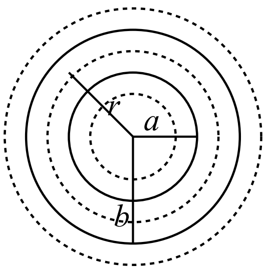
1. If $\vec{\mathrm{E}} = kr^3\hat{\mathrm{r}}$ in a region find the charge density in the region.
   soln
   By Gauss' law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$.
   $\therefore \quad \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 k r^3) = 5k\epsilon_0 r^2$.

2. A hollow spherical shell carries a charge density $\rho = k/r^2$ in the region $a \leq r \leq b$. Find the electric field in the three regions , $r < a, \quad a < r < b, \quad r > b$.
   soln
   Consider a Gausian surface inside the shell. Then



$$E \times 4\pi r^2 = \frac{q_{enc}}{\epsilon_0} = 0$$

$\therefore E = 0$.
For a gausian surface in the shell we have

$$\begin{aligned} E \times 4\pi r^2 &= \frac{1}{\epsilon_0} \int \rho dV = \frac{1}{\epsilon_0} \int_a^r \frac{k}{r^2} 4\pi r^2 dr \\ &= \frac{4\pi k}{\epsilon_0}(r - a) \end{aligned}$$

$\therefore \quad E = \frac{k(r-a)}{\epsilon_0 r^2}$.
For $r > b$ the enclosed charge is $\frac{4\pi k}{\epsilon_0}(b - a)$.

$$\therefore \quad E = \frac{k(b-a)}{\epsilon_0 r^2}$$

3. A spherically symmetric charge distribution is given as $\rho = \rho_0$ for $r \leq a$ and $\rho = 0$ for $r > a$. In addition a point charge q is placed at the origin. Find the electric field in the region using the differntial form of Gauss's law.

   soln:
   The differential form of Gauss' Law is $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$.
   Since the situation is spherically symmetric here, we will only have the radial component of the electric field E_r. So the differential equation we have to solve is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 E_r\right) = \frac{\rho}{\epsilon_0}$$

For $r \leq a$ this gives

$$E_r = \frac{\rho_0 r}{3\epsilon_0} + \frac{c}{r^2}$$

where c is a constant to be determined from known physical conditions.
$\frac{c}{r^2}$ corresponds to the electric field due to a point charge at the origin which has infinite charge density.

$$\therefore\ c = \frac{q}{4\pi\epsilon_0}$$

$$\therefore\ E_r(r < a) = \frac{\rho_0 r}{3\epsilon_0} + \frac{q}{4\pi\epsilon_0 r^2}$$

For $r > a$ we solve the differential equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = 0$$

This gives $E_r(r > a) = \frac{d}{r^2}$.
The integration constant d can be determined by matching the elctric fields at $r = a$ from inside and outside. This gives

$$\frac{\rho_0 a}{3\epsilon_0} + \frac{q}{4\pi\epsilon_0 a^2} = \frac{d}{a^2}$$

This gives $d = \frac{\rho_0 a^3}{3\epsilon_0} + \frac{q}{4\pi\epsilon_0}$.

$$\therefore\ E_r(r > a) = \frac{\rho_0 a^3}{3\epsilon_o r^2} + \frac{q}{4\pi\epsilon_0 r^2}$$

4. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Find the electric field in the three regions, into which the planes partition the space.
soln
We have seen the electric field due to an infinite plane of uniform charge density σ. The electric field will be perpendicular to the plane and on either side it will be directed away from the plane. The magnitude of the field is $E = \frac{\sigma}{2\epsilon_0}$. Due to an infinite plane with surface charge density $-\sigma$ the electric field will be equal and opposite everywhere. When these two palnes are placed parallel to each other the electric field outside the region will cancel while between the plates they add up. So the electric field between the planes will be $\frac{\sigma}{\epsilon_0}$. The direction will be perpendicular to the planes and directed from the positively charged plane to the negatively charged plane. Outside the planes the field will be 0.

