Properties of continuous functions

If f and g are continuous at $x=c$, then

1. $f+g$, $f-g$ are continuous at c.
2. Kf is continuous at c for any scalar K.
3. fg is continuous
4. $\frac{f}{g}$ is continuous provided $g(c) \neq 0$.
5. f^n is continuous, n is positive integer
6. $\sqrt[n]{f}$ is continuous provided it is defined on an open interval containing c, n positive ~~integer~~
7. If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Exp Is $\left|\frac{x\sin x}{x^2+2}\right|$ continuous?

Sol^n $x\sin x$ continuous as product of two continuous function

x^2+2 continuous

as $x^2+2 \neq 0$ on $\mathbb{R}$

$\Rightarrow \frac{x\sin x}{x^2+2}$ is continuous on $\mathbb{R}$

Also $|\cdot|$ function is continuous on $\mathbb{R}$.

$\rightarrow \left|\frac{x\sin x}{x^2+2}\right|$ is continuous.

Property (8)

If g is continuous at b and $\lim_{x \to c} f(x) = b$,

then $\lim_{x \to c} g(f(x)) = g\left(\lim_{x \to c} f(x)\right) = g(b)$.

That means limit can enter into a continuous function.

EXP $\lim_{x \to \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

$= \cos\left(\lim_{x \to \pi/2} 2x + \lim_{x \to \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right)$

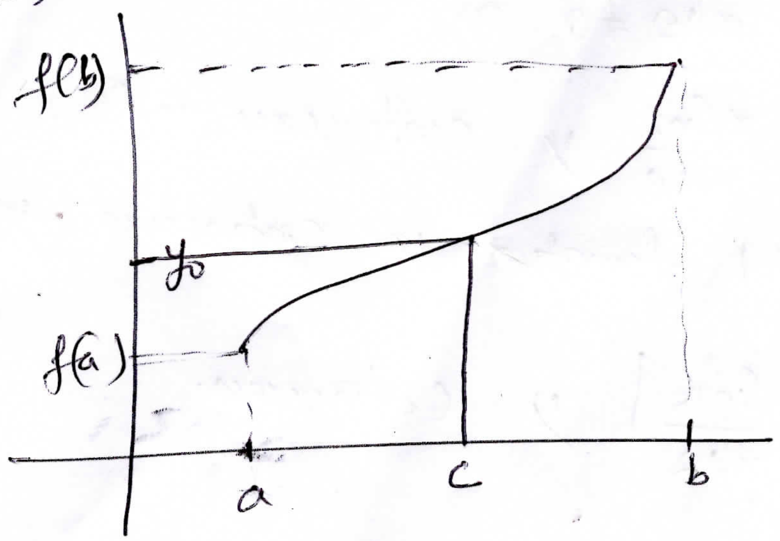
$= \cos\left(\pi + \sin 2\pi\right) = \cos(\pi) = -1$

Property :

(9) Intermediate value property of continuous function

If f is continuous on a closed interval $[a,b]$ and if y_0 is any value between $f(a)$ and $f(b)$, then there exists some $c \in (a,b)$ such that

