SC223 - Linear Algebra

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Lecture 8

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LU Decomposition

● Any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed into a product of lower and upper triangular matrices, with appropriate permutations:

$$PA = LU,$$

where $P \in \mathbb{R}^{m \times m}, L \in \mathbb{R}^{m \times m}, U \in \mathbb{R}^{m \times n}$.

● Solving linear equations with LU decomposition:
$Ax = b \to LUx = b$.

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▶ In the example, where is y?

$$\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 0 & -\mathbf{23} & -16 & 2 & -12 \\ 0 & 0 & 0 & -\mathbf{137} & -95 & 64 \end{array}\right]$$

Example

- $U = \begin{bmatrix} \mathbf{1} & -1 & 2 & 3 & -1 \\ 0 & \mathbf{2} & 5 & 4 & 2 \\ 0 & 0 & -\mathbf{23} & -16 & 2 \\ 0 & 0 & 0 & -\mathbf{137} & -95 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 2 \\ -12 \\ 64 \end{bmatrix}.$

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LU Decomposition Algorithm

- Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$
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  - Let $L_1 A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^1 & \dots & a_{2n}^1 \\ 0 & a_{32}^1 & \dots & a_{3n}^1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & a_{n2}^1 & \dots & a_{nn}^1 \end{bmatrix}$

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- If $L_1 x_1 = e_1$, then $L_2 L_1 x_1 = e_1$.
- Thus, $L = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \dots & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a_{32}^1}{a_{22}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{a_{n1}}{a_{11}} & \frac{a_{n2}^1}{a_{22}} & \dots & 1 \end{bmatrix}$

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- Why should one use LU decomposition?