1. Calculate the laplacian of the following:
(i) $F = x^2 + 2xy + 3z + 4$ (ii) $F = \sin(\hat{\mathrm{k}} \cdot \vec{\mathrm{r}})$ (iii) $F = \frac{1}{r}$

soln:
(i) $\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 2$
(ii)

$$\begin{aligned}
\nabla^2 F &= \frac{\partial^2}{\partial x^2}\sin(\vec{k}\cdot\vec{r}) + \frac{\partial^2}{\partial y^2}\sin(\vec{k}\cdot\vec{r}) + \frac{\partial^2}{\partial z^2}\sin(\vec{k}\cdot\vec{r}) \\
&= -k_x^2\sin(\vec{k}\cdot\vec{r}) - k_y^2\sin(\vec{k}\cdot\vec{r}) - k_z^2\sin(\vec{k}\cdot\vec{r}) \\
&= -k^2\sin(\vec{k}\cdot\vec{r})
\end{aligned}$$

(iii)

$$\begin{aligned}
\nabla^2\left(\frac{1}{r}\right) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right) \\
\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) &= \frac{\partial}{\partial x}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{1}{r^2}\frac{1}{2r}\cdot 2x\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) \\
&= -x\left(-\frac{3}{r^4}\frac{1}{2r}\cdot 2x\right) - \frac{1}{r^3} \\
&= \frac{3x^2}{r^5} - \frac{1}{r^3}
\end{aligned}$$

$$\therefore\ \nabla^2\left(\frac{1}{r}\right) = \frac{3}{r^5}\left(x^2+y^2+z^2\right) - \frac{3}{r^3} = 0$$

This is valid only for $r \neq 0$. At $r = 0$ the function is not differentiable.

2. Evaluate $(\hat{r}\cdot\vec{\nabla})r$ and $(\hat{r}\cdot\vec{\nabla})\hat{r}$ **soln:**

$$\begin{aligned}
(\hat{r}\cdot\vec{\nabla})r &= \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right) r \\
&= \frac{x}{r}\frac{\partial r}{\partial x} + \frac{y}{r}\frac{\partial r}{\partial y} + \frac{z}{r}\frac{\partial r}{\partial z} \\
&= \frac{x}{r}\frac{2x}{2r} + \frac{y}{r}\frac{2y}{2r} + \frac{z}{r}\frac{2z}{2r} \\
&= \frac{x^2+y^2+z^2}{r^2} = 1
\end{aligned}$$

$$
\begin{aligned}
(\hat{r}\cdot\vec{\nabla})\hat{r} &= \left(\frac{x}{r}\frac{\partial}{\partial x}+\frac{y}{r}\frac{\partial}{\partial y}+\frac{z}{r}\frac{\partial}{\partial z}\right)\left[\hat{i}\frac{x}{r}+\hat{j}\frac{y}{r}+\hat{k}\frac{z}{r}\right] \\
&= \hat{i}\left(\frac{x}{r}\frac{\partial}{\partial x}+\frac{y}{r}\frac{\partial}{\partial y}+\frac{z}{r}\frac{\partial}{\partial z}\right)\left(\frac{x}{r}\right)+\hat{j}\left(\frac{x}{r}\frac{\partial}{\partial x}+\frac{y}{r}\frac{\partial}{\partial y}+\frac{z}{r}\frac{\partial}{\partial z}\right)\left(\frac{y}{r}\right)+\hat{k}\left(\frac{x}{r}\frac{\partial}{\partial x}+\frac{y}{r}\frac{\partial}{\partial y}+\frac{z}{r}\frac{\partial}{\partial z}\right) \\
&= \hat{i}0+\hat{j}0+\hat{k}0 \\
&= 0
\end{aligned}
$$

3. Find the volume of an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using the tripple integral $\int\int\int dxdydz$ with appropriate limits.

soln:
We will calculate the volume in the first quadrant of the coordinate system which is 1/8 the volume of the ellipsoid. We will first integrate over z at a fixed (x, y). The lower limit is 0 while the upper limit is decided by the equation of the ellipsoid and is given as a function of (x, y). This will be $c\sqrt{1-x^2/a^2-y^2/b^2}$. The ellipsoid cuts the xy plane along an ellipse whose equation is obtained by putting $z = 0$ in the equation of the ellipsoid. This gives the equation of the ellipse as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Next we do the integration over y. The limits will be given as 0 and $b\sqrt{1-x^2/a^2}$. Finally the limits on x will be from 0 to a.

