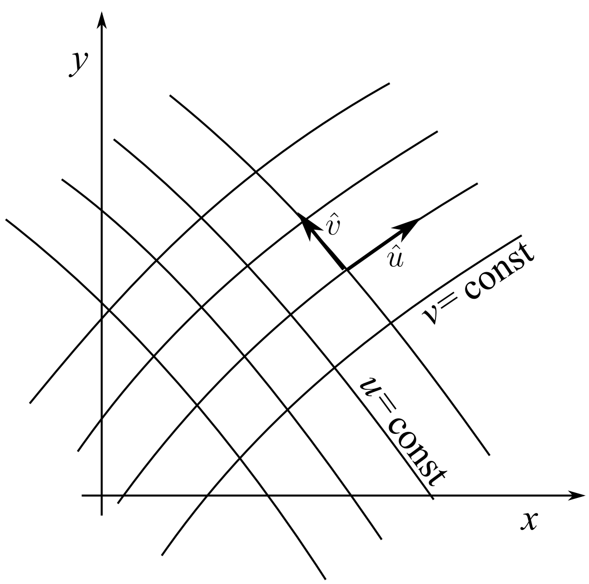
Curvilinear Co-ordinate Systems

Cartesian co-ordinate system is simple to understand and convenient in many problems. Due to certain symmetries, some other coordinate system may be convenient for certain computation. We must be able to do calculus in these co-ordinate systems. In the cartesian system, the axes are straight lines. This makes many aspects of calculus rather straight forward and simple to understand. For e.g the unit vectors $\hat{i}$ and $\hat{j}$ are constant and hence immune to differentiation. If the coordinate axes are not straight lines, we call the co-ordinate system curvilinear. We must have a one-one correspondence from any co-ordinate system to the cartesian system.

Let (u, v) be a curvilinear co-ordinate system. We must be able to write x and y in terms of u and v, i.e, we have functions $x(u, v)$ and $y(u, v)$. Now if we keep v constant and only change u we will trace a curve in the x-y plane. These curves are denoted by the constant values of v. Likewise if we keep u constant and change v we trace another set of curves.



Now if we increase u by an amount du, (v =constant), we will generate an infinitesimal displacement in the x-y plane given by (see figure)

$$\begin{aligned}\vec{dl}_u &= \frac{\partial x}{\partial u}du\hat{i} + \frac{\partial y}{\partial u}du\hat{j} \\ &= \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}\right)du\end{aligned}$$

Similarly if we increase v by an amount dv keeping u constant, we will generate an infinitesimal displacement

$$\vec{dl}_v = \left(\frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}\right) dv$$

We will concentrate on those co-ordinate system (u, v) which has $\vec{dl}_u$ and $\vec{dl}_v$ orthogonal.

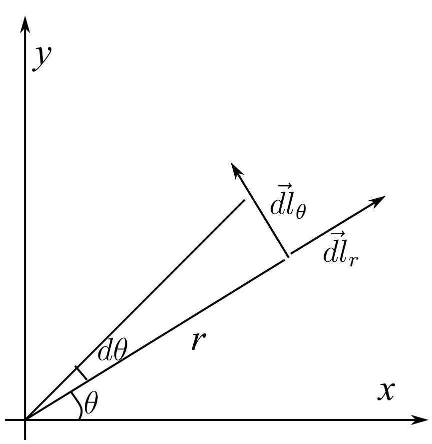
$$\begin{aligned} \text{Now} \quad |\vec{dl}_u| &= \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2} du = h_u du \\ \text{and} \quad |\vec{dl}_v| &= \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2} dv = h_v dv \end{aligned}$$

So the unit vectors along $\vec{dl}_u$ and $\vec{dl}_v$ are

$$\begin{aligned} \hat{u} = \frac{\vec{dl}_u}{|\vec{dl}_u|} &= \frac{1}{h_u}\left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}\right) \\ \hat{v} = \frac{\vec{dl}_v}{|\vec{dl}_v|} &= \frac{1}{h_v}\left(\frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}\right) \end{aligned}$$

We can extend this to three or higher dimension. All the three types of differentiations, gradient, divergence and curl can be expressed in terms of the functions $h_u, h_v, ...$

<u>Polar co-ordinates</u>:



