the upper link.

2. Minimum Singular value

    - Minimum singular value of 0.0521 is associated with right singular (input) vector V(:,2) and left singular vector U(:,2).
    - Since the first component of V is much larger than its second component, it follows that maximum singular value is primarily associated with first control τ_1
    - Since first component of U(:,1) is much larger than its second component, it follows that maximum singular value is primarily associated with the first output θ_1.
    - The minimum singular value is therefore associated with τ_1 and θ_1 - the lower link.

3.6.2 Plant analysis at ω = 5 and 15 (rad/sec)

From SVD analysis at DC, 5 rad/sec and 15 rad/sec, we infer that for two dof robotic arm with increase in frequency ω, there is an increase in coupling effect, where τ_2 will increase its influence on θ_1 and τ_1 will increase its influence on θ_2

3.7 Plant frequency response

- In order to visualize the manipulator's multivariable input-output frequency response properties, we consider singular values of its transfer function plotted versus frequency.

- System has poor gain margin at low frequency.One channel having gain of -25dB and another channel having gain of -10 dB.

- Channel with -25dB margin has gain of 0.06, we need to have gain of 100, to make it good at low frequency, to atleast make it at range of 20dB. Hence we will add integrator to this channel.

- Channel with -10dB margin has gain of 0.32, we need to have gain of around 500, to make it good at low frequency, to make it to range of 50dB. Hence we will also add integrator to this channel.

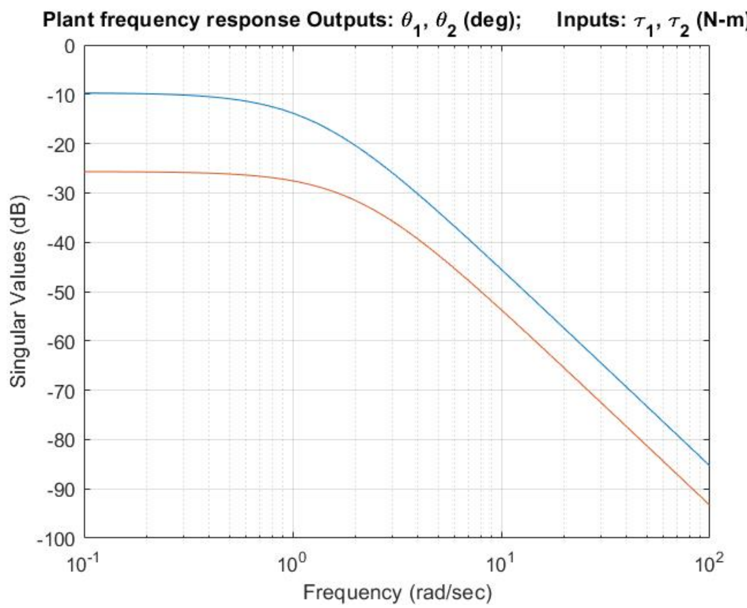

Figure 8: Plant frequency response

4 Design of Control System

Control system will be designed based on following three methodologies

1. Linear Quadratic Regulator (LQR)
2. LQG/LTR

4.1 Linear Quadratic Regulator (LQR)

4.1.1 Augmenting plant with integrators

$$A_{LQ} = \begin{pmatrix} 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 4.0421 & 0.6419 & 0 & 0 \\ 0 & 0 & 0.4017 & 1.7479 & 0 & 0 \end{pmatrix} \tag{16}$$

$$B_{LQ} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.2482 & -0.0983 \\ -0.0983 & 0.5066 \end{pmatrix} \tag{17}$$

Updated state vector $x = [z, y, x_r]^T$
z –> integrator states
$y = [\theta_1, \theta_2]$
$x_r = [\dot{\theta}_1, \dot{\theta}_2]$

4.1.2 LQR Design Parameters

$Q = M^T M$ –> State weighting matrix
R–> Control weighting matrix
The design parameters M and R is used to trade-off the speed at which x goes to zero (bandwidth) versus the control action used.

- To drive the state x to zero faster (increase bandwidth), one chooses M "larger" or R "smaller . (expensive control)

- To reduce the control action used, one chooses R "larger" or M "smaller" (cheap control)

I selected 4 different choices for Q matrix based on the plant dynamics. It is selected in such a way that (A,M) should be detectable.

1. Q1=eye(6,6) Penalizing all the states

2. Q2 =diag([1 1 1 1 0 0]) Penalizing the states we care about

3. Q3 =diag([1 1 1.5 2 0 0]) Penalizing θ_1 by 1.5 and θ_2 by 2

4. Q4 = diag([1 1 500 25 0 0]) Excessive penalizing θ_1 by 500 and θ_2 by 25

I selected 3 different choices for R matrix based on plant dynamics.

