1. The plane $x = 0$ has a constant surface charge density σ_1 and the plane $x = a$ has a constant charge density σ_2. Find the electric field in the three regions $x < 0,\ \ 0 < x < a$ and $x > a$ by solving the Laplace's equation.
   soln:
   By symmetry along the y and the z coordinates, the potential varies only along x. In the three regions, the volume charge density is 0. Let the potential in region 1, $x < 0$, be Φ_1, region 2, $0 < x < a$, be Φ_2 and in region 3, $x > 0$, be Φ_3.
   Then we have
   $$\frac{\partial^2 \Phi_1}{\partial x^2} = 0, \quad \frac{\partial^2 \Phi_2}{\partial x^2} = 0, \quad \frac{\partial^2 \Phi_3}{\partial x^2} = 0$$
   The solutions are of the same type,
   $$\Phi_1 = c_1 x + d_1, \quad \Phi_2 = c_2 x + d_2, \quad \Phi_3 = c_3 x + d_3$$
   The 6 constants have to be determined from the boundary conditions at the interfaces and infinities. The electric fields in the three regions are
   $$E_{1x} = -c_1, \quad E_{2x} = -c_2, \quad E_{3x} = -c_3$$
   Using the boundary condition on electric fields at he interface we get
   $$\begin{aligned} E_{2x} - E_{1x} &= \frac{\sigma_1}{\epsilon_0} \\ \therefore \quad c_1 - c_2 &= \frac{\sigma_1}{\epsilon_0} \\ \text{similarly} \quad c_2 - c_3 &= \frac{\sigma_2}{\epsilon_0} \end{aligned} \tag{1}$$
   This gives
   $$c_2 = c_3 + \frac{\sigma_2}{\epsilon_0} \quad \text{and} \quad c_1 = c_3 + \frac{\sigma_2 + \sigma_1}{\epsilon_0}$$
   Equating the potentials at $x = 0$ gives $d_1 = d_2$. At $x = a$ $\Phi_2(a) = \Phi_3(a)$ gives
   $c_2 a + d_2 = c_3 a + d_3$
   $\therefore \quad (c_2 - c_3)a = d_3 - d_2$
   $\therefore \quad d_1 = d_3 - \frac{\sigma_2}{\epsilon_0} a.$
   All the constants are obtained in terms of c_3 and d_3. They correspond to a constant background potential and electric field which can be arbitrary. So the potential in the three region is
   $$\Phi_1 = \left(c_3 + \frac{\sigma_1 + \sigma_2}{\epsilon_0}\right) x + d_3 - \frac{\sigma_2}{\epsilon_0} a, \quad \Phi_2 = \left(c_3 + \frac{\sigma_2}{\epsilon_0}\right) x + d_3 - \frac{\sigma_2}{\epsilon_0} a, \quad \Phi_3 = c_3 x + d_3$$

2. The points on the xy plane is maintained at potential $V_0 \sin(\alpha x + \beta)$. The potential goes to 0 as $z \to \pm\infty$. Find the potential at all the points above and below the xy plane.

soln:
In this problem it is clear that the potential doesn't vary along y.
Using the Variable separation method we get. $V(x,z) = X(x)Z(z)$ where

$$\begin{aligned} X(x) &= A\cos kx + B\sin kx \\ Z(z) &= Ce^{kz} + De^{-kz} \end{aligned}$$

We assume $k > 0$.
As $z \to \infty \quad V \to 0 \quad \implies \quad C = 0$.
So for $z > 0$ we have $Z(z) = De^{-kz}$. Similarly for $z < 0$ we have $Z(z) = Ce^{kz}$.
Since the twopotentials match at $z = 0$ we have $C = D$.
So we have

$$\begin{aligned} V &= Ce^{-kz}(A\cos kx + B\sin kx) \quad \text{for} \quad z > 0 \\ &= Ce^{kz}(A\cos kx + B\sin kx) \quad \text{for} \quad z < 0 \end{aligned} \tag{2}$$

