Divide & Conquer Algo & Rec. Relation (D & C.)

Binary Search

When n is even

# of comparisons in search seq. of size n $\quad f(n) = f\left(\frac{n}{2}\right) + 2$ → needed to implement the reduction

One to determine which half of the list to use

one to determine whether any terms of the list remain

In general, Problem of size n divided into a subproblem of size $\frac{n}{b}$

$$f(n) = a\, f(n/b) + g(n)$$ → extra operations to combine soln. of subproblems

$\boxed{\text{D \& C Rec. Relation}}$

Suppose n is divisible by b. Let $n = b^k$, $k \in \mathbb{Z}^+$

$$\Rightarrow f(n) = a f(n/b) + g(n)$$
$$= a^2 f(n/b^2) + a\, g(n/b) + g(n)$$
$$= a^3 f(n/b^3) + a^2 g(n/b^2) + a\, g(n/b) + g(n)$$
$$\cdots \quad \cdots \quad \cdots$$

$$\Rightarrow f(n) = a^k f(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j)$$

Since $\frac{n}{b^k} = 1$

$$\Rightarrow \quad f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j g(n/b^j) \qquad (1)$$, Thus we obtain

$\boxed{\text{Th-1}}$ Let f be increasing fn. that satisfies the rec. relation $f(n) = a f(n/b) + c$, whenever n is divisible by b, where $a \geq 1$ & $b > 1$ & $c \in \mathbb{R}^+$ Then

$$f(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$

, Also when $n = b^k$

$f(n) = c_1 n^{\log_b a} + c_2$, $c_1 = f(1) + c/(a-1)$ & $c_2 = -c/(a-1)$

☐ let $n = b^k$ then by eqn (1), for $g(n) = c$

$\Rightarrow f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c = a^k f(1) + \sum_{j=0}^{k-1} a^j$

Case ① $a = 1$

$f(n) = f(1) + ck$ now $n = b^k \Rightarrow k = \log_b n$

$\Rightarrow \overset{= f(b^k)}{f(n)} = f(1) + c \log_b n$

When $n \neq b^k$ we have $b^k < n < b^{k+1}$ for $k \in \mathbb{Z}^+$

$\because$ f is increasing

$f(n) \le f(b^{k+1}) = f(1) + c(k+1)$

$= (f(1) + c) + ck \le (f(1)+c) + c \log_b n$

$\therefore$ When $\underline{a = 1}$ $\boxed{f(n) = O(\log n)}$

Case ② $a > 1$, let $n = b^k$

$f(n) = f(b^k) = a^k f(1) + c \frac{(a^k - 1)}{(a-1)}$

$= a^k \left[f(1) + \frac{c}{(a-1)}\right] - \frac{c}{(a-1)}$

Now $a^k = a^{\log_b n} = n^{\log_b a}$

$\therefore f(n) = c_1 n^{\log_b a} + c_2$, $c_1 = f(1) + \frac{c}{(a-1)}$

& $c_2 = \frac{c}{(a-1)}$

Now when $n \neq b^k$, $\Rightarrow b^k < n < b^{k+1}$, $k \in \mathbb{Z}^+$

$\because$ f is increasing

$f(n) \le f(b^{k+1}) = c_1 a^{k+1} + c_2$

$\le (c_1 a) \cdot n^{\log_b n} + c_2$

$f(n) \le (c_1 a) n^{\log_b a} + c_2$

$\because k \le \log_b n < k+1$

$\therefore \underline{f(n) = O(n^{\log_b a})}$ ▥

Example let $f(n) = 5f(n/2) + 3$ & $f(1) = 7$

find $f(2^k)$, $k \in \mathbb{Z}^+$, Also estimate $f(n)$, if f is increasing fn.

