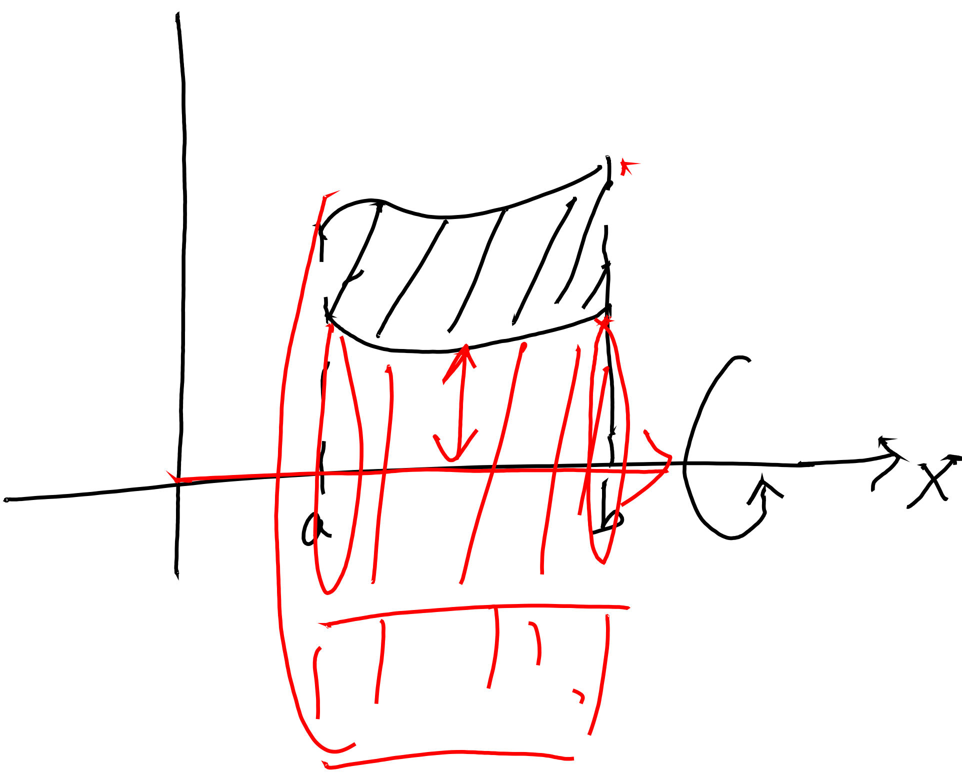
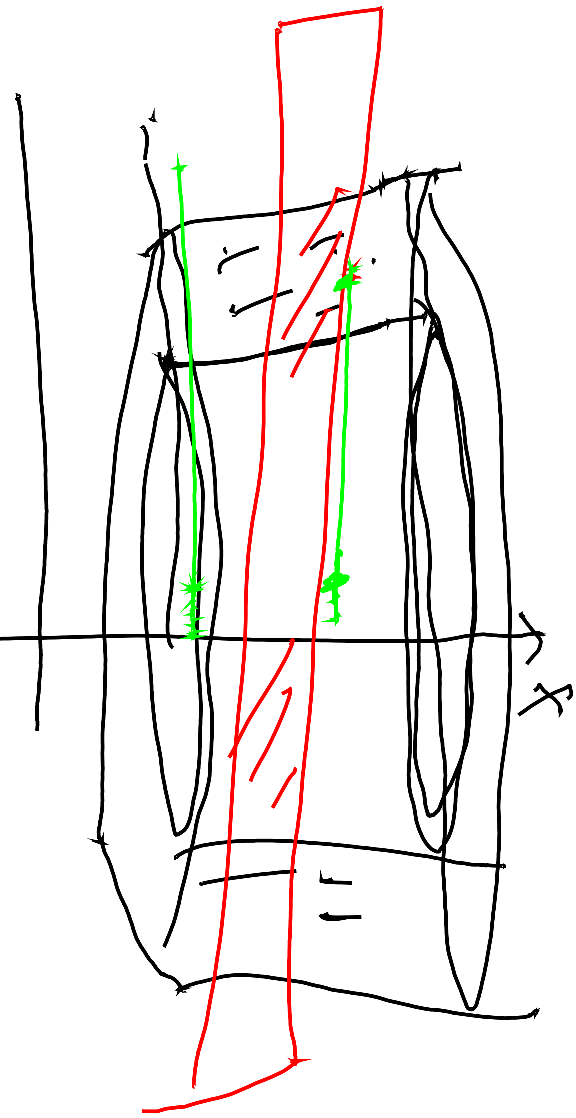
Solids of revolution: The Washer method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, then the solid has hole in it.

