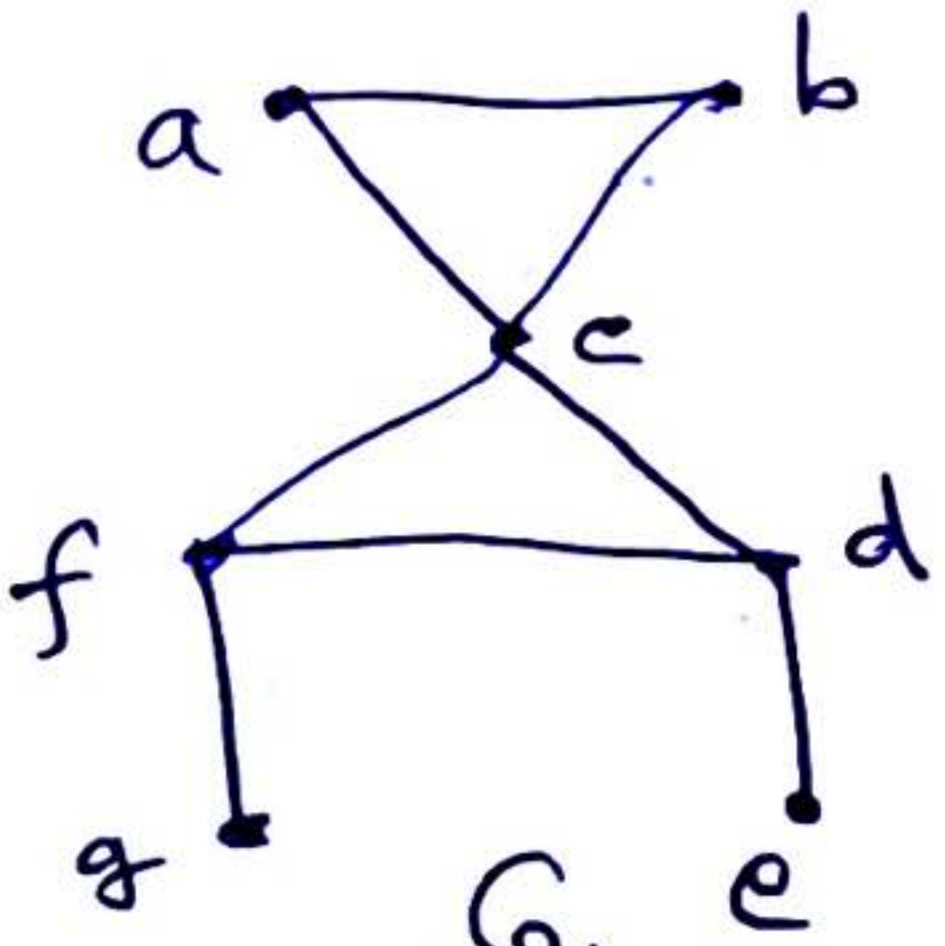
For directed graph G similar def.

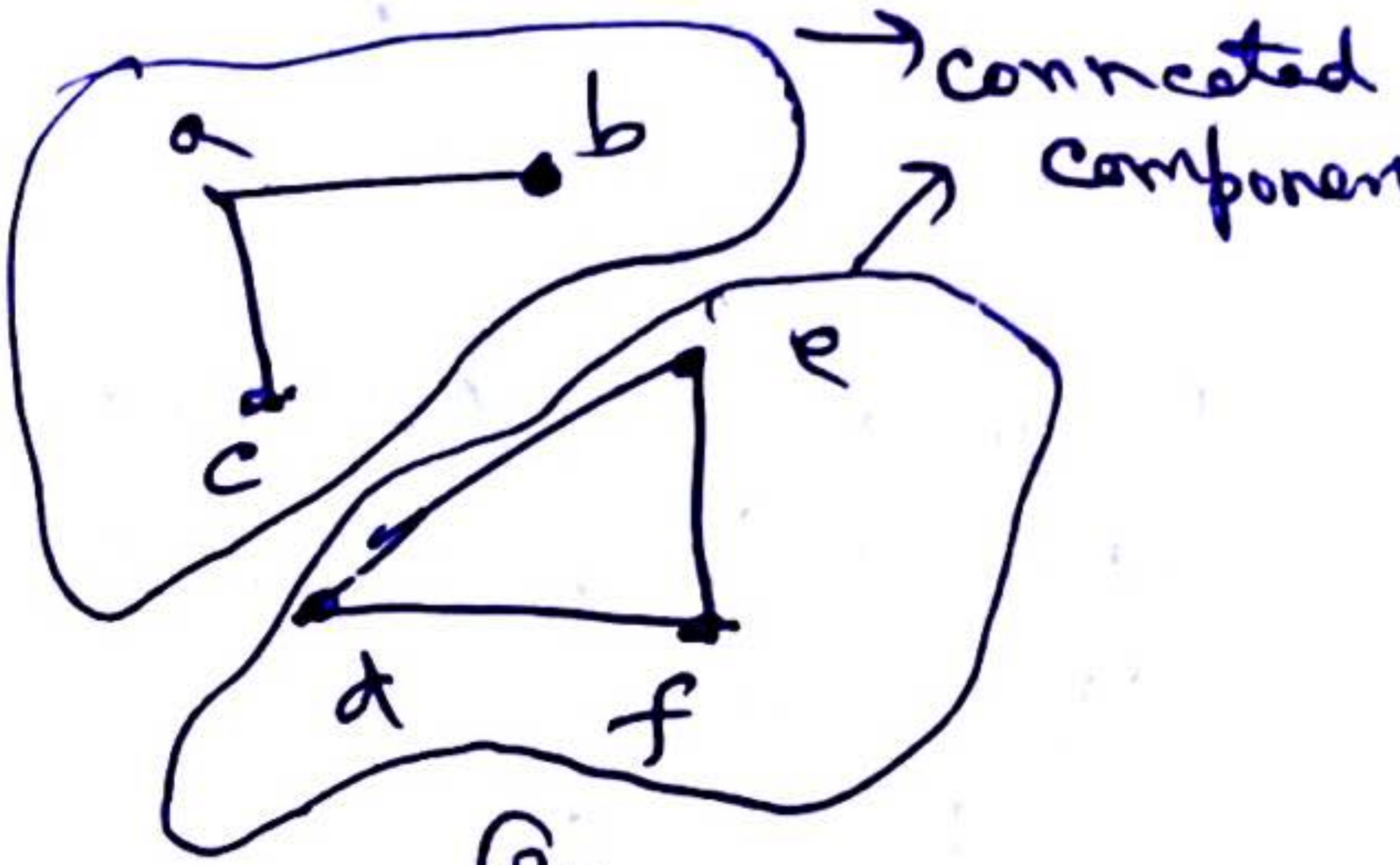
— An undirected graph is called connected iff there is a path between every pair of distinct vertices of the graph.

