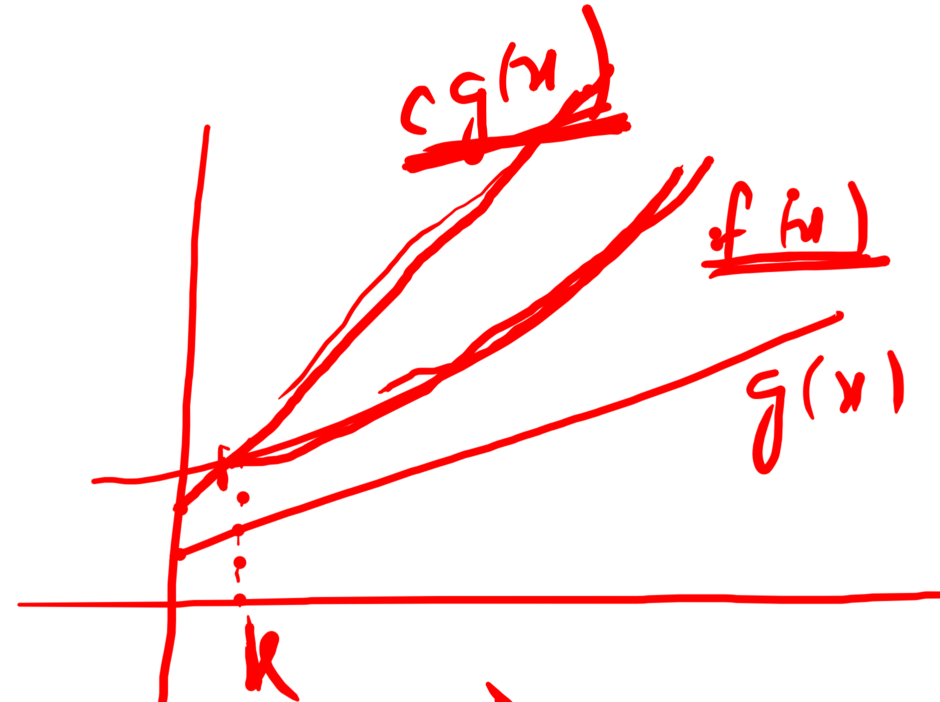
The Growth of functions

Def^n $f, g : \mathbb{Z} \to \mathbb{R}$

We say $f(x)$ is $O(g(x))$ if $x > k$

there exists constants c and k s.t.

$$|f(x)| \leq c\,|g(x)| \quad \text{for all } x > k$$



Show that

Exp $f(x) = \underbrace{x^2 + 2x + 1}_{f(x)}$ is $O(\underset{\sim}{x^2})$

$g(x) = x^2$

$f(x)$ is $O(g(x))$

$f(x) = \underline{x^2 + 2x + 1} \leq x^2 + 2x^2 + x^2 = 4x^2$

for $x > 1$

$1 < x$

$x^2 + 2x + 1 = O(x^2)$

$f(x)$ is $O(g(x))$

$\underbrace{x^2 + 2x + 1}_{f(x)} \leq \underbrace{4}_{c}\underbrace{x^2}_{g(x)}$

$x > \frac{1}{k}$

$$f(x) = \underbrace{x^2+2x+1}_{f(x)} \le x^2+x^2+x^2 = \underset{\sim}{3}\,\underset{\sim}{x^2}$$

$\underset{c}{}\ \underset{g(x)}{}$

for $x > \underset{k}{\underline{2}}$ $\quad \underline{2x \le x^2}$, $\underline{1 \le x^2}$

for $\underbrace{x^2+2x+1}_{f(x)} \le \underset{c}{\underset{\sim}{3}}\ \underset{g(x)}{\widetilde{x^2}}$ for $x > \underset{k}{\underset{\sim}{2}}$

