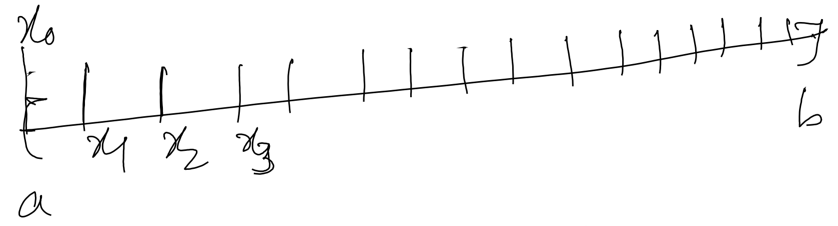
Definite Integrals

$$\int_a^b f(x)\,dx$$

Consider the interval $[a,b]$

Partition $[a,b]$



$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Partition P devides $[a,b]$ into n closed subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \cdots,\ [x_{n-1}, x_n]$$

Let kth subinterval is $[x_{k-1}, x_k]$

Length of the kth subinterval

$$\Delta x_k = x_k - x_{k-1}$$

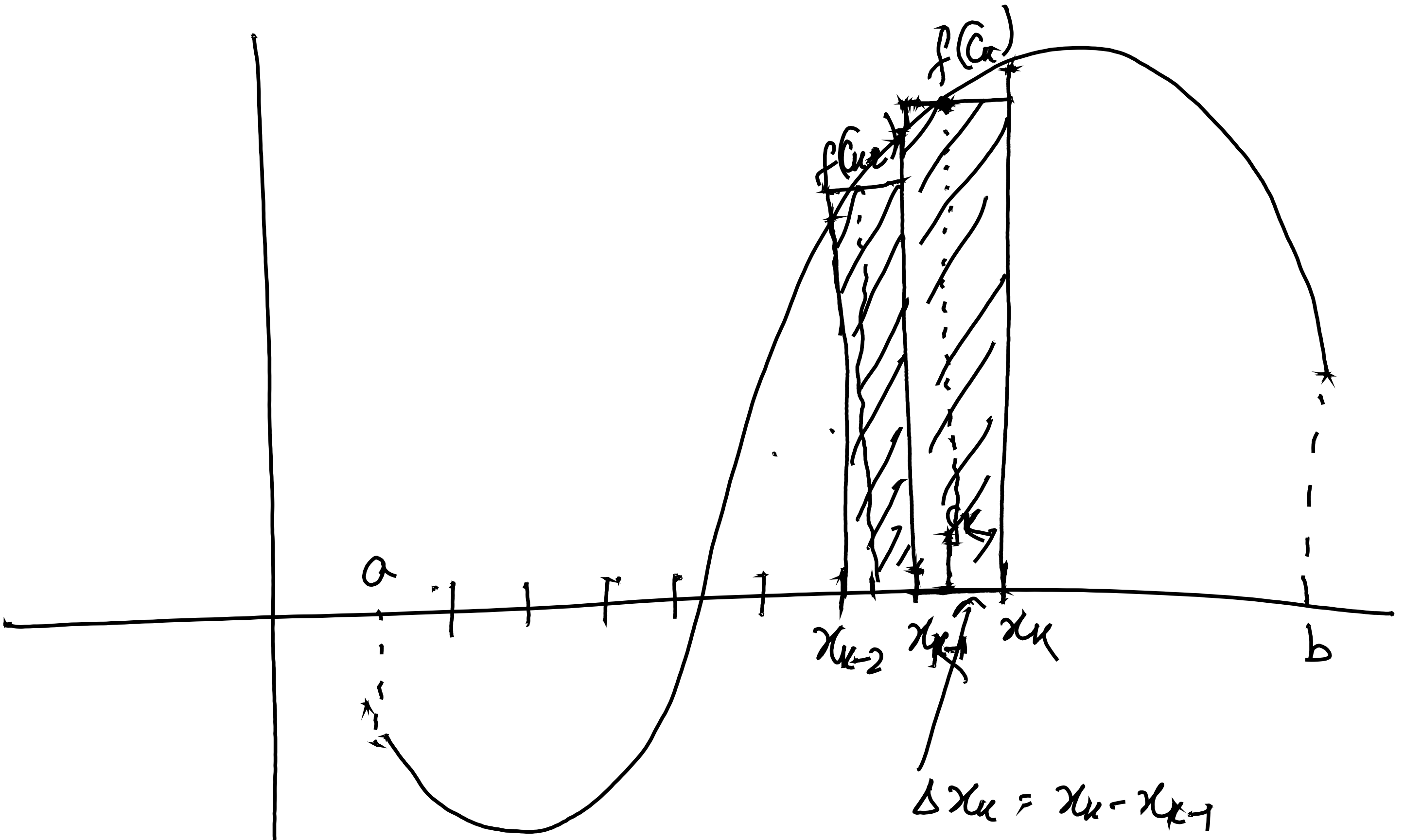
In each subinterval we select a point.

