SC223 - Linear Algebra

Aditya Tatu

Lecture 4

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When does a solution exist?

- Solution to $Ax = b$ exists if and only if b belongs to the set of *all possible linear combinations* of columns of A.
- **Column Space:** The set of all possible linear combinations of columns of A is called the Column space of matrix A, and is denoted by $C(A)$.

$$C(A) = \{Ax \mid \forall x \in \mathbb{R}^n\}, C(A) \subseteq \mathbb{R}^m$$

- *Properties:*
- Let $\mathbf{0}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$. $\mathbf{0}_m \in C(A)$ for any matrix A
- If $b_1, b_2 \in C(A)$, $\forall p, q \in \mathbb{R}, p \cdot b_1 + q \cdot b_2 \in C(A)$.

What about multiple solutions?

- Let $Ax = b$ and $Ay = b$, with $x \neq y$.
- Then, $A(x - y) = \mathbf{0}_m$.
- Similarly, let $z \in \mathbb{R}^n$ be such that $Az = \mathbf{0}_m$. Then, if $Ax = b$, $A(x + z) = b \Rightarrow$ Multiple Solutions!
- **Nullspace:** For a matrix $A \in \mathbb{R}^{m \times n}$, the *Nullspace* is the set of vectors that get mapped to $\mathbf{0}_m$, and is denoted by $N(A)$.

$$N(A) := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}_m\}, N(A) \subseteq \mathbb{R}^n$$

- $Ax = \mathbf{0}_m$ are also called **Homogeneous equations**.
- *Properties:*
- $\mathbf{0}_n \in N(A)$.
- If $x, y \in N(A)$, $\forall p, q \in \mathbb{R}, p \cdot x + q \cdot y \in N(A)$.
- If $\exists z \in N(A), z \neq \mathbf{0}_n$, then $Ax = b$ will have infinitely many solutions, if one exists!

- If there are multiple solutions to $Ax = b \Rightarrow \exists z \neq \mathbf{0}_n$ such that $Az = \mathbf{0}_m$.
- Re-writing, we get $\sum_{i=1}^{n} z_i a_{*i} = \mathbf{0}_m$, with at least one non-zero entry in z, say z_k.

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- **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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\begin{bmatrix} a_{1*}^T \\ a_{2*}^T \\ \vdots \\ a_{m*}^T \end{bmatrix} = \begin{bmatrix} a_{*1} & \dots & a_{*r} \end{bmatrix}_{m \times r} \begin{bmatrix} c_{1*}^T \\ c_{2*}^T \\ \vdots \\ c_{r*}^T \end{bmatrix}_{r \times n}
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