Electrostatic potential

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|^3}(\vec{r}-\vec{r'})d^3\vec{r'}$$

$$\vec{\nabla}\times\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r'})\left[\vec{\nabla}\times\frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}\right]d^3\vec{r'}$$

$$\vec{\nabla}\times\frac{(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3} = \frac{1}{|\vec{r}-\vec{r'}|^3}\vec{\nabla}\times(\vec{r}-\vec{r'}) + \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r'}|^3}\right)\times(\vec{r}-\vec{r'}) = 0$$

$\therefore$ $\vec{\nabla}\times\vec{E}(\vec{r}) = 0$ for any charge distribution.
$\therefore$ $\vec{E} = \vec{\nabla}F$ for some scalar function $F(\vec{r})$.
So the direction of $\vec{E}$ is in the direction of maximum increase of the function F and it is perpendicular to the surface over which F is constant. Conventionally $\vec{E}$ is considered to be directed along the fastest decrease of a function rather than fastest increase. Such a function is obviously $-F$. Let $\Phi(\vec{r}) = -F(\vec{r})$.
Then $\vec{E} = \vec{\nabla}F = -\vec{\nabla}\Phi$.
The function $\Phi(\vec{r})$ is called the potential function of the charge configuration. It is often a more convenient quantity to handle than $\vec{E}$ since it is a scalar. Ofcourse, once we know $\Phi(\vec{r})$ we can easily find $\vec{E}(\vec{r})$.
Note that if $\Phi' = \Phi + c$ where c is a constant then $\vec{\nabla}\Phi' = \vec{\nabla}\Phi$. So Φ' is also a valid potential for the charge configuration. So the potential is always arbitrary upto a constant value. However the potential difference is independent of this arbitrariness.

$$\Phi'(b) - \Phi'(a) = (\Phi(b)+c) - (\Phi(a)+c) = \Phi(b) - \Phi(a)$$

If we know the potential due to charges $q_1, q_2, ..., q_n$ as $\Phi_1, \Phi_2, ..., \Phi_n$, then the potential for the configuration can be obtained as a linear superposition of these potentials, i.e,

$$\Phi = \Phi_1 + \Phi_2 + ... + \Phi_n$$

This is obtained from the linear superposition of electric fields

$$\begin{aligned}\vec{E} &= \vec{E}_1 + \vec{E}_2 + ... + \vec{E}_n \\ &= -\vec{\nabla}\Phi_1 - \vec{\nabla}\Phi_2 ..... - \vec{\nabla}\Phi_n \\ &= -\vec{\nabla}(\Phi_1 + \Phi_2 + ... + \Phi_n) \\ &= -\vec{\nabla}\Phi\end{aligned}$$

Now if Φ' is any other potential satisfying $\vec{E} = -\vec{\nabla}\Phi'$, then we have $0 = -\vec{\nabla}(\Phi - \Phi')$.
This implies the scalar function $\Phi - \Phi'$ is constant. As we stated earlier $\Phi' = \phi + c$ are

equivalent. So Φ is indeed the potential for the given charge configuration.

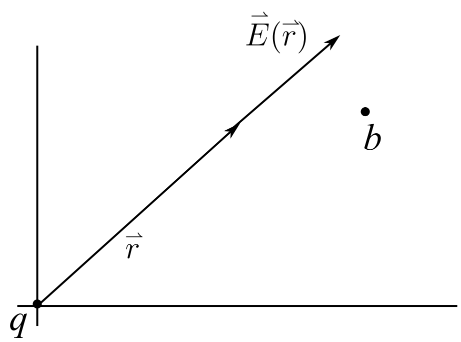
In M.K.S unit potential is measured in Nm/coulomb = Joule/coulomb. This is called the volt.
So 1 volt $= 1\frac{\text{Joule}}{\text{coulomb}}$.
The arbitrary constant that can be added to a potential can be used to measure the potential with respect to some convenient point. So if we say

$$\Phi(\vec{r}) = -\int_{b}^{\vec{r}} \vec{F} \cdot \vec{dl}$$

then the potential at every point $\vec{r}$ is measured with respect to the the point b. In this form $\Phi(b) = 0$. Generally when the charge configuration is confined within a bounded region, the potential at infinity is considered to be 0 and hence the infinity becomes the reference, i.e,

$$\Phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \cdot \vec{dl}$$

1 Potential due to a point charge



A point charge q is at the origin

$$\vec{E}(\vec{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

The potential at the point $\vec{r}$ is

$$\begin{aligned}
\Phi(\vec{r}) &= -\int_{b}^{\vec{r}} \vec{E}(\vec{r}) \cdot \vec{dl} \\
&= -\int_{r_b}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2} dr'\hat{r}' \cdot \hat{r}' \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r_b}\right)
\end{aligned}$$

If the reference point is at ∞ then $r_b \to \infty$

$$\therefore\ \Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}$$

Note: The potential function is generally denoted by the greek symbol Φ or the english symbol V.

