Rules of Inference for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

Universal instantiation

$$\frac{P(c_1) \quad P(c_2) \ldots P(c_n)}{P(c) \text{ for an arbitrary } c}$$

$c \in D$

$D = \{c_1, c_2 \ldots c_n\}$

$$\frac{P(c_1) \wedge P(c_2), \ldots, \wedge P(c_n)}{\therefore\ \forall x\, P(x)}$$

Universal generalization

$$\frac{\exists x\, P(x)}{P(c) \text{ for some element } c}$$ Existential Instantiation

$P(c_2)$

$P(c_1) \quad P(c_2) \; \cdots \; P(c_n)$

$D = \{c_1, \ldots, c_n\}$

$P(c_1)$

$$\frac{P(c) \text{ for some element } c}{\exists x\, P(x)}$$ Existential generalization

3. D(Marla) → C(Marla)

4. D(Marla)

5. C(Marla)

$$\frac{\forall x\, P x}{P(c)}$$

Exp

P_1 : Everyone in this discrete mathematics class has taken a course in computer science.

P_2 : Marla is a student in this class.

C : Marla has taken a course in CS.
C(Marla)

D: set of people

Sol^n

D(x) : x is in this discrete math class

C(x) : x has taken a course in CS.

1. $\forall x(D(x) \to C(x))$

× $\forall x(D(x) \wedge C(x))$

2. D(Marla)

2. D(Marla)

∴ C M

Normal Forms

Conjunctive Normal form formula (cnf)

Disjunctive Normal form formula (dnf)

literal : p , $\neg p$

elementary conjunction : conjunction of literals

elementary product

$p \wedge q \wedge \neg r$, $p \wedge r$, $\neg q \wedge p$

elementary disjunction : disjunction of literals

elementary sum $p \vee \neg q$, $\neg p \vee q \vee r$

Defn A formula that is disjunction of elementary conjunctions is called a disjunctive normal form formula

$\cancel{(p \wedge q) \vee (\neg p \wedge q \wedge q)}) \vee$

$$\underline{(p \wedge q)} \vee \underline{(\neg p \wedge q \wedge r)} \vee \underline{(p \wedge \neg r)}$$

dnf

Defn A formula that is conjunction of elementary disjunction is called conjunctive normal form form . cnf

$$(p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge (p \vee \neg r)$$

Exp $(p \to q) \wedge \neg q$

To convert it to dnf

$\equiv (\neg p \vee q) \wedge \neg q$

$\equiv \underline{(\neg p \wedge \neg q)} \vee \underline{(q \wedge \neg q)}$ dnf

Exp

$p \leftrightarrow q \equiv \underline{(p \wedge q)} \vee (\neg p \wedge \neg q)$

$p \to q \equiv \neg p \vee q$

2. $\neg (p \wedge q) \leftrightarrow (p \vee q)$ To dnf

~~To cnf~~

1. $\neg(p \wedge q) \leftrightarrow (p \vee q)$

$\equiv [\neg(p \wedge q) \wedge (p \vee q)] \vee [\neg\neg(p \wedge q) \wedge \neg(p \vee q)]$

$\equiv [(\neg p \vee \neg q) \wedge (p \vee q)] \vee [(p \wedge q) \wedge (\neg p \wedge \neg q)]$

$\equiv (\neg p \wedge p) \vee (\neg q \wedge p) \vee (\neg p \wedge q) \vee (\neg q \wedge q)$

$\vee (p \wedge q \wedge \neg p \wedge \neg q)$ dnf

The dnf form is not unique

$$(p \wedge q) \vee r$$

$$\equiv (p \vee r) \wedge (q \vee r)$$

$$\equiv (p \wedge q) \vee (r \wedge q) \vee (p \wedge r) \vee (r \wedge r)$$

dnf

cnf

$$\underline{p \leftrightarrow q} \equiv \underline{(p \to q) \wedge (q \to p)}$$

Boolean Algebra

$B = \{0, 1\}$ $\qquad \overline{0} = 1$, $\overline{1} = 0$

Boolean sum : $1+1=1$, $1+0=1$, $0+1=1$,
$0+0=0$

Boolean product : $1 \cdot 1 = 1$, $1 \cdot 0 = 0$
$0 \cdot 1 = 0$, $0 \cdot 0 = 0$

