

Institute of
Data



2021



Data Science and AI

Module 1

Part 1:

Mathematics & Statistics



Agenda: Module 1 Part 1

- Linear algebra
- Calculus
- Multivariable calculus
- Statistics
- Probability



What is Linear Algebra and *why is it important for Data Science?*

- Linear Algebra is the branch of mathematics that deals with ***linear equations*** and their representations
- Linear Algebra is used extensively in science and engineering to '***model***' many systems such as in economics, health and finance
- Although many systems are '***non-linear***', Linear Models can be effective ***first-order approximation***. This is crucial, because non-linear models are ***very difficult*** to represent and manipulate.
- One excellent way to better understand Linear Algebra is use the ***geometrical representations*** of its constructs
- Another way to better understand Linear Algebra is through ***programming***. This is crucial for a Data Scientist.
- Key constructs of Linear Algebra are: Scalar, Vector, Matrix and Tensor



Linear Algebra

- Vectors
 - definitions
 - vector arithmetic (adding, subtracting and multiplying vectors), dot products and cross products
- Matrices
 - definitions
 - matrix arithmetic
 - inverses, determinants, transposes
- Solving systems of linear equations
- Eigenvalues and eigenvectors



Linear Algebra

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Mapping and usage of Linear Algebra in Data Science

Concept	Definition	Mapping to Data Science	Examples
Scalar	A 'zero-dimensional' dataset. A number, value, magnitude. Geometrically, it's a point on on a line.	A single data point	Age of a customer
Vector	A one-dimension dataset. A two or more values. Geometrically it represent a vector in a plane that has magnitude and direction.	A number of data points (usually about a single entity)	Attributes (or features) of one customer: Age, income, marital status, postcode, ..., etc. In Deep Learning a vector could be the input to a Neural Network.
Matrix	A two-dimensional dataset. Geometrically, it represents a transformation of two or more vectors.	A set of observations for multiple entities. A transformation of a dataset from one representation to another.	Information about all customers. In Deep Learning a matrix may represents the mapping and weights on hidden layer.
Tensor	An n-dimensional dataset.	A number of sets of observations	Information about all customers. TensorFlow is built around tensors.



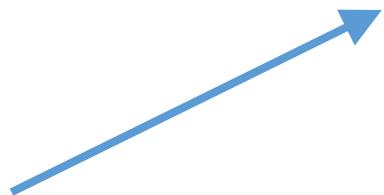
Vectors

def: ?

a directed quantity

examples: ?

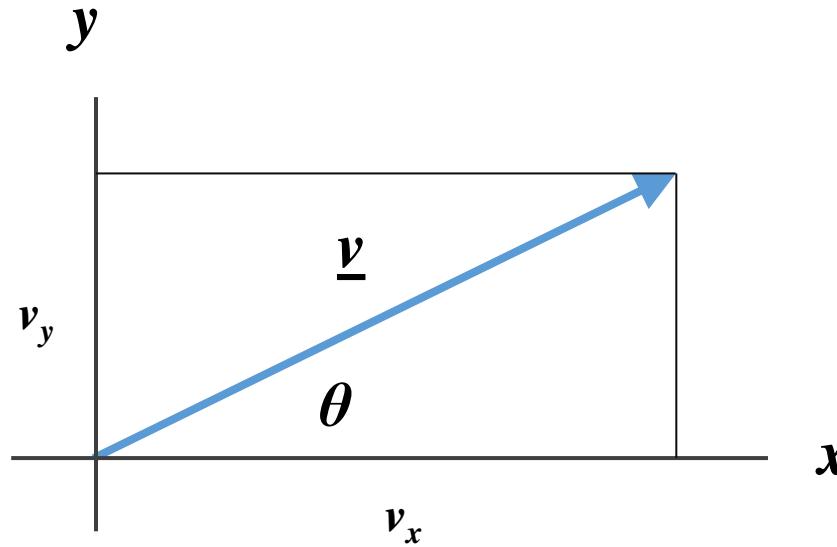
- displacement (*not* length)
- velocity (*not* speed)
- acceleration
- force
- weight (*not* mass)



dimensionality > 1



Vector Decomposition: 2D



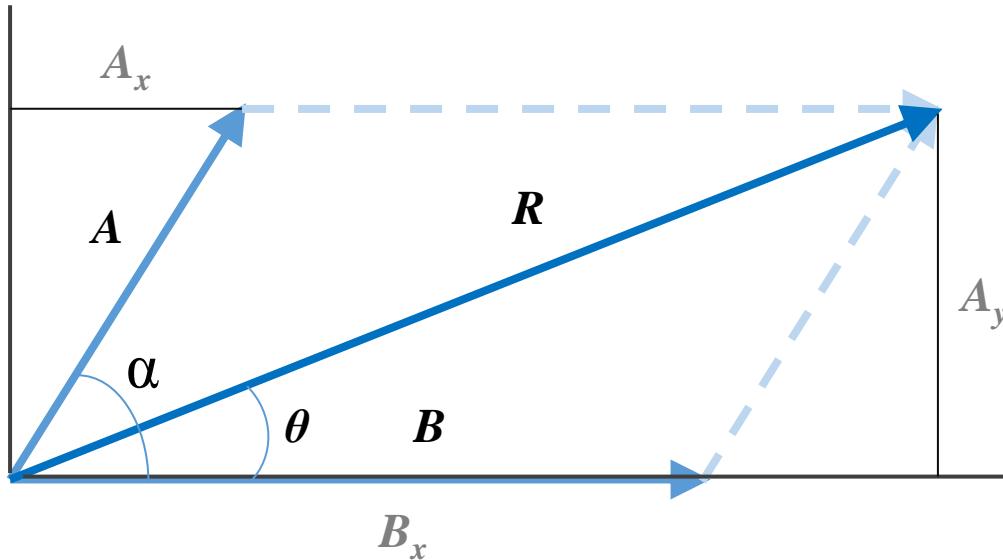
$$v_x = v \cos \theta$$

$$v_y = v \sin \theta$$

$$|\underline{v}| = (v_x^2 + v_y^2)^{1/2}$$



Vector Addition



in general:

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

in this example:

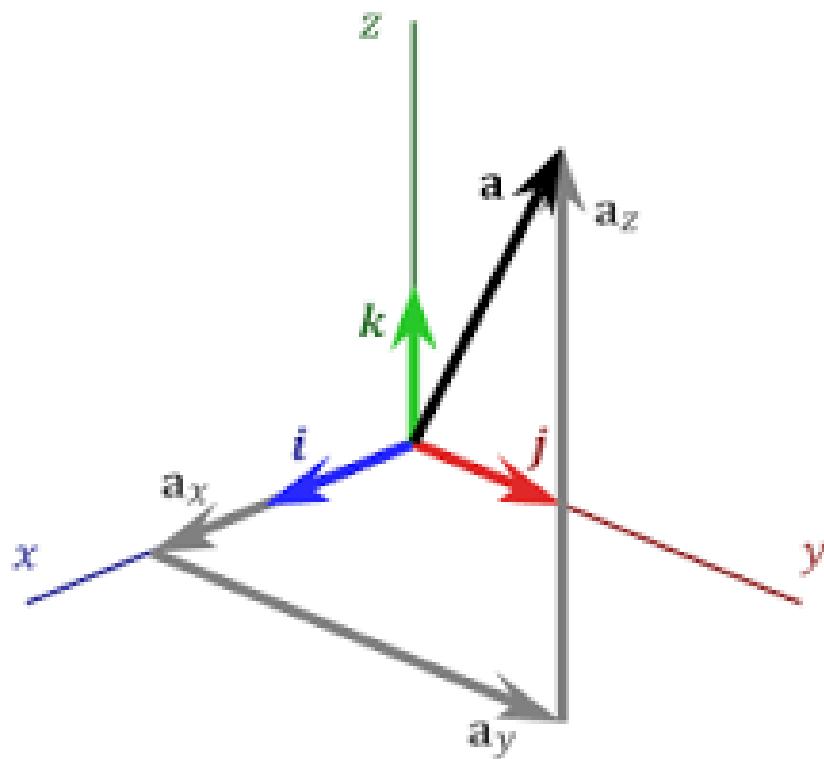
$$R_x = A \cos \alpha + B$$

$$R_y = A \sin \alpha$$

$$\theta = \tan^{-1} (R_y / R_x)$$



3D Vectors



$$|\underline{a}| = (\underline{a}_x^2 + \underline{a}_y^2 + \underline{a}_z^2)^{1/2}$$

Note:

i, j, k are unit vectors



Scalar Multiplication of Vectors

aka inner product, dot product
result is a *scalar*

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= (a_1, a_2, \dots, a_n) \bullet (b_1, b_2, \dots, b_n) \\ &= a_1b_1 + a_2b_2 + \dots + a_nb_n \\ &= |\mathbf{a}| |\mathbf{b}| \cos(\theta)\end{aligned}$$



