Differentiation

Rate of change

The derivative of $f(x)$ at x_0 is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$ provided the limit exists.

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

→ It is the slope of the graph $y=f(x)$ at $x=x_0$.
→ The slope of the tangent to the curve $y=f(x)$ at $x=x_0$
→ The rate of change of $f(x)$ w.r.t. x at $x=x_0$.
→ The derivative $f'(x_0)$ at a point x_0.

Left hand derivative (LHD)

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

Right hand derivative (RHD) $\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$

When LHD = RHD, both exist and are equal, then the function is said to differentiable i.e. $f'(x)$ exists.

Exp $f(x) = |x|$ at 0

LHD $= \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

RHD $= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$

LHD ≠ RHD

⇒ $f(x) = |x|$ is not differentiable at 0.
Elsewhere it is differentiable.

Ex Prove that differentiability $\Rightarrow$ Continuity

But Continuity $\not\Rightarrow$ differentiability

[$f(x) = |x|$ is continuous at 0 but not differentiable at 0]

Chain rule:

Implicit differentiation

$\Big\}$ You know

$y^2 = x^2 + \sin xy$

Find $\frac{dy}{dx}$

Sol^n $2y \frac{dy}{dx} = 2x + (\cos xy) \frac{d}{dx}(xy)$

$\Rightarrow 2y \frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$

$\Rightarrow 2y \frac{dy}{dx} - (\cos xy)\, x \frac{dy}{dx} = 2x + y(\cos xy)$

$\Rightarrow (2y - x\cos xy) \frac{dy}{dx} = 2x + y\cos xy$

$\Rightarrow \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$

Applications of derivatives

1. Extreme values of functions

Let f be a function with domain D. Then f has a global maximum value in D at a point c if $f(x) \leq f(c)$ for all $x \in D$.

f has a global minimum value in D at c if
$$f(x) \geq f(c) \text{ for all } x \in D.$$

Exp

	Domain	
$f(x) = x^2$	$(-\infty, \infty)$	No global maximum global minimum 0 at $x=0$.
$f(x) = x^2$	$[0,2]$	global maximum 4 at $x=2$ global minimum 0, at $x=0$
$f(x) = x^2$	$(0,2]$	global maximum at $x=2$. No global minimum
$f(x) = x^2$	$(0,2)$	No global minimum, no global maximum.

Extreme value theorem

If f be a continuous function on a closed interval $[a,b]$, then f attains both global maximum and global minimum in $[a,b]$

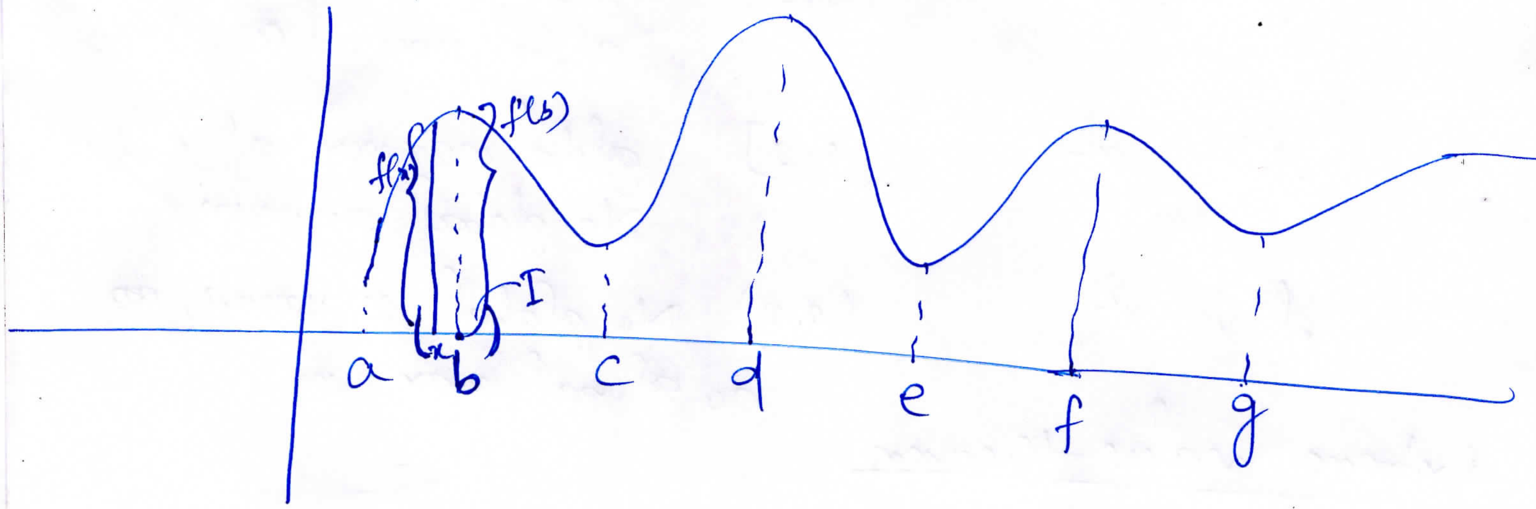
That is there are real numbers $x_1, x_2 \in [a,b]$ with $f(x_1) = m$, $f(x_2) = M$ such that
$$m \leq f(x) \leq M \text{ for } \cancel{all} \text{ every other } x \in [a,b]$$