☐ Using theorem for $a=5$, $b=2$ & $c=3$ if $n = 2^k$

$$f(n) = a^k\left[f(1) + \frac{c}{a-1}\right] + \left[-\frac{c}{(a-1)}\right]$$

$$= 5^k\left[7 + \frac{3}{4}\right] - \frac{3}{4} = 5^k \cdot \frac{31}{4} - \frac{3}{4}$$

$\because f$ is increasing $f(n) = O(n^{\log_b a}) = O(n^{\log 5})$ ☒

Ex. Find # of comparisons used to locate max or min element in a seq...

☐ $\because f(n) = 2f(\frac{n}{2}) + 2$ n is even

Using Th-1 $f(n) = O(n^{\log 2}) = O(n)$ $\because b = 2$

Master Theorem Let f be an increasing fn. that satisfies the rec. relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, $k \in \mathbb{Z}^+$, $a \geq 1$, $b(>1) \in \mathbb{Z}$

& $c, d \in \mathbb{R}_+ \cup \{0\}$. Then

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Ex. For multiplying two $n \times n$ matrices

$$f(n) = 7f(n/2) + \frac{15n^2}{4} \quad n \text{ even}$$

$\log 7 \simeq 2.8$

$\Rightarrow f(n) = O(n^{\log 7})$

$\boxed{\text{Generating functions}}$ (G.F.)

For the seq. $a_0, a_1, \ldots, a_k$ of real nos

the G.F. is $G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$

Ex. G.F. for $\{a_k\}$, $a_k = 3$

is $\sum_{k=0}^{\infty} 3x^k$

$a_k = k+1$ $\quad \sum_{k=0}^{\infty} (k+1)x^k$

Ex. G.F. of 1 1 1 1 1 1 is

$1 + x + x^2 + x^3 + x^4 + x^5 = \frac{x^6 - 1}{x - 1}$

Ex. G.F. of $1, 1, 1, 1, \ldots$ is $\frac{1}{1-x}$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots$, $|x| < 1$

Th. let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ & $g(x) = \sum_{k=0}^{\infty} b_k x^k$

Then $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

& $f(x)\,g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$

Th. $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$ $\quad$ Extended Binomial Th.

$u \in \mathbb{R}$

$x \in \mathbb{R}, |x| < 1$ $\quad \binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$

$\boxed{\text{Lemma}}$ $\quad \binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$

Graphs

$G = (V, E)$ V — Non empty set of vertices (or nodes)
E — a set of edges

- If $|V| = \infty$ Infinite graph
  $|V| < \infty$ finite graph

- Simple graph: Each edge connects two different vertices & no two edge connect the same pair of vertices
- Multigraph: Graphs that have multiple edges connecting the same vertices
- Loops: Edges that connect a vertex to itself
- Undirected graph
- Directed graph: $G = (V, E)$, E - set of directed edges
  $(u, v) \in E$ start u & ends at v
- Simple directed graph: Directed graph with no loops & no multiple edges
- Pseudographs: G that may include loops & possibly multiple edges connecting the same pair of vertices.

GRAPH MODELS

1. Niche Overlap Graphs in Ecology
2. Acquaintanceship graph
3. Influence graph
4. The Hollywood graph
5. Round-Robin Tournaments
6. Collaboration Graph (Erdos graph)
7. Call graph
8. Web graph (Bow-tie str)
9. Roadmaps/flightmaps

GRAPH TERMINOLOGY & SPECIAL TYPES OF GRAPHS

— Two vertices u & $v \in V$ G (undirected graph)

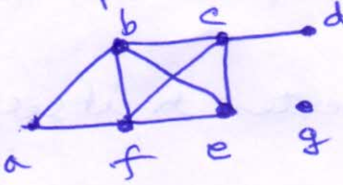
adjacent in G if u & v are end pts. of an edge of G.

— e is associated with $\{u, v\}$ edge e is called incident with vertices u & v.

— G (undirected graph) $v \in V$

$\deg(v) = \#$ of edges incident with it except that a loop at a vertex contributes twice to the deg.

Ex.



