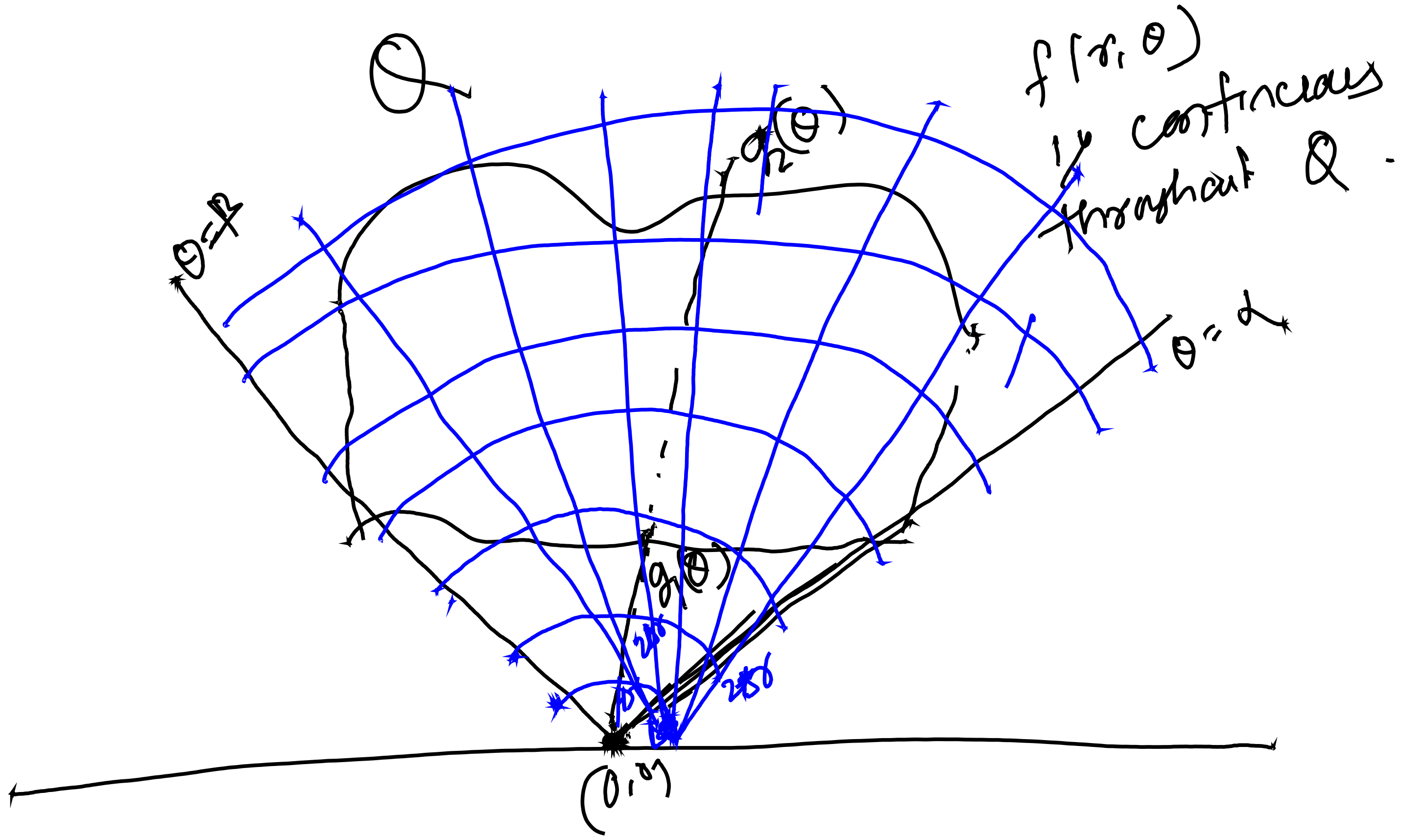
Double integrals in polar form

Suppose that a function $f(r,\theta)$ is defined over a region Q that is bounded by the rays $\theta=\alpha$ and $\theta=\beta$ and by the curves $r=g_1(\theta)$ and $r=g_2(\theta)$.