Local maximum and local minimum

A function f has a local maximum (relative maximum) value at a point C within its domain if $f(x) \le f(c)$ for all $x \in D$ lying in some open interval containing C.

A function f has a local minimum (relative minimum) value at a point C if there exist an open interval I, ~~containing C and~~ contained in D such that

$f(x) \geqslant f(c) \quad \forall x \in I \subset D$.



b, d, f. local maximum

a, c, e, g local minimum

a global minimum

d global maximum

$f : [a,b] \to \mathbb{R}$

f has a local maximum at a if $f(x) \le f(a)$ for all x in some open interval $[a, a+\delta) \subset [a,b]$, $\delta > 0$.

f has a local maximum at an interior point c if $f(x) \le f(c)$ $\forall x$ in $(c-\delta, c+\delta) \subset [a,b]$.

The first derivative theorem

If f has a local maximum or local minimum at an interior point c of its domain, and if f' exists at c.

Then $f'(c) = 0$.

Proof

Let f has a local maximum at c.

$\Rightarrow f(x) \le f(c) \quad \forall x \in (c-\delta, c+\delta)$

$\Rightarrow f(x) - f(c) \le 0 \quad \forall x \in (c-\delta, c+\delta)$

c is an interior point of f' exists.

$$f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$$

$$\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0$$

c x

as $f(x)-f(c) \le 0$ of $x > c$.

Also $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$ as $f(x)-f(c) \le 0$ of $x < c$.

Since $f'(c)$ exists both LHD of RHD have to be equal.

$$\Rightarrow \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = 0$$

$\Rightarrow f'(c) = 0$.

Similarly one can prove the result if f has a local minimum at c.

Note

f can possibly have an extreme value ~~above~~ at

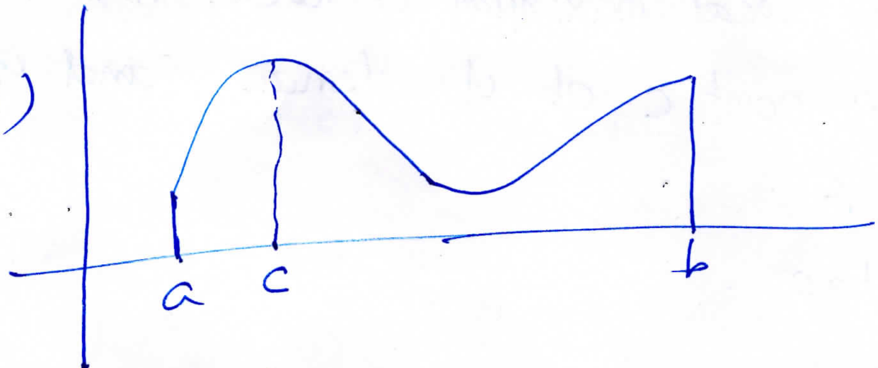
(i) interior points where $f'=0$

(ii) interior points where f' does not exist

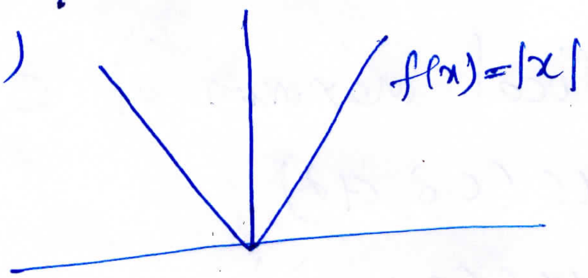
(iii) endpoints of the domain of f.

