1. Find the equation of the tangent plane to the ellipsoid
   $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_0, y_0, z_0) on the ellipsoid.

   soln:
   Consider a scalar function $f(x, y, z)$. Consider a surface over which f is constant, say, $f(x, y, z) = k$. $\vec{\nabla} f$ at a point on this surface is normal to this surface. This normal will also be normal to the tangent plane at this point.
   We have $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\vec{\nabla} f = 2 \left(\hat{i} \frac{x}{a^2} + \hat{j} \frac{y}{b^2} + \hat{k} \frac{z}{c^2} \right) \tag{1}$$

   At (x_0, y_0, z_0), $\vec{\nabla} f = 2 \left(\hat{i} \frac{x_0}{a^2} + \hat{j} \frac{y_0}{b^2} + \hat{k} \frac{z_0}{c^2} \right)$.
   Every vector on the tangent plane is perpendicular to this vector. So the equation of the tangent plane is given by

$$(x - x_0)\frac{x_0}{a^2} + (y - y_0)\frac{y_0}{b^2} + (z - z_0)\frac{z_0}{c^2} = 0$$
$$\text{i.e} \qquad \frac{x_0}{a^2}x + \frac{y_0}{b^2}y + \frac{z_0}{c^2}z = 1$$

2. Prove that for any vector field $\vec{A}$, $\vec{\nabla} \cdot \vec{A}$ is a scalar.

   soln:
   Let us denote the coordinates before and after rotation as

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

   $X' = RX$ where R is the rotation matrix.

$$\therefore \ x_i' = \sum_j R_{ij} x_j \quad \text{and} \quad x_i = \sum_j (R^T)_{ij} x_j' = \sum_j R_{ji} x_j' \tag{2}$$

   Here we have used the fact that $R^{-1} = R^T$ i.e. R is an orthogonal matrix. We have to show that $\vec{\nabla}' \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A}$, i.e

$$\sum_i \frac{\partial A_i'}{\partial x_i'} = \sum_i \frac{\partial A_i}{\partial x_i}$$

$$\frac{\partial}{\partial x_i'} = \sum_j \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x_i'} = \sum_j R_{ij} \frac{\partial}{\partial x_j} \quad \text{from Eq.(2)}$$

   $A_i' = \sum_k R_{ik} A_k$ since $\vec{A}$ is a vector.

$$\vec{\nabla}' \cdot \vec{A}' = \sum_i \frac{\partial A_i'}{\partial x_i'} = \sum_i \sum_j R_{ij} \frac{\partial}{\partial x_j} \left(\sum_k R_{ik} A_k \right)$$

$$= \sum_j \sum_k \left(\sum_i R_{ij} R_{ik} \right) \frac{\partial A_k}{\partial x_j}$$

Now $\sum_i R_{ij} R_{ik} = \sum_i (R^T)_{ji} R_{ik} = \delta_{jk}$ since $R^T R = \mathbb{I}$.
So only $j = k$ terms survive in the above summation

$$\therefore \quad \vec{\nabla}' \cdot \vec{A}' = \sum_j \frac{\partial A_j}{\partial x_j} = \vec{\nabla} \cdot \vec{A}$$

So the divergence of a vector is invariant under rotation. Hence it is a scalar quantity. This proof is valid for the divergence operator in any dimension since we have only used the orthogonality of R.
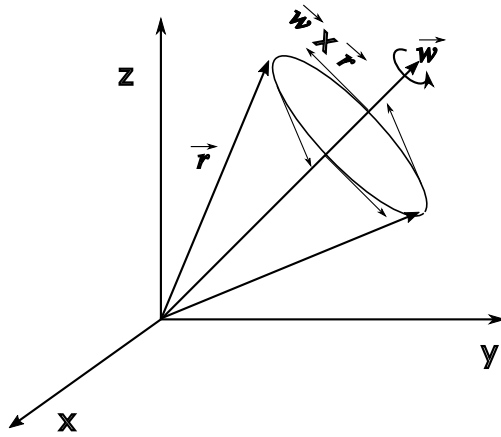
3. Let $\vec{A} = \vec{\omega} \times \vec{r}$ where $\vec{\omega}$ is a fixed vector in space. Find $\vec{\nabla} \times \vec{A}$.

   soln:

$$\vec{A} = \vec{\omega} \times \vec{r} = \hat{i}(\omega_y z - \omega_z y) + \hat{j}(\omega_z x - \omega_x z) + \hat{k}(\omega_x y - \omega_y x)$$

$$\therefore \vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \hat{i}\left(\omega_x - (-\omega_x)\right) + \hat{j}\left(\omega_y - (-\omega_y)\right) + \hat{k}\left(\omega_z - (-\omega_z)\right) = 2\vec{\omega}$$

Interpretation:
The vector field $\vec{\omega} \times \vec{r}$ curls around the vector $\vec{\omega}$. It is the velocity vector of the particles of a rigid body rotating with angular velocity $\vec{\omega}$.