1. $R1$=0.0001*eye(2,2);

2. $R1$=0.000001*eye(2,2); (Expensive control but good properties at output)

3. $R1$=0.01*eye(2,2); (cheap control but bad properties at input)

Selection of Q and R is done as a trade off between properties like getting negligible overshoot, good command following, higher bandwidth, cheap control etc.
LQR design methodology will be performed with matrix Q as diag([1 1 1 1 0 0]) and R as

0.0001*eye(2,2)

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

$$R = \begin{pmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{pmatrix} \tag{19}$$

The solution to LQR problem is a linear full-state feedback control law.
$u = -Gx$
where G is control gain matrix given by
$G = R^{-1}B^TK$
K is unique symmetric positive (atleast) semidefinite solution of following Control Algebraic Riccati Equation (CARE)
$0 = KA + A^TK + M^TM - KBR^{-1}B^TK$
Following are solutions to LQR problem

$$G = \begin{pmatrix} 99.9999 & 0.1717 & 150.4323 & 8.7327 & 36.1025 & 4.9309 \\ -0.1717 & 99.9999 & 8.2977 & 125.1246 & 4.8458 & 22.9240 \end{pmatrix} \tag{20}$$

$$K = \begin{pmatrix} 1.3245 & 0.0424 & 0.3609 & 0.0489 & 0.0436 & 0.0084 \\ 0.0424 & 1.2085 & 0.0491 & 0.2293 & 0.0085 & 0.0214 \\ 0.3609 & 0.0491 & 0.4365 & 0.0661 & 0.0663 & 0.0145 \\ 0.0489 & 0.2293 & 0.0661 & 0.2578 & 0.0144 & 0.0275 \\ 0.0436 & 0.0085 & 0.0663 & 0.0144 & 0.0162 & 0.0041 \\ 0.0084 & 0.0214 & 0.0145 & 0.0275 & 0.0041 & 0.0053 \end{pmatrix} \tag{21}$$

Index	Pole	Damping ζ	Frequency (rad/sec)	Time Constant (sec)
1	-5.25e+00 + 5.14e+00i	7.15e-01	7.35e+00	1.90e-01
2	-5.25e+00 - 5.14e+00i	7.15e-01	7.35e+00	1.90e-01
3	-3.56e+00 + 3.01e+00i	7.64e-01	4.66e+00	2.81e-01
4	-3.56e+00 - 3.01e+00i	7.64e-01	4.66e+00	2.81e-01
5	-9.89e-01	1.00e+00	9.89e-01	1.01e+00
6	-1.00e+00	1.00e+00	1.00e+00	1.00e+00

Table 3: Time domain characteristic of closed loop poles

4.1.3 Loop Properties

1. **Open Loop Frequency Response at Input**

   - Transfer function matrix
     $T_{diup} = [I + KP]^{-1}$

   - Singular value plot suggest that input disturbances d_i with frequency below 0.1rad/sec will be attenuated by about 35dB.(Figure 9)

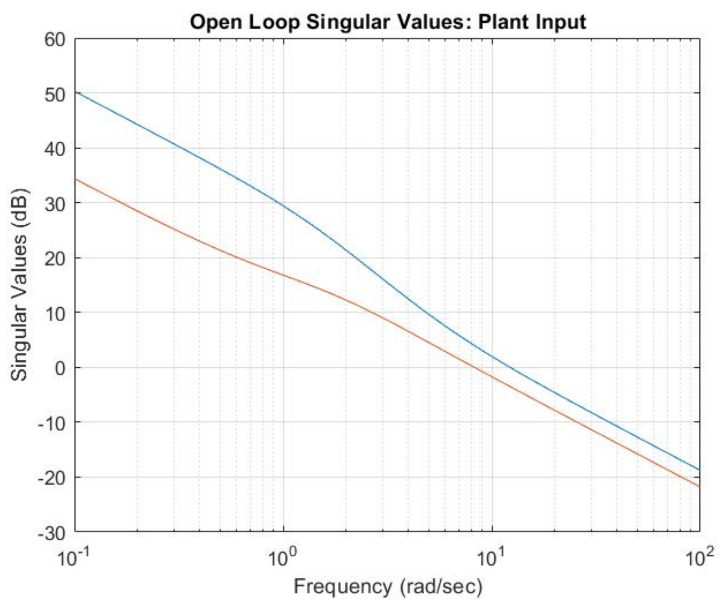

Figure 9: Open Loop Singular values at Input

2. **Sensitivity Frequency Response at Input**

   - Transfer function matrix
     $S_u = [I + KP]^{-1}$

   - Transfer function matrix from d_i to u_p

   - Singular value plot suggest that input disturbances d_i with frequency below 0.1rad/sec will be attenuated by about 35dB.(Figure 10)