$\deg(g) = 0$ $\deg(b) = \deg(c) = \deg(f) = 4$

$\deg(e) = 3$

$\deg(a) = 2$

— A vertex of deg 0 is called isolated

— A vertex is pendant iff it has deg 1.

Vertex d is pendant.

Theorem -1 The Handshaking Th.

$G = (V, E)$ undirected graph with e edges.

Then $2e = \sum_{u \in V} \deg(u)$ (true even if multiple edges & loops are prst)

□ Each edge contributes 2 to the sum of degrees. ▨

Ex. How many edges are there in a graph with 10 vertices each of deg 6.

□ $2e = 60 \Rightarrow e = 30$. ▨

Theorem-2 An undirected graph has even # of vertices of odd degree.

□ $V_1 :=$ set of vertices of even degree $G = (V, E)$

$V_2 :=$ set of " of odd degree

Then $\underbrace{2e}_{\text{even}} = \sum_{u \in V} \deg(u) = \underbrace{\sum_{u \in V_1} \underbrace{\deg(u)}_{\text{even}}}_{\text{even}} + \underbrace{\sum_{u \in V_2} \deg(u)}_{\text{even}}$ $\Rightarrow$

$\Rightarrow$ even # of vertices of odd degree. ▨

G – directed graph (u, v) — u is adjacent to v

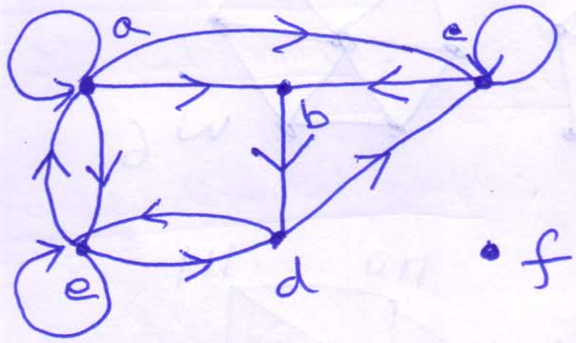
u: initial vertex; v: terminal or end vertex

For a loop initial vertex = terminal vertex

— $\deg^-(v) :=$ # of edges with v as their terminal vertex
in-degree of a vertex v

— $\deg^+(v) :=$ # of edges with v as their initial vertex
out-deg. of a vertex v

$\boxed{\text{Note: A loop at a vertex contributes 1 to both in-deg \& out-deg of the vertex}}$



$\deg^-(a) = 2$
$\deg^-(c) = 3$
$\deg^+(d) = 2$
$\deg^+(a) = 4$ $\deg^-(f) = 0$
$\deg^+(b) = 1$
$\deg^+(f) = 0$

Theorem-3

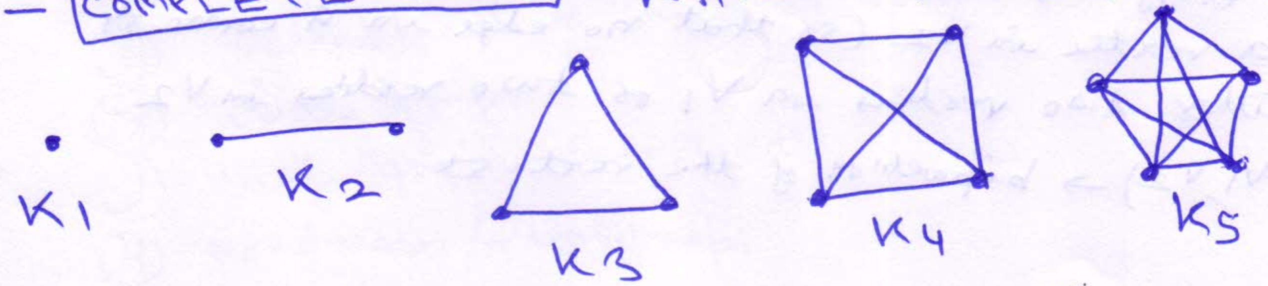
$G = (V, E)$ directed graph then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

□ Each edge has initial vertex & a terminal vertex.

