1. Verify divergence theorem for the vector function $\vec{A} = \vec{r}$. The region is a spherical surface of radius a with the center at the origin.

   soln:
   On the surface of the sphere the normal is along $\hat{r}$. So the surface integral is

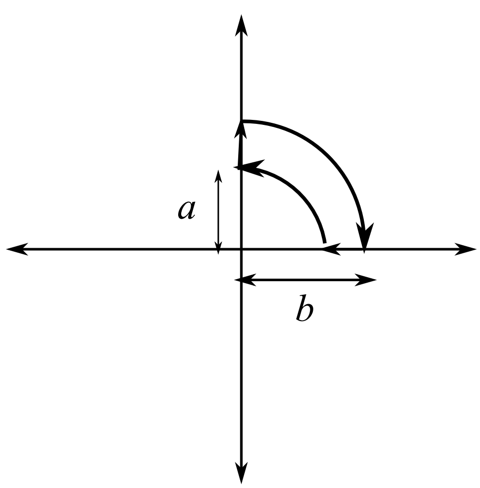
$$\oint_S \vec{A}\cdot\hat{n}ds = \oint_S \vec{r}\cdot\hat{r}ds = \oint_S ads = a\oint_S ds = 4\pi a^3$$

   $\vec{\nabla}\cdot\vec{r} = 3$. So

$$\int_V \vec{\nabla}\cdot\vec{A}dV = \int_V 3dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$

$$\therefore \quad \int_V \vec{\nabla}\cdot\vec{A}dV = \oint_S \vec{A}\cdot\hat{n}ds$$

2. Verify stokes' theorem for the vector field $\vec{A} = (y\hat{i} - x\hat{j})/(x^2+y^2)$ over the region shown in the figure. The loop consists of a quarter arc of two concentric circles of radii a and b and two straight paths along the y and the x axes.



   soln
   At every point on the xy plane the vector field has a magnitude $\sqrt{y^2+x^2}/(x^2+y^2) = 1/\sqrt{x^2+y^2}$.
   Along the inner circular arc the magnitude of $\vec{A}$ is $1/a$ while along the outer circle the magnitude is $1/b$.
   The direction of $\vec{A}$ is tangential to the circular arcs.
   Along the inner circle it is opposite to the direction in which we traverse the circle, i.e,

opposite to $\vec{dl}$.
So along the inner circle

$$\int \vec{A}\cdot\vec{dl} = \int -\frac{1}{a}dl$$

Along the circular arc $dl = ad\theta$

$$\therefore \quad \int_{C_1} \vec{A}\cdot\vec{dl} = \int_0^{\pi/2} -\frac{1}{a}ad\theta = -\frac{\pi}{2}$$

Along the outer arc $\vec{A}$ is along $\vec{dl}$.
$|\vec{A}| = 1/b$ and $dl = bd\theta$.

$$\therefore \quad \int_{C_2} \vec{A}\cdot\vec{dl} = \int_0^{\pi/2} \frac{1}{b}bd\theta = \frac{\pi}{2}$$

So the line integral along these two arcs cancel each other. Along the y axis $\vec{A} = \hat{\imath}/y$. But $\vec{dl} = \hat{\jmath}dl$. So $\vec{A}\cdot\vec{dl} = 0$.
Similarly along the straight path along the x axis $\vec{A}\cdot\vec{dl} = 0$. So the two straight paths don't contribute anything to the loop integrals. So we have

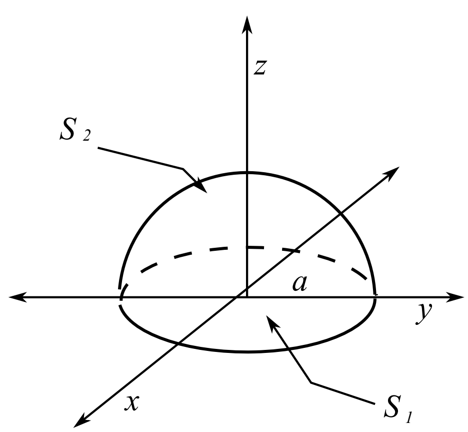
$$\oint \vec{A}\cdot\vec{dl} = -\frac{\pi}{2} + \frac{\pi}{2} = 0$$

$\vec{\nabla}\times\vec{A} = 0$ everywhere except the origin.

$$\therefore \quad \int (\vec{\nabla}\times\vec{A})\cdot\hat{n}ds = 0$$

This satisfies the Stokes' theorem.

3. Verify Stokes' Theorem for the vector field $\vec{A} = (y\hat{\imath} - x\hat{\jmath})$ over a region bounded by a circle of radius a on the xy plane in the following two cases as shown in the figure:



(a) The region is S_1 the flat circular disk of radius a on the xy plane.

(b) The region is S_2 the hemisphere over the xy plane with center at the origin

soln

Let us traverse the circular loop clockwise. Then $\vec{A}$ is along $\vec{dl}$. $|\vec{A}| = \sqrt{x^2+y^2} = a$ along the circle.

$$\therefore \quad \oint \vec{A}\cdot\vec{dl} = \int_0^{2\pi} a^2 d\phi = 2\pi a^2$$

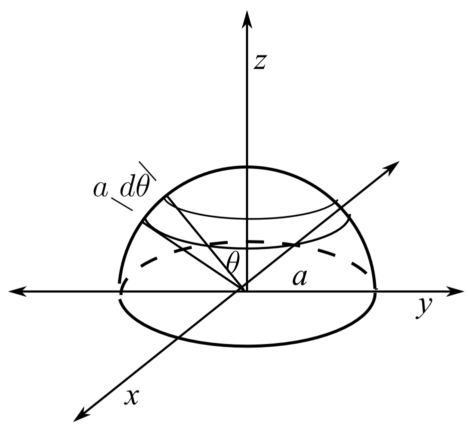
We note that $\vec{\nabla}\times\vec{A} = -2\hat{k}$ everywhere. This is common for both the parts

(a) Along the surface S_1 the normal is along $-\hat{k}$ everywhere.

$$\therefore \quad \int_{S_1} (\vec{\nabla}\times\vec{A})\cdot\hat{n}ds = \int_{S_1} 2ds = 2\pi a^2$$

So Stokes' theorem is valid when we consider surface S_1 bounded by the loop

(b) Over the surface S_2 the normal every where is along $-\hat{r}$. To do the surface integral over this surface consider a narrow strip (see fig) of the sphere parallel to the equator of width $ad\theta$. The points on this strip makes an angle θ with the z axis.



