Functions of ~~Severlar~~ Several variables

$$z = f(x_1, x_2, \ldots, x_n)$$

$$f : D \to \mathbb{R}, \quad D \subset \mathbb{R}^n$$

__Exp__ radius of a sphere with center at the origin in $\mathbb{R}^3$.

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

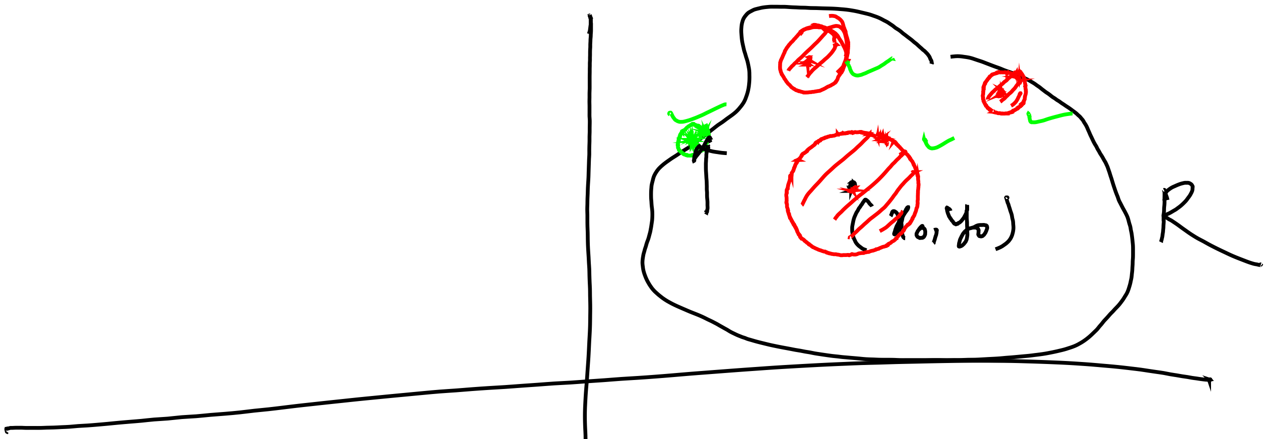
$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

__Exp__ Volume of a cone of base radius r and height h.

$$V(r, h) = \frac{1}{3}\pi r^2 h$$

Functions of two variables

__Interior point__ A point (x_0, y_0) in a region R in the xy-plane is an interior point of R if it is the center of a disk that lies entirely in the region R.



Interior point

(x_0, y_0) is an interior point of R if there exists a disk $D((x_0,y_0), r)$ such that

$$(x_0, y_0) \in D((x_0, y_0), r) \subset R.$$

Boundary point

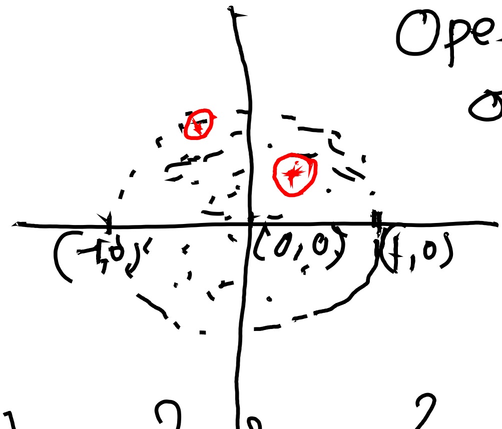
A point (x_0, y_0) is said to be a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R.

A region is said to be open if it consists of entirely of interior points. or each of ~~the~~ its points are interior points.

A region is said to be closed if it contains all its boundary points.