$$
\begin{aligned}
V &= \int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dzdydx \\
&= \int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-x^2/a^2-y^2/b^2}dydx \\
&= c\int_0^a\int_0^{b\alpha}\sqrt{\alpha^2-y^2/b^2}dydx \quad \text{where} \quad \alpha=\sqrt{1-x^2/a^2} \\
&= c\int_0^a\alpha\int_0^{b\alpha}\sqrt{1-\left(\frac{y}{b\alpha}\right)^2}dydx \\
&= bc\int_0^a\alpha^2\frac{\pi}{4}dx \\
&= \frac{\pi bc}{4}\int_0^a(1-x^2/a^2)dx \\
&= \frac{\pi bc}{4}(a-a/3) \\
&= \frac{\pi abc}{6}
\end{aligned}
$$

The volume of the whole ellipsoid is $8\times\frac{\pi abc}{6}=\frac{4}{3}\pi abc$.

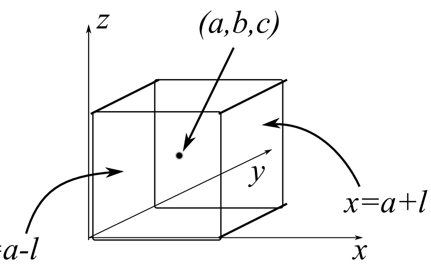
4. Consider $\vec{\mathrm{A}}=x^2\hat{\mathrm{i}}+y^2\hat{\mathrm{j}}+z^2\hat{\mathrm{k}}$

(a) Evaluate $\oint_S \vec{\mathrm{A}}\cdot\vec{\mathrm{da}}$ where S is a cubical surface given by the planes $x = a\pm l;\quad y = b\pm l;\quad z = c\pm l.$

(b) Verify that at the point (a,b,c),

$$\vec{\nabla}\cdot\vec{A} = \lim_{l\to 0}\frac{1}{8l^3}\oint_S \vec{\mathrm{A}}\cdot\vec{\mathrm{da}}$$

soln:



(a)

The surface of the cube consists of 6 planes. Let S_1 be the surface $x = a+l$. Over S_1, $\vec{A} = (a+l)^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and $\vec{da} = \hat{i}dydz$.

$$\begin{aligned}\therefore\quad \int_{S_1}\vec{A}\cdot\vec{da} &= \int_{c-l}^{c+l}\int_{b-l}^{b+l}(a+l)^2 dydz\\ &= 4l^2(a+l)^2\end{aligned}$$

Over the surface $S_2:\quad x = a-l$,
$\vec{A} = (a-l)^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and $\vec{da} = -\hat{i}dydz$

$$\begin{aligned}\therefore\quad \int_{S_2}\vec{A}\cdot\vec{da} &= -\int_{c-l}^{c+l}\int_{b-l}^{b+l}(a-l)^2 dydz\\ &= -4l^2(a-l)^2\end{aligned}$$

$\therefore$ net flux from S_1 and S_2 is $4l^2[(a+l)^2-(a-l)^2] = 16al^3$.
Similarly from the other two pair of surfaces we will have $16bl^3$ and $16cl^3$.
So the total flux of the vector field $\vec{A}$ through the given cube is $16l^3(a+b+c)$.

(b) The volume of the cube is $8l^3$.

$$\begin{aligned}\therefore\quad \lim_{l\to 0}\frac{1}{V}\oint_S\vec{A}\cdot\vec{da} &= \lim_{l\to 0}\frac{1}{8l^3}16l^3(a+b+c)\\ &= 2(a+b+c)\end{aligned}$$

This is same as the value of $\vec{\nabla}\cdot\vec{A}$ at (a,b,c).
This limit will be true for volume of any shape enclosing the point (a,b,c).

5. Evaluate $\int_P^Q \vec{\mathrm{A}}\cdot\vec{\mathrm{dl}}$ for $\vec{\mathrm{A}} = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$ along the following paths : $P\equiv(-a,0);\quad Q\equiv(a,0)$.

(a) $(-a,0)\to(0,a)\to(a,0)$

(b) $(-a, 0) \to (0, -a) \to (a, 0)$

(c) a loop, forward along (a) and backward along (b)

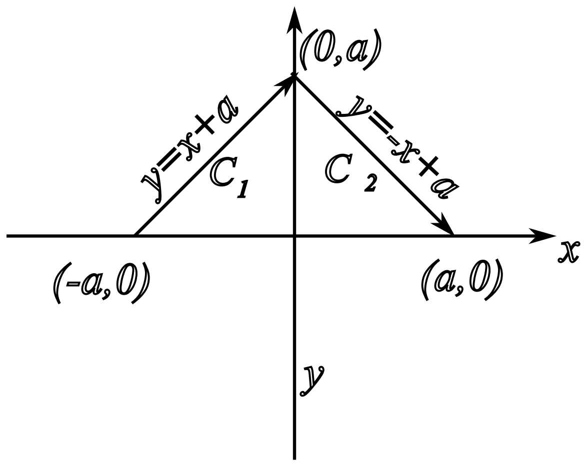
(d) Let I be the value of the loop integral evaluated in (c). Let S be the flat area enclosed by the loop. Verify that at the origin

$$(\vec{\nabla} \times \vec{A}) = \left[\lim_{a\to 0} \frac{I}{S}\right](-\hat{k})$$

(e) Can we find a scalar function F such that $\vec{\nabla}F = y\hat{\imath} - x\hat{\jmath}$?

soln:

(a) The path is made up of two straight curves C_1 and C_2.



