# SC223 - Linear Algebra

Aditya Tatu

Lecture 6

August 23, 2022

# Gaussian Elimination/Row Reduction

- Let us use the following example:

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2x_2 + 5x_3 + 4x_4 + 2x_5 &= 2 \\
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- When are linear equations easy to solve? As few variables (ideally 1) as possible.
- Representation as a matrix

$$\left[\begin{array}{ccccc|c} 0 & 2 & 5 & 4 & 2 & 2 \\ 1 & -1 & 2 & 3 & -1 & 1 \\ 2 & 1 & 0 & 4 & 2 & -1 \\ 3 & 1 & 3 & 2 & -2 & 3 \end{array}\right]$$

- This is the called the **Augmented matrix**.

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● **Elementary row operations**: Exchanging two rows, adding a multiple of one row to another, and multiplying a constant to all entries of a row. All of them preserve solutions to linear equations.

● Idea is to use row operations to get an **upper triangular** AM:

$$\left[\begin{array}{ccccc|c} * & - & - & - & - & - \\ 0 & * & - & - & - & - \\ 0 & 0 & * & - & - & - \\ 0 & 0 & 0 & * & - & - \end{array}\right]$$

● The $*$ positions are called **leading entries**, and are the left-most non-zero entry of each row.

● **Definition:**(*Echelon form*) A matrix is said to be in Echelon form if

► All non-zero rows are above any zero rows (if any).

► Each leading entry (leftmost non-zero entry) in a row is in a column to the right of the leading entry of the row above it.

● Examples of Echelon forms:

$$\begin{bmatrix} * & - & - & - & - & - \\ 0 & 0 & * & - & - & - \\ 0 & 0 & 0 & * & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & - & - & - & - \\ 0 & 0 & * & - & - \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & - & - \\ 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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● Examples of non-Echelon forms:

$$\begin{bmatrix} * & - & - & - & - & - \\ 0 & 0 & * & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & - & - \end{bmatrix} \begin{bmatrix} * & - & - & - & - & - \\ 0 & 0 & 0 & * & - & - \\ 0 & 0 & 0 & * & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & - & - \\ 0 & 0 & * \\ 0 & * & - \\ 0 & 0 & 0 \end{bmatrix}$$

● **Definition:** *(Row-reduced Echelon form (RREF))* A matrix in Echelon form is said to be in a Row-Reduced Echelon Form if

- If the leading entry in each row is 1
- If the leading term is the only non-zero term in that column.

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● Examples of RREF:

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# Solving Linear Equations

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0 & 4 & -3 & -7 & 1 & 0
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- Thus, $P_{12}A = L_{43}^{-1}L_{32}^{-1}L_{42}^{-1}L_{31}^{-1}L_{41}^{-1}U = LU$.

# LU Decomposition

- In general, any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed into a product of lower and upper triangular matrices, with appropriate permutations:

$$PA = LU,$$

where $P \in \mathbb{R}^{m \times m}$, $L \in \mathbb{R}^{m \times m}$, $U \in \mathbb{R}^{m \times n}$.

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● Solving linear equations with $LU$ decomposition:
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● First let $Ux = y$ and solve $Ly = b$, and next solve for $x$ in $Ux = y$.