Algorithms

Feb 17, 2020

Algo ① Find max. element in a $< \infty$ seq.

```
procedure max (a1, a2, ..., an : integers)
max := a1
for i := 2 to n
        if max < ai then max := ai
        {max is the largest element}
```

Pseudo Code

Properties

① Input: (seq. of integers)

② Output: (the largest integer in the seq.)

③ Definitness: Each step is precicely defined as only assignments, a finite loop & conditional statement occur

④ Correctness: In order to show the correctness we must show when algo terminates the value of the variable max equals the maximum of the terms of the seq. Note initial value is the first term a_1 as successive terms are examined max is updated to the value of the term if the term exceeds the max. of the terms previously examined ∴ when all the terms are examined max = the value of the largest term.

⑤ Finiteness: The algo uses finite no. of steps.

⑥ Effectiveness: The algo can be carried out in $< \infty$ amount of time as each step is either a comparison or an assignment.

⑦ Generality: The algo is general as it can be used to find max. of any $< \infty$ seq. of integers.

Searching Algos

1. Linear Search or sequential search algo

```
procedure linear search (x : integer, a1, a2, ..., an : distinct integers)
i := 1
while (i ≤ n and x ≠ ai)
    i := i+1
if i ≤ n then location := i
  else location := 0
(location is the subscript of the term that equals to
    x or is 0 if x is not found)
```

2. Binary Search can be used when the list has terms in order of increasing size (or lexicographic in case of words).

Example Search for 19 in the list

1 2 3 5 6 7 8 10 12 13 15 16 18 19 20 22

↑ first split

1 2 3 5 6 7 8 10   12 13 15 16 18 19 20 22

As $10 < 19$ choose other list & split

12 13 15 16   18 19 20 22

As $16 < 19$ choose other list

18 19   20 22

As 19 is $\cancel{} \leq$ largest term of first list choose first list

18 19

18   19

as $18 < 19$ choose second list

Search is narrowed down to 1 term
comparision is made 19 is located as 14th term

In general,

To search x in $a_1, a_2, \ldots, a_n$

where $a_1 < a_2 < \cdots < a_n$

begin by comparing x with middle term of the seq. a_m, $m = \lfloor \frac{n+1}{2} \rfloor$

The Binary Search Algo

```
procedure binary search (x: integer, a1, a2, ..., an: increasing integers)
i := 1 {i is left endpoint of search interval}
j := n {j is right end point of    "    "  }
while i < j
begin m := ⌊(i+j)/2⌋
    if x > am then i := m+1
       else j := m
    end
  if x = ai then location := i
    else location := 0
```

Sorting Algo. ~400 pages 15 different sorting Algo's

Bubble sort

```
procedure bubblesort (a1, a2, ..., an: real nos. with n ≥ 2)
for i := 1 to n-1
    for j := 1 to n-i
        if aj > aj+1 then interchange aj & aj+1
{a1, ..., an is in increasing order}
```

Example: Sort 3 2 4 1 5 into increasing order using bubble sort

First pass

3	2	2	2
2	3	3	3
4	4	4	1
1	1	1	4
5	5	5	5

(pair in correct order

↕ interchange

Second pass

2	2	2
3	3	1
1	1	3
4	4	4
5	5	5

3rd pass

2	1
1	2
3	3
4	4
5	5

4th pass

1
2
3
4
5

Insertion sort

procedure insertion sort ($a_1, a_2, \ldots, a_n$: real nos with $n \geq 2$)

for $j := 2$ to n

begin

$i := 1$

while $a_j > a_i$

$i := i + 1$

$m := a_j$

for $k := 0$ to $j - i - 1$

$a_{j-k} := a_{j-k-1}$

$a_i := m$

end {$a_1, a_2, \ldots, a_n$ are sorted}

Example Use insertion sort to put elements 3, 2, 4, 1, 5 in increasing order

I First compare 2 & 3, as $3 > 2$ place 2 at 1st place

2 3 4 1 5

Compare 4 with 2 & with 3
4 is placed at 3rd position

$4 > 2$ & $4 > 3$

list is 2 3 4 1 5

compare $1 < 2$, $1 < 3$, $1 < 4$ $\therefore$ 1 is placed at 1st position

1 2 3 4 5 as $5 > 4$ it goes to end position

1 2 3 4 5 sorted list

Greedy Algo

That makes best choice at each step

GROWTH OF FUNCTIONS

Feb 19, 2020

Big-O-notation

Order of ∞ - G. H. Hardy

— f & g are fns from $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$

$f(x) = O(g(x))$ if $\exists$ constants c & K s.t.

$$|f(x)| \leq c\,|g(x)| \quad \text{whenever} \quad x > K$$

↳ Bord const.

