$$f(z) = e^{i\theta} = \underbrace{\cos\theta}_{u} + i\underbrace{\sin\theta}_{v}$$

is analytic

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$0 = \frac{1}{r}\cos\theta$$

e^{iz}

Entire function

A function which is analytic in the whole complex plane is called an entire function.

e^{z}

$\cos z$

$\sin z$

$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

$$\ln e^{iz} = \cancel{e^{i(x+iy)}} = \cos z + i \sin z$$

$$= e^{i r e^{i\theta}}$$

analytic

$$e^{i\theta} = \cos\theta + i\sin\theta$$

X

analytic

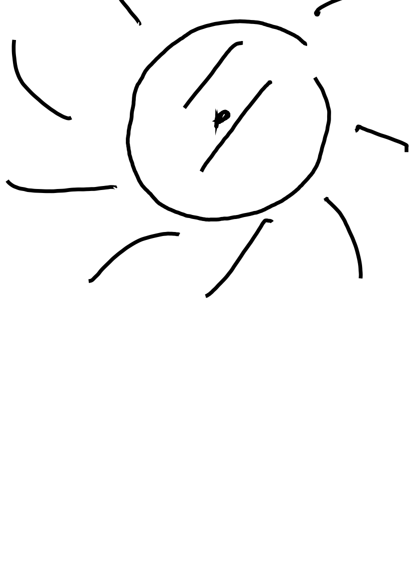
$$f(z) = \frac{1}{z} \; : \; \boxed{|z|>1}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\sinh z = \frac{e^{z}-e^{-z}}{2}$$



$$\ln z = \ln r e^{i\theta} = \ln r + i\theta$$

$|z| > 0$

Many valued

θ argument

$$\operatorname{Ln} z = \ln r + i \operatorname{Arg}(z)$$ single valued function

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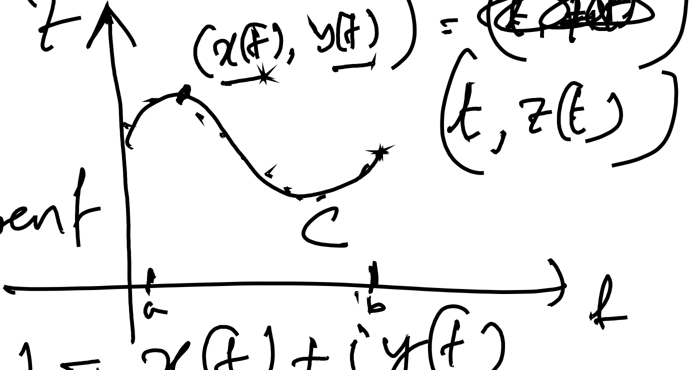
Complex integration

Line integral in complex plane

$$\int_C f(z)\, dz$$

C is a given curve called the path of integration.



We can represent this curve

$$z(t) = x(t) + i y(t)$$

$$a \le t \le b$$

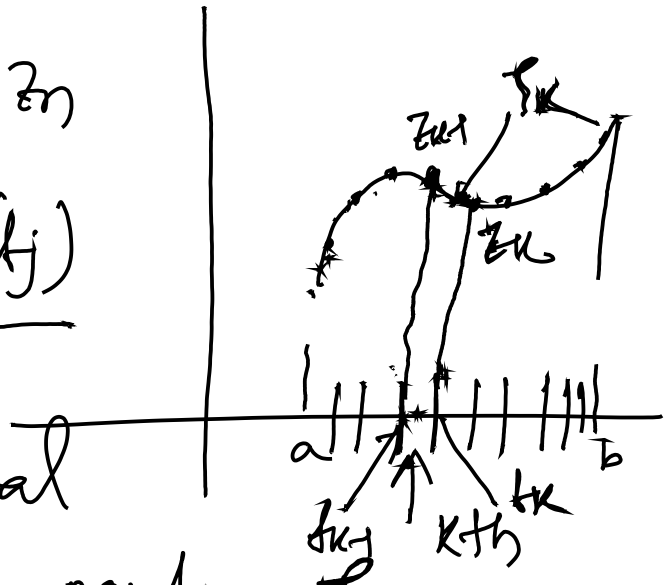
$C : z(t) = x(t) + i\,y(t) \qquad \underline{a \le t \le b}$

