SC223 - Linear Algebra

Aditya Tatu

Lecture 7

August 25, 2022

Solving Linear Equations

$$
\left[\begin{array}{ccccc|c} 0 & 2 & 5 & 4 & 2 & 2 \\ 1 & -1 & 2 & 3 & -1 & 1 \\ 2 & 1 & 0 & 4 & 2 & -1 \\ 3 & 1 & 3 & 2 & -2 & 3 \end{array}\right]
\xrightarrow{R_1 \leftrightarrow R_2}
\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & 2 & 5 & 4 & 2 & 2 \\ 2 & 1 & 0 & 4 & 2 & -1 \\ 3 & 1 & 3 & 2 & -2 & 3 \end{array}\right]
\xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_4 \leftarrow R_4 - 3R_1}
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\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 3 & -4 & -2 & 4 & -3 \\ 0 & 4 & -3 & -7 & 1 & 0 \end{array}\right]
\xrightarrow[R_3 \leftarrow 2R_3 - 3R_2]{R_4 \leftarrow R_4 - 2R_2}
\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 0 & \mathbf{-23} & -16 & 2 & -12 \\ 0 & 0 & -13 & -15 & -3 & -4 \end{array}\right]
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\xrightarrow{R_4 \leftarrow 23R_4 - 13R_3}
\underbrace{\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 0 & \mathbf{-23} & -16 & 2 & -12 \\ 0 & 0 & 0 & \mathbf{-137} & -95 & 64 \end{array}\right]}_{U \text{ (except the last column)}}
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- Any Elementary row operation can be represented as a matrix.
- These matrices will be called *Elementary Row Transformations*.

$$(R_1 \leftrightarrow R_2) \rightarrow P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- $U = L_{43} L_{32} L_{42} L_{31} L_{41} P_{12} A$.

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- Are all elementary row transformations invertible?
- Moreover, the inverse is also lower triangular! (except for permutation RT)
- What matrix will you get if you multiply two lower triangular matrices?$\to$ Lower Triangular!
- Thus, $P_{12}A = L_{43}^{-1}L_{32}^{-1}L_{42}^{-1}L_{31}^{-1}L_{41}^{-1}U = LU$.

LU Decomposition

- Any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed into a product of lower and upper triangular matrices, with appropriate permutations:

$$PA = LU,$$

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$Ax = b \rightarrow LUx = b$.

- First let $Ux = y$ and solve $Ly = b$, and next solve for x in $Ux = y$.

Example

- $U = \begin{bmatrix} \mathbf{1} & -1 & 2 & 3 & -1 \\ 0 & \mathbf{2} & 5 & 4 & 2 \\ 0 & 0 & -\mathbf{23} & -16 & 2 \\ 0 & 0 & 0 & -\mathbf{137} & -95 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 2 \\ -12 \\ 64 \end{bmatrix}.$

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- Solve for x in

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$$R_k \cdot R_{k-1} \cdot \dots R_1 \left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

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- $\mathcal{O}(n^3)$
- Why should one use LU decomposition?