Let the point chosen in the kth subinterval is c_k.

At c_k, the corresponding function value is $f(c_k)$

Consider the sum

$$S_p = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

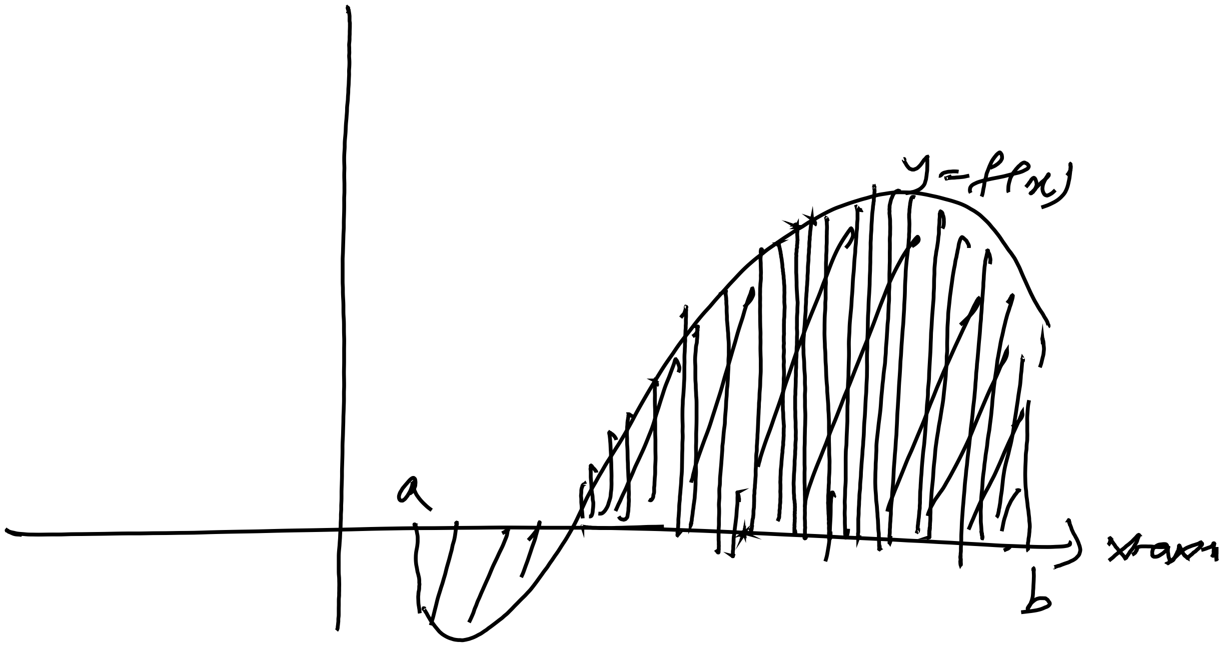


$\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ is the sum of the areas of all the rectangles.

This sum S_p is called Riemann sum

There are many such sums depending on the partitions and depending on ~~the~~ how we have chosen the point c_k.