Example



G_1

connected graph



G_2

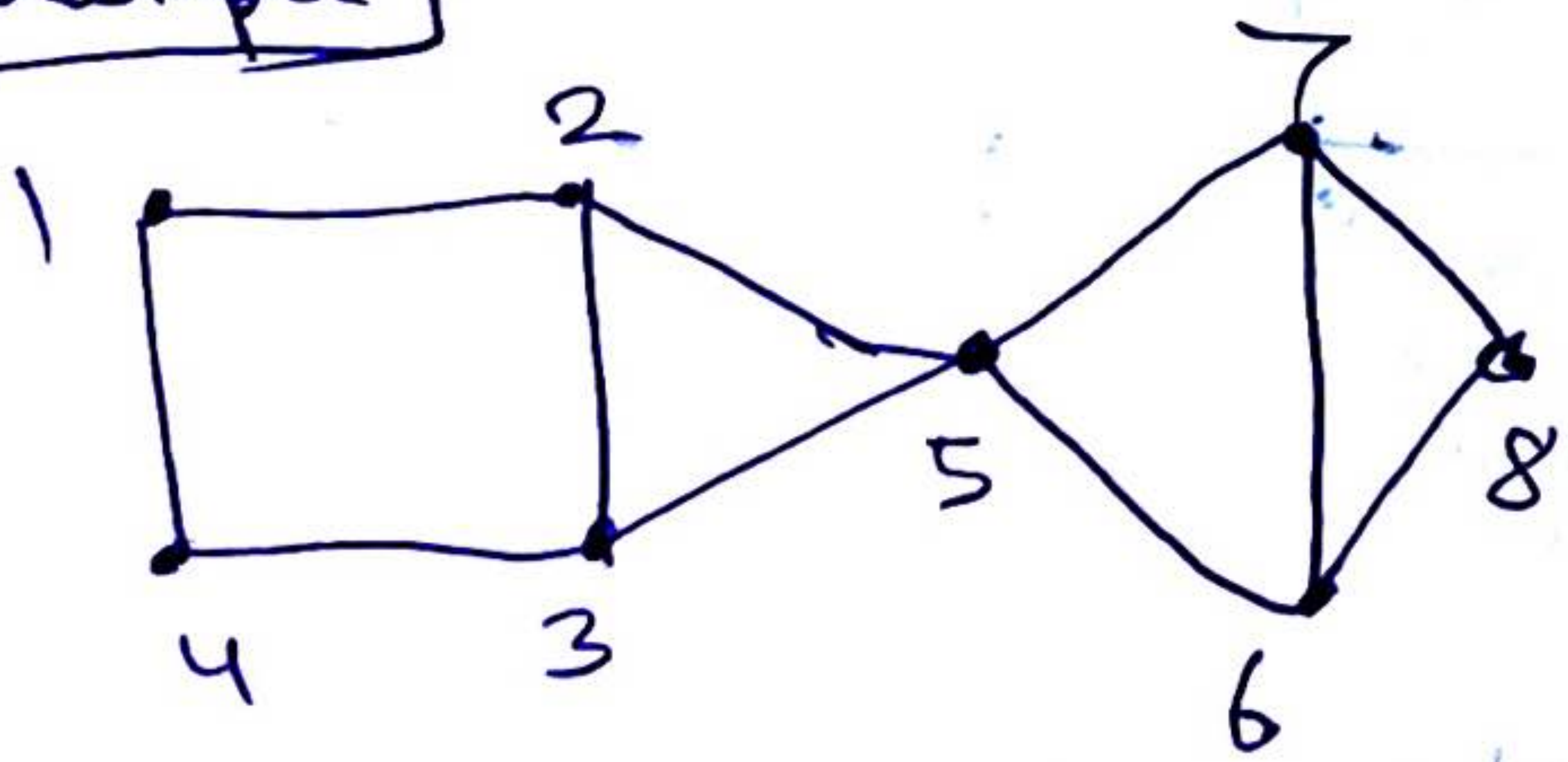
disconnected graph

Theorem There is a simple path between every pair of distinct vertices of a connected undirected graph.

~~Cut edge or bridge~~

— In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected provided removal of no proper subset of these edges disconnects G.

