Libnitz Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions whose values lies in $[a, b]$, then

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)\,dt\right)$$

$$= f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

Exp If $F(x) = \int_{\frac{1}{x}}^{x} \frac{1}{t}\,dt$, $[a, b]$

$a > 0$

Here $f(t) = \frac{1}{t}$

Find $F'(x)$.

Sol^n $F'(x) = \frac{d}{dx}\left(\int_{\frac{1}{x}}^{x} \frac{1}{t}\,dt\right)$

$$= f(x)\cdot 1 - f\left(\tfrac{1}{x}\right)\cdot\left(-\tfrac{1}{x^2}\right) = \frac{1}{x} + x\cdot\frac{1}{x^2}$$

$$= 2/x$$

Second fundamental theorem of Calculus

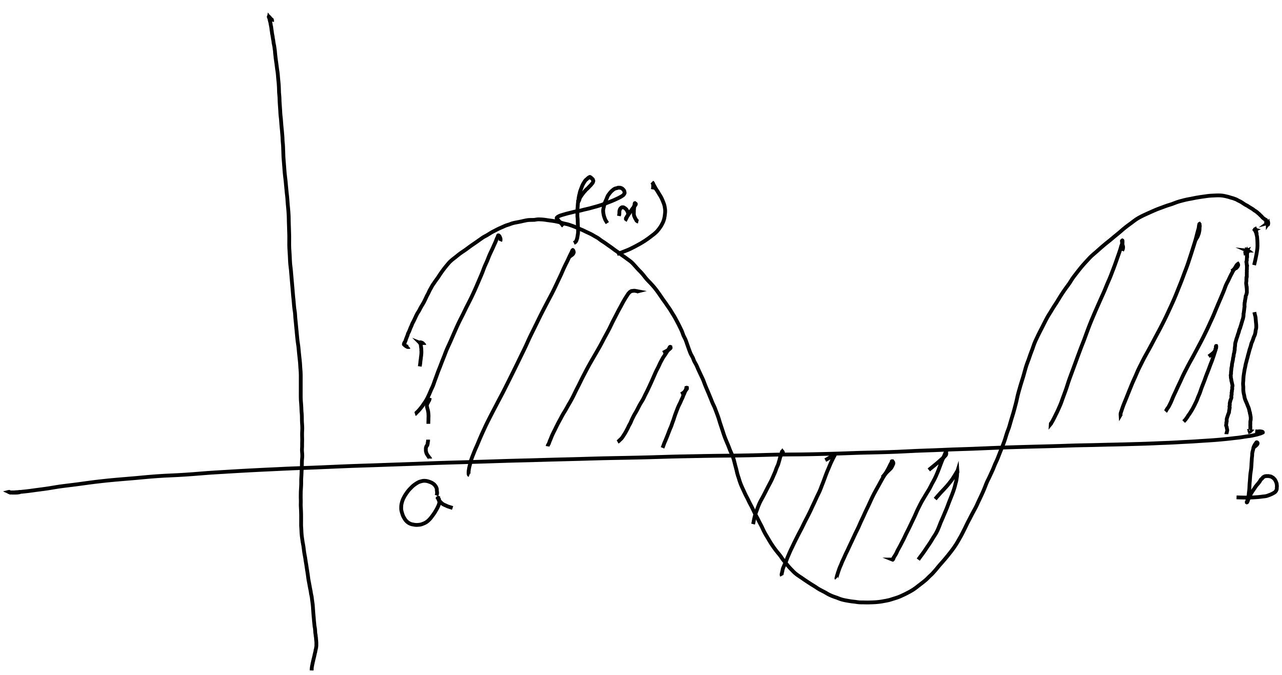
If f is continuous on $[a,b]$ and F is antiderivative of f in $[a,b]$

then $\int_a^b f(x)\,dx = F(b) - F(a)$.