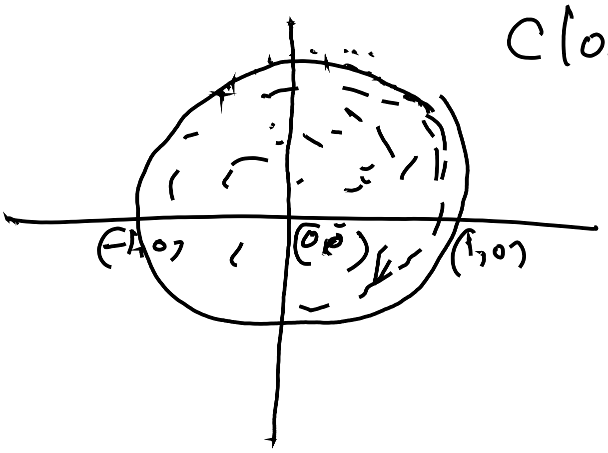
EXP $R : \{(x, y) : x^2 + y^2 < 1\}$

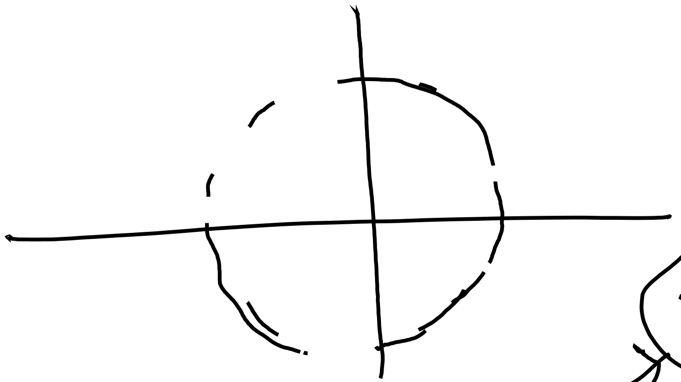
Open disk of radius 1.



EXP $R : \{(x, y) : x^2 + y^2 \leq 1\}$

Closed disk





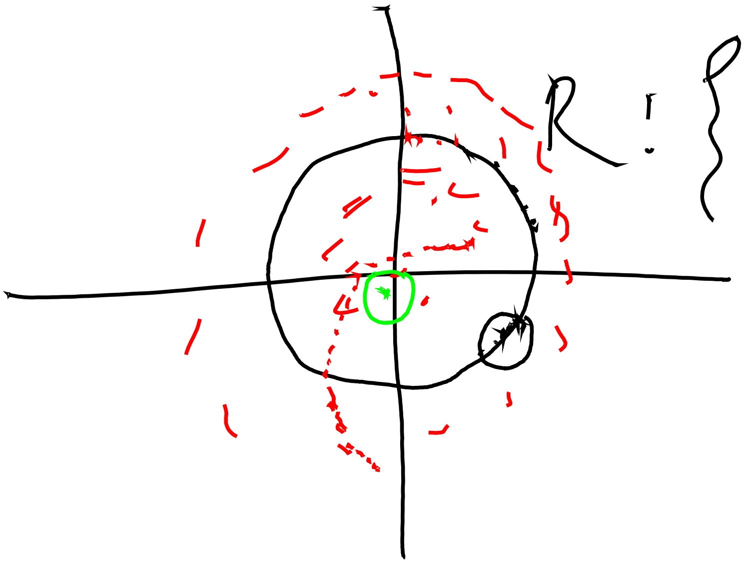
R_1

$x^2+y^2<1$

$\neq$

R_2 $x^2+y^2 \le .99999$

~~(0, .9999)~~



$R: \{(x,y): x^2+y^2=1\}$

closed region.