$\boxed{\text{Example}}$ 1 $\quad f(x) = x^2 + 2x + 1 = O(x^2)$

When $x > 1 \Rightarrow x < x^2$ & $1 < x^2$

$\Rightarrow f(x) = x^2 + 2x + 1 < x^2 + 2x^2 + x^2 = 4x^2$

$\therefore c = 4$ & $K = 1$

$\Rightarrow f(x) = O(x^2)$

Alternat. when $x > 2 \quad 2x < x^2$ & $1 \leq x^2$

$\therefore f(x) = x^2 + 2x + 1 \leq x^2 + x^2 + x^2 = 3x^2$

$c = 3$ & $K = 2$

Note $f(x) = O(x^3)$ or $f(x) = O(x^2 + x + 7)$

Also $x^2 = O(x^2 + 2x + 1)$

$x^2 < x^2 + 2x + 1$ whenever $x > 1$

$c = 1$ & $K = 1$

$\boxed{\text{Example 2}}$ $\quad 7x^2 = O(x^3)$

When $x > 7 \Rightarrow 7x^2 < x^3 \quad \therefore c = 1$ & $K = 7$

$\therefore 7x^2 = O(x^3)$

Alter: When $x > 1 \quad 7x^2 < 7x^3 \quad \therefore c = 7$ & $K = 1$.

Example 3 Show that $n^2 \neq O(n)$

We need to show that no pair of constts. c & K exist s.t. $n^2 \leq cn$ whenever $n > K$

Observe that when $n > 0$, $n^2 \leq cn$ gives $n \leq c$

$n \leq c$ cannot hold for all $n > K$ if we choose $k =$.

when $n > \max\{K, c\}$ it is not true $n \leq c$

Example 4 We know $7x^2 = O(x^3)$ does it true that $x^3 = O(7x^2)$

□ We need to find c & K s.t. $x^3 \leq c \cdot 7x^2$ where $x > K$

$x \leq 7c \quad \forall x > K$ no such c exist. as x can be made arbitr. large.

$\therefore \quad x^3 \neq O(7x^2)$

Properties

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \forall a_i \in \mathbb{R}$$

$$f(x) = O(x^n)$$

□ If $x > 1$ we have

$$\begin{aligned}
|f(x)| &= |a_n x^n + \cdots + a_1 x + a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\
&= x^n \left(|a_n| + \frac{|a_{n-1}|}{x} + \cdots + \frac{|a_1|}{x^{n-1}} + \frac{|a_0|}{x^n} \right) \\
&\leq x^n \left(|a_n| + |a_{n-1}| + \cdots + |a_0| \right)
\end{aligned}$$

$\therefore \quad c = \sum_{i=0}^{n} |a_i|$ where $x > 1$ & $K = 1$

$\therefore \quad f(x) = O(x^n)$

$\boxed{\text{Example}}$ $1+2+3+\cdots+n = O(n^2)$

$\square$ $\because$ $1+2+\cdots+n \le n+n+\cdots+n = n^2$

$c=1$ & $k=1$

$\boxed{\text{Example}}$ $n! = O(n^n)$

$\square$ $n! = 1\cdot 2\cdot 3 \cdots n \le n\cdot n\cdot n \cdots n = n^n$

$c=1$ & $k=1$

Take log on both sides

$\log n! \le n \log n$

$\Rightarrow \log n! = O(n \log n)$ taking $c=1$ & $k=1$

$\boxed{\text{Example}}$ $n < 2^n$ when n is +ve integer

$n = O(2^n)$

$\square$ $\because$ $n < 2^n$ $\therefore$ for $c=1$ & $k=1$

Take log base 2

$\log_2 n < n \Rightarrow \log_2 n = O(n)$

$c=k=1$

$\because \log_b n = \dfrac{\log n}{\log b} < \dfrac{n}{\log b}$ $\therefore c = \dfrac{1}{\log b}$ & $k=1$

$\log_b n = O(n)$

$\boxed{\text{Sum}}$ If $f_1(x) = O(g_1(x))$ & $f_2(x) = O(g_2(x))$

then $(f_1+f_2)(x) = O(\max |g_1(x)|, |g_2(x)|)$

$\square$ $|f_1(x)| \le c_1 |g_1(x)|$ & $|f_2(x)| \le c_2 |g_2(x)|$

$x > k_1$ $\quad x > k_2$

$(f_1+f_2)(x) = |f_1(x) + f_2(x)| \le |f_1(x)| + |f_2(x)|$

$\le c_1 |g_1(x)| + c_2 |g_2(x)|$

when $x >$ both k_1 & k_2

$\le c_1 |g(x)| + c_2 |g(x)| = (c_1+c_2) |g(x)|$

$= c|g(x)|$

$c = c_1 + c_2$ & $g(x) = \max\{|g_1(x)|, g_2(x)|\}$ $\quad x > \max\{k_1, k_2\}$