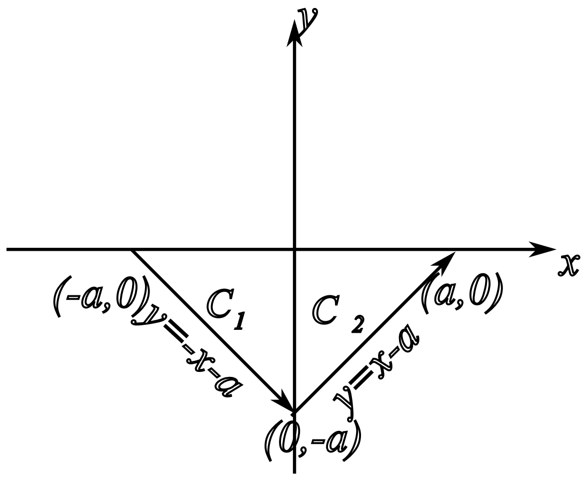
Along C_1, $\quad y = x + a$.
$\therefore\ dy = dx$.
$\vec{A} \cdot \vec{dl} = (y\hat{\imath} - x\hat{\jmath}) \cdot (dx\hat{\imath} + dy\hat{\jmath}) = ydx - xdy$.
Along C_1 we have $\vec{A} \cdot \vec{dl} = (x + a)dx - xdx = adx$.

$$\therefore \quad \int_{C_1} \vec{A} \cdot \vec{dl} = \int_{-a}^{0} adx = a^2$$

Along C_2, $\quad y = -x + a$.
$\therefore\ \ dy = -dx$.
So we have $\vec{A} \cdot \vec{dl} = (-x + a)dx + xdx = adx$.

$$\therefore \quad \int_{C_2} \vec{A} \cdot \vec{dl} = \int_{0}^{a} adx = a^2$$

$$\therefore \int_{P}^{Q} \vec{A} \cdot \vec{dl} = \int_{C_1} + \int_{C_2} = 2a^2$$



(b) The path is made up of two straight curves C_1 and C_2.
Along C_1, $y = -x - a$.
$\therefore\ dy = -dx$.
$\vec{A} \cdot \vec{dl} = (y\hat{i} - x\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = ydx - xdy$.
Along C_1 we have $\vec{A} \cdot \vec{dl} = (-x - a)dx + xdx = -adx$.

$$\therefore \quad \int_{C_1} \vec{A} \cdot \vec{dl} = \int_{-a}^{0} (-a)dx = -a^2$$

Along C_2, $y = x - a$.
$\therefore\ dy = dx$.
So we have $\vec{A} \cdot \vec{dl} = (x - a)dx - xdx = -adx$.

$$\therefore \quad \int_{C_2} \vec{A} \cdot \vec{dl} = \int_{0}^{a} (-a)dx = -a^2$$

$$\therefore \quad \int_{P}^{Q} \vec{A} \cdot \vec{dl} = \int_{C_1} + \int_{C_2} = -2a^2$$

(c) Along the loop the value of the integral will be $2a^2 - (-2a^2) = 4a^2$.

(d) We have $I = 4a^2$ and $S = 2a^2$
$\therefore\ \frac{I}{S} = 2$.
$\therefore\ \lim_{a\to 0} \frac{I}{S} = 2$.
$\vec{\nabla} \times \vec{A} = -2\hat{k}$ everywhere.
So we have $\vec{\nabla} \times \vec{A} = \left[\lim_{a\to 0} \frac{I}{S}\right] (-\hat{k})$ at the origin.

(e) $\int_a^b \vec{\nabla} F \cdot \vec{dl} = \int_a^b dF = F(b) - F(a)$.
This is independent of any path we take from a to b.
We just saw in the previous part that for $\vec{A} = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$ the integral $\int_a^b \vec{A} \cdot \vec{dl}$ is path dependent.
So we can't have any scalar function F such that $\vec{\nabla} F = y\hat{\mathrm{i}} - x\hat{\mathrm{j}}$.