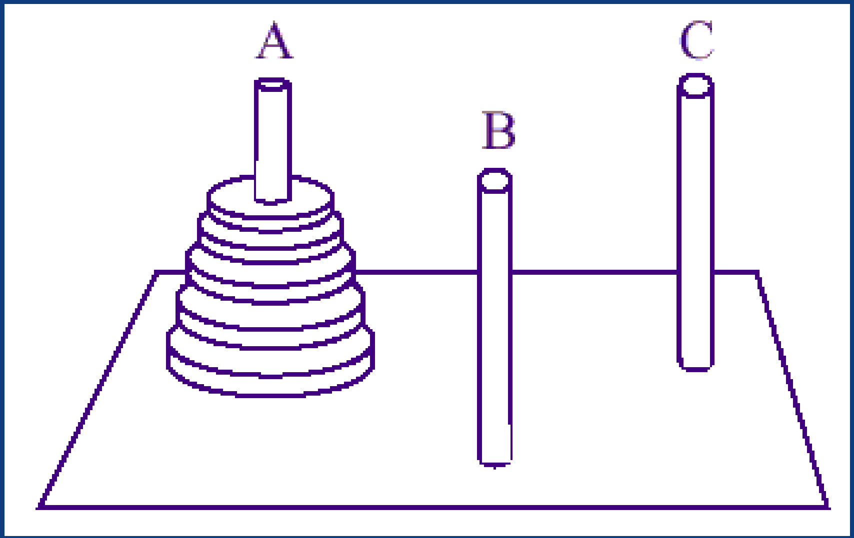
Recurrence Relations

Tower of Hanoi

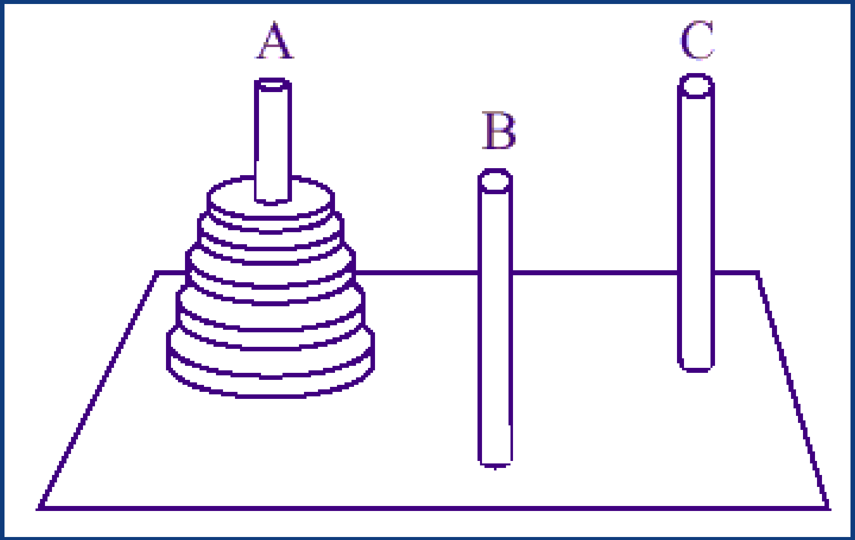
- Invented by French mathematician Edouard Lucas in 1883.

Towers of Hanoi



- There are three pegs: A, B and C
- Peg A has stack of disks
- Disks are stored in decreasing size.

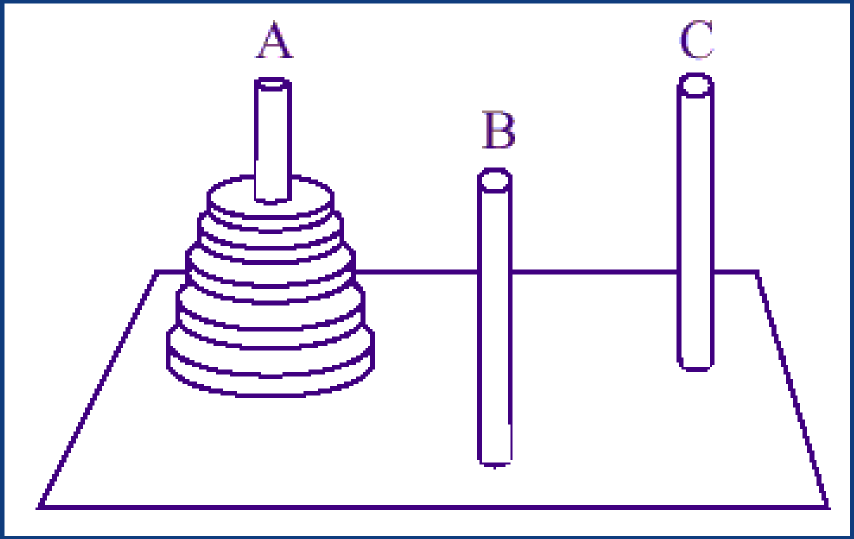
Tower of Hanoi



Problem : How will you move all the disks from peg A to peg C ?