Ex:

(i)

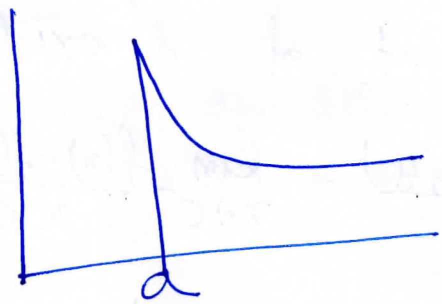


(ii)



$|x|$ is not differentiable at 0, but it has a local minimum, in fact a global minimum at $x=0$.

(iii)



At the point a, f has ~~an extreme value~~ global maximum.

Def^n

An interior point of the domain of a function f where f' is zero or undefined is called a critical point of f

Using Taylor's theorem we can prove the sufficient condition for the existence of extreme values.

Theorem

Let $f: D \to \mathbb{R}$

c is an interior point in the domain D.

Let f' is differentiable continuously n times

If $f'(c) = 0$ and $f''(c) > 0$ then at c there is a local minimum

If $f'(c) = 0$ and $f''(c) < 0$ then at c there is a local maximum.

If $f'(c) = 0 = f''(c) = \cdots = f^{(n-1)}(c) = 0$

and if $f^{(n)}(c) \neq 0$ and n is even, then

c is a local minimum if $f^{(n)}(c) > 0$

c is a local maximum if $f^{(n)}(c) < 0$.

If n is odd, c is neither a local maximum nor a local minimum.

Exp

$f(x) = (x-1)^3$ critical point $x = 1$

$f'(x) = 3(x-1)^2 = 0$

$f'(1) = 0$

$f''(x) = 6(x-1)$, $f''(1) = 0$

$f'''(x) = 6$ $f'''(1) \neq 0$ and n is odd.

So 1 is neither a local maximum nor a local minimum. 1 is called a point of inflection. ~~inflection~~

Proof of Sufficient condition

Taylor's theorem say that

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(c) + \underbrace{\frac{h^n}{n!}f^{(n)}(c+\theta h)}_{\text{remainder term}} \quad 0<\theta<1$$

$$\Rightarrow f(c+h) - f(c) = \frac{h^n}{n!}f^{(n)}(c+\theta h)$$

as $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$

As $f^{(n)}(c) \neq 0$, $\Rightarrow f^{(n)}(c+\theta h)$ and $f^{(n)}(c)$ have same sign. (and it is continuous)

$\Rightarrow f(c+h) - f(c) > 0$ if n is even and $f^{(n)}(c) > 0$

$\Rightarrow f(c+h) > f(c)$ $\forall h$ if n is even and $f^{(n)}(c) > 0$.

$\Rightarrow c$ is a local minimum if n is even and $f^{(n)}(c) > 0$.

and $f(c+h) - f(c) < 0$ if n is even and $f^{(n)}(c) < 0$

$\Rightarrow f(c+h) < f(c)$ $\forall h$ if n is even and $f^{(n)}(c) < 0$

$\Rightarrow c$ is a local maximum if n is even and $f^{(n)}(c) < 0$

If n is odd, then $f(c+h) - f(c) > 0$ or < 0 depend on h.

So when n is odd, f has neither a local maximum nor a local minimum at c.

Application-2 (Mean value theorem)

Constant functions has derivative zero, but there could be more complicated functions whose derivative is also zero.

Rolle's theorem

Suppose that $y = f(x)$ is

(i) continuous at every point of the closed interval $[a, b]$

(ii) differentiable at every point of the open interval (a, b)

(iii) $f(a) = f(b)$

Then there is at least one $c \in (a, b)$ such that $f'(c) = 0$.

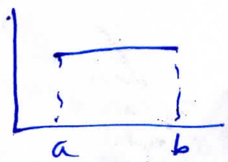
Proof By the hypothesis f has derivative at every interior point.

f is continuous on $[a, b]$

So maximum and minimum occur. ~~by extreme~~

If either the maximum or minimum occurs at the interior point c, then by previous result $f'(c) = 0$.

