1. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Find $\vec{\nabla} f$. Find the rate of change of f at the point $(1,1,0)$ along a direction specified by the unit vector $\frac{1}{\sqrt{2}}(\hat{\mathrm{i}} - \hat{\mathrm{j}})$.

soln
$f = \sqrt{x^2+y^2+z^2} = r$.

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\begin{aligned} \vec{\nabla} f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r} \end{aligned}$$

At (1,1,0), $\vec{\nabla} f = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$.
Let $\hat{n} = \frac{\hat{i}-\hat{j}}{\sqrt{2}}$.
Let $\vec{dr} = dr\hat{n}$.
$\therefore\ df = \vec{\nabla} f \cdot \vec{dr} = \vec{\nabla} f \cdot \hat{n} dr$.
$\therefore\ \frac{df}{dr} = \vec{\nabla} f \cdot \hat{n} = \frac{\hat{i}+\hat{j}}{\sqrt{2}} \cdot \frac{\hat{i}-\hat{j}}{\sqrt{2}} = 0$

2. Let $\vec{\mathrm{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z). Show that
(a) $\vec{\nabla}(1/r) = -\hat{\mathrm{r}}/r^2$
(b) Evaluate $\vec{\nabla}(r^n)$

soln
(a)
$f(\vec{r}) = 1/r$.
$\therefore\quad \vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$.
$\frac{\partial f}{\partial x} = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\left(\frac{x-x'}{r}\right) = -\frac{x-x'}{r^3}$
Similarly $\frac{\partial f}{\partial y} = -\frac{y-y'}{r^3}$ and $\frac{\partial f}{\partial z} = -\frac{z-z'}{r^3}$
$\therefore\ \vec{\nabla} f = -\frac{(x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}}{r^3} = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$.

(b)
Let $f(\vec{r}) = r^n$. Then

$$\frac{\partial f}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x-x'}{r} = nr^{n-2}(x-x')$$

$\therefore\ \vec{\nabla} f = nr^{n-2}((x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}) = nr^{n-1}\hat{r}$.

3. Find the gradient of the function $f(\vec{\mathrm{r}}) = \sin(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})$ where $\vec{\mathrm{k}}$ is a fixed vector. Why do you think is the direction of gradient vector fixed in space?

soln:

$$\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial(\vec{k} \cdot \vec{r})}\frac{\partial(\vec{k} \cdot \vec{r})}{\partial x} \\ &= \cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})k_x \end{aligned} \tag{1}$$

Similarly $\frac{\partial f}{\partial y} = \cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})k_y$ and $\frac{\partial f}{\partial z} = \cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})k_z$.
Putting all these together we get

$$\begin{aligned} \vec{\nabla} f &= \hat{i}\cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})k_x + \hat{j}\cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})k_y + \hat{k}\cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})k_z \\ &= \cos(\vec{\mathrm{k}} \cdot \vec{\mathrm{r}})\vec{k} \end{aligned}$$

The magnitude of the gradient of f changes from point to point. But the direction of the gradient is along $\vec{k}$ which is a fixed vector.
The value of f doesn't change over a surface defined by $\vec{k} \cdot \vec{r}$ =constant. This is the equation of a plane whose normal is along $\vec{k}$. So the function changes at the maximum rate along $\vec{k}$. That is the direction of the gradient. This is how plane wave fronts are made.

4. A real square matrix M is orthogonal if $M^{-1} = M^T$. Using the fact that the magnitude of a vector doesn't change under rotation prove that a rotation matrix is orthogonal.

soln:
Let R be a rotation matrix.
Let $\vec{A}' = R\vec{A}$ and $\vec{B}' = R\vec{B}$. Let us denote the matrix representation of $vecA$ as $[A] = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$. Then $[A'] = R[A]$ and $[B'] = R[B]$.
In the matrix representation $\vec{A} \cdot \vec{B} = [A]^T[B]$.
Since $\vec{A} \cdot \vec{B}$ is a scalar we have $\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$, i.e , $[A']^T[B'] = [A]^T[B]$. So we have

$$[A]^T R^T R[B] = [A]^T[B]$$

If this has to be true for any arbitrary vector A and B then the only possibility is $R^T R = \mathbb{I}$, i.e $R^{-1} = R^T$. So R is an orthogonal matrix.

5. *This question tries to give an idea of what a scalar quantity is.*
The electric potential at a point on a horizontal plate with respect to a given coordinate

system is given as $V(x, y) = xy$. If someone work with a coordinate system that is rotated by 45°, the new coordinates (x', y') are given in terms of the old ones as $x' = \frac{x+y}{\sqrt{2}}$ and $y' = \frac{y-x}{\sqrt{2}}$. Let's write this as $\vec{r'} = R\vec{r}$. Potential is a scalar quantity. If $V'(x', y')$ is the functional form of the potential function in the new coordinate system then $V'(x', y') = V(x, y)$.

(a) Find the form of the function $V'(x', y')$.

(b) Verify that $\vec{\nabla}'V' = R\vec{\nabla}V$, i.e., components of a gradient transform as a vector quantity.

soln:

(a) The relation between the coordinates (x', y') and (x, y) is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$

Inverting the above relation we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{aligned} V(x, y) &= xy = \frac{1}{2}(x' - y')(x' + y') = \frac{1}{2}(x'^2 - y'^2) \\ \therefore \quad V'(x', y') &= \frac{1}{2}(x'^2 - y'^2) \end{aligned}$$

(b)

$$\vec{\nabla}V(x, y) = y\hat{i} + x\hat{j} \equiv \begin{pmatrix} A_x \\ A_y \end{pmatrix}, \quad \vec{\nabla}'V'(x', y') = x'\hat{i}' - y'\hat{j}' \equiv \begin{pmatrix} A'_x \\ A'_y \end{pmatrix}$$

So we have

$$\begin{aligned} \begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} x' \\ -y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x + y \\ x - y \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} A_y + A_x \\ A_y - A_x \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \end{aligned}$$

$\therefore \ \vec{\nabla}'V' = R\vec{\nabla}V$

6. Let

$$D = \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix}$$

Under a rotation of the coordinate system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$$

show that

$$D' = \begin{pmatrix} \frac{\partial A'_x}{\partial x'} & \frac{\partial A'_y}{\partial x'} \\ \frac{\partial A'_x}{\partial y'} & \frac{\partial A'_y}{\partial y'} \end{pmatrix} = RDR^T$$

soln
D can be wrtten as

$$D = \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} (A_x \ A_y)$$

The first column matrix in the above product is the $\vec{\nabla}$ operator which we have seen transforms as a vector under the rotation R. So

$$\vec{\nabla}' = \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = R \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

Also the vector $\vec{A}$ transforms as $\vec{A}' = R\vec{A}$.

$$\begin{aligned} \therefore \quad D' &= \begin{pmatrix} \frac{\partial A'_x}{\partial x'} & \frac{\partial A'_y}{\partial x'} \\ \frac{\partial A'_x}{\partial y'} & \frac{\partial A'_y}{\partial y'} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} (A'_x \ A'_y) \\ &= R \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} (A_x \ A_y) R^T \\ &= R \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} \end{pmatrix} R^T = RDR^T \end{aligned}$$