Vector Multiplication of Vectors

aka cross product

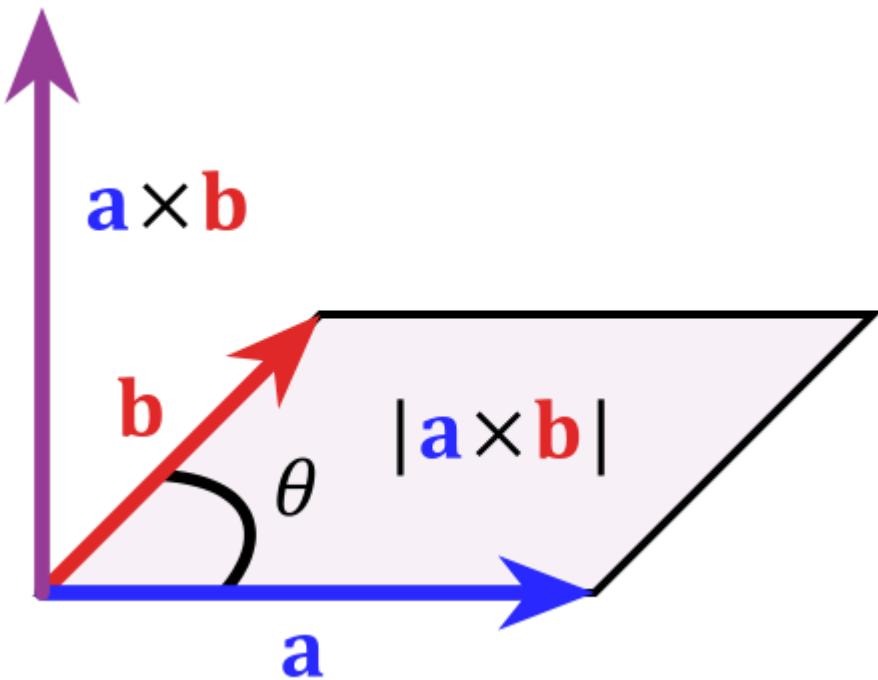
result is a *vector*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= (u_y v_z - u_z v_y) \mathbf{i} + (u_z v_x - u_x v_z) \mathbf{j} + (u_x v_y - u_y v_x) \mathbf{k}$$



Cross Product – cont'd



magnitude of $\mathbf{a} \times \mathbf{b}$
= area of parallelogram



Vector Operations

Entry-wise multiplication:

aka Hadamard product

result is a *vector*

$$\mathbf{a} * \mathbf{b} = (a_1b_1, a_2b_2, \dots a_nb_n)$$

Transpose

$$(a_1, a_2, \dots a_n)^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$



Matrices

def: ?

a rectangular array of numbers

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$



Matrix Arithmetic

addition

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \end{aligned}$$



Matrix Arithmetic

multiplication

$$\begin{aligned} A B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \end{aligned}$$

in general:

A is $m \times n$ B is $n \times p$

$A B$ is $m \times p$



Matrix Operations

Transpose

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

the rows of A^T are the columns of A



Determinant of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$



Identity Matrix

Define the $n \times n$ identity I_n :

$$A I_n = I_n A = A$$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$



Matrix Inversion

Define the inverse A^{-1} of an invertible matrix A :

$$A A^{-1} = I_n = A^{-1}A$$

only exists if...

A is $n \times n$

$\det(A) \neq 0$

Methods:

- Gaussian (Gauss-Jordan) elimination
- LU decomposition (orthogonalisation)
- Eigen decomposition



Solving Systems of Linear Equations

Simultaneous linear equations in n unknowns (x_i):

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

in matrix form:

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$



Solving Systems of Linear Equations – cont'd

Problem:

$$A \mathbf{x} = \mathbf{b}$$

Solution:

$$\mathbf{x} = A^{-1} \mathbf{b}$$

So, we only need to invert the matrix of coefficients A and multiply with vector \mathbf{b}



Eigenvalues and Eigenvectors

def: A nonzero vector that scales linearly in response to a linear transformation

$$T(\boldsymbol{v}) = \lambda \boldsymbol{v}$$

T is a linear transformation

λ is a scalar = ‘eigenvalue’ (*aka* characteristic value, characteristic root)

\boldsymbol{v} is a vector = ‘eigenvector’ (*aka* characteristic vector)

eigenbasis: a set of eigenvectors of T that forms a basis of the domain of T



What is bigger than a matrix?

tensor

- represented by an n -dimensional array
- examples
 - stress tensor (mechanics)
 - spacetime tensor (general relativity)



Lab 1.1.1: Vector and Matrix Operations

Purpose:

- To apply the definitions of vector and matrix operations by designing code that implements them.

Materials:

- See Notebook:
 - ‘Lab 1.1.1 – Vector - Matrix Operations’





Calculus

- Limits and continuity
- Taking derivatives
- Integration
- Sequences and series



What is Calculus and why is it important for Data Scientists?

- Calculus is the mathematical study of **continuous change**. It is used extensively in many science and engineering domains such as business, economics and medicine.
- All key concepts in calculus can be mapped directly to **geometrical concepts**. For example, differentiation is the slope of a curve and integration is the area under a curve.
- Calculus is usually used with linear algebra to find the "**best fit**" linear approximation for a set of points in a domain. Therefore it is essential for Data Science as it underpins all model optimisation.

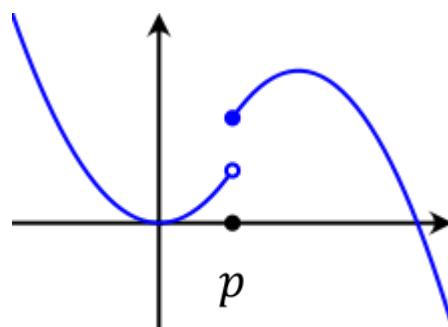


Limits and Continuity

If a function $f(x)$ approaches a value L ('limit') as x approaches p , then

$$\lim_{x \rightarrow p} f(x) = L$$

Note: The limits of a discontinuous function are directional



$$\lim_{x \rightarrow p^-} f(x) \neq \lim_{x \rightarrow p^+} f(x)$$



Limit Theorems

$$\lim_{n \rightarrow \infty} k a_n = k \lim_{n \rightarrow \infty} a_n$$

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

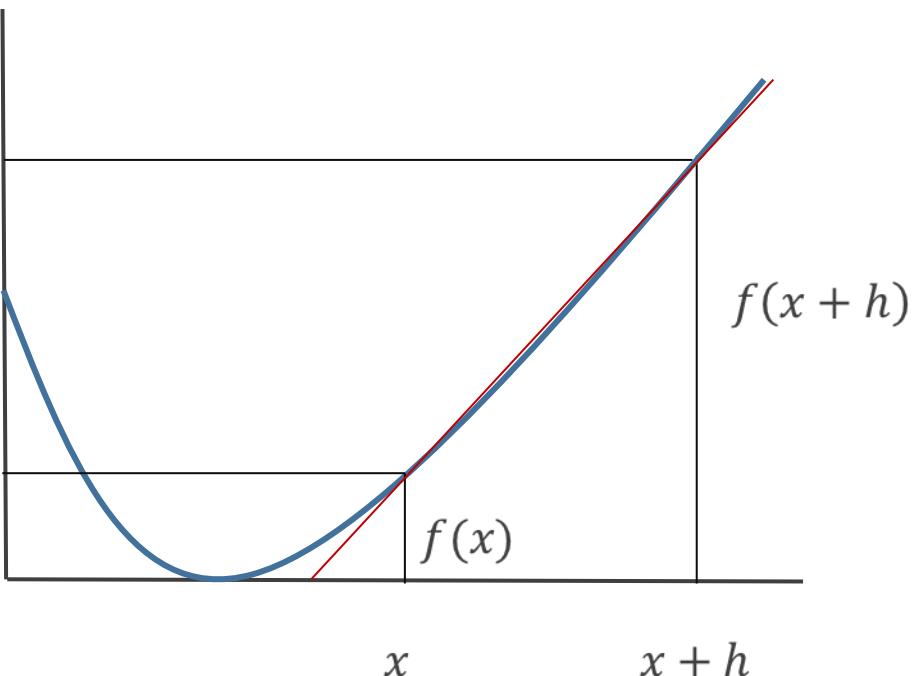
$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$



Differentiation

Rate of change of a continuous function $f(x)$:



Derivative of $f(x)$:

$$\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$



Rules of Differentiation

Let f, g, h be functions of x , and let a, b be constants ...

Linearity:

$$\frac{d(af+bg)}{dx} = a \frac{df}{dx} + b \frac{dg}{dx}$$

Product rule:

$$\frac{d(fg)}{dx} = g \frac{df}{dx} + f \frac{dg}{dx}$$

Chain rule:

if

$$h(x) = f(g(x))$$

then

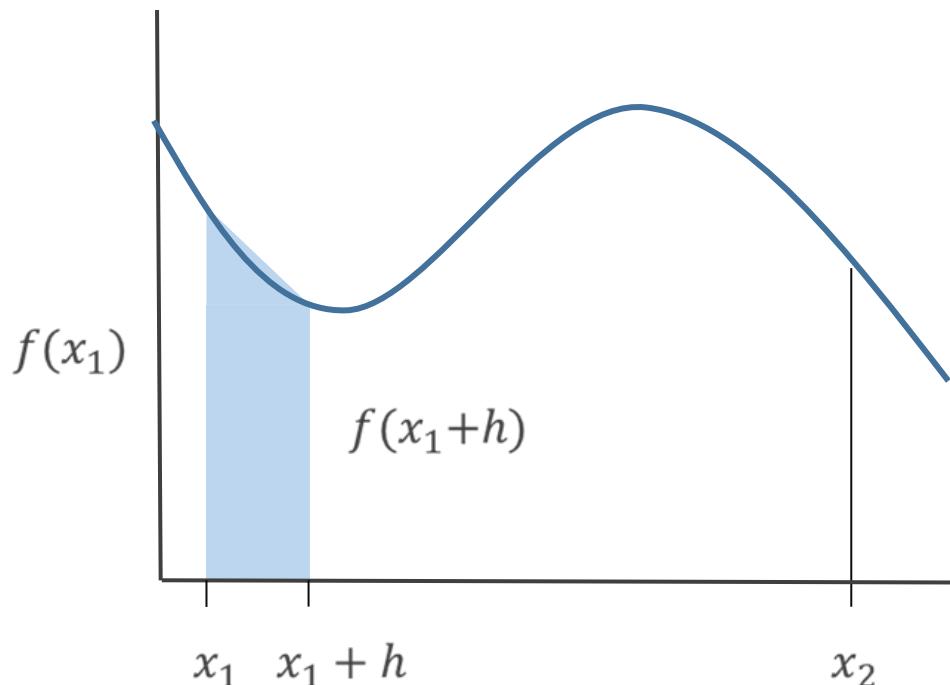
$$\frac{dh}{dx} = \frac{dh}{dg} \frac{dg}{dx}$$



