Differentiation

Rate of change

The derivative of $f(x)$ at a point x_0 is

$$f'(x_0) = \boxed{\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$

provided the above limit exists.

→ It is the slope of the graph $y = f(x)$ at the point x_0.

→ The slope of the tangent to the curve $y = f(x)$ at $x = x_0$

→ The rate of change of $f(x)$ with respect to x at $x = x_0$.

→ The derivative $f'(x_0)$ at the point x_0.

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{--- (1)}$$

The left hand limit

$$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

if they exist, then it is called the left hand derivative of $f(x)$ at x_0.

The right hand limit

$$\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$, if it exists, then it is called the right hand derivative of $f(x)$ at x_0.

The limit (1) can also be written as

$$\boxed{f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \quad \text{--- (2)}$$

Exp $f(x) = |x|$ at 0

L.H.D $= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0^-} \frac{|0+h| - |0|}{h}$

$= \lim_{h \to 0^-} \frac{|h|}{h} = -1$

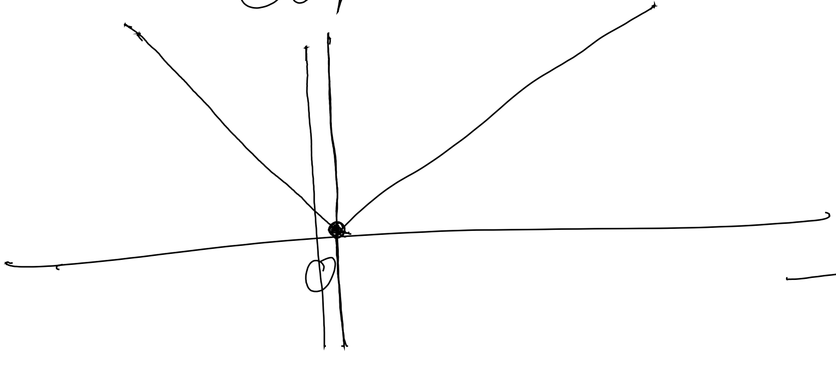
R.H.D $= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$

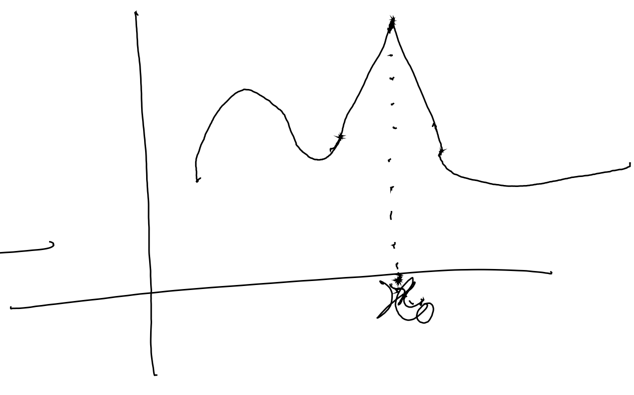
$= \lim_{h \to 0^+} \frac{|0+h| - |0|}{h}$

$= \lim_{h \to 0^+} \frac{|h|}{h} = 1$

LHD $\neq$ RHD

So $f(x) = |x|$ is not differentiable at 0.





Ex Prove that

if a function is differentiable at x_0, then it is continuous at x_0.

But converse is not true.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exist.}$$

To prove $\lim_{x \to x_0} f(x) = f(x_0)$

Chain rule

Implicit differentiat

Ex1 $y^2 = x^2 + \sin xy$

$\frac{dy}{dx} = ?$

$$2y \frac{dy}{dx} = 2x + \cos(xy) \frac{d}{dx}(xy)$$

$$= 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = 2x + \cancel{y\cos(xy)} + y\cos(xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

Applications of derivatives

① Extreme values of functions

Let f be a function with domain D.
Then f has a global maximum value in D at a point c if
$f(x) \le f(c)$ for all $x \in D$.

f has a global minimum value in D at c if
$f(x) \geq f(c)$ for all $x \in D$.

EXP

	Domain	
$f(x) = x^2$	$(-\infty, \infty)$	No global maximum global minimum 0 at $x = 0$
$f(x) = x^2$	$[0, 2]$	global maximum value is 4 at $x=2$ global minimum value is 0 at $x = 0$.
$f(x) = x^2$,	$(-\infty, 0]$	

$f(x) = x^2$ domain $[0, 2]$

$f(x) =$ has global maximum value 4 at $x = 2$.