Exp show that $7x^2$ is $O(x^3)$

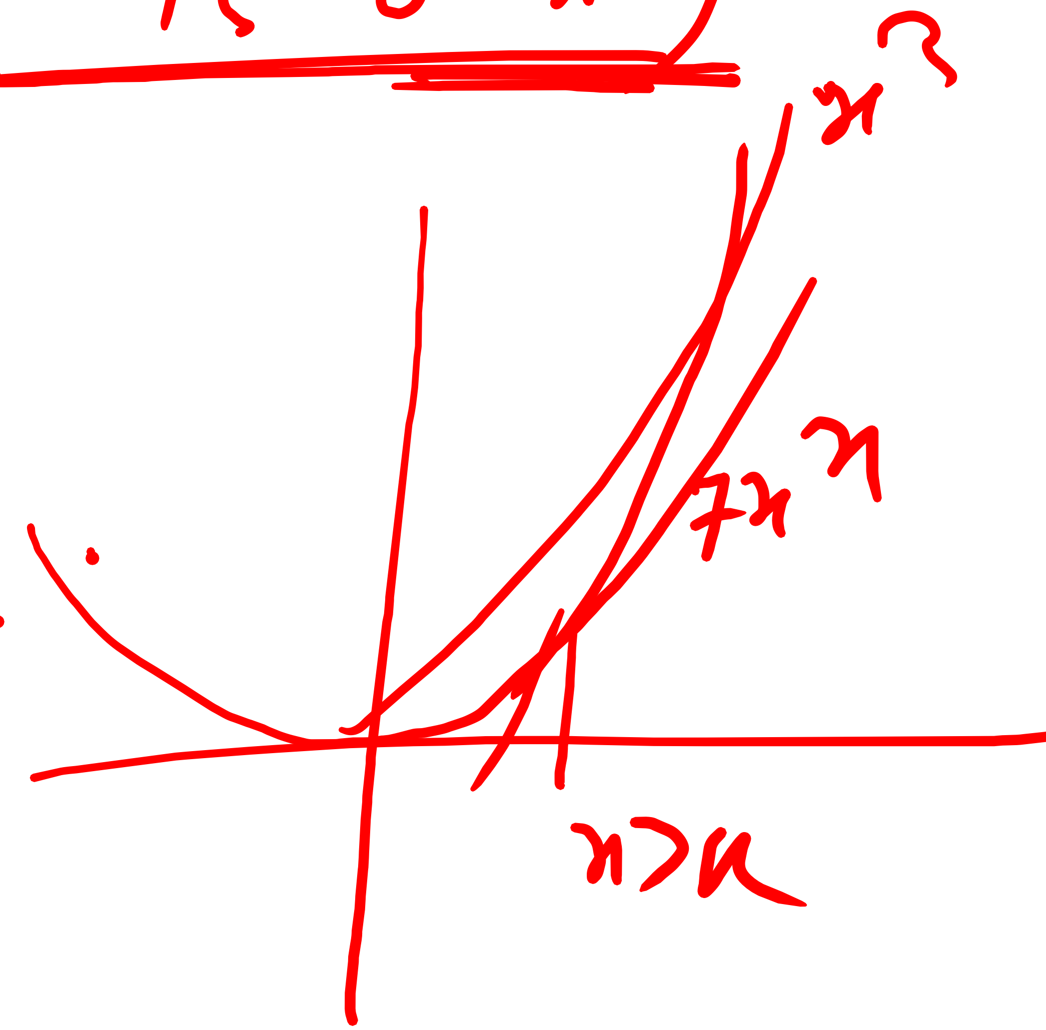
$7x^2$ is $O(x^2)$

$7x^2 < 1\cdot x^3$ for $\underbrace{x > 7}_{k}$

$\le 1\cdot x^3$

$7x^2 = O(x^3)$

$\underbrace{7x^2}_{f(x)} < \underbrace{7}_{c}\,\underbrace{x^3}_{g(x)}$ $\qquad x > \underbrace{1}_{k}$



