SC223 - Linear Algebra

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Lecture 5

August 18, 2022

● **Linear Independent set of vectors:** A set of vectors $\{a_{*i}, i = 1, \ldots, n\}$ is said to be linearly independent if none of the vectors can be written as a linear combination of the remaining vectors.

● A set of vectors $\{a_{*i}, i = 1, \ldots, n\}$ is said to be linearly independent if

$$\sum_{i=1}^{n} z_i a_{*i} = \mathbf{0}_m \quad \Rightarrow z_1 = z_2 = \ldots = z_n = 0$$

● **Column Rank of a Matrix:** The number of linearly independent columns of a matrix is called the Column Rank.

- **Matrix Transpose:** For $A \in \mathbb{R}^{m \times n}$ given by

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

the matrix

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● **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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- Since rows of A are columns of A^T, notation for rowspace is $C(A^T)$, and can be written as

$$C(A^T) = \{A^T y \mid \forall y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

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● Properties:

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- If $x_1, x_2 \in C(A^T)$, then $\forall p, q \in \mathbb{R}, p \cdot x_1 + q \cdot x_2$

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$$\begin{aligned} I_r(i,j) &= \quad 1, \text{ if } i = j \\ &= \quad 0, \text{ otherwise} \end{aligned}$$

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- Since each row of the two matrices also have to be the same:

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- **Rank of a matrix:** The number of linearly independent columns or rows of a matrix is called the *Rank of the matrix*.

Gaussian Elimination/Row Reduction

- Let us use the following example:

$$\begin{aligned} 2x_2 + 5x_3 + 4x_4 + 2x_5 &= 2 \\ x_1 - x_2 + 2x_3 + 3x_4 - x_5 &= 1 \\ 2x_1 + x_2 + 4x_4 + 2x_5 &= -1 \\ 3x_1 + x_2 + 3x_3 + 2x_4 - 2x_5 &= 3 \end{aligned}$$

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- When are linear equations easy to solve? As few variables (ideally 1) as possible.
- Representation as a matrix

$$
\left[\begin{array}{ccccc|c}
0 & 2 & 5 & 4 & 2 & 2 \\
1 & -1 & 2 & 3 & -1 & 1 \\
2 & 1 & 0 & 4 & 2 & -1 \\
3 & 1 & 3 & 2 & -2 & 3
\end{array}\right]
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- This is the called the **Augmented matrix**.

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● Idea is to use row operations to get an **upper triangular** AM:

$$\left[\begin{array}{ccccc|c} * & - & - & - & - & - \\ 0 & * & - & - & - & - \\ 0 & 0 & * & - & - & - \\ 0 & 0 & 0 & * & - & - \end{array}\right]$$

● The $*$ positions are called **leading entries**, and are the left-most non-zero entry of each row.

● **Definition:**(*Echelon form*) A matrix is said to be in Echelon form if

- All non-zero rows are above any zero rows (if any).
- Each leading entry (leftmost non-zero entry) in a row is in a column to the right of the leading entry of the row above it.
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● Examples of Echelon forms:

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● Examples of non-Echelon forms:

$$\begin{bmatrix} * & - & - & - & - & - \\ 0 & 0 & * & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & - & - \end{bmatrix} \begin{bmatrix} * & - & - & - & - & - \\ 0 & 0 & 0 & * & - & - \\ 0 & 0 & 0 & * & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & - & - \\ 0 & 0 & * \\ 0 & * & - \\ 0 & 0 & 0 \end{bmatrix}$$

● **Definition:** *(Row-reduced Echelon form (RREF))* A matrix in Echelon form is said to be in a Row-Reduced Echelon Form if

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- Examples of RREF:

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Solving Linear Equations

$$\left[\begin{array}{ccccc|c} 0 & 2 & 5 & 4 & 2 & 2 \\ 1 & -1 & 2 & 3 & -1 & 1 \\ 2 & 1 & 0 & 4 & 2 & -1 \\ 3 & 1 & 3 & 2 & -2 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & 2 & 5 & 4 & 2 & 2 \\ 2 & 1 & 0 & 4 & 2 & -1 \\ 3 & 1 & 3 & 2 & -2 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_4 \leftarrow R_4 - 3R_1}$$

$$\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 3 & -4 & -2 & 4 & -3 \\ 0 & 4 & -3 & -7 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow 2R_3 - 3R_2]{R_4 \leftarrow R_4 - 2R_2} \left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 0 & \mathbf{-23} & -16 & 2 & -12 \\ 0 & 0 & -13 & -15 & -3 & -4 \end{array}\right]$$

$$\xrightarrow{R_4 \leftarrow 23R_4 - 13R_3} \underbrace{\left[\begin{array}{ccccc|c} \mathbf{1} & -1 & 2 & 3 & -1 & 1 \\ 0 & \mathbf{2} & 5 & 4 & 2 & 2 \\ 0 & 0 & \mathbf{-23} & -16 & 2 & -12 \\ 0 & 0 & 0 & \mathbf{-137} & -95 & 64 \end{array}\right]}_{U}$$

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$$R_1 \leftrightarrow R_2 \rightarrow P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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- Thus, $P_{12}A = L'_{43}L'_{32}L'_{42}L'_{31}L'_{41}U = LU$.

LU Decomposition

- In general, any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed into a product of lower and upper triangular matrices, with appropriate permutations:

$$PA = LU,$$

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● Solving linear equations with LU decomposition:
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● First let $Ux = y$ and solve $Ly = b$, and next solve for x in $Ux = y$.