Morera's Theorem

If $f(z)$ is continuous in a simply connected domain D and if

$\oint_C f(z)dz = 0$ for every simple closed curve in D, then

$f(z)$ is analytic in D.

Liouville's Theorem

Entire function: If $f(z)$ analytic in the whole complex plane

If $f(z)$ is entire function and $f(z)$ bounded ie $|f(z)| \leq M$, then $f(z)$ is a constant function.

$\sin z = \sin(x+iy)$ (It is not bounded)

$= (\sin x \cos iy + \cos x \sin iy)$

Power Series

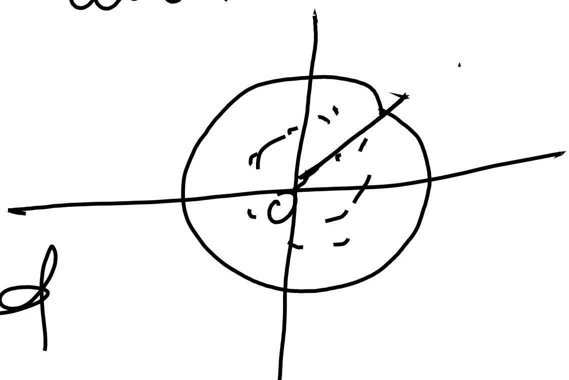
$$\sum a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

This is a power series about z_0.

$a_0, a_1, \ldots$ complex coefficients.

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

$$= \frac{1}{1-z} \quad \text{when } |z| < 1$$



$|z| = 1$ is called the circle of convergence

1 is called the radius of convergence.

If $z = 2i$

$$1 + 2i + (2i)^2 + (2i)^3 + \cdots$$

$$= 1 + 2i + -4 - 8i + 16 + 32i + \cdots$$

1) Ratio Test

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

If R is radius of convergence

$$\boxed{R = \frac{1}{\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|}}$$

Or $$\boxed{R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

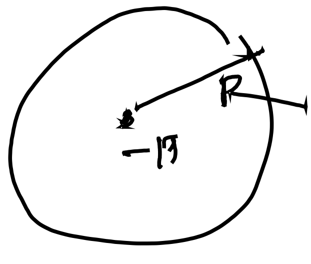
2) Root Test

$$\boxed{R = \frac{1}{\lim_{n \to \infty} |a_n|^{1/n}}}$$

Exp $\sum_{n=0}^{\infty} z^n$ $\quad R = 1$

Exp $\sum \frac{1}{n^n}(z+\pi)^n$

Center of power series is $-\pi$



$a_n = \frac{1}{n^n}$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0$$

$R = \infty$

Exp $\sum \frac{n!}{n^n}(z+\pi)^n$ center is $-\pi$

$a_n = \frac{n!}{n^n}$

$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!}\frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$\boxed{R = e}$

Ex

(A) Find center and radius of convergence

(1) $\sum_{n=0}^{\infty} n(z+i\sqrt{2})^n$

(2) $\sum_{n=0}^{\infty} \frac{2^{2n}}{n!}(z-3)^n$

(3) $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n (z-\pi i)^n$

(4) $\sum_{n=0}^{\infty} \frac{i^n n^3}{2^n} z^{2n} = \sum_{n=0}^{\infty} \frac{i^n n^3}{2^n} z^{2n}$

(5) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$

(6) $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} (z-3i)^n$

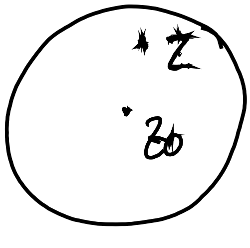
$$\boxed{R = \frac{1}{4}}$$

Note: Every analytic function can be represented in the form of a power series about some point.

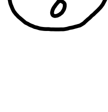
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

About z_0 — Taylor series

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \dots$$



$$\frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$



$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad |z| < 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots -$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$\tan^{-1} z$

$|z^2| < 1$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 \cdots$$

Integrating both sides

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

Laurent Series

If $f(z)$ is analytic inside an annular disk by two concentric circles C_1 and C_2 with center z_0, then $f(z)$ can be represented by a Laurent series

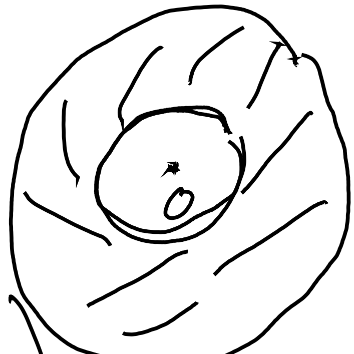
$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part}}.$$

$$= a_0 + a_1(z-z_0) + \cdots$$

$$+ b_1 (z-z_0)^{-1} + b_2 (z-z_0)^{-2} + \cdots$$

b_1 is called the residue of $f(z)$ at z_0.

Exp $f(z) = z^2 e^{\frac{1}{z}}$



$$= z^2\left(1+\frac{\frac{1}{z}}{1!}+\frac{\left(\frac{1}{z}\right)^2}{2!}+\cdots\right)$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} + \cdots$$

$$= \frac{1}{2}+z+z^2 + \underbrace{\frac{1}{3!}z^{-1}+\frac{1}{4!}z^{-2}+\cdots}_{\text{principal part}}$$

analytic part

Residue of $f(z)$ at $z=0$

is $\frac{1}{3!} = \frac{1}{6}$.