Also let $0 \le g_1(\theta) \le g_2(\theta) \le a$

for all θ, $\alpha \le \theta \le \beta$



We cover Q by a grid of circular arcs and rays.

The arcs are cut from the circles centered at the origin with radii $\Delta r, 2\Delta r, \ldots, m\Delta r$

where $\Delta r = \frac{a}{m}$

The rays are given by

$\theta = \alpha, \cancel{\alpha}, \alpha + \Delta\theta, \alpha + 2\Delta\theta, \ldots, \alpha + m\Delta\theta = \beta$

where $\Delta\theta = \frac{\beta - \alpha}{m}$

The arc and the rays partition Q into small patches called polar rectangles.

Let $\Delta A_1, \Delta A_2, \ldots \Delta A_n$ be the area of the polar rectangles.

Let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k.

We then form the sum

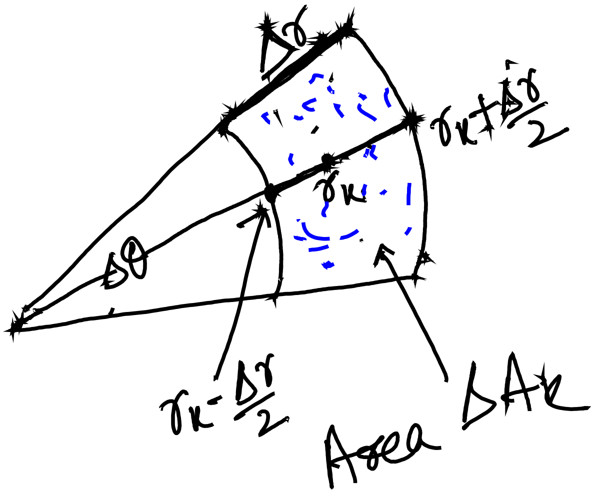
$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) \, \Delta A_k$$

Letting $\Delta r \to 0$ & $\Delta \theta \to 0$,
that means $n \to \infty$

$$\lim_{n \to \infty} S_n = \iint_Q f(r, \theta) \, dA$$

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta \theta$.

$$\frac{\pi r^2}{\frac{2\pi}{\theta}} = \frac{1}{2}\theta r^2$$



$$\Delta A_k = \frac{1}{2}\Delta\theta\left(r_k+\frac{\Delta r}{2}\right)^2$$

$$-\frac{1}{2}\Delta\theta\left(r_k-\frac{\Delta r}{2}\right)^2$$

$$= \frac{\Delta\theta}{2}\left[\cancel{r_k^2}+\cancel{\frac{\Delta r^2}{4}}+\cancel{2}r_k\frac{\Delta r}{\cancel{2}} - \cancel{r_k^2}-\cancel{\frac{\Delta r^2}{4}}+\cancel{2}r_k\frac{\Delta r}{\cancel{2}}\right]$$

$$= \frac{\Delta\theta}{2}\,2r_k\Delta r = r_k\Delta r\Delta\theta$$

Combining this result with the sum defining S_n

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) \Delta A_k$$

$$= \sum_{k=1}^{n} f(r_k, \theta_k) \underline{r_k \Delta r \Delta \theta}$$

As $n \to \infty$ all the values of

$(\Delta r, \Delta \theta) \to (0, 0)$

$$\lim_{n \to \infty} S_n = \iint_R f(r, \theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

Finding the limits of integration

1. Sketch the region and label the boundary curves.

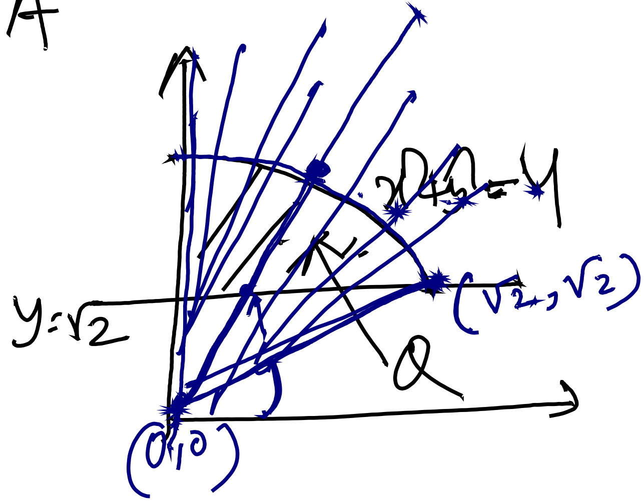
2. Find the r-limits of integration.