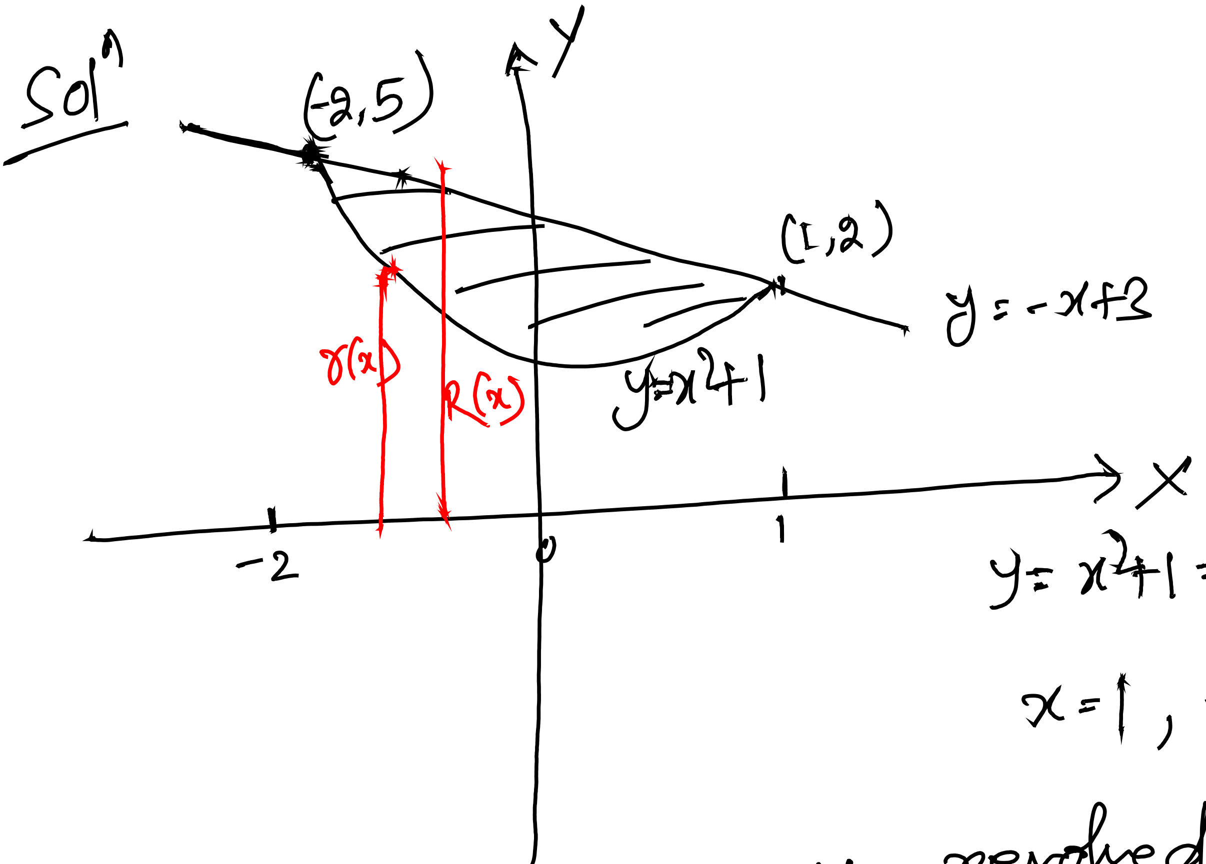
The cross section perpendicular to the axis of revolution are washers instead of disks.