Tower of Hanoi

- Rule 1: You can only move one disk at a time
- Rule 2: You can make use of all the pegs
- Rule 3: At no point should a larger disk be placed on a smaller one



Solution :

- Transfer the top n-1 disks from A to B
- Transfer the largest disk from A to C
- Transfer n-1 disks from B to C

What is the cost ?

Let T(n) denote the number of moves required to transfer a stack of size n

What is T(0), T(1), and T(2)

Tower of Hanoi

Let $T(n)$ denote the number of moves required to transfer a stack of size n

What is $T(0)$, $T(1)$, and $T(2)$

$T(0)=0$, $T(1)=1$, and $T(2)= 3$

Let T(n) denote the number of moves required to transfer a stack of size n

Solution :

- Transfer the top n-1 disks from A to B
- Transfer the largest disk from A to C
- Transfer n-1 disks from B to C

Let T(n) denote the number of moves required to transfer a stack of size n

Solution :

- Transfer the top n-1 disks from A to B----T(n-1)
- Transfer the largest disk from A to C--- 1
- Transfer n-1 disks from B to C ----- T(n-1)

Upper bound on the cost

$T(n) <= T(n-1) + 1 + T(n-1)$

$T(n) <= 2. T(n-1) + 1$

What about the lower bound ?

$T(n) >= ??$

Is it possible to accomplish the task
In less than $2.T(n-1) + 1$ moves ?

What about the lower bound ?

T(n) >= ??

Is it possible to accomplish the task
In less than 2.T(n-1) + 1 moves ?

No it is not possible. But why ?

Tower of Hanoi

- Given $T(0)=0$ and $T(1)=1$ we have:

- $T(n) <= 2. T(n-1) + 1$

  and

- $T(n) >= 2. T(n-1) + 1$

Given $T(0)=0$ and $T(1)=1$ we have: $T(n) = 2. T(n-1) + 1$

- There are multiple ways to solve it
- First one is make a guess and then verify the guess.

Tower of Hanoi

- T(0)=0
- T(1)=2. 0 + 1 = 1
- T(2)= 2.1 + 1 = 3
- T(3)= 2.3 + 1 = 7

- It might be that $T(n) = 2^n - 1$

- Verify it !

Tower of Hanoi

How to verify: Remember that recurrences are ideally suited for Mathematical Induction.

Pizza cutting problem

- The problem was solved by Jacob Steiner in 1826.

Problem Statement:

- How many slices of pizza can a person obtain by making n straight cuts with a pizza knife ?

Let’s rephrase it mathematically !

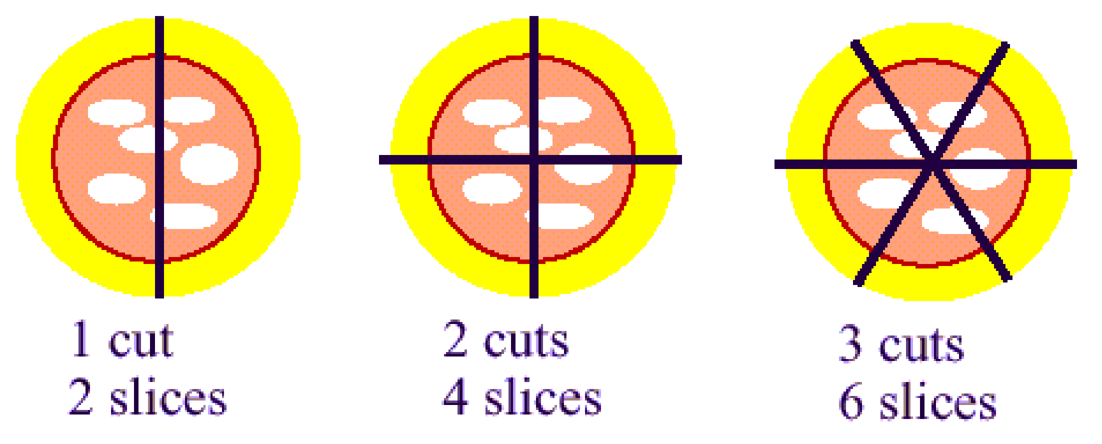
- What is the maximum number of regions defined by n lines in a plane ?

