First order ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{——} (1)$$

y is dependent variable

x is independent variable.

1st order because it involves only 1st order derivative.

A differentiable function $y(x)$ is said to be a solution of (1) if it satisfies (1).

The general solution to a 1st order differential equation is a solution that contains all possible solutions.

Separable equation

$$\frac{dy}{dx} = f(x,y) = g(x)\,h(y) \quad \text{——— (1)}$$

If $f(x,y)$ can be written as $g(x)\,h(y)$ then it is called a separable equation.

We can ~~written~~ write (1) as

$$\frac{dy}{h(y)} = g(x)\,dx$$

and integrate it to get the general solution of (1).

EXP $\quad \frac{dy}{dx} = \frac{1}{y}\log x \quad \text{——— (2)}$

$\Rightarrow \quad y\,dy = \log x\,dx$

Integrating $\Rightarrow \quad \frac{y^2}{2} = x\log x - x + C$

general solution of (2)

Homogeneous equations

$$\frac{dy}{dx} = f(x, y) \quad \text{—— (3)}$$

$f(x,y)$ is said to be homogeneous function of degree n if

$$f(tx, ty) = t^n f(x, y)$$

where t is independent of x and y.

The equation (3) is said to be a homogeneous differential equation.

Solution method

put $y = vx$ to the given homogeneous differential equation. It reduces the given equation to a separable equation.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Exp Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Sol^n

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} = f(x,y) \quad \text{——} (A)$$

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$f(tx, ty)$

$$= \frac{(tx)^2 ty}{(tx)^3+(ty)^3}$$

$$= \frac{\cancel{t^3}\, x^2y}{\cancel{t^3}(x^3+y^3)} = t^0 f(x,y)$$

putting in (A)

$$v + x\frac{dv}{dx} = \frac{x^2\,vx}{x^3+(vx)^3} = \frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v - v^4}{1+v^3}$$

$$= -\frac{v^4}{1+v^3}$$

$$\Rightarrow -\frac{1+v^3}{v^4}\,dv = \frac{dx}{x}$$

separable equation

$$\left(-\frac{1}{V^4} - \frac{1}{V}\right) dv = \frac{dx}{x}$$

Integrate it

$$\frac{1}{3V^3} - \log V = \log x + \log c$$

$$\Rightarrow \frac{1}{3V^3} = \log (Vxc)$$

$$\Rightarrow Vxc = e^{\frac{1}{3V^3}}$$

$$\Rightarrow yc = e^{\frac{1}{3\left(\frac{y}{x}\right)^3}}$$

$$\Rightarrow \boxed{y = \frac{1}{c} e^{\frac{x^3}{3y^3}}}$$

Special cases

Case-1

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad (1) \quad \text{where } \boxed{\frac{a}{a'} \neq \frac{b}{b'}}$$

The pair of lines $ax+by+c=0$ (2)
and $a'x+b'y+c'=0$ (3)

intersect each other.

If (h,k) is the point of intersection, then shift the origin to (h,k)

by setting
$$x = h+X$$
$$y = k+Y$$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Equation (1) reduces to

$$\frac{dY}{dX} = \frac{a(h+X)+b(k+Y)+c}{a'(h+X)+b'(k+Y)+c'}$$

$$= \frac{aX+bY+\overbrace{(ah+bk+c)}^{=0}}{a'X+b'Y+\underbrace{(a'h+b'k+c')}_{=0}}$$

So eq^n (1) can be written as

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$$ This is a homogeneous equati

Apply $Y = VX$ and solve for Y. Hence we get y.

Exp

$$\frac{dy}{dx} = \frac{x+y+4}{x-y-6} \quad \text{—— (1)}$$

Sol^n

$x = h+X$

$y = k+Y$

$$\frac{dY}{dX} = \frac{X+Y+\overbrace{(h+k+4)}^{=0}}{X-Y+\underbrace{(h-k-6)}_{=0}}$$ for $h=1$, $k=-5$

The point of intersection of the two lines $\begin{cases} h+k+4=0 \\ h-k-6=0 \end{cases}$

is $(h,k) = (1,-5)$ $\quad h=1$, $k=-5$

Eq^n (1) can be written as

$$\frac{dy}{dx} = \frac{x+y}{x-y} \quad \text{(homogeneous)}$$

put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{1+v^2}\,dv = \frac{dx}{x} \quad \text{(separable)}$$

$$\Rightarrow \frac{1}{1+v^2}\,dv - \frac{v}{1+v^2}\,dv = \frac{dx}{x}$$

Integrate it

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c$$

$$\Rightarrow \tan^{-1} v = \log\left(x\sqrt{1+v^2}\right) + c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \log\left(x\sqrt{1+\frac{y^2}{x^2}}\right) + c$$

$$= \log\sqrt{x^2+y^2} + c$$

$X = x - h = x - 1$ $\qquad x = h + X$

$Y = y - k = y + 5$ $\qquad y = k + Y$

$h = 1$

$k = -5$

So $\tan^{-1}\dfrac{y+5}{x-1} = \log\sqrt{(x-1)^2 + (y+5)^2} + C$ — (1)

~~Case-2~~ $\dfrac{dy}{dx} = \dfrac{\boxed{ax+by}+c}{\boxed{a'x+b'y}+c'}$ where $\boxed{\dfrac{a}{a'} = \dfrac{b}{b'}}$

$\left.\begin{array}{l} ax+by+c=0 \\ \text{and } a'x+b'y+c'=0 \end{array}\right\}$ parallel lines.

Let us say $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{\lambda}$

$\Rightarrow a' = a\lambda \qquad b' = b\lambda$

So $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a\lambda x + b\lambda y + c'} = \dfrac{ax+by+c}{\lambda(ax+by)+c'}$ — (2)

put $ax+by = z$

$y = \dfrac{1}{b}(z - ax)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{b}\left(\dfrac{dz}{dx} - a\right)$

Thus eq^n (2) reduces to

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = \frac{z+c}{\lambda z + c'}$$

$$\Rightarrow \frac{dz}{dx} = b\left(\frac{z+c}{\lambda z+c'}\right) + a$$

Separable equation.

Exp Solve $\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$ — (1)

Sol^n put $x+y = z$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

put in equation (1)

$$\frac{dz}{dx} - 1 = \frac{z+4}{z-6}$$

$$\Rightarrow \frac{dz}{dx} = 1 + \frac{z+4}{z-6} = \frac{z-6+z+4}{z-6}$$

$$= \frac{2(z-1)}{z-6}$$

$$\Rightarrow \frac{z-6}{z-1}\,dz = 2dx$$

$\Rightarrow$

$$\Rightarrow \frac{z-1-5}{z-1}\,dz = 2dx$$

$$\Rightarrow \left(1-\frac{5}{z-1}\right)dz = 2dx$$

Integrating we get

$$z - 5\log(z-1) = 2x + C$$

$$\Rightarrow x+y-5\log(x+y-1) = 2x + C$$

$$\Rightarrow \cancel{x} \boxed{y-x-5\log(x+y-1) = C}$$