$$\int_0^1 e^x\,dx = e^1 - e^0$$

Area

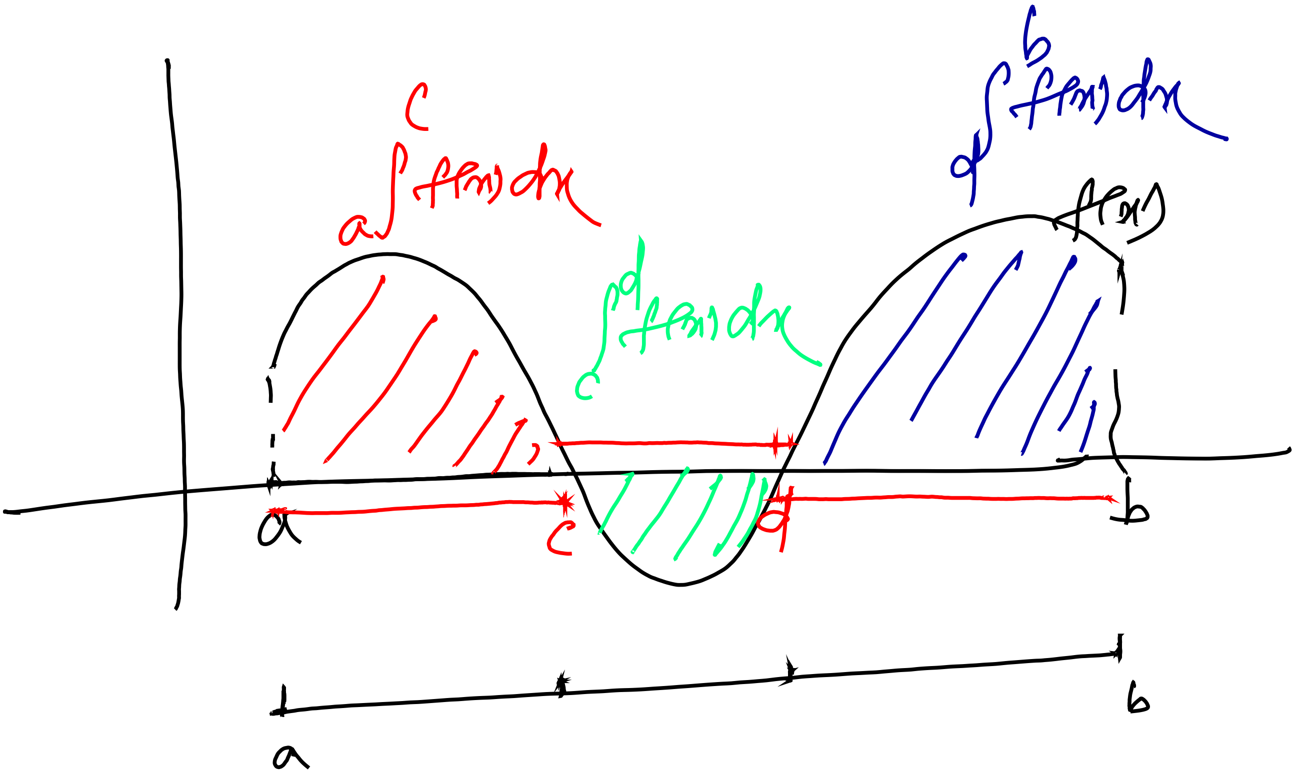
To find the area of the region bounded by $f(x)$ and the x-axis over the interval $[a,b]$



① Subdivide [a,b] at the zeros of f.

② Integrate f over each subinterval

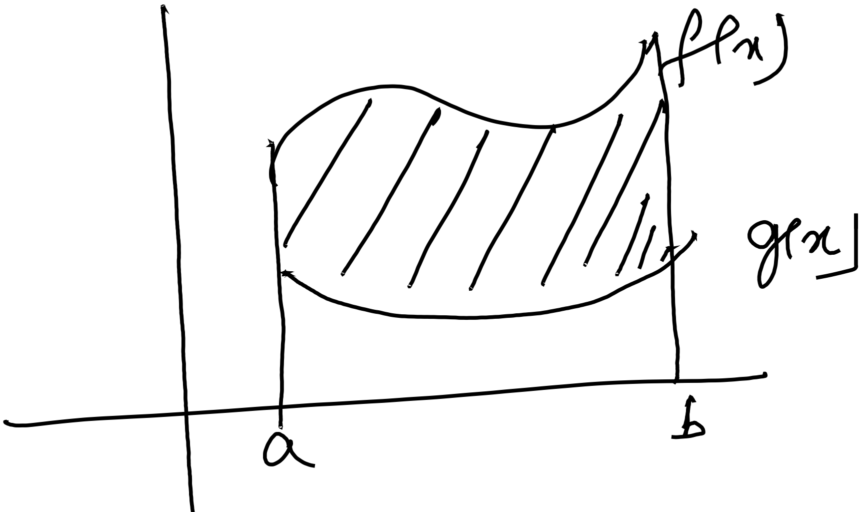
③ Add the absolute values of the integrals.

