Directional derivative $z = f(x,y)$

at a point $P(x_0, y_0)$ in the direction of $u = u_1\hat{i} + u_2\hat{j}$ is

$$\left(D_u f\right)_P = \left(\frac{df}{ds}\right)_{u,P}$$

$$= \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

When $u = \hat{i}$, the directional derivative of f at P is $\frac{\partial f}{\partial x}$ at P.

When $u = \hat{j}$, $\left(D_u f\right)_P = \left(\frac{\partial f}{\partial y}\right)_P$.

Gradient

The gradient vector of $f(x,y)$ at a point $P(x_0, y_0)$ is the vector $\nabla f\Big|_P = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\Big|_P$

Relationship between gradient and directional derivative

$$x = x_0 + s u_1$$
$$y = y_0 + s u_2$$

~~$(a_1\hat{i} + a_2\hat{j}) \cdot (b_1\hat{i} + b_2)$~~

$$D_u f\Big|_P = \frac{df}{ds}\Big|_{P_0}$$

$$= \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \cdot \frac{dy}{ds}$$

$$= \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right) \cdot (u_1\hat{i} + u_2\hat{j})$$

$$= \nabla f \cdot u$$

$$\Rightarrow \boxed{\frac{df}{ds} = \nabla f \cdot u}$$

$$D_u f = \nabla f \cdot u = |\nabla f|\,|u| \cos\theta$$

$$= |\nabla f| \cos\theta$$

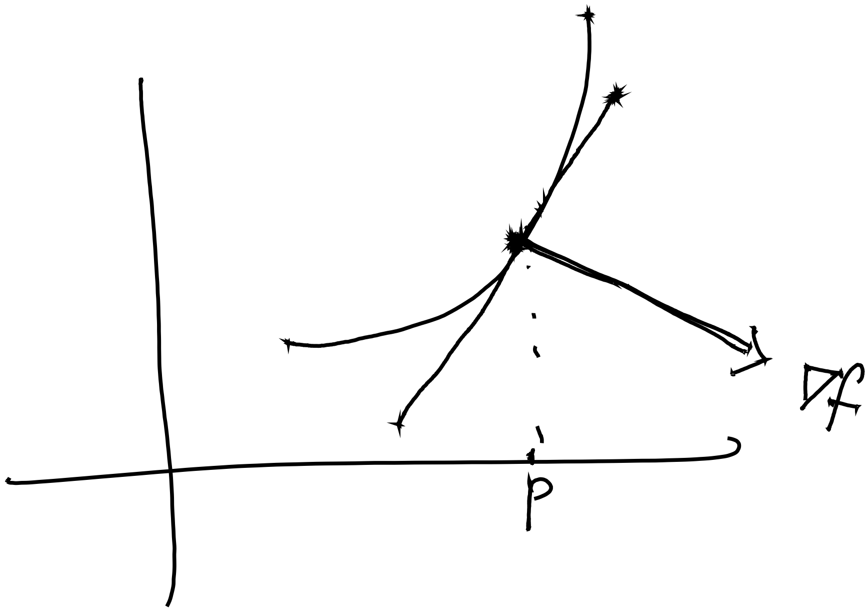
(1) The function f increases most rapidly when $\cos\theta = 1$ or $\theta = 0^\circ$

That is when u is in the direction of ∇f.

So in the gradient direction, the rate of change of a surface at a point is maximum.

(2) Similarly the rate of change of a surface at a point is minimum in the direction of $-\nabla f$.

(3) Any direction u orthogonal to ∇f is a direction of no change in f.

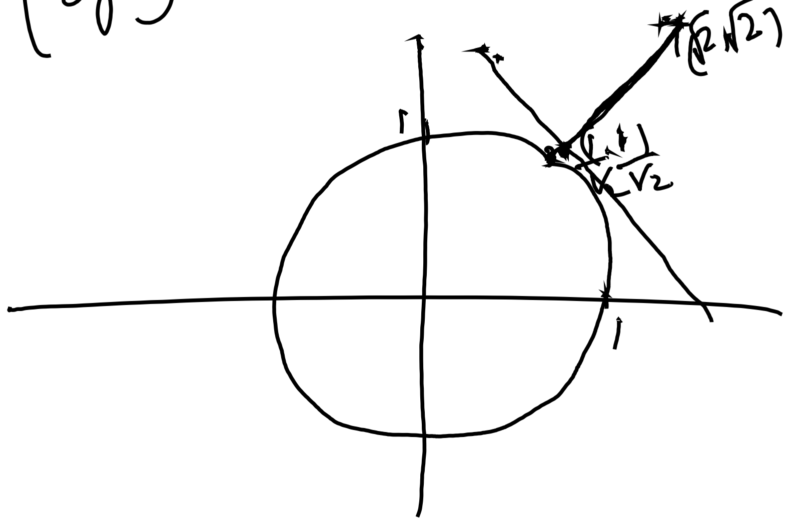


$$f(x, y) = x^2 + y^2$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$P \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\nabla f = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$



