SC223 - Linear Algebra

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Lecture 12

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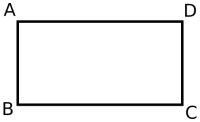
Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.

Example

- Symmetry group of a rectangle.
  - What all can we do to the rectangle so that we get the same (not necessarily same vertices) rectangle?



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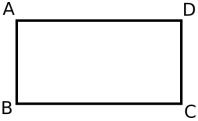

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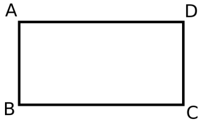

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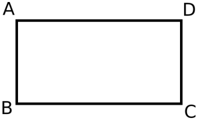

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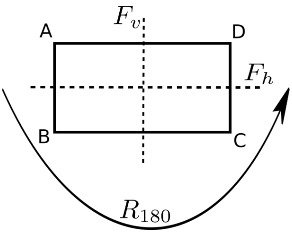

Figure: Symmetries of a rectangle

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- Rotate by 180 deg: R_{180}
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- Flip along horizontal axis: F_h.

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R_{180}	R_{180}	I	F_v	F_h
F_h	F_h	F_v	I	R_{180}
F_v	F_v	F_h	R_{180}	I

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- $\circ$ is associative.
- $\circ$ is commutative. Thus $(S, \circ)$ forms an Abelian group.

- Consider the set $S' = 00, 01, 10, 11$ with bitwise addition modulo-2 operation: $+_2$.

$+_2$	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

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- Both are examples of the *Klein*-4 group.

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- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$,

and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.

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- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
- **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

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  - $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The mutiplicative identity will be denoted by 1.
  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c, a \times (b +_F c) = a \times b +_F a \times c$

Examples of Fields

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- $(\mathbb{R}[x], +, \times)$, where $\mathbb{R}[x]$ is the set of all rational polynomials of the form $\frac{p(x)}{q(x)}$, with $q \neq 0$, and p and q are polynomials in one variable with real coefficients.

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- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ forms a vector space over $\mathbb{F}$.

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● Any element of the vector space $(V, +, \cdot)$ will be referred to as a **vector**, and any element $a \in \mathbb{F}$ will be referred to as a **scalar**.

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- $(\mathbb{R}^{\mathbb{Z}}, +, \cdot)$ over $\mathbb{R}$, where $\mathbb{R}^{\mathbb{Z}}$ is the set of all doubly-infinite sequences.

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- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.

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- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.
- $(\mathbb{L}_2(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathbb{L}_2(\mathbb{R})$ denotes the set of all square-integrable functions $f : \mathbb{R} \to \mathbb{R}$.