The four spaces associated with a real matrix & elementary row operations

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Column space of an $m \times n$ matrix A is all possible $b \in R^m$ which are solutions to the equation $Ax = b$.

Left null space of an $m \times n$ matrix A is all possible $x \in R^m$ which are solutions to the equation $A^T x = 0$. Here, A^T is the transpose of matrix A and the 0 on the right hand side represents the all zero vector in R^n.

Row space of an $m \times n$ matrix A is all possible $b \in R^m$ which are solutions to the equation $A^T x = b$.

Null space of an $m \times n$ matrix A is all possible $x \in R^n$ which are solutions to the equation $Ax = 0$. Here, the 0 on the right hand side represents the all zero vector in R^m.

The **column rank** of a matrix A is the maximum number of linearly independent columns in A.

The **row rank** of a matrix A is the maximum number of linearly independent rows in A.

A result, to be proved in a later lecture, is that column rank = row rank. Thus, it is simply referred to as the **rank** of the matrix.

The only common element between the column space and the left null space is the all zeroes vector in R^m. Let $v \in \mathcal{C}(A) \cap \mathcal{N}(A^T)$.

Membership in $\mathcal{C}(A) \Rightarrow, \exists x \in R^n, Ax = v$.

Membership in $\mathcal{N}(A^T) \Rightarrow A^T v = 0$.

Using the first result in the second, we get,

$A^T A x = 0$.

We see that the dimansions are compatible to left multiply both sides by x^T.

Thus $x^T A^T A x = x^T 0$

The right hand side evaluatees to 0. Thus,

$(x^T A^T)(Ax) = 0$.

The two bracketed terms on the left hand side are respectively v^T and v. Taking,

$v = (v_1, \ldots, v_n,$

we see that,

$v_1^2 + \cdots + v_n^2 = 0.$

Since all the components are real numbers, the only possibility is identically 0 on all coordinates. This completes the proof.

We presented the following **Elementary Row Operations**

1. multiply a row by a non-zero constant
2. subtract another row's entries from some row

We established that the solution to a system before applying either of these rules, is identical to the solution of the system after applying either of these rules. Since it is true for one application of these rules, by mathematical induction, it follows that it is true for any series of such operations.

Two algorithms that use these elementary steps in simplifying a matrix represented system of equations are:

1. Guass elimination
2. Guass-Jordan elimination

We also defined the **coefficient matrix** and the **augmented matrix**.