

Vector Potential

In electrostatics since $\vec{\nabla} \times \vec{E} = 0$ we can express $\vec{E} = -\vec{\nabla}\Phi$ where Φ is the electrostatic potential. Once Φ is found we can find the electric field at all points. The equivalent equation in magnetostatics is

$$\vec{\nabla} \cdot \vec{B} = 0$$

Since divergence of a curl of a vector field is 0, we claim that \vec{B} can be expressed as the curl of a vector field, i.e.,

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (1)$$

Indeed we can always find a vector field \vec{A} that satisfies Eq. 1.

\vec{A} is called the magnetic vector potential for the magnetic field \vec{B} . Just like the electrostatic scalar potential Φ , the magnetic vector potential is not unique. We can always add a constant to the scalar potential Φ and it still gives the same electric field. For the case of magnetic vector potential \vec{A} , the quantity we can add is more complicated. We can add the gradient of a scalar, i.e, if

$$\vec{A}' = \vec{A} + \vec{\nabla}F$$

where F is a scalar field. Then

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}F) = \vec{\nabla} \times \vec{A} = \vec{B}$$

So \vec{A}' is also a valid vector potential.

To specify \vec{A} more precisely we must also know $\vec{\nabla} \cdot \vec{A}$. So far we don't have any condition on $\vec{\nabla} \cdot \vec{A}$. If we specify some value for $\vec{\nabla} \cdot \vec{A}$ say $\vec{\nabla} \cdot \vec{A} = 0$ then \vec{A} can be uniquely determined. But this is purely a matter of choice as fixing a reference point to the scalar potential. So let us have

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad \text{and} \quad \vec{\nabla} \cdot \vec{A} = 0$$

These equations look identical to the following

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{and} \quad \vec{\nabla} \cdot \vec{B} = 0$$

We know that given \vec{J} ,

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{z}}{r^2} d\tau$$

So for a given magnetic field \vec{B} , the vector potential \vec{A} can be written in analogy as

$$\vec{A} = \frac{1}{4\pi} \int \frac{\vec{B} \times \hat{z}}{r^2} d\tau$$

The source of a magnetic field is the current density \vec{J} in the region. So we would like to evaluate the vector potential for a given source. This can be obtained as follows:

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2(\vec{A}) = \mu_0 \vec{J} \\ \therefore \vec{\nabla}^2(\vec{A}) &= -\mu_0 \vec{J}\end{aligned}$$

This is a set of three equations along the three components of the vector x, y and z , i.e,

$$\vec{\nabla}^2 A_x = -\mu_0 J_x, \quad \vec{\nabla}^2 A_y = -\mu_0 J_y, \quad \vec{\nabla}^2 A_z = -\mu_0 J_z$$

We know the solution to these equations from the Poisson's Equation in electrostatics- $\vec{\nabla}^2 \Phi = -\frac{\rho}{\epsilon_0}$.

Reading off the solution from there we have

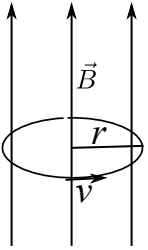
$$A_x = \frac{\mu_0}{4\pi} \int \frac{J_x}{r} d\tau$$

Likewise for A_y and A_z . Combining all the components we have

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r} d\tau$$

The vector potential doesn't have a straightforward interpretation as the scalar potential in electrostatics.

The scalar potential is the potential energy or the work done by a unit charged particle in moving from infinity to a point. Dimensionally the vector potential is equal to that of momentum per unit charge. In certain cases we can indeed relate it to the momentum of a charges particle in a magnetic field. Let us consider a simple example.



Consider a uniform magnetic field \vec{B} along \hat{z} . Any charged particle moving perpendicular to \vec{B} will perform a circular motion. The radius of this circular orbit is given by

$$\begin{aligned}qvB &= \frac{mv^2}{r} \\ \therefore mv &= qBr\end{aligned}$$

mv is the momentum of the particle. Considering the vectors we have $\vec{p} = q(\vec{B} \times \vec{r})$.

Let us now find a vector potential for this magnetic field.