4. Find the divergence of the following:

(a) $\vec{A} = \hat{r}$,

(b) $\vec{A} = \frac{\hat{r}}{r}$ in 2 dimension

(c) $\vec{A} = \frac{\hat{r}}{r}$ in 3 dimension

(d) $\vec{A} = \frac{\hat{r}}{r^2}$ in 3 dimension. Plot this field.

(e) $\vec{A} = \frac{\hat{r}}{r^3}$ in 3 dimension

soln:

(a) $\vec{A} = \hat{r} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}; r = \sqrt{x^2 + y^2 + z^2}$
$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
$\frac{\partial A_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = -\frac{x^2}{r^3} + \frac{1}{r}$
Similarly $\frac{\partial A_y}{\partial y} = -\frac{y^2}{r^3} + \frac{1}{r}$ and $\frac{\partial A_z}{\partial z} = -\frac{z^2}{r^3} + \frac{1}{r}$.
$\therefore \vec{\nabla} \cdot \vec{A} = -\frac{x^2+y^2+z^2}{r^3} + 3/r = 2/r.$

For the remaining parts we evaluate the following:

$$
\begin{aligned}
\vec{\nabla} \cdot (r^n \hat{r}) &= r^n \vec{\nabla} \cdot \hat{r} + \vec{\nabla}(r^n) \cdot \hat{r} \\
&= r^n \left(\frac{2}{r}\right) + n r^{n-1} \hat{r} \cdot \hat{r} \quad \text{(in 3 dim.)} \\
&= (n+2)r^{n-1} \\
\vec{\nabla} \cdot (r^n \hat{r}) &= r^n \left(\frac{1}{r}\right) + n r^{n-1} \hat{r} \cdot \hat{r} \quad \text{(in 2 dim.)} \\
&= (n+1)r^{n-1}
\end{aligned}
$$

(b) $\vec{A} = \frac{\hat{r}}{r}$ in 2 dim. $(n = -1)$
$\vec{\nabla} \cdot \vec{A} = (-1+1)r^{-2} = 0.$
(This is true only for $r \neq 0$. At $r = 0, \vec{\nabla} \cdot \vec{A} \to \infty$)

(c) $\vec{A} = \frac{\hat{r}}{r}$ in 3 dim. $(n = -1)$
$\vec{\nabla} \cdot \vec{A} = (-1+2)r^{-2} = 1/r^2$

(d) $\vec{A} = \frac{\hat{r}}{r^2}$ in 3 dim. $(n = -2)$
$\vec{\nabla} \cdot \vec{A} = (-2+2)r^{-3} = 0.$
(This is true only for $r \neq 0$. At $r = 0, \vec{\nabla} \cdot \vec{A} \to \infty$)

(e) $\vec{A} = \frac{\hat{r}}{r^3}$ in 3 dim. $(n = -3)$
$\vec{\nabla} \cdot \vec{A} = (-3+2)r^{-4} = -1/r^4$

All the fields above are radially outward. But the divergence changes from positive to 0 to negative. Explain this.

5. Find the curl of the following:

(a) $\vec{A} = y\hat{i} - x\hat{j}$

(b) $\vec{A} = \frac{1}{\sqrt{x^2+y^2}}(y\hat{i} - x\hat{j})$

(c) $\vec{A} = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$

(d) $\vec{A} = (x^2 + y^2)\hat{k}$

soln:

(a) $\vec{A} = y\hat{i} - x\hat{j}$
   soln
   $\vec{\nabla} \times \vec{A} = \hat{k}(-1 - 1) = -2\hat{k}$

(b) $\vec{A} = \frac{1}{\sqrt{x^2+y^2}}(y\hat{i} - x\hat{j})$
   soln:
   We first evaluate $\vec{\nabla} \times \left[(x^2 + y^2)^n (y\hat{i} - x\hat{j}) \right]$ which will help us work out the other parts.

