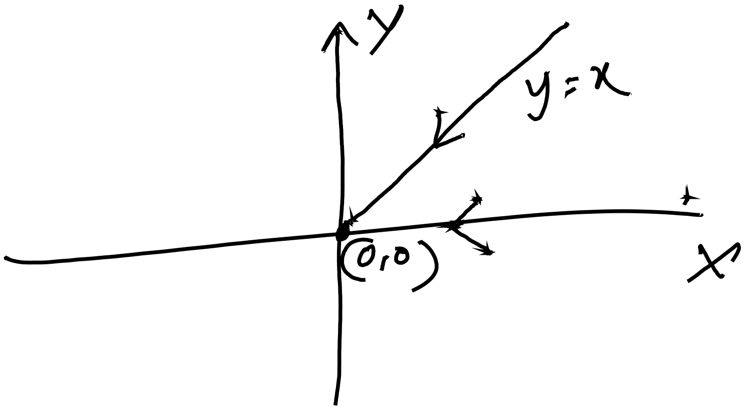
Limits of functions of two variables

Exp Does $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x}$ exist ?



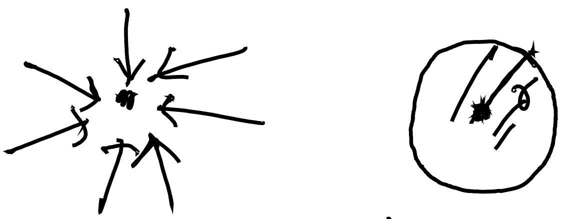
Along x-axis, $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{y}{x} = 0$$

Along $y=x$ line

$$\lim_{(x,y)\to(0,0)} \frac{y}{x} = \lim_{(x,y)\to(0,0)} 1 = 1$$

So $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x}$ does not exist.

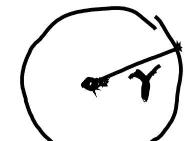


$\lim\limits_{r\to 0}$

Changing to polar co-ordinates

Substitute $x = r\cos\theta$, $y = r\sin\theta$ and investigate the limit of the resulting expression as $r \to 0$

- Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ



$r \to 0$

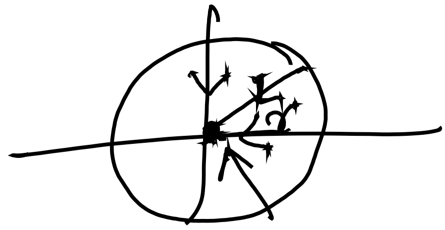
$$0 < |r| < \delta \Rightarrow |F(r,\theta) - L| < \epsilon .$$

$$\boxed{\lim_{r \to 0} F(r,\theta) = L}$$

If such an L exists then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} F(r,\theta) = L$$

Exp) $\lim_{(x,y)\to(0,0)} \frac{y}{x}$



$x = r\cos\theta$
$y = r\sin\theta$

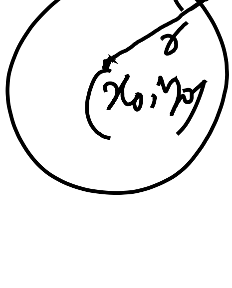
$$\lim_{(x,y)\to(0,0)} \frac{y}{x} = \lim_{r\to 0} \frac{r\sin\theta}{r\cos\theta}$$

$$= \lim_{r\to 0} \tan\theta = \tan\theta$$

For different values of θ we get different limiting valu

So limit does not exist.

$\lim_{(x,y)\to(x_0,y_0)}$ $\quad x = x_0 + r\cos\theta$
$y = y_0 + r\sin\theta$
$r \to 0$