Exp The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Sol^n



$y = x^2+1 = -x+3$

$x = 1, \quad x = -2$

If the region is revolved about x-axis

$R(x) = -x+3$

$r(x) = x^2+1$

Area of the washer

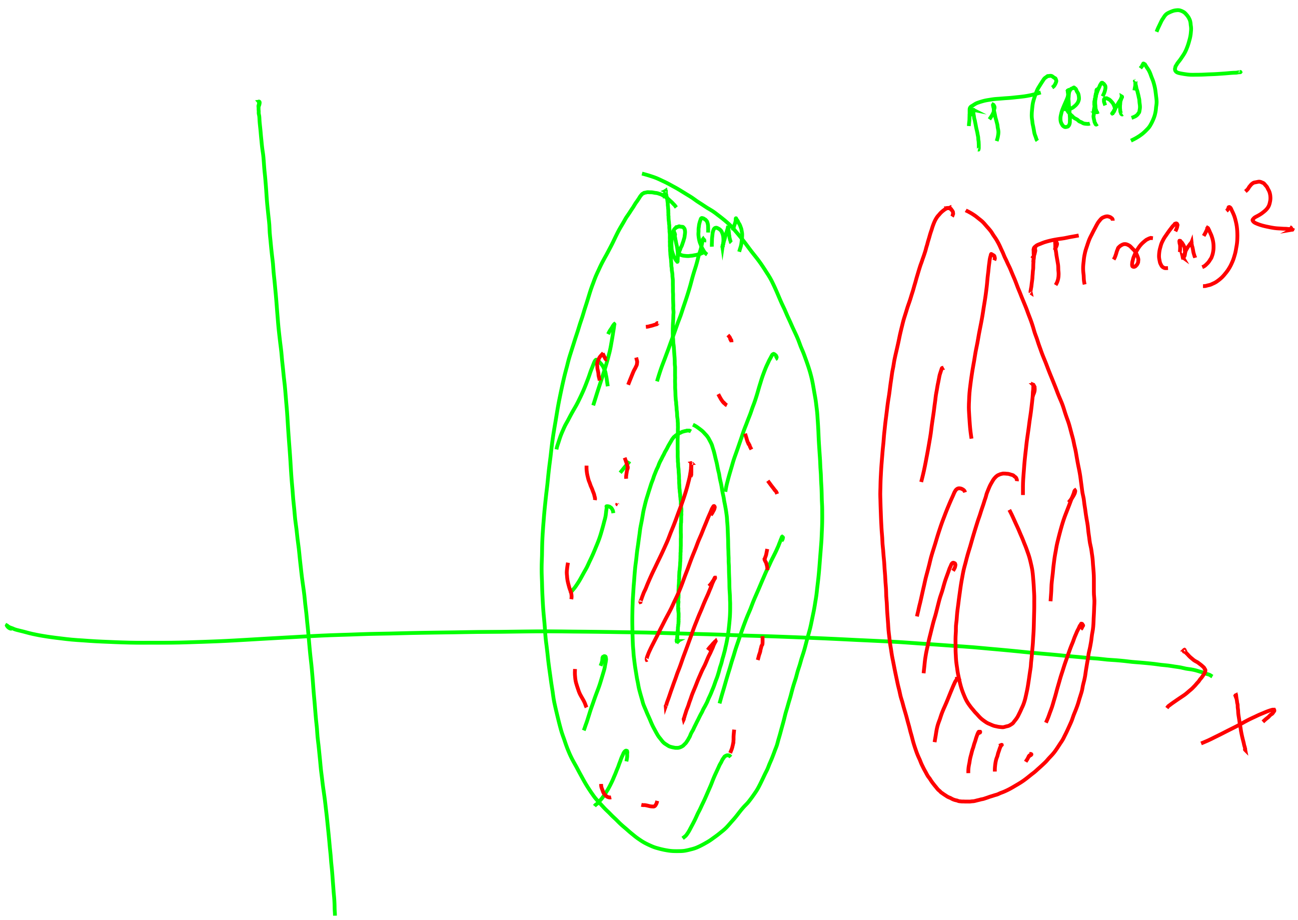
$= \pi (R(x))^2 - \pi (r(x))^2$

$$V = \int_{-2}^{1} \pi\left((R(x))^2 - (r(x))^2\right) dx$$

$$= \int_{-2}^{1} \pi\left((-x+3)^2 - (x^2+1)^2\right) dx$$