The constant C can be absorbed into A and B.
At $z = 0$ we equate V to the given potential function.

$$\begin{aligned} \therefore \quad A\cos kx + B\sin kx &= V_0\sin(\alpha x + \beta) \\ &= V_0(\sin\beta\cos\alpha x + \cos\beta\sin\alpha x) \end{aligned} \tag{3}$$

$\therefore \quad A = V_0\sin\beta, \quad B = V_0\cos\beta$ and $k = \alpha$.

$$\begin{aligned} \therefore \quad V &= V_0e^{-kz}(\sin\beta\cos kx + \cos\beta\sin kx) = V_0e^{-kz}\sin(\alpha x + \beta) \quad ; \quad z > 0 \\ &= V_0e^{kz}\sin(\alpha x + \beta) \quad ; \quad z < 0 \end{aligned} \tag{4}$$

3. A conducting sphere of radius R has an amount of charge Q over it. This sphere is placed in an otherwise uniform electric field $\vec{E}_0$. The potential of the sphere is found to be V_0. Find the potential in the region outside the sphere.

soln:
The given electrostatic configuration has an azimuthal symmetry if we consider the uniform electric field along $\hat{z}$. The general variable separated form of the potential is given as

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Far away from the sphere the influence of the sphere gets negligible. So the electric field is $E_0\hat{z}$. This corresponds to an electric potential function $-E_0 z + c = -E_0 r\cos\theta + c$

where c is a constant
Now as $r \to \infty, \quad V(r,\theta) \to \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$. So we have

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta + c$$

This gives $A_0 = c, A_1 = -E_0, \quad A_2 = A_3 = ... = 0$. Let us rewrite the potential using these values of the constants A_l.

$$V(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) + A_0$$

The sphere is an equipotential surface. So $V(R,\theta)$ must be independent of θ.

$$\begin{aligned} V(R,\theta) &= -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) + A_0 \\ &= A_0 + \frac{B_0}{R} + \left(-E_0 R + \frac{B_1}{R^2}\right)\cos\theta + \sum_{l=2}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \end{aligned}$$

If this has to be independent of θ then we must have $B_1 = E_0 R^3$ and $B_2 = B_3 = ... = 0$. Let us rewrite the potential again

$$V(r,\theta) = A_0 + \frac{B_0}{r} + \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta$$

The total charge on the sphere is Q. Q/ϵ_0 will be equal to the total flux of the electric field over the surface of the sphere. Over the surface of the sphere the electric field is normal, which is the radial direction $\hat{r}$. The radial component of the electric field can be derived from $V(r,\theta)$. This is $-\frac{\partial V}{\partial r}$ given by

$$E_r(r,\theta) = \frac{B_0}{r^2} + \left(E_0 + \frac{2E_0 R^3}{r^3}\right)\cos\theta$$

At $r = R$ we get

$$E_r(R,\theta) = \frac{B_0}{R^2} + 3E_0\cos\theta$$

This is the normal component of the electric field over the surface of the sphere. Integrating this over the surface of the sphere will give the total charge on the sphere. The constant term $\frac{B_0}{R^2}$ when integrated over the sphere will give $\frac{B_0}{R^2} 4\pi R^2 = 4\pi B_0$. We integrate the other term

$$\begin{aligned} \therefore \quad \frac{Q}{\epsilon_0} &= 4\pi B_0 + \int_0^{2\pi}\int_0^{\pi} 3E_0\cos\theta R^2 \sin\theta d\theta d\phi \\ &= 4\pi B_0 + 6\pi R^2 E_0 \int_0^{\pi} \cos\theta\sin\theta d\theta \\ &= 4\pi B_0 \end{aligned}$$

$$\therefore \quad B_0 = \frac{Q}{4\pi\epsilon_0}$$

$$\therefore \quad V(r,\theta) = A_0 + \frac{Q}{4\pi\epsilon_0 r} + \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta$$