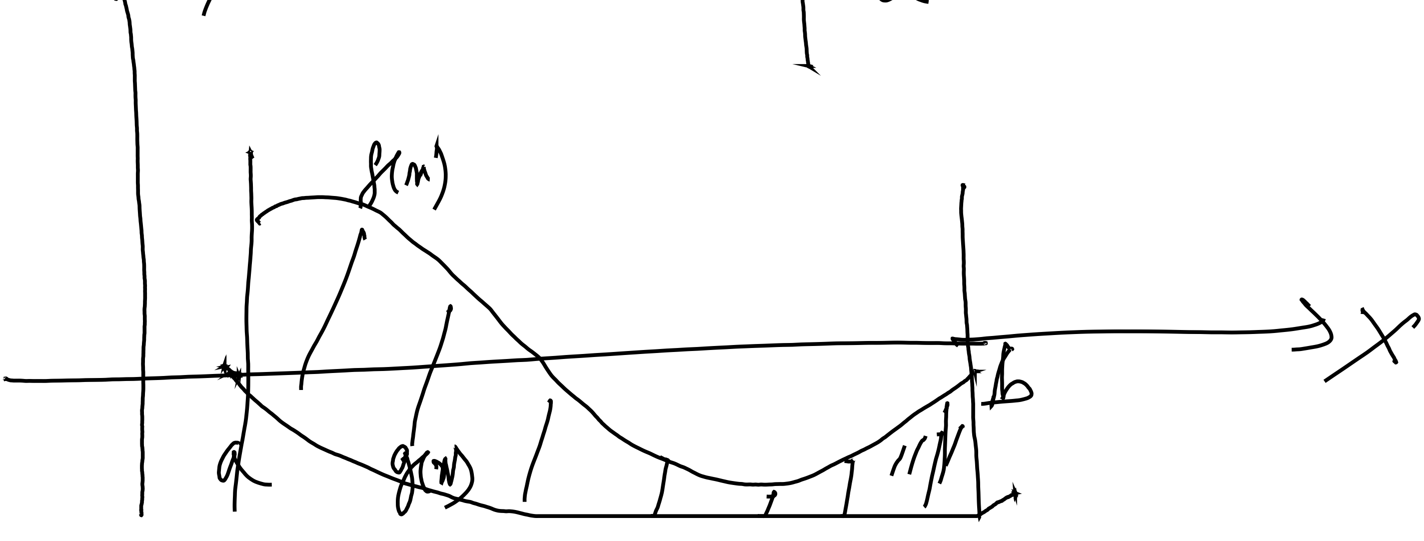


Area bounded between two curves

If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is

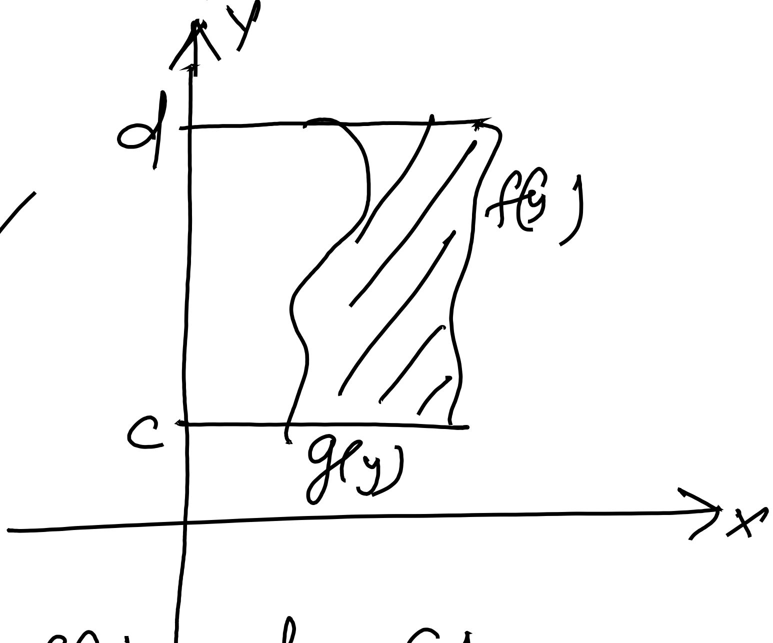
$$A = \int_a^b \left(f(x) - g(x) \right) dx$$





Similarly with respect to y-axis

$$\int_c^d \left(f(y) - g(y) \right) dy$$



f(y) and g(y)
continuous functions.