$y_0 = f(c)$



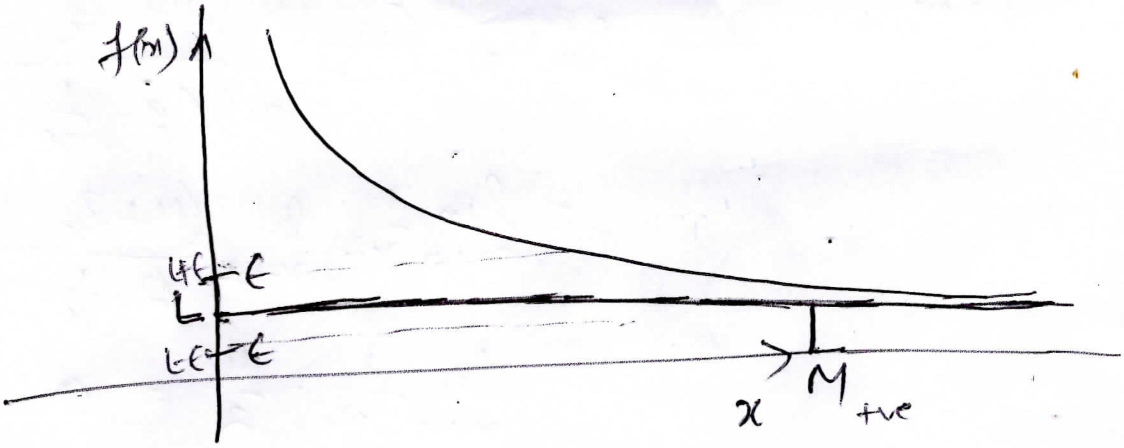
Limits involving infinity

Asymptotes of graphs

$\lim_{x \to \infty} f(x) = L$

If for ~~every~~ any $\epsilon > 0$, there exists a real number $M > 0$ such that for all x, $x > M \Rightarrow |f(x) - L| < \epsilon$.

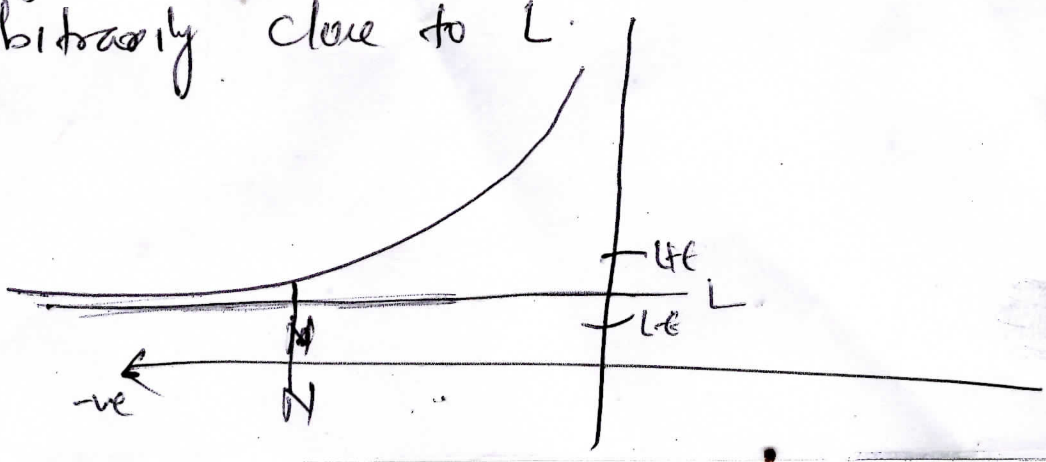
Meaning: As x moves increasingly far from origin in the positive direction, $f(x)$ gets arbitrarily close to L.



$\lim_{x \to -\infty} f(x) = L$

If for any $\epsilon > 0$, there exist a real number N such that for all x, $x < N \Rightarrow |f(x) - L| < \epsilon$

Meaning: As x moves increasingly far from origin in the negative direction, $f(x)$ gets arbitrarily close to L.



Horizontal Asymptotes

A line $y=b$ is called a horizontal asymptote of the graph of a function $y=f(x)$ if either $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$.

Exp $f(x)=\dfrac{11x+2}{2x^3-1}$

Find the horizontal asymptote of the curve $y=f(x)$.