OK When the partition becomes finer and finer we get exactly the area bounded by the curve $y=f(x)$ and x-axis in the interval $[a,b]$



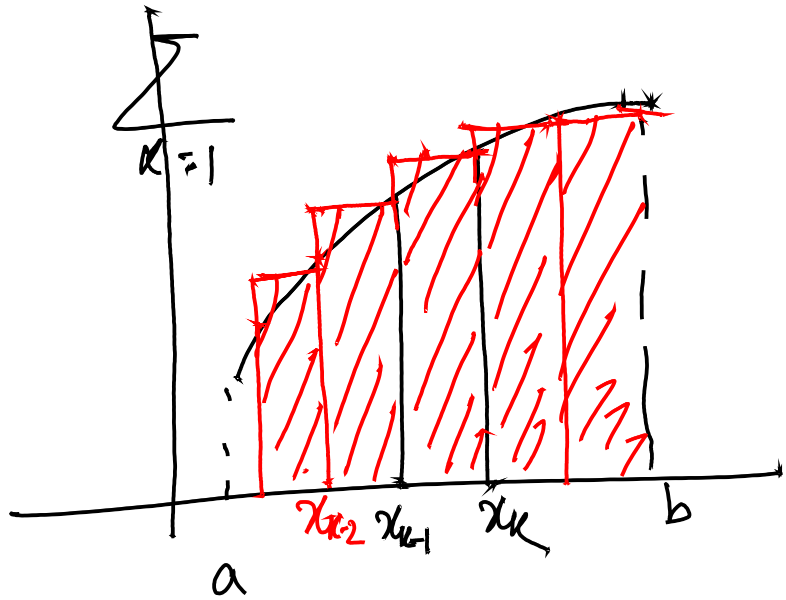
As $n \to \infty$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx$$

