SC223 - Linear Algebra

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Lecture 2

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Row Picture

- Let us look at each row of the system:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Possibilities for a 2×2 system

Possibilities for a 3×3 system

Column Picture

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

can be re-written as

$$x \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix},$$

where $x \cdot \begin{bmatrix} a \\ b \end{bmatrix} := \begin{bmatrix} ax \\ bx \end{bmatrix}$, and $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} := \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.

- In general, for an $m \times n$ system of linear equations $Ax = b$,

$$x_1 \cdot \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{a_{*1} \in \mathbb{R}^n} + x_2 \cdot \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{a_{*2} \in \mathbb{R}^n} + \ldots + x_n \cdot \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{a_{*n} \in \mathbb{R}^n} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- **Linear combination** of a_{*i} and a_{*j} with real numbers x_i and x_j is defined as

$$x_i \cdot \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} + x_j \cdot \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} x_i a_{1i} + x_j a_{1j} \\ x_i a_{2i} + x_j a_{2j} \\ \vdots \\ x_i a_{mi} + x_j a_{mj} \end{bmatrix}$$

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- If $x, y \in N(A)$, $\forall p, q \in \mathbb{R}, p \cdot x + q \cdot y \in N(A)$.
- If $\exists z \in N(A), z \neq \mathbf{0}_n$, then $Ax = b$ will have infinitely many solutions, if one exists!

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- **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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- Since this is the same as Nullspace of the matrix A^T, left nullspace is denoted by $N(A^T)$.

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Thus $C(A^T) \cap N(A) = \mathbf{0}_n$.

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Thus $C(A^T) \cap N(A) = \mathbf{0}_n$.
- Similarly, $C(A) \cap N(A^T) = \mathbf{0}_m$.

- Are Column and Row rank of a matrix different?

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$$= \begin{bmatrix} a_{*1} & \dots & a_{*r} \end{bmatrix}_{m\times r} \underbrace{\begin{bmatrix} I_{r\times r} & b_{*r+1} & \dots & b_{*n} \end{bmatrix}_{r\times n}}_{C}$$

$$\begin{bmatrix} a_{1*}^T \\ a_{2*}^T \\ \vdots \\ a_{m*}^T \end{bmatrix} = \begin{bmatrix} a_{*1} & \dots & a_{*r} \end{bmatrix}_{m\times r} \begin{bmatrix} c_{1*}^T \\ c_{2*}^T \\ \vdots \\ c_{r*}^T \end{bmatrix}_{r\times n}$$