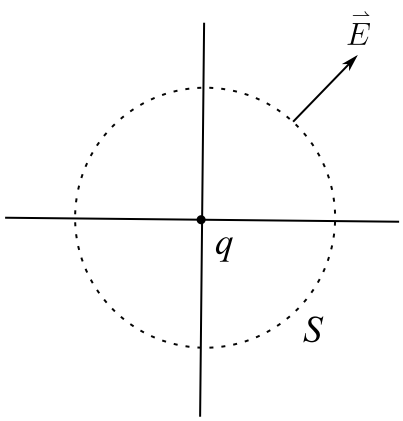
Dirac Delta Function



Consider the simplest electrostatic configuration. A point charge q at the origin. The electric field due to this charge is given as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

The flux of $\vec{E}$ over a sphere of radius a will be

$$\oint_S \vec{E}\cdot\hat{n}da = \frac{1}{4\pi\epsilon_0}\frac{q}{a^2}4\pi a^2 = \frac{q}{\epsilon_0}$$

This is consistent with the integral form of the Guss' law. Now

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0 \quad \text{for} \quad r > 0$$

By divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{E}dV = \oint_S \vec{E}\cdot\hat{n}da = \frac{q}{\epsilon_0}$$

The contribution to the volume integral only comes from the origin since $\vec{\nabla}\cdot\vec{E} = 0$ at all other points. If $\vec{\nabla}\cdot\vec{E}$ is finite at $r = 0$ then its contribution to the volume integral on the l.h.s is 0 since the volume tends to 0. So $\vec{\nabla}\cdot\vec{E} \to \infty$ as $r \to 0$. So we have the following

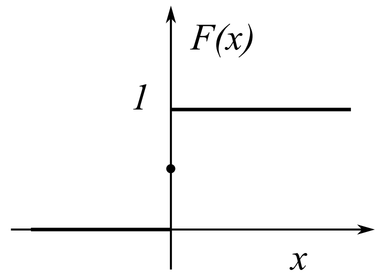
$$\begin{aligned}\int_V \vec{\nabla}\cdot\vec{E}dV &= 0 \quad \text{if} V \text{doesn't include the origin}\\ &= \frac{q}{\epsilon_0} \quad \text{if} V \text{includes origin within it}\end{aligned}$$

$\vec{\nabla}\cdot\vec{E}$ is described by a function called the Dirac delta function.

1 Dirac delta function

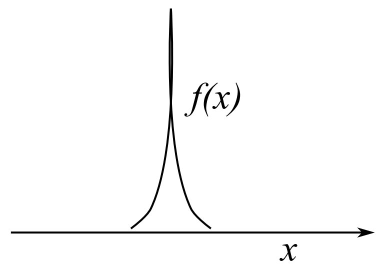
Consider the following fumction

$$\begin{aligned} F(x) &= 1 \text{ for } x>0 \\ &= 0 \text{ for } x<0 \\ &= \frac{1}{2} \text{ at } x=0 \end{aligned}$$



Let $f(x) = \frac{dF}{dx}$.
Now $f(x) = 0$ for $x > 0$ and $x < 0$
$\lim_{x\to 0} f(x) \to \infty$
The integral

$$\int_{-a}^{a} f(x)dx = [F(x)]_{-a}^{a} = 1 - 0 = 1$$



The value of this integral is 1 if the region includes $x = 0$. Otherwise the value of the integral is 0 . Functions with such integral preperties as that of $f(x)$ are called dirac delta function denoted as $\delta(x)$. This can be defined only in terms of its integral property as seen above. Another important property which is also sometimes considered as its defining property is the following.
For any continuous function $g(x)$ at the origin

$$\int_{-a}^{a} g(x)\delta(x)dx = g(0)$$

We say that a delta function fires only at $x = 0$ and selects the value of $g(x)$ at $x = 0$. We can also have delta function firring at $x = a$ denoted as $\delta(x - a)$.
For such a function we will have

$$\int_{-\infty}^{\infty} g(x)\delta(x-a)dx = g(a)$$

Eg 1) Find $\delta(3x)$ in terms of $\delta(x)$.
conside $\int_{-\infty}^{\infty} \delta(3x)dx$.
Let $t = 3x$. Then $dt = 3dx$.
As $x \to \pm\infty \quad t \to \pm\infty$

$$\therefore \quad \int_{-\infty}^{\infty} \delta(3x)dx = \frac{1}{3}\int_{-\infty}^{\infty} \delta(t)dt = \frac{1}{3}$$