At $r = R, V = V_0$

$$\therefore \quad A_0 + \frac{Q}{4\pi\epsilon_0 R} = V_0$$

$$\therefore \quad A_0 = V_0 - \frac{Q}{4\pi\epsilon_0 R}$$

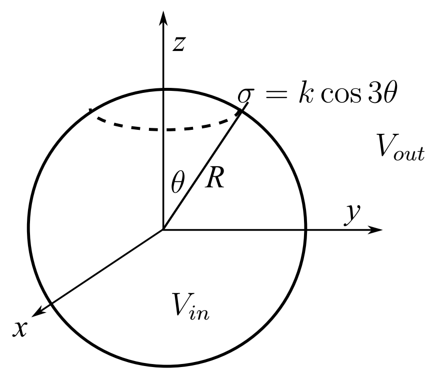
Now we have fixed all the constants. The potential is given as

$$V(r,\theta) = V_0 - \frac{Q}{4\pi\epsilon_0 R} + \frac{Q}{4\pi\epsilon_0 r} + \left(-r + \frac{R^3}{r^2}\right) E_0 \cos\theta$$

We can have any amount of charge Q on the conducting sphere and maintain it at any potential V_0.

4. A sphere of radius R has a surface charge given by the surface charge density $\sigma = k\cos 3\theta$ where k is a constant. Find the potential inside and outside the sphere.

soln



For a problem with azimuthal symmetry potential is

$$\begin{aligned} V_{out} &= \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \\ V_{in} &= \sum_{l=0}^{\infty}\left(A'_l r^l + \frac{B'_l}{r^{l+1}}\right) P_l(\cos\theta) \end{aligned}$$

Since there are no sources at infinity, we will demand $V_out \to 0$ as $r \to \infty$.
This implies $A_l = 0$ for $l = 0, 1, 2, ...$ So we have

$$V_{out} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

As $r \to 0$,

$$V_{in} \to \sum_{l=0}^{\infty} \frac{B'_l}{r^{l+1}} P_l(\cos\theta) \to \infty$$

So for V_{in} to be finite inside we must have $B'_l = 0$ for $l = 0, 1, 2, ...$

$$\therefore \quad V_in = \sum_{l=0}^{\infty} A'_l r^l P_l(\cos\theta)$$

At $r = R$, $\;V_{in}(R) = V_{out}(R)$.

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A'_l R^l P_l(\cos\theta)$$

Since $P_l(\cos\theta)$ are orthogonal functions and hence linearly independent we have

$$\frac{B_l}{R^{l+1}} = A'_l R^l \quad \Longrightarrow \quad B_l = A'_l R^{2l+1} \tag{5}$$

Over the surface of the sphere we have

$$E_{r \;\; out} - E_{r \;\; in} = \frac{\sigma}{\epsilon_0} = \frac{k\cos 3\theta}{\epsilon_0}$$

So we calculate the radial component of the electric field inside and outside.

$$\begin{aligned}
E_{rout} = -\frac{\partial V_{out}}{\partial r} &= -\frac{\partial}{\partial r}\left[\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)\right] \\
&= \sum_{l=0}^{\infty} (l+1)\frac{B_l}{r^{l+2}} P_l(\cos\theta) \\
E_{rin} &= \sum_{l=0}^{\infty} (-l) A'_l r^{l-1} P_l(\cos\theta)
\end{aligned}$$

So at the surface $r = R$ we will have

$$\begin{aligned}
(E_{rout} - E_{rin})|_{r=R} &= \sum_{l=0}^{\infty} \left[(l+1)\frac{B_l}{R^{l+2}} + l A'_l R^{l-1}\right] P_l(\cos\theta) \\
&= \frac{k\cos 3\theta}{\epsilon_0}
\end{aligned}$$

Substituting for B_l from eq.(5) we get

$$\sum_{l=0}^{\infty} (2l+1) A'_l R^{l-1} P_l(\cos\theta) = \frac{k\cos 3\theta}{\epsilon_0} \tag{6}$$

Multiplying both sides by $P_m(\cos\theta)$ and integrating from $\theta = 0$ to π will give A'_l. But let us do it differently.