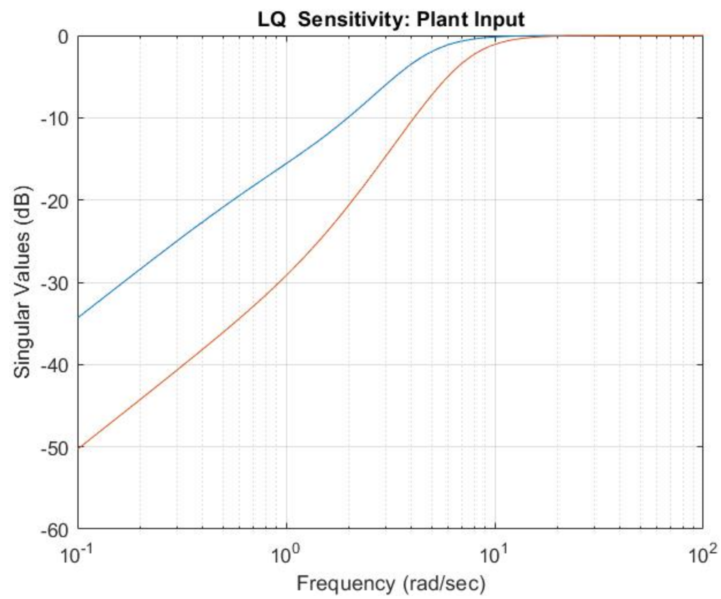

Figure 10: LQ sensitivity at Input

3. **Sensitivity Frequency Response at Error**

- Transfer function matrix
  $S_e = [I + PK]^{-1} = [I + L_e]^{-1}$
- Singular value plot suggest that low frequency reference comman will be followed and low frequency output disturbances will be attenuated. More precisely, reference commands r with frequency content below 0.3 rad/sec should be followed to be within about 20dB; that is steady state error of about 12 percent (Figure 11)

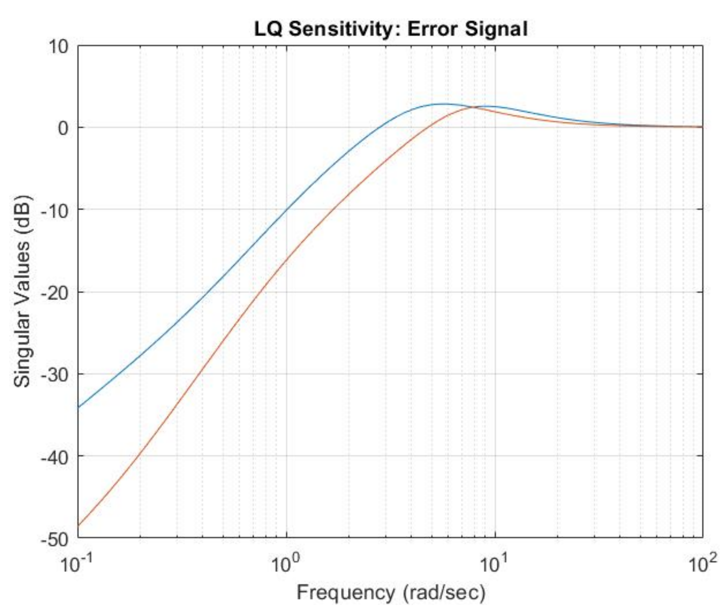

Figure 11: Sensitivity at Input

4. **Complementary sensitivity at plant output**

   - Transfer function matrix
     $T_e = PK[I + PK]^{-1}$
   - Singular value are important for assessing the impact of a unfiltered reference command r on the output y.
   - The bump in the picture suggests that a prefilter is essential to prevent excessive overshoot to step reference commands. (Figure 12)

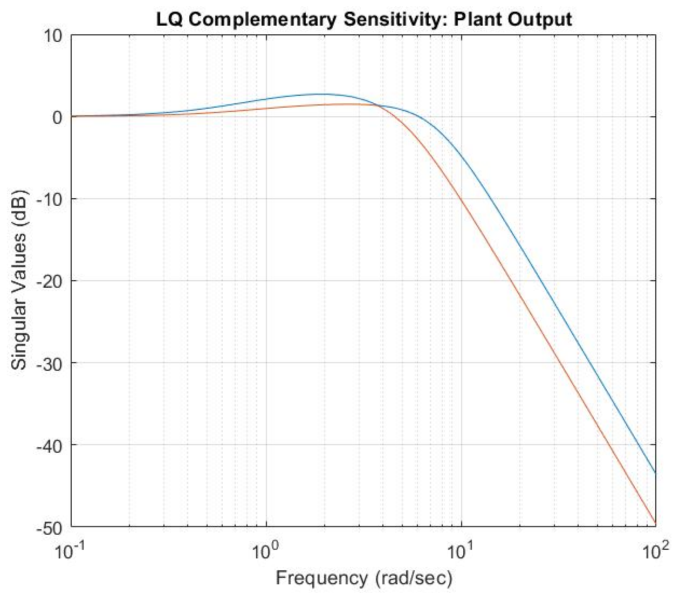

Figure 12: Complementary sensitivity at Plant Output

4.1.4 Response to step Reference Command

1. **Response to θ_1 Step Reference Command**

   - θ_1 follows the step command with 30 % overshoot and settling time is around 3 seconds.(Figure 13)
   - Using pre filter W=

$$\frac{1.5}{s+.5}$$

will reduce overshoot and settling time.

corresponding control plots is plotted in Figure 14 and they are of acceptable size.