Example

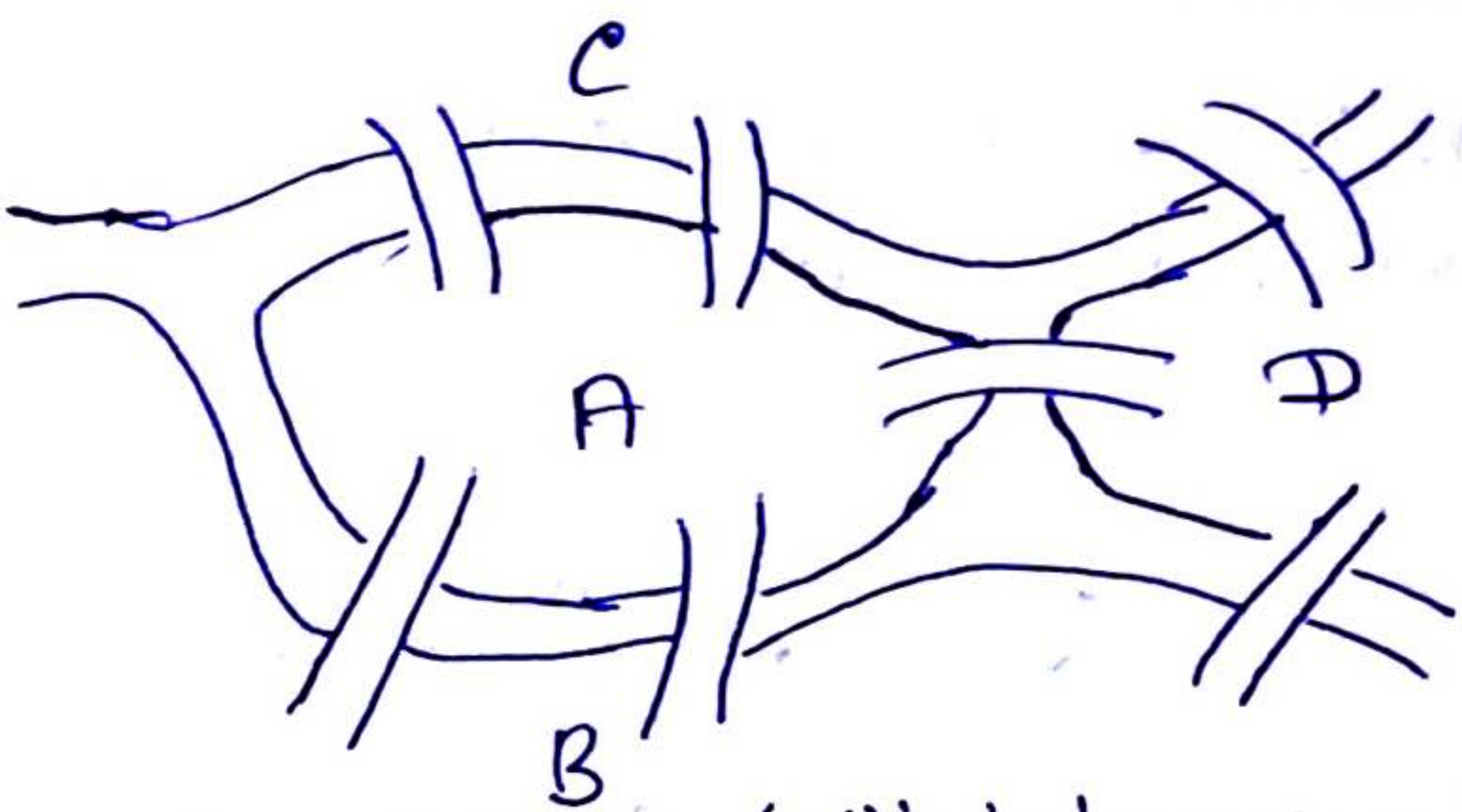


$\{(1,2), (2,3), (3,5), (2,5)\}$ is not cut set.