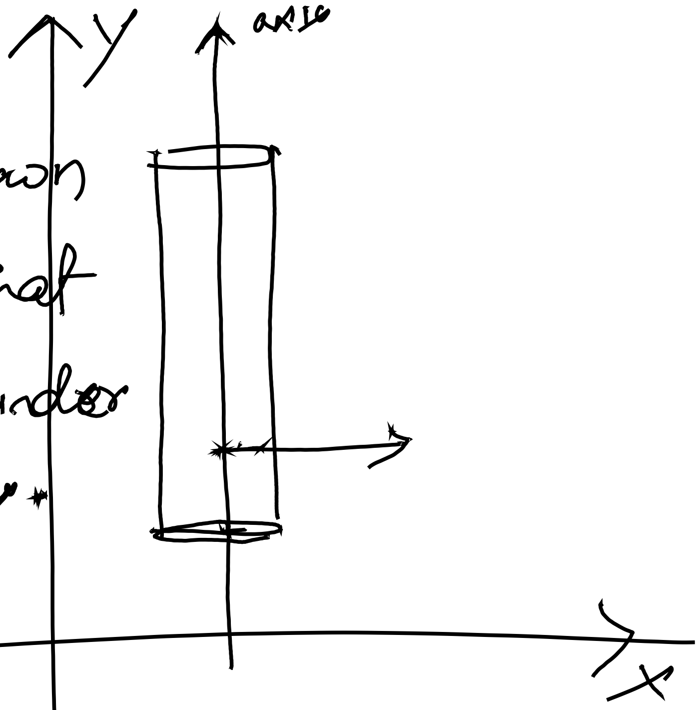
$$= \pi \int_{-2}^{1} \left(x^2+9-6x-x^4-1-2x^2\right) dx$$

$$= \frac{117}{5}\pi$$



Volume using cylindrical shells

Suppose we slice through the solid using circular cylinders of increasing radii, like cookies cutters.



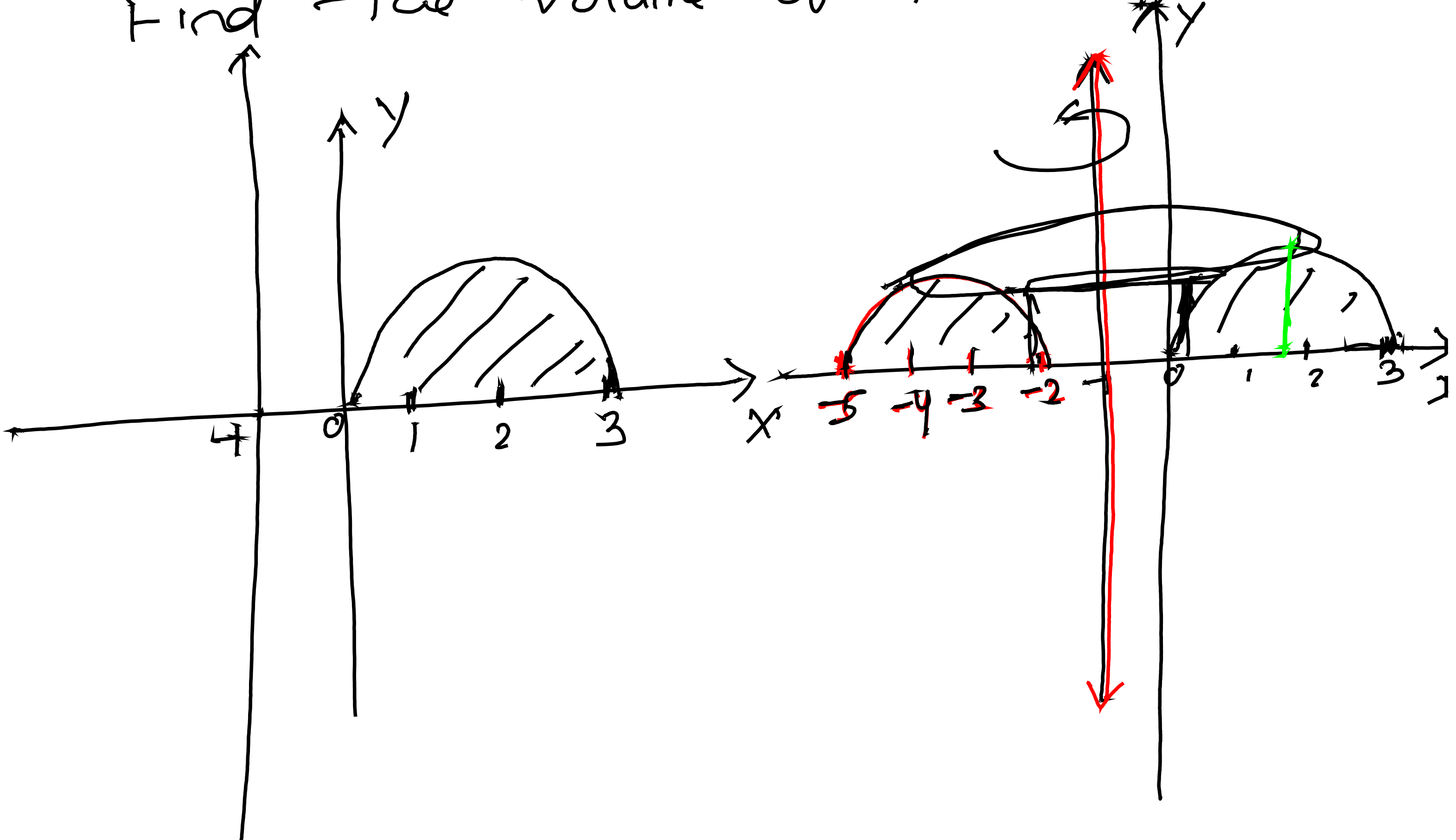
We slice straight down through the solid so that the axis of each cylinder is parallel to y-axis.

