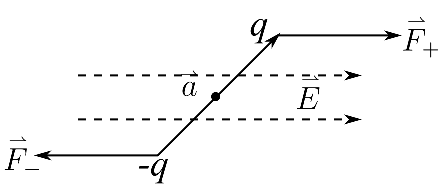
Dielectrics

Electric field in a dielectric medium: Material that don't have free electrons are insulators. They don't conduct. When we place these material in an electric field, they don't make the electric field inside 0. But these material also respond to the external electric field and modify the electric field in the whole region, within and outside it. These material are called dielectrics.

1 Polarization

Dielectrics don't have free electrons. The electrons are bound to the atoms or molecules. However when the material is placed in an external electric field, the average position of the electrons and the positive ions shift to produce a tiny dipole. Sometime the atoms and molecules have inherrent dipole moment but they are randomly oriented in the material due to symmetry. An external electric field breaks this symmetry and a large number of these tiny dipoles orient along the electric field.

Let us see what kind of effects a dipole has in an electric field. Consider a dipole $\vec{p} = q\vec{a}$ kept in an electric field $\vec{E}$. The forces on the charges $+q$ and $-q$ are as shown. These two forces will create a torque on the dipole to orient it along $\vec{E}$. This torque is given by



$$
\begin{aligned}
\vec{\tau} &= \frac{\vec{a}}{2} \times \vec{F}_+ + \left(-\frac{\vec{a}}{2}\right) \times \vec{F}_- \\
&= \frac{\vec{a}}{2} \times q\vec{E} - \frac{\vec{a}}{2} \times (-q)\vec{E} = q\vec{a} \times \vec{E}
\end{aligned}
$$

The tortal force on the dipole in auniform electric field is 0. However if the electric field $\vec{E}$ is not uniform there will be a small change in the force on q and $-q$. This will exert a force on $\vec{p}$ given by

$$\vec{F}_p = q(\vec{E}_+ - \vec{E}_-)$$

If $\vec{E} = \hat{i}E_x + \hat{j}E_y + \hat{k}E_z$ then

$$
\begin{aligned}
\vec{E}_+ - \vec{E}_- &= \hat{i}\left[\vec{a} \cdot (\vec{\nabla}E_x)\right] + \hat{j}\left[\vec{a} \cdot (\vec{\nabla}E_y)\right] + \hat{k}\left[\vec{a} \cdot (\vec{\nabla}E_z)\right] \\
&= (\vec{a} \cdot \vec{\nabla})(\hat{i}E_x + \hat{j}E_y + \hat{k}E_z) \\
&= (\vec{a} \cdot \vec{\nabla})\vec{E}
\end{aligned}
$$

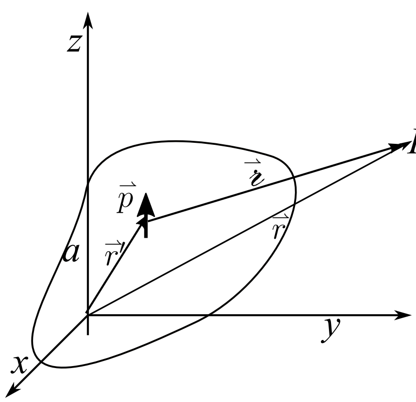
In simple words this is the change in the electric field $\vec{E}$ from $-q$ to $+q$, i.e through the small displacement $\vec{a}$.

$$\therefore \quad \vec{F}_p = q(\vec{a}\cdot\vec{\nabla})\vec{E} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

If $\vec{E}$ is uniform $(\vec{p}\cdot\vec{\nabla})\vec{E}$ is 0 and hence the force on the dipole is 0.

In a dielectric material the tiny dipoles respond to these external electric field and largely orient along the electric field $\vec{E}$. This creates a net large dipole moment in the di-electric material. This dipole moment is measured in terms of a quantity called polarization $\vec{P}$, defined as net dipole moment per unit volume.

As the dielectric material gets polarized, it produces its own electric field. The total electric field at any point is equal to the sum of the external and the internal electric field created by the dipoles of the dielectric.



The potential at point P due to the tiny dipole inside a dielectric is

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}\cdot\vec{p}(\vec{r}')}{\mathscr{r}^2} \\ &= \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\hat{\mathscr{r}}\cdot\vec{P}(\vec{r}')d\tau'}{\mathscr{r}^2} \end{aligned}$$

where $\vec{P}(\vec{r}')$ is the polarization at $\vec{r}'$.