Here $x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le \theta \le 2\pi$

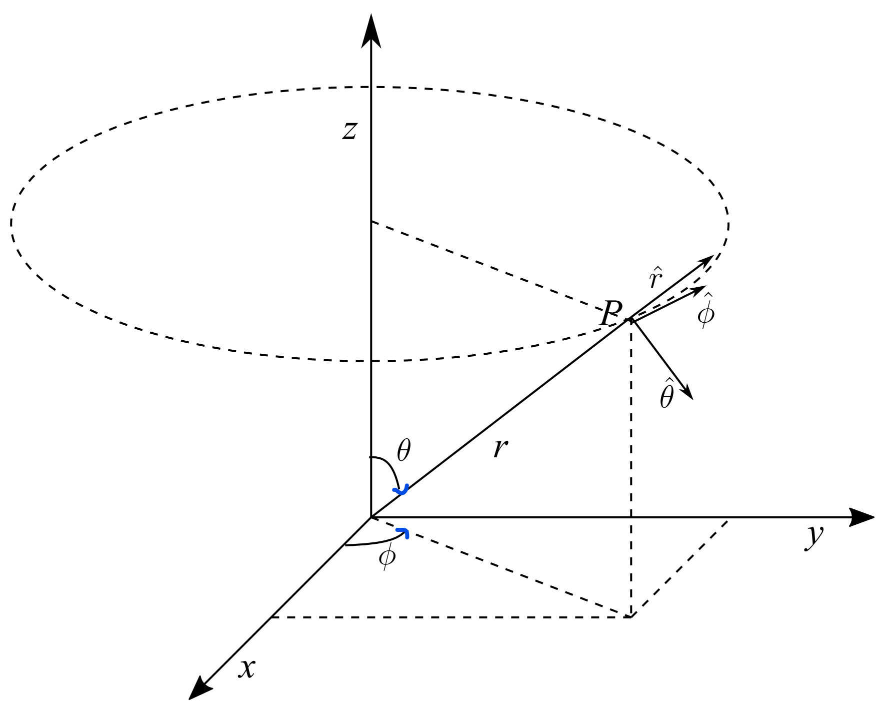
$$\begin{aligned} \vec{dl}_r &= \left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j}\right) dr \\ \vec{dl}_\theta &= \left(\frac{\partial x}{\partial \theta}\hat{i} + \frac{\partial y}{\partial \theta}\hat{j}\right) d\theta \end{aligned}$$

$$
\begin{aligned}
\therefore \quad \vec{dl}_r &= \left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) dr \\
\vec{dl}_\theta &= \left(-r\sin\theta\hat{i} + r\cos\theta\hat{j}\right) d\theta
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad |\vec{dl}_r| &= dr \implies h_r = 1 \\
|\vec{dl}_\theta| &= rd\theta \implies h_\theta = r
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \hat{r} &= \frac{1}{h_r}\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) = \cos\theta\hat{i} + \sin\theta\hat{j} \\
\text{and} \quad \hat{\theta} &= \frac{1}{h_\theta}\left(-r\sin\theta\hat{i} + r\cos\theta\hat{j}\right) = -\sin\theta\hat{i} + \cos\theta\hat{j}
\end{aligned}
$$

1 Spherical Polar co-ordinate



$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Verify that

$$
\begin{aligned}
\hat{r} &= \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \\
\hat{\theta} &= \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} \\
\hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j}
\end{aligned}
$$

$\hat{\theta} = \frac{\partial \hat{r}}{\partial \theta}$

$$0 \le \theta \le \pi \quad \text{and} \quad 0 \le \phi < 2\pi$$

Note that in both these co-ordinate systemsthe unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ depends upon the co-ordinates. So differentiations like $\frac{\partial \hat{r}}{\partial \theta}, \frac{\partial \hat{\theta}}{\partial \theta}$ etc. are non zero. For e.g. $\frac{\partial \hat{r}}{\partial \theta} = hat\theta, \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$.

<u>Gradient</u>:
Let $F(r, \theta, \phi)$ be a scalar function. Then

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial \phi} d\phi \tag{1}$$

Now the increments $dr, d\theta, d\phi$corresponds to an infinitesimal displacement vector.

$$\vec{dl} = h_r dr \hat{r} + h_\theta d\theta \hat{\theta} + h_\phi d\phi \hat{\phi}$$

Let the component of $\vec{\nabla} F$ along $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ at the point (r, θ, ϕ) be $(\vec{\nabla} F)_r, (\vec{\nabla} F)_\theta$ and $(\vec{\nabla} F)_\phi$. Then

$$\begin{aligned} dF &= \vec{\nabla} F \cdot \vec{dl} \\ &= (\vec{\nabla} F)_r h_r dr + (\vec{\nabla} F)_\theta h_\theta d\theta + (\vec{\nabla} F)_\phi h_\phi d\phi \qquad (2) \end{aligned}$$

