# SC223 - Linear Algebra

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Lecture 9

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# $LU$ Decomposition Algorithm

- Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, $L_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & \dots & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ -\frac{a_{n1}}{a_{11}} & 0 & \dots & 1 \end{bmatrix}$
  - Let $L_1 A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^1 & \dots & a_{2n}^1 \\ 0 & a_{32}^1 & \dots & a_{3n}^1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & a_{n2}^1 & \dots & a_{nn}^1 \end{bmatrix}$
  - $L_2 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & -\frac{a_{32}^1}{a_{22}^1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & -\frac{a_{n2}^1}{a_{22}^1} & 0 & \dots & 1 \end{bmatrix}$, $L_2 L_1 A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^1 & a_{23}^1 & \dots & a_{2n}^1 \\ 0 & 0 & a_{33}^2 & \dots & a_{3n}^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & a_{n3}^2 & \dots & a_{nn}^2 \end{bmatrix}$
  - Final Step: $L_{n-1} \cdot \dots \cdot L_1 A = U$.

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- Thus, $L = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \ldots & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a^1_{32}}{a_{22}} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{a_{n1}}{a_{11}} & \frac{a^1_{n2}}{a_{22}} & \ldots & 1 \end{bmatrix}$

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- Why should one use $LU$ decomposition?