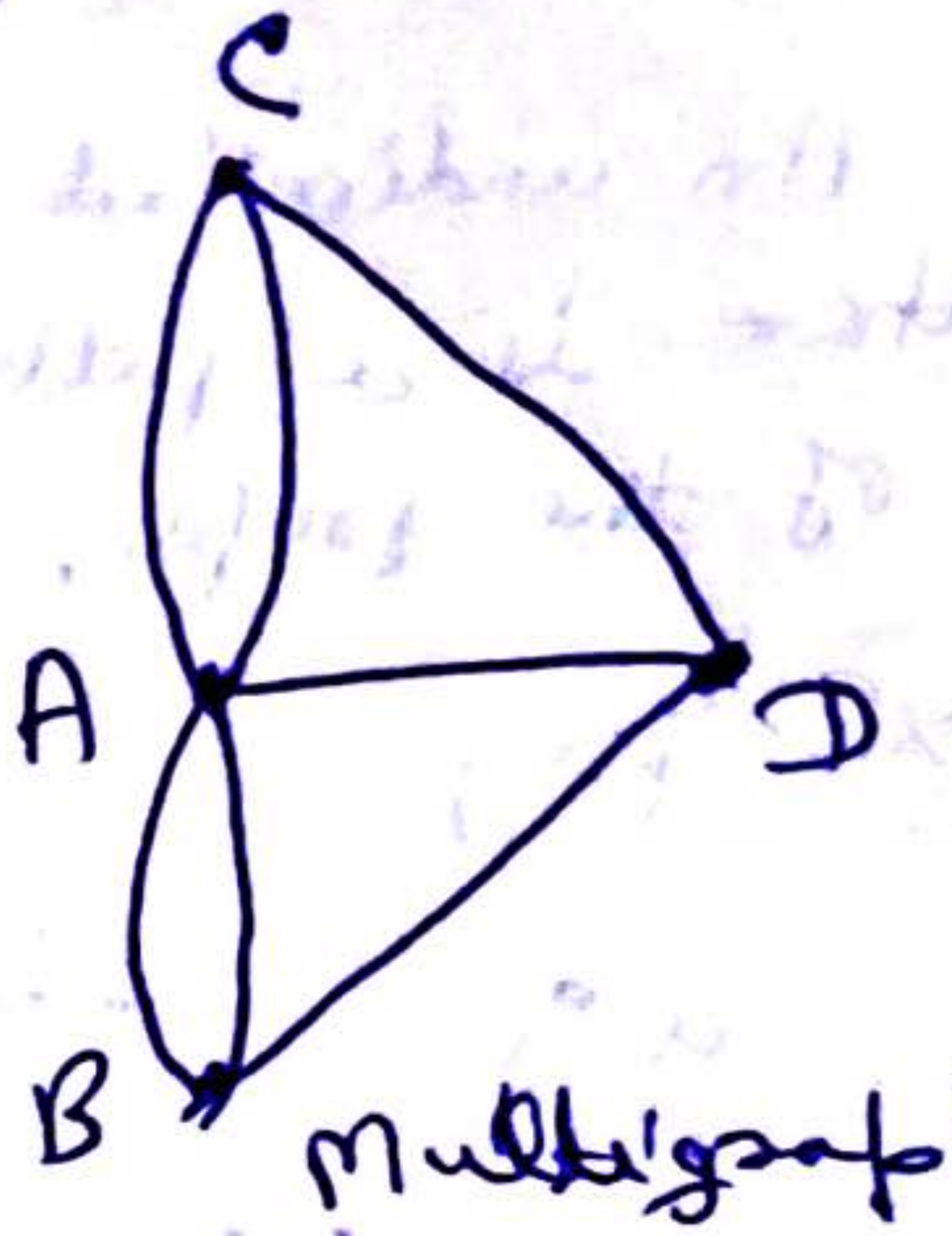
$\{(2,5), (3,5)\}$ is a cut set

$\{(1,2), (2,3), (3,5)\}$ is also a cut set

Euler & Hamilton Paths



Pregel River (Königsberg, Russia)



Multigraph

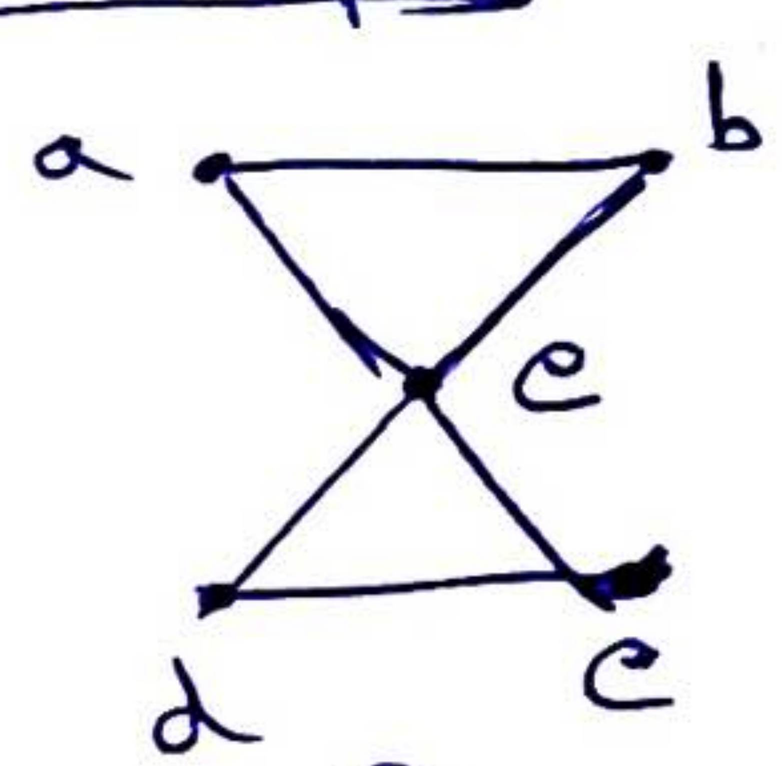
Whether it is possible to start at some location in town travel across all the bridges without crossing any bridge twice and return to the starting point?

OR

Is there a simple circuit in the multigraph that contains every edge? (Euler solved this)

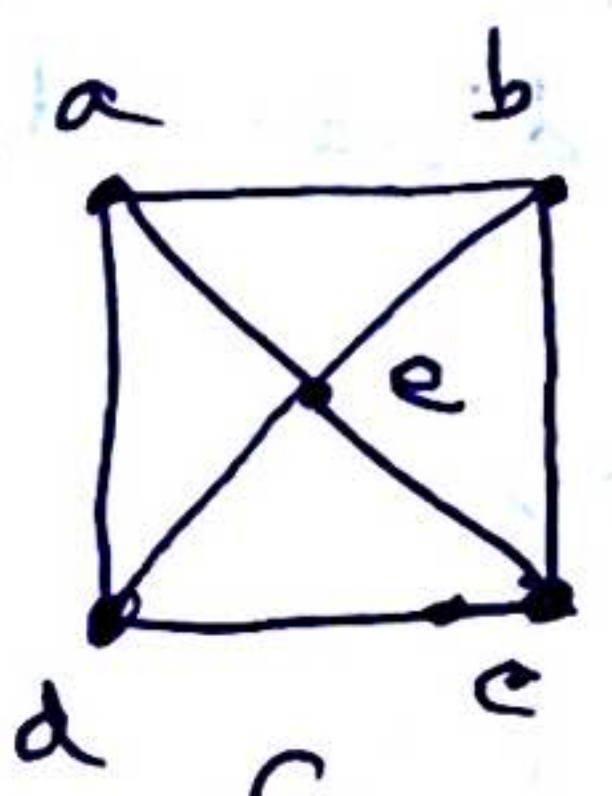
— An Euler circuit (E.C.) in a graph G is a simple circuit containing every edge of G. An Euler path (E.P.) in G is a simple path containing every edge of G.