Subdivide the interval $[a, b]$

$t_0 = a < t_1 < t_2 < \cdots < t_n = b$

$z_0, z_1, z_2, \ldots, z_n$

$z_j = \underline{x(t_j)} + i\,\underline{y(t_j)}$



In each subinterval we choose arbitrary point ξ_k

$t_{k-1} \to z_{k-1}$

$t_k \to z_k$

$$S_n = \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k$$

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz$$

ξ_K

Properties

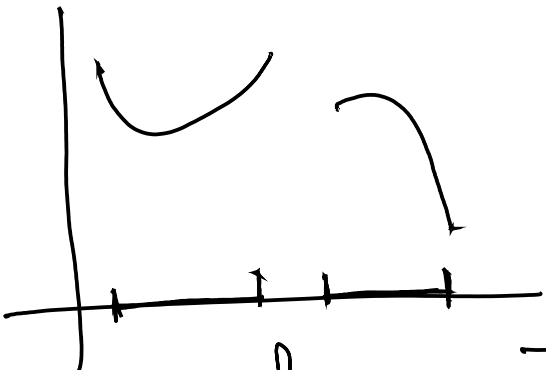
(1) $$\int_C \left(K_1 f_1(z) + K_2 f_2(z) \right) dz$$

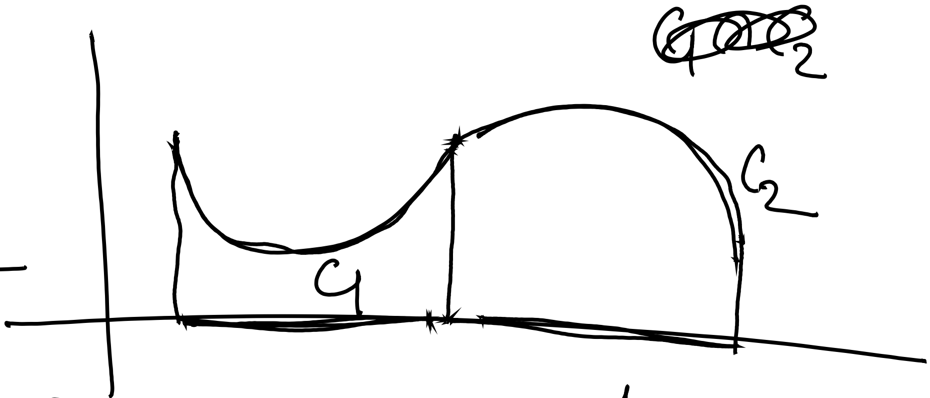
$$= K_1 \int_C f_1(z)\, dz + K_2 \int_C f_2(z)\, dz$$

(2) $$\int_{z_0}^{z_1} f(z)\, dz = -\int_{z_1}^{z_0} f(z)\, dz$$

(3) $$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

$$C = C_1 \cup C_2$$





Existence

If $f(z)$ is continuous and C is piecewise smooth then $\int_C f(z)\, dz$ exists.

Result

If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral of $f(z)$ in the domain D, that is an analytic function $F(z)$ such that $\underline{F'(z) = f(z)}$ in D.

$$\int_{z_0}^{z_1} \underline{f(z)}\, dz = F(z_1) - F(z_0).$$

Exp

$$\int_{0}^{1+i} z^2\, dz = \left[\frac{z^3}{3}\right]_0^{1+i}$$

$$= \frac{(1+i)^3}{3} - 0$$

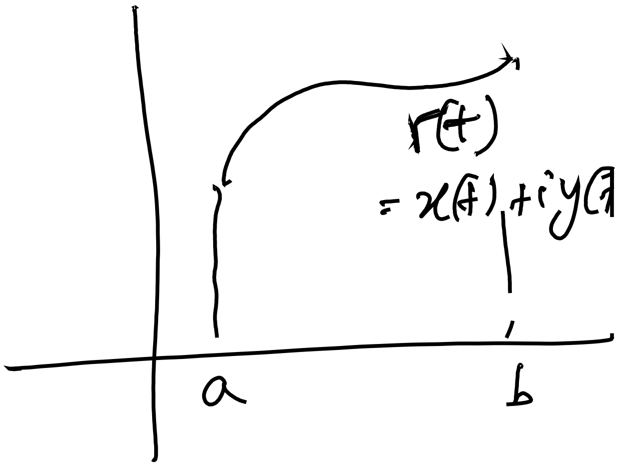
$$= -\frac{2}{3} + \frac{2}{3} i$$

Integration by use of path

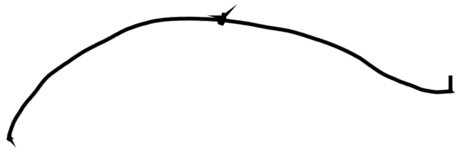
Let C be a piecewise smooth curve represented by $z = \Gamma(t)$, $a \le t \le b$.