If both the maximum and minimum occur at the end point, then because $f(a) = f(b)$.

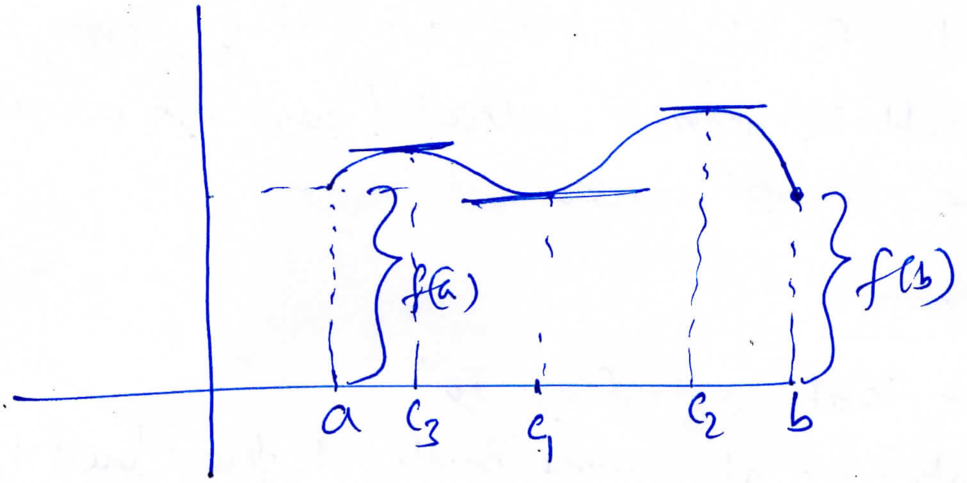


it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ $\forall x \in [a, b]$.

$\Rightarrow f'(x) = 0$.

So there exist at least one real number $c \in (a, b)$ at which $f'(c) = 0$

Geometrical meaning of Rolle's Thm



f is continuous on $[a,b]$

$f(a)=f(b)$

f is differentiable on (a,b)

There exists c_1, c_2, c_3 at which $f'=0$.

Prob

Show that the equation $x^3+3x+1=0$ has exactly one real root

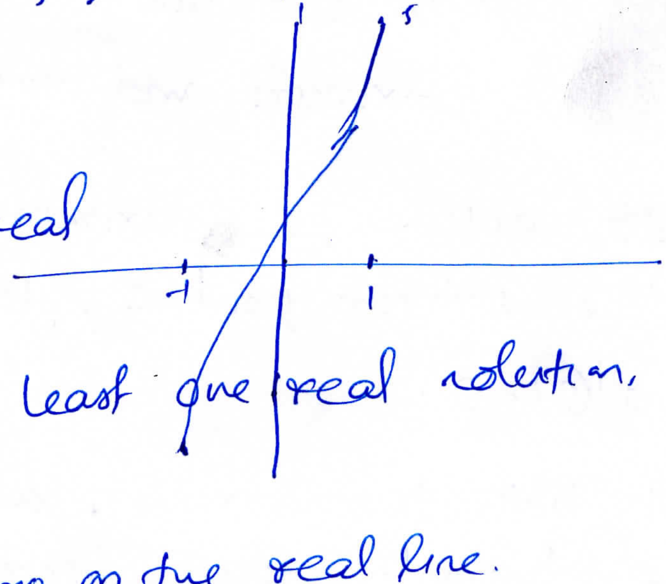
Sol

$f(x) = x^3+3x+1$ $\qquad f(-1) = -3$

$\qquad\qquad\qquad\qquad\qquad f(1) = 5$

f is continuous

and at one value f is +ve, and at another value it is -ve.



So it must cross the real line at least once.

So therefore it has at least one real solution.

$f'(x) = 3x^2+3$

$f'(x)$ is never zero on the real line.

