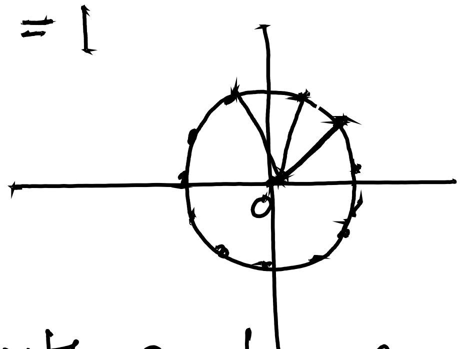
The unit circle $|z| = 1$

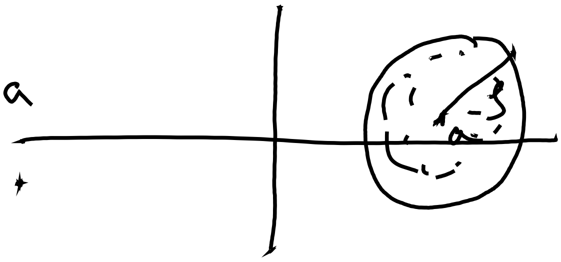
$\{z : |z| = 1\}$

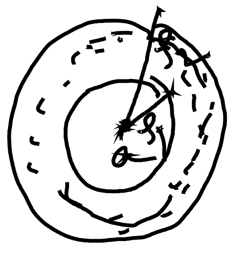


$|z - a| = \rho$ circle with center a radius ρ.

$|z - a| \leq \rho$ closed disk with center a radius ρ.

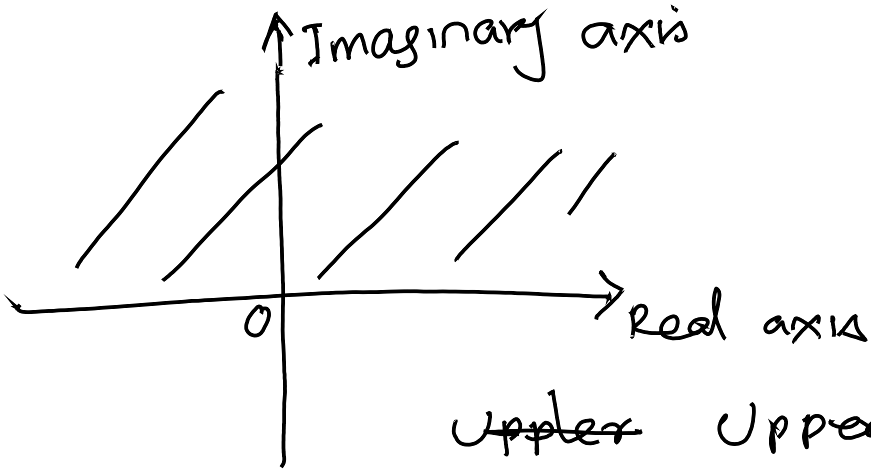
$|z - a| < \rho$ open disk with center a radius ρ.





$\{z : \rho_1 \leq |z - a| \leq \rho_2\}$ closed annular ~~disk~~ region

$\{z : \rho_1 \leq |z - a| < \rho_2\}$ open annular region.



