Matrix rank theorem

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We establish the following theorem, for real valued matrices $A_{m \times n}$.

Theorem 1 *For any matrix with entries as real numbers, the column rank is the same as the row rank. This is just called rank of the matrix.*

Proof

We provide the following algorithm.

Step 1. List all the linearly independent columns of a matrix and move them to the left by column exchanges. This places all the independent columns to the left and all the dependent columns to the right, thereafter. Let us denote the number of linearly independent columns by r. Thus the left most r columns consist of a maximum set of linearly independent columns, while the right most $n - r$ columns are linear combinations of the first r.

Step 2. We now rearrange rows such that the first s (from the top) are linearly independent and the last $m - s$ are linear combinations of the first s rows.

Step 3. Eliminate all the dependent columns.

The number of rows in the reduced matrix cannot be less than the number of columns, based on dimension and linear independence considerations. Let us call the original matrix A and the reduced matrix A_{red}.

Then,

$$\textit{Row-rank}(A_{red}) \leq \textit{number-of-columns}(A_{red}) = \textit{col-rank}(A_{red}) = \textit{col-rank}(A)$$

Consider any row in A that is generated as a linear combination of other rows in A. Clearly, ignoring the eliminated columns, the same linear combination generates the truncated row, from the truncated rows. Thus, linear dependence of a row on other rows prior to truncation, continues to hold after truncation.

Let us come to linear independence. Suppose a set of rows independent in A become dependent in A_{red}. This means the homogeneous equation

$$x_1 R_1 + \cdots + x_s R_s = 0$$

has a non-trivial solution. Using this solution in the original matrix A, we conclude that it yields 0 on the untruncated columns. Since the truncated columns are all linear combinations of the untruncated columns, and the untruncated columns, all evaluate to zero, the truncated columns will also evaluate to 0.

Let us take for example column $r + 1$.

$$C_{r+1} = a_1 \times C_1 + \cdots + a_r \times C_r$$

Suppose, $(y_1, \ldots, y_s)$ is a non-trivial solution to the homogeneous equation we set up on the newly dependent rows. Using the same values, on the original matrix A, we see that column C_{r+1} also evaluates to 0. This is because:

$$y_1 \times A_{1,r+1} + \cdots + y_s \times A_{s,r+1} = \Sigma_{i=1}^{r} a_i \times (y_1 \times A_{1,i} + \cdots + y_s \times A_{s,i}) = 0$$

This is a contradiction to these rows being independent in A. Thus both dependence and independence are preserved.

We have thus established that $row\text{-}rank(A) \leq col\text{-}rank(A)$. By identical arguments, we can establish that $col\text{-}rank(A) \leq row\text{-}rank(A)$. This establishes the theorem.