$\therefore \ (\vec{\nabla}\times\vec{A})\cdot\hat{n} = -2\hat{k}\cdot(-\hat{r}) = 2\cos\theta$.

The contribution to the surface integral from this strip is

$$2\cos\theta \times \text{area of the strip} = 2\pi a^2 \sin 2\theta d\theta$$

$$\int_{S_2} (\vec{\nabla}\times\vec{A})\cdot\hat{n}ds = \int_0^{\pi/2} 2\pi a^2 \sin 2\theta d\theta = 2\pi a^2$$

So Stokes theorem is valid also for the curved surface S_2.

4. If $\vec{\nabla}\times\vec{A} = 0$ then show that there is a scalar function $F(\vec{r})$ such that $\vec{\nabla}F = \vec{A}$.
Consider the origin and some point in space $\vec{r}$.
Let C_1 and C_2 be two different curves from the origin to the position $\vec{r}$. Consider a

loop forward along C_1 and backward along C_2. The line integral $\int \vec{A} \cdot \vec{dl}$ is zero over this closed loop by stokes' theorem since $\vec{\nabla} \times \vec{A} = 0$. Hence we have

$$\int_{C_1} \vec{A} \cdot \vec{dl} = \int_{C_2} \vec{A} \cdot \vec{dl}$$

This shows that the value of the line integral from origin to the point $\vec{r}$ is independent of the curve. So we can write

$$\int_0^{\vec{r}} \vec{A} \cdot \vec{dl} = F(\vec{r})$$

Now if $\vec{r}$ changes by a small amount $\vec{dr}$ then the change in F, is given as $\vec{\nabla} F \cdot \vec{dr}$. The small change in the integral on the l.h.s is $\vec{A}(\vec{r}) \cdot \vec{dr}$. Since this is true for any arbitrary $\vec{dr}$ we have

$$\vec{\nabla} F = \vec{A}(\vec{r})$$

An explicit form of F can be evaluated as follows:

$$F(x, y, z) = \int_0^x A_x(x', y', z')dx' + \int_0^y A_y(x', y', z')dy' + \int_0^z A_z(x', y', z')dz'$$

We have considered the lower limit of the integral to be the origin. But one can choose any point as the lower limit. This will only add a constant to the function we obtained. The scalar function can be determined upto a constant as usual in any integral calculus.

5. Use the divergence theorem and the stokes' theorem to show that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ for any vector field $\vec{A}$.

   soln
   Consider a closed surface S enclosing a volume V. Let us calculate

$$\oint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} ds$$

over this surface. We can break the closed surface into two parts like we break a coconut shell. Let us call the two parts of the broken surfaces S_1 and S_2.
Then we have the integral as

$$\int_{S_1} (\vec{\nabla} \times \vec{A}) \cdot \hat{n} ds + \int_{S_2} (\vec{\nabla} \times \vec{A}) \cdot \hat{n} ds$$

Let C be the closed curve along which lies the boundary of the surfaces S_1 and S_2.
By Stokes' theorem we have

$$\oint_C \vec{A} \cdot \vec{dl} = \int_{S_1} (\vec{\nabla} \times \vec{A}) \cdot \hat{n} ds = \int_{S_2} (\vec{\nabla} \times \vec{A}) \cdot (-\hat{n}) ds$$

Here we have to invert the direction of the normals in one of the integrals over the surfaces S_1 and S_2. This depends upon the sense of traversing along the curve C.
Thus the value of the integral in Eq.(5) is 0.

$$\therefore \quad \oint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} ds = 0$$

If V is the volume enclosed by the surface S then by divergence theorem

$$\int_V \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) dV = \oint_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} ds = 0$$

Since the volume integral will be 0 for any volume we conclude the function in the integrand in identically 0 $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.

6. Show that

$$\int_S (\vec{\nabla} \times \vec{A}) f \cdot \hat{n} da = \int_S (\vec{A} \times \vec{\nabla} f) \cdot \hat{n} da + \oint_C f\vec{A} \cdot \vec{dl}$$

where C is a closed curve enclosing a surface S.

soln

$$\vec{\nabla} \times (f\vec{A}) = \vec{\nabla} f \times \vec{A} + f(\vec{\nabla} \times \vec{A})$$

Using Stokes' theorem we get

$$\begin{aligned} \oint_C f\vec{A} \cdot \vec{dl} &= \int_S (\vec{\nabla} \times (f\vec{A})) \cdot \hat{n} da \\ &= \int_S (\vec{\nabla} f \times \vec{A}) \cdot \hat{n} da + \int_S f(\vec{\nabla} \times \vec{A}) \cdot \hat{n} da \\ &= -\int_S (\vec{A} \times \vec{\nabla} f) \cdot \hat{n} da + \int_S f(\vec{\nabla} \times \vec{A}) \cdot \hat{n} da \end{aligned} \tag{1}$$

Rearranging the terms we get the required expression.