Since \vec{B} is along \hat{z} , if we want to find \vec{A} at each point on the orbit of the particle we can use the Stokes' theorem.

$$\oint_C \vec{A} \cdot d\vec{l} = \int_S \vec{\nabla} \times \vec{A} \cdot \hat{n} da = \int_S \vec{B} \cdot \hat{n} da$$

where S is the surface perpendicular to the z axis enclosed by the circuit C .

$$\therefore \oint_C \vec{A} \cdot d\vec{l} = B \times \pi r^2$$

Along the chosen curve $\vec{dl} = r d\phi \hat{\phi}$. By symmetry of the problem A_ϕ is constant over ϕ . So

$$\begin{aligned}\oint_C \vec{A} \cdot \vec{dl} &= \int_0^{2\pi} A_\phi r d\phi = B\pi r^2 \\ \therefore A_\phi &= \frac{1}{2}Br \\ \therefore \vec{A} &= \frac{1}{2}(\vec{B} \times \vec{r}) = \frac{\vec{p}}{2q}\end{aligned}$$

So our choice of vector potential is $\frac{1}{2}$ that of the momentum of a unit charge performing a circular motion in the magnetic field. The interpretation of the vector potential as momentum is very interesting because in relativity the four components (t, x, y, z) has t and the three space components. They become dimensionally equivalent if we consider (ct, x, y, z) . The electric and magnetic fields (\vec{E}, \vec{B}) also become dimensionally equivalent if we consider $(\frac{\vec{E}}{c}, \vec{B})$.

Now if we consider the four quantities (Φ, A_x, A_y, A_z) , they indeed form a four dimensional vector in relativistic formulation. Here Φ is the scalar potential of electrostatics and \vec{A} is the magnetic vector potential. In fact Φ/c has the same dimension as \vec{A} .

In relativity the space time gets mixed, i.e, transform into each other for moving observers. Likewise, the four components of the potential (Φ, A_x, A_y, A_z) get mixed for a moving observer producing the appropriate electromagnetic field for the observer. In fact the energy and the momentum (E, p_x, p_y, p_z) transform in the same way.

Eg. 1:

What current density would produce the vector potential $\vec{A} = k\hat{\phi}$?

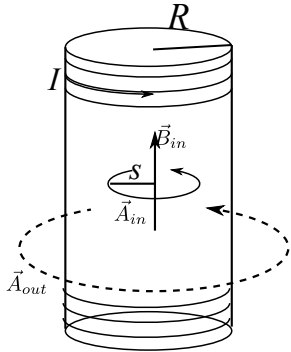
$$\begin{aligned}\vec{B} = \vec{\nabla} \times \vec{A} &= \frac{1}{s} \frac{\partial}{\partial s}(sk)\hat{z} = \frac{k}{s}\hat{z} \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} \\ \therefore \vec{J} = \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) &= \frac{1}{\mu_0} \left[-\frac{\partial}{\partial s} \left(\frac{k}{s} \right) \right] \hat{\phi} = \frac{1}{\mu_0} \frac{k}{s^2} \hat{\phi}\end{aligned}$$

Eg. 2

Let us find the vector potential for the case of an infinite solenoid of radius R . The solenoid has n turns per unit length carrying current I . We have found magnetic field inside the solenoid

$$\vec{B}_{in} = \mu_0 n I \hat{z}$$

Consider a loop of radius s inside the solenoid as shown



$$\begin{aligned}\oint_C \vec{A}_{in} \cdot \vec{dl} &= \int_S (\vec{\nabla} \times \vec{A}_{in}) \cdot \hat{n} da = \int_S \vec{B}_{in} \cdot \hat{n} da \\ \therefore \vec{A}_{in} \times 2\pi s &= \mu_0 n I \times \pi s^2 \\ \therefore A_{in} &= \frac{\mu_0 n I s}{2} \implies \vec{A}_{in} = \frac{\mu_0 n I s}{2} \hat{\phi}\end{aligned}$$

