Feb 10, 2020

— $m \times n$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots a_{ij} & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

n columns, m rows, $i \to$ row, j^{th} column

$$A = (a_{ij}) \qquad 1 \le i \le m, \ 1 \le j \le n$$

— $m = n$ (square matrix)

$$tr(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

— Basic Matrix opn^s

(1) Addition $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$ $\quad 1 \le i \le m$, $1 \le j \le n$

(2) $\times$ by scalar $\lambda [a_{ij}] = [\lambda a_{ij}]$

(3) $\times$ of two matrices

$$A = [a_{ij}]_{m \times n} \qquad B = [b_{ij}]_{n \times p}$$

then $C = AB$ $m \times p$ and $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$1 \le i \le m, \ 1 \le j \le p$$

(4) A & B commute

if $AB = BA$, in general $AB \ne BA$

— Diagonal matrix $A = [a_{ij}]$ is diagonal if

$a_{ij} = 0 \ (i \ne j)$, $A = \mathrm{diag}[a_{11} \ a_{22} \cdots a_{nn}]$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \qquad I_n = \mathrm{diag}[1 \ 1 \cdots 1]$$

$$A I_n = I_n A = A_{n \times n}$$

unit or Identity matrix

(5) Transpose $A = [a_{ij}]$ transpose $A^t = [a_{ji}]$ $\quad 1 \le j \le n$, $1 \le i \le m$

If $a_{ij} \in \mathbb{C}$ $\quad \bar{A} = [\bar{a}_{ij}]$

conjugate transpose $A^* = (\bar{A})^t = [\bar{a}_{ji}]$.

(6) Properties

$(A^t)^t = A$, $(AB)^t = B^t A^t$, $(A^*)^* = A$

$(AB)^* = B^* A^*$

$\boxed{\text{Determinant}}$ $A = (a_{ij})_{n\times n}$

$$\det(A) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\, a_{i\sigma(i)}$$

$n=2$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$\boxed{\text{Inv.}}$ A $n\times n$ If $\det(A) \neq 0$, $A \to$ non-singular

$A^{-1}A = AA^{-1} = I_n$ If $\det(A) = 0$ $A \to$ singular

$\boxed{\text{Explicit form}}$ $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$

$\boxed{\text{Properties}}$

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (A^t)^{-1} = (A^{-1})^t, \quad (A^*)^{-1} = (A^{-1})^*$$

$\boxed{\text{Rank}}$ A $m\times n$ matrix

- Submatrix: Any k ($k \le m$) rows & l ($l \le n$) columns of A form a $k\times l$ submatrix of A
- Rank(A) := the order of the largest non singular submatrix of A

In particular, when $m=n$ then $R(A) = n$ iff A is non singular

$\boxed{\text{Properties}}$ $R(A) = 0$ iff $A = 0$

- $R(A^t) = R(A)$
- $R(A+B) \le R(A) + R(B)$
- $R(A) + R(B) - n \le R(AB) \le \min\{R(A), R(B)\}$ B $n\times p$ matrix
- If $B = PAQ$ with P & Q non singular then $R(B) = R(A)$ & $B \cong A$

Linear Equations

$$A\bar{x} = \bar{b}$$

$$A = (a_{ij})_{m\times n} \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \& \quad \bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad , \quad 1 \le i \le m$$

$B = [A, b]_{m\times(n+1)}$ then $R(A) \le R(B)$

(i) $\exists$! soln. iff $R(A) = R(B) = n$

(ii) An ∞ # of solns iff $R(A) = R(B) < n$

(iii) No soln iff $R(A) < R(B)$

In case (i), when A is $\square$ matrix the soln. is

$$\bar{x} = A^{-1}\bar{b}$$

Eigenvalues & Eigen $V^{r(s)}$

Cha. Poly.

$$\det(A - \lambda I_n) = 0$$

— If roots of $\det(A-\lambda I_n) = 0$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$ (Known as Eigen values of A) then

— $tr(A) = \sum_{i=1}^{n} \lambda_i$ & $\det(A) = \prod_{i=1}^{n} \lambda_i$

Eigen V^r A V^r $\bar{x}\,(\neq 0)$ is called Eigen V^r if

$$A\bar{x} = \lambda \bar{x}$$

— $A = (a_{ij})_{2\times 2}$ Eigen values λ_1 & λ_2

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4\det(A)}}{2}$$

— If $\det(A) = 0 \Rightarrow \lambda_1 = tr(A)$ & $\lambda_2 = 0$

Block Matrix

- $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{*} = \left(\begin{array}{c|c} A^{*} & C^{*} \\ \hline B^{*} & D^{*} \end{array}\right)$

