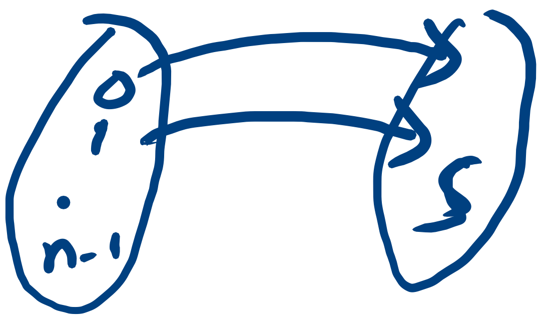
Cardinality

Defn No. of elements in a set is called cardinality of a set

Defn A set S is finite with cardinality $n \in \mathbb{N}$ if there is a bijection from $\{0, 1, \cdots n-1\}$ to S. A set S is infinite if it is not finite



Def^n Two set A and B have same cardinality if there is a bijection from A to B.

Def^n Countable

A set S is either finite or has the same cardinality as the set of positive integers is called countable

either $\aleph$ aleph null

$\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$

when S is infinite $|S| = \aleph_0$

Ex 1 The set of odd positive integers is a countable set

$$\mathbb{Z}^+ \xrightarrow{f} \{1, 3, 5, 7, 9, \dots\}$$

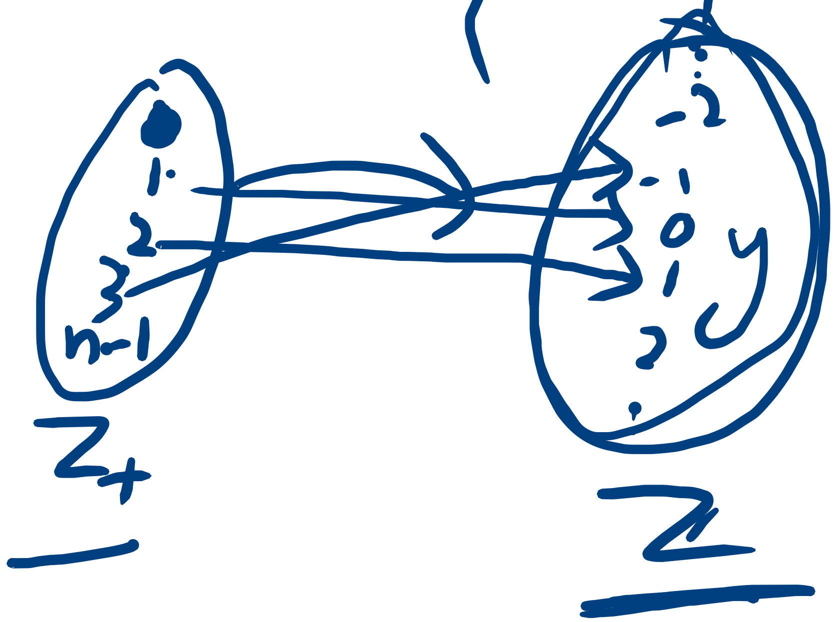
$f(x) = y$

Define $f : \mathbb{Z}_+ \to$ set of odd positive integers by $f(n) = 2n - 1$

$$\begin{array}{ccccccccc}
\mathbb{Z}^+ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \; \dots \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15
\end{array}$$

Exp set of all integers is countable

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$



Define $f : \mathbb{Z}_+ \to \mathbb{Z}$

by
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -\dfrac{(n-1)}{2} & \text{if } n \text{ odd} \end{cases}$$

$1 \to 0$

$2 \to 1$

$3 \to -1$

$4 \to 2$

$5 \to -2$

Exp set of positive rational numbers countable.

$$Q_+ = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}_+ \right\}$$

$$
\begin{array}{c|ccccc}
 & p \to & & & & \\
\hline
q & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
\downarrow & \frac{1}{2} & \cancel{\frac{2}{2}} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\
 & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$\mathbb{R}$ is uncountable