- **Notation:** Let L_n be the number of regions obtained by drawing n lines.

- **Base Case:** We know what is L_0

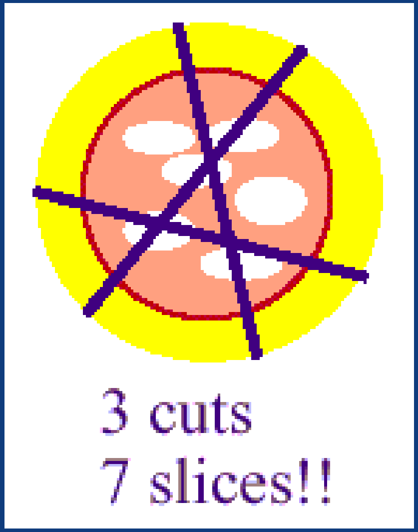
What is L_1, L_2 and L_3

- Case 1: We cut through the center each time



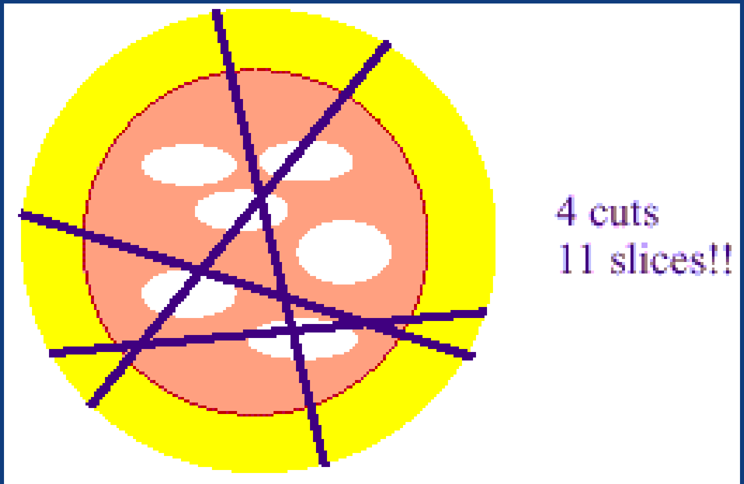
- In Case 1, each cut adds two regions.
- Thus, $L_n = 2.n \quad (n > 0)$

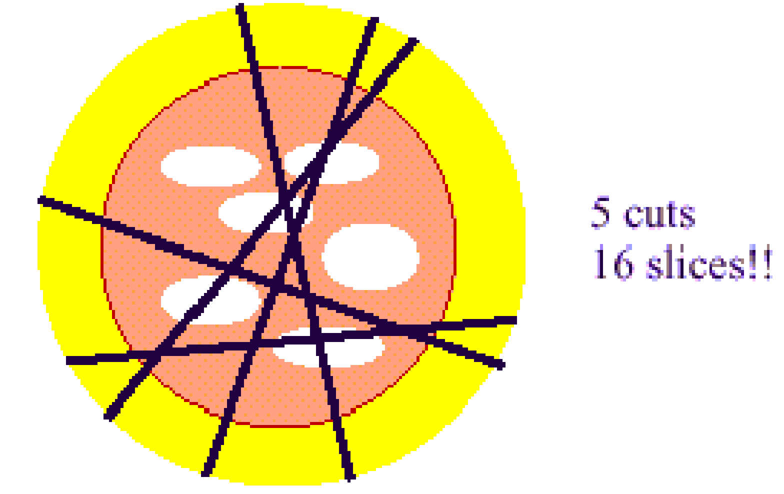
Case 2: You don’t need to cut through the center each time



A Better Slicing Method …

- What is common that you see between these two figures ?





An Important Observation

The nth line increases the number of regions by k iff it splits k of the old regions iff it intersects previous lines in k-1 different places.

Upper bound

$$L_n <= L_{n-1} + n$$

Is it possible that $L_n = L_{n-1} + n$

- It's possible to attain upper bound always, provided you draw the new line such that it intersects with all the previous lines.


QUESTIONS?