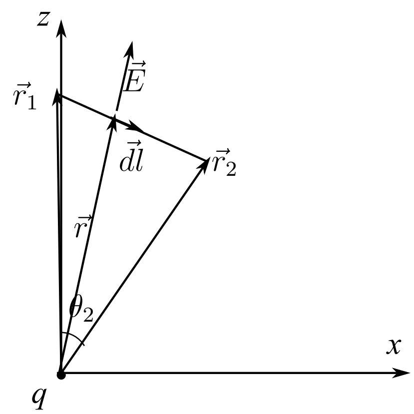
1. Consider a point charge q at the origin. Find the electric potential at a point $\vec{r}_2 : (r = r_2, \theta = \theta_2, \phi = 0)$ with respect to the potential at $\vec{r}_1 : r = r_1, \theta = 0, \phi = 0$ as reference by evaluating the integral $-\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot \vec{dl}$ along a straight line joining $\vec{r}_1$ to $\vec{r}_2$.



soln:
Since $\phi = 0$ both the points are on the xz plane. The equation of the line joining $\vec{r}_1$ and $\vec{r}_2$ is

$$\frac{r\cos\theta - r_1}{r\sin\theta} = \frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}, \quad \phi = 0$$

This gives the relation between r and θ as

$$r\left[\cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right] = r_1 \tag{1}$$

$$\therefore \quad r\left[-\sin\theta - \cos\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right]d\theta + dr\left[\cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right] = 0$$

$$\therefore \quad \frac{dr}{d\theta} = \frac{r\left[\sin\theta + \cos\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)\right]}{\cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)} = -r\frac{f'(\theta)}{f(\theta)} \tag{2}$$

where $f(\theta) = \cos\theta - \sin\theta\left(\frac{r_2\cos\theta_2 - r_1}{r_2\sin\theta_2}\right)$. The potential difference between $\vec{r}_1$ and $\vec{r}_2$ is now

$$\begin{aligned}
-\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot \vec{dl} &= -\frac{q}{4\pi\epsilon_0}\int_{\vec{r}_1}^{\vec{r}_2} \frac{1}{r^2}\hat{r} \cdot (\hat{r}dr + rd\theta\hat{\theta} + r\sin\theta d\theta\hat{\phi}) \\
&= -\frac{q}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{1}{r^2}dr \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{1}{r^2}r\frac{f'(\theta)}{f(\theta)}d\theta \quad \text{from eq.(2)} \\
&= -\frac{q}{4\pi\epsilon_0}\int_0^{\theta_2} \frac{f'(\theta)}{rf(\theta)}d\theta
\end{aligned}$$

From Eq.(1) we see that $rf(\theta) = r_1$.

$$\begin{aligned} -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{l} &= \frac{1}{r_1}\int_0^{\theta_2} f'(\theta)d\theta \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r_1}\left[f(\theta)\right]_0^{\theta_2} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{r_1}\left(f(\theta_2) - f(0)\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{r_1}\left(\frac{r_1}{r_2} - 1\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \end{aligned}$$

If the reference point $\vec{r}_1$ is at ∞ then the potential at $\vec{r}_2$ is $\frac{q}{4\pi\epsilon_0 r_2}$. We see that the potential is independent of θ_2. It can be shown to be also independent of ϕ_2. It only depends on r_2, the distance from the origin.

2. (a) A charge distribution $\rho_1(\vec{r})$ produces a potential $\phi_1(\vec{r})$ in a region τ and another charge distribution $\rho_2(\vec{r})$ produces a potential $\phi_2(\vec{r})$ in the region. Prove that

$$\int_\tau \rho_1\phi_2 d^3\vec{r} = \int_\tau \rho_2\phi_1 d^3\vec{r}$$

How do you interpret this result.

soln:
Poisson's equation gives $\frac{\rho_1}{\epsilon_0} = -\vec{\nabla}^2\phi_1$.

$$\therefore \quad \int_\tau \rho_1\phi_2 d^3\vec{r} = -\epsilon_0\int_\tau (\vec{\nabla}^2\phi_1)\phi_2 d^3\vec{r}$$