$\therefore$ $7x^2$ is $O(x^3)$

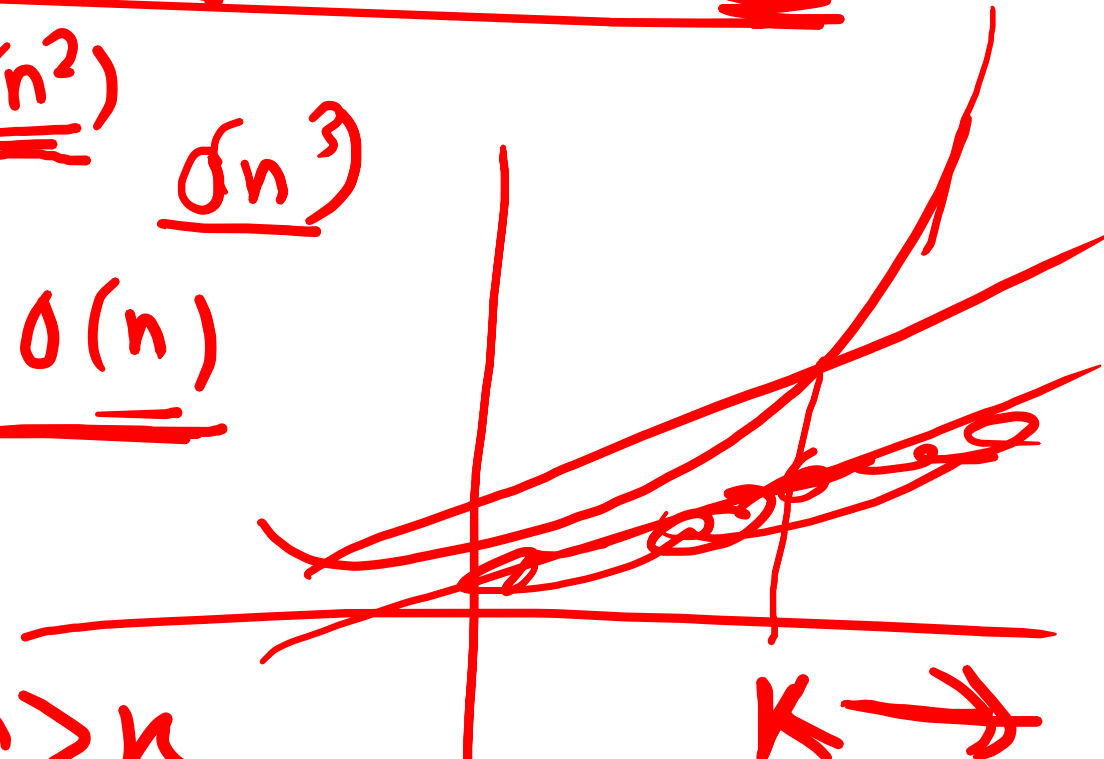
Exp Show that $8n^2+2n-3$ is not $O(n)$

$8n^2+2n-3$ is $O(n^2)$

$O(n^3)$

Suppose

$8n^2+2n-3$ is $O(n)$



$\Rightarrow \exists\ c, k$ s.t.

$8n^2+2n-3 \le c\,n$ for $n > k$

$8n + 2 - \frac{3}{n} \le C$

$\Rightarrow \lim_{n\to\infty}\left(8n+2-\frac{3}{n}\right) \le C$

$\infty \le c$

Exp $7x^2$ is $O(x^3)$

Is it true that x^3 is $O(7x^2)$

suppose x^3 is $O(7x^2)$

$\Rightarrow \exists\ C$ and k s.t. $x^3 \le C 7x^2$ for $x > k$

$\Rightarrow x \le 7C$ $x > k$

x^3 is not $O(7x^2)$.

$\underline{\text{Th}^m}$ $f(x) = \underline{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$

where $\underline{a_0}, \underline{a_1}, \ldots \underline{a_{n-1}}, \underline{a_n}$ are $\underline{\text{real numbers}}$

Then $f(x)$ is $O(\underbrace{x^n}_{g(x)})$.

$\underline{\text{Pf}}$ $\underline{x > 1}$

$|f(x)| = |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0|$

$\leq |a_n x^n| + |a_{n-1} x^{n-1}| + \cdots + |a_1 x| + |a_0|$

$= |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0|$

$= x^n \left(\underline{|a_n| + \frac{|a_{n-1}|}{x}} + \cdots + \frac{|a_1|}{x^{n-1}} + \frac{a_0}{x^n} \right)$

$$x > 1$$
$$K$$

$$|f(x)| \le x^n\left(|a_n| + \frac{|a_{n-1}|}{x} + \frac{|a_{n-2}|}{x^2} \cdots + \frac{|a_1|}{x^{n-1}} + \frac{|a_0|}{x^n}\right)$$

$$\le x^n \underbrace{\left(|a_n| + |a_{n-1}| + |a_{n-2}| + |a_1| + |a_0|\right)}_{c}$$

$$= c\, x^n$$

$$x > 1 \quad = c$$

when

$c = (|a_n| + |a_{n-1}| + \dots$

$\dots + |a_1| + a_0)$

$$x^2 > 1 \qquad \frac{|a_{n-2}|}{x^2} < |a_{n-2}|$$

$$\frac{1}{x^2} < 1$$

$$x > 1 \qquad \frac{1}{x} < 1$$

$$\frac{|a_{n-1}|}{x} < |a_{n-1}|$$

$$\underbrace{|f(x)| \le c\, x^n = c|g(x)|}$$

$f(x) = \underbrace{x^5} + x^4 + 1 \qquad O(x^5)$

<u>Exp</u> big O estimate of $n!$

$n! = 1 \cdot 2 \cdot 3 \cdots n$

$\leq \underbrace{n \cdot n \cdot n \cdots n}_{n \text{ term}}$

$= n^n = 1 \cdot n^n$

$n!$ is $O(n^n)$

$n > 1$

$\log n! = \log n^n = 1 \cdot n \log n$

$\log n!$ is $O(n \log n)$

$n > 1$

$n < 2^n$ $\quad$ n is $O(2^n)$

$\log n < n$ $\quad$ $\log n$ is $O(n)$

The growth of Combinations of functions

Thm Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ Then $(f_1+f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$