$\|P\| \to 0$ $\qquad$ $\|P\|$ norm

P : $[x_0, x_1]$, $[x_1, x_2]$, - - , $[x_{n-1}, x_n]$

$\|P\| = \min\{ \; |x_k - x_{k-1}| \}$

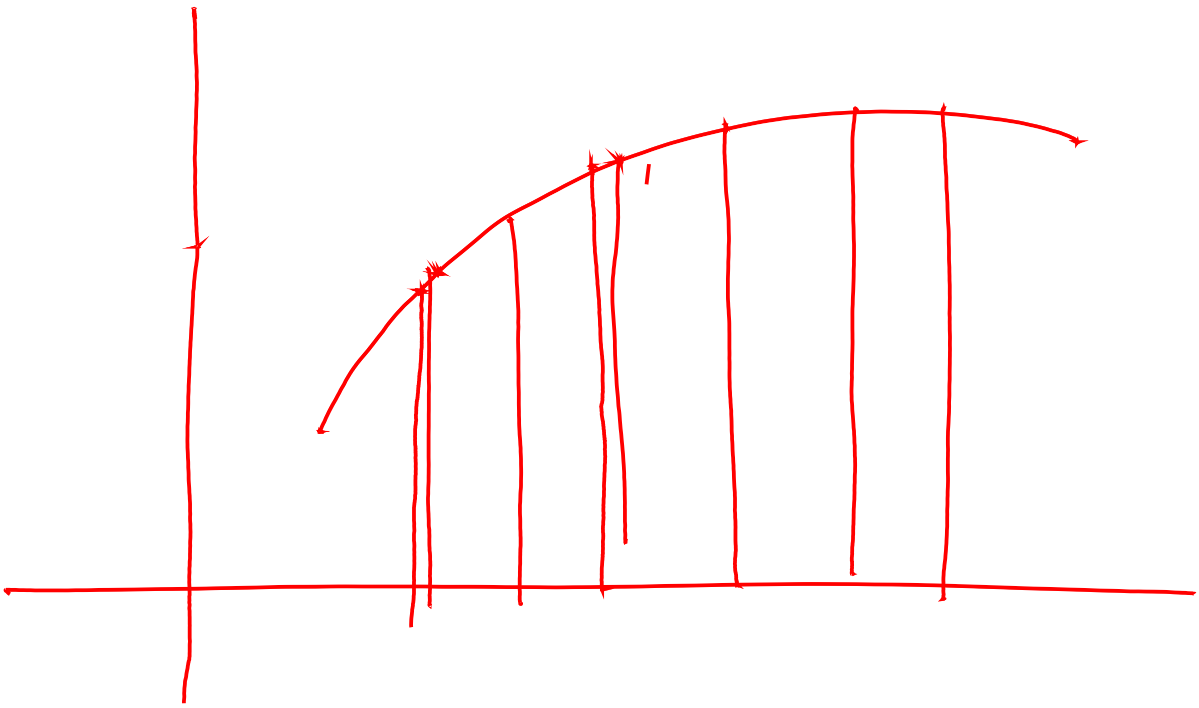


$[x_{k-1} \quad x_{k-?}]$

$$x_k = \max\{x_k, x_{k-1}\}$$

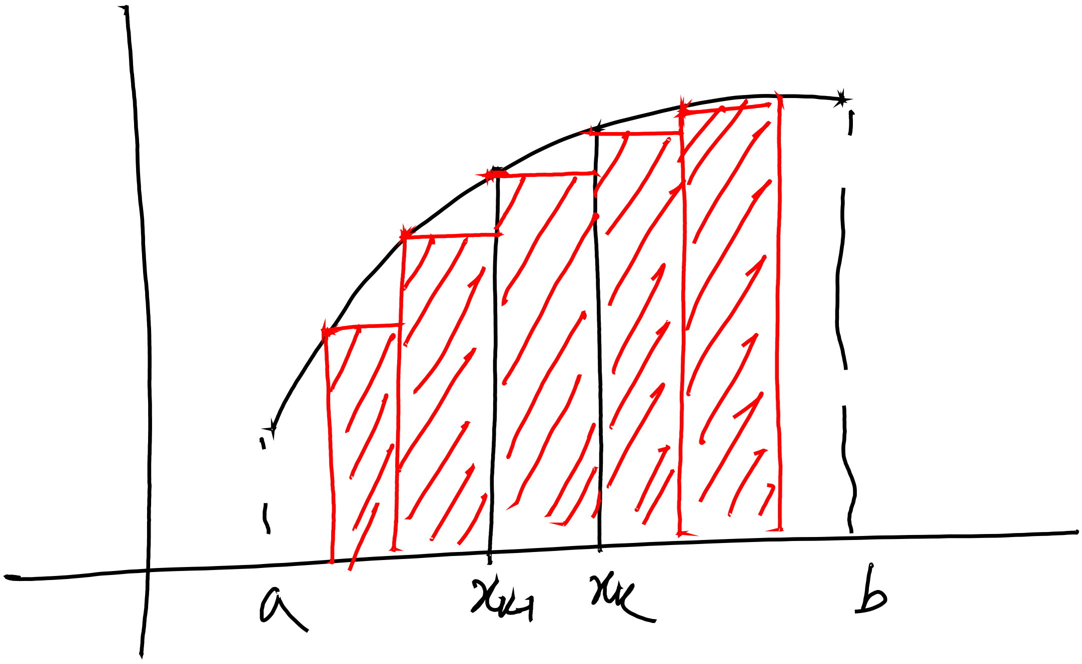
$$x_{k-1} = \min\{x_{k-1}, x_k\}$$

$$\sum_{k=1}^{n} f(x_k)\,\Delta x_k$$ = Upper sum



$$\sum_{k=1}^{n} f(x_{k-1})\,\Delta x_k$$

Lower sum



As $n \to \infty$, upper sum = Lower sum

$$= \int_a^b f(x)\,dx$$

Exp Compute $\int_0^b x\,dx$ and find the area A under the line $y = x$ over the interval $[0, b]$.

Sol^n We calculate $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k$

Note It does not matter how we choose the partition and the point c_k as long as $\|P\| \to 0$ or $n \to \infty$.

Consider the partition of $[0, b]$

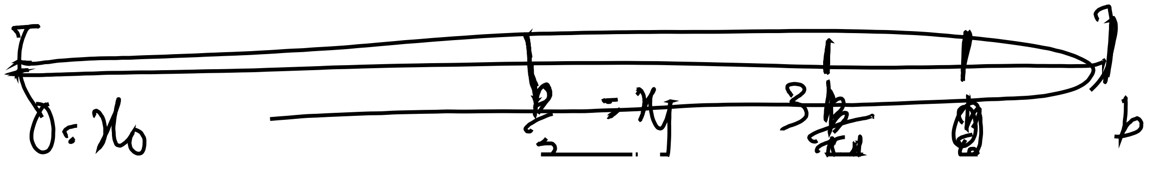
$$P = \left\{ 0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \ldots, \frac{nb}{n} \right\}$$