Using the product rule $\vec{\nabla}\cdot(\phi_2\vec{\nabla}\phi_1) = \phi_2\vec{\nabla}^2\phi_1 + \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2$ we get

$$\begin{aligned} \int_\tau \rho_1\phi_2 d^3\vec{r} &= -\epsilon_0\int_\tau \vec{\nabla}\cdot(\phi_2\vec{\nabla}\phi_1)d^3\vec{r} + \epsilon_0\int_\tau \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 d^3\vec{r} \\ &= -\epsilon_0\int_S \phi_2\vec{\nabla}\phi_1\cdot\hat{n}da + \epsilon_0\int_\tau \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 d^3\vec{r} \end{aligned}$$

We can extend the integral on the l.h.s beyond the region τ. This doesn't change the value of the integral on the l.h.s as $\rho_1 = 0$ outside this region. But on the r.h.s both the surface and the volume integral contributes and their contribution changes as we extend the region of integration. But for a charge configuration which is confined in a finite region τ, both ϕ_2 and $\vec{\nabla}\phi_1$ goes to 0 as the surface S tends to infinity. So we get

$$\int_\tau \rho_1\phi_2 d^3\vec{r} = \epsilon_0\int_{\text{all space}} \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 d^3\vec{r}$$

Similarly we can prove

$$\int_\tau \rho_2\phi_1 d^3\vec{r} = \epsilon_0\int_{\text{all space}} \vec{\nabla}\phi_2\cdot\vec{\nabla}\phi_1 d^3\vec{r}$$

So we have

$$\int_\tau \rho_1\phi_2 d^3\vec{r} = \int_\tau \rho_2\phi_1 d^3\vec{r}$$

The integral $\int_\tau \rho_1\phi_2 d^3\vec{r}$ gives the work done in bringing the charge configuration specified by $\rho_1(\vec{r})$ in the electric field created by the charge distribution $\rho_2(\vec{r})$. Similarly the other integral gives the work done in bringing the charge configuration specified by $\rho_2(\vec{r})$ in the electric field created by the charge distribution $\rho_1(\vec{r})$. This two work done must be the same since the final charge configuration at the end of the two processes is the same, i.e a combined charge configuration given by ρ_1 and ρ_2.

(b) The interaction energy of two point charges q_1 and q_2 placed at $\vec{r}_1$ and $\vec{r}_2$ is given as $\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3\vec{r}$ where the integration is done over the whole space. Prove that this is equal to $\frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$ where $r_{12} = |\vec{r}_2 - \vec{r}_1|$ as is expected.

soln:

$$\begin{aligned}
\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3\vec{r} &= \epsilon_0 \int \vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 d^3\vec{r} \\
&= \epsilon_0 \int \vec{\nabla} \cdot (\phi_1 \vec{\nabla}\phi_2) d^3\vec{r} - \epsilon_0 \int \phi_1 \vec{\nabla}^2 \phi_2 d^3\vec{r} \\
&= \epsilon_0 \int_S (\phi_1 \vec{\nabla}\phi_2) \cdot \hat{n} da - \epsilon_0 \int \phi_1 \vec{\nabla}^2 \phi_2 d^3\vec{r}
\end{aligned}$$

The surface S of the surface integral is at infinity since we do the integral over the whole space. ϕ_1 goes as $1/r$ and $\vec{\nabla}\phi_2$ goes as $1/(r^2)$. So the integrand goes as $\frac{1}{r^3}$ while the surface area goes as r^2. So the surface integral goes to 0. In the other integral we know $\phi_1 = \frac{q_1}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|}$ and $\vec{\nabla}^2\phi_2 = -\frac{q_2\delta^3(\vec{r} - \vec{r}_2)}{\epsilon_0} d^3\vec{r}$

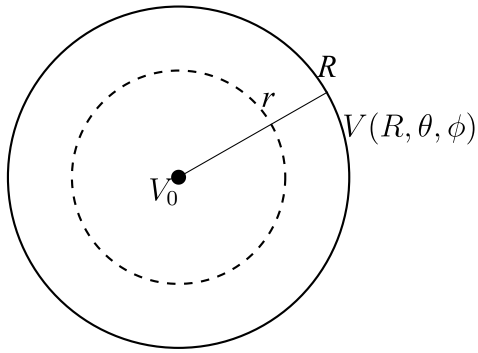
$$\begin{aligned}
\epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3\vec{r} &= -\epsilon_0 \int \frac{q_1}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|} \frac{(-q_2\delta^3(\vec{r} - \vec{r}_2))}{\epsilon_0} d^3\vec{r} \\
&= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{|\vec{r}_2 - \vec{r}_1|}
\end{aligned}$$