No global minimum.

$f(x) = x^2$ domain $[0, 2)$

global minimum at $x = 0$ but no global maximum

$f(x) = x^2$ domain $(0, 2)$

No global maximum

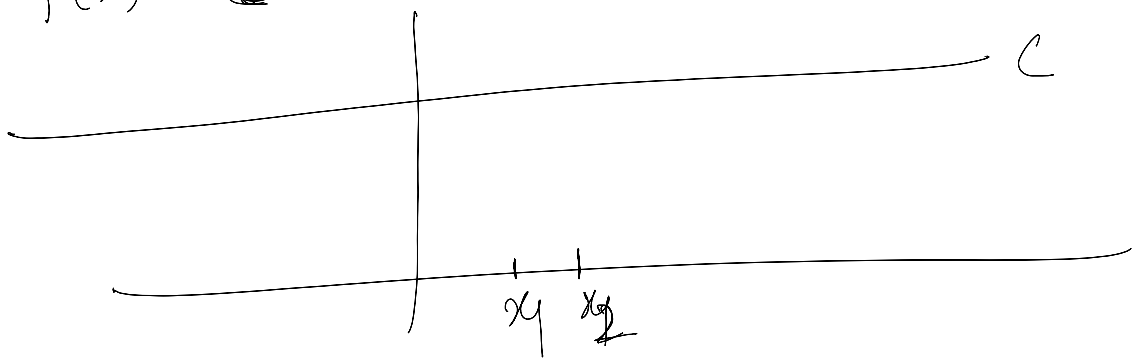
No global minimum.

Extreme value theorem

If a function f is continuous on a closed interval $[a, b]$, then f attains both global maximum and global minimum on $[a, b]$.

In otherwords, there are real numbers $x_1, x_2 \in [a, b]$ with $f(x_1) = m$, $f(x_2) = M$ such that $m \le f(x) \le M$ for all $x \in [a, b]$.

$f(x) = C$



$f(x) \leq f(x_1) \quad \forall x \in D$

Local maximum and local minimum

A function f has a local maximum (relative maximum) value at a point c within its domain if

$f(x) \leq f(c)$ for all x lying in some open interval containing c and contained in D.

f has a local maximum at an interior point c if

$f(x) \leq f(c) \quad \forall x$ in $(c-\delta, c+\delta) \subset D$.

$(\delta > 0)$

A function f has a local minimum (relative minimum) value at a point c if there exists an open interval I, contained in D such that