$$x_0 = 0, \quad x_1 = \frac{b}{n}, \quad x_2 = \frac{2b}{n}, \ldots$$

$$x_k = \frac{kb}{n}, \qquad x_{k-1} = (k-1)\frac{b}{n}$$

$$\Delta x_k = \frac{b}{n}$$

Let us choose $c_k = \frac{kb}{n}$



$$\sum_{k=1}^{n} f(c_k)\Delta x_k \qquad \int_0^b x\,dx$$

$$f(x) = x$$

$$= \sum_{k=1}^{n} f\left(\frac{kb}{n}\right)\frac{b}{n}$$

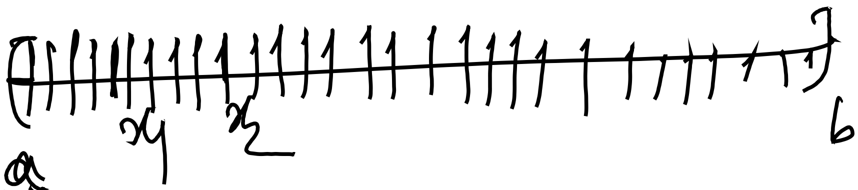
$$= \sum_{k=1}^{n} \frac{kb}{n}\cdot\frac{b}{n}$$

$$= \frac{b^2}{n^2}\sum_{k=1}^{n} k = \frac{b^2}{n^2}\,\frac{n(n+1)}{2}$$

$$= \frac{b^2}{2}\,\frac{n+1}{n} = \frac{b^2}{2}\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$$

$$= \lim_{n\to\infty}\frac{b^2}{2}\left(1+\frac{1}{n}\right) = \frac{b^2}{2}$$



$x_0 = a$

$x_1 = a + \frac{b-a}{n}$

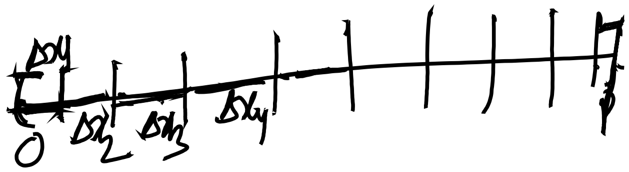
$x_2 = a + 2\frac{b-a}{n}$

$\vdots$

$\frac{b-a}{n} = h$

$x_0,\ x_0+h,\ x_0+2h, \dots$

Exp-2



$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ rational} \\ 0 & \text{if } x \in \mathbb{R}-\mathbb{Q} \text{ irrational} \end{cases}$$

Is $f(x)$ Riemann integrable on $[0,1]$.

Sol^n We pick a partition P on $[0,1]$ and choose c_k to be the point giving maximum value for f on $[x_{k-1}, x_k]$

$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} 1 \times \Delta x_k = \sum_{k=1} \Delta x_k = 1-0 = 1$$

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = 1}$$

On the other hand if we pick c_k to be the point giving minimum value to f on $[x_{k-1}, x_k]$.