Integration

Area under a continuous function $f(x)$:

$$F(x) = \lim_{h \rightarrow 0} \sum [f(x_i) + f(x_i + h)] h/2$$



If $f(x) = dF/dx$

then the integral of $f(x)$ between x_1 and x_2 is:

$$\int_{x_1}^{x_2} f(x) dx = F(x_2) - F(x_1)$$



Sequences and Series

Infinite sequence

$$a_n = f(n)$$

typically: $0 \leq n \leq \infty$

$-\infty \leq n \leq \infty$

examples:

$$a_n = n$$

$$a_n = 1/n$$

$$a_n = 1/n^2$$

$$a_n = (-1)^n$$



Multivariate Calculus

- Partial derivatives
- Multivariate differentiation
- Multivariate integration
- Optimising multivariate functions



Partial Derivatives

If f is a function of several variables, we can calculate the partial derivative with respect to any single variable by treating the others as constants:

example:

$$f(x, y) = 3x^2 + 2xy$$

$$\frac{\partial f}{\partial x} = 6x + 2y$$

$$\frac{\partial f}{\partial y} = 2x$$



Partial Derivatives – cont'd

For a function operating on a 3-dimensional Euclidean space, the partial derivatives define the ***gradient*** of the function:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

The *del* operator is often written as:

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$



Multivariate Optimisation

Given a function $f: A \rightarrow \mathbb{R}$

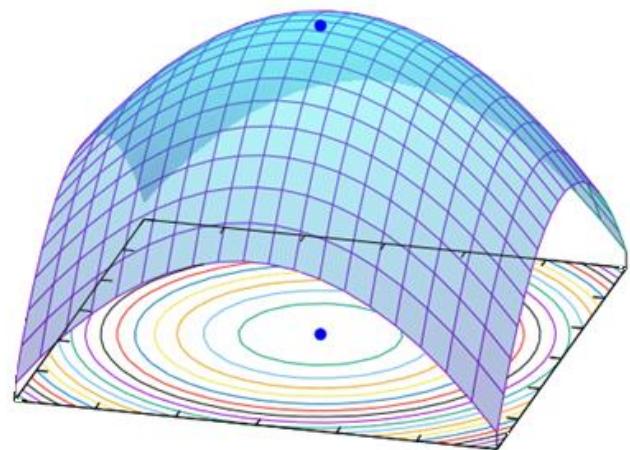
minimisation:

find $x_0 \in A$ such that $f(x_0) \leq f(x) \forall x \in A$

maximisation:

find $x_0 \in A$ such that $f(x_0) \geq f(x) \forall x \in A$

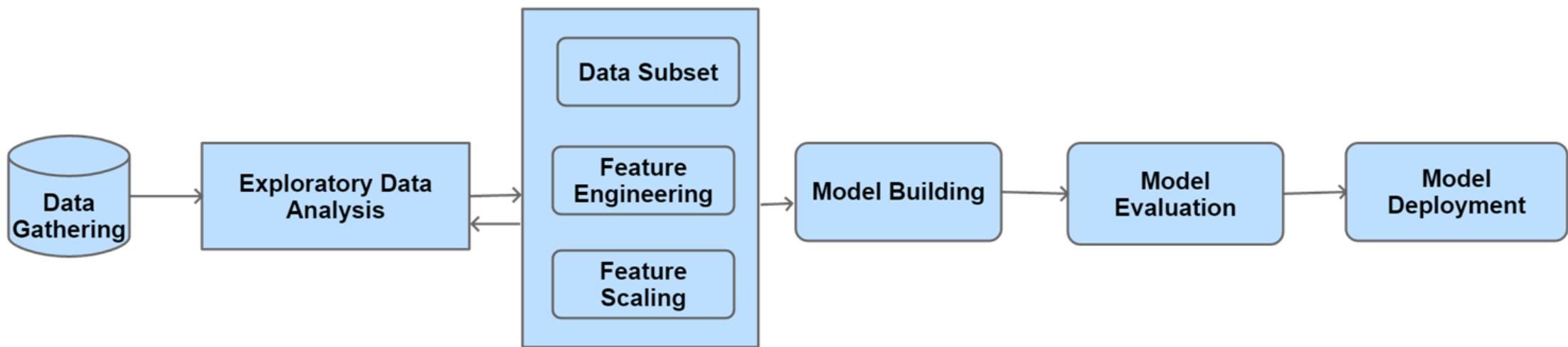
f = [objective | loss | cost | utility | fitness | energy]
function





Discussion

- Why do data scientists need to be proficient at calculus, infinite series, and linear algebra?
- Considering the illustrative Data Science process, where would you use calculus?





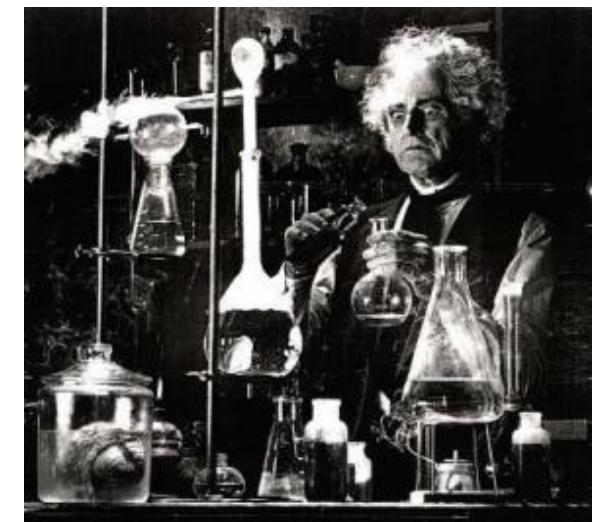
Homework: Optimisation

Purpose:

- To introduce several methods of multivariate optimisation via working examples.

Materials:

- An Interactive Tutorial on Numerical Optimization
 - <https://www.benfrederickson.com/numerical-optimization/>
- Prepare to discuss:
 - trade-off: convergence speed vs accuracy
 - faster convergence requires lower resolution
 - prefer methods with adaptive resolution





Statistics

- Statistical Thinking
- Categorical data
- Continuous variables
- Summarising quantitative data
- Modelling data distributions
- Confidence intervals
- Significance tests and hypothesis testing
- Statistical Inference



Why statistics is important for a Data Scientist?

- **Statistical Thinking** is an essential component of a data-driven mindset which is crucial for a Data Scientist
 - Statistical analysis must start with the appropriate **data** (sample)
 - Statistical Inference (reasoning) should start with measurement, ideally, via **controlled experiments**
 - Statistics uses samples (a small subset of the population) and therefore always has a degree of **uncertainty**
 - Sampling must be **random, and preferably, independent**
- The best way to learn statistics is by **experimenting with data using Python code and visualisation**



Statistics – Part 1

- Analysing categorical data



Categorical Variables

Examples

- FALSE / TRUE (alt: 0 / 1)
- colour
- size
- class
 - e.g. species, occupation, degree program, disease category
- tier
 - e.g. age range, income range, frequency range



Analysing Categorical Variables

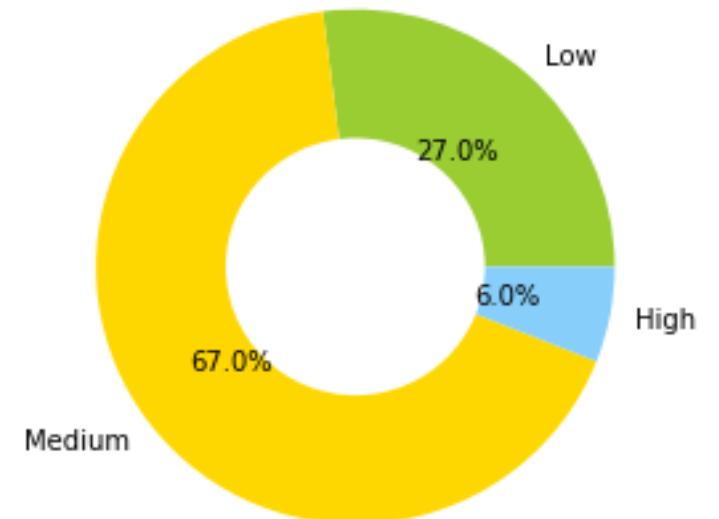
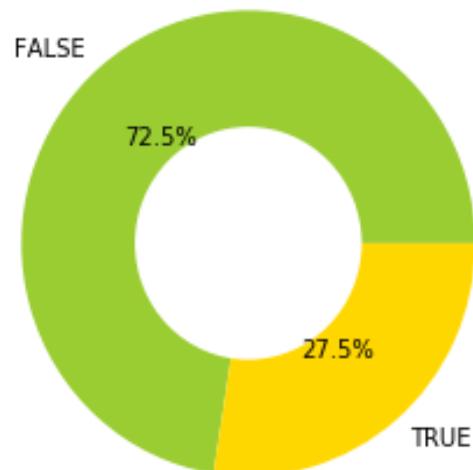
Frequency Tables (aka contingency tables)

category incidence for a single variable within a population

Passed Exam	
FALSE	37
TRUE	14

Income Bracket	
Low	0.27
Medium	0.67
High	0.06

Donut Charts





Analysing Categorical Variables – cont'd

Two-Way Frequency Tables

- *for a Single Variable within Two Populations*

Income Bracket:	Low	Medium	High	Total
Male	27	75	6	108
Female	32	59	3	94
TOTAL	59	134	9	202

- totals row, column: marginal frequencies (*aka* marginal distribution)



Analysing Categorical Variables – cont'd

Dummy Variables (*aka* dummy coding)

- *allows categorical variables to be treated like continuous variables*

Passed Exam	
0	37
1	14

Treatment	T1	T2
Control	0	0
Drug 1	1	0
Drug 2	0	1



Statistics – Part 2

- Continuous variables
- Summarising quantitative data



Continuous Variables

Examples

- height
- dose
- temperature
- concentration
- revenue
- clicks

“Continuous”?