Exp $f(z) = \frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)}$

$$= \frac{1}{z^3}(1-z)^{-1} = \frac{1}{z^3}(1+z+z^2+\cdots)$$

$$= \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z} + 1+z+z^2+\cdots$$

principal part analytic part

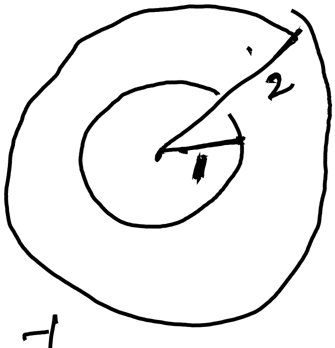
Residue of $f(z)$ at 0 $= 1$

Exp $f(z) = \dfrac{-2z+3}{z^2-3z+2}$

Find the Laurent series about $z=0$.

Sol^n $$f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-1}{z-1} - \frac{1}{z-2}$$

$$\frac{-1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z|<1 \qquad \text{—— (1)}$$



$$\frac{-1}{z-2} = \frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad \text{for } \left|\frac{z}{2}\right|<1$$

$$\Rightarrow |z|<2 \qquad \text{—— (2)}$$

For $|z|<1$

$$f(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

For $1 < |z| < 2$

$$-\frac{1}{z-1} = \frac{-1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

for $\left|\frac{1}{z}\right| < 1$

$\Rightarrow |z| > 1$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right)$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \qquad \text{——— (3)}$$

$$-\frac{1}{z-2} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z \qquad \text{for } |z| < 2$$

from (2)

For $1 < |z| < 2$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}}$$

For $|z| > 2$

$$-\frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad \text{for } |z| > 1 \quad \text{—— (3)}$$

$$-\frac{1}{z-2} = \frac{-1}{z\left(1-\frac{2}{z}\right)} = -\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots \right)$$

$$= -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \quad \text{—— (4)} \quad \text{for } \left|\frac{2}{z}\right| < 1$$

$$\Rightarrow |z| > 2$$

For $|z|>2$
both (3) & (4) are valid

$$\text{So } f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

Singularity

A function $f(z)$ has a singular point at $z = z_0$ if it is not analytic at z_0, but every neighborhood of z_0 contains points at which $f(z)$ is analytic.

Ex:

$f(z) = \frac{1}{z}$, $z = 0$ is a singular point of $f(z)$

$f(z) = \frac{1}{z - \pi}$, $z = \pi$ is singular point.

$f(z) = \frac{1}{\sin z}$, $z = \pm n\pi$, $n = 0, 1, 2, \ldots$ are singular points.

$f(z) = e^{1/z}$, $z = 0$ is a singular point.

$f(z) = \frac{1}{z(z - \sqrt{2})}$, $z = 0$, $z = \sqrt{2}$ are singular points.

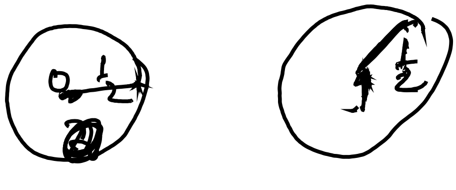
$f(z) = \operatorname{Re} z$ No singular point.

Isolated singular point

z_0 is said to be isolated singular point of $f(z)$ if z_0 has a neighbourhood in which there is no other singular point.

$f(z) = \frac{1}{z}$, $z = 0$ is isolated singular point.

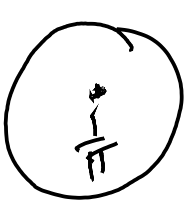
$f(z) = \frac{1}{z(z-1)}$, $z = 0$, $z = 1$ are isolated singular points.



Non isolated

$f(z) = \frac{1}{\sin\frac{1}{z}}$

$\sin\frac{1}{z} = 0$



Non isolated

$\frac{1}{z} = \pm n\pi$, $n = 0, 1, 2 \ldots$

$\Rightarrow z = \pm \frac{1}{n\pi}$

$\ldots, -\frac{1}{2\pi}, -\frac{1}{\pi}, \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$

Classification of singular points

1. Removable singular points
2. pole
3. Essential singular points

Using Laurent Series

Let z_0 be an isolated singular point of $f(z)$.

Let Laurent series of $f(z)$ about z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

(1) If there is no principal part in the Laurent series, then z_0 is removable singular point.

Exp $f(z) = \dfrac{\sin z}{z}$

$z = 0$ is a singular point.

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$z = 0$ is a removable singular point.

Pole

If the principal part has only finite no. of terms, that is

$$b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots + b_m(z-z_0)^{-m}$$

Then z_0 is called a pole of order m.

Exp $f(z) = \dfrac{1}{(z-1)(z+1)}$

$+1, -1$ are singular points.

$$f(z) = \frac{1}{(z-1)(z+1)}$$

about $z=1$

$$f(z) = \frac{1}{(z-1)(z-1+2)}$$

$$= \frac{1}{(z-1)\,2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{1}{2(z-1)}\left(1+\frac{z-1}{2}\right)^{-1}$$

$$= \frac{1}{2(z-1)}\left(1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \cdots\right)$$

$$= \frac{1}{2(z-1)} - \frac{1}{4} + \frac{1}{8}(z-1) + \cdots$$

$z=1$ is a pole of order 1

Similarly $z=-1$ is also a pole of order 1.

Essential singular point

$\sin \frac{1}{z}$

$z = 0$ is a singular point.

$$\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - - -$$

$$= \frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - - -$$

$z = 0$ is called essential singular point.