Example



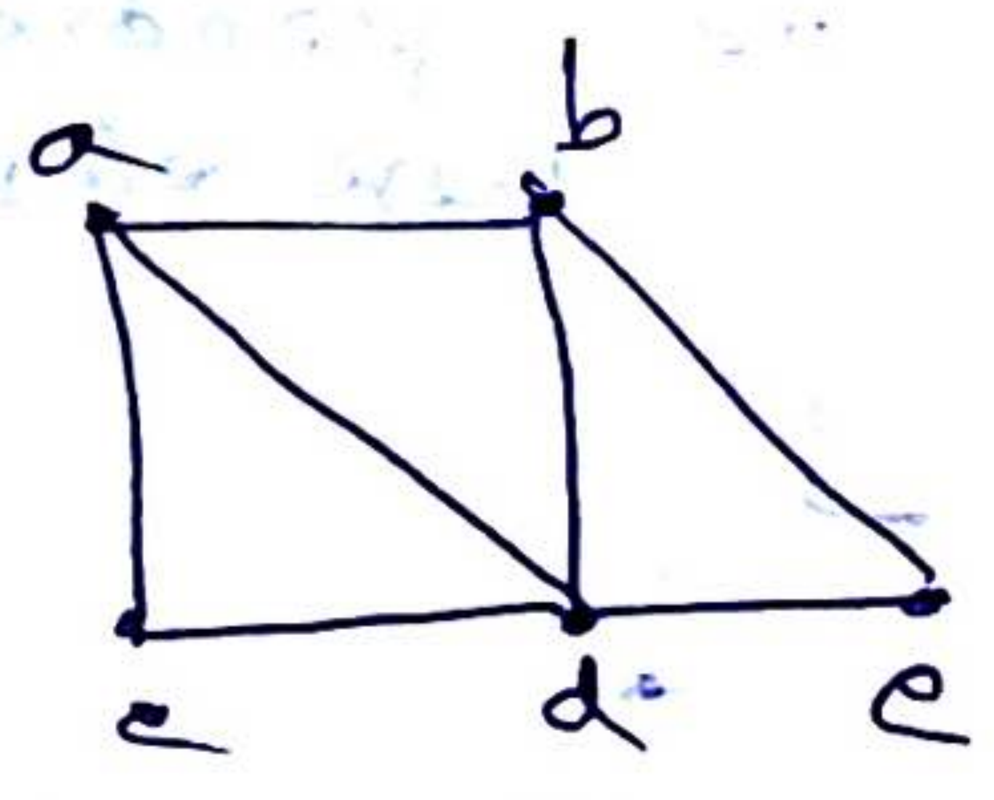
G_1

a e c d e b a
E.C.



G_2

has no E.C.



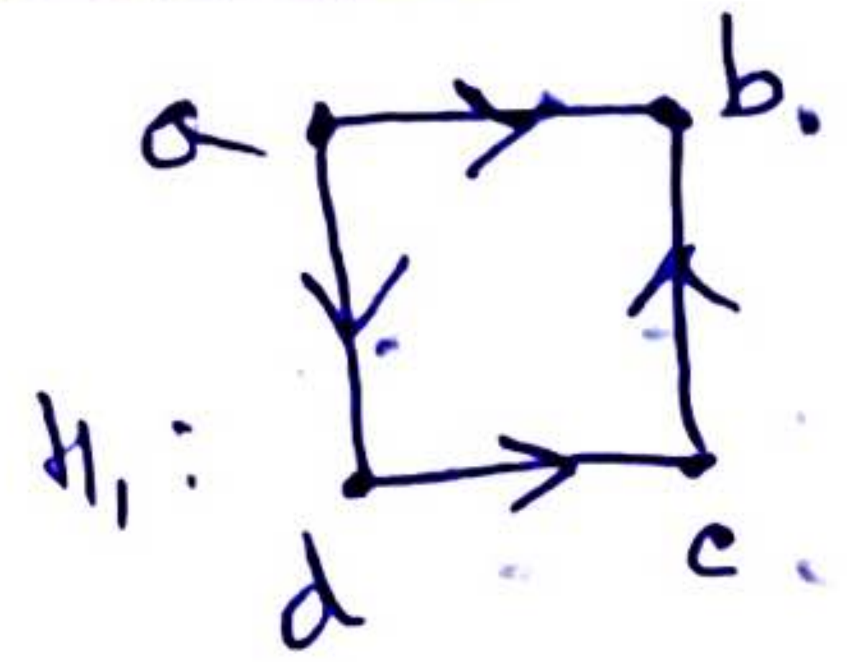
G_3

has no E.C.

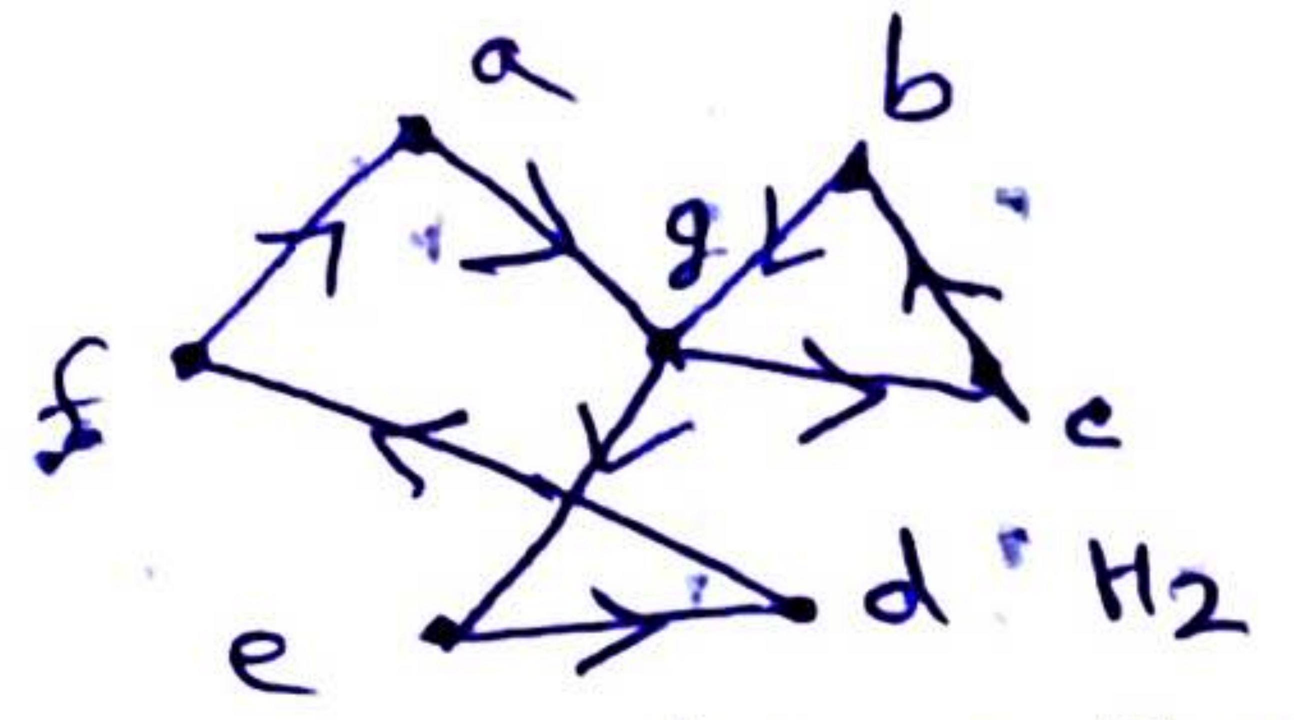
G_3 has an E.P.