$$\vec{\nabla} \times \left[(x^2 + y^2)^n (y\hat{i} - x\hat{j}) \right] = (x^2 + y^2)^n \vec{\nabla} \times (y\hat{i} - x\hat{j}) + \vec{\nabla}(x^2 + y^2)^n \times (y\hat{i} - x\hat{j}) \quad \text{product rules}$$
$$= (x^2 + y^2)^n(-2\hat{k}) + n(x^2 + y^2)^{n-1}(2x\hat{i} + 2y\hat{j}) \times (y\hat{i} - x\hat{j})$$
$$= -2(x^2 + y^2)^n\hat{k} + n(x^2 + y^2)^{n-1}(-2x^2 - 2y^2)\hat{k}$$
$$= -2(x^2 + y^2)^n(1 + n)\hat{k}$$

For $\vec{A} = (y\hat{i} - x\hat{j})/\sqrt{x^2 + y^2}$, $\quad n = -1/2$.

$$\vec{\nabla} \times \vec{A} = -\frac{1}{\sqrt{x^2 + y^2}}\hat{k}$$

This is not differentiable at $x = y = 0$, i.e along the z axis.

(c) $\vec{A} = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$
   soln
   Here $n = -1$.
   $$\therefore \quad \vec{\nabla} \times \vec{A} = 0$$

This is not valid along the z axis.

(d) $\vec{A} = (x^2 + y^2)\hat{k}$
   soln

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y}\right) + \hat{j}\left(-\frac{\partial A_z}{\partial x}\right)$$
$$= 2y\hat{i} - 2x\hat{j} = 2(y\hat{i} - x\hat{j})$$

6. For any vector field $\vec{A}$ and any scalar field F show that
   (i) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$;   (ii) $\vec{\nabla} \times (\vec{\nabla}F) = 0$.

soln:

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(i) $\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

$\therefore\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0.$

(ii) $\vec{\nabla} \times \vec{\nabla} F = \hat{i}\left(\frac{\partial^2 F}{\partial y \partial x} - \frac{\partial^2 F}{\partial x \partial y}\right) + \ldots\ldots = 0$

7. Can we find a scalar function F such that $\vec{\nabla}F = y\hat{i} - x\hat{j}$?
What about $\vec{\nabla}F = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$?

soln
$\vec{\nabla} \times \vec{\nabla}F = \vec{\nabla} \times (y\hat{i} - x\hat{j}) = -2\hat{k}$
But curl of the gradient of any scalar field must be zero. So there exist no such F such that $\vec{\nabla}F = y\hat{i} - x\hat{j}$

If $\vec{\nabla}F = \frac{1}{x^2+y^2}(y\hat{i} - x\hat{j})$ it can be shown that $\vec{\nabla} \times (\vec{\nabla}F) = 0$ at almost all places but the result is not applicable when $x = y = 0$ which is the z axis. In fact it can be shown that the $\vec{\nabla} \times (\vec{\nabla}F) \neq 0$ along the z axis. The scalar function F, that we find, steadily increases along $\vec{\nabla}F$. Hence along a circle around the origin, the plot of given $\vec{\nabla}F$ shows that F steadily increases. After completing a full circle when we come to the same point we arrive at a different value of F. This makes F discontinuous, and hence not differentiable. Let us find a F and demonstrate this phenomenon. We have the following system of differential equations

$$\frac{\partial F}{\partial x} = \frac{y}{x^2 + y^2} \qquad ; \qquad \frac{\partial F}{\partial y} = \frac{-x}{x^2 + y^2}$$

The first one gives $F(x,y) = \tan^{-1}(x/y) + g(y)$ while the second equation gives $F(x,y) = -\tan^{-1}(y/x) + h(x) = -(\pi/2 - \tan^{-1}(x/y)) + h(x)$.
For consistency we need $g(y)$ and $h(x)$ to be constants, say, c_1 and c_2.
$c_2 - c_1 = \pi/2$. We can take $c_1 = 0$ and $c_2 = \pi/2$. This gives
$F(x,y) = \tan^{-1}(x/y)$.
So it appears that This function is not continuous. To see this we reparametrize $x = r\sin\theta$, $y = r\cos\theta$. Then we get $F(x,y) = \theta$. As we go around a circle of radius 1, starting at $(0,1)$ and come back to the same point the value of the function changes from 0 to 2π. So it is discontinuous at $(0,1)$, hence, not differentiable at this point which lies on the y axis. In fact it is not differentiable at any point on the y axis. Hence the given vector field cannot be the gradient of a scalar function.