3. Prove the mean value theorem in electrostatics which states that in a chargeless region, the average of the potential over the surface of any sphere is equal to the potential at the center of the sphere.
This is true for any regular polyhedron. If the faces of a regular polyhedron having n faces are maintained at potentials $V_1, V_2, ..., V_n$ then the potential at the center of the polyhedron is $(V_1 + V_2 + ... + V_n)/n$. How many such regular polyhedron do you think are possible? Look for platonic solids. Tetrahedron, cube, octahedron, dodecahedron and icosahedron.

soln
Consider a sphere of radius R in a chargeless region. The average value of potential

over this sphere is

$$V_{avg} = \frac{1}{4\pi R^2} \int_0^{2\pi} \int_0^{\pi} V(R,\theta,\phi) R^2 \sin\theta d\theta d\phi$$



Let V_0 be the potential at the center of the sphere. Then

$$V(R,\theta,\phi) = V_0 + \int_0^R \vec{\nabla} V \cdot \hat{r} dr$$

The integral in the above step is independent of the path since $\vec{\nabla} V = -\vec{E}$ is a curlless field. So we do the integral along the radial direction from 0 to R, θ, ϕ.
So average potential is

$$\begin{aligned}
V_{avg} &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left[V_0 + \int_0^R \vec{\nabla} V \cdot \hat{r} dr \right] \sin\theta d\theta d\phi \\
&= \frac{1}{4\pi} V_0 \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi + \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^R \vec{\nabla} V \cdot \hat{r} dr \sin\theta d\theta d\phi \\
&= V_0 + \frac{1}{4\pi} \int_0^R \left[\int_0^{2\pi} \int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi \right] dr
\end{aligned} \tag{3}$$

The integral over θ and ϕ is at a constant r. We can write this as a surface integral over a sphere of radius r as follows:

$$\int_0^{2\pi} \int_0^{\pi} \vec{\nabla} V \cdot \hat{r} \sin\theta d\theta d\phi = \frac{1}{r^2} \int_0^{2\pi} \int_0^{\pi} (\vec{\nabla} V \cdot \hat{r}) r^2 \sin\theta d\theta d\phi = \frac{1}{r^2} \oint_S -\vec{E} \cdot \hat{r} da$$

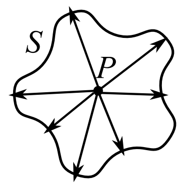
This surface integral is equal to the total charge enclosed inside the sphere of radius r. In a chargeless region this is 0. So

$$V_{avg} = V_0$$

4. Prove that in a chargeless region electrostatic potential cannot have a maxima or a minima.

   soln:
   In a chargeless region the potential at any point is equal to the average potential over any sphere around it. Since the average can't be a maxima or a minima in a region we conclude that the electrostatic potential can't have a maxima or a minima.

   A proof independent of the mean value theorem is physically more interesting.
   Suppose there is a maxima of the potential at a point P. Then there exist a neighbourhood of the point over which the potential is lower than that at the point. Let S be the closed surface of this neighbourhood. The potential at every point over the surface S is lower than the potential at P. The electric field lines over the surface is thus directed outward every where as shown in the figure. So $\oint_S \vec{E} \cdot \hat{n} da > 0$ since $\vec{E}$



   is directed outward everywhere over the surface. By Gauss' law this implies there is a non-zero charge enclosed within the surface S. This contradicts the fact that the region is chargeless. So we can't have a maxima at the point P. Similarly we can't have a minima at P.

5. A chargeless region is bounded by two conducting surfaces.

   (a) If a charge Q_1 is placed on conductor 1 while 2 is chargeless the potential in the region is given by the function $\Phi_1(x, y, z)$. If a charge Q_2 is placed on conductor 2 while 1 is chargeless the potential in the region is given by the function $\Phi_2(x, y, z)$. Now if charge Q_1 is placed on conductor 1 and charge Q_2 is placed on 2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.