Exp Find the directions in which

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

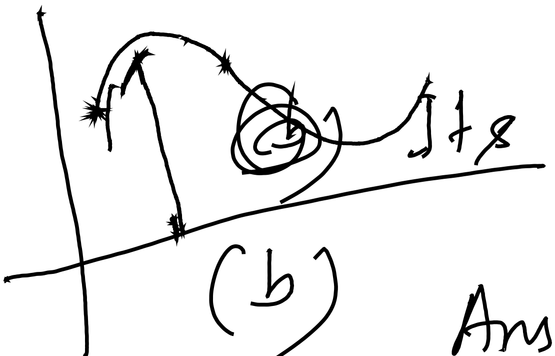
(a) increases most rapidly at the point $(1,1)$.

(b) decreases most rapidly at $(1,1)$

(c) What are the directions of no change in f at $(1,1)$.

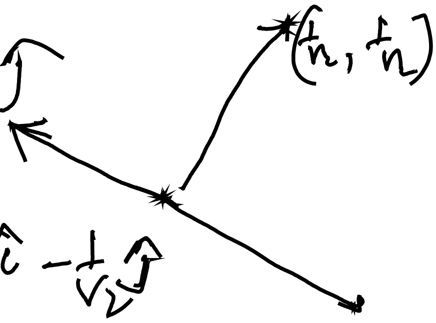
Sol^n (a) $\nabla f\Big|_{(1,1)} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}_{(1,1)} = \begin{bmatrix} x \\ y \end{bmatrix}_{(1,1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \hat{i} + \hat{j}$