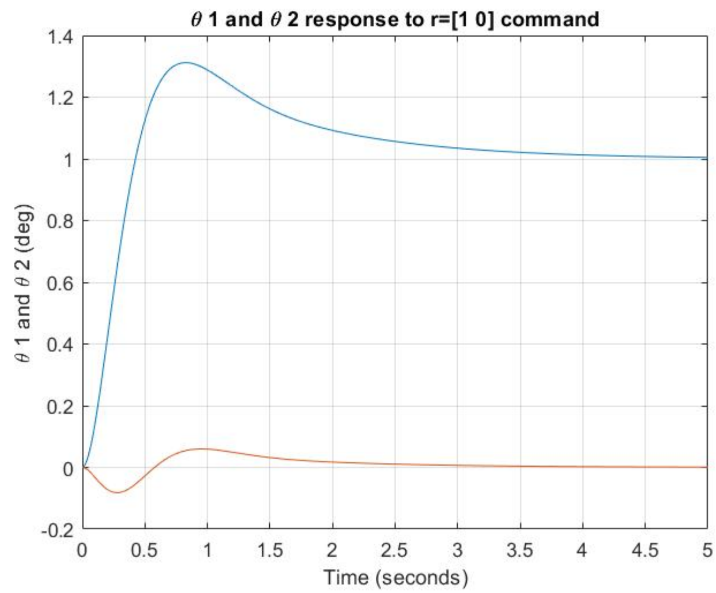

Figure 13: θ_1 and θ_2 response to step input r=[1 0] command

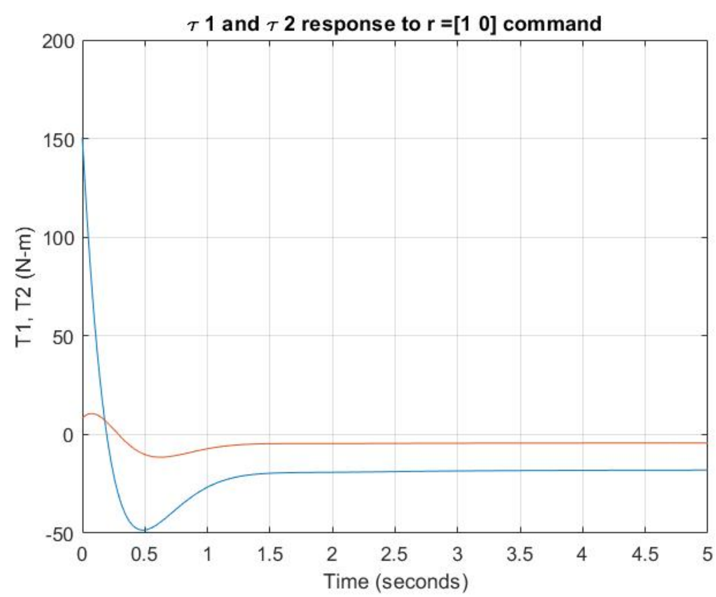

Figure 14: τ_1 and τ_2 response to step input r=[1 0] command

2. **Response to θ_2 Step Reference Command**

   - θ_1 follows the step command with 20 % overshoot and settling time is around 2.5 seconds.(Figure 15)

   - Using pre filter W=

$$\frac{1.5}{s+.5}$$

will reduce overshoot and settling time.

- The corresponding control plots is plotted in Figure 16 and they are of acceptable size.