Notice that $\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}'\left(\dfrac{1}{\mathscr{r}}\right)$

where $\vec{\nabla}'$ is the gradient w.r.t $\vec{r}'$.

Note that $\vec{\nabla}'\left(\dfrac{1}{\mathscr{r}}\right) = -\vec{\nabla}\left(\dfrac{1}{\mathscr{r}}\right)$ (verify)

$$\therefore \quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \vec{P}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

Now $\vec{\nabla}'\cdot\left(\dfrac{\vec{P}}{\mathscr{r}}\right) = \vec{P}\cdot\vec{\nabla}'\left(\dfrac{1}{\mathscr{r}}\right) + \dfrac{1}{\mathscr{r}}\vec{\nabla}'\cdot\vec{P}$

$$\therefore \quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_\tau \vec{\nabla}'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right)d\tau' - \int_\tau \frac{1}{\mathscr{r}}\vec{\nabla}'\cdot\vec{P}d\tau'\right]$$

Using divergence theorem on the first term gives

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\vec{P}}{\mathscr{r}}\cdot\hat{n}da' - \frac{1}{4\pi\epsilon_0}\int_\tau \frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})d\tau'$$

Define $\rho_b = -\vec{\nabla}'\cdot\vec{P}$ and $\sigma_b = \vec{P}\cdot\hat{n}$. Then

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho_b(\vec{r}')}{\mathscr{r}}d\tau' + \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b(\vec{r}')}{\mathscr{r}}da'$$

The potential due to the polarization of the dielectric is due to an effective volume charge density $\rho_b = -(\vec{\nabla}' \cdot \vec{P})$ and an effective surface charge density $\sigma_b = \vec{P} \cdot \hat{n}$ on the surface of the dielectric. The suffix b stands for the bound charge. The nomenclature signifies that these charges are bound to the dielectric material. They can't be moved. Since the charges are created due to the polarization the total bound charge on the dielectric is 0, i.e,

$$\int_\tau \rho_b d\tau' + \oint_S \sigma_b da' = 0$$

This is trivial by divergence theorem.

2 Gauss' law

The bound charges in the dielectric ρ_b is due to the external fields that affect it. So we have ρ_b and ρ_f, called the free charges. Generally we know the ρ_f. ρ_f causes ρ_b. The total charge density in the region is $\rho = \rho_f + \rho_b$. By Gauss' law

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f + \rho_b}{\epsilon_0} &= \frac{\rho_f - \vec{\nabla} \cdot \vec{P}}{\epsilon_0} \\
\therefore \quad \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} &= \rho_f \\
\therefore \quad \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) &= \rho_f \qquad (1)
\end{aligned}$$

We don't know ρ_b, i.e, how the dielectric will respond to ρ_f. So Eq. 1 is more useful in a dielectric rather than Gauss' law applied to the electric field $\vec{E}$. For a dielectric we define a new quantity in place of $\vec{E}$ i.e,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

called the electric displacement. Interms of this we have

$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad (2)$$

In fact this is also applicable for free space where $\vec{P} = 0$. So $\vec{D} = \epsilon_0 \vec{E}$ and hence Gauss' law itself becomes

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

By divergence theorem we have

$$\oint_S \vec{D} \cdot \hat{n} da = Q_f \text{ enclosed}$$

Eq. 2 gives the divergence of $\vec{D}$. A field is completely known if we know both, its divergence and curl (Helmholtz Theorem). We have $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\vec{\nabla} \times \vec{E} = 0$. So $\vec{E}$ is completely specified once ρ is specified.
We have not obtained $\vec{\nabla} \times \vec{D}$ yet. At this stage unless we know how the dielectric responds to the electric field we will not know how is $\vec{P}$ related to $\vec{E}$ and hence we will not know what is $\vec{\nabla} \times \vec{P}$.

$$\vec{\nabla} \times \vec{D} = \epsilon_0 \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \vec{P} = \vec{\nabla} \times \vec{P} = ?$$