- $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) = \left(\begin{array}{c|c} AA' + BC' & AB' + BD' \\ \hline CA' + DC' & CB' + DD' \end{array}\right)$

Note
# of rows in A = # of rows in B
" " " C = " " " D
of columns in A = # of colums in C
" " " B = " " " in D

$$M = \left(\begin{array}{c|c} A_{m\times n} & B_{m\times n'} \\ \hline C_{m'\times n} & D_{m'\times n'} \end{array}\right)_{(m+m')\times(n+n')}$$

- $A_{n\times n}$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ Each appear $\mu_A(\lambda_i)$ times in the list
alg. multiplicity

1. $tr(A) = \sum_{i=1}^{n} a_{ii} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
2. $\det(A) = \prod_{i=1}^{n} \lambda_i$
3. Eigenvalues of A^K (for +ve K) will be $\{\lambda_1^K, \lambda_2^K, \ldots, \lambda_n^K\}$
4. A is invertible iff every eigenvalue is nonzero
5. If A is invertible then eigenvalues of A^{-1} are $\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\}$
6. If $A = A^{*}$ every eigenvalue is real
(also true for $A = A^{*}$)
7. If A is unitary ($A^{*}A = AA^{*} = I$) then $|\lambda_i| = 1$
8. If $A_{n\times n}$ $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ then for $I + A$ eigenvalues are $\{\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1\}$

$\boxed{\text{Hadamard Matrix}}$

$$H = (\pm 1)_{n \times n} \quad \text{s.t. } HH^t = nI$$

$$\langle R_i, R_j \rangle = 0 \quad (\because H^{-1} = (\tfrac{1}{n})H^t \Rightarrow H^t H = nI)$$

$$\therefore \langle c_i, c_j \rangle = 0$$

— Normalized Hadamard Matrix $\quad R_1 = (11\cdots 1)$, $c_1 = (11\cdots 1)$

$n=1 \quad H_1 = (1) \qquad n=2 \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$n = 4 \quad H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{pmatrix}$

$\boxed{\text{Theorem}}$ If a Hadamard matrix H of order n exists then $n = 1, 2$ or $n \equiv 0 \pmod 4$ i.e. $n = 4K$

$\square$ W.L.O.G. Suppose H is normalized. Suppose $n \geq 3$ $(n \geq 3)$

$$\begin{array}{llccc}
 & 111\cdots1 & 11\cdots1 & 11\cdots1 & 11\cdots1 \\
R_1 & 11\cdots1 & 11\cdots1 & --\cdots- & --\cdots- \\
R_2 & 11\cdots1 & --\cdots- & 11\cdots1 & --\cdots- \\
R_3 & \underbrace{11\cdots1}_{i} & \underbrace{--\cdots-}_{j} & \underbrace{11\cdots1}_{K} & \underbrace{--\cdots-}_{l}
\end{array}$$

$\because$ Rows are orthogonal $\Rightarrow$

$$\left.\begin{array}{l} i + j - k - l = 0 \\ i - j + k - l = 0 \\ i - j - k + l = 0 \end{array}\right\} \Rightarrow i = j = k = l \Rightarrow n = 4i$$

a multiple of 4

$\boxed{\text{Conjecture}}$ Hadamard matrix exist whenever order is a multiple of 4 (No proof is known yet).

$\boxed{\text{Construction (1)}}$ If H_n is a Hadamard matrix of order n (Slyvester) then $H_{2n} = H_n \otimes H_2 = \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}$ (of order $2n$)

$\therefore$ if u start with $H_1 = (1)$ this gives $H_2, H_4, H_8 \ldots$

$\therefore$ Hadmard matrix of all orders which are powers of 2.

