Total derivative

$$z = f(x, y)$$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$

Th^m If $z = f(x,y)$ is defined on a region R. Suppose f_x and f_y exist throughout R and are continuous at the point (x_0, y_0) in R. If $(x_0+\Delta x, y_0+\Delta y)$ is in R, then

$$\Delta z\Big|_{(x_0, y_0)} = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy, and $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

In this case $f(x,y)$ is said to be differentiable and

$$dz = df\Big|_{(x_0, y_0)} = \frac{\partial f}{\partial x}(x_0, y_0)\,dx + \frac{\partial f}{\partial y}(x_0, y_0)\,dy$$

Exp

$$z = f(x,y) = 3x^2 - xy$$

(x_0, y_0) any point in the domain of $f(x,y)$.

$$\Delta z\Big|_{(x_0,y_0)} = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= 3(x_0+\Delta x)^2 - (x_0+\Delta x)(y_0+\Delta y) - 3x_0^2 + x_0 y_0$$

$$= \cancel{3x_0^2} + 3\Delta x^2 + 6x_0\Delta x - \cancel{x_0 y_0} - x_0\Delta y$$
$$- y_0\Delta x - \Delta x\Delta y - \cancel{3x_0^2} + \cancel{x_0 y_0}$$

$$= (6x_0 - y_0)\Delta x + (-x_0)\Delta y$$
$$+ 3\Delta x^2 - \Delta x\Delta y$$

$$= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$
$$+ 3\Delta x \cdot \Delta x + (-\Delta x)\Delta y$$

$\epsilon_1 = 3\Delta x$ $\quad \epsilon_1 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

$\epsilon_2 = -\Delta x$ $\quad \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

So $f(x,y)$ is differentiable at (x_0, y_0).

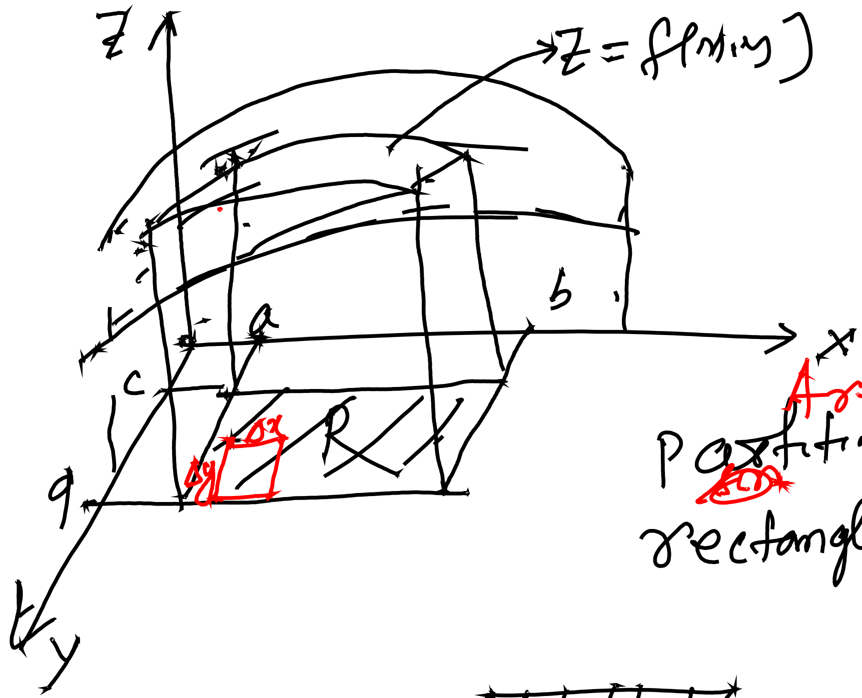
$$df\big|_{(x_0,y_0)} = (6x_0 - y_0)\,dx - x_0\,dy$$

Multiple Integrals

Double integral as volume of a solid

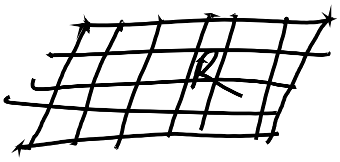
✓ double integral over a rectangular region

$z = f(x,y)$ $a \le x \le b$
$c \le y \le d$



Area of the total

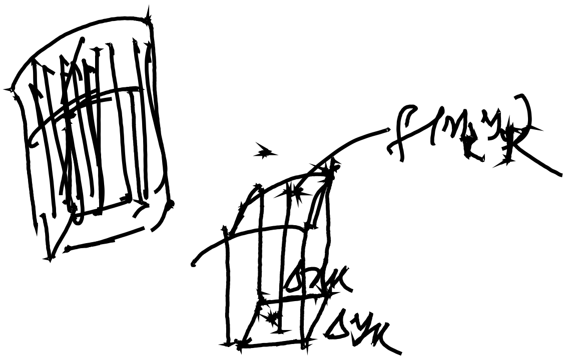
partition the plane rectangle R



The area of the small kth rectangle

$$\Delta A_k = \Delta x_k \Delta y_k$$

The volume of the small strip over the kth rectangle is



$$= f(x_k, y_k)\, \Delta A_k$$

$$= f(x_k, y_k)\, \Delta x_k \Delta y_k$$

Sum of the volume of all these small strips

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta x_k \Delta y_k$$

When the partition is very small, that is $\|P\| \to 0$ or $n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta x_k \Delta y_k$$

$$= \iint_R f(x,y)\, dx\, dy$$

= volume of that solid.