$\therefore \quad \delta(3x) = \frac{1}{3}\delta(x) mmm.$
Now let us find $\delta(-3x)$.
Let $t = -3x$. Then $dt = -3dx$.
As $x \to \pm\infty, \quad t \to \mp\infty$

$$\therefore \quad \int_{-\infty}^{\infty} \delta(-3x)dx = -\frac{1}{3}\int_{\infty}^{-\infty} \delta(t)dt = \frac{1}{3}$$

$\therefore \quad \delta(-3x)$ is also $\frac{1}{3}\delta(x)$.
So we see that $\delta(kx) = \frac{1}{|k|}\delta(x)$.
In three dimensions the delta function is denoted as $\delta^3(\vec{r})$ and defined as

$$\int_{\text{all space}} \delta^3(\vec{r}) = 1$$

In cartesian coordinate we can write it as

$$\int_{\text{all space}} \delta^3(\vec{r})dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

Just like one dimensional delta function we have

$$\int_V f(\vec{r})\delta^3(\vec{r} - \vec{a})dV = f(\vec{a})$$

where the volume V includes the point $\vec{a}$.
We have seen that $\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 0$ everywhere except the origin.
By the divergence theorem

$$\int_V \vec{\nabla} \cdot \frac{\hat{r}}{r^2} dV = \oint_S \frac{\hat{r}}{r^2} \cdot \vec{da}$$

We consider the volume V as a sphere of radius R and S the surface of the sphere. Then $\vec{da} = \hat{r}R^2 \sin\theta d\theta d\phi$

$$\begin{aligned} \therefore \quad \oint_S \frac{\hat{r}}{r^2} \cdot \vec{da} &= \int_0^{2\pi}\int_0^{\pi} \frac{\hat{r}\cdot\hat{r}}{R^2} R^2 \sin\theta d\theta d\phi \\ &= 4\pi \end{aligned}$$

$$\therefore \quad \int_V \left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right) = 4\pi$$

over any volume enclosing the origin.

$$\therefore \quad \vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})$$

If $\vec{A}(\vec{r}) = 2x\hat{i} + 4\hat{j} + e^{-r}\hat{k}$ then

$$\int_{\text{all space}} \vec{A}(\vec{r})\vec{\nabla} \cdot \frac{\hat{r}}{r^2} dV = 4\pi\vec{A}(0) = 4\pi(4\hat{j} = \hat{k})$$

Eg.2

$$\begin{aligned} \vec{E}(\vec{r}) &= \frac{ca^2}{\epsilon_0}\frac{\hat{r}}{r^2} \quad ; \qquad r \geq a \\ &= 0 \qquad\quad ; \qquad r < a \end{aligned}$$

Find the charge density that causes this $\vec{E}$.

The volume charge density $\rho(\vec{r})$ is obtained from the differential form of the Gauss' law.

$$\frac{\rho(\vec{r})}{\epsilon_0} = \vec{\nabla} \cdot \vec{E}$$

For $r > a$ and $r < a$, $\quad \vec{\nabla} \cdot \vec{E} = 0$.
$\therefore \quad \rho(\vec{r}) = 0$ for $r > a$ and $r < a$.
At $r = a$, $\vec{E}$ is discontinuous (check). So we can't compute $\vec{\nabla} \cdot \vec{E}$. We will have to work with the integral property of $\vec{\nabla} \cdot \vec{E}$. For a sphere with radius $R_1 < a$

$$\int_{V1} (\vec{\nabla} \cdot \vec{E}) dV = \int_0^{R_1} (\vec{\nabla} \cdot \vec{E}) 4\pi r^2 dr = 0$$

But if $R_2 > a$ then

$$\int_{V2} (\vec{\nabla} \cdot \vec{E}) dV = \int_0^{R_2} (\vec{\nabla} \cdot \vec{E}) 4\pi r^2 dr = \oint (\vec{E} \cdot \hat{n}) da = \frac{4\pi c a^2}{\epsilon_0}$$

So $\vec{\nabla} \cdot \vec{E} = k\delta(r-a)$

$$\int_0^{R_2} (\vec{\nabla} \cdot \vec{E}) 4\pi r^2 dr = \int_0^{R_2} k\delta(r-a) 4\pi r^2 dr = \frac{4\pi c a^2}{\epsilon_0}$$

$$\therefore \quad k = \frac{c}{\epsilon_0} \quad \Longrightarrow \quad \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = c\delta(r-a)$$

From this the surface charge density is obtained as follows

$$\sigma = \int_{a-\epsilon}^{a+\epsilon} \rho(r) dr = \int_{a-\epsilon}^{a+\epsilon} \delta(r-a) dr = c$$