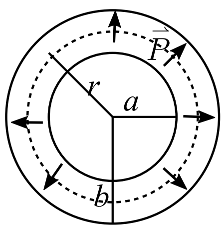
This affects the conditions on the electric displacement at the boundary of the dielectric. While we can easily say
$D_1^\perp - D_2^\perp = \sigma_f$ at the boundary between two dielectric material we can't say
$D_1^\parallel = D_2^\parallel$. Instead we will have

$$D_1^\parallel - D_2^\parallel = P_1^\parallel - P_2^\parallel$$

Since the parallel components don't involve any charge, it is easy to directly apply the boundary condition on the electric fields

$$E_1^\parallel = E_2^\parallel$$

Eg.:
A thick spherical shell with inner radius a and outer radius b is made up of a dielectric material with frozen in polarization $\vec{P}(\vec{r}) = \frac{k}{r}\hat{r}$. Find the electric field in the three partitioned regions



Since the total charge is 0 the field outside the shell is 0

$$\therefore \quad E(r > b) = 0$$

By Gauss' law and spherical symmetry $E = 0$ for $r < a$.
When $a < r < b$, let the electric displacement be D_2 which must be radial by symmetry. If D_1 is the electric displacement for $r < a$ then $D_2 = D_1 \implies \quad D_2 = 0$ since $D_1 = \epsilon_0 E_1 = 0$
$D_2 = \epsilon_0 E_2 + P \quad \implies \quad E_2 = -\frac{P}{\epsilon_0}$
$\therefore \quad$ For $a < r < b$, $\vec{E}_2 = -\frac{\vec{P}}{\epsilon_0} = -\frac{k}{r\epsilon_0}\hat{r}$.

3 Linear Dielectrics

Generally a polarization in a material is caused by an electric field. So the polarization in a material at a point is dependent on the local net electric field at the point. If the dependence of the polarization $\vec{P}$ on the electric field $\vec{E}$ is linear then

$$\vec{P} = \alpha\vec{E}$$

We call such material as linear dielectric. The proportionality constant α has the dimension same as ϵ_0. One generally writes

$$\vec{P} = \epsilon_0\chi_e\vec{E}$$

The constant χ_e is dimensionless. It is called the electric susceptibility. Now the electric displacement $\vec{D}$ is given as

$$\begin{aligned}\vec{D} = \epsilon_0\vec{E} + \vec{P} &= \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} \\ &= \epsilon_0(1+\chi_e)\vec{E} = \epsilon\vec{E}\end{aligned}$$

So in a linear dielectric , the electric displacement $\vec{D}$ is proportional to $\vec{E}$. The proportionality constant ϵ is called the permittivity of the dielectric. To remove dimension from this constant one talks about relative permitivity

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

This is called the dielectric constant of the material . So

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\epsilon_r\vec{E}$$

The electric displacement $\vec{D}$ is a very useful quantity for a dielectric. As we have seen earlier

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

is only dependent on the free charge that we build up. It doesn't depend upon the bound charges due to the polarization of the material, on which we have no control. So in this respect $\vec{D}$ replaces the role of $\vec{E}$. But let us calculate $\vec{\nabla}\times\vec{D}$ in a linear dielectric.

$$\begin{aligned}\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\epsilon_0\epsilon_r\vec{E}) &= \epsilon_0\left[\epsilon_r\left(\vec{\nabla}\times\vec{E}\right) + \left(\vec{\nabla}\epsilon_r\right)\times\vec{E}\right] \\ &= \epsilon_0\left(\vec{\nabla}\epsilon_r\right)\times\vec{E}\end{aligned}$$

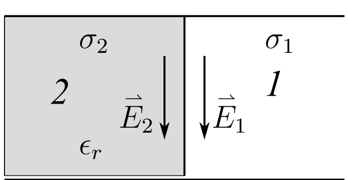
If the region where we are conputing $\vec{\nabla}\times\vec{D}$ is homogeneous, then $\vec{\nabla}\epsilon_r = 0$ and we will have $\vec{\nabla}\times\vec{D} = 0$. In this case we can treat $\vec{D}$ exactly like the electric field. But in region which is not homogeneous we may have $\vec{\nabla}\times\vec{D} \neq 0$. In such a case $\vec{D}$ cannot be seen exactly like a modified $\vec{E}$. Typically the difference of ϵ_r in a region occurs at the boundary between two dielectric material. In such places we certainly have $\vec{\nabla}\times\vec{D} \neq 0$. So at the boundary between two dielectric material we have

$$\begin{aligned}D_1^{\perp} - D_2^{\perp} &= \sigma_f \\ \text{and}\quad E_1^{\parallel} &= E_2^{\parallel}\end{aligned}$$

since we are certain that $\vec{\nabla}\times\vec{E} = 0$ everywhere in electrostatics.