$B = \{0, 1\}$, $\quad B^n = \{(x_1, \ldots x_n) \mid x_i \in B, \ 1 \le i \le n\}$

$B^2 = \{0,1\} \times \{0,1\}$ $\qquad F(x,y,z) = xy + z$

$\underline{B^3} = \{0,1\} \times \{0,1\} \times \{0,1\}$ $\qquad \underline{\text{Ex}} \quad \underline{F(x,y) = xy}$

$\qquad\quad x \qquad\quad y \qquad\quad z \qquad\qquad (x,y) \in B^2$

A variable x is called a Boolean variable if it assumes values from B

A function from $\underline{B^n \to B}$ is called a Boolean function of degree n

$F(x, y) = \underline{xy}$ a BF of degree 2

$(x, y) \in B^2$

$f : \underline{B^3} \to \underline{B}$

$\underline{(x_1 x_2) + \overline{x_3}}$

Boolean Expressions

It is defined recursively as follows

(1) $0, 1, x_1, x_2, x_3 \ldots x_n$ are BE

(2) If E_1, E_2 are BE

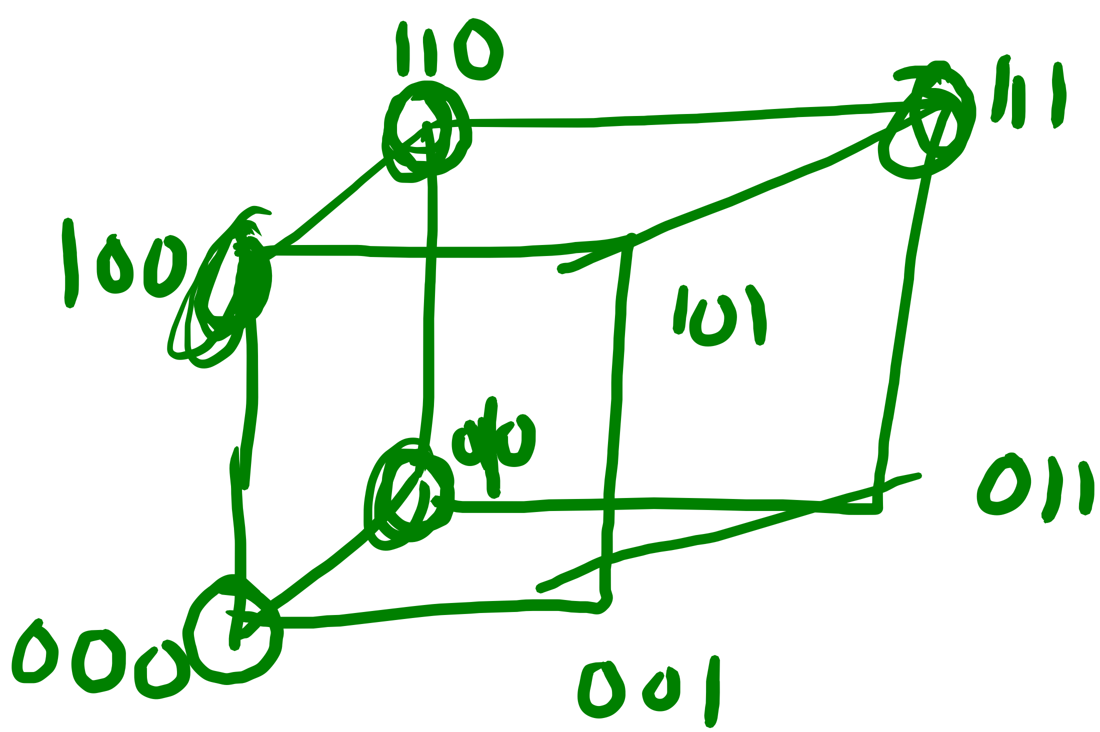
then $\overline{E_1}$, $(E_1 E_2)$, $(E_1 + E_2)$ are BE

Exp find the value of the BF

$$F(x,y,z) = xy + \overline{z} \qquad (x,y,z) \in B^3$$

x	y	z	xy	$\overline{z}$	$xy+\overline{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Each Boolean function can be represented by a n cube that corresponds to the n tuple of bits when the function has value 1.