Pf $f_1(x)$ is $O\,g_1(x)$

$\Rightarrow \exists\ c_1, k_1$ s.t.

$|f_1(x)| \le c_1|g_1(x)|$ $\quad x > k_1$

$f_2(x)$ is $O(g_2(x))$

$\exists\ c_2, k_2$ s.t.

$|f_2(x)| \le c_2|g_2(x)|$

$x > k_2$

$$\boxed{|(f_1+f_2)(x)|} = |f_1(x) + f_2(x)|$$

$$\leq |f_1(x)| + |f_2(x)|$$

$$\leq c_1 \underline{|g_1(x)|} + c_2 \underline{|g_2(x)|}$$

$$\leq c_1 \underbrace{\max(g_1(x), g_2(x))}_{g(x)} + c_2 \underline{\max(g_1(x), g_2(x))}$$

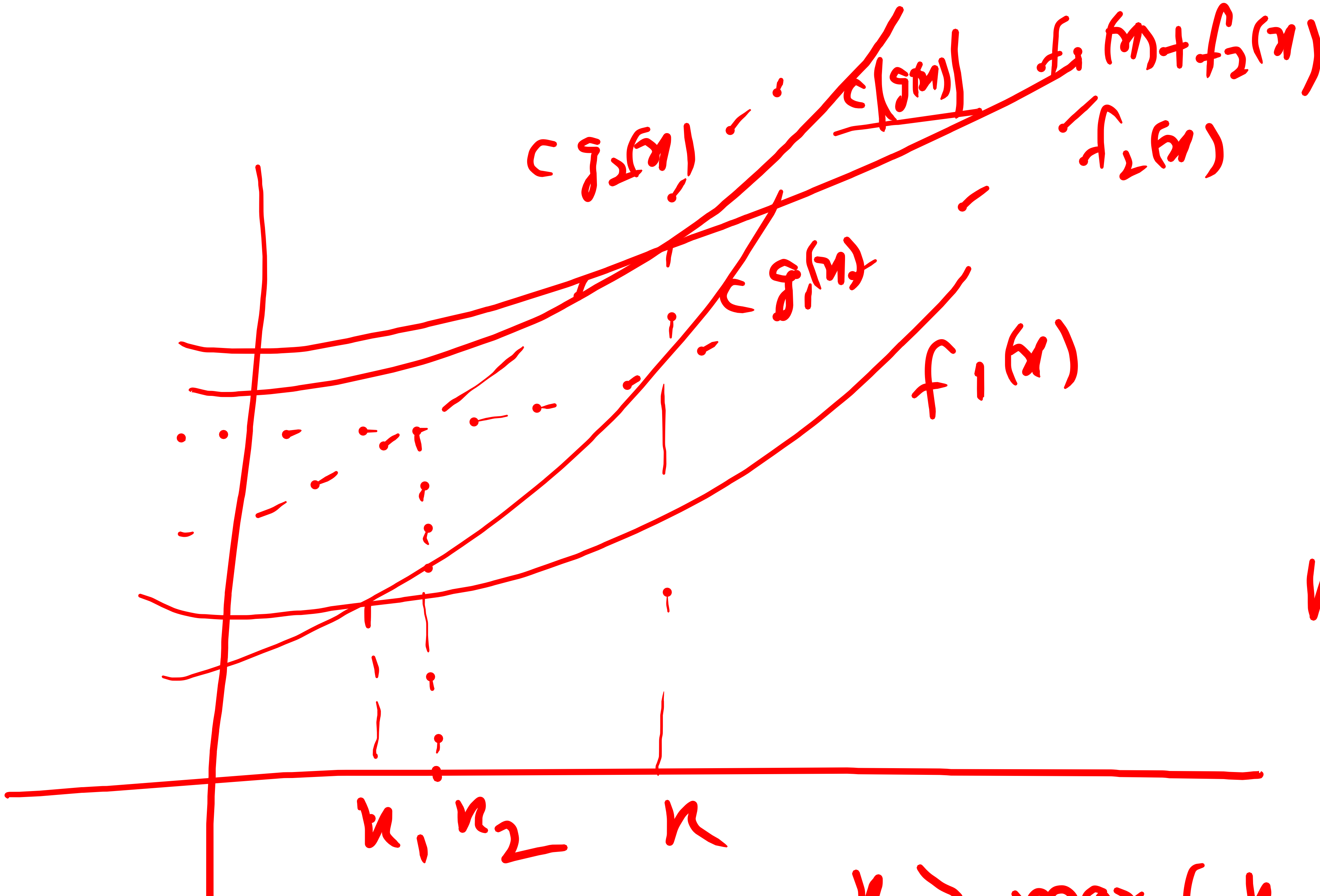
$$\underline{|g(x)| = \max(|g_1(x)|, |g_2(x)|)}$$

