Boolean Algebra

Set

$B = \{0, 1\}$ $\bar{0} = 1$ $\bar{1} = 0$ complement

Boolean sum (+ or OR) $1+1=1$, $1+0=1$, $0+1=1$, $0+0=0$

Boolean × (· or AND) $1\cdot1=1$, $1\cdot0=0$, $0\cdot1=0$, $0\cdot0=0$

Example $1\cdot0+\overline{(0+1)} = 0+\bar{1} = 0+0 = 0$

Boolean Expressions and Boolean functions

$B = \{0,1\}$ $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B\}$, $1 \le i \le n$

— A variable x is called Boolean variable if it assumes values only from B. i.e, its only possible values are 0 & 1.

— A fn. from $B^n \to B$ is called a Boolean fn. of deg n. (B.F.)

Example $F(x,y) = xy$ is a B.F. of deg 2 $\forall (x,y) \in B^2$

— Boolean Expressions (B.E.) in variables $x_1, x_2, \ldots, x_n$ are defined recursively as

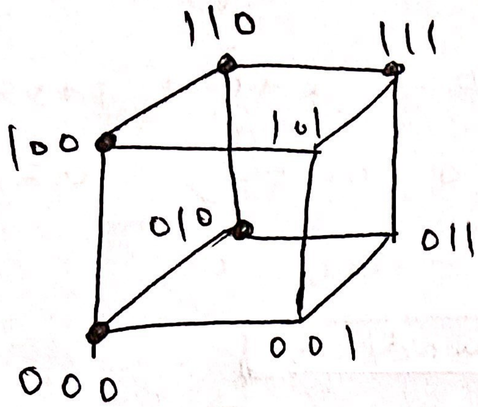
① $0, 1, x_1, x_2, \ldots, x_n$ are B.E.

② If E_1 & E_2 are B.E. $\Rightarrow \bar{E_1}$, $(E_1 E_2)$ & (E_1+E_2) are B.E.

Example Find the values of the B.F. $F(x,y,z) = xy + \bar{z}$

x	y	z	xy	$\bar{z}$	$F(x,y,z) = xy+\bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

— B.F. can be rep^d by a n-cube that corresponds to the n-tuples of bits where the fn has value 1.



$F(x,y,z) = xy + \bar{z}$ ←

— B.F.s F & G of n-variables are equal iff $F(b_1,\dots,b_n) = G(b_1,\dots,b_n)$, whenever $b_1, b_2, \dots, b_n \in B$

— The complement of B.F. F is the fn. $\bar{F}$
$$\bar{F}(x_1,\dots,x_n) = \overline{F(x_1,x_2,\dots,x_n)}$$

— The Boolean sum F+G is defined as
$$(F+G)(x_1,\dots,x_n) = F(x_1,x_2,\dots,x_n) + G(x_1,\dots,x_n)$$

— The Boolean × FG is defined as
$$(FG)(x_1,x_2,\dots,x_n) = F(x_1,\dots,x_n)\, G(x_1,\dots,x_n)$$

$\boxed{\text{Example}}$ How many B.F. of deg n there?
$f: B^n \to B$ Since $|B^n| = 2^n$ each of these n-tuples assigned 0 or 1 $\Rightarrow$ there are 2^{2^n} B.F. of deg n. ■

$\boxed{\text{Identities of Boolean Algebra}}$

$\boxed{\text{Example}}$ Show that $x(y+z) = xy + xz$

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

∵ col 7 = col 8 identity is valid.