Cor. $f_1(x) = O(g_1(x))$ & $f_2(x) = O(g_2(x))$

$\Rightarrow$ $(f_1+f_2)(x) = O(g(x))$

$\because \max\{g(x), g(x)\} = g(x)$ ▨

Th. $f_1(x) = O(g_1(x))$ & $f_2(x) = O(g_2(x))$

then $(f_1 f_2)(x) = O(g_1(x) g_2(x))$

P $|f_1 f_2(x)| = |f_1(x)|\,|f_2(x)| \le c_1 |g_1(x)| . c_2 |g_2(x)|$

$\le \underbrace{c_1 c_2}_{c} |(g_1 g_2)(x)|$

$|f_1 f_2(x)| \le c |g_1(x) g_2(x)|$ & $k = \max\{k_1, k_2\}$ ▨

$\boxed{\text{Example}}$ $f(n) = 3n \log n! + (n^2+3)\log n = O(n^2 \log n)$

P $\log n! = O(n \log n)$ & $3n = O(n)$

$\therefore 3n \log n! = O(n^2 \log n)$

now

$\underline{n^2+3 < 2n^2}$ when $n > 2 \Rightarrow (n^2+3)\log n \le 2n^2 \log n$

$c = 2$ $k = 2$

$\therefore n^2+3 = O(n^2)$ ✓

$\therefore$ ~~$n^2+3 \log n = O(n^2 \log n)$~~

$\therefore f(n) = O(n^2 \log n)$ ✓ ▨

$\boxed{\text{Example}}$ $f(x) = (x+1)\log(x^2+1) + 3x^2 = O(x^2)$

$x+1 = O(x)$ $\quad x^2+1 \le 2x^2$ when $x > 1$

$\therefore \log(x^2+1) \le \log(2x^2) = \log 2 + 2\log x$

$\le 3 \log x$

$\therefore$ for $x > 2$ $\log(x^2+1) = O(\log x)$

$\therefore (x+1)\log(x^2+1) = O(x \log x)$

$3x^2 = O(x^2)$

$\therefore f(x) = O(\max(x \log x, x^2))$

$\because x \log x \le x^2$ for $x > 1$

$f(x) = O(x^2)$

▨

$\boxed{\text{Big } \Omega\text{-notation}}$ f & $g: \mathbb{Z}$ or $\mathbb{R} \to \mathbb{R}$

$f(x) = \Omega(g(x))$ if $\exists$ +ve constants c & k s.t.

$$|f(x)| \geq c\,|g(x)| \quad \text{whenever } x > k$$

$\boxed{\text{Example}}$ $f(x) = 8x^3 + 5x^2 + 7 = \Omega(\underset{g(x)}{x^3})$

$\because \; 8x^3 + 5x^2 + 7 \geq 8x^3 \;\; \forall$ +ve reals

i.e., $g(x) = O(8x^3 + 5x^2 + 7)$

$\boxed{\text{Standard Ref. fns.}}$ x^n, $n > 0$ or c^x $c > 0$

$\boxed{\text{Big } \Theta\text{-fn.}}$ f & $g: \mathbb{Z}$ or $\mathbb{R} \to \mathbb{R}$

$f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$
& $f(x) = \Omega(g(x))$

$\boxed{\text{Ref. fns}}$ $x^n, c^x, \log x, \ldots$

$\boxed{\text{Examples}}$ $1 + 2 + 3 + \cdots + n = \Omega(n^2)$

$\therefore \; 1 + 2 + 3 + \cdots + n = \Theta(n^2)$

☐ We know that $1 + 2 + \cdots + n = O(n^2)$

$$1 + 2 + 3 + \cdots + n \geq \lceil \tfrac{n}{2} \rceil + (\lceil \tfrac{n}{2} \rceil + 1) + \cdots + n$$

(ignore first half & sum only the terms $\geq \lceil \frac{n}{2} \rceil$)

$$\geq \lceil \tfrac{n}{2} \rceil + \cdots + \lceil \tfrac{n}{2} \rceil$$

$$= (n - \lceil \tfrac{n}{2} \rceil + 1) \lceil \tfrac{n}{2} \rceil$$

$$\geq \left(\tfrac{n}{2}\right)\left(\tfrac{n}{2}\right) = \frac{n^2}{4}$$

$\therefore \; 1 + 2 + \cdots + n = \Omega(n^2) \;\; \therefore \; f(n) = \Theta(n^2)$

$$f(x) = \Theta(g(x))$$

$$c_1|g(x)| \leq |f(x)| \leq c_2|g(x)| \qquad x > k$$

$$f(x) = O(g(x)) \; \& \; f(x) = \Omega(g(x))$$