Ans
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x\left(11+\frac{2}{x}\right)}{x^3\left(2-\frac{1}{x^3}\right)}$$

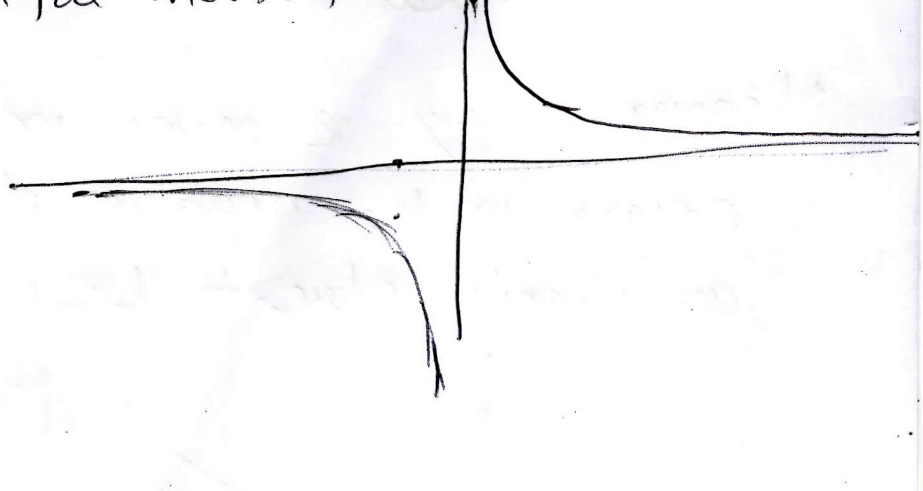
$$= \lim_{x\to\infty} \frac{1}{x^2} \lim_{x\to\infty} \frac{\left(11+\frac{2}{x}\right)}{\left(2-\frac{1}{x^3}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} \cdot \frac{11}{2} = 0$$

$y=0$ (X-axis) is the horizontal asymptote.

Exp $y=f(x)=\frac{1}{x}$

X-axis is the horizontal asymptote.



Oblique Asymptote

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, then the graph ~~of~~ the function has an oblique asymptote.

Exp $f(x) = \dfrac{x^2-3}{2x-4}$

$$f(x) = \frac{x^2-3}{2x-4} = \underbrace{\left(\frac{x}{2}+1\right)}_{g(x)} + \underbrace{\frac{1}{2x-4}}_{\text{remainder}}$$

As $x \to \pm\infty$, the remainder term goes to 0.

Thus as $x \to \pm\infty$ the difference between the functions $f(x)$ and $g(x)$ become arbitrarily small.

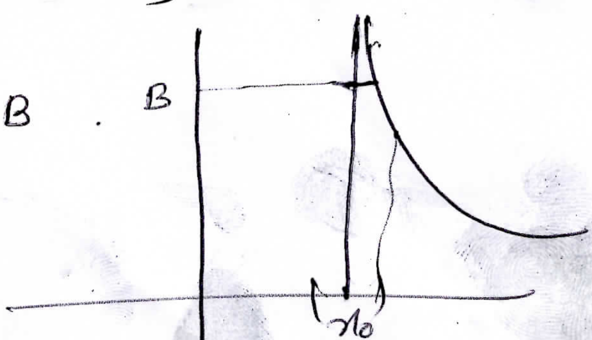
So $\boxed{g(x) = \frac{x}{2}+1}$ is an oblique asymptote to the graph of $f(x) = \dfrac{x^2-3}{2x-4}$

Infinite limits

$$\lim_{x \to x_0} f(x) = \infty$$

If for any positive number $B>0$, there exist $\delta>0$ such that for all x

$$0<|x-x_0|<\delta \Rightarrow f(x)>B.$$



$\lim_{x \to x_0} f(x) = -\infty$

If for every negative real number $-B$, there exists a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow f(x) < -B.$$

Vertical asymptote

A line $x = a$ is called a vertical asymptote of the graph $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty$$

or $\lim_{x \to a^-} f(x) = \pm\infty$.

Ex. $y = f(x) = \frac{1}{x}$

$\lim_{x \to 0^+} \frac{1}{x} \to +\infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

So y-axis $(x = 0)$ is a vertical asymptote to $y = \frac{1}{x}$