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{8}{5} P_3(\cos\theta) - \frac{3}{5} P_1(\cos\theta)$$

Substituting in eq.(6) we get

$$\sum_{l=0}^{\infty}(2l+1)A'_l R^{l-1} P_l(\cos\theta) = \frac{k}{5\epsilon_0}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right]$$

We see that only the $l = 1$ and $l = 3$ terms survive on l.h.s.

$$\therefore \quad 3A'_1 = -\frac{3k}{5\epsilon_0} \qquad \Longrightarrow \qquad A'_1 = -\frac{k}{5\epsilon_0}$$
$$7A'_3 R^2 = \frac{8k}{5\epsilon_0} \qquad \Longrightarrow \qquad A'_3 = \frac{8k}{35\epsilon_0 R^2}$$

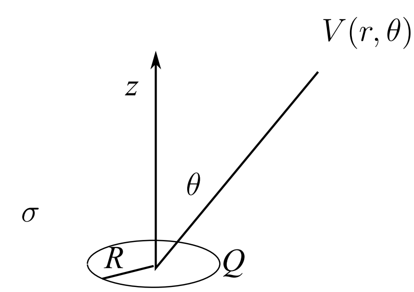
All other $A'_l = 0$.
From Eq.(5) we get

$$\begin{aligned} B_1 &= A'_1 R^3 = -\frac{kR^3}{5\epsilon_0} \\ B_3 &= A'_3 R^7 = \frac{8kR^5}{35\epsilon_0} \end{aligned}$$

$$\begin{aligned} \therefore \quad V_{out} &= \frac{kR^3}{5\epsilon_0}\frac{\cos\theta}{r^2} + \frac{8kR^5}{35\epsilon_0}\frac{P_3(\cos\theta)}{r^4} \\ V_{in} &= \frac{k}{5\epsilon_0} r\cos\theta + \frac{8kR^5}{35\epsilon_0 R^2} r^3 P_3(\cos\theta) \end{aligned}$$

Check that the potential is continuous at $r = R$.

5. A ring of radius R has a charge Q uniformly spread along it. The ring is placed on the x-y plane with the z-axis coinciding with its axis. Find the potential $V(r,\theta)$ in the region surrounding the ring.



soln:
It is easy to calculate the potential due to the ring at a distance r from the center of the ring along the z axis. This is given by

$$V(r,0) = \frac{Q}{4\pi\epsilon_0\sqrt{R^2 + r^2}}$$

First consider $r > R$. We rewrite $V(r,0)$ as

$$V(r,0) = \frac{Q}{4\pi\epsilon_0 r}\left(1+\frac{R^2}{r^2}\right)^{-\frac{1}{2}} = \frac{Q}{4\pi\epsilon_0 r}(1+x)^{-\frac{1}{2}}$$

where $x = \frac{R^2}{r^2}$. We do a taylor expansion of $V(r,0)$ around $x = 0$.

$$\begin{aligned} V(r,0) &= \frac{Q}{4\pi\epsilon_0 r}\left(1 - \frac{1}{2}x + \frac{1}{2}\frac{3}{2}\frac{x^2}{2!} - \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{x^3}{3!} + ...\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(1 - \frac{1}{2}\frac{R^2}{r^3} + \frac{3}{8}\frac{R^4}{r^5} - \frac{10}{16}\frac{R^6}{r^7} + ...\right) \end{aligned}$$

Using the fact that $P_l(0) = 1 \;\forall\; l$ and from the general form of the potential in an azimuthal symmetric electrostatic configuration

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

we can write down the potential at all points (r,θ) as

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left(1 - \frac{1}{2}\frac{R^2}{r^3}P_2(\cos\theta) + \frac{3}{8}\frac{R^4}{r^5}P_4(\cos\theta) - \frac{10}{16}\frac{R^6}{r^7}P_6(\cos\theta) + ...\right)$$