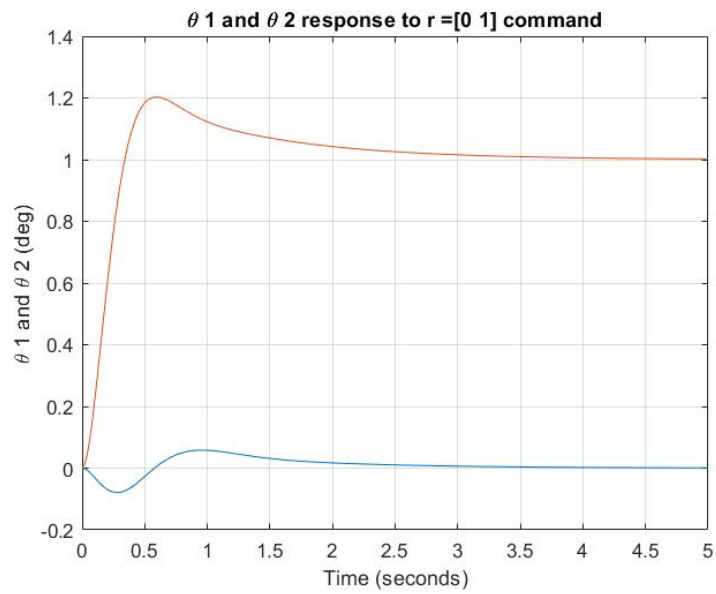

Figure 15: θ_1 and θ_2 response to step input r=[0 1] command

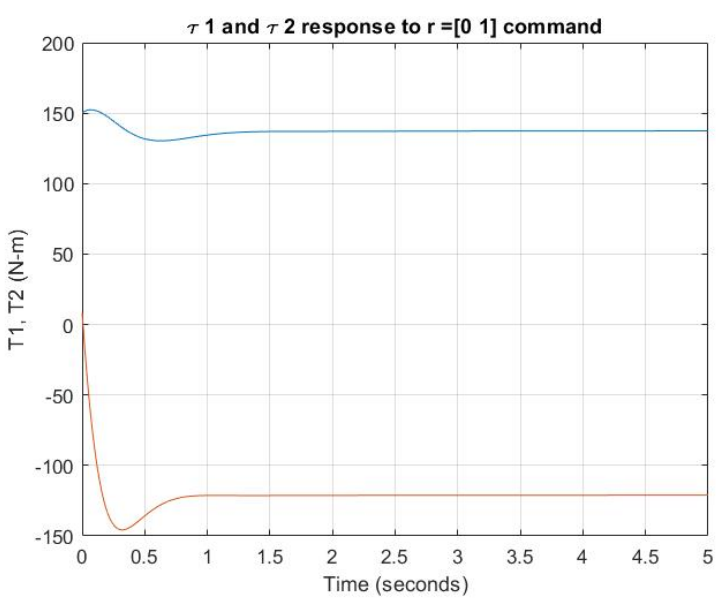

Figure 16: τ_1 and τ_2 response to step input r=[0 1] command

4.2 LQG/LTR

4.2.1 Augmenting plant with integrators

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0.2482 & -0.0983 & 4.0421 & 0.6419 & 0 & 0 \\ -0.0983 & 0.5066 & 0.4017 & 1.7479 & 0 & 0 \end{pmatrix} \tag{22}$$

$$b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{23}$$

updated state vector
$x = [x_i, x_p]^T$
x_i –> integrator states
$x_p = [\theta_1,\ \theta_2, \dot{\theta}_1, \dot{\theta}_2]$

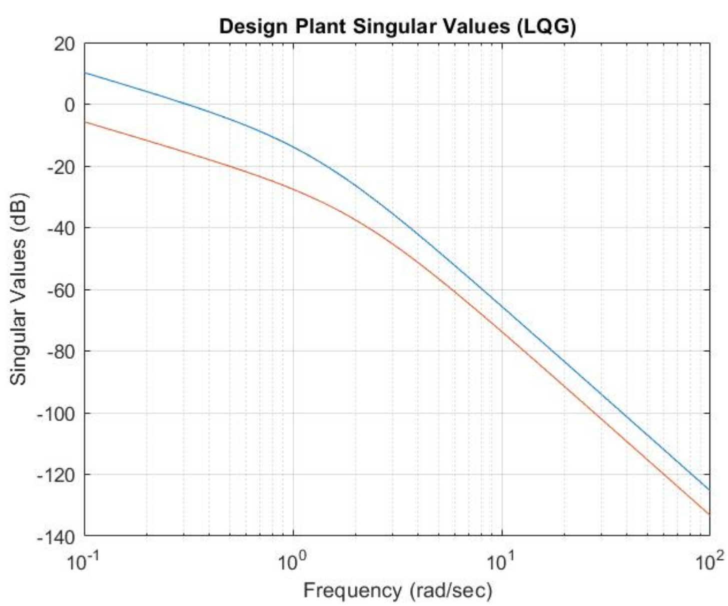

Figure 17: Design Plant Singular values

The Singular value plot exhibit a slope of -20dB at low frequency as expected. The minimum singular value is below 0dB and maximum singular value crosses 0dB at 0.2 rad/sec. Gains are low in both channel.

4.2.2 Design Target loop Transfer Function Matrix using Kalman Filtering

Consider augmented system shown in figure 18.

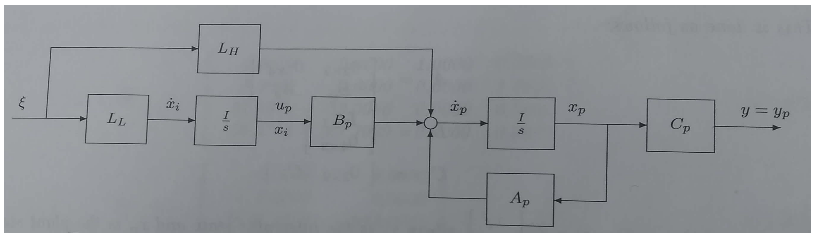

Figure 18: Design Plant Singular values

- It will be used to design a target loop transfer function matrix $L_o = G_{KF}$ with desirable closed loop properties at the output.
- Augment system $G_{FOL} = C(sI - A)^{-1}L$ with