F and G are equal

iff $F(b_1 \ldots b_n) = G(b_1 \ldots b_n)$

when $b_1, \ldots b_n \in B$

The complement of BF F is $\bar{F}$

$\bar{F}(x_1 \ldots x_n) = \overline{F(x_1 \ldots x_n)}$

The Boolean sum F+G is

$(F+G)(x_1 \ldots x_n) = F(x_1 \ldots x_n) + G(x_1 \ldots x_n)$

The Boolean product $(FG)(x_1 \ldots x_n) = F(x_1 \ldots x_n) G(x_1 \ldots x_n)$

$$F : B^n \to B \qquad B = \{0, 1\}$$

$$F : \underbrace{B \times B \cdots \times B}_{n} \to \underset{\{0,1\}}{B} \qquad B \times B$$

$$F : B^2 \to B$$

$$F : \underbrace{\{(0\,0), (0\,1), (1\,0), (1\,1)\}}_{2^2 = 4} \to \underbrace{\{0, 1\}}_{2} \qquad 2^4 \qquad 2^{2^2}$$

$$F : 2^n \to 2 \qquad 2^{2^n}$$

Identities of Boolean Algebra

~~Ex~~ $\bar{\bar{x}} = x$ Law of double complement

$x + x = x$
$x \cdot x = x$ } Idempotent

$\underline{\underbrace{x(y+z)} = \underbrace{xy + xz}}$

$x + 0 = x$
$x \cdot 1 = x$ } Identity

$x + y = y + x$
$xy = yx$ } Commutative

$$x + (y+z) = (x+y) + z$$

$$x(yz) = (xy)z$$

Associative

$$x + yz = (x+y)(x+z)$$

$$x(y+z) = xy + xz$$

Distributive

$$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$$

$$\overline{xy} = \bar{x} + \bar{y}$$

$$\overline{(x+y)} = \bar{x}\,\bar{y}$$

DeMorgan's Laws

$x + xy = x$

$x(x+y) = x$ Absorption

$x + \bar{x} = 1$ Unit property

$x\bar{x} = 0$ Zero property.

Boolean Algebra

A BA is a set $B = \{0,1\}$ with two binary operations $\vee$ and $\wedge$, elements, 0 & 1, and unary operation $-$ s.t. following properties holds $\forall x, y, z \in B$.

$$\langle B, \vee, \wedge, \bar{\ } \rangle$$

$x \in B$

1. $x \vee 0 = x$
   $x \wedge 1 = x$ Identity

2. $x \vee \bar{x} = 1$

$x \wedge \bar{x} = 0$ Complement

3. $(x \vee y) \vee z = x \vee (y \vee z)$

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ } Associative

4. $x \vee y = y \vee x$

$x \wedge y = y \wedge x$ } Commutative

5. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ } Distributive

Exp: find BF for BF $f(x, y, z)$

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1 ✓
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1 ✓
0	0	1	0	0
0	0	0	0	0

$1 = T$

$0 = F$

$F = 1$ when $x = z = 1,\ y = 0$

$x\bar{y}z = 1$ iff $x = \bar{y} = z = 1$

$G = 1$ when $x = y = 1,\ z = 0$ or $x = z = 0,\ y = 1$

$G = xy\bar{z} + \bar{x}y\bar{z}$ iff $x = z = 1,\ y = 0$

$f(x, y, z) = x\bar{y}z$

Exp

$$x(y+z) = xy + xz$$

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1				1	1
1	1	0				1	1
1	0	1				1	1
1	0	0				0	0
0	1	1				0	0
0	1	0				0	0
0	0	1				0	0
0	0	0				0	0

Exp $x(y+z) = xy + xz$

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1				1	1
1	1	0				1	1
1	0	1				1	1
1	0	0				0	0
0	1	1				0	0
0	1	0				0	0
0	0	1				0	0
0	0	0				0	0