SC223 - Linear Algebra

Aditya Tatu

Lecture 11

September 2, 2022

What is the Structure?

- We have seen linear combinations of elements from
  - $\forall x, y \in \mathbb{R}^2, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$

What is the Structure?

- We have seen linear combinations of elements from
  - $\forall x, y \in \mathbb{R}^2, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$
  - $$\forall x, y \in \mathbb{R}^3, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix}$$

What is the Structure?

- We have seen linear combinations of elements from
  - $\forall x, y \in \mathbb{R}^2, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$
  - $$\forall x, y \in \mathbb{R}^3, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix}$$
  - $$\forall x, y \in \mathbb{R}^n, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}$$

What is the Structure?

- We have seen linear combinations of elements from
  - $\forall x, y \in \mathbb{R}^2, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$
  - $$\forall x, y \in \mathbb{R}^3, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix}$$
  - $$\forall x, y \in \mathbb{R}^n, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}$$
  - $$\forall x, y \in \mathbb{R}^{\mathbb{Z}}, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y :=$$

What is the Structure?

- We have seen linear combinations of elements from
  - $\forall x, y \in \mathbb{R}^2, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$
  - $$\forall x, y \in \mathbb{R}^3, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix}$$
  - $$\forall x, y \in \mathbb{R}^n, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}$$
  - $$\forall x, y \in \mathbb{R}^{\mathbb{Z}}, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := (\ldots, ax_{-1} + by_{-1}, ax_0 + by_0, ax_1 + by_1, \ldots)$$

What is the Structure?

- We have seen linear combinations of elements from
  - $\forall x, y \in \mathbb{R}^2, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$
  - $$\forall x, y \in \mathbb{R}^3, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ ax_3 + by_3 \end{bmatrix}$$
  - $$\forall x, y \in \mathbb{R}^n, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := a \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}$$
  - $\forall x, y \in \mathbb{R}^{\mathbb{Z}}, \forall a, b \in \mathbb{R}, a \cdot x + b \cdot y := (\ldots, ax_{-1} + by_{-1}, ax_0 + by_0, ax_1 + by_1, \ldots)$
  - $\forall f, g \in \{h : \mathbb{R} \to \mathbb{R}\}, \forall a, b \in \mathbb{R}, a \cdot f + b \cdot g, (a \cdot f + b \cdot g)(x) = a \cdot f(x) + b \cdot g(x), \forall x \in \mathbb{R}$.

Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
- $(V, +)$ is an **Abelian group**:

Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$

Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta.$

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.

Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot \mathbb{F} \times V \to V$.

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot \mathbb{F} \times V \to V$.
- **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.

Vector Spaces

- **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:
- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot \mathbb{F} \times V \to V$.
- **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.

Vector Spaces

● **Definition:** A Vector space is a set V with a **field** $(\mathbb{F}, +_F, \times)$, and two binary operations, vector addition $+$ and scalar multiplication $\cdot$ that satisfy the following axioms:

- $(V, +)$ is an **Abelian group**:
  - $\forall x, y \in V, x + y \in V$
  - $\exists \theta \in V, \forall x \in V, x + \theta = \theta + x = x$
  - $\forall x \in V, \exists y \in V, x + y = y + x = \theta$. We will denote y by $-x$.
  - $\forall x, y, z \in V, (x + y) + z = x + (y + z)$.
  - $\forall x, y \in V, x + y = y + x$.
- **Closure with respect to Scalar multiplication:** $\cdot \mathbb{F} \times V \to V$.
- **Scalar Multiplication identity:** $\exists 1 \in \mathbb{F}$ such that $1 \cdot v = v, \forall v \in V$.
- **Distributivity:** $\forall a \in \mathbb{F}, \forall u, v \in V, a \cdot (u + v) = a \cdot u + a \cdot v$, and $\forall a, b \in \mathbb{F}, \forall u \in V, (a +_F b) \cdot u = a \cdot u + b \cdot u$.
- **Compatibility of field and scalar multiplication:** $\forall a, b \in \mathbb{F}, \forall u \in V, (a \times b) \cdot u = a \cdot (b \cdot u)$.

Field

- **Definition:**(Field). A field is a set $\mathbb{F}$ with two binary operations, addition $+_F$ and multiplication $\times$ that satisfy the following axioms:

Field

- **Definition:**(Field). A field is a set $\mathbb{F}$ with two binary operations, addition $+_F$ and multiplication $\times$ that satisfy the following axioms:
  - $(\mathbb{F}, +_F)$ is an **Abelian group**. The additive identity will be denoted by 0.