$$L = [L_L L_H]$$
$$L_L = [C(-A)^{-1}B]^{-1}$$

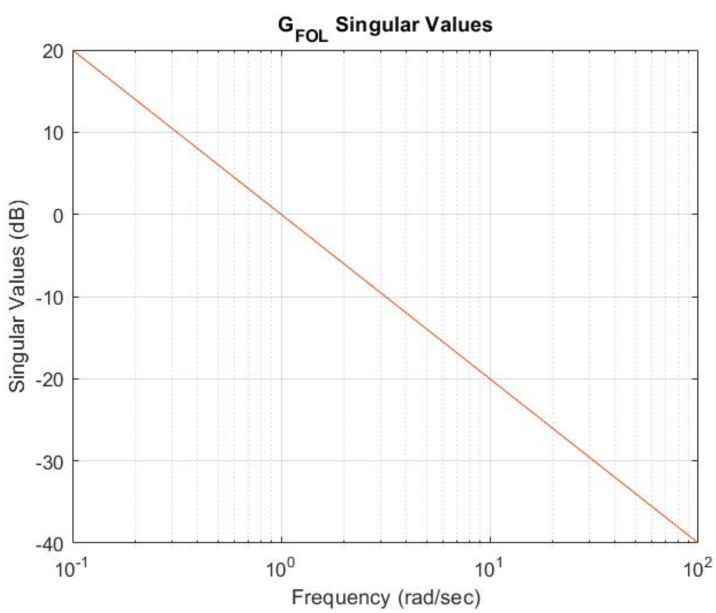

Figure 19: G_{FOL} Singular values

From Figure 19 we can conclude that matrix L_H matches the singular values at high frequencies. Together L_L and L_H match the singular values of G_{FOL} at all frequencies.

- Selection of μ (Target Loop Bandwidth parameter)
- Decreasing μ will raise the bandwidth
- Increasing μ will decrease the bandwidth

There are 3 Different choices of μ

- $\mu = 0.1$
- $\mu = 10$
- $\mu = 0.01$

Solve the FARE with $\Theta = \mu I$
$AX + XA^t + LL^T - XC^T\Theta^{-1}CX = 0$
$H_f = XC^T\Theta^{-1}$

Target closed loop poles and zeros ($\lambda(A - H_fC)$)

$$t_{poles} = \begin{pmatrix} -2.0370 \\ -2.0370 \\ -1.2808 \\ -1.2808 \\ -3.1623 \\ -3.1623 \end{pmatrix} \tag{24}$$

$$t_{zeros} = \begin{pmatrix} -2.0370 \\ -0.888 \\ -0.7088 \\ -1.2808 \end{pmatrix} \tag{25}$$

4.2.3 Loop Properties

1. **Target Loop G_{KF} Singular values**
   - Transfer function matrix
     $L_o = G_{KF} = C(sI - A)^{-1}H_F$

- Singular values are matched at low frequencies with a slope of -20dB /dec .(Figure 19)