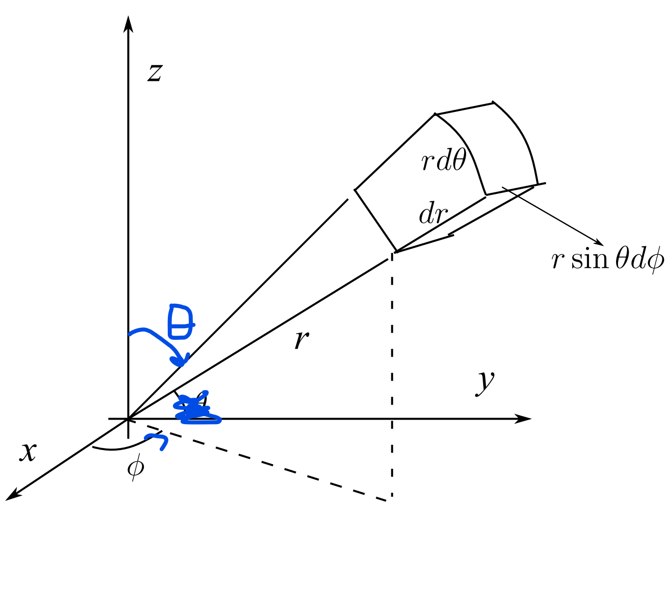
Comparing 1 and 2 we get

$$(\vec{\nabla} F)_r = \frac{1}{h_r} \frac{\partial F}{\partial r}, \quad (\vec{\nabla} F)_\theta = \frac{1}{h_\theta} \frac{\partial F}{d\theta}, (\vec{\nabla} F)_\phi = \frac{1}{h_\phi} \frac{\partial F}{d\phi}$$

$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$

$$\therefore \quad \vec{\nabla} F = \hat{r} \frac{\partial F}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial F}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial F}{\partial \phi}$$

A volume element in spherical polar system is



$$\begin{aligned} dV = |\vec{dl}_r||\vec{dl}_\theta||\vec{dl}_\phi| &= h_r h_\theta h_\phi dr d\theta d\phi \\ &= r^2 \sin\theta dr d\theta d\phi \end{aligned}$$

The three surface elements are

$$\begin{aligned}
h_\theta h_\phi d\theta d\phi \hat{r} &= r^2 \sin\theta d\theta d\phi \hat{r} \quad \text{on the surface } r = \text{constant} \\
h_r h_\theta dr d\theta \hat{\phi} &= r dr d\theta \hat{\phi} \quad \text{on the surface } \phi = \text{constant} \\
h_r h_\phi dr d\phi \hat{\theta} &= r \sin\theta dr d\phi \hat{\theta} \quad \text{on the surface } \theta = \text{constant}
\end{aligned}$$

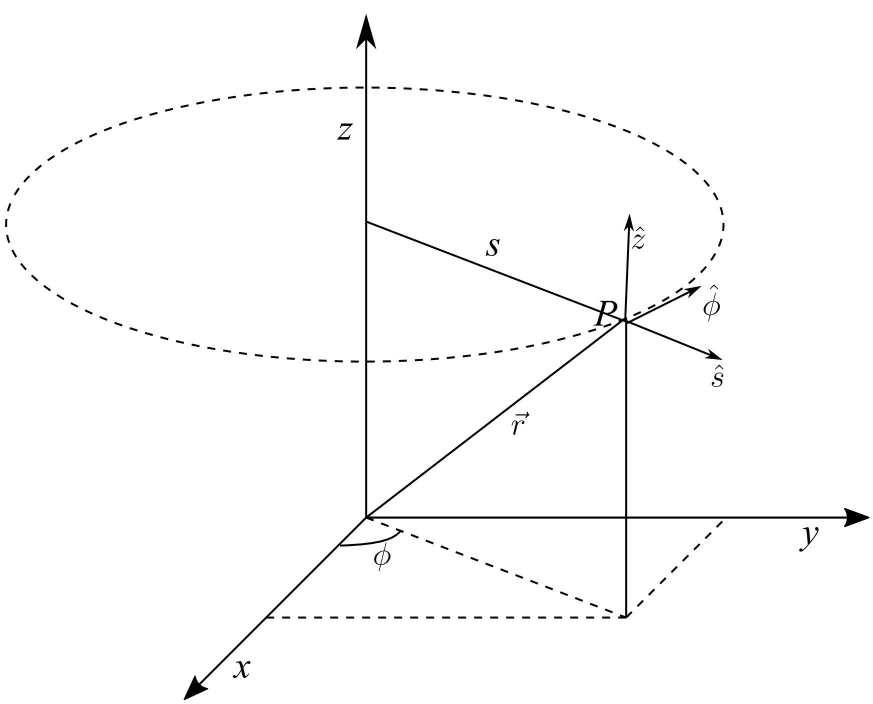
Using these we can evaluate the expression for divergence and the crl. These are

$$\begin{aligned}
\vec{\nabla} \cdot \vec{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(A_\phi\right) \\
\vec{\nabla} \times \vec{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}
\end{aligned}$$

