Non homogeneous second order linear equations

$$ay'' + by' + cy = G(x) \quad \text{--- } (1)$$

a, b, c are constants.
$G(x)$ is a continuous function.

The associated homogeneous equation

$$ay'' + by' + cy = 0 \quad \text{--- } (2)$$

Let the solution of the associated homogeneous equation is $y_c = c_1 y_1 + c_2 y_2$.

y_c is called complementary solution of (1).

Suppose we some how know one particular solution of equation (1). Let us say it y_p.

We see that $y_p + y_c$ is also a solution of (1).

$$y = y_c + y_p$$

$$ay'' + by' + cy$$

$$= a(y_c + y_p)'' + b(y_c + y_p)' + c(y_c + y_p)$$

$$= ay_c'' + ay_p'' + by_c' + by_p' + cy_c + cy_p$$

$$= \underbrace{(ay_c'' + by_c' + cy_c)}_{=0} + ay_p'' + by_p' + cy_p$$

$$= ay_p'' + by_p' + cy_p$$

$$= G(x).$$

y_c

$ay'' + by' + cy = 0$

y_p

$ay'' + by' + cy = G(x)$

$y = y_c + y_p$ is also a solution of (1)

$y_c + y_p$ is general solution of (1).

Note

The general solution y of the nonhomogeneous equation ① has the form $y = y_c + y_p$

where y_c is ~~com~~ solution of associated homogeneous equation ②

y_p is any particular solution of equation ①.

The method of undetermined coefficients

$$ay'' + by' + cy = G(x) \quad \text{—} ①$$

If $G(x)$ has some particular type, then we can use the this method of undetermined coefficients.

If $G(x)$ has a term that is a constant multiple of	and if	then include this expression in the trial ~~solution~~ function for y_p
e^{rx}	① r is not a root of Auxiliary equation	Ae^{rx}
	② r is a single root of Auxiliary equation	Axe^{rx}
	③ r is a double root of Auxiliary equation	Ax^2e^{rx}
$\sin kx$ or, $\cos kx$	ki is not a root of the auxiliary equation	$B\cos kx + C\sin kx$
px^2+qx+r	0 is not root of Auxiliary eq^n.	Ax^2+Bx+C
	0 is a single root of Auxiliary equation	Ax^3+Bx^2+Cx
	0 is a double root of Auxiliary equation	$Ax^4+Bx^3+Cx^2$

Exp $y'' - 2y' - 3y = 1 - x^2$ ——— (1)

$G(x)$

Sol^n The associated homogeneous equation

$y'' - 2y' - 3y = 0$ ——— (2)

Auxiliary equation $m^2 - 2m - 3 = 0$

$\Rightarrow (m+1)(m-3) = 0$

$m_1 = -1, \; m_2 = 3$

$$\boxed{y_c = c_1 e^{-x} + c_2 e^{3x}}$$

$G(x) = 1 - x^2$ (polynomial of degree 2)

Trial solution y_p is of the form

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

Put these in (1)

$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$

$\Rightarrow -3Ax^2 + (-4A - 3B)x + 2A - 2B - 3C$

$= 1 - x^2$

$-3A = -1 \Rightarrow A = 1/3$

$-4A - 3B = 0 \rightarrow -\frac{4}{3} - 3B = 0 \Rightarrow B = -\frac{4}{9}$

$2A - 2B - 3C = 1 \quad \frac{2}{3} + \frac{8}{9} - 3C = 1$

$\Rightarrow C = \frac{5}{27}$

$$\boxed{y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}}$$ particular solution.

The general solution is

$$\boxed{\begin{aligned} y(x) &= y_c + y_p \\ &= c_1 e^{-x} + c_2 e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27} \end{aligned}}$$

Exp

$$y'' - y' = 2\sin x \quad \text{--- } (1)$$

$$G(x) = 2\sin x$$

$$\boxed{\begin{aligned} y_p &= A\sin x \\ y_p &= B\cos x \end{aligned}}$$

$$y_p = A\sin x + B\cos x$$

$$y_p' = A\cos x - B\sin x$$

$$y_p'' = -A\sin x - B\cos x$$

put in equation (1)

$$-A\sin x - B\cos x - A\cos x + B\sin x = 2\sin x$$

$$(-A + B) = 2 \qquad B = 1$$

$$-B - A = 0 \qquad A = -1$$

$$\boxed{y_p = -\sin x + \cos x}$$

$y'' - y' = 0$

Auxiliary equation $m^2 - m = 0$

$\Rightarrow m(m-1) = 0$

$\Rightarrow m_1 = 0, \; m_2 = 1$

$G(x) = 2\sin x$

$K = 1$

$$\boxed{Ki = i \quad \text{not a root of Auxiliary equation.}}$$

$y_c = c_1 + c_2 e^x$

General solution y

$y_c + y_p = c_1 + c_2 e^x$ y_p

$-\sin x + \cos x$

Exp Find the general solution of

$$y'' - y' = 5e^{x} - \sin 2x$$

Sol^n Associated homogeneous equation

$$y'' - y' = 0$$

Auxiliary equation $m^2 - m = 0$

$\Rightarrow m_1 = 0, \ m_2 = 1$

$$y_c = c_1 e^{0.x} + c_2 e^{1.x} = c_1 + c_2 e^{x}$$

$$G(x) = 5e^{x} - \sin 2x$$

$5e^{x}$ $r = 1$ Here and r is a single root of Auxiliary equation.