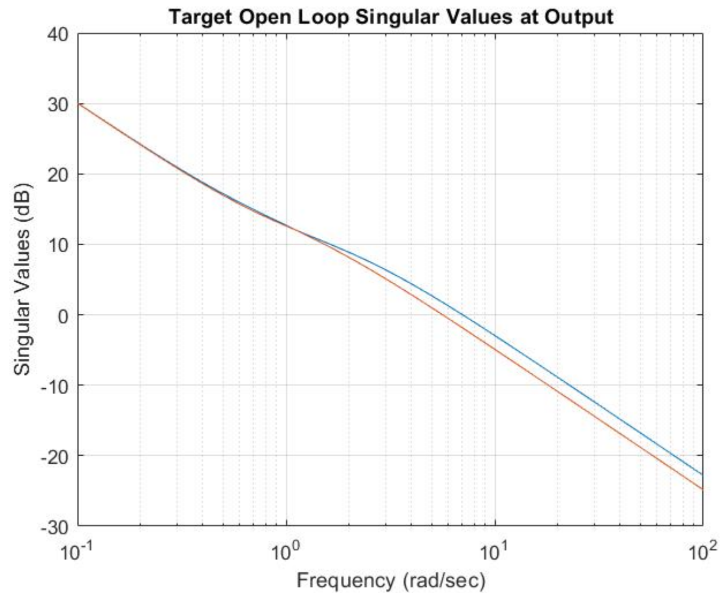

Figure 20: Target Loop G_{KF} Singular values

2. **Target Sensitivity Singular Values at Output**

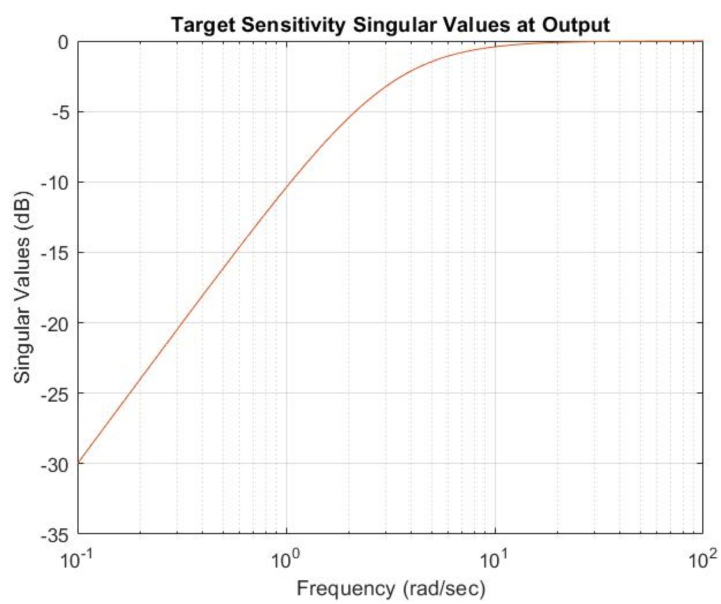

Figure 21: Target Sensitivity Singular Values at Output

3. **Target Complementary Sensitivity Singular Values at Output**

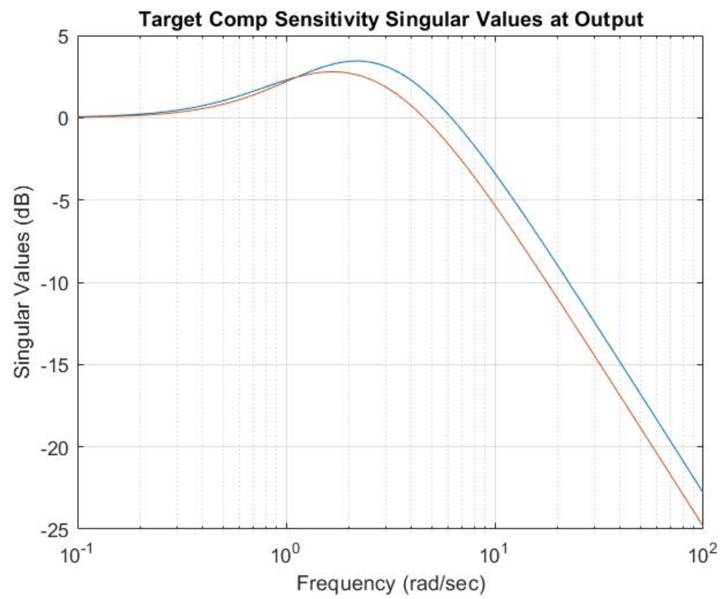

Figure 22: Target Complementary Sensitivity Singular Values at Output

4. **Target Closed Loop Singular Values (r to y)**

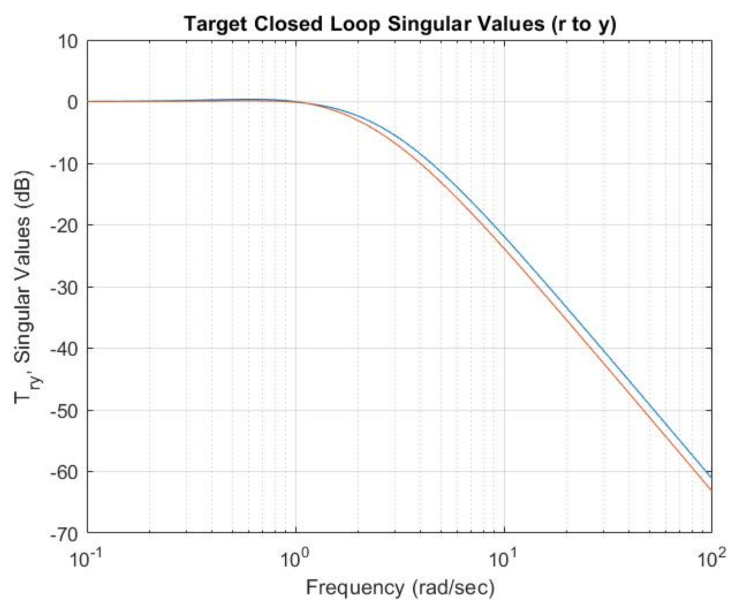

Figure 23: Target Closed Loop Singular Values (r to y)

4.2.4 Response to step Reference Command

1. **Response to θ_1 Step Reference Command**

    - θ_1 follows the step command with less than 10 % overshoot and settling time is around 2 seconds.(Figure 24)

- The corresponding control plots is plotted in Figure 25 and they are of acceptable size.