Exp

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

put $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r\,\cos^3\theta = 0$$

exists

$$\boxed{\lim_{r\to 0} \frac{r}{\cos\theta}}$$

does not exist for $\theta = \frac{\pi}{2}$

Exp $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$x = r\cos\theta$

$y = r\sin\theta$

$$f(r,\theta) = \frac{2r^2\cos^2\theta \cdot r\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta}$$

$$= \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}$$

$$\lim_{r\to 0} \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}$$

Along the path $y = mx^2$

$x = r\cos\theta \qquad y = mr^2\cos^2\theta$

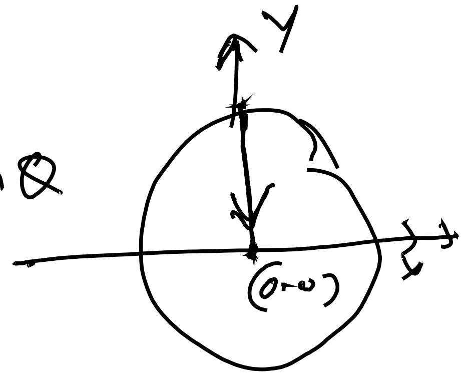
$$F(r,\theta) = \frac{2r^2\cos^2\theta \cdot mr^2\cos^2\theta}{r^4\cos^4\theta + m^2r^4\cos^4\theta}$$

$$= \frac{2m}{1+m^2}$$ different for different values of m

So limit does not exist.

Exp $\quad f(x,y) = \dfrac{2x}{x^2+x+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ exist ?}$$



$x = r\cos\theta \qquad y = r\sin\theta$

$$F(r,\theta) = \frac{2r\cos\theta}{r^2\cos^2\theta + r\cos\theta + r^2\sin^2\theta}$$

$$= \frac{2r\cos\theta}{r^2 + r\cos\theta} = \frac{2\cos\theta}{r+\cos\theta}$$

$$\lim_{r\to 0} F(r,\theta) = \lim_{r\to 0} \frac{2\cos\theta}{r+\cos\theta}$$

$$= \frac{2\cos\theta}{\cos\theta}$$

This does not exist when $\theta = \pi/2$

So limit does not exist.

Continuity

A function $f(x,y)$ is continuous at a point (x_0, y_0) if

(i) f is defined at (x_0, y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

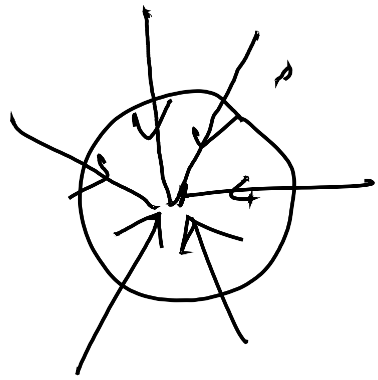
(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Exp Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at every point except at the origin.

$$f(x,y) = \frac{2xy}{x^2+y^2}$$



Approach $(0,0)$ along path $y = mx$

$$f(x,y)\Big|_{\text{along } y=mx} = \frac{2x \cdot mx}{x^2+m^2x^2}$$

$$= \frac{2m}{1+m^2}$$

which is different for different values of m

As $(x,y)\to(0,0)$ we get different limiting value.

So limit does not exist at $(0,0)$

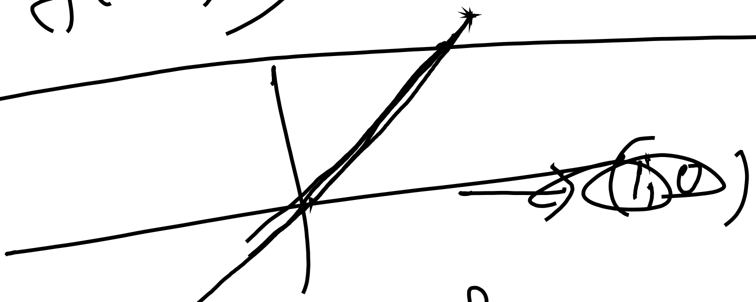
So $f(x,y)$ is not continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$$