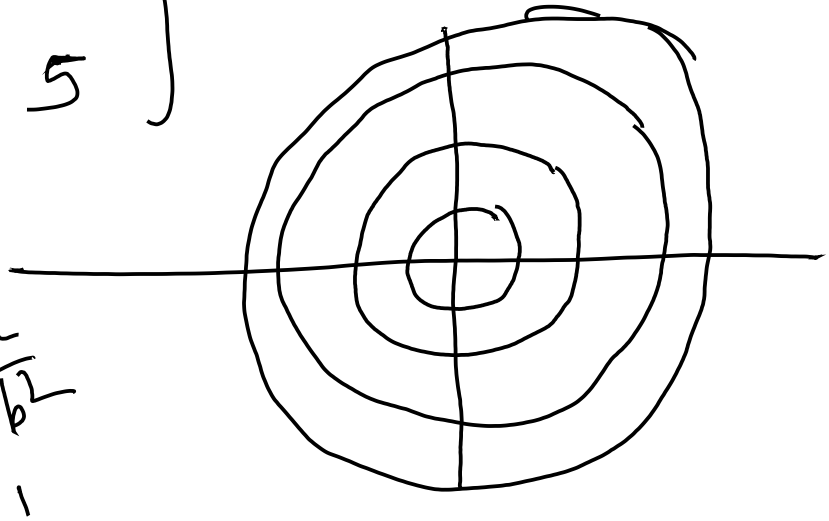
(x_0, y_0) in R

Level curves

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a level curve.

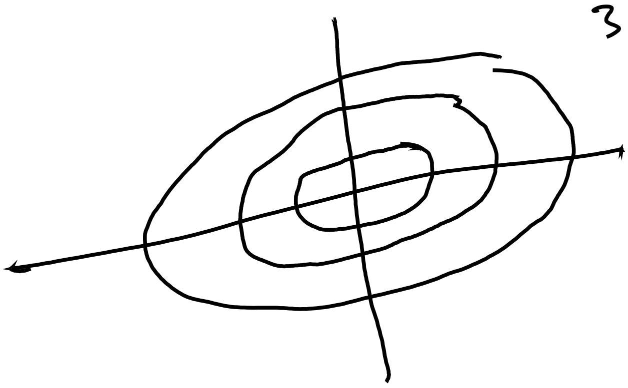
Exp $z = f(x,y) = x^2+y^2$

$$\left.\begin{array}{l} x^2+y^2 = 1 \\ x^2+y^2 = 2 \\ x^2+y^2 = 5 \end{array}\right\} \text{level curves}$$



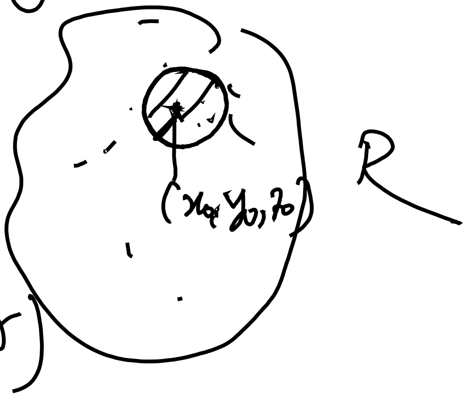
Exp $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

1
2
3



$\mathbb{R}^3$

A point (x_0, y_0, z_0) in a region R in space is called an interior point of if is the center of a solid ball that lies entirely in R.



(x_0, y_0, z_0) is an interior point if there exists a ball $B((x_0, y_0, z_0), r)$

such that

$$(x_0, y_0, z_0) \in B((x_0, y_0, z_0), r) \subset R.$$

Boundary point

(x_0, y_0, z_0) is a boundary point of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie in R.

Limit of a function of two variables

If the values of $f(x,y)$ lie arbitrarily close to a fixed real number L for all points (x,y) sufficiently close to a point (x_0, y_0), we say that $f(x,y)$ approaches the limit L as (x,y) approaches (x_0, y_0).