For $r < R$, $V(r,0)$ can be written as

$$V(r,0) = \frac{Q}{4\pi\epsilon_0 R}\left(1+\frac{r^2}{R^2}\right)^{-\frac{1}{2}} = \frac{Q}{4\pi\epsilon_0 R}(1+x)^{-\frac{1}{2}}$$

where $x = \frac{r^2}{R^2}$.
The taylor expansion of $V(r,0)$ around $x = 0$.

$$\begin{aligned} V(r,0) &= \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{1}{2}x + \frac{1}{2}\frac{3}{2}\frac{x^2}{2!} - \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{x^3}{3!} + ...\right) \\ &= \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{1}{2}\frac{r^2}{R^2} + \frac{3}{8}\frac{r^4}{R^4} - \frac{10}{16}\frac{r^6}{R^6} + ...\right) \end{aligned}$$

Comparing with the general form of an azimuthal symmetric potential we can write

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{1}{2}\frac{r^2}{R^2}P_2(\cos\theta) + \frac{3}{8}\frac{r^4}{R^4}P_4(\cos\theta) - \frac{10}{16}\frac{r^6}{R^6}P_6(\cos\theta) + ...\right)$$

Note that the two potentials for $r < R$ and $r > R$ match at $r = R$.
The potential only contains terms with even l. The given problem has symmetry about the x-y plane, i.e, $\theta = \frac{\pi}{2}$. $P_l(\cos\theta)$ has this property for even l. For odd l the legendre polynomials are antisymmetric about $\theta = \frac{\pi}{2}$. So those terms are not present.

6. Solve Laplace's equation by separation of variables in cylindrical co-ordinates, assuming there is no dependence on z.

soln

In cylindrical coordinates, the Laplace's Equation is written as

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Since there is no dependence on z we have

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Let us consider $V(s,\phi) = f(s)g(\phi)$. Then

$$g\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{f}{s^2}\frac{\partial^2 g}{\partial \phi^2} = 0$$

Dividing throughout by V and multiplying by s^2 gives

$$\frac{s}{f}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{g}\frac{\partial^2 g}{\partial \phi^2} = 0$$

Each term above is a constant. The angular function $g(\phi)$ is expected to be periodic. Hence we consider the constant related to it as negative.

Let $\frac{1}{g}\frac{\partial^2 g}{\partial \phi^2} = -k^2$ and $\frac{s}{f}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) = k^2$.

$$\therefore \quad g(\phi) = A\sin(k\phi) + B\cos(k\phi) \tag{7}$$

Since $g(\phi) = g(\phi + 2\pi)$ for any physically meaningful solution, k must be an integer.

The differential Eqn in s is

$$s\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) = k^2 f$$

$f = s^l$ satisfies the equation. Substituting , we get

$l^2 s^l = k^2 f$, i.e $l^2 f = k^2 f \quad \Longrightarrow \quad l = \pm k$.

$\therefore \quad f = Ds^k + Es^{-k}$.

We note that this is valid when $k \neq 0$. If $k = 0$ then

$$\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) = 0 \quad \Longrightarrow \quad s\frac{\partial f}{\partial s} = c_1$$

$\therefore \quad f(s) = c_1 \ln s + c_2$.

Since k has to be an integer, l is also an integer. Hence the general solution in cylindrical coordinates is

$$V(s,\phi) = (c_1 \ln s + c_2)B_0 + \sum_{k=1}^{\infty}\left(D_k s^k + E_k s^{-k}\right)\left(A_k \sin k\phi + B_k \cos k\phi\right)$$

$c_1, c_2, B_0, A_k, B_k, D_k$ and E_k are constants to be determined from the boundary conditions. It is useful to write this potential as

$$V(s,\phi) = (c_1 \ln s + c_2) + \sum_{k=1}^{\infty} s^k\left(A_k \sin k\phi + B_k \cos k\phi\right) + s^{-k}\left(C_k \sin k\phi + D_k \cos k\phi\right)$$