- variability is treated as infinite
 - precision is determined by data acquisition methodology
- range usually has practical limits
 - outliers can be defined statistically or heuristically
- *frequency (contingency) is not meaningful*



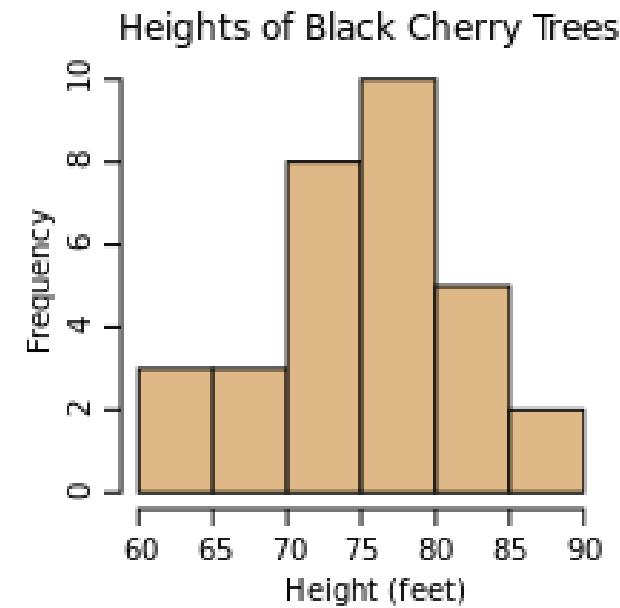
Analysing Continuous Variables

Distribution of binned data:

- choose an appropriate bin width
- ‘cut’ the data into bins
- count the number of samples that fall into each bin

what is the resulting plot called?

- histogram





Summarising Quantitative Data

Measuring the centre of the data

mean

the average value of the variable

median

the value that separates the 50% lowest values from the rest

mode

the most frequently occurring value



Summarising Quantitative Data – cont'd

Quantiles

- inverse of binning data for a histogram:
 - specify proportions of samples we want in each bin
 - compute bin boundaries that correspond

example: 4 quantiles from a random sample (mean = 0, variance = 1):

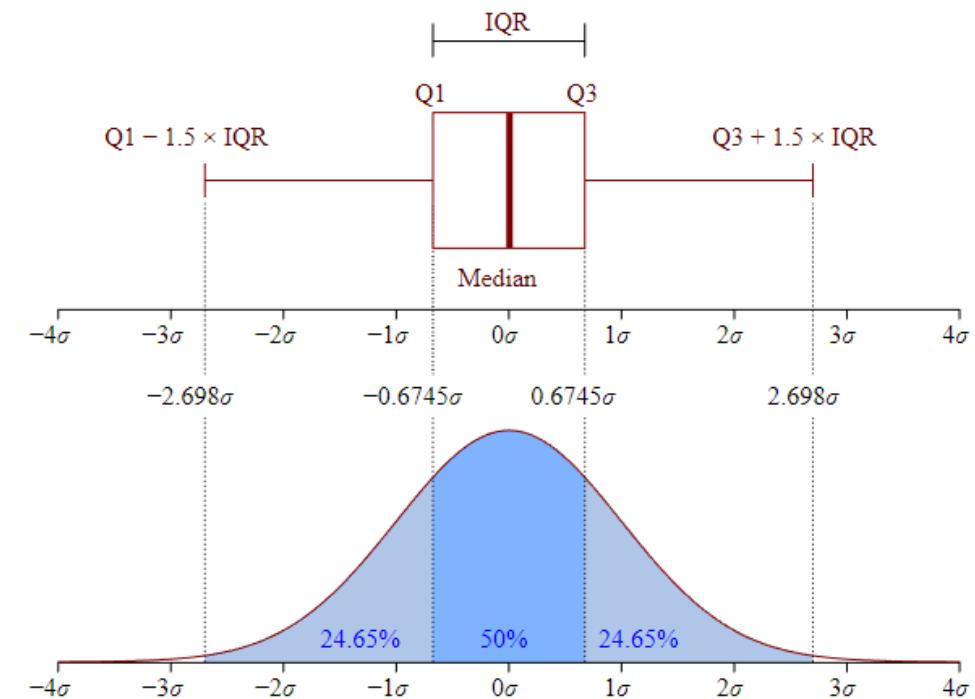
Quantile	Boundaries	Count
(0 - 0.25]	(-3.135, -1.61]	82
(0.25 - 0.50]	(-1.61, -0.0913]	82
(0.50 - 0.75]	(-0.0913, 1.427]	82
(0.75 - 1.0]	(1.427, 2.946]	82



Summarising Quantitative Data – cont'd

Interquartile range (IQR)

- $IQR = [0.25, 0.75]$
- box plots are drawn with whiskers extending $1.5 \times IQR$ beyond the 0.25 and 0.75 quantiles (i.e. the 1st and 3rd quartiles)
- **outliers** are typically defined as lying outside this range



By Jhguch at en.wikipedia, CC BY-SA 2.5,
<https://commons.wikimedia.org/w/index.php?curid=14524285>



Summarising Quantitative Data – cont'd

Moments of a Sample

mean

$$\frac{1}{n} \sum_{i=1}^n x_i$$

variance

$$\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

skewness

$$\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^3$$

kurtosis

$$\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^4 - 3$$



Lab 1.1.3: Simple data visualisation

Purpose:

- Use various plot types to visualise statistical observations.

Materials:

- Notebook: 'Lab 1.1.3 Statistics - part 1 Lab'

Note:

- There may not be enough time to complete this lab in the class.
Please complete it as a part of your homework.
This should apply to all labs.