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

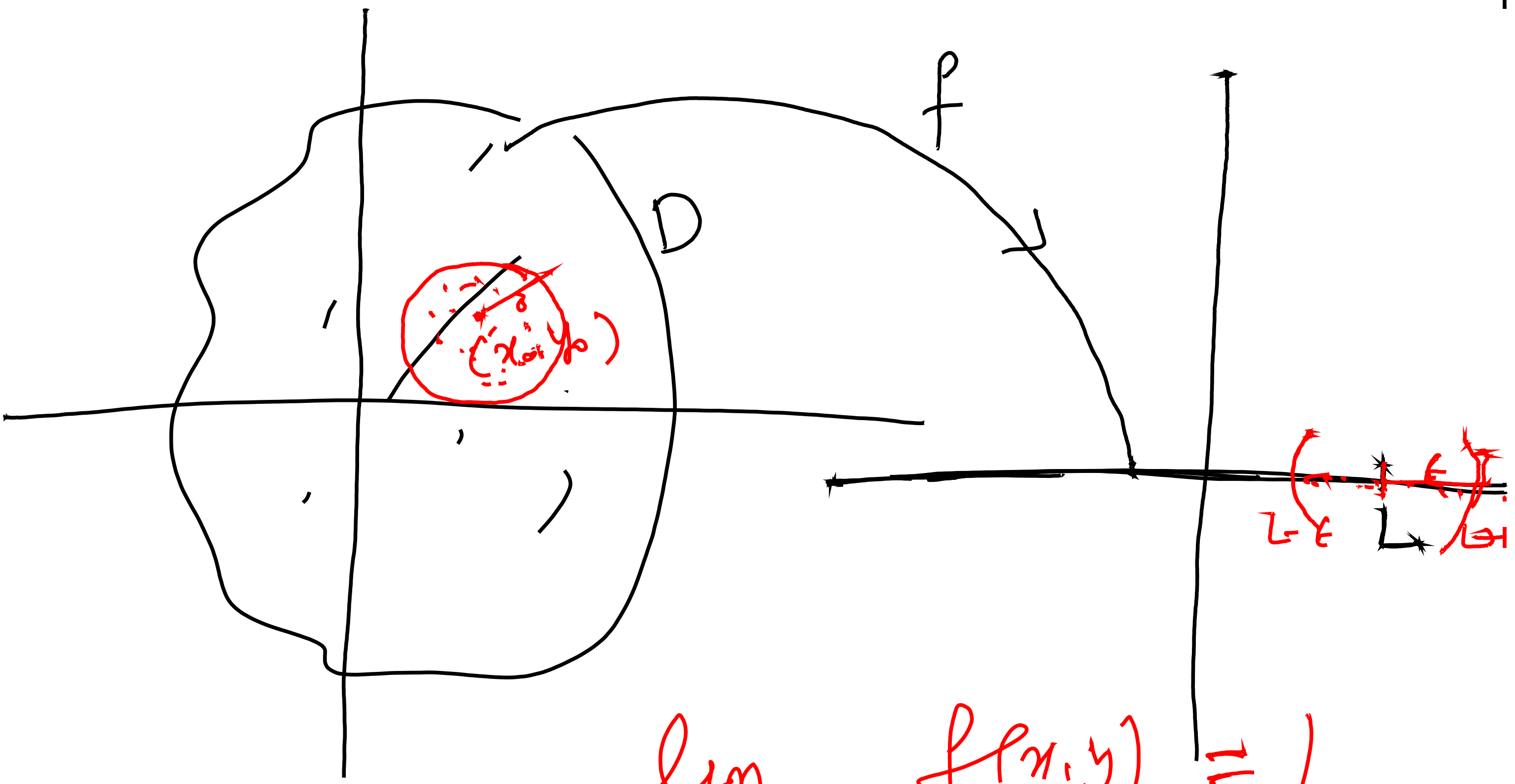
Assumption (x_0, y_0) is an ~~the~~ interior point of the domain of the function $f(x,y)$.

If for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all (x,y) in the domain of f.

$$|f(x,y) - L| < \epsilon \quad \text{whenever}$$

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

$f : D \subset \mathbb{R}^2 \to \mathbb{R}$



$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$

Then

① $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm g(x,y) = L \pm M$

② $\displaystyle\lim_{(x,y)\to(x_0,y_0)} K f(x,y) = KL$

K is any scalar.