Let $f(z)$ be continuous function on C

then
$$\int_C f(z)\,dz = \int_{t=a}^{t=b} f(\Gamma(t))\,\Gamma'(t)\,dt$$



$$\boxed{\int_C f(z)\,dz = \int_{t=a}^{b} f(\Gamma(t))\,\Gamma'(t)\,dt}$$

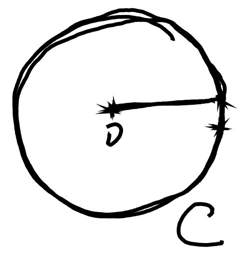


$z = \Gamma(t)$

$dz = \Gamma'(t)\,dt$

Exp

$$\oint_C \frac{dz}{z} \qquad C:\ r(t) = e^{it},\quad 0 \le t \le 2\pi$$



$$= \int_0^{2\pi} f(r(t))\, r'(t)\, dt$$

$$f(z) = \frac{1}{z}$$

$$f(r(t)) = \frac{1}{r(t)} = \frac{1}{e^{it}}$$

$$= \int_0^{2\pi} i\, e^{-it} e^{it}\, dt$$

$$= i \int_0^{2\pi} dt = 2\pi i$$

$$\int_C$$

Exp

$$\oint_C (z - z_0)^m\, dz$$

C: is a circle with center z_0 and radius r.

$$r(t) = z_0 + r e^{it},\quad 0 \le t \le 2\pi$$

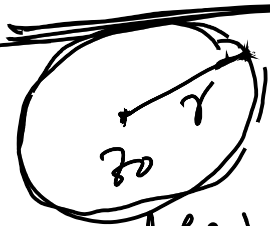
$$= \int_0^{2\pi} f(r(t))\, r'(t)\, dt$$

$$f(z) = (z - z_0)^m$$

$$f(r(t)) = (\cancel{z_0} + r e^{it} - \cancel{z_0})^m$$

$$r'(t) = i r e^{it}$$

$$\boxed{r(t) = z_0 + r e^{it}}$$



$$f(z) = (z - z_0)^m$$

$$\int_0^{2\pi} f(r(t))\, r'(t)\, dt$$

$$f(r(t)) = r^m e^{imt}$$

$$= \int_0^{2\pi} r^m e^{imt} \cdot i r e^{it}\, dt$$

$$= i r^{m+1} \int_0^{2\pi} e^{i(m+1)t}\, dt$$

$$= i r^{m+1} \left[\frac{e^{i(m+1)t}}{i(m+1)} \right]_0^{2\pi}$$

$$= \frac{r^{m+1}}{m+1} \left[e^{i(m+1)2\pi} - e^{i(m+1)\cdot 0} \right]$$

$$= \frac{r^{m+1}}{m+1} [1 - 1] = 0$$

when $m \neq -1$

For $m = -1$

$$\int_C (z - z_0)^{-1} \, dz$$

$$= \boxed{\int_C \frac{1}{z - z_0} \, dz}$$

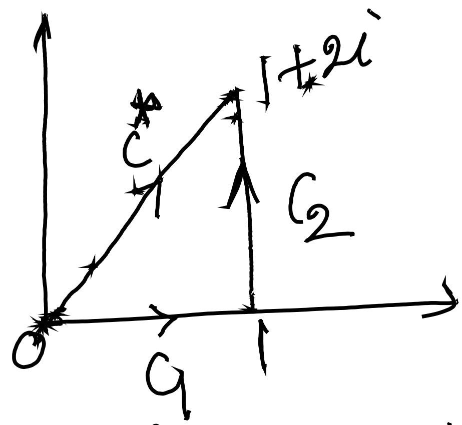
$\gamma(t) = z_0 + re^{it}$

$$= \int_0^{2\pi} \frac{1}{\cancel{z_0} + re^{it} - \cancel{z_0}} \, ire^{it} \, dt$$

$$= i \int_0^{2\pi} dt = 2\pi i$$

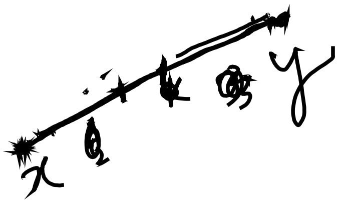
Integral of non analytic function is path dependent.

$$\int_C Re(z)\, dz$$



There are two paths from 0 to $1+2i$

C^* and $C_1 \cup C_2$.



$(1-t)x + ty$

$0 \le t \le 1$

Path C^*

$$\int_{C^*} Re(z)\, dz$$

$$= \int_0^1 f(r(t))\, r'(t)\, dt$$

C^* : $(1-t)\cdot 0 + t(1+2i)$

$0 \le t \le 1$

$f(z) = Re(z)$

$f(r(t)) = Re(r(t))$

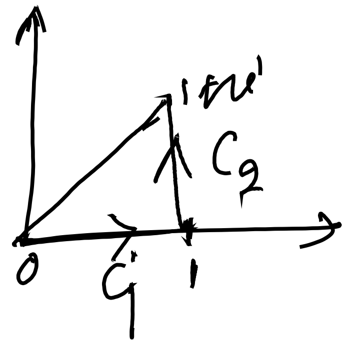
$= t$

$$= \int_0^1 t\,(1+2i)\, dt$$

$$= (1+2i)\left[\frac{t^2}{2}\right]_0^1 = \frac{1+2i}{2}$$

Over C_1

$$\int_{C_1} f(z)\,dz$$



$$= \int_0^1 f(r(t))\,r'(t)\,dt$$

$C_1: \ (1-t)\cdot 0 + t\cdot 1$

$0 \le t \le 1$

$= t$

$r(t) = t$

$r'(t) = 1$

$$= \int_0^1 t\cdot 1\,dt$$

$$= \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$f(z) = \operatorname{Re} z$

