Cholesky decomposition

When the matrix A is symmetric, then there exists a lower triangular matrix L such that

$$A = LL^T.$$

$$\overbrace{\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}}^{L} \overbrace{\begin{bmatrix} l_{11} & l_{21} & l_{31} & \cdots & l_{n1} \\ 0 & l_{22} & l_{32} & \cdots & l_{n2} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & l_{nn} \end{bmatrix}}^{L^T}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$l_{11}^2 = a_{11}$$

$$\boxed{l_{11} = \sqrt{a_{11}}}$$

$$l_{11} l_{21} = a_{12}$$

$$\Rightarrow \boxed{l_{21} = \frac{a_{12}}{l_{11}}} \quad \cdots \quad \boxed{l_{31} = \frac{a_{13}}{l_{11}}} \quad \cdots \quad \boxed{l_{n1} = \frac{a_{1n}}{l_{11}}}$$

$$l_{n1} = \frac{a_{1n}}{l_{11}}.$$

$$l_{21} l_{11} = a_{21} = a_{12}$$

$$l_{21} = \frac{a_{12}}{l_{11}}$$

$$l_{21}^2 + l_{22}^2 = a_{22} \Rightarrow l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$$l_{21} l_{31} + l_{22} l_{32} = a_{23}$$

$$\Rightarrow l_{32} = \frac{1}{l_{22}}\left(a_{23} - l_{21} l_{31}\right)$$

$$l_{21} l_{41} + l_{22} l_{42} = a_{24}$$

$$\Rightarrow l_{42} = \frac{1}{l_{22}}\left(a_{24} - l_{21} l_{41}\right)$$

$$\vdots$$

$$l_{n2} = \frac{1}{l_{22}}\left(a_{2n} - l_{21} l_{n1}\right)$$

In general

$$\boxed{l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}}$$

and

$$\boxed{l_{ji} = \frac{1}{l_{ii}}\left(a_{ji} - \sum_{k=1}^{i-1} l_{jk}\, l_{ik} \right) \quad \text{for } j = i+1, i+2, \ldots, n}$$

Q Whether every symmetric matrix has Cholesky factorization?

No

~~Def^n~~

Result If A is symmetric positive definite matrix then it has Cholesky factorization.

Defn

A square matrix A is said to be positive definite if
$$x^T A x > 0 \quad \text{for all } \underline{x \neq 0}$$

$$[x_1, x_2]\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \underset{1\times 2}{[x_1, x_2]} \underset{2\times 1}{\begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 3x_2 \end{bmatrix}} = x_1(2x_1 + x_2) + x_2(-x_1 + 3x_2)$$

$$= 2x_1^2 + \cancel{x_1 x_2} - \cancel{x_1 x_2} + 3x_2^2 = 2x_1^2 + 3x_2^2 > 0$$

for $X \neq 0$

$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ is a positive definite.

Characterization of positive definite matrices

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① If all the eigenvalues of A are > 0, then A is positive definite.

(2)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

3×3

$$A_1 = a_{11}$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\vdots$

$$A_n = A$$

If $A_j > 0$ for all $j = 1, 2, \ldots, n$

then A is positive definite.

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{bmatrix}$$

whether A positive definite ?

$A_1 = 4 > 0$

$$A_2 = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0$$

$$A_3 = \begin{vmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{vmatrix} = 4(58-25) - 2(58-30) + 6(10-12)$$

$$= 4 \times 33 + 2 \times 28 + 6(-2)$$

$$= 132 - 56 - 12 = 74 > 0$$

A is positive definite.

Negative definite matrix

$A_{n \times n}$ is said to be negative definite if $x^T A x < 0$ for all $x \neq 0$.

Characterization of negative definite matrices

(1) If all eigenvalues of A are $-ve$, then A is negative definite.

(2) If the sign of $A_j = (-1)^j$ for all $j = 1, 2, \dots, n$ then A is negative definite.

Exp

$$A = \begin{bmatrix} -2 & 1 \\ -5 & 1 \end{bmatrix}$$

$A_1 = -2 < 0$

$$A_2 = \begin{vmatrix} -2 & 1 \\ -5 & 1 \end{vmatrix} = -2+5 = 3 > 0$$

A is $\boxed{\text{negative definite}}$

Exp

$$A = \begin{bmatrix} -2 & 1 \\ 5 & 1 \end{bmatrix}$$

$A_1 = -2 < 0$

$$A_2 = \begin{vmatrix} -2 & 1 \\ 5 & 1 \end{vmatrix} = -2-5 = -7 < 0$$

$\boxed{\text{Indefinite matrix}}$.

Positive semi-definite matrix

$A_{n \times n}$ is said to be positive semi-definite if

$$x^T A x \geq 0 \quad , \quad x \neq 0 .$$

Negative semi-definite matrix

$A_{n \times n}$ is said to be negative semi-definite if

$$x^T A x \leq 0 \quad , \quad x \neq 0 .$$

Indefinite matrix

If $A_{n \times n}$ is not in the above categories, then it is called an indefinite matrix.

Orthogonal vectors

Two vectors x and y are said to be orthogonal if $x^T y = 0$

A vector x for which $\|x\| = 1$ then it is called normalized or unit norm vector.

Defn A set of vectors $\{x_1, x_2, \ldots x_n\}$ form an orthonormal set if they are mutually orthogonal and normalized

that is $\quad x_i^T x_j = 0 \quad$ for $i \neq j$

$\qquad\qquad = 1 \quad$ for $i = j$

$$x_i^T x_i = \|x_i\|^2$$

For matrices $A_{n\times n}$, $B_{n\times n}$ (square)

Two o matrices A and B are said to be orthogonal if $A^T B = 0$ (matrix)

Unitary matrix

A square matrix Q is said to be unitary if $\underline{Q^T Q = Q Q^T = I}$

If Q is unitary matrix, then Qx is called a unitary transformation of x.

Properties of unitary matrices

(1) Unitary matrices are easily inverted since

$$Q^{-1} = Q^T$$

(2) Unitary transformation preserve its norm.

$$\|Qx\|^2 = (Qx)^T Qx = x^T Q^T Q x = x^T x = \|x\|^2$$

$$\Rightarrow \|Qx\| = \|x\|$$

Since a unitary transformation doesn't change the magnitude of X, any error in X will not magnified by a unitary transformation.

This indicates that any algorithm that involves only unitary transformation tend to be stable numerically.

QR factorization

An $m \times n$ matrix A is said to have a QR factorization if

$$A = QR$$

where Q is an $m \times m$ unitary matrix and R is an $m \times n$ upper triangular matrix.

Why such a factorization is useful ?
For a

Suppose we want to solve

$$Ax = b$$

If A has QR factorization,

$$QRx = b$$

Premultiplying by Q^T

$$Q^T Q R x = Q^T b$$

$$\Rightarrow \quad Rx = Q^T b$$

Which can be easily solved because R is an upper triangular matrix.

Note $A_{m \times n}$, $m \geq n$
and if A has n linearly independent columns
then it can be factored as $A = QR$.

Note

an $m \times n$ matrix with $m \geq n$,

the term upper triangular matrix

means that $\boxed{a_{ij} = 0 \text{ for } i > j}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A = QR$

$Q_{m \times m}$

$R_{m \times n}$