2 Laplace's Equation

The divergence theorem states that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

where ρ is the charge density at the point where $\vec{\nabla} \cdot \vec{E}$ is calculated. Since $\vec{E} = -\vec{\nabla}\Phi$ we have

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot (\vec{\nabla}\Phi) = \frac{\rho}{\epsilon_0}$$

$$\therefore \quad -\nabla^2\Phi = \frac{\rho}{\epsilon} \tag{1}$$

Equation 1 is called the Poisson's Equation. It is a partial differential equation. If we are given a charge distribution , i.e, the density function $\rho(\vec{r})$, then we can get Φ by solving the Poisson's Equation. From Φ we can find the electric field at all points. This is the central problem of electrostatics.

When the charge density at a point is 0 we have

$$\nabla^2\Phi = 0 \tag{2}$$

This is called the Laplace's Equation. This is a homogeneous differential equation.
We will study methods to solve Laplace's Equation in three dimension in various coordinate systems. If we know any one solution to the Poisson's equation, called a particular solution, we can find any other solution by adding the solution of Laplace's Equation to it. So our essential job is to find all possible solutions to the Laplace's equation. All this works because ∇^2 is a linear operator. Hence the solutions to the Laplace's Equation form a linear vector space.

Why do we need Laplace's Equation when we are mainly interested in the Poisson's Equation ?

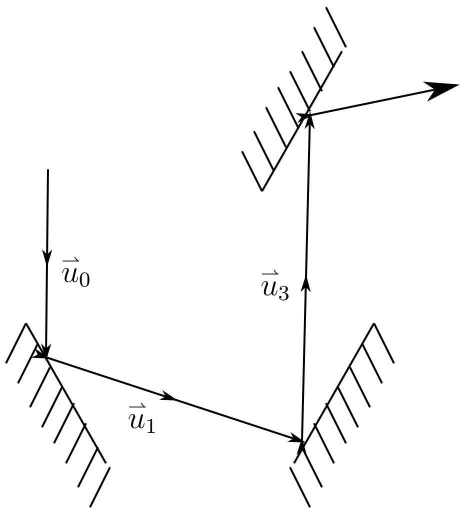
Generally we we would be requiring to find the Electric field $\vec{E}$ and the electric potential Φ caused due to a given charge distribution given by the function $\rho(\vec{r})$.
To understand the importance of Laplace's equation in electrostatics let us compare it with a more familiar problem. This is the Newton's equation of motion.

Electrostatics	Mechanics
$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$	$\frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m}$
$\nabla^2\Phi = 0$	$\frac{d^2\vec{r}}{dt^2} = 0$

Newton's Equation of motion is similar to the Poisson's Eqn. Both are 2nd order differential Equations. In Newton's Eqns, the R.H.S has the source or the cause of the motion. Once we know $\vec{F}$ we can find the trajectory $\vec{r}(t)$ by solving Newton's Equation. Same is the case with Poisson's Equation in Electrostatics.
The second equations are also equivalent. In mechanics if we have $\vec{F} = 0$ in a region the particle follows a straight line motion given by

$$\vec{r}(t) = \vec{u}t + \vec{a}$$

where $\vec{u}$ and $\vec{a}$ are constant vectors to be determined from certain initial conditions, say the position or velocities at certain instants of time. Sometimes we don't have well defined forces acting on the particle. The particle collides with certain barriers at certain times and bounces. In between these collisions the particle travels straight like a free particle.



At every collision the velocity of the particle changes from $\vec{u}_0$ to $\vec{u}_1$ to $\vec{u}_2$ to ..... These changes are instantaneous taking place in infinitesimal times. The forces involved during collisions are the δ functions in time. They are called impulsive forces. We can obtain such trajectories by solving the free particle equation in the various time intervals and then match the trajectories at the instants $t_1, t_2, t_3, ...$ to get $\vec{u}_0, \vec{u}_1, \vec{u}_2, ...$the Even in electrostatics often the source term on the R.H.S of Poisson's Equation are not well-defined as a volume density function $\rho(\vec{r})$. But they may be some surface charges or line charges. Such distributions are delta functions if written in volume density as we have discussed and seen in various situations. These are exactly like the source of impulsive forces in mechanics. So here we rather solve the free space Equation, viz, the Laplace's Equation in various regions and then match the solutions at the boundaries of these regions containg the surface and line charges. Shortly we will see in detail how to do this. This will be the central problem in Electrostatics for some time.