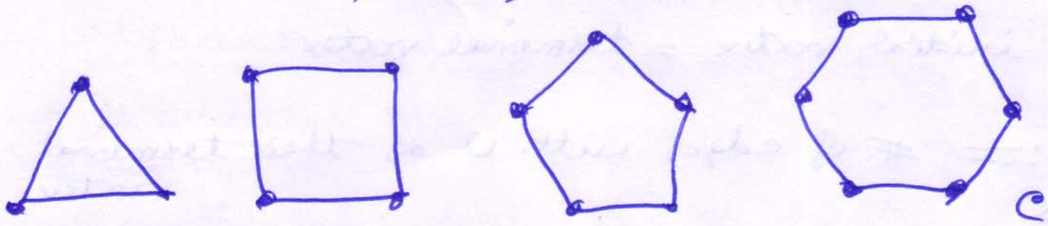
SPECIAL GRAPHS

— **COMPLETE GRAPHS** $K_n :=$ Complete graph on n vertices



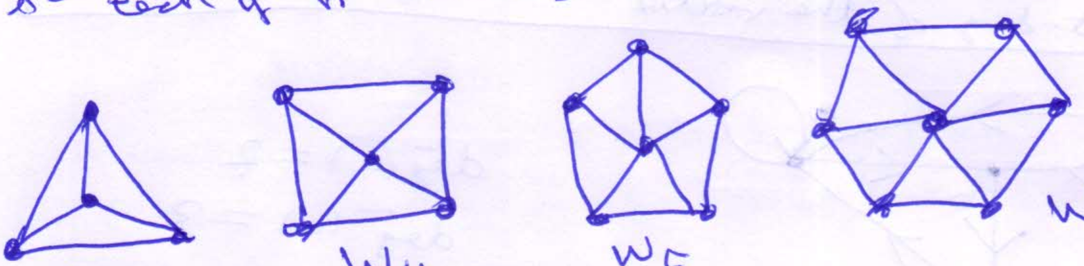
Simple graph that contains exactly one edge between each pair of distinct vertices

Cycles $C_n, n \geq 3$ on n vertices and edges
$\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$ & $\{n, 1\}$



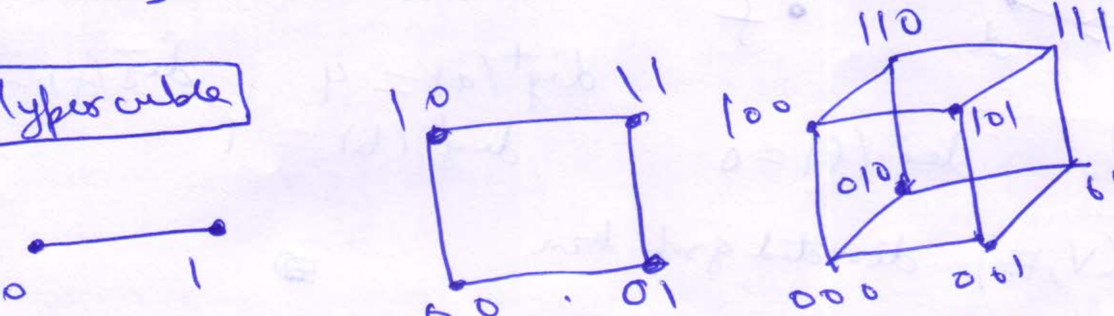
C_3 C_4 C_5 C_6

Wheels W_n when we add additional vertex to the cycle C_n for $n \geq 3$ & connect this new vertex to each of n vertices in C_n by new edges.



W_3 W_4 W_5 W_6

Hypercube



Q_1 Q_n Q_2 Q_3

Bipartite Graph

- G (simple graph) $\rightarrow$ Bipartite $V = V_1 \dot{\cup} V_2$ s.t. every edge in the graph connects a vertex in V_1 & a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2)
$(V_1, V_2) \rightarrow$ bipartition of the vertex set.