$\boxed{\text{Quadratic residues}}$ $p \neq 2$ prime

— Non zero squares mod p i.e. the numbers
$1^2, 2^2, 3^2, \ldots$ (mod p) called the q.r. mod p

$\because (p-a)^2 \equiv (-a)^2 \equiv a^2 \pmod{p}$

We consider $1^2, 2^2, \ldots, \left(\frac{p-1}{2}\right)^2 \pmod{p}$

$p = 11$

$1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16 = 5$

$5^2 = 25 \equiv 3$

$\therefore$ Q.R. $= \{1, 3, 4, 5, 9\}$

Q.N.R. $= \{2, 6, 7, 8, 10\}$

$A \equiv B \pmod{c}$
$\Rightarrow c \mid A - B$
or $A-B$ is $\times$ of c.

$\boxed{\text{Properties}}$ (1) q.r. $\times$ q.r $=$ q.r.
q.n.r $\times$ q.n.r $=$ q.r
q.r $\times$ q.n.r $=$ q.n.r

(2) If $p = 4k+1$ then -1 is a q.r. mod p
if $p = 4k+3$ then -1 is a q.n.r. mod p

(3) $p\ (\neq 2)$ prime $\chi \Rightarrow$ Legendre symbol
$\chi(i) = 0$ if i is a multiple of p
$\chi(i) = 1$ if the remainder when $i \pmod{p}$ is q.r.
$= -1$ if $i \pmod{p}$ is q.n.r.

Jacobsthal matrix

$Q = (q_{ij})_{p \times p} \qquad q_{ij} = \chi(j-i)$

$p = 7$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & - & 1 & - & - \\ - & 0 & 1 & 1 & - & 1 & - \\ - & - & 0 & 1 & 1 & - & 1 \\ 1 & - & - & 0 & 1 & 1 & - \\ - & 1 & - & - & 0 & 1 & 1 \\ 1 & - & 1 & - & - & 0 & 1 \\ 1 & 1 & - & 1 & - & - & 0 \end{bmatrix} \end{array}$$

$n = p+1 \equiv 0 \pmod 4$

$q_{ij} = \chi(j-i) = \chi(-1)\,\chi(i-j) = -q_{ji}$

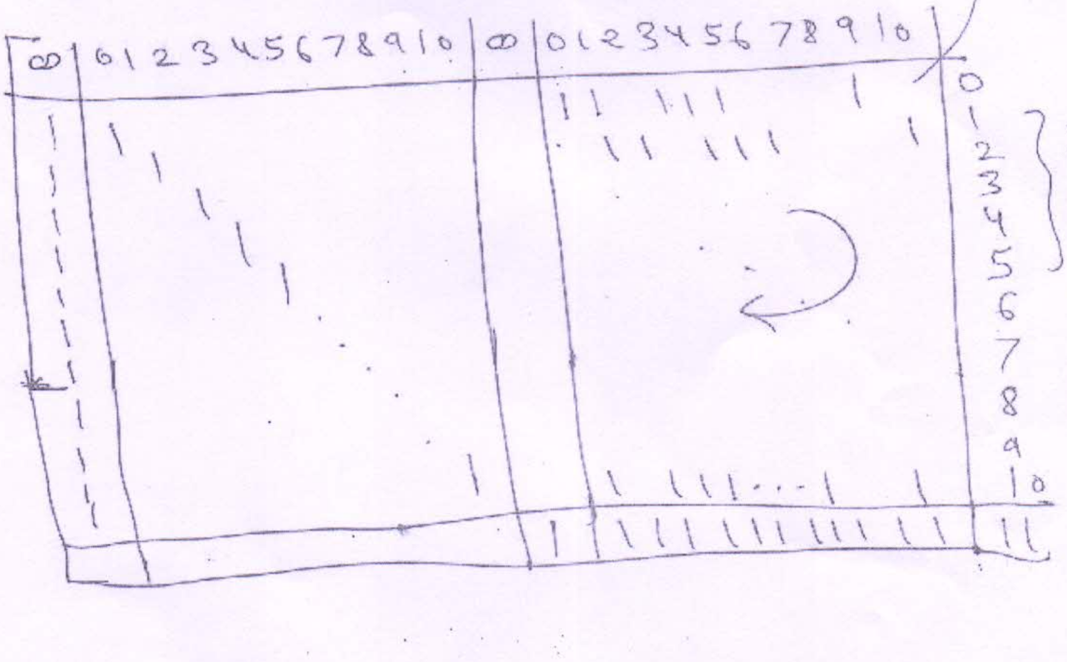
$$H = \begin{pmatrix} 1 & \top \\ \top^t & Q - I \end{pmatrix} \qquad HH^t = (p+1) I_p$$

normalized Hadamard matrix

Paley type Hadamard matrix

Extended Golay Code : $[24, 12]$

$G =$



All sum of any two rows has wt. = 6

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$$

Fourier Matrix

Let $n \geq 1$ $\quad \omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \quad i = \sqrt{-1}$

n^{th} root of unity

1. $\omega^n = 1$
2. $\omega\bar{\omega} = 1$
3. $\bar{\omega} = \omega^{-1}$
4. $\bar{\omega}^k = \omega^{-k} = \omega^{n-k}$
5. $1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$