The vertical axis of each cylinder is the same, but the radii of the cylinders increase with each slice.

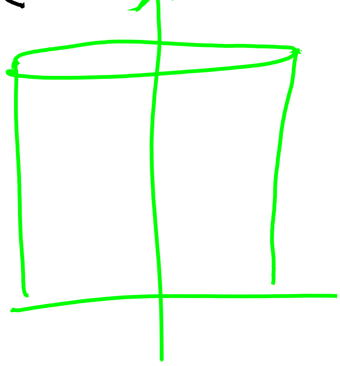
In this way the solid is sliced up into thin cylindrical shells of constant thickness (very small).

EXP

ΔV_K The region enclosed by the x-axis and the parabola $y = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of the solid.



A cylindrical shell of height y_k is obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$



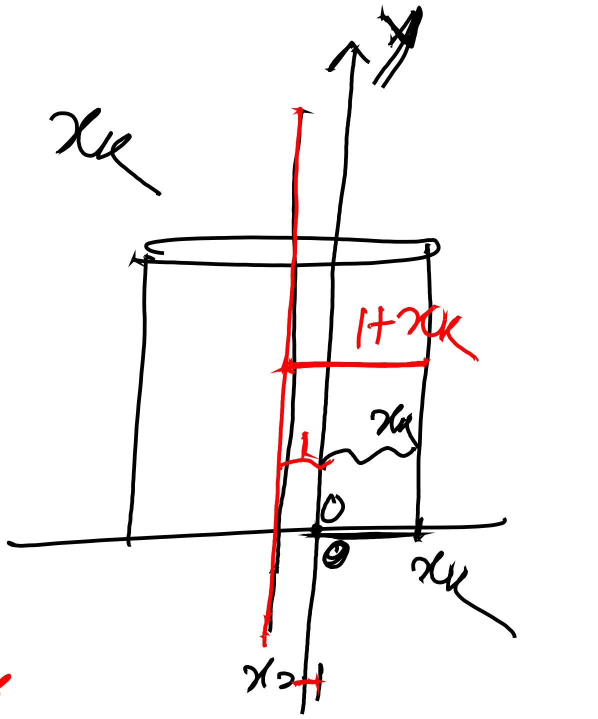
~~The volume of the cylindrical~~

The volume of the cylindrical shell

$\Delta V_k =$ circumference $\times$ height $\times$ thickness