$\mathbb{Q}_+$ is countable

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1, & \frac{1}{2}, & 2, & 3, & \frac{1}{3}, & \frac{1}{4}, \; \frac{2}{3} \cdots
\end{array}
$$

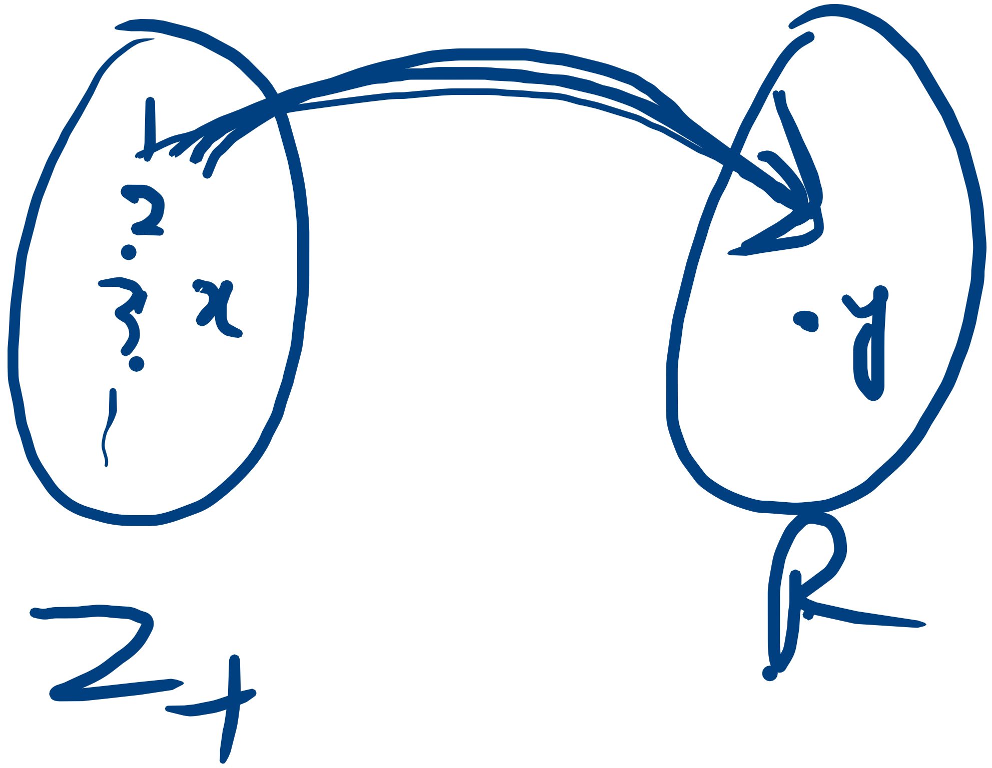
$p+q=1$

$p+q=2$

$p+q=3$

$p+q=4$

$$
\begin{array}{ccccccc}
\mathbb{Z}_+ & 1 & 2 & 3 & 4 & 5 & 6 \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Q}_+ & 1 & \frac{1}{2} & \frac{2}{1} & \frac{3}{1} & \frac{1}{2} & \frac{1}{4}
\end{array}
$$



$\nexists x \quad \underline{f(x) = y}$

Algorithms

Defn An algorithm is a finite set of precise instructions for performing a computation or solving a problem.

pseudocode

~~Alg~~ Algorithm

$a_1 \quad a_2 \quad \cdot \quad \cdot\cdot \quad a_n$

finding the maximum element in the list

procedure max $(a_1 \cdots a_n)$

max $= a_1$

for $i = 2$ to n

if max $< a_i$ then max $= a_i$.

Properties

1. Input - An algorithm has input values from a specific set
2. Output - From each set of input values algorithm produces output values
3. Definiteness - The steps of an algorithm must be defined precisely

Correctness - for each input value it gives correct output values

Finiteness - An algorithm produces the desired output after a finite number of steps.

