Generalized Libnitz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx}\left(\int_{u(x)}^{V(x)} f(t)\,dt\right) = f(V(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

$y_1 = x+3$, $y_2 = x-3$, $y_3 = 2x$

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix}$$

$\neq 0$ L.I.

$= 0$ L.D

Homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0 \quad \text{———} \ (1)$$

a, b, c constants.

One function that behaves like (1) is $y = e^{mx}$, where m is constant.

$m = ?$

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

putting these in (1) we get

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$\Rightarrow (am^2 + bm + c)e^{mx} = 0$$

$$\Rightarrow \boxed{am^2 + bm + c = 0} \quad (\text{As } e^{mx} \neq 0) \quad \text{———} \ (2)$$

This equation (2) is called

~~Auxiliary~~

Equation (2) is called the auxiliary equation or characteristic equation of the given differential equation.

$$\boxed{am^2+bm+c=0} \quad \text{--- (2)}$$

Two roots

$$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Case-I When $b^2-4ac > 0$

In this case the auxiliary equation (2) has two real and unequal roots m_1 and m_2.

$y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are two linearly independent solutions of (1).

So the general solution is

$$\boxed{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Exp $y'' - y' - 6y = 0$ —— (1)

Sol^n Put $y = e^{mx}$ in this equation.

$$m^2 e^{mx} - m e^{mx} - 6 e^{mx} = 0$$

$$\Rightarrow (m^2 - m - 6) e^{mx} = 0$$

$$\Rightarrow \boxed{m^2 - m - 6 = 0}$$ Auxiliary equation

$$(m-3)(m+2) = 0$$

$$\Rightarrow m_1 = 3, \quad m_2 = -2$$

e^{3x} and e^{-2x} are two linearly independent solutions of (1)

So the general solution is

$$\boxed{y(x) = c_1 e^{3x} + c_2 e^{-2x}}$$

Case-2 When $b^2-4ac=0$

In this case $m_1 = m_2 = \frac{-b}{2a}$

We have only one solution

$$\boxed{y_1 = e^{m_1 x}}$$

$$\cancel{y_2 = c_1 e^{m_1 x}}$$

Choose $\boxed{y_2 = x e^{m_1 x}}$

This is also a solution of (1)

$$y_2 = x e^{m_1 x}$$

$$y_2' = e^{m_1 x} + m_1 x e^{m_1 x}$$

$$y_2'' = m_1 e^{m_1 x} + m_1 e^{m_1 x} + m_1^2 x e^{m_1 x}$$

$$= m_1^2 x e^{m_1 x} + 2m_1 e^{m_1 x}$$

$$a y_2'' + b y_2' + c y_2$$

$$= a m_1^2 x e^{m_1 x} + 2am_1 e^{m_1 x} + b e^{m_1 x}$$

$$+ b m_1 x e^{m_1 x} + c x e^{m_1 x}$$

$$= \left(am_1^2 + bm_1 + c\right)^{=0} x e^{m_1 x} + \left(2am_1 + b\right)^{=0} e^{m_1 x}$$

$$= 0$$

$am^2 + bm + c = 0$

$y_2 = x e^{m_1 x}$ is also a solution of (1)

$m_1 = m_2 = -\frac{b}{2a}$

$\Rightarrow 2am_1 + b = 0$

$$\left.\begin{aligned} y_1 &= e^{m_1 x} \\ y_2 &= x e^{m_1 x} \end{aligned}\right\}$$

So the general solution is

$$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$
$$= (c_1 + c_2 x) e^{m_1 x}.$$

$$a y''(x) + b y'(x) + c y(x) = 0 \quad (1)$$

Exp $\quad y'' + 4y' + 4y = 0$

Putting $\quad y = e^{mx}$

$$(m^2 + 4m + 4)\, e^{mx} = 0$$

Auxiliary equation is

$$\boxed{m^2 + 4m + 4 = 0}$$

$$(m+2)^2 = 0$$

$$m_1 = -2 = m_2$$

$y_1 = e^{-2x}$ is one solution.

Another L.I. solution is $y_2 = x e^{-2x}$

So the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$= (c_1 + c_2 x)\, e^{-2x}.$$

Case - 3 When $b^2 - 4ac < 0$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The roots of the auxiliary equation are complex roots.

$$m_1 = \alpha + i\beta$$

$$m_2 = \alpha - i\beta$$

α, β are real numbers.

$$y_1 = e^{m_1 x} = e^{(\alpha + i\beta)x}$$

$$= e^{\alpha x} \cdot e^{i\beta x}$$

$$= e^{\alpha x} (\cos\beta x + i \sin\beta x)$$

$$y_2 = e^{m_2 x} = e^{\alpha x} (\cos\beta x - i \sin\beta x)$$

Let $y_3 = \frac{1}{2}(y_1 + y_2) = e^{\alpha x} \cos\beta x$ ✓

$y_4 = \frac{1}{2i}(y_1 - y_2) = e^{\alpha x} \sin\beta x$ ✓

Hence the general solution is

$$y(x) = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

$$\boxed{y(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)}$$

Summary

If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are two complex roots of the auxiliary equation $am^2 + bm + c = 0$, then $y(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$ is the general solution of

~~$ay'' + by' + c = 0$~~

$$ay'' + by' + cy = 0$$

Exp Find a particular solution to the initial value problem

$y'' - 2y' + y = 0 \quad \text{--- (1)}, \quad y(0) = 1, \; y'(0) = -1.$

Solⁿ

The auxiliary equation is

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m_1 = m_2 = 1$

General solution is

$y(x) = c_1 e^x + c_2 x e^x$

$y'(x) = c_1 e^x + c_2 e^x + c_2 x e^x$

$y(0) = 1$

$\Rightarrow c_1 e^0 + c_2 \cdot 0 \cdot e^0 = 1 \Rightarrow \boxed{c_1 = 1}$

$y'(0) = -1$

$c_1 e^0 + c_2 e^0 + c_2 \cdot 0 e^0 = -1$

$\Rightarrow c_1 + c_2 = -1 \Rightarrow 1 + c_2 = -1$

$\Rightarrow \boxed{c_2 = -2}$

So the particular solution is $\boxed{y(x) = e^x - 2xe^x}$

Find a particular solution

Exp $y'' + 4y = 0$, $y(0) = 0$

$y(\pi/12) = 1$

Sol^n Auxiliary eq^n is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$ $\quad \alpha + i\beta$, $\alpha - i\beta$

$\alpha = 0$, $\beta = 2$

General solution is

$y(x) = c_1 e^{0.x} \cos 2x + c_2 e^{0.x} \sin 2x$

$= c_1 \cos 2x + c_2 \sin 2x$

$y(0) = 0$

$\Rightarrow c_1 \cos 2.0 + c_2 \sin 2.0 = 0$

$\Rightarrow \boxed{c_1 = 0}$

$y(\pi/12) = 1$

$\Rightarrow c_1 \cos 2.\frac{\pi}{12} + c_2 \sin 2.\frac{\pi}{12} = 1$

$\Rightarrow c_1 \cos \pi/6 + c_2 \sin \pi/6 = 1$

$\Rightarrow c_2 . \frac{1}{2} = 1 \Rightarrow \boxed{c_2 = 2}$

$\boxed{y(x) = 2 \sin 2x}$