SC223 - Linear Algebra
Autumn 2022
Tutorial 4: Matrix Inverse and Groups

1. Compute the inverse of the following matrices, wherever possible using the Gauss-Jordan method.

   (a) $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & 3 \\ 4 & -2 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2. Let A, B, C be square matrices of size n. If $AB = AC$, then is it necessary that $B = C$?

3. Which of the following sets along with the given binary operation form a group? If they form a group, is the group Abelian?

   (a) The set of integers $\mathbb{Z}$ with addition.

   (b) The set of all invertible matrices of size n with matrix multiplication. Use the fact that $det(AB) = det(A)det(B)$.

   (c) The set of rational numbers $\mathbb{Q}$ with multiplication.

   (d) Let S denote the set of transformations that map an equilateral triangle to itself, i.e., set of its symmetries. Consider the composition of transformations as the binary operation.

   (e) $\mathbb{R}^n$ with entry-wise addition.

   (f) Let S denote any set. Consider the set F_S of all functions from S to $\mathbb{R}$, with the binary operation $\circ$ defined as: $\forall f, g \in F_S, (f \circ g)(x) = f(x) + g(x)$.

4. Consider all possible semantically distinct boolean formulae over three propositional variables. There are 256 of them. You can think of them as the corresponding truth tables. Call this set of formulae S. Together with the following logical operators, do they form a group? If so, prove that the group axioms hold. If not, state which of the axioms are violated. For those that form a group, determine of the group is Abelian.

   (a) $(S, \vee)$

   (b) $(S, \wedge)$

   (c) $(S, \oplus)$

   (d) $(S, \Rightarrow)$

5. As you might recall, we introduced matrices with a view to solving simultaneous linear equations. There is a special class of simultaneous linear equations where we try to get solutions modulo a specific positive integer. One could have different moduli for different variables, and different equations, but let us keep it simple. Thus, for example, we could solve a system of equations where each of the variables is a number modulo 7. Is there a concept of identity, when considering matrix multiplication of numbers modulo 7? It goes without saying, that the scalar addition as well scalar multiplication within the matrix multiplication are also done modulo 7. We have moved from the domain of real numbers to not just integers, but the integers modulo 7. Can you use Guassian eliminaton to solve such a system? Try it with the following system of equations.

$$4x + 3y + z = 6$$
$$x + 2y + 4z = 5$$
$$3x + y + 5z = 1$$

6. Do the notions of linear dependence and linear independence of a set of vectors over R^3 extend to the corresponding restriction to $\{0, 1, 2, 3, 4, 5, 6\}^3$? Assume that the vectors we consider in R^3 have their entries all in $\{0, 1, 2, 3, 4, 5, 6\}$.