Boolean Identities

$\overline{\overline{x}} = x$ Law of the double complement

$x + x = x$, $x \cdot x = x$ } Idempotent laws

$x + 0 = x$, $x \cdot 1 = x$ } Identity laws

$x + 1 = 1$, $x \cdot 0 = 0$ } Domination laws

$x + y = y + x$, $xy = yx$ } Commutative laws

$x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$ } Associative laws

$x + yz = (x + y)(x + z)$, $x(y + z) = xy + xz$ } Distributive laws

$\overline{(xy)} = \overline{x} + \overline{y}$, $\overline{(x + y)} = \overline{x}\,\overline{y}$ } De Morgan's laws

$x + xy = x$, $x(x + y) = x$ } Absorption laws

$x + \overline{x} = 1$ } Unit property

$x\overline{x} = 0$ } Zero property

$\boxed{\text{Example}}$ $x + yz = (x + y)(x + z)$

We can convert it into logical equivalence of prop. variables $x \to p$, $y \to q$, $z \to r$

$+ \to$ disjunction

$\cdot \to$ conjunction

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Boolean Algebra

A Boolean Algebra (B.A.) is a set B with two binary operations $\vee$ & $\wedge$, elements 0 & 1 and a unary operation $-$ s.t. following properties holds $\forall\, x, y, z \in B$

$$\langle \underset{\{0,1\}}{\underset{\shortparallel}{B}}, \vee, \wedge, - \rangle$$

① $\left.\begin{array}{l} x \vee 0 = x \\ x \wedge 1 = x \end{array}\right\}$ Identity laws

② $\left.\begin{array}{l} x \vee \bar{x} = 1 \\ x \wedge \bar{x} = 0 \end{array}\right\}$ Complement laws

③ $\left.\begin{array}{l} (x \vee y) \vee z = x \vee (y \vee z) \\ (x \wedge y) \wedge z = x \wedge (y \wedge z) \end{array}\right\}$ Associative laws

④ $\left.\begin{array}{l} x \vee y = y \vee x \\ x \wedge y = y \wedge x \end{array}\right\}$ Commutative laws

⑤ $\left.\begin{array}{l} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{array}\right\}$ Distributive laws

Using above laws one can prove

(i) $x \vee x = x$ and $x \wedge x = x \quad \forall\, x \in B$

(ii) $\bar{0} = 1$ & $\bar{1} = 0$

(iii) $\overline{(x \vee y)} = \bar{x} \wedge \bar{y}$ and $\overline{(x \wedge y)} = \bar{x} \vee \bar{y}$

Example Find B. Expression for B. F. $F(x,y,z)$ & $G(x,y,z)$

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

An expression for F has value 1 when $x = z = 1$ & $y = 0$ & zero otherwise.

$x\bar{y}z = 1$ iff $x = \bar{y} = z = 1$

iff $x = z = 1$ & $y = 0$

$\therefore F(x,y,z) = x\bar{y}z$

An expression for $G = 1$ when $x = y = 1$ & $z = 0$ or when $x = z = 0$ & $y = 1$

Now $xy\bar{z} = 1$ iff $x = y = 1$ & $z = 0$

and

~~$\bar{x}y$~~ $\bar{x}y\bar{z} = 1$ iff $x = z = 0$ & $y = 1$

$\therefore G(x,y,z) = xy\bar{z} + \bar{x}y\bar{z}$

$\because G = 1$ iff $x = y = 1$ & $z = 0$ or $x = z = 0$ & $y = 1$

— A **literal** is a B. variable or its complement

— A **minterm** ~~in a~~ of the B. variables $x_1, x_2, \ldots, x_n$ is a B. product $y_1 y_2 \ldots y_n$, where $y_i = x_i$ or $y_i = \bar{x}_i$

$\therefore$ A minterm is a $\times$ of n literals

Example Find the sum-of-products expansion for the fn. $F(x,y,z) = (x+y)\bar{z}$.

$F(x,y,z) = (x+y)\bar{z}$

$= x\bar{z} + y\bar{z}$ Dist. law

$= x1\bar{z} + 1y\bar{z}$ Identity law

$= x(y+\bar{y})\bar{z} + (x+\bar{x})y\bar{z}$ Unit property

$= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z}$ (Dist law)

$= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$ Idempotent law