a, c, d, e, b, d, a, b

Example

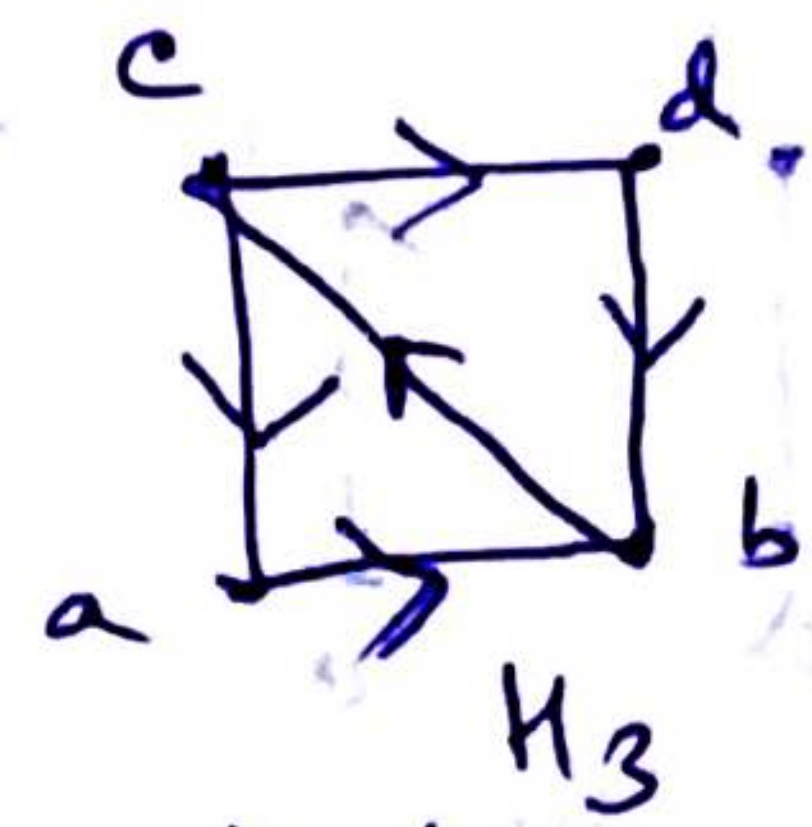


H₁ has no E.C.



H_2 has an E.C.

a, g, c, b, g, e, d, f, a



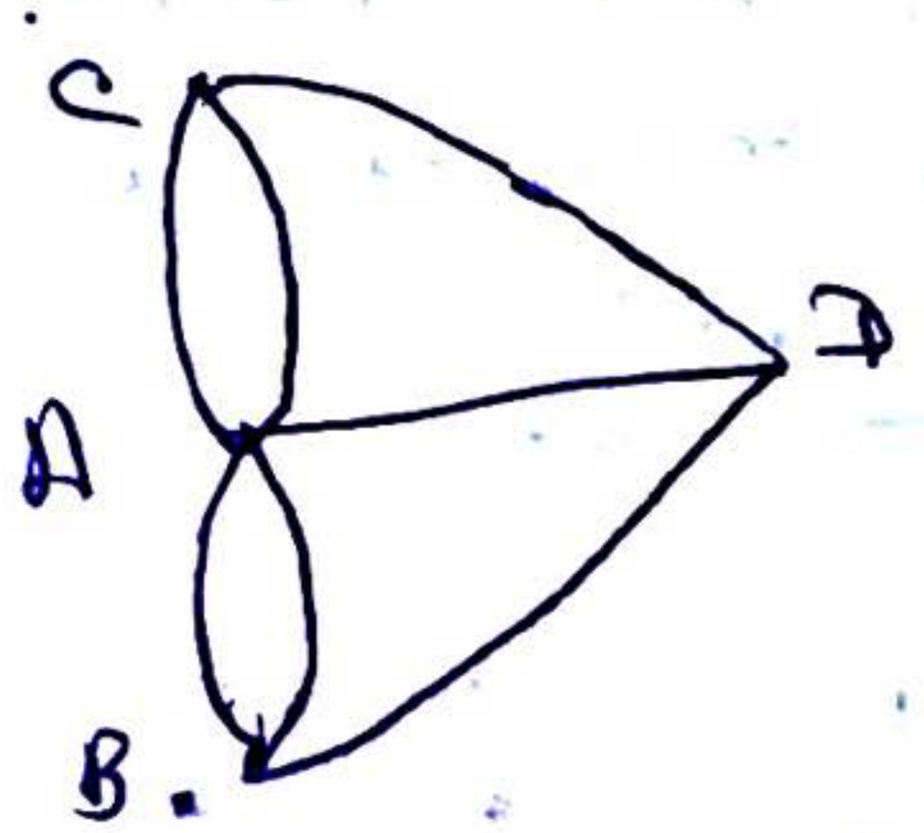
H_3 has no E.C.
H_3 has an E.P.
c, a, b, c, d, b

Theorem: A connected multigraph with at least 2 vertices has an E.C. iff each of its vertices has even deg.

Proof: If G has an E.C. that starts at vertex a and ends at a $\Rightarrow$ deg(a) = even. Now any other deg is also even as every time circuit passes through the deg it contributes 2 to vertex deg. Thus $\forall$ a vertex $a \in G$ deg a = even.