   soln
   Both Φ_1 and Φ_2 makes the surfaces of the two conductors S_1 and S_2 equipotential. Hence the potential function $\Phi = \Phi_1 + \Phi_2$ also makes the two surfaces equipotential. This is a necessary boundary condition for any solution of the electrostatic problem. Now if this function also gives the given charges on the two conductors then this will satisfy all the required physical boundary conditions of the problem. For this we need the electric field.
   Let $\vec{E}_1 = -\vec{\nabla}\Phi_1$. Then over the surfaces S_1 and S_2 of conductor 1 and 2 we have

$$\oint_{S_1} \vec{E}_1 \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E}_1 \cdot \hat{n} da = 0$$

Similarly for $\vec{E}_2 = -\vec{\nabla}\Phi_2$ we have

$$\oint_{S_1} \vec{E}_2 \cdot \hat{n} da = 0 \quad \text{and} \quad \oint_{S_2} \vec{E}_2 \cdot \hat{n} da = Q_2/\epsilon_0$$

Let $\vec{E} = \vec{E}_1 + \vec{E}_2 = -\vec{\nabla}\Phi$. From the above eqns. we can see that

$$\oint_{S_1} \vec{E} \cdot \hat{n} da = Q_1/\epsilon_0 \quad \text{and} \quad \oint_{S_2} \vec{E} \cdot \hat{n} da = Q_2/\epsilon_0$$

So the electric field caused by the potential Φ satisfies the required boundary conditions. Hence $\Phi = \Phi_1 + \Phi_2$ satisfies all the required physical conditions, i.e, it makes the conducting surfaces equipotential and gives the correct charges over the conductors. The uniqueness theorem suggests that such solutions are unique and hence nature adopts this potential function in the region bounded by the conductors. Hence we have found the required solution for the problem.

(b) If conductor 1 is maintained at potential V_1 and 2 is grounded the potential in the region is given by the function $\Phi_1(x, y, z)$. If conductor 2 is maintained at potential V_2 and 1 is grounded the potential in the region is given by the function $\Phi_2(x, y, z)$.
Now if conductor 1 is maintained at potential V_1 and conductor 2 is maintained at potential V_2 prove that the potential in the region will be given by the function $\Phi = \Phi_1 + \Phi_2$.

soln:
In the chargeless region

$$\nabla^2\Phi = \nabla^2\Phi_1 + \nabla^2\Phi_2 = 0 + 0 = 0$$

So $\Phi = \Phi_1 + \Phi_2$ satisfies the Laplace's equation in the region. We have to verify whether Φ matches the boundary condition, i.e it matches the potential on the two conductors.
Over conductor 1, $\Phi_1 = V_1, \Phi_2 = 0$. So $\Phi = V_1$. Over conductor 2, $\Phi_1 = 0, \Phi_2 = V_2$. So $\Phi = V_2$. So this potential satisfies the potentials at the two bounding surfaces. Hence $\Phi = \Phi_1 + \Phi_2$ is a solution to the given electrostatic problem.
Here we assume that the potential at ∞ is 0.

This result can be extended to a region bounded by any number of conductors.

6. A conducting sphere of radius a is concentrically surrounded by another conducting spherical shell of radius b.

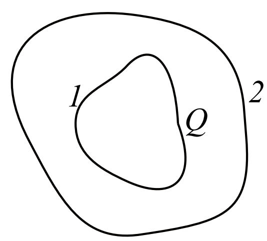
(a) A charge Q is placed on the inner conducting sphere. What will be the potential over the outer sphere.

soln:
For points outside the inner sphere the potential is $\frac{Q}{4\pi\epsilon_0 r}$. So the potential over the outer sphere will be $\frac{Q}{4\pi\epsilon_0 b}$.

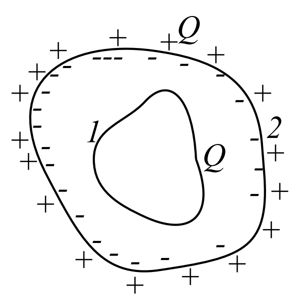
(b) Instead if the charge Q is placed over the outer shell, what will be the potential of the inner sphere?

soln:
If the charge is placed on the outer sphere the potential of the outer sphere will be $\frac{Q}{4\pi\epsilon_0 b}$. Since there is no charge within the outer sphere, the electric field within the sphere everywhere will be 0 and the potential will be constant. This potential will be same as that of the outer sphere and hence it will be $\frac{Q}{4\pi\epsilon_0 b}$.