For outside,

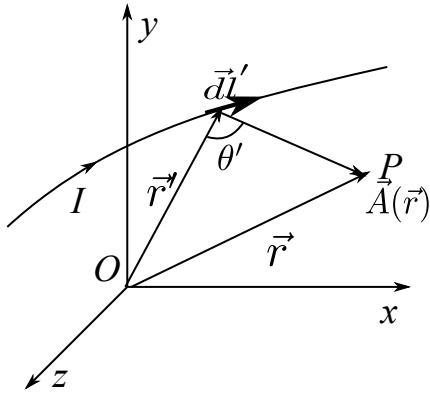
$$A_{out} \times 2\pi s = \int_S \vec{B} \cdot \hat{n} da = \mu_0 n I \times \pi R^2 \quad (\vec{B}_{out} = 0)$$

$$\therefore A_{out} = \frac{\mu_0 n I R^2}{2s} \implies \vec{A}_{out} = \frac{\mu_0 n I R^2}{2s} \hat{\phi}$$

So there exist a vector potential outside, though $\vec{B}_{out} = 0$

1 Multipole Expansion of the vector potential

Just like expanding the potential at a point in terms of monopole, dipole, quadrupole etc. in electrostatics, we can expand the vector potential \vec{A} in electrostatics. For a wire carrying a current I ,



$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{z}$$

$$\frac{1}{z} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}}$$

$$= \frac{1}{r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r} \cos \theta'}}$$

$$= \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos \theta')$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos \theta') d\vec{l}'$$

In this we are assuming that $r > r'$ all along the wire. This will certainly not be true for a very long wire stretching to infinity. So this expression is valid only for confined currents. This is the case for current loops which are confined in some region and we are observing $\vec{A}(\vec{r})$ at a large distance. So under these conditions the integral becomes a closed loop integral

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \oint (r')^l P_l(\cos \theta') d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\vec{l}' + \frac{1}{r^2} \oint r' \cos \theta' d\vec{l}' + \dots \right]$$

For a closed loop $\oint d\vec{l}' = 0$. So the monopole term doesn't contribute to the magnetic vector potential. This bolsters the fact that the Biot-Savart Law only gives magnetic fields for dipoles and other higher multipoles. There are no magnetic monopoles so far in nature.

So unlike electrostatics the most dominant term of magnetic field is always the dipole term and not the monopole term. The dipole term is written as

$$\begin{aligned}
\vec{A}_{dip} &= \frac{\mu_0 I}{4\pi r^2} \oint r' \cos \theta' d\vec{l}' \\
&= \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}') d\vec{l}' \\
&= \frac{\mu_0 I}{4\pi r^2} \left(\oint \frac{1}{2} (\vec{r}' \times d\vec{l}') \right) \times \hat{r} \\
&= \frac{\mu_0 I}{4\pi r^2} \vec{a} \times \hat{r}
\end{aligned} \tag{2}$$

Where $\vec{a} = \oint \frac{1}{2} (\vec{r}' \times d\vec{l}')$ is the vector area of the loop. The quantity $\vec{m} = I\vec{a}$ is called the dipole moment of the loop.

$$\therefore \vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} (\vec{m} \times \hat{r})$$

$$\begin{aligned}
\vec{B}_{dip} &= \vec{\nabla} \times \vec{A}_{dip} = \frac{\mu_0}{4\pi} \left[\vec{\nabla} \times \left(\vec{m} \times \frac{\hat{r}}{r^2} \right) \right] \\
&= \frac{\mu_0}{4\pi} \left[-(\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} + \vec{m} \left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right) \right] \\
&= \frac{\mu_0}{4\pi} \left[-(\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} \right] \quad \text{for } r \neq 0
\end{aligned}$$

Consider $\vec{m} = m\hat{z}$, i.e, a dipole along \hat{z} . Then

$$\begin{aligned}
(\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} &= m \frac{\partial}{\partial z} \left(\frac{\vec{r}}{r^3} \right) = -\vec{r} \left(-\frac{3mz}{r^5} \right) + \frac{\vec{m}}{r^3} \\
&= (-3(\vec{m} \cdot \hat{r})\hat{r} + \vec{m}) \frac{1}{r^3} \\
\therefore \vec{B}_{dip} &= \frac{\mu_0 I}{4\pi r^2} (-3(\vec{m} \cdot \hat{r})\hat{r} + \vec{m})
\end{aligned}$$