If there ~~were~~ are two points a & b at which $f(x)$ is zero. $\quad f(a)=f(b)=0$

By Rolle's Thm $\exists\, c \in (a,b)$ s.t. $f'(c)=0$

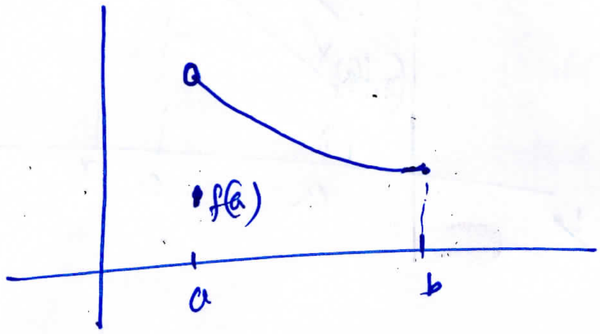
But $f'(x) \neq 0$ on any $x \in \mathbb{R}$.

So there does not exist two points at which $f(x)$ is zero.

$\Rightarrow$ The function $f(x) = x^3+3x+1$ has exactly one real solution.

Note If they fail at even one point, the graph may not have a horizontal tangent.

Ex-1

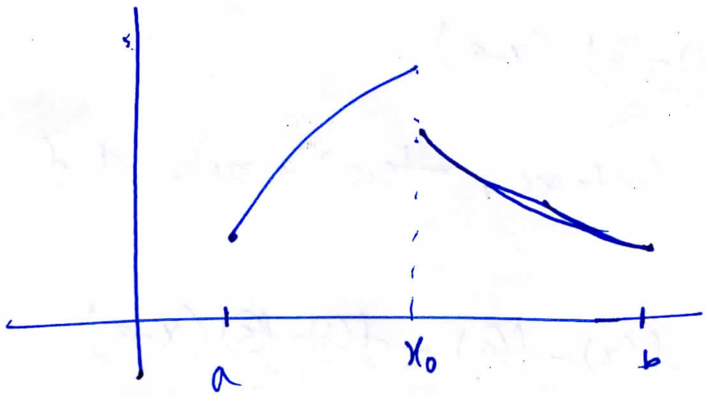


$f(a) = f(b)$

f is discontinuous at a, at an end point.

There exists no point c at which $f'(c) = 0$.

Ex-2

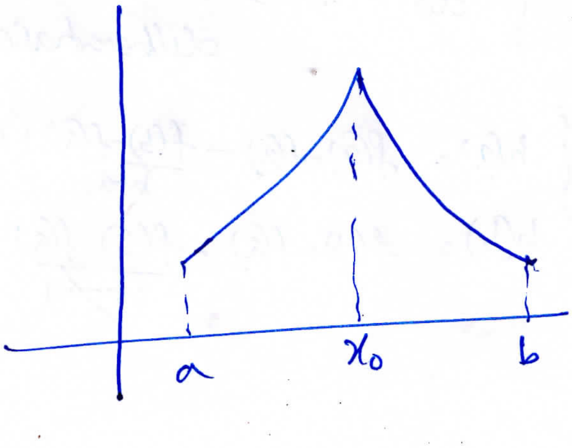


$f(a) = f(b)$

$f(x)$ is discontinuous at an interior point x_0.

There exist no point c at which $f'(c) = 0$.

Ex-3



$f(a) = f(b)$

f is continuous on $[a, b]$

f is not differentiable at x_0.

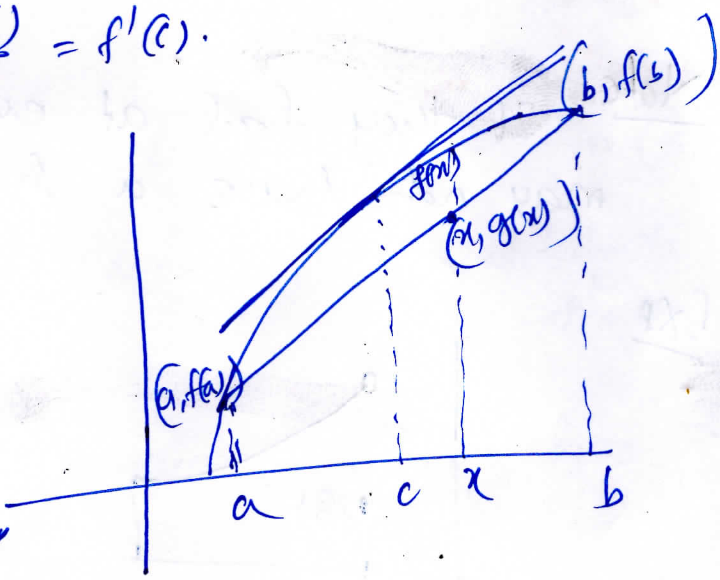
There exists no point $c \in (a, b)$ such that $f'(c) = 0$.