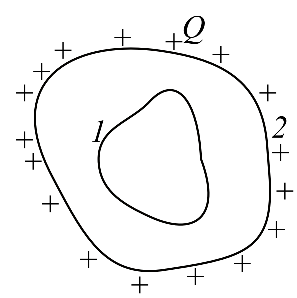
(c) How will your answer change if the shapes of the conductors were not spherical but arbitrary.



soln:



Configuration 1

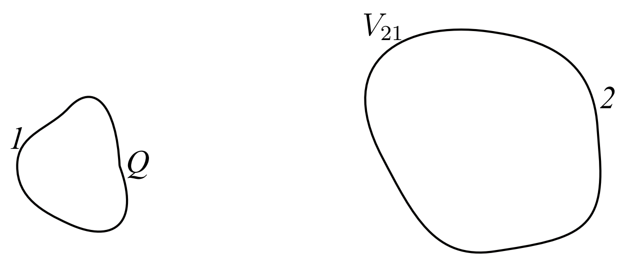


Configuration 2

In part (a) and (b) we see that whether the charge Q is placed on the inner or the outer sphere, the potential on the other sphere happens to be the same viz. $\frac{Q}{4\pi\epsilon_0 b}$. We could explicitly calculate the potential in the case of spherical conductors. When the shape of the conductors is arbitrary, we can't calculate the potential but we can make certain statements about the potentials on the two conducting surfaces in these configurations.
First let us consider the charge Q placed on the outer conductor. This is shown in configuration 2 in the figure. This will make the potential over the surface of the outer conductor V. As there is no charge within this surface, there is no electric field within and hence this region is equipotential with the value V. So the potential over the surface of the inner conductor is also V.

When the charge Q is placed on the inner conductor as shown in configuration 1, surface charge $-Q$ is induced over the inner surface of the outer conductor. This shields the outer world of the charge Q placed on the inner conductor. Since the total charge on the outer conductor is 0, an amount of charge Q spreads itself over the outer surface of the outer conductor. Since this charge can't see the existence of the charge Q on the inner conductor due to shielding by the negative charges, it spreads itself over the outer surface exactly in the same fashion as in the first case where wee placed an amount of charge Q on the outer conductor. So the potential on the outer conductor will be V. Note however that now the potential on the inner conductor is not V as the whole region within the outer conductor is not equipotential.

(d) In general if we have two conducting surfaces S_1 and S_2, when a charge Q is placed on conductor 1, the potential on conductor 2 is found to be V_{21}. Whereas when the charge Q is placed on conductor 2 the potential on conductor 1 is found to be V_{12}. Prove that $V_{12} = V_{21}$.



soln:

Let Φ_1 be the potential function in the region bounded by the two conductors when the charge Q is placed on the first conductor and Φ_2 be the potential function when the charge is placed on the second conductor. If we denote the region bounded by the two conductors as τ then

$$\begin{aligned}
\int_\tau \vec{\nabla}\cdot(\Phi_1\vec{\nabla}\Phi_2)d\tau &= \int_\tau \vec{\nabla}\Phi_1\cdot\vec{\nabla}\Phi_2 d\tau + \int_\tau \Phi_1\vec{\nabla}^2\Phi_2 d\tau \\
\therefore \quad \oint_{S_1+S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da &= \int_\tau \vec{\nabla}\Phi_1\cdot\vec{\nabla}\Phi_2 d\tau + \int_\tau \Phi_1\vec{\nabla}^2\Phi_2 d\tau
\end{aligned}$$

Since there are no volume charge density in the region τ, the second integral on the r.h.s is 0 as $\vec{\nabla}^2\Phi_2 = 0$ by Gauss' law. The integral on the l.h.s is easy to evaluate since over the surfaces S_1 and S_2 the potentials are constant they being surfaces of conductors.

$$\begin{aligned}
\oint_{S_1+S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da &= \oint_{S_1} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da + \oint_{S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da \\
&= 0 + V_{12}Q
\end{aligned}$$

The first integral n r.h.s is 0 since $\oint_{S_1}(\vec{\nabla}\Phi_2)\cdot\hat{n}da$ gives the total charge on the first conductor. Since the charge Q is placed on the second conductor this integral is 0. Over the surface S_2 of the second conductor, $\oint_{S_2}\Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da = V_{12}Q$ as $\Phi_1 = V_{12}$ is the potential on the second conductor when Q is placed on the first conductor.