Statistics – Part 3

- Modelling data distributions



Summarising Quantitative Data – cont'd

Summary statistics

standard deviation

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2}$$

square root of variance

same units as mean



Modelling Data Distributions

Sample vs Population

μ = mean of population

\bar{x} = mean of sample

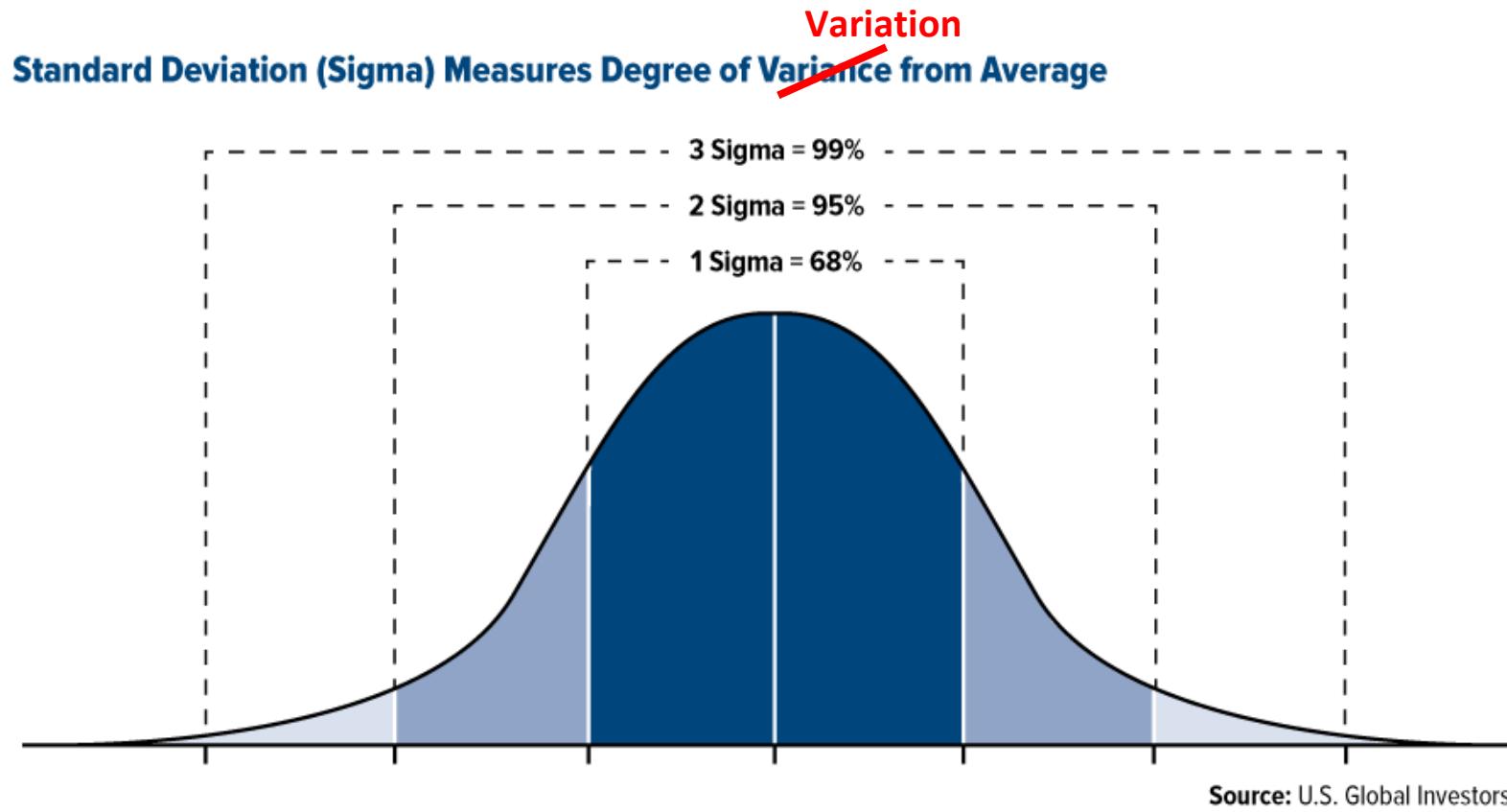
σ = standard deviation of population

s = standard deviation of sample



Modelling Data Distributions – cont'd

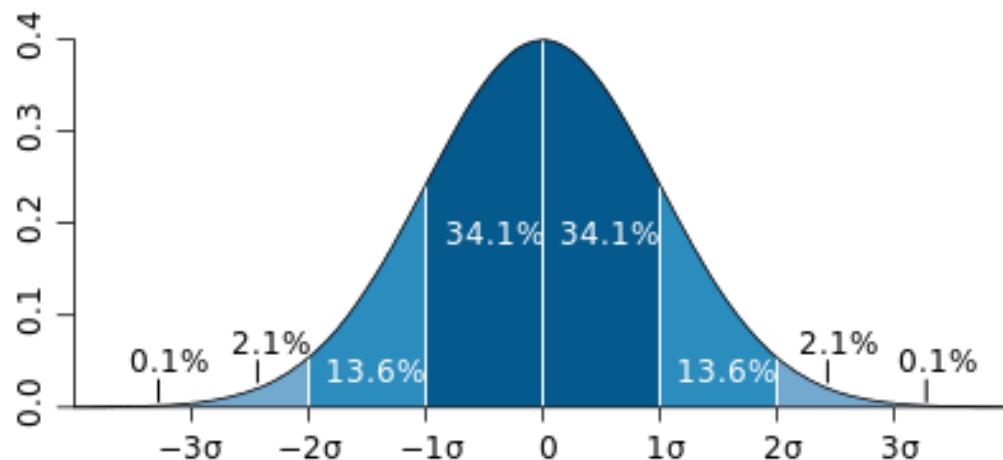
Mean and Standard Deviation of a Population



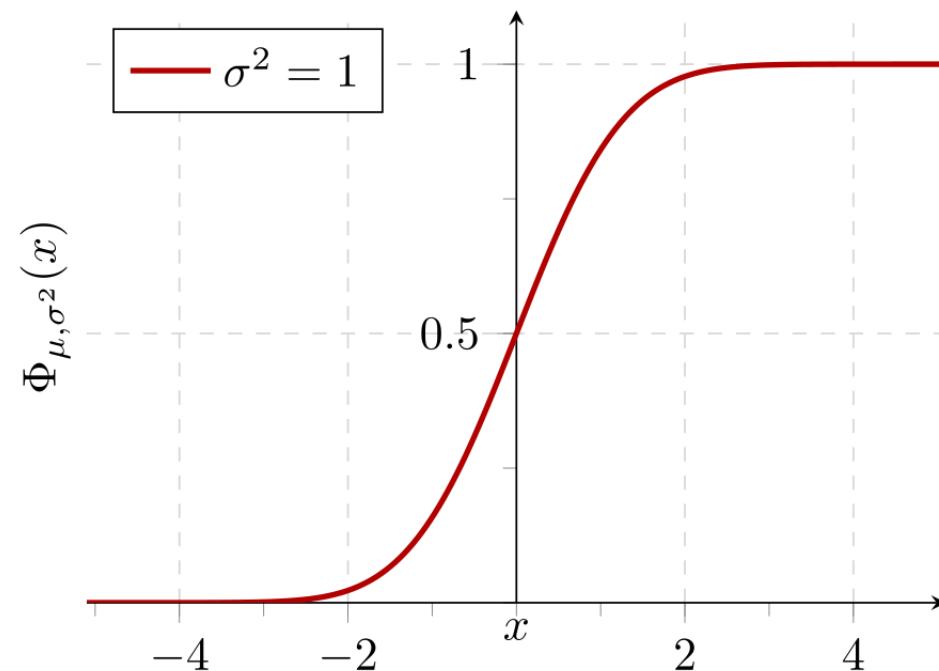


Modelling Data Distributions – cont'd

Probability Density Function



Cumulative Probability



By M. W. Toews - Own work, based (in concept) on figure by Jeremy Kemp, on 2005-02-09, CC BY 2.5, <https://commons.wikimedia.org/w/index.php?curid=1903871>

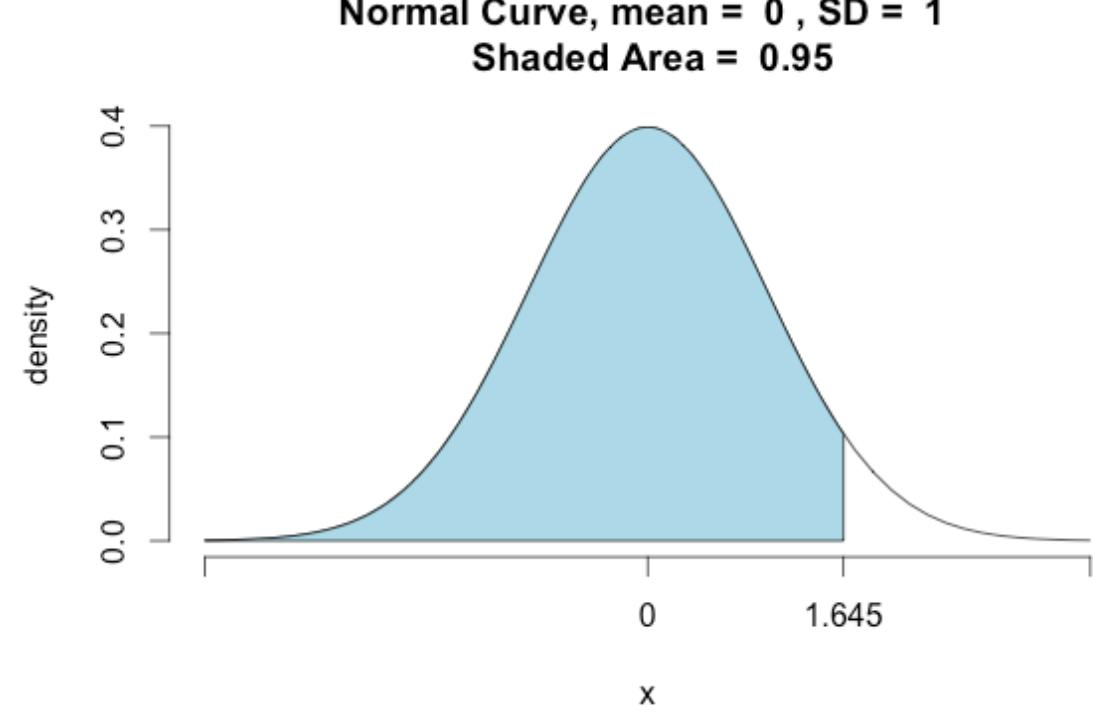


Modelling Data Distributions – cont'd

z-score

measures how far a sample lies
from the population mean:

$$z = \frac{x - \mu}{\sigma}$$





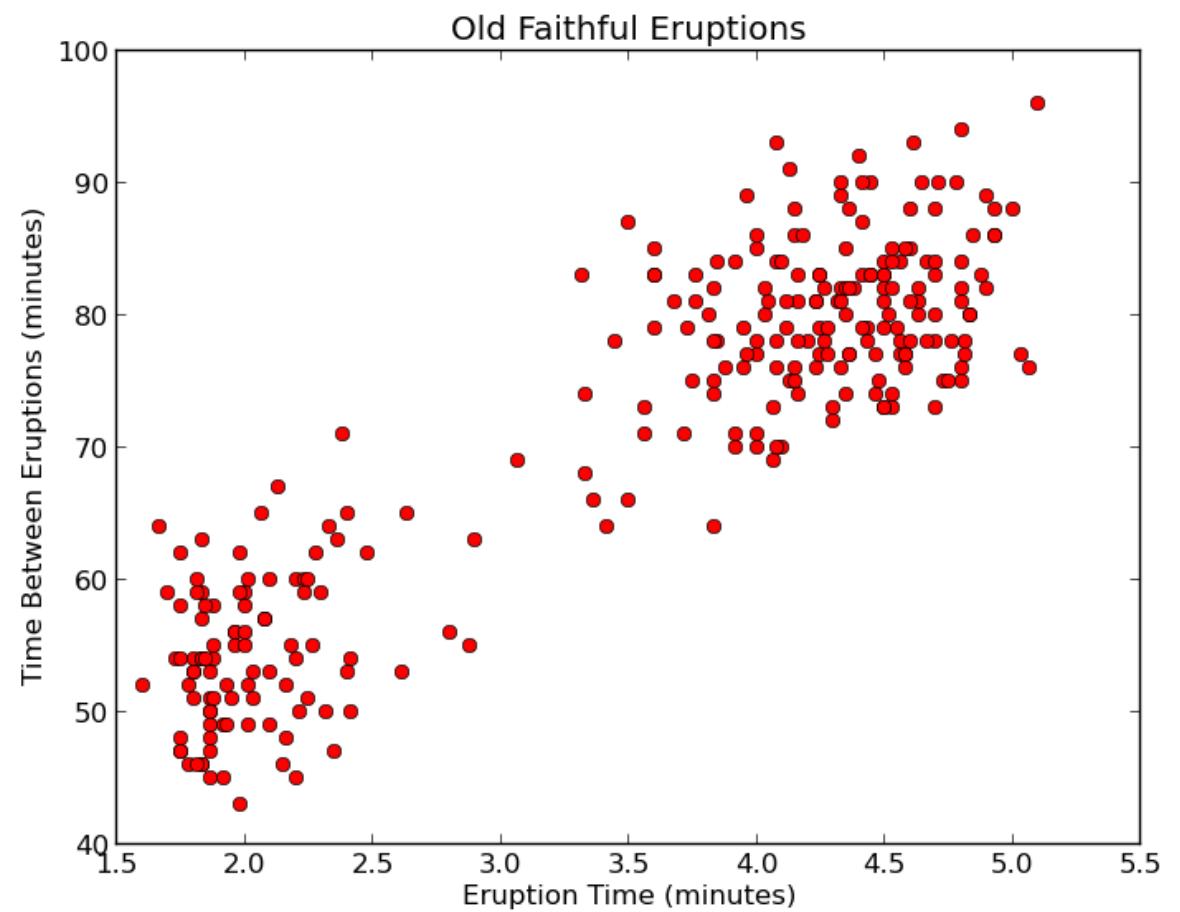
Statistics – Part 4

- Exploring bivariate numerical data



Scatter Plots

- 2D: plots one variable against another
- demonstrates a relation (or lack thereof) between two variables
- *assumption: data pairs are sampled simultaneously*