Effectiveness - Each step of an algorithm must be performed in a finite amount of time.

Searching algorithms

$\underline{a_1} \quad \underline{a_2} \quad \cdots\cdots \quad \underline{a_n}$

search for x . in the list

Linear search

$\underline{a_1 \quad a_2 \quad \cdots \quad a_n}$

$i = 1$

while ($i \leq n$ and $\underline{x} \neq a_i$)

$i = i + 1$

if $i \leq n$ then location $= i$

else location $= 0$.

Binary search

Elements are listed in increasing order

Exp

ϕ

1 2 3 5 6

Exp

~~Sear~~

1 2 3 5 6 7 8 10 12 13 15 16 18 19 20 22

↑ 14th item

Search for 19

split the list into two

1 2 3 5 6 7 8 10   12 13 15 16 18 19 20 22

19 > 10

12 13 15 16   18 19 20 22

19 > 16   split

18 19   20 22

split   19 $\not>$ 19

18   19

$a_1 \quad a_2 \quad \cdots \quad a_n$ $\qquad a_1 < a_2 < \cdots < a_n$

middle term $\quad m = \left\lfloor \frac{n+1}{2} \right\rfloor$

Algorithm

$i = 1$
$j = n$
while $i < j$
begin
$\quad m = \left\lfloor \frac{i+j}{2} \right\rfloor$
$\quad$ if $x > a_m$ then $i = m+1$
$\quad$ else $j = m$
end

$[a_1 \; a_2 \cdots a_m][a_{m+1} \cdots a_{n-1} \; a_n]$

$i = 1 \qquad a_i \; i = \qquad j = n$

a_m

$j = m$

if $x = a_i$ location $= i$
else location $= 0$

$a_1 \quad a_2 \quad \cdots \quad a_m \quad a_{m+1} \quad \cdots \quad a_{n-1} \quad a_n$

$\underline{i=1}$ $\quad m \quad$ $j = n$

$\underline{i = m+1}$

$\underline{j = m}$

$x > a_m$

$x \not> a_m$

Sorting — putting the elements in a sequence in increasing order.

The Bubble sort — $a_1, a_2, \ldots, a_n$

for $i = 1$ to $n-1$

  for $j = 1$ to $n-i$

    if $a_j > a_{j+1}$ then interchange a_j and a_{j+1}

for $i = 1$ to $n-1$

for $j = 1$ to $n-i$

if $a_j > a_{j+1}$ then interchange a_j and a_{j+1}

$a_1 \quad a_2 \quad a_3 \quad a \quad a_4 \quad a_n$

Exp $\quad 3 \quad 2 \quad 4 \quad 1 \quad 5$ $\qquad i = 2$

$i = 1 \qquad i = 2$

$j = 1$, to $n-1$ $\qquad j = 1$ to $n-2$

First pass $\qquad 3 > 2$

2 3 4 1 5

2 3 4 1 5

2 3 1 4 5

Second pass

2 3 1 4 5

2 3 1 4 5

2 ~~3~~ 1 3 4 5

2 1 3 4 5

$i = 2$

$j = 1$ ~~to~~ to $n-2$

Third pass

2 1 3 4 5

1 2 3 4 5

1 2 3 4 5

Fourth pass

1 2 3 4 5

Insertion Sort

for $j = 2$ to n

begin

$i = 1$

while $a_j > a_i$

$i = i + 1$

$m = a_j$

for $k = 0$ to $j - i - 1$

$a_{j-k} = a_{j-k-1}$

$a_i = m$

Exp

3 2 4 1 5

$2 \ngtr 3$

2 3 4 1 5

2 3 4 1 5

1 2 3 4 5

$m = 1$

$a_4 = a_3$

$a_3 = a_2$

$a_2 = a_1$

$a_1 = m$