$$\therefore \quad \oint_{S_2} \Phi_1(\vec{\nabla}\Phi_2)\cdot\hat{n}da = V_{12}\oint_{S_2}\vec{\nabla}\Phi_2\cdot\hat{n}da = V_{12}Q$$

Putting everything together we see that

$$V_{12}Q = \int_\tau \vec{\nabla}\Phi_1 \cdot \vec{\nabla}\Phi_2 d\tau$$

Similarly considering $\int_\tau \vec{\nabla} \cdot (\Phi_2 \vec{\nabla}\Phi_1) d\tau$ we can show that

$$V_{21}Q = \int_\tau \vec{\nabla}\Phi_2 \cdot \vec{\nabla}\Phi_1 d\tau$$

As the integrals on the r.h.s are the same we conclude

$$V_{12}Q = V_{21}Q \quad \Longrightarrow V_{12} = V_{21}$$

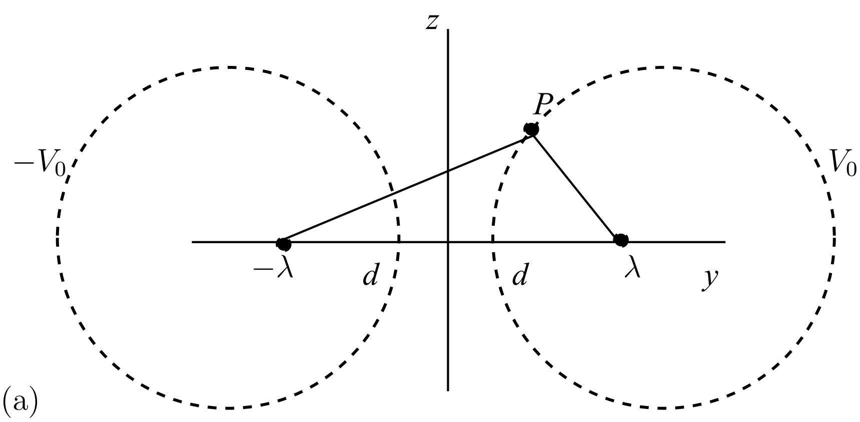
7. Two infinitely long wires running parallel to the x axis carry uniform charge densities $+\lambda$ and $-\lambda$.

   (a) Find the potential at any point using the origin as the reference.

   (b) Show that the equipotential surfaces are circular cylinders. Locate the axis and radius of the cylinder corresponding to a given potential V_0.

soln
The two line charges $+\lambda$ and $-\lambda$, parallel to the x axis cuts the y axis at $y = -d$ and $y = d$ respectively. Consider a point $P(x, y, z)$. The distance of P from the line charges are s_1 and s_2 given by

$$s_1 = \sqrt{(y+d)^2 + z^2} \qquad \text{and} \qquad s_2 = \sqrt{(y-d)^2 + z^2}$$



(a)

The potential at point P is

$$V = \frac{-\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_1}{k_1}\right) + \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s_2}{k_2}\right) = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\left(\frac{s_2}{s_1}\right) + \ln\left(\frac{k_1}{k_2}\right)\right]$$

If we want $V = 0$ at the origin where $s_1 = s_2 = d$ then $k_1 = k_2$. So

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\sqrt{(y-d)^2+z^2}}{\sqrt{(y+d)^2+z^2}}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{(y-d)^2+z^2}{(y+d)^2+z^2}\right)$$

(b) Consider a surface of constant potential V_0. Then from the above eqn. we have

$$(y-d)^2 + z^2 = \left[(y+d)^2 + z^2\right] K \qquad \text{where} \qquad K = \exp\left[\frac{4\pi\epsilon_0 V_0}{\lambda}\right]$$

$$\begin{aligned} \therefore \quad y^2(1-K) + z^2(1-K) - 2yd(1+K) + d^2(1-K) &= 0 \\ \therefore \quad y^2 + z^2 - 2yd\left(\frac{1+K}{1-K}\right) + d^2 &= 0 \end{aligned}$$

This is the equation of a circle $(y-y_0)^2 + z^2 = R^2$ with center at $(y_0, 0) = (d\left(\frac{1+K}{1-K}\right), 0)$ and radius $R = \frac{2d\sqrt{K}}{1-K}$.