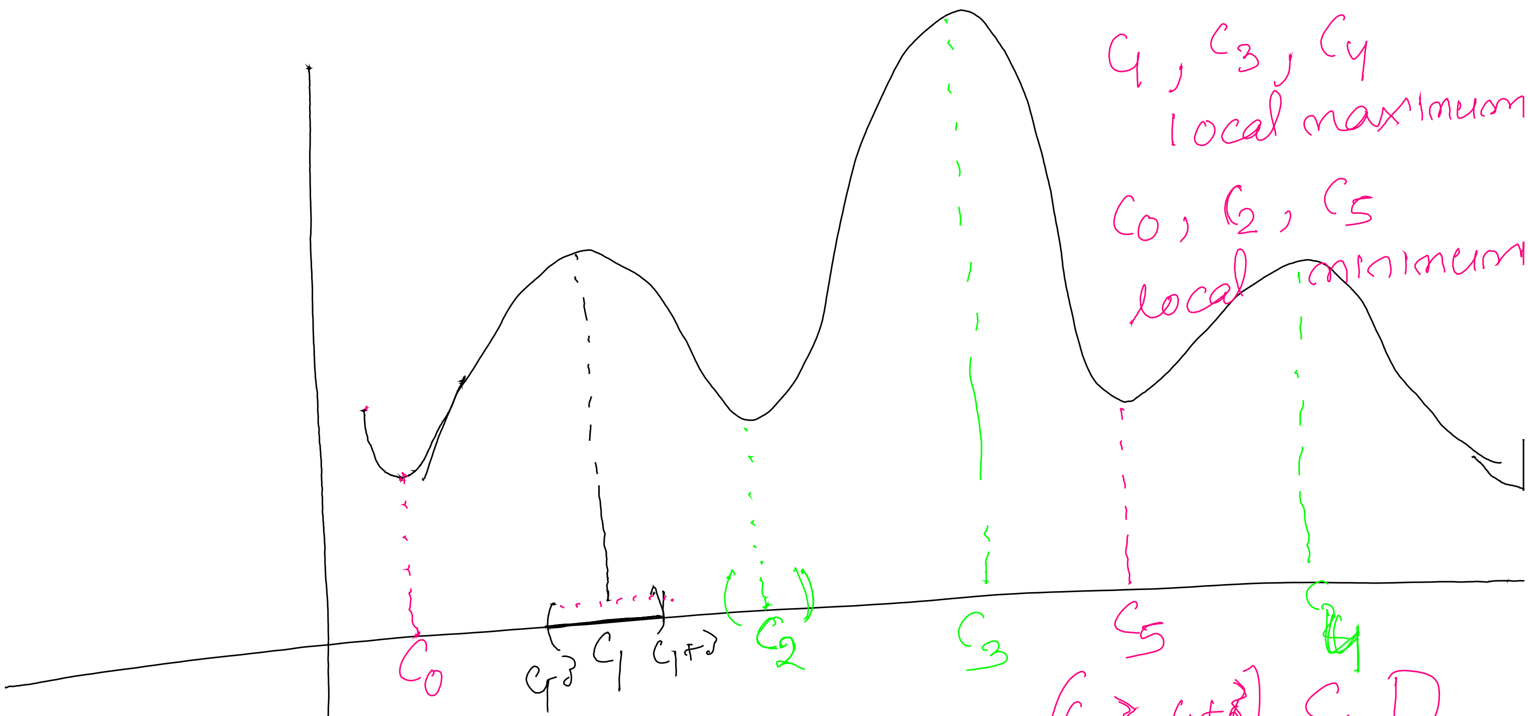
$$f(x) \geq f(c) \quad \forall\, x \in I \subset D .$$

f has a local minimum at an interior point c if $f(x) \geq f(c)\ \forall\, x \in (c-\delta, c+\delta) \subset D$.

c_1,

c_1, c_3, c_4 local maximum

c_0, c_2, c_5 local minimum



$f(c_1) \geq f(x)\ \forall\, x \in (c_1-\delta, c_1+\delta) \subset D$

c_1 is a local maximum

$f(c_2) \leq f(x)\ \forall\, x \in (c_2-\delta_2, c_2+\delta_2) \subset D$

c_2 is a local minimum.

The first derivative theorem

$f : [a,b] \to \mathbb{R}$

f has a local maximum at a

if $f(x) \le f(a)$ for all x in $[a, a+\delta) \subset [a,b]$

$\delta > 0$.

f has a local minimum at a

if $f(x) \geq f(a)$ for all x in $[a, a+\delta) \subset [a,b]$

Similarly for other end point b.

f has a local maximum at b

if $f(x) \le f(b)$ for all x in $(b-\delta, b] \subset [a,b]$

f has a local minimum at b

if $f(x) \geq f(b)$ for all x in $(b-\delta, b] \subset [a,b]$.

The first derivative theorem

If f has a local maximum or local minimum at an interior point c of its domain, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof Let f has a local maximum at c

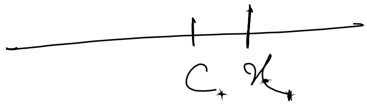
$\Rightarrow f(x) \le f(c) \quad \forall x \in (c-\delta, c+\delta)$

$\Rightarrow f(x) - f(c) \le 0 \quad \forall x \in (c-\delta, c+\delta)$ ——— (1)

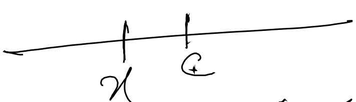
As c is an interior point and $f'(c)$ exists

So $f'(c) = \lim_{x \to c^+} \dfrac{f(x)-f(c)}{x-c} = \lim_{x \to c^-} \dfrac{f(x)-f(c)}{x-c}$ ——— (2)

$\lim_{x \to c^+} \dfrac{f(x)-f(c)}{x-c} \le 0$



From (1) and as $x > c$



$\lim_{x \to c^-} \dfrac{f(x)-f(c)}{x-c} \ge 0$ From (1) and as $x < c$

From (2) $\lim_{x \to c^+} \dfrac{f(x)-f(c)}{x-c} = \lim_{x \to c^-} \dfrac{f(x)-f(c)}{x-c} = 0$