$$= 2\pi(1+x_k) \times (3x_k - x_k^2) \times \Delta x_k$$

If the point is at x_k



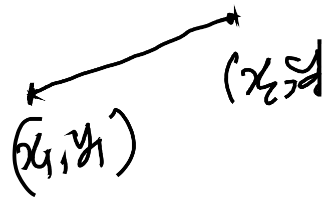
We get many such cylindrical shells

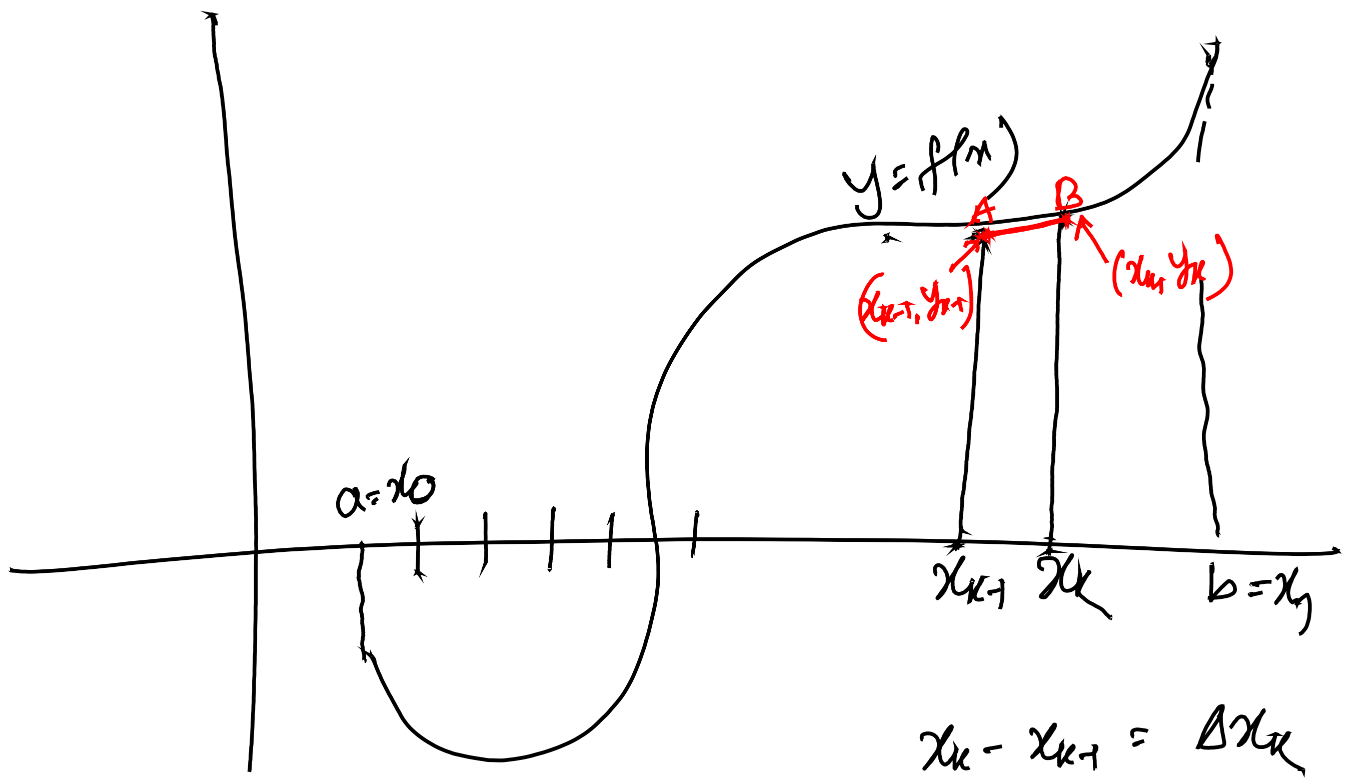
Summing together

$$\lim_{n\to\infty}\sum_{k=1}^{n} \Delta V_k = \lim_{n\to\infty}\sum_{k=1}^{n} 2\pi(1+x_k)(3x_k-x_k^2)\Delta x_k$$

$$= \int_0^3 2\pi(1+x)(3x-x^2)\,dx = \frac{45}{2}\pi$$

Arc length of a curve





$x_k - x_{k-1} = \Delta x_k$

$y_k - y_{k-1} = \Delta y_k$

The length of the chord AB

is

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Sum of length of all such chords

