# LOGIC

## PROPOSITIONAL LOGIC (Deals with propositions)

- Proposition: It's a declarative sentence that is either T or F but not both.

**Example**
① Toronto is the capital of Canada F
② $2+2=3$ F
③ $1+1=2$ T
} Propositions

④ What time is it? (not a Proposition) as not declarative sentence
⑤ $x+1=2$ (not a proposition) neither T nor false

- Propositional variables or statement variables are denoted by $p, q, r, s, \ldots$
- Truth value is denoted by T or F
- Let $p$ be a proposition. The negation of $p$ denoted by $(\neg p)$ or $(\bar{p})$ is the statement "It is not the case that $p$" (not $p$)
- The truth value of $\neg p$ is opposite of the truth value of $p$

**Example** $p$: Today is Friday"
$\neg p$: It is not the case that today is Friday
or Today is not Friday.

## TRUTH TABLE

① $\neg p$:

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

② $p \wedge q$ (conjuction) proposition $p$ and $q$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Example** $p$: Today is Friday $q$: It is raining today

$p \wedge q$: Today is Friday & it is raining today.
T on rainy fridays & is false on any day that is not a friday & on fridays when it does not rain.

③ **Disjunction** $p \vee q$ : p or q

T.T.

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Example (same)**

Today is friday or it is raining today.

T on any day that is either Friday or a rainy day (including rainy fridays), F on days that are not fridays when it also does not rain.

④ **Exclusive : $p \oplus q$**

T.T.

| p | q | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

$p \oplus q$ is T when exactly one of them is T and is false otherwise.

**CONDITIONAL STATEMENTS** p & q

- $p \rightarrow q$ is the proposition "if p, then q"

$p \rightarrow q$ is F when p is T & q is F & true otherwise.

hypothesis, conclusion or consequence

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

$p \rightarrow q$

"if p, then q"

"if p, q"

"p is sufficient for q"

"q if p"

"q when p"

"a necessary condition for p is q"

"q unless ¬p"

"p implies q"

"p only if q"

"a sufficient condition for q is p"

"q whenever p"

"q is necessary for p"

"q follows from p"

## Converse, contrapositive & inverse

- $q \to p$ is converse of $p \to q$
- The contrapositive of $p \to q$ is $\neg q \to \neg p$
- $\neg p \to \neg q$ is called inverse of $p \to q$

**Biconditionals** $p \leftrightarrow q$ is the proposition "p if and only if q"

T.T.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

## T.T. of compound propositions

**Example** $(p \vee \neg q) \to (p \wedge q)$

| $p$ | $q$ | $\neg q$ | $p \vee \neg q$ | $p \wedge q$ | $(p \vee \neg q) \to (p \wedge q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | T | T | F | F |
| F | T | F | F | F | T |
| F | F | T | T | F | F |

## Precedence of logical operators

| Operator | Precedence |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\to$ | 4 |
| $\leftrightarrow$ | 5 |

## Propositional Equivalences

- A compound prop. that is always true no matter what the truth values of the prop.s that occur in it is called <u>tautology</u>.
- A compound prop. that is always false ~~no matter~~ is called <u>contradiction</u>.
- A compound prop. that is neither tautology nor a contradiction is called a <u>contingency</u>.

**Example** $p \vee \neg p$ is tautology & $p \wedge \neg p$ is a contradiction.

T.T.

| $p$ | $\neg p$ | $p \vee \neg p$ | $p \wedge \neg p$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

## LOGICAL EQUIVALENCES

Compound props $p$ & $q$ are <u>logically equivalent</u> if $p \leftrightarrow q$ is a tautology. $p \equiv q$ or $p \Leftrightarrow q$ (denoted by)

**Example** $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

$$\left.\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned}\right\} \text{Demorgan's Law}$$

T.T.

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

Example $p \to q \equiv \neg p \vee q$

| $p$ | $q$ | $\neg p$ | $\neg p \vee q$ | $p \to q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

Example $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ distributive law

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$
$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$$

## SOME LOGICAL EQUIVALENCES

$p \wedge T \equiv p$
$p \vee F \equiv p$ } Identity laws

$p \vee T \equiv T$
$p \wedge F \equiv F$ } Domination laws

$p \vee p \equiv p$
$p \wedge p \equiv p$ } Idempotent laws

$\neg(\neg p) \equiv p$ } Double negation law

$p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$ } Commutative laws

$(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ } Associative laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ } Distributive laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ } De Morgan's laws

$p \vee (p \wedge q) \equiv p$
$p \wedge (p \vee q) \equiv p$ } Absorption laws

$p \vee \neg p \equiv T$
$p \wedge \neg p \equiv F$ } Negation laws

$\boxed{\text{Conditional Logi Equi}}$

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(q \to \neg p)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

$\boxed{\text{Biconditional} \equiv}$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

$\boxed{\text{Example}}$ Show that $\neg(p \to q) \equiv p \wedge \neg q$

□ $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv \neg(\neg p) \wedge \neg q$
$\equiv p \wedge \neg q$