On the other hand one can construct an E.C. in a graph G of even deg. (Verify!) ▨

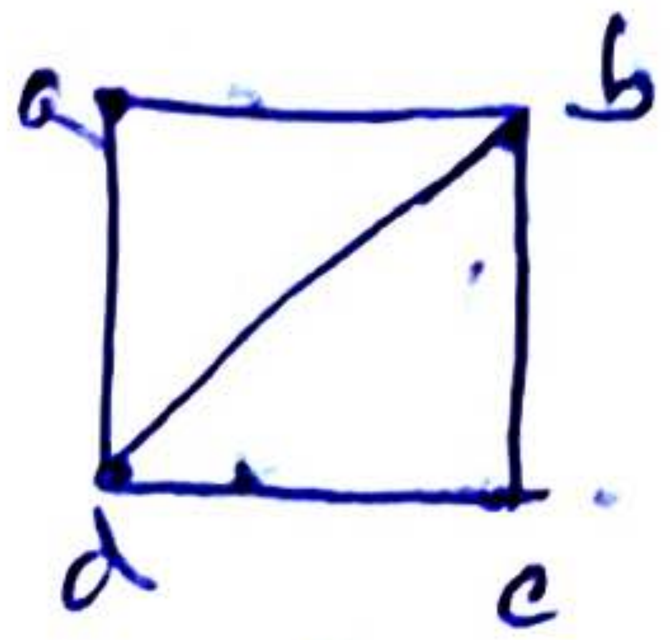
Example



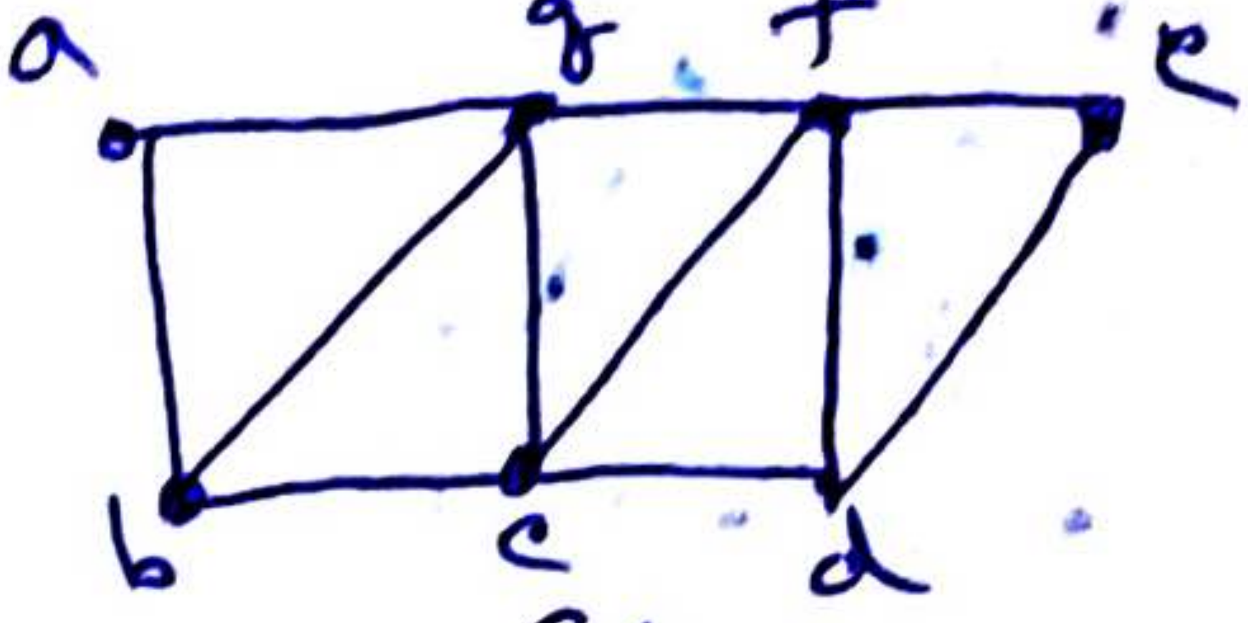
This graph has no E.C.

Theorem A connected multigraph has an E.P. but not an E.C. iff it has exactly two vertices of odd degrees.