5. The electric field in a region is cylindrically symmetric, given as follows:

$$\begin{aligned} \vec{\mathrm{E}}(\vec{\mathrm{r}}) &= \frac{c}{s}\hat{s}; \quad \text{when} \quad s \geq a \\ &= \frac{cs}{a^2}\hat{s}; \quad \text{when} \quad s < a \end{aligned}$$

Find the charge distribution in the region using Gauss' law.
soln

The charge density is given by the differential form of Gauss' law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$.
Due to cylindrical symmetry of the problem the partial differentiation w.r.t z and ϕ is zero. So we have

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s)$$

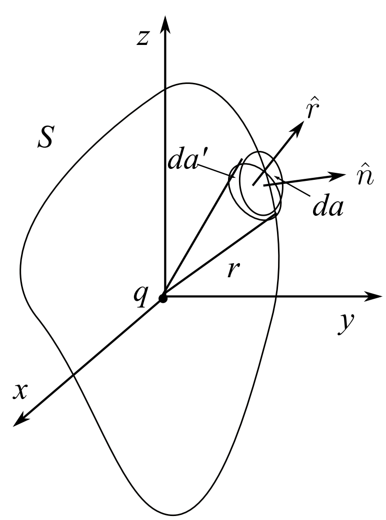
For $s > a, E_s = c/s \quad \Longrightarrow \quad \vec{\nabla} \cdot \vec{E} = 0.$
For $s < a, E_s = \frac{cs}{a^2}\hat{s} \quad \Longrightarrow \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{cs}{a^2}) = \frac{2c}{a^2}.$
By differential form of Gauss's law

$$\begin{aligned} \rho &= 0; \qquad s \geq a \\ &= \frac{2c\epsilon_0}{a^2}; \quad s < a \end{aligned}$$

So the charge density is uniform over a cylinder of radius a and 0 outside this.
As the given $\vec{E}$ is continuous at the interface $s = a$ (verify), we don't have to worry about infinite charge densities like, point charge, line charge or surface charges at the interface.

6. The coulomb's law of forces between two point charges and the Gauss's law of electrostatics are equivalent. Starting from the Coulomb's law derive the Gauss's law. Start with the electric field due to a point charge q at the origin given according to Coulomb's law as $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$.

soln:
Let us calculate the flux of the electric field over a surface S enclosing a point charge



q within it. Let us consider the origin of the coordinate system at the location of the point charge.

$$\oint_S \vec{E} \cdot \hat{n} da = \oint \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \cdot \hat{n} da$$

As shown in the figure $\hat{n}da$ is an infinitesimal area element over the surface S. As the surface is not spherical in general we consider the projection of this area element locally over the surface of a sphere of radius r passing through the point P and center at the origin. The local normal to the sphere is $\hat{r}$ and the projected infinitesimal area element on this sphere is given as $da' = r^2 d\Omega = (\hat{r} \cdot \hat{n} da)$ where $d\Omega$ is the solid angle subtended by the area element da' at the center of the sphere. Putting all these in the above expression for flux we get

$$
\begin{aligned}
\oint_S \vec{E} \cdot \hat{n} da &= \oint \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \cdot \hat{n} da \\
&= \oint_S \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} da' \\
&= \oint_S \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} r^2 d\Omega \\
&= \frac{q}{4\pi\epsilon_0} \oint_S d\Omega \\
&= \frac{q}{4\pi\epsilon_0} \times 4\pi \\
&= \frac{q}{\epsilon_0}
\end{aligned}
$$

We see that the total flux is independent of the shape of the enclosing surface. It depends only upon the magnitude of the charge. Hence it will be independent of the location of the charge within the surface. We can consider several such charges $q_1, q_2, ..., q_n$ within the surface. All of them will contribute "linearly" to the electric flux. This leads to the statement of the Gauss' law that the total flux of the electric field over a closed surface is proportional to the amount of charge within the surface.