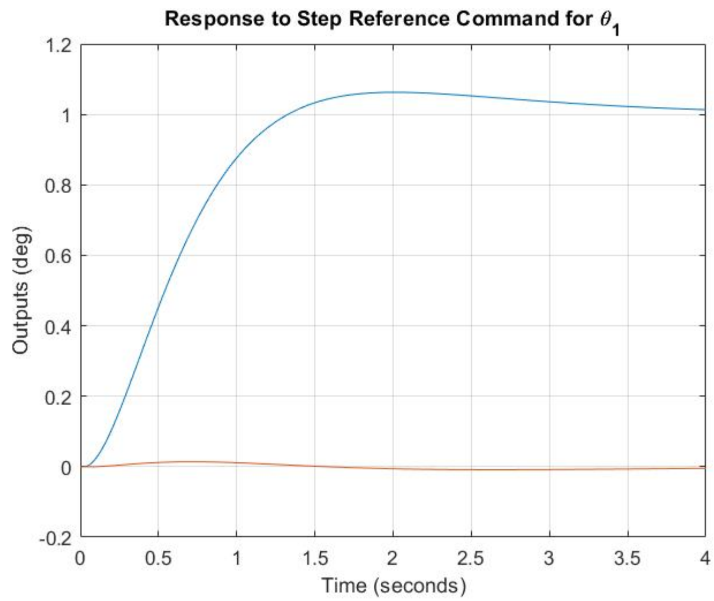

Figure 24: θ_1 and θ_2 response to step input r=[1 0] command

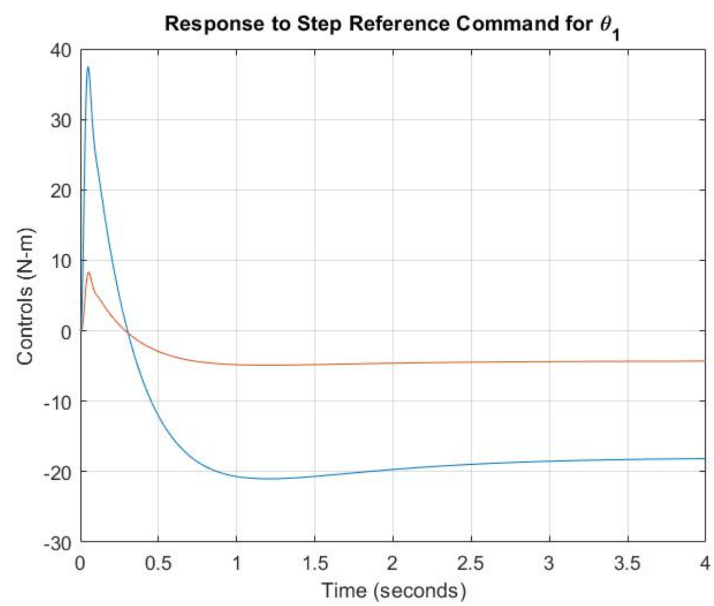

Figure 25: τ_1 and τ_1 response to step input r=[1 0] command

2. **Response to θ_2 Step Reference Command**

   - θ_2 follows the step command with less than 10 % overshoot and settling time is around 2.5 seconds.(Figure 26)

- The corresponding control plots is plotted in Figure 27 and they are of acceptable size.

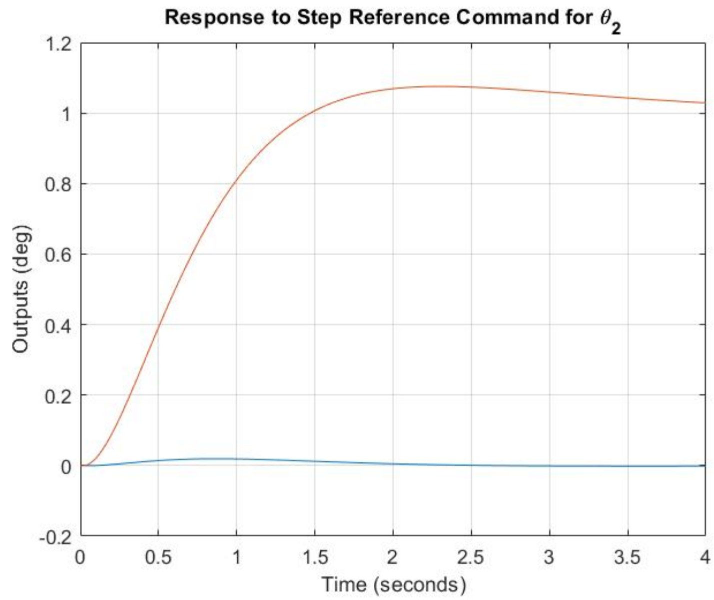

Figure 26: θ_1 and θ_2 response to step input r=[0 1] command

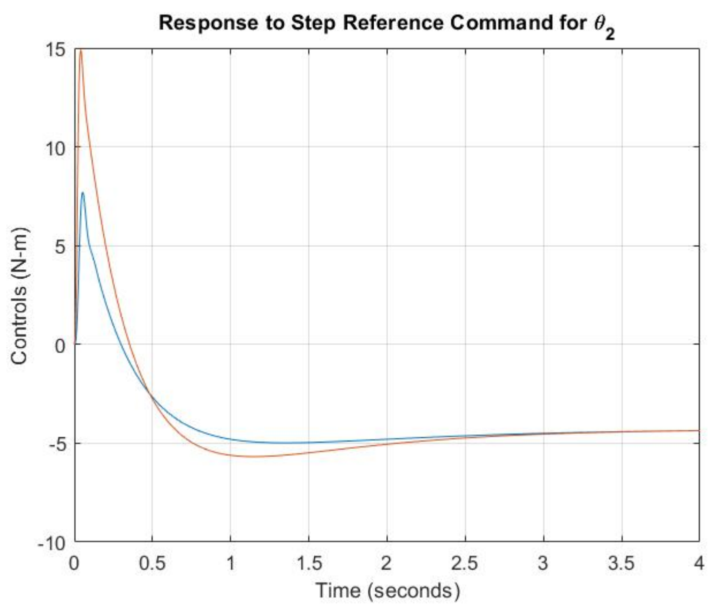

Figure 27: τ_1 and τ_1 response to step input r=[0 1] command

References

[1] Bruno Siciliano and Lorenzo Sciavicco. Robotics Modelling, Planning and Control

[2] Armando A. Rodriguez Analysis and Design of Multivariable Feedback Control system