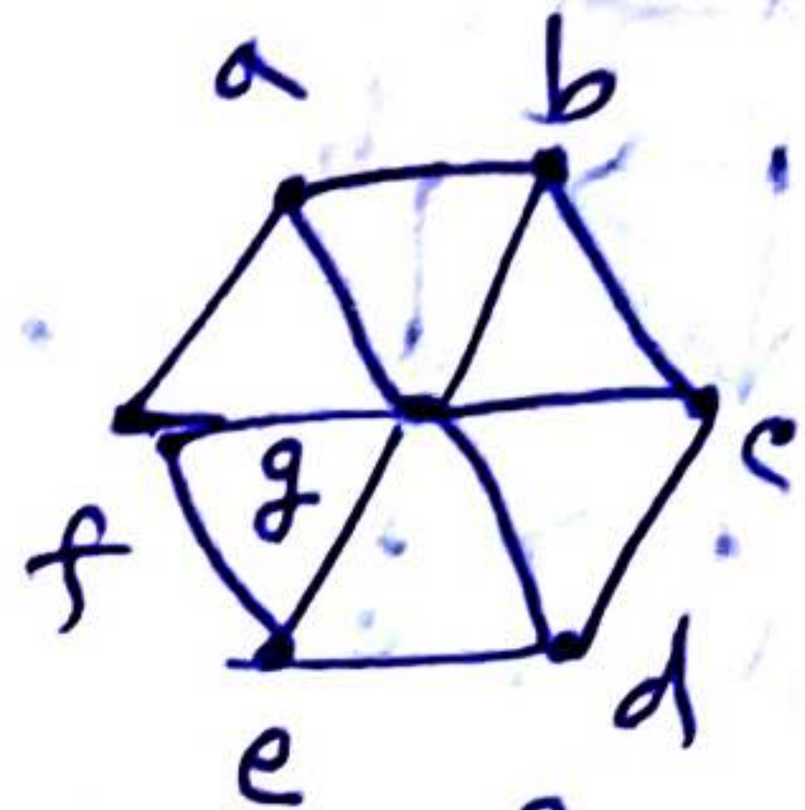
Example



G_1



G_2



G_3

G_1 has 2 vertices of odd deg. b & d
∴ it has E.P.
d, a, b, c, d, b

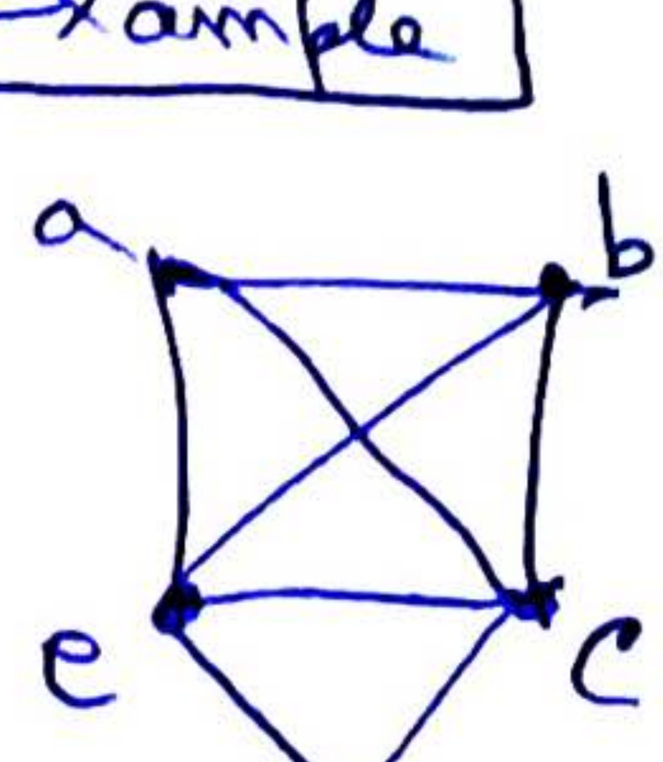
G_2 has 2 vertices of odd deg b & d
E.P. is
b, a, g, f, e, d, c, g, b, c, f, d

G_3 has 6 ~~deg~~ vertices of odd dg
No E.P.

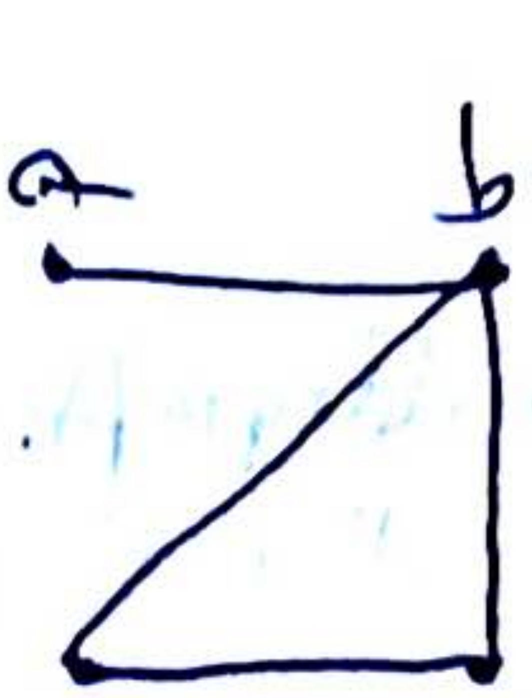
HAMILTON PATHS & CIRCUITS

— A simple path in a graph G that passes through every vertex exactly once is called a <u>Hamilton path (H.P.)</u> and a simple circuit in a graph G that passes through every vertex exactly once is called a <u>Hamilton circuit (H.C.)</u>.

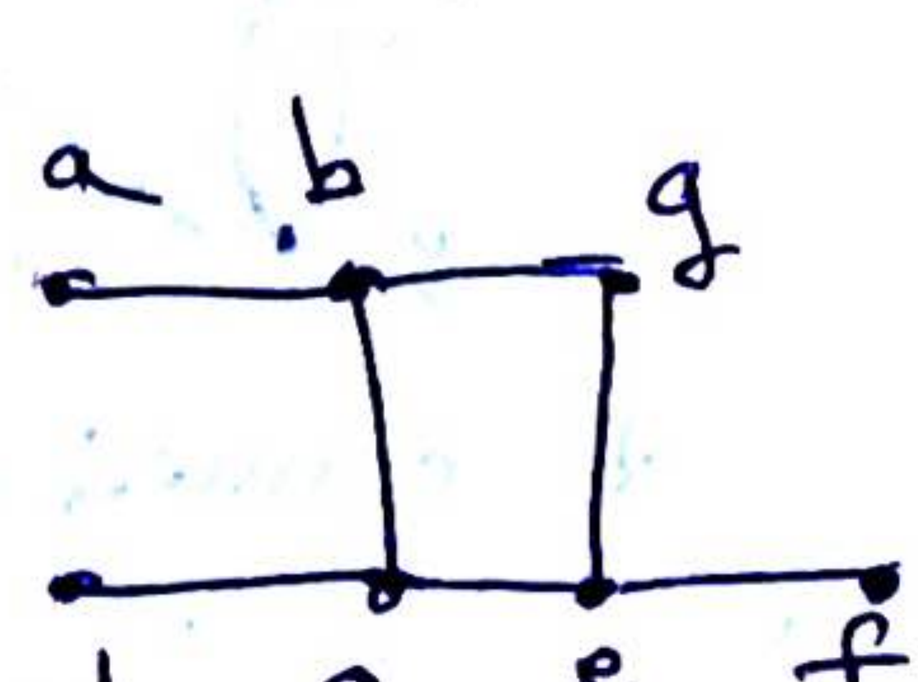
Example



G_1



G_2



G_3

G_1 has H.C.
a, b, c, d, e, a

G_2 has no H.C.
G_2 has H.P.
a, b, c, d

G_3 has no H.C.
" " " H.P.