Matrices

A matrix is a rectangular array of symbols.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$a_{ij} \in F$

$F = \mathbb{R}$
$\mathbb{C}$

$A = [a_{ij}]$

$i = 1, 2, \ldots, m$

$j = 1, 2, \ldots, n$

row vector $\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$

Column vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ or $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$

Identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

Transpose

$$A = [a_{ij}] \qquad A^t = [a_{ji}]$$

$$1 \le i \le m$$
$$1 \le j \le n$$

square matrix of order n

$$A = [a_{ij}]$$

$$1 \le i \le n$$
$$1 \le j \le n$$

$$\bar{A} = [\bar{a}_{ij}]$$

Conjugate transpose

$$A^* = (\bar{A})^t = [\bar{a}_{ji}]$$

Eigen Values and Eigen Vectors

$A = [a_{ij}]_{n\times n}$

Defⁿ Let A be an $n\times n$ matrix.
A scalar $\underline{\lambda}$ is said to be an ~~eg~~ eigen value of A if there exists a non zero vector X s.t. $\underline{Ax = \lambda X}$.
The vector x is called eigen vector belonging to λ.

The characteristic polynomial

$$\det(A - \lambda I)$$

The characteristic eqn for A

$$\det(A - \lambda I) = 0$$

roots $\lambda_1, \lambda_2 \ldots \lambda_n$ called eigen value.

$Ax = \lambda x$ can be written as

$$(A - \lambda I)\underline{x} = 0$$

If it has a nontrivial soln. the λ is an eigen value of A.

Exp

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

Characteristic eqn

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 12 = 0$$

$$\underline{\lambda_1} = 4, \quad \underline{\lambda_2} = -3$$

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$$\begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - \lambda - 12 = 0$$

$$\underline{\lambda_1} = 4, \quad \underline{\lambda_2} = \underline{-3}$$

$$(A - 4I)x = 0$$

$$\left(\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 2x_2 = 0$$

$$3x_1 - 6x_2 = 0$$

$$-x_1 + 2x_2 = 0$$

$$(x_1, x_2)^T$$

$$= (2x_2, x_2)$$

$$x_1 = 2x_2$$

$$x_2(2, 1)$$

$\lambda_2 = -3$

$(\underline{A} + 3\mathbb{I})\, x = 0$

$$\left(\begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

any nonzero multiple of $(-1, 3)$ is an eigen vector belonging to λ_2.

$\lambda_2 = -3$

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Properties

$A_{n \times n}$ $\lambda_1 \cdots \lambda_n$ eigen values

(1) trace of A

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \lambda_1 + \lambda_2 + \cdots + \lambda_2$$

(2)
$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

A singular $\det(A) = 0$

A nonsingular $\det(A) \neq 0$

(3) Eigen value of A^K will be $\{\lambda_1^K, \lambda_2^K \dots \lambda_n^K\}$

(4) A $\lambda_1 \dots \lambda_n$

A^{-1} have eigen value $\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots \frac{1}{\lambda_n}\}$

(5) If $A = A^*$ every eigen vector is real.

Block Matrix

$$M = \left(\begin{array}{c|c} A_{\underline{m}\times\underline{n}} & B_{\underline{m}\times n'} \\ \hline C_{\underline{m'}\times\underline{n}} & D_{\underline{m'}\times\underline{n'}} \end{array}\right)_{(m+m')\times(n+n')}$$

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^t = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

Hadamard Matrix

Defn A Hadamard matrix H is a square matrix whose entries must be ± 1

s.t. $HH^T = nI$.

$H = (\pm 1)_{n\times n}$

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st $HH^T = \underline{nI}$.

$n=1$ $\qquad H_1 = \begin{pmatrix} 1 \end{pmatrix}$

$n=2$ $\qquad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$n=4$ $\qquad H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$$\overset{H}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \overset{H^T}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} =$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}$$

$$= 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 1 I$$

Properties of a Hadamard matrix of order n exists then $n = 1, 2,$ and $4m$ $m \in \mathbb{N}$

Construction of Hadamard matrix

$GF(s)$ s is finite

Let $GF(s)$ be a Galois field of order s whe $s = p^n$. p prime $n \in \mathbb{N}$

$2 \quad 2^2 \quad 2^3 \quad 2^4$ $\quad 5$

$3 \quad 3^2 \quad 3^3 \quad 3^4$

An element a in GF(5) is a quadratic residue iff there exists b in GF(5) s.t.

$$a = b^2$$

$$GF(5) = \{0, 1, 2, 3, 4\}$$

GF(11)

$$1^2 = 1$$

$$b^2 = 2^2 = 4 = 4$$

$$3^2 = 9 = 4$$

$$4^2 = 16 = 1$$

1, 4 are quadratic residues

~~$0 = 0^2$~~

2, 3 are ~~not~~ qnr

qnr: quadratic non residue

The quadratic character $\chi(a)$ is defind as

$$\chi(a) = \begin{cases} 1 & \text{if } a \text{ is a QR in } GF(5) \\ -1 & \text{if } a \text{ is a QNR in } GF(5) \\ 0 & \text{if } a = 0 \end{cases}$$

$\chi(1) = \chi(4) = 1$

$\chi(2) = \chi(3) = -1$

$\chi(0) = 0$

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$$\chi(0) = 0$$

Jacobsthal matrix $\underline{Q}$ for $\underline{GF(q)}$ is a $q \times q$ matrix s.t. entry in $\underline{\text{row } a}$ and $\underline{\text{column}}$ b is $\underline{\chi(a-b)}$

$\underline{GF(5)}$

$0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\underline{Q} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & + & - & - & + \\ 1 & + & 0 & + & - & - \\ \underline{2} & - & + & 0 & + & - \\ 3 & - & - & + & 0 & + \\ 4 & + & - & - & + & 0 \end{array}$$

$\chi(2-3)$

$\chi(-1) = \chi(4)$

$+1 \quad +$

$-1 \quad -1$

Paley construction

If $q \equiv 3 \mod 4.$

$$H = I + \begin{bmatrix} 0 & J^T \\ -J & Q \end{bmatrix}$$

GF(11)

= 8

GF(q) GF(5)

GF(5) — $q \equiv 1 \mod 4$

$q \equiv 3 \mod 4$

GF(11)

J is a column vector of all 1s

GF(5)

$$j = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

GF(11)

$11 \equiv 3 \mod 4$

Case II

$\underline{GF(5)}$

$5 \equiv 1 \mod 4$

$$\underline{H} = \underline{\begin{bmatrix} 0 & j^T \\ j & Q \end{bmatrix}}$$

$$\begin{array}{ccccccc} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \cdot & \cdot & \cdot \\ \hline \end{array}$$

$$\begin{pmatrix} \underline{0} & \pm & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{pmatrix}$$

S.F replace 0 by $\underline{\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}}$

$\underline{+1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$ $\quad -1 \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

replace ± 1 by

$\pm \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Fourier Matrix

nth roots of unity

Let $n \geq 1$, $\omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$i = \sqrt{-1}$.

(1) $\omega^n = 1$

$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

$\omega\bar{\omega} = 1$

Fourier matrix of order n $F = F_n$

s.t. $F^* = \frac{1}{\sqrt{n}} \omega^{(i-1)(j-1)}$

$$= \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

ω^k is periodic

$k = 0, 1, 2, \ldots$

$$F^* = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega & \omega^4 & \cdots & \omega^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \end{pmatrix}$$

for $n=2$

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$n=4$

$$F = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$