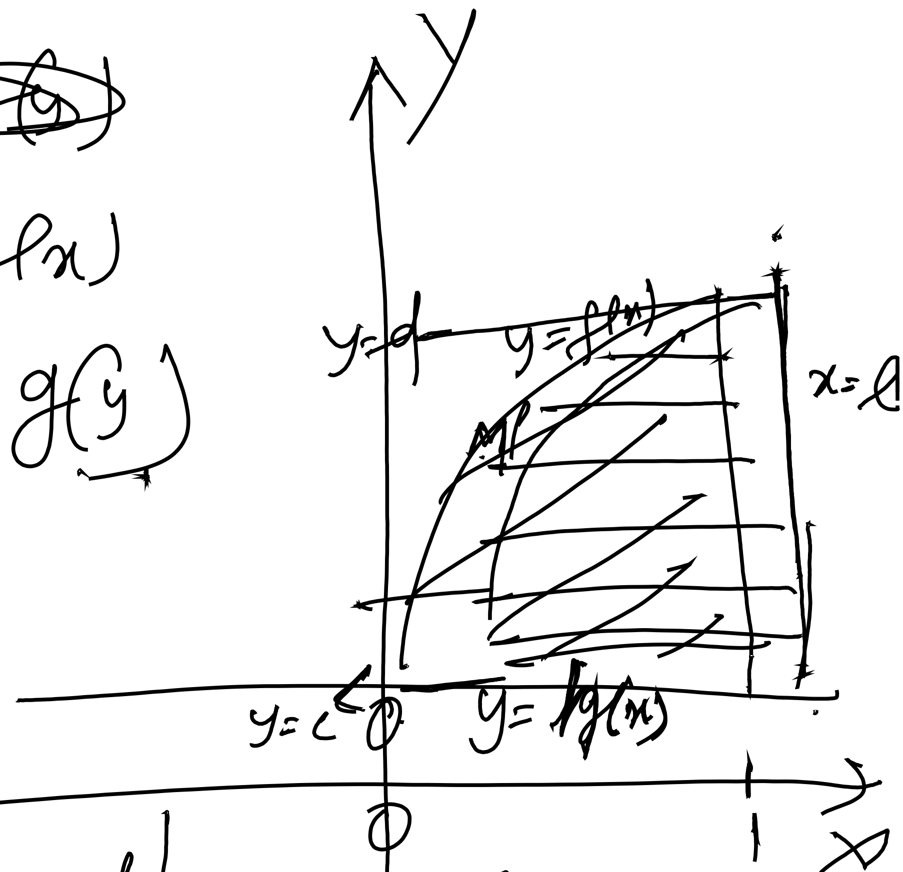
$x^2 + 3x =$

~~x = f(y)~~

$y = f(x)$

$x = g(y)$

$$\int_{y=c}^{y=d} l(x) -$$



$$\int_0^1 \left(f(x) - h(x) \right) dx$$

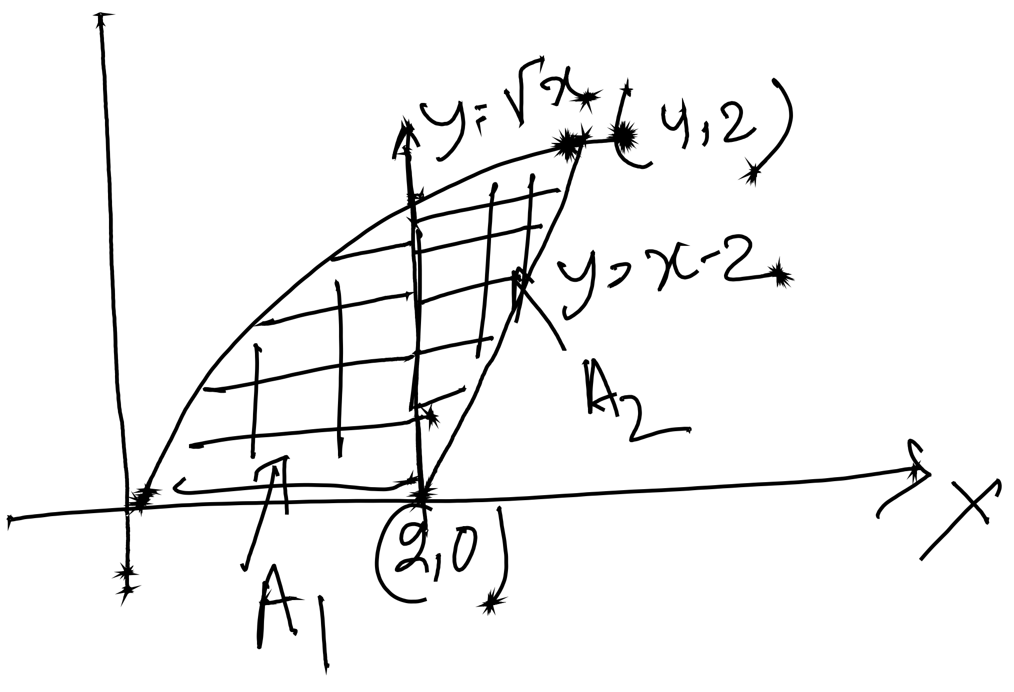
$y = x^2 \quad y = f(x) = x^2$

$f^{-1}(x) = \sqrt{y}$

Exp