Fourier matrix of order n $\quad F = F_n$

s.t. $F^* = \frac{1}{\sqrt{n}}\left(\omega^{(i-1)(j-1)}\right)$

$$= \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

ω^k is periodic
$(k = 0, 1, 2, \dots)$

$$\Rightarrow F^* = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{n-2} & & \omega \end{pmatrix}$$

$n = 2$

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$n = 4$

$$F = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \qquad \omega = e^{-2\pi i/4} = -i$$

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix}$$

① $F = F^t$ ③ $F = \overline{F^*}$

② $F^* = (F^*)^t = \overline{F}$

Th. F is unitary

$$FF^* = F^*F = I \quad \text{or} \quad F^{-1} = F^*$$

$$F\overline{F}^t = \overline{F}^t F = I \quad \text{or} \quad F^{-1} = \overline{F}^t$$

Th. $F^{2*} = F^* F^* = F^2$

Eigenvalues of F are $\pm 1, \pm i$

$\boxed{\text{Circulant Matrix}}$

$$C = \text{circ}(c_1\ c_2 \cdots c_n)$$

$$= \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}$$

$$C = (c_{jk}) = (c_{k-j+1}) \quad \text{sub mod } n$$

— ~~A is circulant iff~~
A is circulant iff A^* is circulant

$$\underset{n\times n}{A} \quad \bar{x}\ (\neq 0)$$

$$A\bar{x} = \lambda\bar{x}$$

$\bar{x}$ → Eigen vr

λ → Eigen value

$$(A - \lambda I_n)\bar{x} = 0$$

$$\boxed{\det(A - \lambda I_n) = 0}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda^2 - \lambda - 3\lambda + 3$$

$$\lambda(\lambda - 1) - 3(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1 \ \& \ \lambda = 3$$

✓

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$\lambda = 1$

$A\bar{x} = \lambda \bar{x}$ $\quad \bar{x} \neq 0$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x} = 1 . \bar{x}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$2x_1 + x_2 = x_1$$
$$x_1 + 2x_2 = x_2$$

$$x_1 = -x_2$$

Soln. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x} = 3 \bar{x}$$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \quad \end{pmatrix}$$

$A_{n\times n}$ $\quad \bar{x} (\neq 0)$

$A\bar{x} = \lambda \bar{x} \rightarrow$ Eigen vr

$\lambda \rightarrow$ Eigen value

$(A - \lambda I_n)\bar{x} = 0$

$$\boxed{\det(A - \lambda I_n) = 0}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0$

$4 - 4\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda^2 - \lambda - 3\lambda + 3$

$\lambda(\lambda-1) - 3(\lambda-1) = 0$

$(\lambda-1)(\lambda-3) = 0$

$\lambda = 1$ & $\lambda = 3$

✓

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$$

A $n \times n$ complex matrix

$\lambda_1, \lambda_2, \dots \lambda_n$

① $tr(A) = \sum_{i=1}^{n} a_{ii} = \lambda_1 + \lambda_2 + \dots + \lambda_n$

② $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

③ A^K $\{\lambda_1^K, \lambda_2^K, \dots, \lambda_n^K\}$

④ A is invertible iff every eigenvalue is nonzero

A^{-1} $\{1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_n\}$

⑤ If $A = A^*$ eigenvalues are real

⑥ $AA^* = A^*A = I \Rightarrow |\lambda_i| = 1$
(A unitary)

Hadamard Matrix

$$H = (\pm 1)_{n \times n}$$

$$HH^t = nI_n \checkmark$$

$n = 1 \quad H = (1)$

$n = 2 \quad H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$HH^t = 2I_2$$

