SC223 - Linear Algebra

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Lecture 1

August 2, 2022

What is Linear algebra?

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- It is the study of *structures* that behave like a *line*.

What is a Structure?

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Figure: Even though all of these are different, but we recognize them as having the *structure* of a chair. Image source: freepik.com

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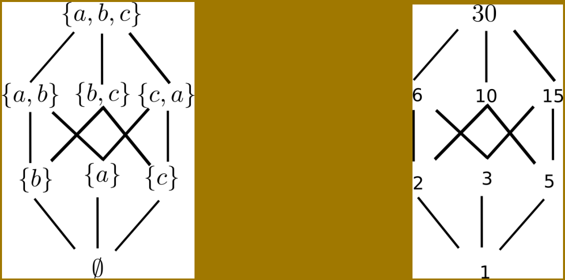
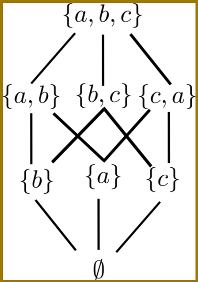
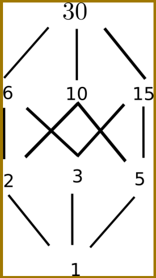


Figure: Hasse Diagram for $(\mathcal{P}(A), \subseteq)$ and $(B, |)$





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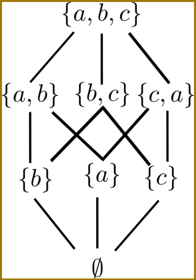

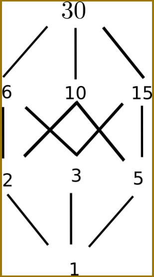

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- They have the structure of a *Poset*.

Linear Algebra

- System of Linear Equations.
- Vector Spaces.
- Linear Transformations and Matrices.
- Eigenvalues and Eigenvectors.
- Inner Products and Norms.
- Complex Vector Spaces.

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- Linear systems in 3 variables: Solve for x, y and z

$$2x + 3y - z = 5$$
$$x - 5y + 2z = 10$$
$$3x + 2y + z = 1$$

Linear Equations

- We will now study them in the form:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -5 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 1 \end{bmatrix}$$

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- In general, solve for x in

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b},$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

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- What can be done if no solution exists?
- What is the Computational cost of the algorithm to solve $Ax = b$.

Row Picture

- Let us look at each row of the system:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Possibilities for a 2×2 system

Possibilities for a 3×3 system

Column Picture

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

can be re-written as

$$x \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix},$$

where $x \cdot \begin{bmatrix} a \\ b \end{bmatrix} := \begin{bmatrix} ax \\ bx \end{bmatrix}$, and $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} := \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.

- In general, for an $m \times n$ system of linear equations $Ax = b$,

$$x_1 \cdot \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{a_{*1} \in \mathbb{R}^n} + x_2 \cdot \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{a_{*2} \in \mathbb{R}^n} + \ldots + x_n \cdot \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{a_{*n} \in \mathbb{R}^n} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- **Linear combination** of a_{*i} and a_{*j} with real numbers x_i and x_j is defined as

$$x_i \cdot \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} + x_j \cdot \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} x_i a_{1i} + x_j a_{1j} \\ x_i a_{2i} + x_j a_{2j} \\ \vdots \\ x_i a_{mi} + x_j a_{mj} \end{bmatrix}$$

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- If $\exists z \in N(A), z \neq \mathbf{0}_n$, then $Ax = b$ will have infinitely many solutions, if one exists!