PREDICATE LOGIC

Example $P(x) : x > 3$ true or False for values of x.

$P(4)$ is T & $P(2)$ is F.

$P(x_1, x_2, \ldots, x_n)$ is the value of the prop. fn. P at the n-tuple $(x_1, x_2, \ldots, x_n)$ & P is also called <u>n-place predicate</u> or <u>n-ary predicate</u>.

Quantifiers Quantification expresses the extent to which a predicate is true over a range of elements.

Universal Quantifiers

- The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain".
- Notation $\forall x\, P(x)$ $\quad \forall \rightarrow$ universal quantifier
- An element for which $P(x)$ is false is called <u>counterexample</u> of $\forall x\, P(x)$

Statement	When true?	When false?
$\forall x\, P(x)$	$P(x)$ is true ~~for~~ for every x	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Example ① $P(x) : x + 1 > x$ what is the truth value of the quantification $\forall x\, P(x)$, domain $\rightarrow \mathbb{R}$ (all real nos)

$\because$ $P(x)$ is true $\forall x \in \mathbb{R}$ the quantification $\forall x\, P(x)$ is true.

② $P(x) : x > 3$ What is the truth value of the quantification $\exists x\, P(x)$ domain $\rightarrow \mathbb{R}$

$\because$ $x > 3$ is true sometime for example when $x = 4$

$\therefore$ $\exists x\, P(x)$ is true.

Other Quantifiers $\exists !\, x\, P(x)$ (or $\exists_1 x\, P(x)$) means There exist a unique x such that $P(x)$ is true.

Precedence of Quantifiers

$\forall$ & $\exists$ have higher precedence than all logical operations. For eg. $\forall x P(x) \vee Q(x)$ is disjunction of $\forall x P(x)$ and $Q(x)$.

$$\forall x P(x) \vee Q(x) = (\forall x P(x)) \vee Q(x) \neq \forall x (P(x) \vee Q(x))$$

LOGICAL EQUIVALENCES INVOLVING QUANTIFIERS

- Statements involving predicates & quantifiers are <u>logically equivalent</u> iff they have the same truth value no matter which predicates are substituted into these statements & which domain of discourse is used for the variables in these prop. fns.

$S \equiv T$

Example $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

$\square$ Suppose $\forall x (P(x) \wedge Q(x))$ is T $\Rightarrow$ if $a \in$ Domain

$P(a) \wedge Q(a)$ is T $\Rightarrow$ $P(a)$ is T and $Q(a)$ is T $\forall a \in D$

$\Rightarrow$ $\forall x P(x)$ and $\forall x Q(x)$ are both true

$\Rightarrow$ $\forall x P(x) \wedge \forall x Q(x)$ is T. ~~Thus LHS.~~

Now, suppose $\forall x P(x) \wedge \forall x Q(x)$ is T $\Rightarrow \forall x P(x)$ is T and $\forall x Q(x)$ is T, Hence if $a \in D$ then $P(a)$ is T and $Q(a)$ is T ($\because P(x)$ & $Q(x)$ are both true $\forall x \in D$)

$\Rightarrow \forall a \in D$ $P(a) \wedge Q(a)$ is true

$\Rightarrow$ $\forall x (P(x) \wedge Q(x))$ is T. $\blacksquare$

NEGATING QUNTIFIED EXPRESSIONS

Example $\forall x P(x)$: Every student in your class has taken a course in calculus

$\neg \forall x P(x)$: It is not the case that every student in your class has taken a course in calculus.

$\simeq$ There is a student in your class who has not taken a course in calculus.

$\Rightarrow$ $\exists x \neg P(x)$

Thus $\neg \forall x P(x) \equiv \exists x \neg P(x)$

De Morgan's Laws for quantifiers

Negation	Equivalent statement	When is Negation true?	When false?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x $P(x)$ is false	There is an x for which $P(x)$ is T
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is F	$P(x)$ is T for every x

NESTED QUANTIFIERS

$\forall x\, Q(x)$, where $Q(x)$ is $\exists y\, P(x,y)$, where $P(x,y)$ is $x+y=0$

Example $D = \mathbb{R}$

$$\forall x \exists y \, (x+y=0)$$

for every real no. x, there is a real no. y s.t. $x+y=0$

Quantification of two variables

Statement	When True?	When False?
$\forall x \forall y\, P(x,y)$ $\forall y \forall x\, P(x,y)$	$P(x,y)$ is true for every pair x, y	There is a pair x, y for which $P(x,y)$ is false
$\forall x \exists y\, P(x,y)$	For every x there is a y for which $P(x,y)$ is true	There is an x s.t. $P(x,y)$ is false for every y
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x there is a y for which $P(x,y)$ is false
$\exists x \exists y\, P(x,y)$ $\exists y \exists x\, P(x,y)$	There is a pair x, y for which $P(x,y)$ is T	$P(x,y)$ is false for every pair x, y

Example $Q(x,y): x+y=0$ What are the truth values of the quantifications $\exists y \forall x\, Q(x,y)$ and $\forall x \exists y\, Q(x,y)$ where $x,y \in \mathbb{R}$

□ ① $\exists y \forall x\, Q(x,y)$

There is real no. y s.t. $\forall$ real no. x $Q(x,y)$

No matter what y is choosen there is only one value of x s.t. $x+y=0$

$\because$ There is no real no. y s.t. $x+y=0$ $\forall$ real nos. x

the statement $\exists y \forall x\, Q(x,y)$ is false.

② $\forall x \exists y\, Q(x,y)$

For every real no x there is a real no. y s.t. $x+y=0$

($\because y=-x$ $\forall x \in \mathbb{R}$)

$\Rightarrow \forall x \exists y\, Q(x,y)$ is true.