Imagine a ray L from the origin cutting through Q in the direction of increasing r.
Mark the values where L enters and leaves Q.
These are r-limits of integration.
(r-limits usually depends on θ).

3. Find the smallest and largest θ-values that bound Q.
These are θ-limits of integration.

Exp

$$\iint_{Q} f(x,y)\,dA$$



$x = r\cos\theta$

$y = r\sin\theta$

$$\iint g(r,\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=\frac{\sqrt{2}}{\sin\theta}}^{r=2} g(r,\theta)\, r\,dr\,d\theta$$

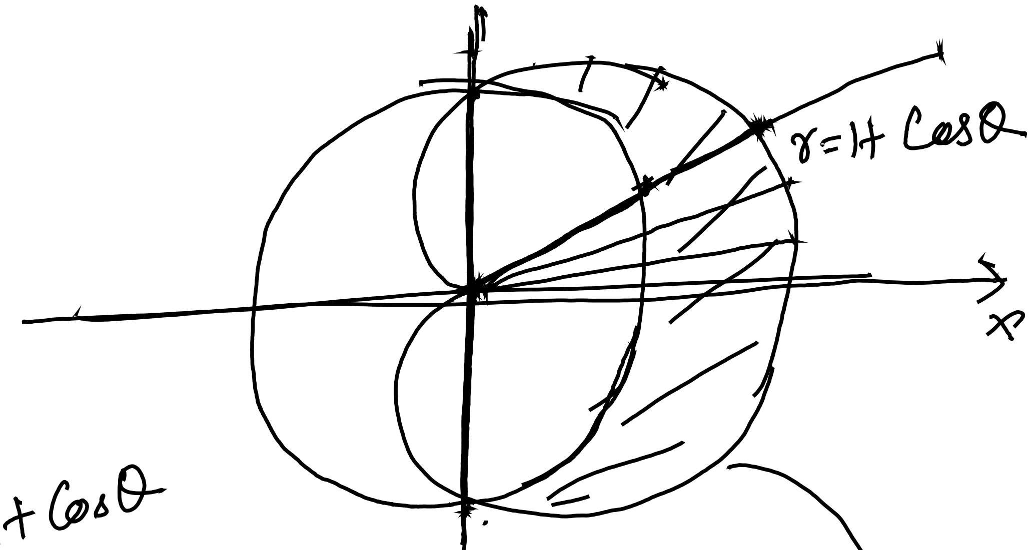
$y = \sqrt{2}$

$\Rightarrow r\sin\theta = \sqrt{2}$

$\Rightarrow r = \dfrac{\sqrt{2}}{\sin\theta}$

$r = 2$

Exp Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1+\cos\theta$ and outside the circle $r=1$.