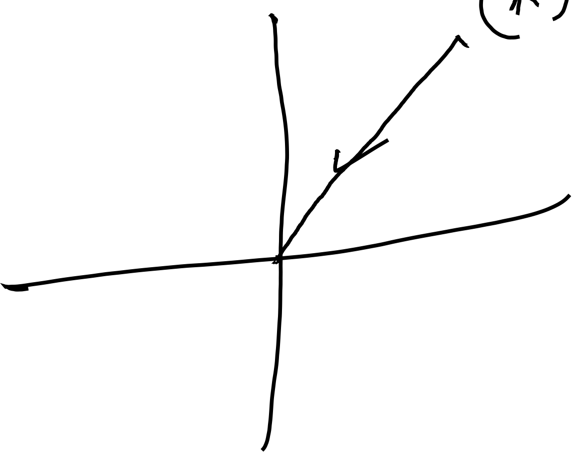
$$f(0,0) = 0$$

So $f(x,y)$ is continuous at $(0,0)$



$(x \neq y)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})}$$



$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$$

Result

~~If f is continuous at (x_0, y_0) single variable function is continuous at~~

If f is continuous at (x_0, y_0) and g is a single variable function continuous at $f(x_0, y_0)$, then the composition $g \circ f = g(f(x,y))$ is continuous.

Exp

$f(x,y) = x-y$

$g(x) = e^x$

e^{x-y}, $\cos\left(\dfrac{xy}{x^2+1}\right)$

$f(x,y) = \dfrac{xy}{x^2+1}$

$g(x) = \cos x$

$\ln(1+x^2+y^2)$

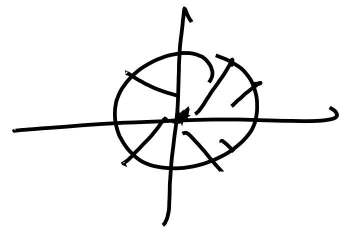
Functions with more than two variables

The definitions of limit and continuity for functions of two variables and the properties (related to limit and continuity) all extend to functions of more than two variables.

In 3-dimensional case we take δ-sphere instead of δ-disk.

$0 < \sqrt{x^2+y^2} < \delta$ δ-disk

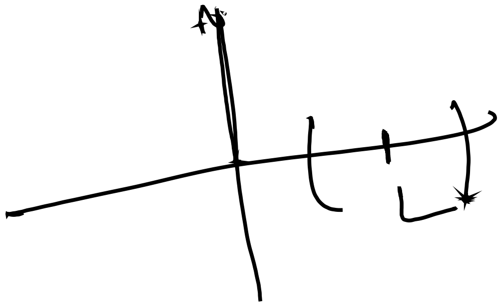
$0 < \sqrt{x^2+y^2+z^2} < \delta$ δ-sphere

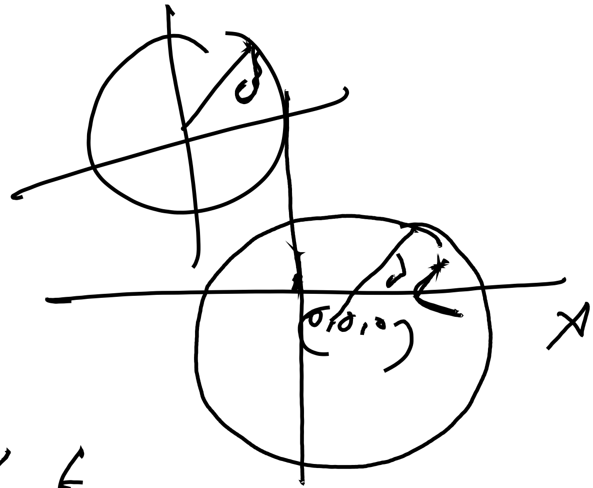


$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$

$x^2+y^2+z^2+w^2 = r^2$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$





$$\left| f(x,y,z) - 0 \right| < \epsilon$$

for $0 < \sqrt{x^2+y^2+z^2} < \delta$

$$\left| \frac{xyz}{x^2+y^2+z^2} \right| \le f\left(\sqrt{x^2+y^2+z^2}\right)$$

$\le$

$$xyz \le \frac{1}{3^{3/2}} (x^2+y^2+z^2)^{3/2}$$

$$\le \frac{C\,(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)} = C\sqrt{x^2+y^2+z^2}$$

$$\sqrt{x^2+y^2+z^2} < \left(\frac{\epsilon}{C} = \delta\right) \epsilon$$