$f(r(t)) = \operatorname{Re} t$

$= t$

Over C_2

$$\int_C f(z)\,dz$$

$C_2: \ (1-t)\,1$

$+ t\cdot(1+2i)$

$= 1 - \cancel{t} + \cancel{t} + 2it$

$= 1 + 2it$

$r(t) = 1 + 2it$

$r'(t) = 2i$

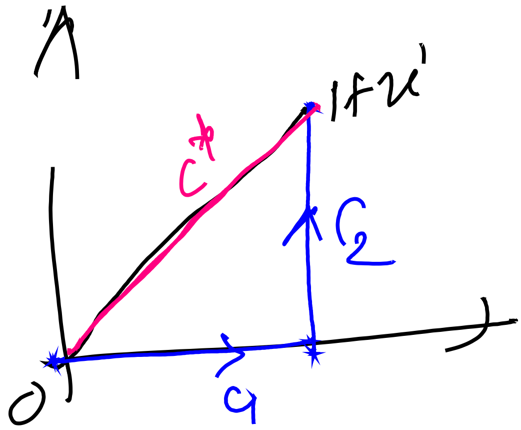
$$= \int_0^1 f(r(t))\,r'(t)\,dt$$

$$= \int_0^1 \operatorname{Re}(r(t))\,r'(t)\,dt$$

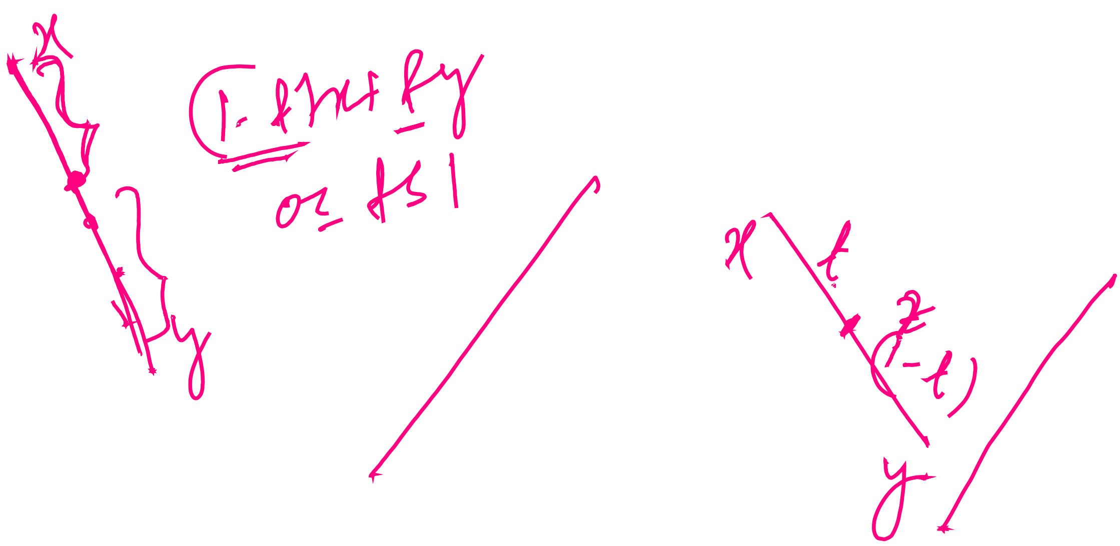
$$= \int_0^1 1\cdot 2i\,dt = 2i$$

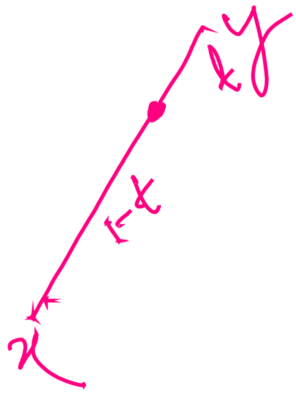
$$\int_{C_1 \cup C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \underline{\underline{\frac{1}{2} + 2i}}$$



$$\int_{C^*} f(z)\,dz = \frac{1+2i}{2}$$




k
y
r-k
x