$n = 4 \quad H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

Normalized Had. matrix

Th. If H is a Hadamard matrix of order n then $n = 1$ or 2 or $n = 4K$

$$\begin{array}{cllll}
 & \overbrace{111\cdots1}^{i} & \overbrace{11\cdots1}^{j} & \overbrace{11\cdots1}^{K} & \overbrace{11\cdots1}^{l} \\
R_1 & 111\cdots1 & 11\cdots1 & 11\cdots1 & 11\cdots1 \\
R_2 & 11\cdots1 & 11\cdots1 & -1-1\cdots-1 & -1-1\cdots-1 \\
R_3 & 11\cdots1 & -1-1\cdots-1 & 111\cdots1 & -1-1\cdots-1
\end{array}$$

$$\left.\begin{array}{l} i + j - K - l = 0 \\ i - j + K - l = 0 \\ i - j - K + l = 0 \end{array}\right\}$$

$\Rightarrow i = j = K = l$

$n = 4K$

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H_4 = H_2 \otimes H_2$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes H_2$$

$$= \begin{pmatrix} 1 \cdot H_2 & 1 \cdot H_2 \\ 1 \cdot H_2 & -1 \cdot H_2 \end{pmatrix}$$

$$= \begin{pmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$\boxed{\text{Matrix}}$

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$$\begin{array}{cc} & n \text{ colns} \\ A = & \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & \cdots & a_{ij} & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

m rows

$\to i^{th}$ row

$\downarrow j^{th}$ coln

$m \times n$

$$A = (a_{ij}) \quad 1 \le i \le m, 1 \le j \le n$$

– $m = n$ $\square$_matrix

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

$\boxed{\text{Basic Matrix opns}}$

① Addition $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$1 \le i \le m, 1 \le j \le n$

② $\times$ by scalar

$$\lambda [a_{ij}] = [\lambda a_{ij}]$$

③ $\times$ of two matrices

$$A = [a_{ij}]_{m \times n} \quad B = [b_{ij}]_{n \times p}$$

$$C = (AB)_{m \times p}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$AB \neq BA$

- If $AB = BA$ A & B commute

$\boxed{\text{Diagonal Matrix}}$

$A = (a_{ij})$ is diagonal

if $a_{ij} = 0$ $(i \neq j)$

$A = \text{diag}(a_{11}\ a_{22}\ \dots\ a_{nn})$

If $a_{ii} = 1\ \forall i$

$I_n = \text{diag}(1\,1\,1 \dots 1)$

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$ Identity or unit matrix

$A \cdot I_n = I_n A = A_{n \times n}$

$\boxed{\text{Transpose}}$ $A = (a_{ij})$

$A^t = (a_{ji})$ $1 \leq j \leq n$, $1 \leq i \leq m$

- If $a_{ij} \in \mathbb{C}$ $\overline{A} = (\overline{a_{ij}})$

Conjugate transpose

$A^* = (\overline{A})^t = [\overline{a_{ji}}]$

Properties

① $(A^t)^t = A$

② $(AB)^t = B^t A^t$

③ $(A^*)^* = A$

④ $(AB)^* = B^* A^*$

—— x ——

$$A = (a_{ij})_{n\times n}$$

Determinant of A

$$\det(A) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\, a_{i\sigma(i)}$$

$n=2$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Inverse $A_{n\times n}$ $\det(A) \neq 0$

$A \to$ non singular

$$A^{-1}A = AA^{-1} = I_n$$

If $\det(A) = 0$ $A \to$ singular

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

Properties

① $(AB)^{-1} = B^{-1}A^{-1}$

② $(A^{t})^{-1} = (A^{-1})^{t}$

③ $(A^{*})^{-1} = (A^{-1})^{*}$

$\boxed{\text{RANK}}$

A $m \times n$ matrix

— Submatrix:
Any k ($k \leq m$) rows
& l ($l \leq n$) columns of A
forms a $k \times l$ submatrix of A

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{3 \times 3}$$

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

— Rank(A) := The order of the largest non-singular submatrix of A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

$\text{rank}(A) = \cancel{2}\ 1$

- $m = n$

$R(A) = n$ iff A is nonsingular

$\boxed{\text{Properties}}$

1. $R(A) = 0$ iff $A = 0$

2. $R(A^*) = R(A)$

3. $R(A+B) \le R(A) + R(B)$

- If $B = PAQ$, P & Q are nonsingular

$R(A) = R(B)$ & $B \cong A$

$\boxed{\text{Linear Equations}}$

$A\bar{x} = \bar{b}$

$A = (a_{ij})_{m \times n}$

$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ & $\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