Field

- **Definition:**(Field). A field is a set $\mathbb{F}$ with two binary operations, addition $+_F$ and multiplication $\times$ that satisfy the following axioms:
  - $(\mathbb{F}, +_F)$ is an **Abelian group**. The additive identity will be denoted by 0.
  - $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The mutiplicative identity will be denoted by 1.

Field

- **Definition:**(Field). A field is a set $\mathbb{F}$ with two binary operations, addition $+_F$ and multiplication $\times$ that satisfy the following axioms:
  - $(\mathbb{F}, +_F)$ is an **Abelian group**. The additive identity will be denoted by 0.
  - $(\mathbb{F} - \{0\}, \times)$ is an **Abelian group**. The mutiplicative identity will be denoted by 1.
  - **Distributivity:** $\forall a, b, c \in \mathbb{F}, (a +_F b) \times c = a \times c +_F b \times c, a \times (b +_F c) = a \times b +_F a \times c$

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$
- $(\mathbb{R}, +, \times)$

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$
- $(\mathbb{R}, +, \times)$
- $(\mathbb{C}, +, \times)$

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$
- $(\mathbb{R}, +, \times)$
- $(\mathbb{C}, +, \times)$
- $(\mathbb{Q}, +, \times)$

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$
- $(\mathbb{R}, +, \times)$
- $(\mathbb{C}, +, \times)$
- $(\mathbb{Q}, +, \times)$
- $(\mathbb{R}[x], +, \times)$, where $\mathbb{R}[x]$ is the set of all rational polynomials of the form $\frac{p(x)}{q(x)}$, with $q \neq 0$, and p and q are polynomials in one variable with real coefficients.

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$
- $(\mathbb{R}, +, \times)$
- $(\mathbb{C}, +, \times)$
- $(\mathbb{Q}, +, \times)$
- $(\mathbb{R}[x], +, \times)$, where $\mathbb{R}[x]$ is the set of all rational polynomials of the form $\frac{p(x)}{q(x)}$, with $q \neq 0$, and p and q are polynomials in one variable with real coefficients.
- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ forms a vector space over $\mathbb{F}$.

Examples of Fields

- $(\mathbb{Z}_2, +_2, \times)$
- $(\mathbb{R}, +, \times)$
- $(\mathbb{C}, +, \times)$
- $(\mathbb{Q}, +, \times)$
- $(\mathbb{R}[x], +, \times)$, where $\mathbb{R}[x]$ is the set of all rational polynomials of the form $\frac{p(x)}{q(x)}$, with $q \neq 0$, and p and q are polynomials in one variable with real coefficients.
- If the 3-tuple $(V, +, \cdot)$ with field $(\mathbb{F}, +_F, \times)$ satisfies all vector space axioms, we say that $(V, +, \cdot)$ forms a vector space over $\mathbb{F}$.

● Any element of the vector space $(V, +, \cdot)$ will be referred to as a **vector**, and any element $a \in \mathbb{F}$ will be referred to as a **scalar**.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{C}^n, +, \cdot)$ over $\mathbb{C}$.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{C}^n, +, \cdot)$ over $\mathbb{C}$.
- $(\mathbb{R}^{\mathbb{Z}}, +, \cdot)$ over $\mathbb{R}$, where $\mathbb{R}^{\mathbb{Z}}$ is the set of all doubly-infinite sequences.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{C}^n, +, \cdot)$ over $\mathbb{C}$.
- $(\mathbb{R}^{\mathbb{Z}}, +, \cdot)$ over $\mathbb{R}$, where $\mathbb{R}^{\mathbb{Z}}$ is the set of all doubly-infinite sequences.
- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.

Examples of Vector spaces

- $(\mathbb{R}, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$.
- $(\mathbb{C}^n, +, \cdot)$ over $\mathbb{C}$.
- $(\mathbb{R}^\mathbb{Z}, +, \cdot)$ over $\mathbb{R}$, where $\mathbb{R}^\mathbb{Z}$ is the set of all doubly-infinite sequences.
- $(\mathcal{P}(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathcal{P}(\mathbb{R})$ is the set of all polynomials of one variable with real coefficients.
- $(\mathbb{L}_2(\mathbb{R}), +, \cdot)$ over $\mathbb{R}$, where $\mathbb{L}_2(\mathbb{R})$ denotes the set of all square-integrable functions $f : \mathbb{R} \to \mathbb{R}$.