(3) $$\lim_{(x,y)\to(x_0,y_0)} f(x,y)\,g(x,y) = LM$$

(4) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$$

provided $M \neq 0$.

(5) $$\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)\right)^n = L^n$$

n positive integer.

(6) $$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

n is positive integer.

$L > 0$.

$$f(x,y) = xy$$

$$\lim_{(x,y)\to(0,0)} xy = 0$$

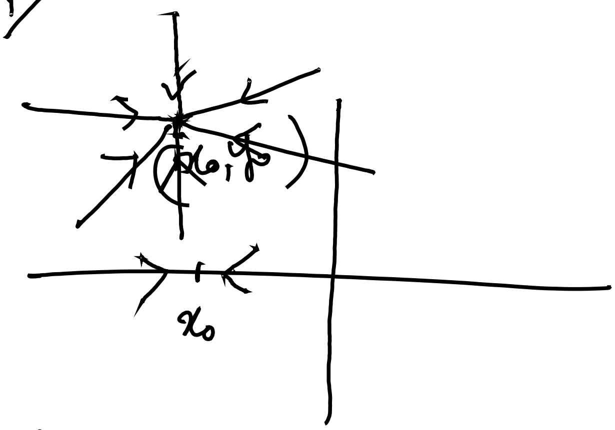
$$\lim_{(x,y)\to(0,0)} x^2y^2 = 0$$

$$\lim_{(x,y)\to(0,0)} \cancel{\frac{1}{xy}} \quad X$$

Exp Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$

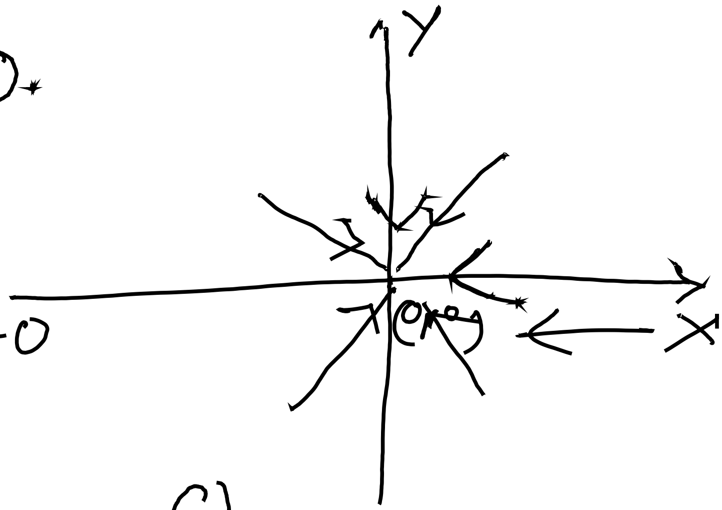
if it exists.

Sol^n



Along the line $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0.$$



Along the line $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$$

So If the limit exists, then it should be equal to 0.

Now we apply the definition of $\lim_{x\to a} f(x)$ for existence.

Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2+y^2} - 0 \right| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta.$$

?

$$\left| \frac{4xy^2}{x^2+y^2} \right|$$

$y^2 \le x^2+y^2$

$$= \frac{4|x|\,y^2}{x^2+y^2} \le \frac{4|x|\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$$= 4|x|$$
$$= 4\sqrt{x^2} \le 4\sqrt{x^2+y^2}$$

$$\left| \frac{4xy^2}{x^2+y^2} \right| \le 4\sqrt{x^2+y^2} < \epsilon$$

If we choose $\delta = \frac{\epsilon}{4}$

then

$$\left|\frac{4xy^2}{x^2+y^2}\right| \le \boxed{4\sqrt{x^2+y^2} < \epsilon}$$

(x, y)