SC223 - Linear Algebra

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Lecture 13

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Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot \mathbb{F} \times V \to V$.
- **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
- **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

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- Any element of the vector space $(V, +, \cdot)$ will be referred to as a **vector**, and any element $a \in \mathbb{F}$ will be referred to as a **scalar**.

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  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c, a \times (b +_F c) = a \times b +_F a \times c$

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- $(\mathbb{L}_2(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathbb{L}_2(\mathbb{R})$ denotes the set of all square-integrable functions $f : \mathbb{R} \to \mathbb{R}$.

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- **Proposition 5:** $\forall v \in V, (-1) \cdot v = -v$.

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▶ $V = \mathcal{L}^2(\mathbb{R}), W = \{f \in V \mid \int_{-\infty}^{\infty} f(t)\, dt = 0\}$.

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- Familiar examples of Subspaces: Let $A \in \mathbb{R}^{m \times n}$. Then, $C(A), N(A^T)$ and $N(A), C(A^T)$ are subspaces of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

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- **Definition:** (Sum of subspaces): Let $U_1, \ldots, U_n$ be subspaces of V. The **sum of subspaces** $U_1, \ldots, U_n$ is defined as:

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- **Proposition 7:** The sum of subspaces $U_1, \ldots, U_n$ of V is a subspace.

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- **Definition:** (Direct Sum of Subspaces) In a VS V with subspaces $U_1, \ldots, U_n$, $W = U_1 + \ldots + U_n$ is said to be a **Direct Sum** if $\forall w \in W$, w is **uniquely** expressed as a sum of elements $w_i \in U_i, i = 1, \ldots, n$.

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- Direct sum notation: $W = U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

- **Proposition 8:** Let $U_1, \ldots, U_n$ be subspaces of V. Then $V = U_1 \oplus \ldots \oplus U_n$ if and only if: (1) $V = U_1 + \ldots + U_n$, and (2) The only representation of $\theta \in V$ is $(\theta, \ldots, \theta)$.

- **Proposition 9:** Let V be a VS with subspaces U_1, U_2. Then $V = U_1 \oplus U_2$ *iff* $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\theta\}$.

Span and Linear Independence

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- **Linearly independent set**: Let V be a vector space and let $W = \{v_1, \ldots, v_n\} \subset V$. We say that the set W is a set of linear independent vectors, if

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- **Proposition 11:** In a FDVS, the number of elements in a linearly independent set of vectors is always less than equal to the number of elements in a spanning set.
- Proof: Let V be a VS and $U = \{u_1, \ldots, u_n\}, W = \{w_1, \ldots, w_m\}$ be its subsets such that $span(U) = V$ and W is LI.
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- Is it possible that $m > n$?
- If so, after n iterations, we will reach a contradiction: $span(\{w_1, w_2, \ldots, w_n\}) = V$

Basis of a Vector space

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- **Proposition 14:** Any set of basis vectors of a VS contains the same number of elements.
- **Dimension of a Vector Space:** Let V be a FDVS. For any set of basis vectors β of V, we define the dimension of V as $dim(V) := |\beta|$.