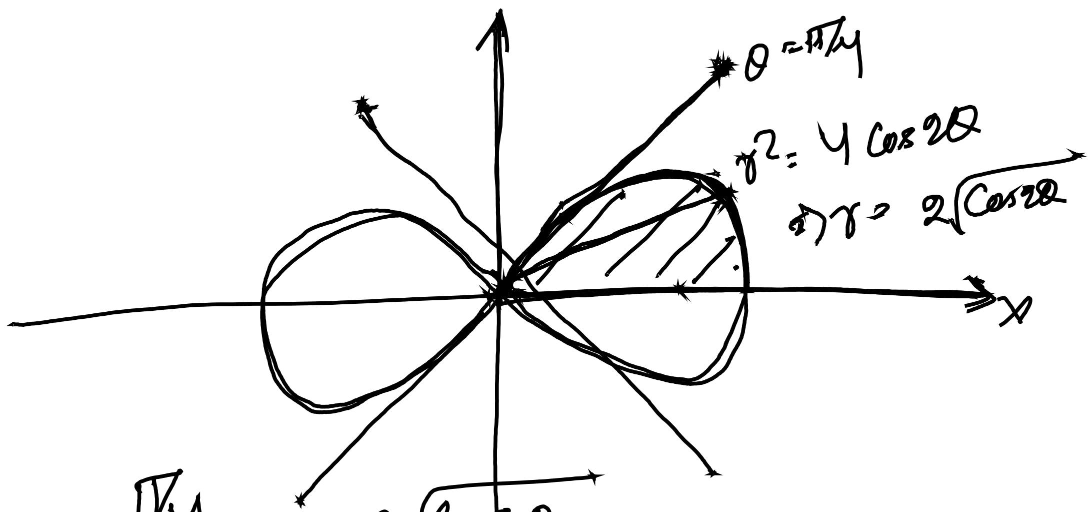


$$\int_{\theta=-\pi/2}^{\pi/2}\int_{r=1}^{r=1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$

Area of a region in polar co-ordinates

$$A = \int_{\theta=\alpha}^{\beta}\int_{r=g_1(\theta)}^{g_2(\theta)} r\, dr\, d\theta = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=1}^{1+\cos\theta} r\, dr\, d\theta$$

Exp Find the area enclosed by the leminacate $r^2 = 4\cos 2\theta$

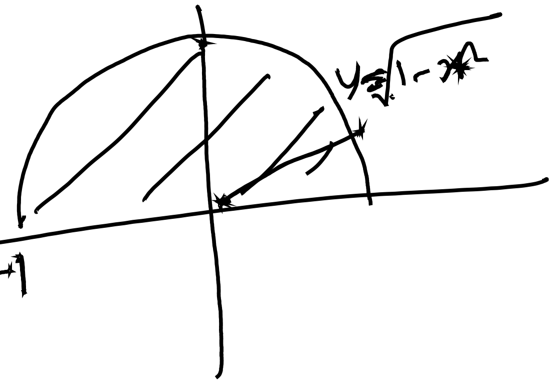


$$A = 4 \int_{\theta=0}^{\pi/4} \int_{r=0}^{r=2\sqrt{\cos 2\theta}} r\, dr\, d\theta$$

$$= 4 \;\big[\quad\;\big]$$

Exp Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$

where R is the ~~semi-circular~~ semi-circular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$



$$\int_{-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$

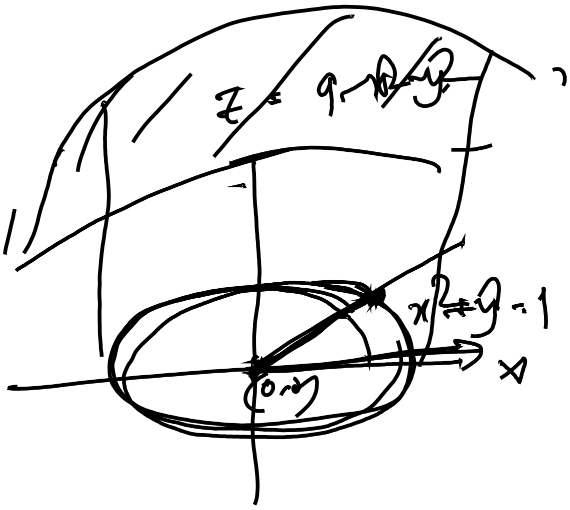
Put $x = r\cos\theta$
$y = r\sin\theta$

$$\iint_R e^{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1} e^{r^2}\, r\,dr\,d\theta$$

$$= \frac{\pi}{2}(e-1)$$

Exp

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane



Solⁿ

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} (9 - r^2)\, r\,dr\,d\theta$$

$$\cancel{= \frac{17\pi}{2}} \qquad \cancel{= \frac{26\pi}{3}}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (9r - r^3)\,dr\,d\theta = \int_{\theta=0}^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{1}{4}\right) d\theta = \frac{17}{4} \times 2\pi = \frac{17}{2}\pi$$