~~Uppler~~ Upper half plane

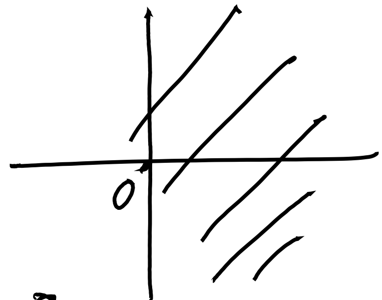
$$\{ z = x + iy \; : \; y \geq 0 \}$$

$$\{ z : \operatorname{Im}(z) \geq 0 \}$$

Lower half plane $\{ z : \operatorname{Im}(z) \leq 0 \}$

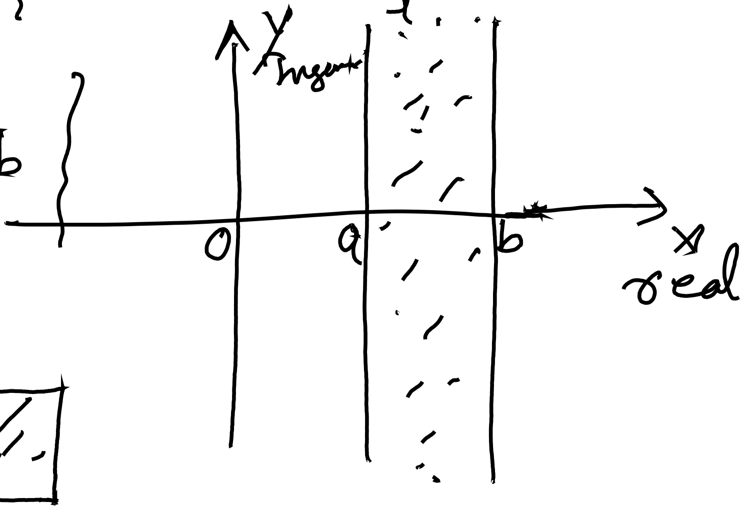
Right half plane:

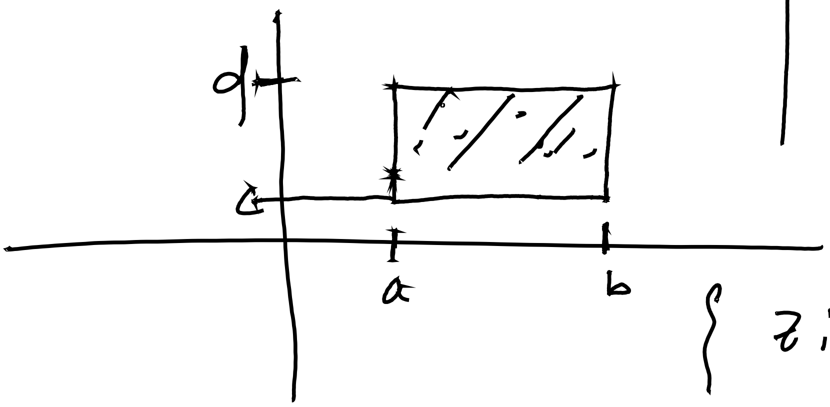
$$\{ z : \operatorname{Re} z \geq 0$$



left half plane $\{ z : \operatorname{Re} z \leq 0 \}$

$$\{ z : \; a \leq \operatorname{Re} z \leq b \}$$





$$\left\{ z ; \; \begin{array}{l} a \leq \operatorname{Re} z \leq b \\ c \leq \operatorname{Im} z \leq d \end{array} \right\}$$

Complex function

$D \subset \mathbb{C}$

$f : D \to \mathbb{C}$ Complex plane

$f(z) \in \mathbb{C}$

Imaginary part of f.