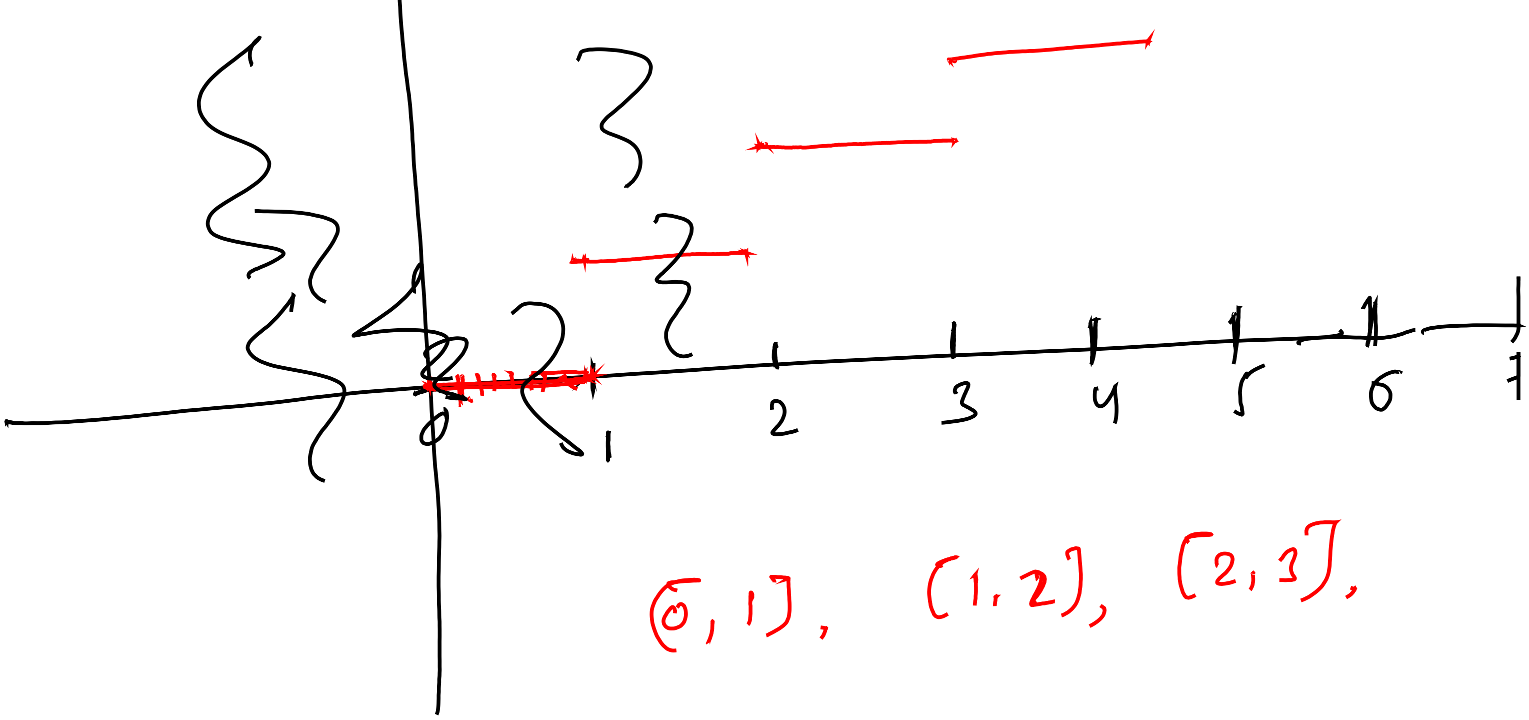
$$\sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} 0 \cdot \Delta x_k = 0$$

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = 0}$$

So $f(x)$ is not Riemann integrable.

$[0, 7]$

$f(x) = [x]$ greatest integer funct



$[0,1]$, $[1,2]$, $[2,3]$,

$\int_a^b f(x)\,dx$

$\int_0^7 [x]\,dx$

$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx$

$+ \dots + \int_6^7 [x]\,dx$

Note

(1) If a function f is continuous over the interval $[a,b]$, then it is Riemann integrable.

(2) If $f(x)$ has at most finitely many jump discontinuity then also it is Riemann integrable.

(3)
$$f(x) = 1 \quad \text{if } x \in \mathbb{Q}$$
$$= 0 \quad \text{if } x \in \mathbb{R} - \mathbb{Q}$$

So $f(x)$ has discontinuity at every real numbers.

Uncountable number of discontinuities.

So $f(x)$ is not Riemann integrable.

The mean value theorem for definite integrals

If f is continuous on $[a,b]$, then ~~a~~there exists a point $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f(x) = F'(x)$$

$$\boxed{F'(c) = \frac{1}{b-a}\left(F(b) - F(a)\right)}$$

First fundamental theorem of calculus

If f is continuous on $[a,b]$, then

$F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$

and differentiable on (a,b) and its

derivative is $f(x)$.

That is $\boxed{F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$

Ex-1

If $y = \int_a^x (t^3+1)\,dt$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^x (t^3+1)\,dt\right) = x^3+1$$

Ex-2

If $y = \int_x^5 3t\sin t\,dt$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_5^x -3t\sin t\,dt\right)$$

$$= -3x\sin x$$

Exp-3 If $y = \int_1^{x^2} \cos t \, dt$

Find $\frac{dy}{dx}$.

Put $u = x^2$ $\quad \frac{du}{dx} = 2x$

$$y = \int_1^{u} \cos t \, dt$$

$$\Rightarrow \frac{dy}{du} = \frac{d}{du}\left(\int_1^{u} \cos t \, dt\right) = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos u \cdot 2x$$

$$= \left(\cos x^2\right) \times 2x$$