$$= c_1 |g(x)| + c_2 |g(x)|$$

$x > k_1$

$\underline{x} > \underline{k_2}$

$$= (c_1 + c_2)|g(x)| = \underline{c\,|g(x)|} \quad \underline{x} > \underline{\max(k_1, k_2)}$$



$x > k$

$k = \max(k_1, k_2)$

$\cancel{\min(k_1, k_2)}$

$k > \max(k_1, k_2)$

Th^m Suppose that $f_1(x)$ is $O(g_1(x))$

$f_2(x)$ is $O(g_2(x))$

$(f_1 f_2)(x)$ is $O(g_1(x) g_2(x))$

$x > k_1$

$x > k_2$

Pf $|(f_1 f_2)(x)| = |f_1(x) \cdot f_2(x)|$

$\leq c_1 |g_1(x)| \; c_2 |g_2(x)|$

$\leq c_1 c_2 |g_1(x)| \; |g_2(x)|$

$\leq c \; |g_1(x) g_2(x)|$

$c = c_1 c_2$

$x > \max(k_1, k_2)$

Exp

$$f(n) = \underbrace{3n \log(n!)}_{f_1(n)} + \underbrace{(n^2+3)\log n}_{f_2(n)}$$

$O(\_)$ $O(\_)$

$O(\max(\_ , \_))$

$3n \log n! + (n^2+3)\log n$

$= O(\underline{\max}(n^2\log n, n^2 \log n))$

$= O(n^2 \log n)$

$\log n$ $O(n)$

$3n$ is $O(n)$ $\log(n!)$ is $O(n \log n)$

$\underline{3n \log n!} = O(n \cdot n\log n) = \underline{O(n^2 \log n)}$

c, k

n^2+3 is $O(n^2)$ $\log n$ is $O(\log n)$

$\underline{(n^2+3)\log n} = O(n^2 \cdot \log n) = \underline{O(n^2 \log n)}$

Exp

$$f(x) = \underbrace{\underbrace{(x+1)}\ \underbrace{\log(x^2+1)}}_{O(x\log x)} + \underbrace{3x^2}_{O(x^2)}$$