$f(z) = u(x,y) + i\,v(x,y)$

real part of f

$f(z) = z^2 + 3z$ $\qquad z = x + iy$

$= (x+iy)^2 + 3(x+iy)$

$= x^2 + 2ixy - y^2 + 3x + 3iy$

$= \underbrace{(x^2 - y^2 + 3x)}_{u(x,y)} + i\underbrace{(2xy + 3y)}_{v(x,y)}$

Re part of z^2+3z $\qquad$ Imaginary part of z^2+3z

Limit

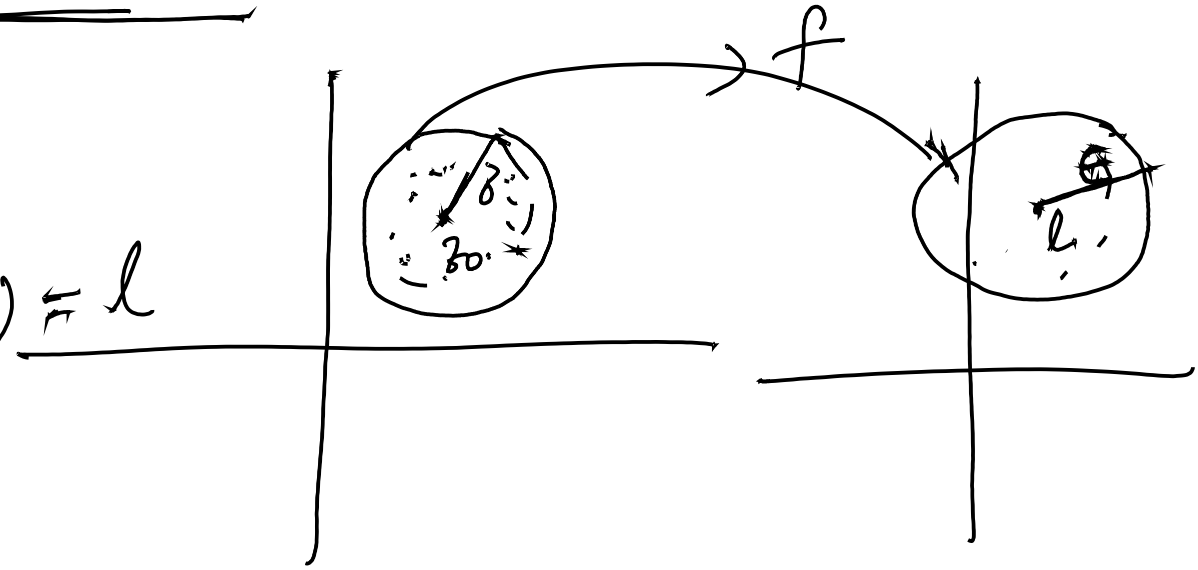
A function $f(z)$ is said to have the limit l as z approaches z_0 written as $\lim_{z \to z_0} f(z) = l$.

if f is defined in a neighbourhood of z_0 (except perhaps at z_0 itself) and if the value of f are close to l for all z close to z_0.

For every $\epsilon > 0$, if there exists $\delta > 0$ such that (for all $z \neq z_0$)

$0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$.

Then $\lim_{z \to z_0} f(z) = l$



Continuous function

A function $f(z)$ is said to be continuous at z_0 if $f(z_0)$ is defined and $\lim_{z \to z_0} f(z) = f(z_0)$ *

$f(z)$ is continuous in a domain D if it is continuous at every point of D.

Derivative

The derivative of a complex function f at a point z_0 written as $f'(z_0)$ is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

provided this above limit exists.

$z_0+\Delta z$

z_0

$z_0 = x_0 + i y_0$ $\quad \underline{\Delta z = \Delta x + i\Delta y}$

$\underline{z_0+\Delta z} = x_0+\Delta x + i(y_0+\Delta y)$

If we write $z - z_0 = \Delta z$

then $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

Properties

1. $\left(f(z) \pm g(z)\right)' = f'(z) \pm g'(z)$

2. $\left(\alpha f(z)\right)' = \alpha f'(z)$

3. $\left(f(z) g(z)\right)' = f'(z) g(z) + f(z) g'(z)$

4. $\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z) g(z) - f(z) g'(z)}{(g(z))^2}$

provided $g(z) \neq 0$ for any z.

$\frac{d}{dz}(z^n) = n z^{n-1}$

$\frac{d}{dz}(1) = 0$

If $f(z)$ is differentiable at z_0, then it is continuous at z_0.

$f(z) = \bar{z}$ is not differentiable

$z = x + iy$

$$f(z) = \bar{z} = x - iy$$

$\Delta z = \Delta x + i\Delta y$

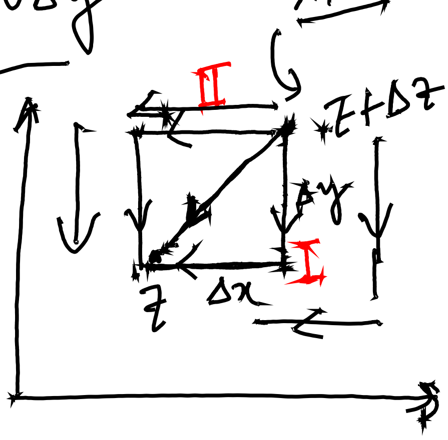
$\Delta z \to 0$

$(\Delta x, \Delta y) \to (0,0)$

$$\boxed{\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$$ does not exist

$$= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{\bar{z}} + \overline{\Delta z} - \cancel{\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



Along path I

$\Delta y \to 0$ first then $\Delta x \to 0$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Along path II $\Delta x \to 0$ first then $\Delta y \to 0$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

limit is different for different path of approach.