$A \quad n \times n$

$\bar{x} \neq 0$

$A\bar{x} = \lambda \bar{x}$

$\Rightarrow \quad (A - \lambda I_n)\bar{x} = 0$

$$\boxed{\det(A - \lambda I_n) = 0}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$$

$$A - \lambda I_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A - \lambda I_2 = \begin{pmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{pmatrix}$$

$\det(A - \lambda I_2) = 0$

$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$

$a_{11}a_{22} - \lambda a_{11} - \lambda a_{22} + \lambda^2 - a_{12}a_{21} = 0$

$\lambda^2 - \lambda\underline{(a_{11} + a_{22})} + \underline{(a_{11}a_{22} - a_{12}a_{21})} = 0$

$$\boxed{\lambda^2 - \lambda\, tr(A) + \det(A) = 0}$$

$\boxed{\text{Generalizations}}$

$$A A^t = pI - J \qquad p \text{ integer}$$

Identity — All 1's

$$BB^t = rI + sJ \qquad r \& s \text{ are integers}$$

$s = 0$ will give Hadamard.

$$HH^t = nI$$

$$\det(HH^t) = \det(H) \cdot \det(H^t)$$
$$= (\det(H))^2 = n^n$$

$\therefore$ H has determinant $\pm n^{n/2}$

Conference Matrix

$$C = (\pm 1)_{n \times n}$$

$$\text{s.t. } CC^t = (n-1) I_n$$

$$C = \begin{pmatrix} 0 & \bar{1} \\ \bar{1} & S \end{pmatrix} \qquad S_{n-1 \times n-1} \text{ s.t.}$$
$$SS^t = (n-1)I - J$$
$$SJ = JS = 0$$

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

Fano Plane

Fourier-Matrix

$(n \geq 1)$

$$\omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$i = \sqrt{-1}$

① $\omega^n = 1$

② $\omega\bar{\omega} = 1$

③ $\bar{\omega}^K = \overline{\omega^K} = \omega^{n-K}$

④ $\bar{\omega} = \bar{\omega}^1$

⑤ $1 + + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

$F = F_n$ s.t.

$$F^* = \frac{1}{\sqrt{n}}\left(\omega^{(i-1)(j-1)}\right)_{n\times n}$$

$n = 2$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H = \left(\begin{array}{c|c} 1 & \mathbb{1} \\ \hline \mathbb{1}^t & Q-I \end{array}\right)$$

$$HH^t = (p+1) I_{p+1}$$

$\boxed{\text{Conference Matrix}}$

$$C = (\pm 1)_{n \times n}$$

s.t. $CC^t = (n-1) I_n$

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

$$\boxed{AA^t = pI - J}$$

Identy

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$BB^t = rI + sJ$$

(-1) is q.r. mod p if $p = 4k+1$

(-1) is q.n.r. mod p if $p = 4k+3$

$\boxed{\text{Legendre symbol}}$

$p\ (\neq 2)$ prime

$\chi(i) = 0$ if i is a multiple of p

$\chi(i) = 1$ if i is q.r. mod p

$\chi(i) = -1$ if i is q.n.r. mod p

$\boxed{\text{Jacobsthal Matrix}}$

$$Q = (q_{ij})_{p \times p}$$

$$q_{ij} = \chi(j-i)$$

$p = 7$

Q.R $= \{1, 2, 4\}$

Q.N.R $= \{3, 5, 6\}$

$p=7$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

$$q_{ij} = \chi(j-i)$$
$$= \chi(-1)\,\chi(i-j)$$
$$= -q_{ji}$$

$$HH^t = nI$$

$$\det(HH^t) = \det(H) \cdot \det(H^t)$$

$$= (\det(H))^2 = n^n$$

$$\det(H) = \pm n^{n/2}$$

$p\ (\neq 2)$ prime

$1^2\ 2^2\ 3^2 \ldots \pmod p$

q.r. mod p

$p = 5$

$1^2 = 1$ ✓ 1 2 3 4

$2^2 = 4$ ✓ q.r.

$3^2 = 4$ Q.R. $= \{1, 4\}$

$4^2 = 1$ Q.N.R. $= \{2, 3\}$

1. q.r. $\times$ q.r $=$ q.r
2. q.n.r $\times$ q.n.r. $=$ q.r.
3. q.r. $\times$ q.n.r. $=$ q.n.r