And the Laplacian is

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}$$

2 Cylindrical co-ordinate system



This system specifies a point a point with three parameters (s, ϕ, z).
s = distance from z-axis
ϕ = angle made by projection of $\vec{r}$ on the x-y plane with the x-axis.
z = the z-co-ordinate.
$0 \le \phi < 2\pi$.

$$x = s\cos\phi, \quad y = s\sin\phi, \quad z = z$$

The infinitesimal length elements are

$$
\begin{aligned}
|\vec{dl}_s| &= ds, \implies h_s = 1 \\
|\vec{dl}_\phi| &= sd\phi \implies h_\phi = s \\
|\vec{dl}_z| &= dz \implies h_z = 1
\end{aligned}
$$

The unit vectors are

$$
\begin{aligned}
\hat{s} &= \cos\phi\hat{i} + \sin\phi\hat{j} \\
\hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j} \\
\hat{z} &= \hat{k}
\end{aligned}
$$

The volume element $dV = sdsd\phi dz$.
The gradient curl and divergence :

$$
\begin{aligned}
\vec{\nabla}F &= \hat{s}\frac{\partial F}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial F}{\partial \phi} + \hat{z}\frac{\partial F}{\partial z} \\
\vec{\nabla}\cdot\vec{A} &= \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\vec{\nabla}\times\vec{A} &= \left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right)\hat{z} \\
\nabla^2 F &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial F}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial z^2}
\end{aligned}
$$

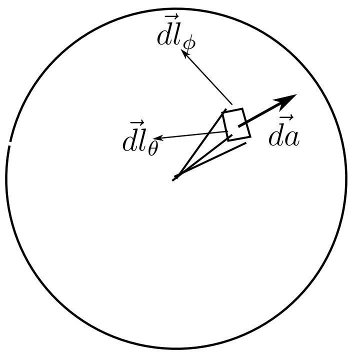
Eg.1 :
$F = \frac{z^2}{x^2+y^2+z^2}$. Find $\vec{\nabla}F$.
In the spherical polar co-ordinates $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$
$\therefore \quad F = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$
$\therefore \quad \vec{\nabla}F = \hat{\theta}\left[\frac{1}{r}\frac{\partial F}{\partial \theta}\right] = \hat{\theta}\frac{-2\cos\theta\sin\theta}{r} = \hat{\theta}\frac{\sin 2\theta}{r}$

Eg.2:
$\vec{A} = r^n\hat{r}$
$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^n) = (n+2)r^{n-1}$
Evaluate $\oint \vec{A}\cdot\vec{da}$ over the surface of a sphere of radius a.

On the surface of the sphere the surface elements are along $\hat{r}$ and $\vec{da}$ is given as

$$
\begin{aligned}
\vec{da} = \vec{dl}_\theta \times \vec{dl}_\phi &= ad\theta\hat{\theta}\times a\sin\theta d\phi\hat{\phi} \\
&= a^2\sin\theta d\theta d\phi\hat{r}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \int_S \vec{A}\cdot\vec{da} &= \int_0^\pi\int_0^{2\pi} a^{n+2}\sin\theta d\theta d\phi \\
&= a^{n+2}4\pi = 4\pi a^{n+2}
\end{aligned}
$$



Eg. 3:

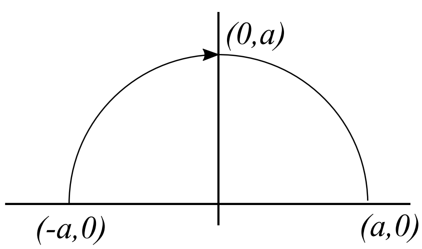
Let $\vec{A} = y\hat{i} - x\hat{j}$. Find $\vec{\nabla} \times \vec{A}$.

We will do this in cylindrical co-ordinates $y = s\sin\phi, \quad x = s\cos\phi$

$\therefore \quad \vec{A} = s(\cos\phi\hat{i} - \sin\phi\hat{j}) = -s\hat{\phi}$

$\vec{\nabla} \times \vec{A} = -\frac{\partial A_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}(-s^2)\hat{z} = -2\hat{z}$

Evaluate $\int_C \vec{A}\cdot\vec{dl}$ where C is the semicircle passing through $(-a, 0), (0, a)$ and $(a, 0)$ $\vec{dl} = ad\phi\hat{\phi}$



$$\begin{aligned}\int_\pi^0 \vec{A}\cdot\vec{dl} &= \int_\pi^0 (-a\hat{\phi})\cdot(ad\phi\hat{\phi}) \\ &= \int_\pi^0 -a^2 d\phi = -a^2\left[0-\pi\right] = \pi a^2\end{aligned}$$