Eg.

Consider a parallel plate capacitor. Half the region between the plates is filled with a dielectric material of dielectric constant ϵ_r as shown. Let us find the change in capacitance.



Due to the dielectric, the surface charge density in region 1 and region 2 are different. Let $\vec{E}_1$ be the electric field in region 1 and $\vec{E}_2$ in region 2. Then by continuity of parallel components we have $\vec{E}_1 = \vec{E}_2$.

$\sigma_1 = \epsilon_0 E_1$.

In region 2 $\vec{D}_2 = \epsilon\vec{E}_2 = \epsilon_0\epsilon_r\vec{E}_2$.

Since σ_2 is the free charge on the plate in region 2, at the interface of the plate and the dielectric we have $D_2 = \sigma_2$

$$\therefore\quad \sigma_2 = \epsilon_0\epsilon_r E_2 = \epsilon_0\epsilon_r E_1$$

So the total charge on the plate is

$$\begin{aligned}Q = \sigma_1\frac{A}{2} + \sigma_2\frac{A}{2} &= (\epsilon_0 E_1 + \epsilon_0\epsilon_r E_1)\frac{A}{2} \\ &= \frac{\epsilon_0 E_1 A}{2}(1+\epsilon_r)\end{aligned}$$

The potential difference between the plates is $V = E_1 d$.

$$\therefore\quad C = \frac{Q}{V} = \frac{\epsilon_0 E_1 A}{2}(1+\epsilon_r)\times\frac{1}{E_1 d} = \frac{\epsilon_0 A}{d}\left(\frac{1+\epsilon_r}{2}\right)$$

Without the dielectric the capacitance would have been $\frac{\epsilon_0 A}{d}$. It changes by a factor $\frac{1+\epsilon_r}{2}$

4 Energy in a dielectric medium

Consider a dielectric medium of dielectric constant ϵ_r. Let there be some free charges in this medium ρ_f. Let us calculate the energy stored in the medium. The energy is given by

$$W = \frac{1}{2}\int_{\mathcal{V}} (\rho_f V) d\tau$$

where we are finding the energy required to accumulate the free charges ρ_f in the volume $\mathcal{V}$. of the dielectric. We can write this in terms of $\vec{D}$ as

$$\begin{aligned} W &= \frac{1}{2}\int_{\mathcal{V}} (\vec{\nabla}\cdot(\vec{D})V d\tau \\ &= \int_{\mathcal{V}} \left[\vec{\nabla}\cdot(\vec{D}V) - \vec{D}\cdot\vec{\nabla}V\right] d\tau \\ &= \frac{1}{2}\oint_S V\vec{D}\cdot\hat{n}da + \frac{1}{2}\int_{\mathcal{V}} (\vec{D}\cdot\vec{E})d\tau \end{aligned}$$

As we go far away from the charge distribution V and $\vec{D}$ tends to 0 and hence the total energy stored in the dielectric is

$$W = \frac{1}{2}\int_{\mathcal{V}} (\vec{D}\cdot\vec{E})d\tau \tag{3}$$

Note that in any region if the electric field is $\vec{E}$, the electrostatic energy is

$$\frac{1}{2}\int_{\mathcal{V}} E^2 d\tau$$

What we got is different from this. This difference is due to the fact that the work done we calculated is only to place the free charges. We didn't bother to calculate the work done to assemble the bound charges. Actually if we started to calculate the energy stored to accumulate $\rho = \rho_f + \rho_b$ then we would have certainly got the universal result of electrostatic energy which is

$$\frac{1}{2}\int_{\mathcal{V}} E^2 d\tau$$

The energy stored in a capacitor filled with a dielectric is

$$W = \frac{1}{2}\int_{\mathcal{V}} (\vec{D}\cdot\vec{E})d\tau = \frac{1}{2}\epsilon_0\epsilon_r \int_{\mathcal{V}} E^2 d\tau$$

If V is the potential difference between the plates then $E = \frac{V}{d}$

$$\begin{aligned} \therefore\ W &= \frac{1}{2}\epsilon_0\epsilon_r\frac{V^2}{d^2}\int_{\mathcal{V}} d\tau = \frac{1}{2}\epsilon_0\epsilon_r\frac{V^2}{d^2}Ad \\ &= \frac{1}{2}\frac{\epsilon_0\epsilon_r A}{d}V^2 = \frac{1}{2}CV^2 \end{aligned}$$