y_p will contain a term Axe^{x}

$\sin 2x$ $k = 2$ Here

$2i$ is not a root of Auxiliary eq^n.

y_p will contain term $A \sin 2x + C \cos 2x$

$$y_p = Axe^x + B\sin 2x + C\cos 2x$$

$$y_p' = Ae^x + Axe^x + 2B\cos 2x - 2C\sin 2x$$

$$y_p'' = Ae^x + Ae^x + Axe^x - 4B\sin 2x - 4C\cos 2x$$

$$y'' - y' = 5e^x - \sin 2x$$

$$2Ae^x + \cancel{Axe^x} - 4B\sin 2x - 4C\cos 2x$$
$$- Ae^x - \cancel{Axe^x} - 2B\cos 2x + 2C\sin 2x$$
$$= 5e^x - \sin 2x$$

$$\Rightarrow Ae^x + (-4B+2C)\sin 2x + (-4C-2B)\cos 2x$$
$$= 5e^x - \sin 2x$$

Comparing

$A = 5$

$$\left.\begin{aligned} -4B+2C &= -1 \\ -4C-2B &= 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} B &= \tfrac{1}{5} \\ C &= -\tfrac{1}{10} \end{aligned}$$

$$\boxed{y_p = 5xe^x + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x}$$

Method of variation of parameter

$$ay'' + by' + cy = G(x) \quad \text{—— (1)}$$

The associated homogeneous equation

$$ay'' + by' + cy = 0 \quad \text{—— (2)}$$

Let $y_c = c_1 y_1(x) + c_2 y_2(x)$

where y_1 and y_2 are L. I. solution of (2).

Assume that the particular solution y_p of the form

$$y_p = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

$c_1(x)$ and $c_2(x)$ are unknown functions of x to be determined.

y_p has to satisfy equation (1) that y_p one condition.

For simple case we take the condition $c_1'(x)\, y_1(x) + c_2'(x)\, y_2(x) = 0$

$$\text{—— (3)}$$

$$y_p = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$$

$$y_p' = \underline{c_1'(x)\,y_1(x)} + c_1(x)\,y_1'(x) + \underline{c_2'(x)\,y_2(x)} + c_2(x)\,y_2'(x)$$

$$= c_1(x)\,y_1'(x) + c_2(x)\,y_2'(x) \qquad (\text{using } (3))$$

$$y_p'' = c_1'(x)\,y_1'(x) + c_1(x)\,y_1''(x) + c_2'(x)\,y_2'(x) + c_2(x)\,y_2''(x)$$

putting all in equation (1)

$$ay'' + by' + cy = G(x)$$

$$a\left(c_1'y_1' + c_1y_1'' + c_2'y_2' + c_2y_2''\right) + b\left(c_1y_1' + c_2y_2'\right) + c\left(c_1y_1 + c_2y_2\right) = G(x)$$

$$\Rightarrow c_1\big(\underbrace{ay_1'' + by_1' + cy_1}_{=0}\big) + c_2\big(\underbrace{ay_2'' + by_2' + cy_2}_{=0}\big) + a\left(c_1'y_1' + c_2'y_2'\right) = G(x)$$

$$\Rightarrow c_1'y_1' + c_2'y_2' = \frac{G(x)}{a} \qquad \text{--- (4)}$$

$$\begin{cases} c_1' y_1 + c_2' y_2 = 0 & \text{--- (3)} \\ c_1' y_1' + c_2' y_2' = \dfrac{G(x)}{a} & \text{--- (4)} \end{cases}$$

Solve for c_1' and c_2'

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{G(x)}{a} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$c_2'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{G(x)}{a} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Find $c_1(x)$ and $c_2(x)$

$$y_p = c_1(x) y_1(x) + c_2(x) y_2(x).$$

Exp $y'' + y = \tan x$

Sol^n The associated homogeneous equation

$$y'' + y = 0$$

Auxiliary equation $m^2 + 1 = 0$

$$\Rightarrow m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$y_1(x) = \cos x$ $\qquad$ $y_2(x) = \sin x$

$y_1'(x) = -\sin x$ $\qquad$ $y_2'(x) = \cos x$

$G(x) = \tan x$

$a = 1$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{G(x)}{a} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}$$

$$c_1'(x) = -\tan x \sin x = \frac{-\sin^2 x}{\cos x}$$

$$c_2'(x) = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{G(x)}{a} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{1}$$

$$= \sin x$$

$$c_1(x) = \int \frac{-\sin^2 x}{\cos x}\, dx = -\log|\sec x + \tan x| + \sin x$$

$$c_2(x) = \int \sin x \, dx = -\cos x$$

$$\cancel{y_p = \ln|\sec x + \tan x|}$$

$$y_p = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$