The Mean value theorem

Suppose (i) $y = f(x)$ is continuous on $[a,b]$

(ii) differentiable on (a,b)

Then there exists at least one point $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.



Proof

Draw the chord joining $(a, f(a))$ and $(b, f(b))$

The equation of the chord is

$$g(x) - f(a) = m(x-a)$$

$$\Rightarrow g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$\Rightarrow g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$

The vertical difference between the graphs of f and g at any point x is

$$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$h(x)$ is continuous on $[a,b]$ } as $f(x)$ is continuous on $[a,b]$
$h(x)$ is differentiable on (a,b) } differentiable on (a,b)

Also $h(a) = h(b) = 0$

$$h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{(\cancel{b-a})}(\cancel{b-a}) = 0$$

By Rolle's theorem there exists $c \in (a,b)$ such that $h'(c) = 0$

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$ (proved)

Geometrical Interpretation

At some point $c \in (a,b)$, the slope of the chord joining $(a, f(a))$ and $(b, f(b))$ is same as the slope of the tangent to the curve $y=f(x)$.

Also $\frac{f(b)-f(a)}{b-a}$ is average change of f over $[a,b]$

$f'(c)$ is instantaneous change.

This means that at some point c, ~~instantaneous~~ instantaneous change is same as the average change of the curve $y=f(x)$.

Note MVT do not require f to be differentiable at end points a and b. Continuity at a and b is enough.

Increasing and Decreasing functions

Suppose f is continuous on $[a,b]$ and differentiable on (a,b).

If $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$

If $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$

Proof Let x_1, x_2 be any two points in $[a,b]$ with $x_1 < x_2$.

The MVT applied to f on $[x_1, x_2]$. It says that there exists $c \in (a,b)$ s.t

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c).$$

~~If~~ $f'(c) > 0 \Rightarrow \dfrac{f(x_2)-f(x_1)}{x_2-x_1} > 0$

$\Rightarrow f(x_2)-f(x_1) > 0$ since $x_2 > x_1$

$\Rightarrow f(x_2) > f(x_1)$ as $x_2 > x_1$

$\Rightarrow f$ is increasing.

If $f'(c) < 0 \Rightarrow \dfrac{f(x_2)-f(x_1)}{x_2-x_1} < 0$

$\Rightarrow f(x_2)-f(x_1) < 0$ as $x_2 > x_1$

$\Rightarrow f(x_2) < f(x_1)$ as $x_2 > x_1$

$\Rightarrow f$ is decreasing.

First derivative test for local extrema

Suppose c is a critical point of a continuous function f, and that f is differentiable at every point in some open interval containing c except possibly at c itself.

① If f' changes from negative to positive at c, then f has a local minimum at c.

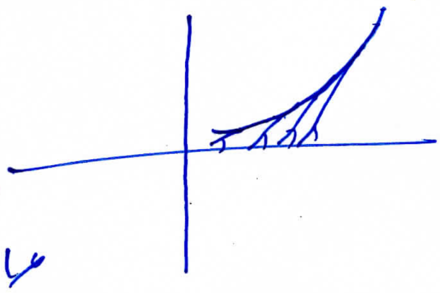
② If f' changes from positive to negative at c, then f has a local maximum at c.

③ If f' does not change sign at c, then f has neither a local max nor a local minimum at c.

Concavity

The graph of a differentiable function $y=f(x)$ is

(i) Concave up on an interval I if f' is increasing on I.



(ii) Concave down on an open interval I if f' is decreasing on I.

If $y=f(x)$ has a second derivative, then

(i) f' increases if $f''>0$ on I

(ii) f' decreases if $f''<0$ on I.

$\Rightarrow$ A twice differentiable function $y=f(x)$ on I is

(i) Concave up if $f''>0$ on I.

(ii) Concave down if $f''<0$ on I.

Point of inflection

A point where the graph of a function has a tangent line and where the concavity changes is a point of inflection.