Correlation

Pearson correlation coefficient

measures strength of covariance between one variable and another:

$$r_{xy} = \frac{1}{(n - 1) s_x s_y} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

- large when big variations in y correspond to big variations in x
- small when small variations in y cancel out big variations in x (or vice versa)

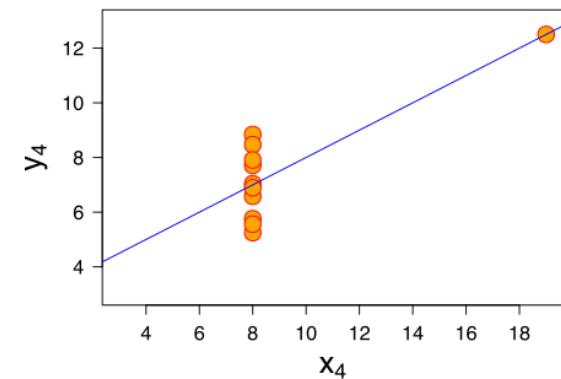
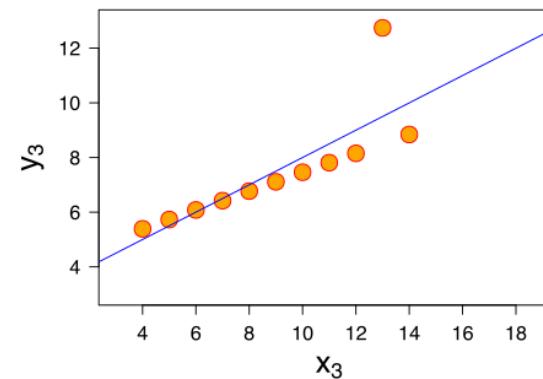
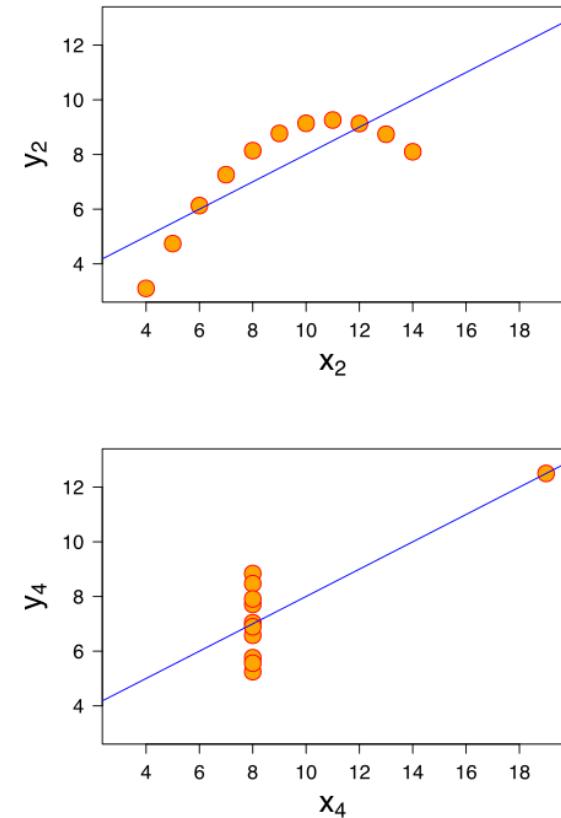
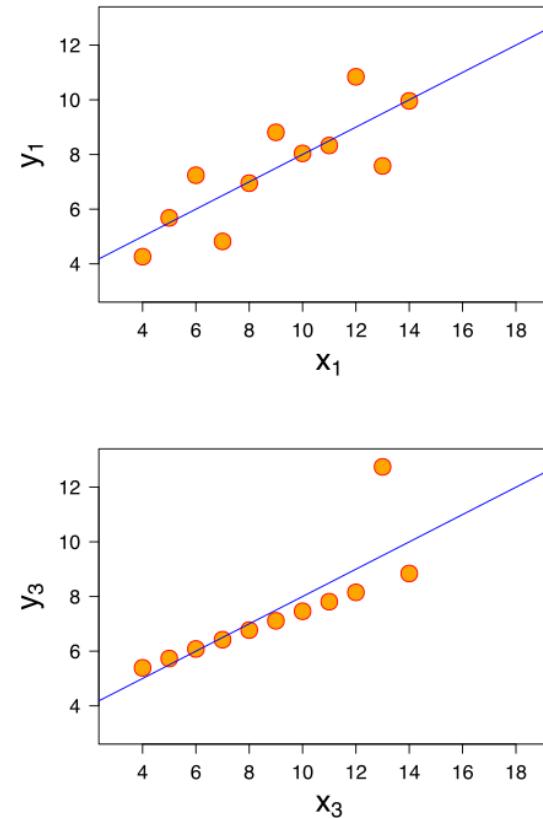


Correlation

Anscomb's quartet

four very different sets of 11 data pairs, each with $r_{xy} = 0.816$

- correlation coefficient
 - assumes a *linear* relation
 - does not completely characterise the relationship between x and y



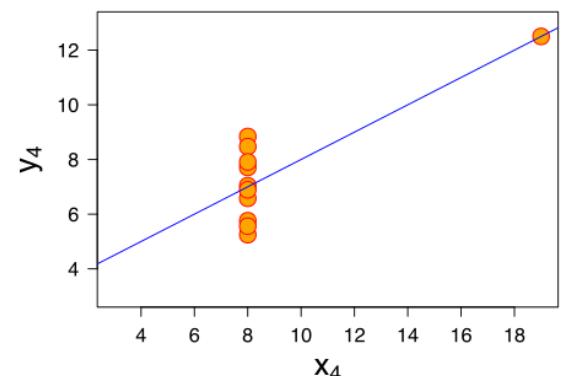
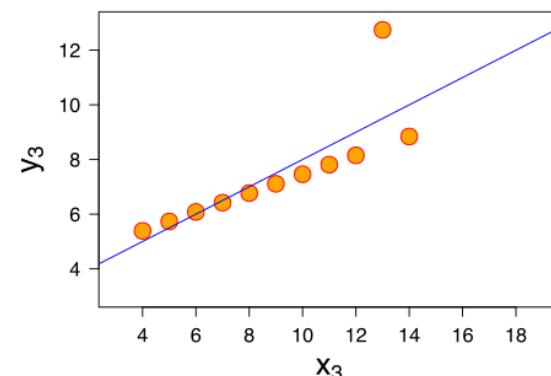
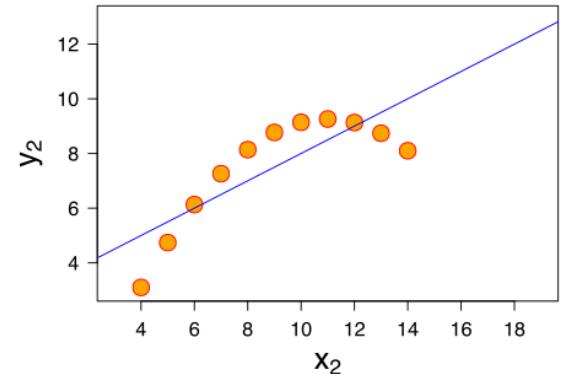
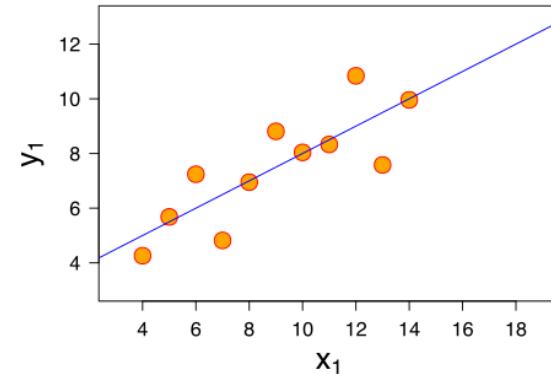
By Anscombe.svg: SchutzDerivative works of this file:(label using subscripts): Avenue - Anscombe.svg, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=9838454>



Trend Lines

best (linear) fit to a 2D scatter plot

- the line that minimises error
by some criterion
- line is specified by
 - slope
 - intercept