Find the area of the region in the 1st quadrant that is bounded above ~~by the x-axis~~

by $y = \sqrt{x}$ and below by the x-axis and the line $y = x-2$.



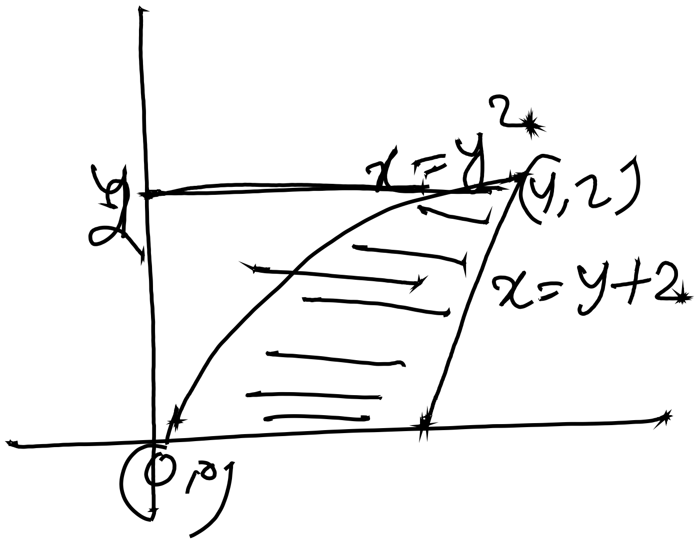
$$A_1 = \int_0^2 (\sqrt{x} - 0)\, dx = \frac{2}{3} \times 2^{3/2}$$