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \quad \text{———} \; (1)$$

Apply MVT on $[x_{k-1}, x_k]$,

There exists $c_k \in (x_{k-1}, x_k)$ such that

$y_k = f(x_k)$

$y_{k-1} = f(x_{k-1})$

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$\Rightarrow \frac{y_k - y_{k-1}}{x_k - x_{k-1}} = f'(c_k)$$

$$\Rightarrow \frac{\Delta y_k}{\Delta x_k} = f'(c_k)$$

$$\Rightarrow \Delta y_k = f'(c_k)\,\Delta x_k \quad \text{--- (2)}$$

Putting the values of Δy_k in (1)

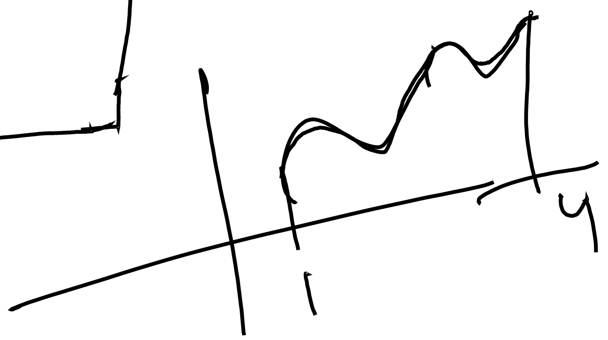
$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + \left(f'(c_k)\,\Delta x_k\right)^2}$$

$$= \sum_{k=1}^{n} \sqrt{1 + \left(f'(c_k)\right)^2}\;\Delta x_k$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} L_k$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{1+(f'(c_k))^2}\;\Delta x_k$$

$$\boxed{L=\int_a^b\sqrt{1+(f'(x))^2}\;dx}$$



Exp Find the length of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}$,

$1\le x\le 4$

Sol^n

$$L=\int_1^4\sqrt{1+(f'(x))^2}\,dx$$

$$\int_1^4\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx=\int_1^4\sqrt{1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2}\,dx$$

$$=6$$

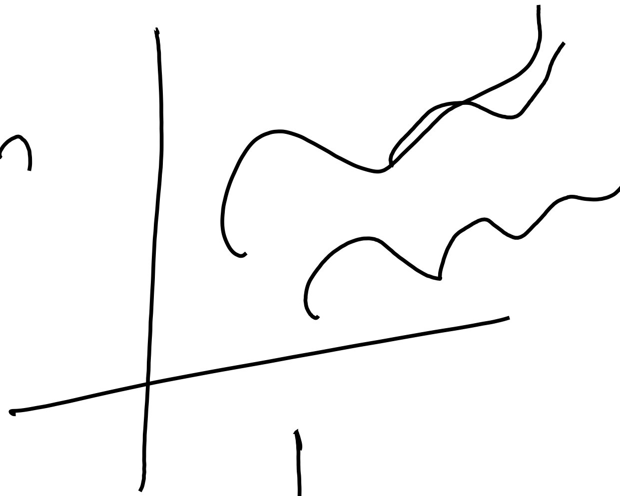
If the curve $x = g(y)$, $c \le y \le d$

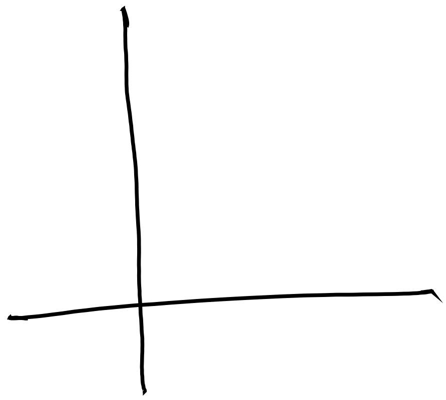
If g' is continuous on $[c, d]$

the length of the curve $x = g(y)$ from

~~$(c, g(c))$~~, $(d, g(d))$ is

$(g(c), c)$, $(g(d), d)$





$$L = \int_c^d \sqrt{\left(1+(g'(y))^2\right)}\,dy$$

$$= \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

$$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$