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Least-Squares Regression

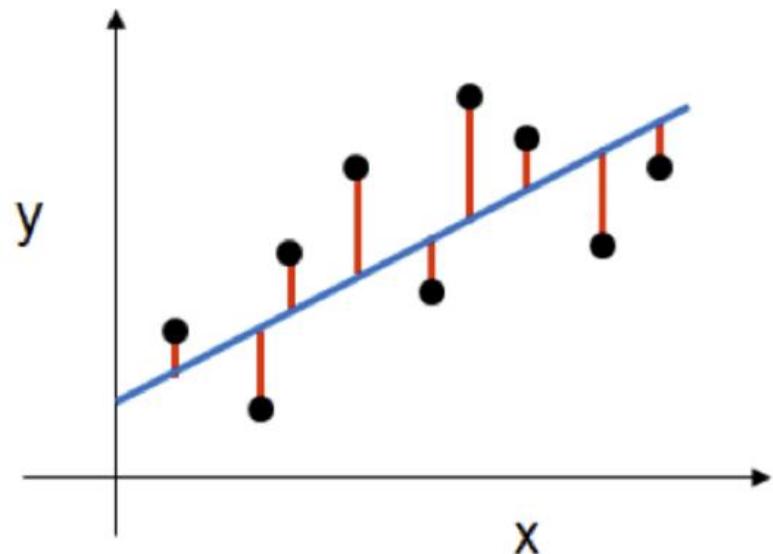
def: residual

the difference between the observed
and predicted values:

$$\varepsilon_i = y_i - \hat{y}_i$$

- least-squares criterion:

$$err = \sum_{i=1}^n \varepsilon_i^2$$





Least-Squares Regression

minimise:

$$\sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2$$

solution:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$



Statistics – Part 5

- Random variables



Random Variables

Discrete random variables

- range is finite or countably infinite
- distribution can be described by a **probability mass function**
 - assigns a probability to each value in the image of X

Continuous random variables

- distribution can be described by a **probability density function**
 - assigns a probability to each specified interval over the range of X



Transforming random variables

standardising variables for analysis:

- *centering* (subtracting the mean)
- *scaling* (dividing by SD)
- allows standard tables to be used to compute percentiles of the sample distribution and probabilities of sampled values

$$x' = \frac{x - \bar{x}}{s}$$

recall: z-score $z = \frac{x-\mu}{\sigma}$



Transforming random variables – cont'd

offset

$$x' = x + 1$$

- datasets based on counts (binning) may contain zeros
 - examples:
 - calls received in each minute at a call centre
 - instances of a keyword in a corpus
 - if the method relies on the logarithm of the count (which many do), it will blow up for $x_i = 0$



Transforming random variables – cont'd

logarithmic rescaling

$$x' = \log(x)$$

- datasets with large dynamic range
 - examples:
 - lifetime value of customer
 - algorithm could be skewed if a small amount data with large values dominates a large amount of data with small values



Transforming random variables – cont'd

Box-Cox transformation

$$x' = x^\lambda \quad \lambda \in \{0, \pm 0.5, \pm 1, \pm 2, \pm 3\}$$

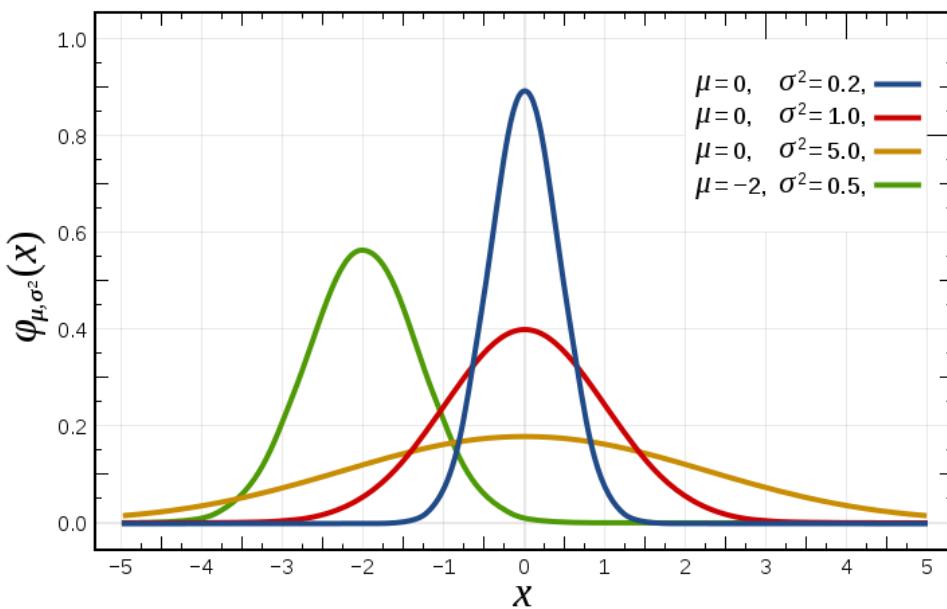
- datasets high skewness or kurtosis
- try different values of λ
 - choose the one that gives the most normal distribution



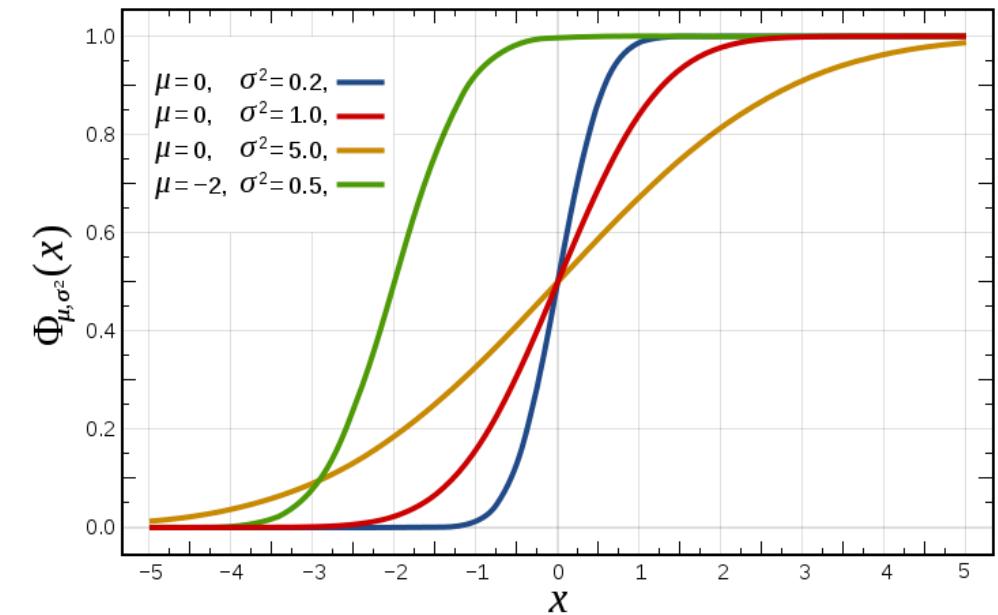
Normal Distribution

aka Gaussian distribution, bell curve

$$PDF = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



CDF



By Inductiveload - self-made, Mathematica, Inkscape, Public Domain, <https://commons.wikimedia.org/w/index.php?curid=3817954>



Other types of Probability Distributions

- Bernoulli distribution
 - The outcome of a single Bernoulli trial (e.g. success/failure, yes/no)
- Binomial distribution
 - The number of "positive occurrences" (e.g. successes, yes votes, etc.) given a fixed total number of independent occurrences
- Geometric distribution
 - Binomial-type observations but where the quantity of interest is the number of failures before the first success; a special case of the negative binomial distribution



Statistics – Part 6

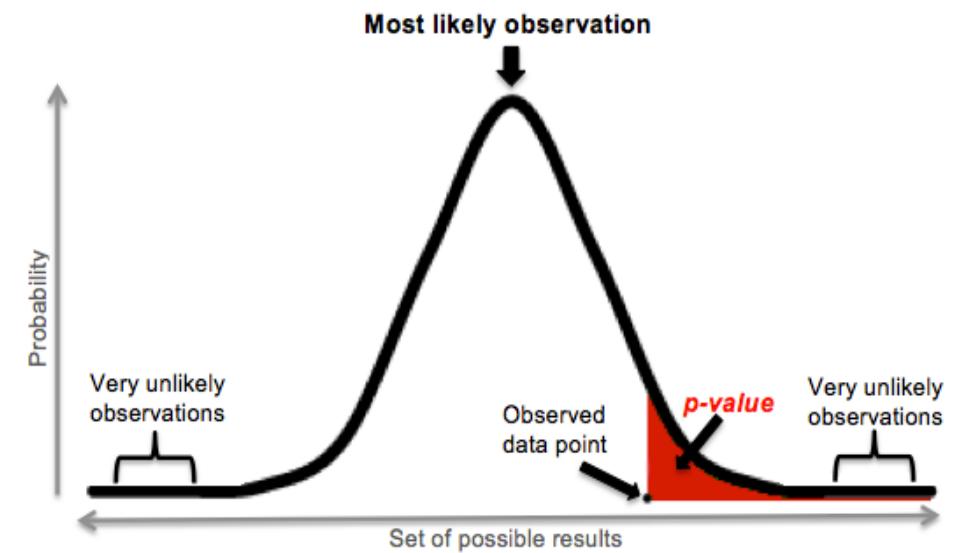
- Confidence intervals
- Significance tests and hypothesis testing
- Inference
- ANOVA



p-Value

P-value measures the probability that a more extreme-valued sample could be randomly drawn from the distribution.

$$p(x > x') = 1 - p(x \leq x')$$



A **p-value** (shaded red area) is the probability of an observed (or more extreme) result arising by chance