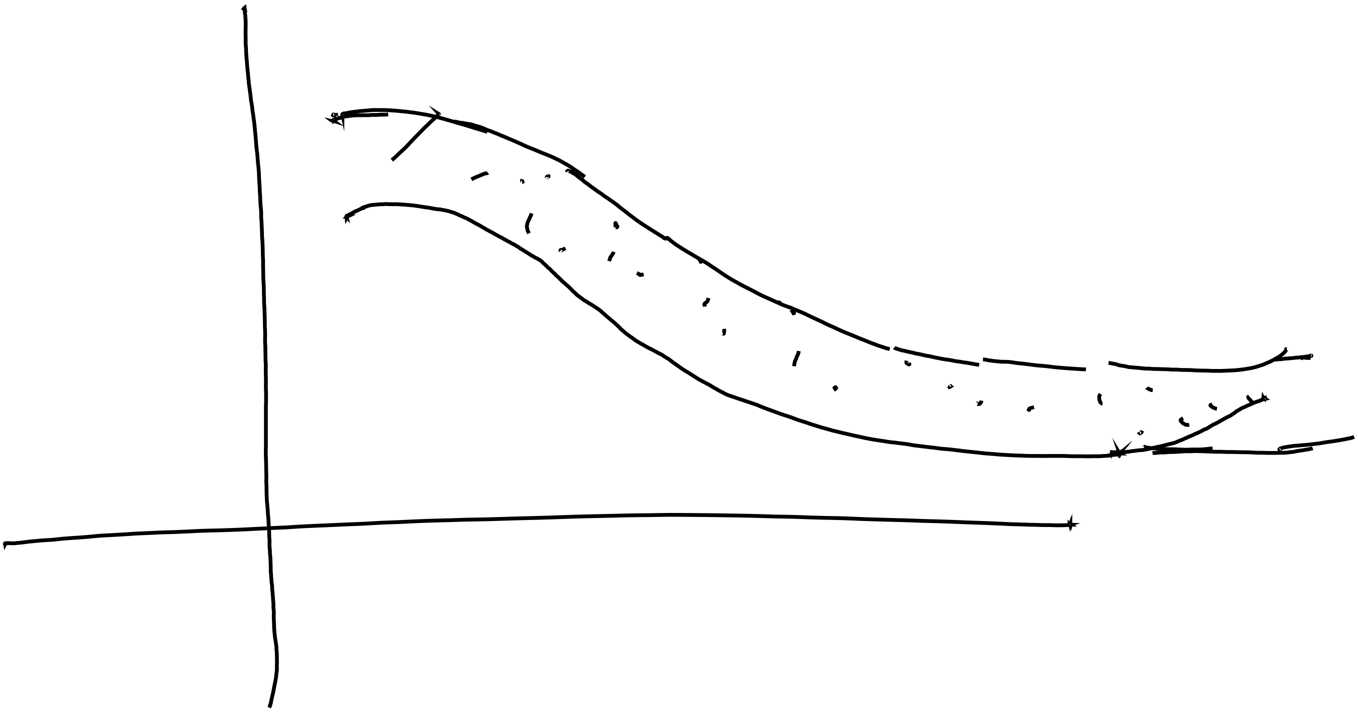


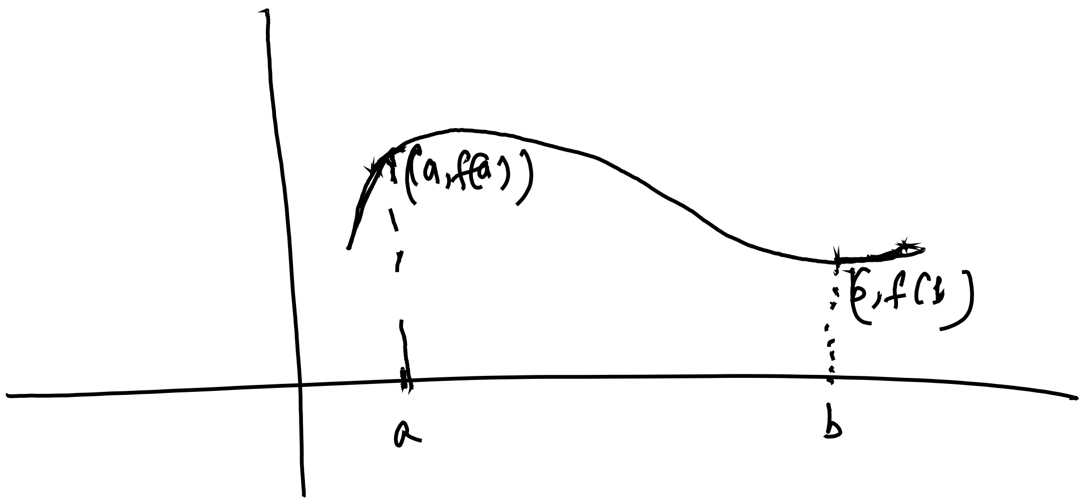
Its direction $= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

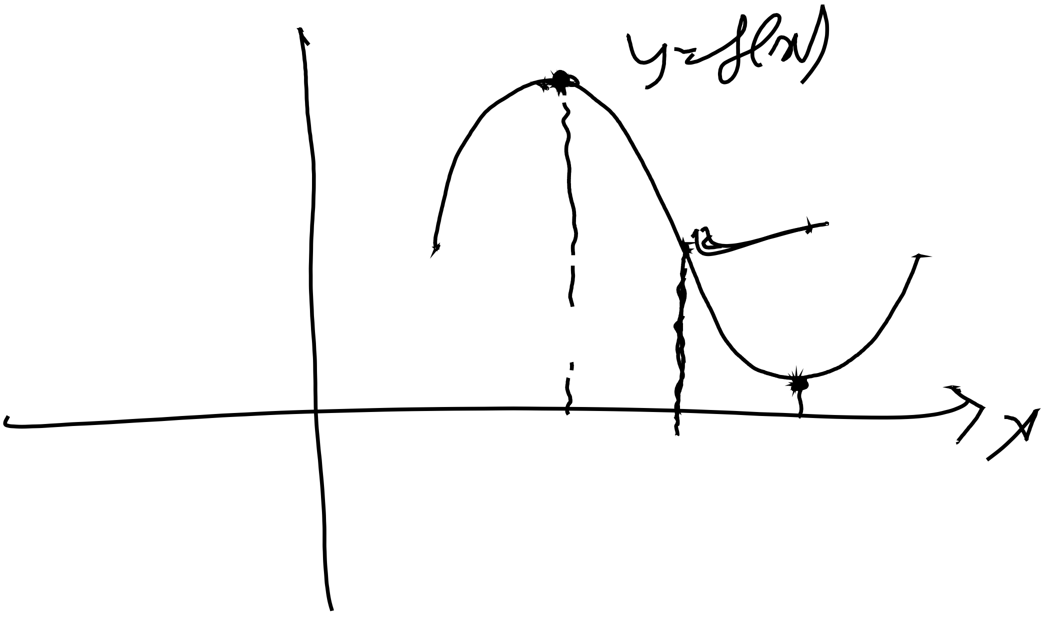
(b) **Ans** $-\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$