$$A_2 = \int_2^4 \left(\sqrt{x} - (x-2)\right) dx$$

$$= \frac{16}{3} - 2 - \frac{2}{3} 2^{3/2}$$

$$A = A_1 + A_2 = \frac{10}{3}$$

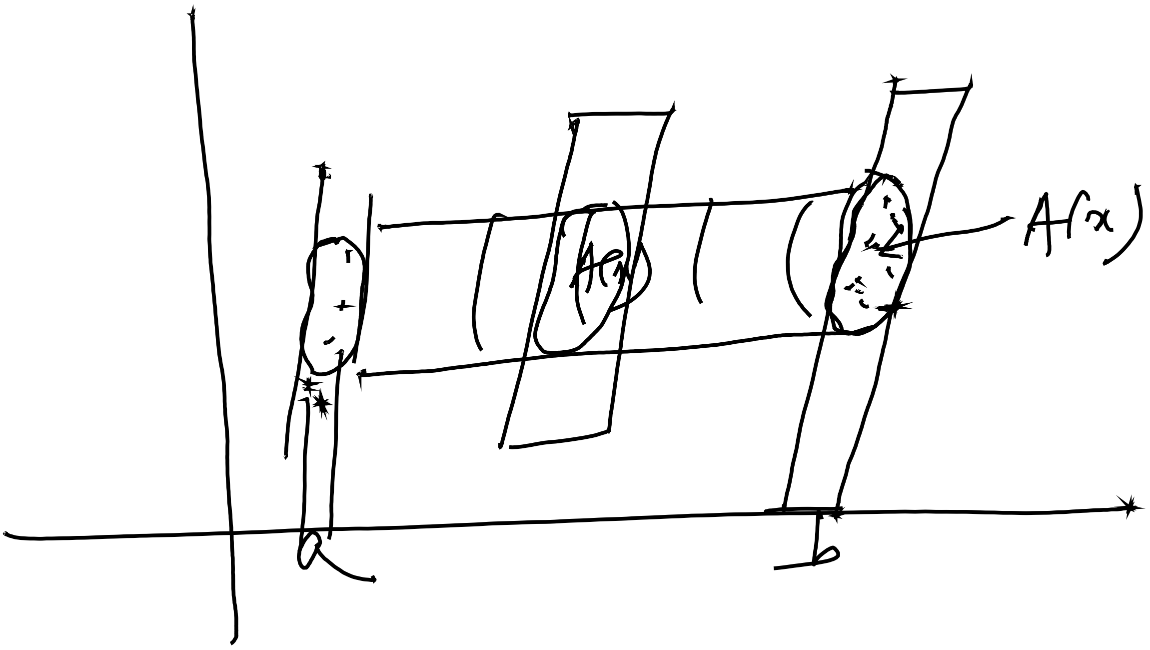
<u>OR</u>



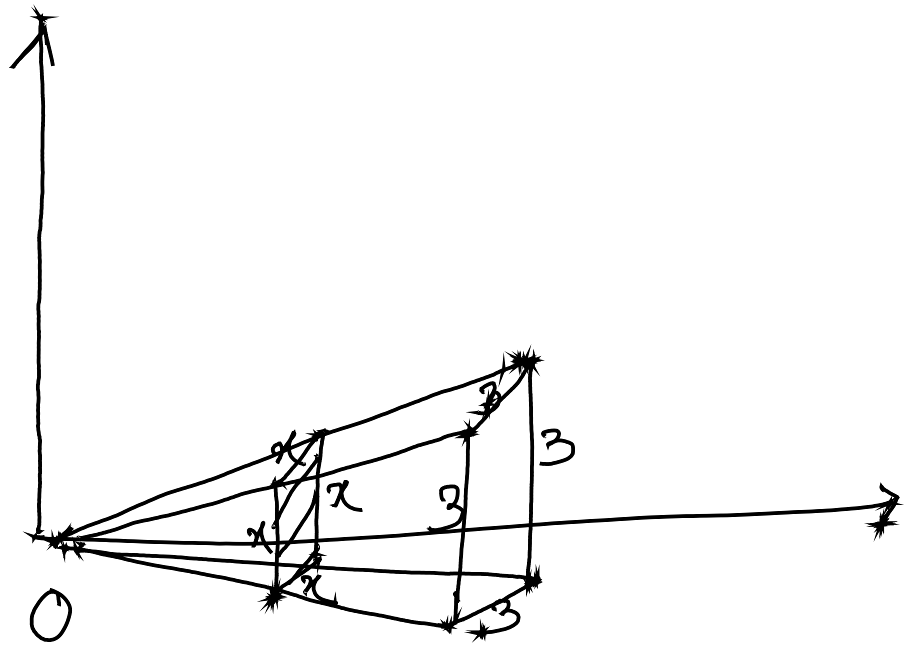
$$A = \int_0^2 (y+2-y^2)\, dy = \frac{10}{3}$$

Calculate the volume of a solid

1. Try to sketch the solid and a typical cross section.
2. Find a formula for the cross sectional area $A(x)$.
3. Find the limits of integration
4. Integrate $A(x)$ to find the volume.



Exp. A pyramid of 3m high has a square base that is 3 m on each side. The cross section of the pyramid perpendicular to the altitude x meter down from the vertex is a square x meter on each side. Find the volume of the solid (pyramid).