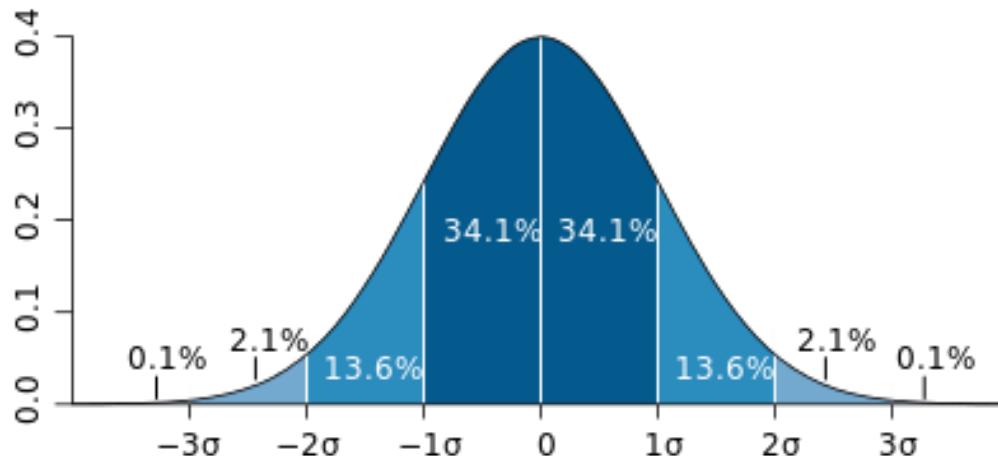
Confidence Intervals

recall: *z-score*

measures how far a sample lies from the population mean:

$$z = \frac{x-\mu}{\sigma}$$

normal distribution:



mean \pm	% population
1σ	68.2
2σ	95.4
3σ	99.7



Confidence Intervals – cont'd

We define confidence intervals in terms of target probability bands:

confidence interval	mean \pm	p-value
0.68	$\sim 1 \sigma$	0.32
0.95	$\sim 2 \sigma$	0.05
0.99	$\sim 3 \sigma$	0.01

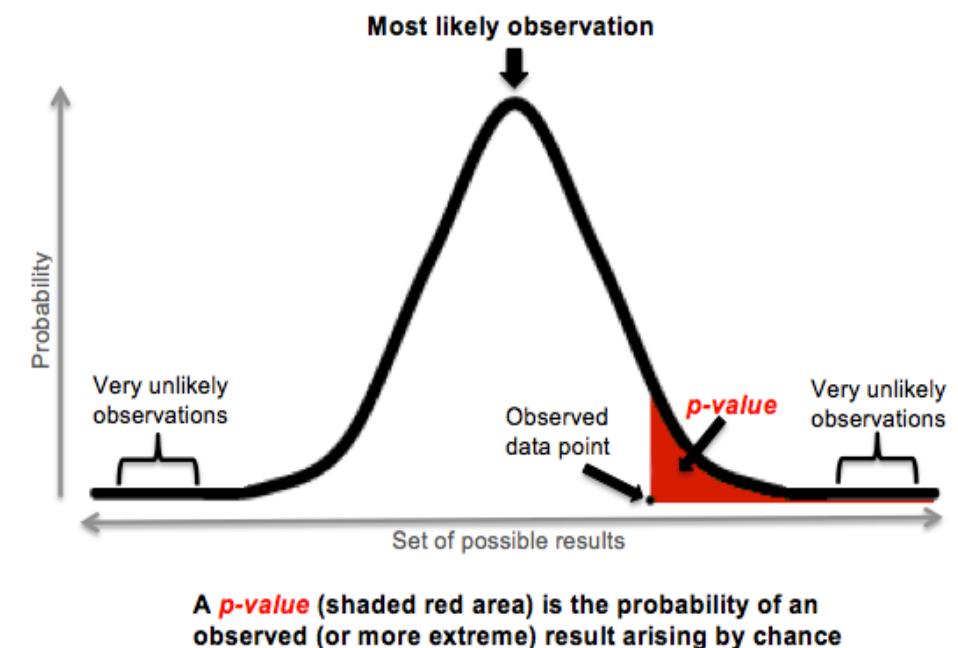


Significance Tests

- Given a specified confidence interval, is a particular sample close enough to the mean that we can confidently presume that comes from the same population?
- Conversely, can we say that the new sample is *different* enough that it probably comes from a different population?

example:

if we choose $p = 0.05$, then
a new sample $x^o > \bar{x} + 2\sigma$
then we can say that x^o is
significantly greater than \bar{x}
(for a 95% confidence interval)



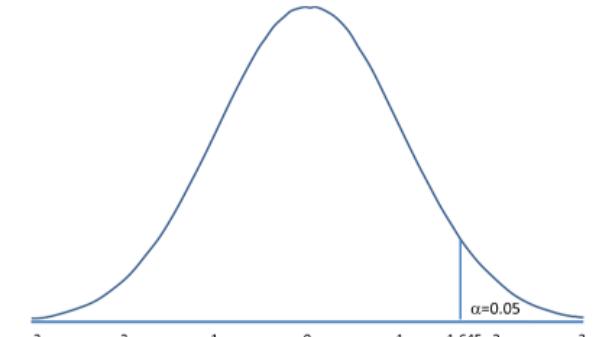


One-Tailed Test vs Two-Tailed Test

One-tailed test: is B greater than A?

a 95% confidence interval would mean we are interested in the last 5% of the right tail

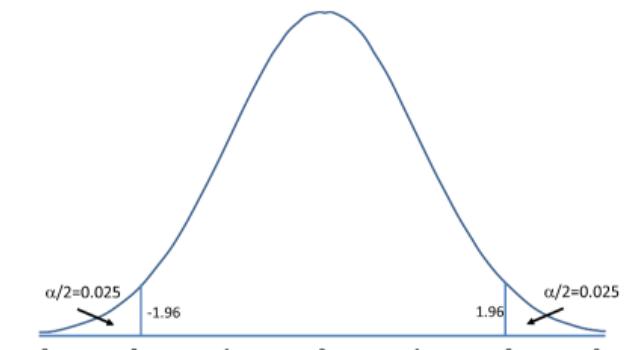
Nb. for “Is B less than A” we would be looking at the left tail instead of the right.



standard
normal
distribution

Two-tailed test: is B different from A?

a 95% confidence interval would mean we are interested in the last 2.5% of each tail





Standard Error of the Mean

Corrects the standard deviation for a finite population (i.e. an acquired dataset):

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \cong \frac{s}{\sqrt{n}}$$

σ is the population SD

s is the sample SD

Central Limit Theorem:

- as sample size increases, SEM approaches the population mean



Student's *t*-Test

Corrects the z-score for a finite population (i.e. an acquired dataset):

$$t = \frac{z}{\sigma_{\bar{x}}} = \frac{(\bar{x} - \mu)}{\sigma/\sqrt{n}}$$

z is the z-score

$\sigma_{\bar{x}}$ is the standard error of the mean

μ, σ are the population mean, SD

\bar{x}, s are the sample mean, SD



Null Hypothesis

If we want to test whether B is different from A , we first assume that it is not. Then we test to see if the difference between A and B is likely to occur by random chance.

If the difference between A and B exceeds the confidence interval, we reject the null hypothesis and infer that B is not from the same population.



ANOVA

- For comparing multiple groups, repeated application of the t -test would randomly give rise to apparent significance
- ANOVA avoids this error by introducing the F -test (analogous to the t -test but for more than 2 groups)
- You can use SciPy to estimate variations between two or more groups



Probability

- Basic theoretical probability
- Bayesian inference
- Probability using sample spaces
- Basic set operations
- Permutations and combinations
- Conditional probability and independence



Probability

If A, B are independent events, the likelihood of ...

A occurring = $P(A)$

A not occurring = $1 - P(A)$

both occurring (and) = $P(A \cap B) = P(A) P(B)$

either occurring (or) = $P(A \cup B) = P(A) + P(B)$

A occurring if B occurs (conditional) = $P(A|B) = \frac{P(A \cap B)}{P(B)}$



Sample Space

def: The set of all possible outcomes of an experiment

- Ordered: sequence is important
- Unordered: sequence is ignored

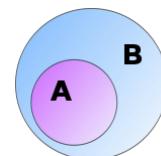
$$\Omega = \{s_1, s_2, \dots, s_n\}$$

$$P(A) = \frac{\text{number of outcomes in event } A}{\text{number of outcomes in sample space } \Omega}$$



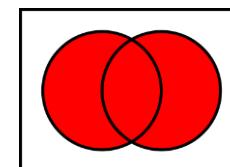
Set Operations

$$A \subseteq B$$



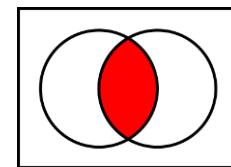
subset

$$A \cup B$$



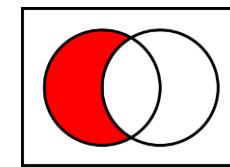
union

$$A \cap B$$



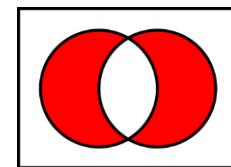
intersection

$$A - B$$



relative complement of B in A

$$A \Delta B$$



symmetric difference of A and B



Permutations and Combinations

Permutation: an ordered set

Combination: an unordered set

number of k -permutations of n :

$$P(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1)$$

number of k -combinations of n :

$$C(n, k) = \frac{n!}{(n - k)! k!}$$



Bayes' inference theorem

- Bayes' inference theorem used to update the probability for a hypothesis as more evidence or information becomes available.
- Theorem:
 - $P(H|E) = P(E|H).P(H)/P(E)$
- Definition:
 - $P(A|B)$: The probability of event A given B



Lab 1.1.4: Applying statistical thinking using Python

Purpose:

- Explore how to use Python (and related packages) to apply Statistical Thinking on data.

Materials:

- Notebook: 'Lab 1.1.4 Statistics - part 2 Lab'

Note:

- There may not be enough time to complete this lab in the class.
Please complete it as a part of your homework.
This should apply to all labs.



Questions?

End of Presentation!