$x+1$ is $O(x)$

$x^2+1 < 2x^2$ when $x > 1$

$\log(x^2+1) < \log(2x^2) = \log 2 + \log(x^2) = \log 2 + 2\log x$

$= O(\log x)$

$(x+1)\log(x^2+1) = O(x \cdot \log x)$

$= O(x\log x)$

$3x^2$ is $O(x^2)$

$x > 2$

$(x+1)\log(x^2+1) + 3x^2 = O($ $= O(x^2)$

$= O(\max(x\log x, x^2))$

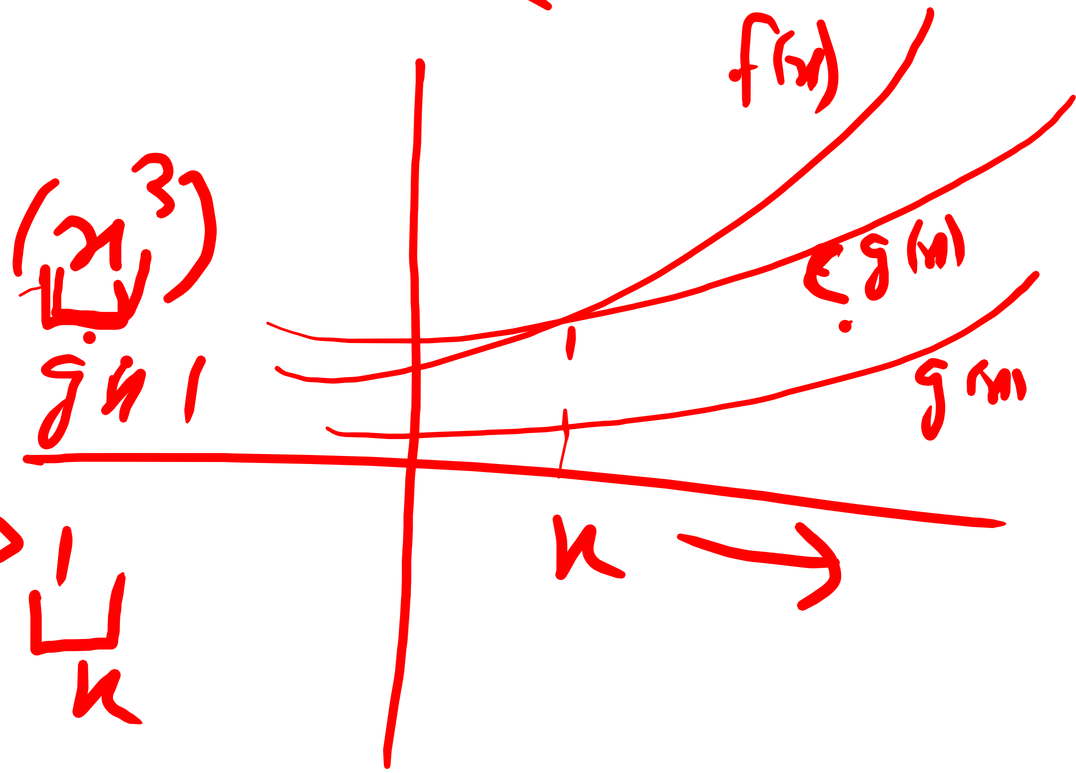
Defn $f, g : \mathbb{Z} \to \mathbb{R}$

$f(x)$ is $\Omega(g(x))$ if

$|f(x)| \geq c\,|g(x)|$ for $x > k$

Exp $\underbrace{8x^3 + 5x^2 + 7}_{f(x)}$ is $\Omega(\underbrace{x^3}_{g(x)})$

$\underbrace{8x^3 + 5x^2 + 7}_{f(x)} \geq \underbrace{8}_{c}\,\underbrace{x^3}_{g(x)}$ $x > \underbrace{1}_{k}$



Defn $f, g : \mathbb{Z} \to \mathbb{R}$

$f(x)$ is $\Theta(g(x))$ iff

$f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$

Exp

$f(n) = 1+2+\cdots+n$ is $\theta(n^2)$

$1+2\cdots+n$ is $O(n^2)$

$1+2\cdots+n$ is $\Omega(n^2)$

$1+2+\cdots+n \le \underbrace{n+n\cdots+n}_{n\ \text{time}} = n.n$ $\quad n>1$

$= n^2$ $\quad \overline{}$ K

$\underbrace{1+2+\cdot+n}_{f(n)} = O(n^2)$

$$f(n) \geq c\, g(n)$$

$n = 9$

$$\underbrace{1 + 2 + \cdots + n}_{f(n)}$$

$f(n)$ is $\Omega(g(n))$

$\frac{n}{2} + 1 = \frac{9}{2} + 1 = 5.5$

$4.5 + 1 = 5.5$

$$= 1 + 2 + \cdots + \frac{n}{2} + \underbrace{\left(\frac{n}{2}+1\right) + \cdots + n}_{n/2 \text{ term}}$$

$\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{9}{2} \right\rceil = \lceil 4.5 \rceil = 5$

$$\geq \underbrace{\left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \cdots + \left\lceil \frac{n}{2} \right\rceil}_{n/2 \text{ term}}$$

$n = 8$

$\frac{n}{2} + 1 = 5$

$$= \frac{n}{2} \cdot \left\lceil \frac{n}{2} \right\rceil \geq \frac{n}{2} \cdot \frac{n}{2} = \underbrace{\frac{1}{4}}_{c} \underbrace{n^2}_{g(n)}$$

$\left\lceil \frac{n}{2} \right\rceil = 4$

$n > \frac{1}{n}$

$1 + 2 \cdots n$

$= \Theta\, n^2$

$\underline{Tn^{m}}$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Θ $\underline{O(x^n)}$ $\underline{\underline{\Omega(x^n)}}$

$$\underline{f(x)} = \underline{\Theta(x^n)}$$