(c) $-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$, $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$




(a,f(a))
(b,f(b))
a
b

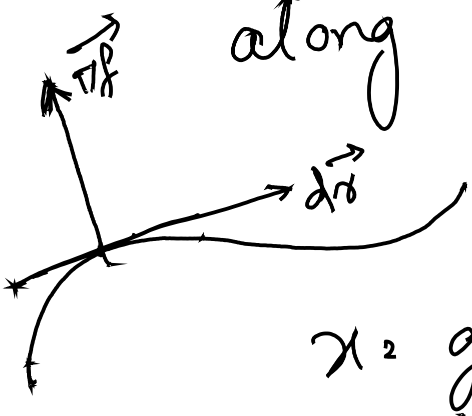

y=f(x)
x

Gradient and tangent to level curve

$$z = f(x,y)$$

Let $f(x,y) = c$ constant value

along a smooth curve



$$r = g(t)\,\hat{i} + h(t)\,\hat{j}.$$

$$x = g(t), \quad y = h(t)$$

$$\frac{d}{dt} f(x,y) = \frac{d}{dt} f\big(g(t), h(t)\big)$$

$$= \frac{d}{dt} c = 0$$

$$\Rightarrow \frac{\partial f}{\partial g} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial h} \cdot \frac{dh}{dt} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial g}\,\hat{i} + \frac{\partial f}{\partial h}\,\hat{j}\right) \cdot \left(\frac{dg}{dt}\,\hat{i} + \frac{dh}{dt}\,\hat{j}\right) = 0$$

$$\Rightarrow \nabla f \cdot \frac{dr}{dt} = 0$$

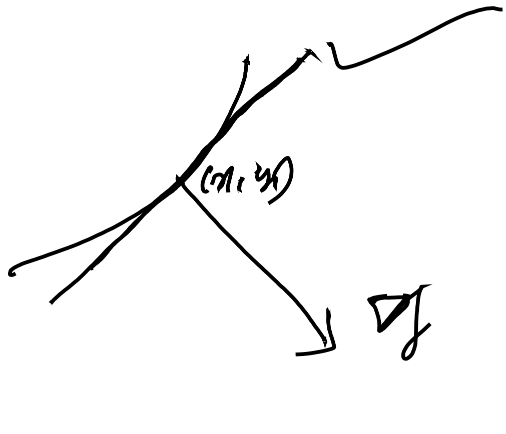
A line through a point (x_0, y_0) normal to the vector $N = A\hat{i} + B\hat{j}$ has the equation

$$\boxed{A(x-x_0) + B(y-y_0) = 0}$$

If N is the gradient vector

So the equation of the tangent line through (x_0, y_0) is

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0) = 0}$$



Exp Find the equation of the tangent line to the ellipse $\frac{x^2}{4} + y^2 = 2$ at $(-2, 1)$.

Sol^n $\quad z = f(x,y) = \frac{x^2}{4} + y^2$

$$\nabla f\Big|_{(-2,1)} = \frac{\partial f}{\partial x}\Big|_{(-2,1)} \hat{i} + \frac{\partial f}{\partial y}\Big|_{(-2,1)} \hat{j}$$

$$= \frac{x}{2}\Big|_{(-2,1)} \hat{i} + 2y\Big|_{(-2,1)} \hat{j}$$

$$= -\hat{i} + 2\hat{j}$$

So the tangent line is

$$\boxed{(-1)(x+2) + 2(y-1) = 0}$$

Properties of gradient

1. $\nabla(f+g) = \nabla f + \nabla g$

2. $\nabla(f-g) = \nabla f - \nabla g$

3. $\nabla(\alpha f) = \alpha \nabla f$ $\quad \alpha$ - scalar

4. $\nabla(fg) = f\nabla g + g \nabla f$

5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f \nabla g}{g^2}$

Equation of tangent plane to a level surface.

$$f(x, y, z) = c$$

Tangent plane at a point (x_0, y_0, z_0)

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0, z_0)} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0, z_0)} (y-y_0) + \left.\frac{\partial f}{\partial z}\right|_{(x_0, y_0, z_0)} (z-z_0) = 0$$

$$x^2+y^2 = 2$$

$$(1,1)$$

$$2x\,\hat{i} + 2y\,\hat{j}$$

$$\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$