$f(z) = z$

$f(z) = z^2$

$f(z) = a_0 + a_1 z + \cdots + a_n z^n$

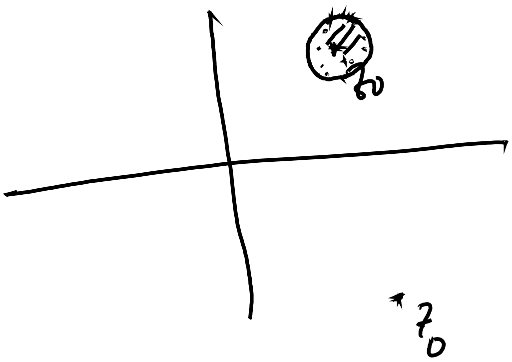
are all differentiable.

$f(z) = e^z$

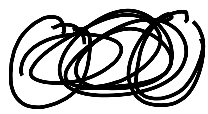
$f(z) = \sin z$

$f(z) = \cos z$

D



$f(z) = |z|^2$



Analytic function

D open connected set

$f : D \to \mathbb{C}$

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

A function $f(z)$ is analytic at a point $z_0 \in D$ if $f(z)$ is differentiable at all points in a neighbourhood of z_0.

Exn $f(z) = \text{Re } z$

$z = x + iy$ $\qquad$ $z + \Delta z$

$\qquad\qquad\qquad\qquad = \underline{x + \Delta x} + i\,\underline{(y + \Delta y)}$

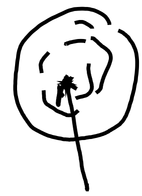
$\text{Re } z = x$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x}{\Delta x + i\Delta y}$$

If $\Delta y \to 0$ first then $\Delta x \to 0$,

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$



If $\Delta x \to 0$ first then $\Delta y \to 0$

$$= \lim_{\Delta y \to 0} 0 = 0$$

$f(z) = \underline{\text{Re } z}$ is not differentiable at any point.

Cauchy-Riemann equation (C-R equation)

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of a point $z = x+iy$ and analytic at z itself. Then at that point the 1st partial derivatives of u and v exist and satisfy the C-R equations

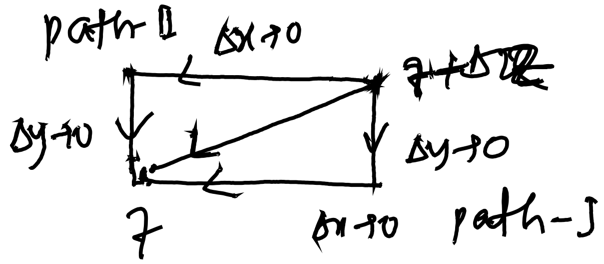
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Proof

Since $f(z)$ is analytic

$\Rightarrow f'(z)$ exists.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists.}$$



$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$z = x+iy$

$\Delta z = \Delta x + i\Delta y$

$$= \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\left[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)\right]}{\Delta x + i\Delta y}$$

Along path-I

$\Delta y \to 0$ first then $\Delta x \to 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{\left[u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{——— (1)}$$

$$\boxed{\begin{gathered} f(x,y) \\ \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h} \end{gathered}}$$

Along path II

$\Delta x \to 0$ first then $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{\left[u(x, y+\Delta y) + i v(x, y+\Delta y) - u(x,y) - i v(x,y)\right]}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y}$$

$$+ \cancel{i} \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\cancel{i} \Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \text{—— (2)}$$

Comparing equation (1) and (2)

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$