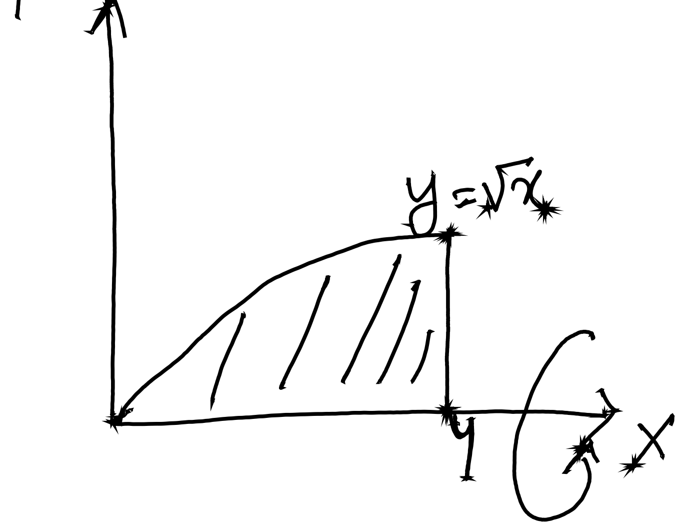
$$A(x) = x^2$$

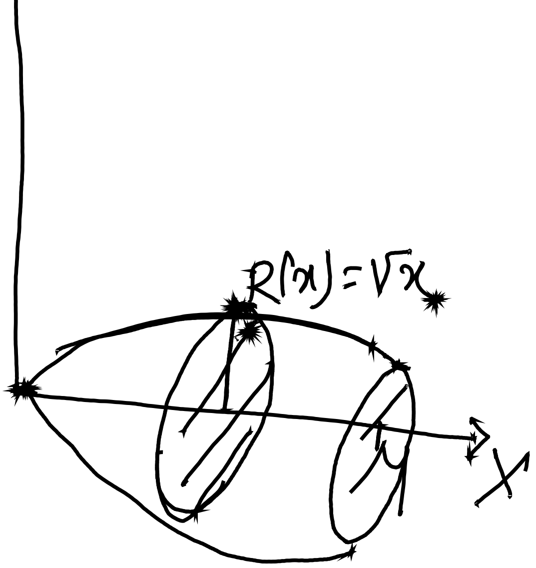
$$V = \int_0^3 A(x)\,dx$$

$$= \int_0^3 x^2\,dx = 9\,m^3$$

Solids of revolution

The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution.



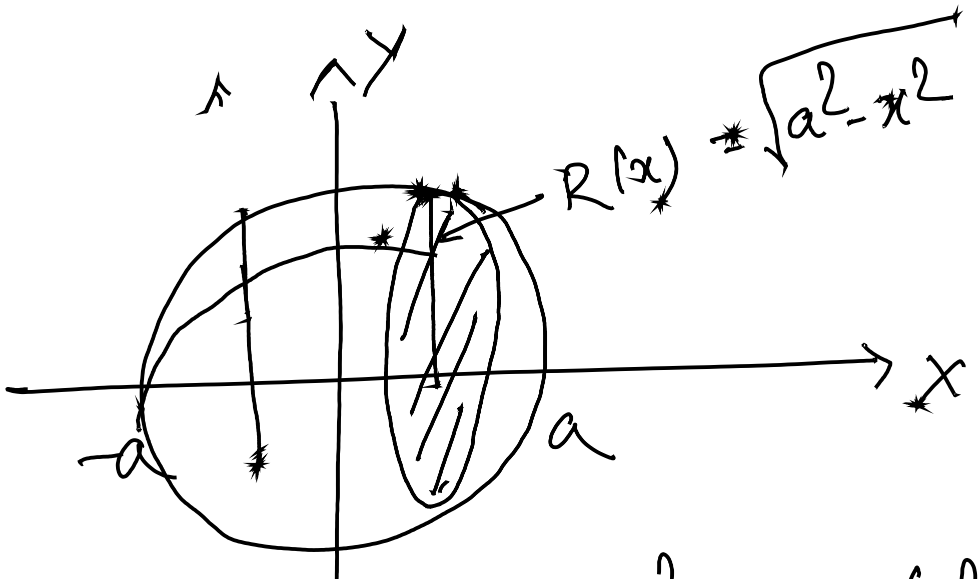


$$A(x) = \pi (R(x))^2$$
$$= \pi x$$

$$\int_{x=0}^{4} \pi x \, dx = 8\pi$$

Exp. The circle $x^2+y^2=a^2$ is rotated about the x-axis to generate a sphere. Find its volume.

Sol^n



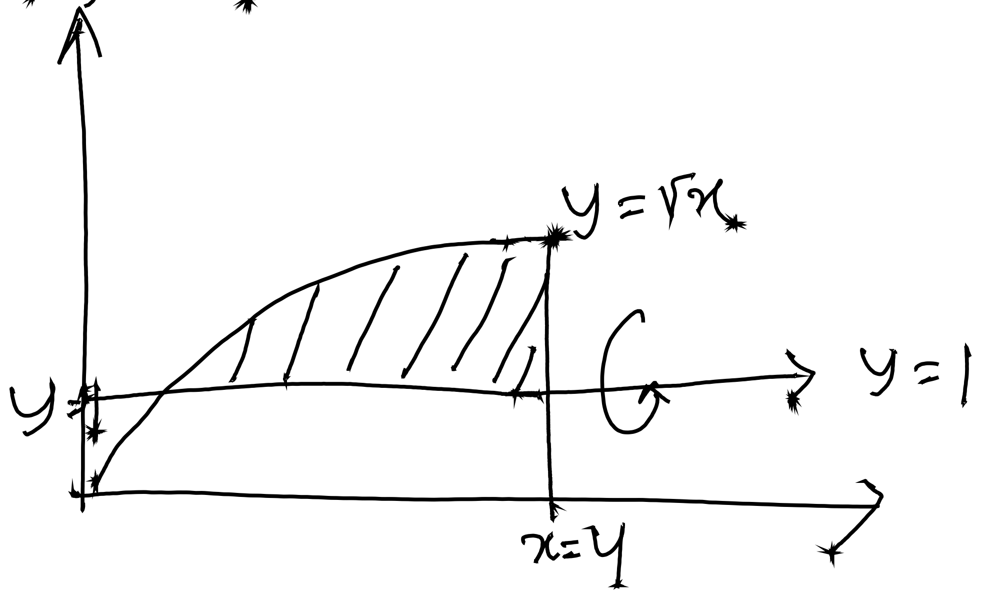
$$A(x) = \pi (R(x))^2 = \pi(a^2-x^2)$$

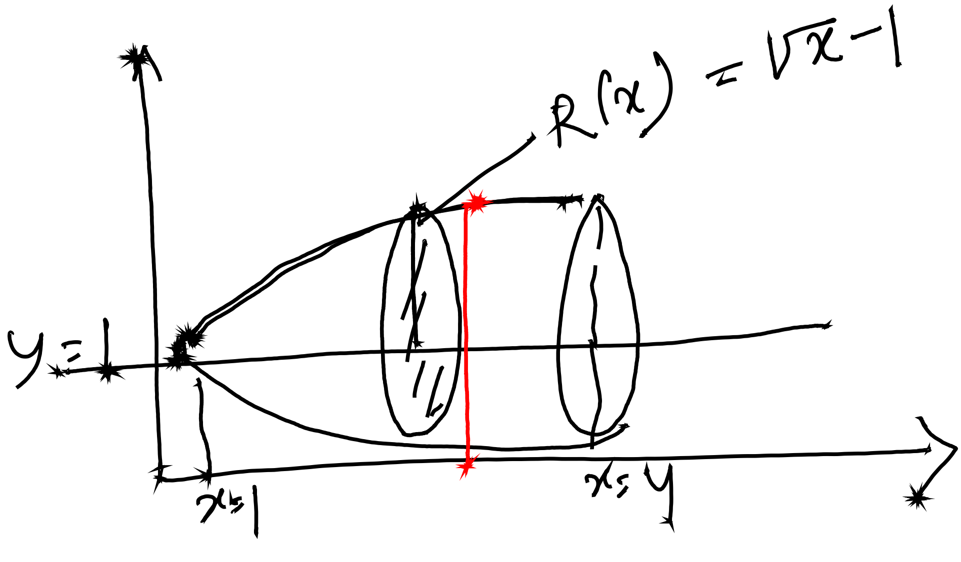
$$V = \int_{-a}^{a} A(x)\,dx = \pi\int_{-a}^{a}(a^2-x^2)\,dx$$

$$= \frac{4\pi a^3}{3}$$

Exp

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$





$$A(x) = \pi(\sqrt{x}-1)^2 = \pi(x+1-2\sqrt{x})$$

$$V = \int_1^4 A(x)\,dx = \frac{7}{6}\pi$$

Volume of solid by rotating a region about y-axis.

$$V = \int_c^d A(y)\,dy = \int_c^d \pi \left(R(y)\right)^2 dy$$