Fubini's Theorem

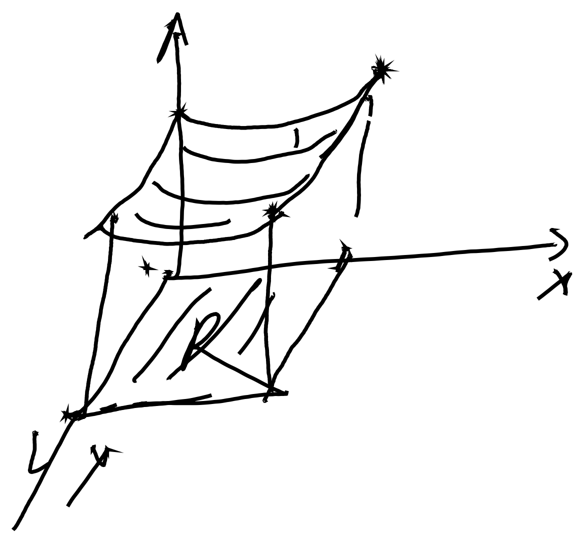
If $f(x,y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$,

then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx .$$

EXP Find the volume of the ~~region~~ solid bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \le x \le 1,\ 0 \le y \le 2$.



$$V = \iint_R (10 + x^2 + 3y^2)\,dA$$

$$= \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3\right]_0^2 dx$$

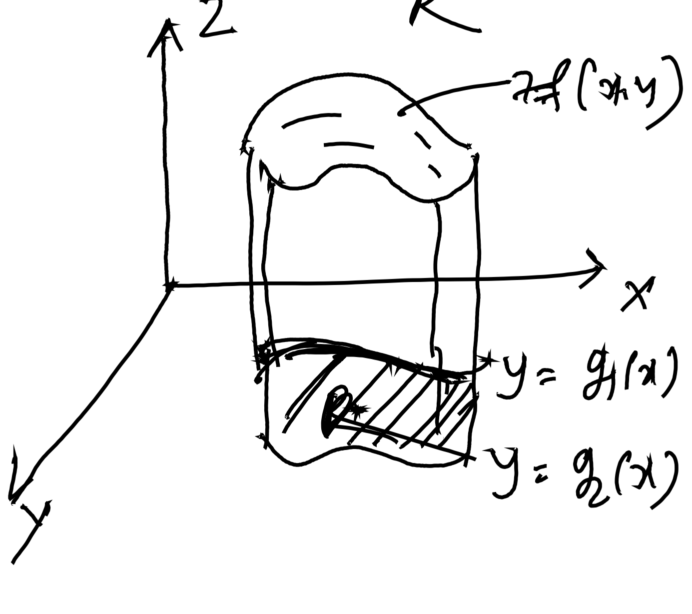
$$= \int_0^1 (20 + 2x^2 + 8)\, dx$$

$$= \left[28x + \frac{2x^3}{3}\right]_0^1 = 28 + \frac{2}{3} = \frac{86}{3}$$

Double integral over a bounded non rectangular region

If $f(x,y)$ is positive and continuous over R, we define the volume of the solid bounded above by $z = f(x,y)$ and below by R is

$$V = \iint_R f(x,y)\, dA$$

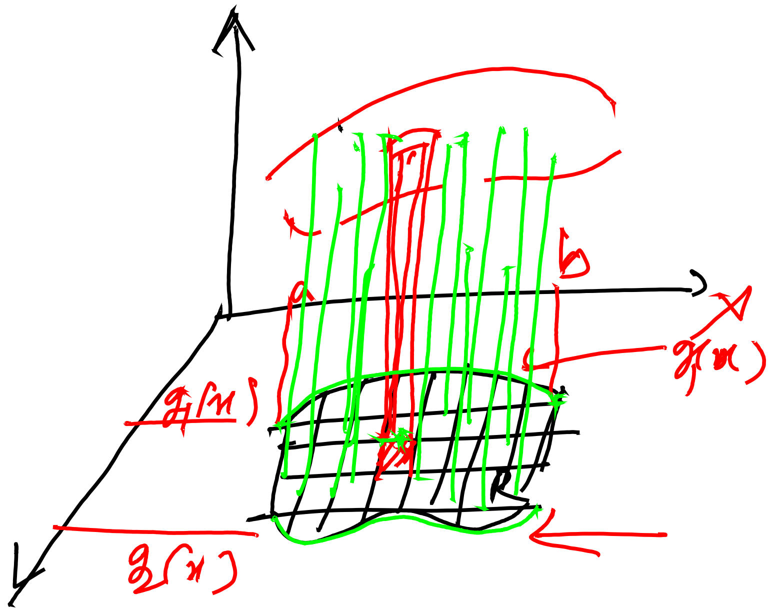


The area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} dy.$$

Then ~~integrate the area~~ the volume.

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



R is the region in xy-plane bounded above and ~~be~~ below by $y = g_2(x)$ and $y = g_1(x)$ respectively and the sides by $x = a$ and $x = b$.

Fubini's Theorem

Let $f(x,y)$ be continuous on a
a region R.

(1) If R is defined by $\left[\begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array}\right.$

with $g_1(x)$ and $g_2(x)$ continuous on $[a,b]$

then

$$V = \iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

(2) If R is defined by $\left.\begin{array}{l} c \le y \le d \\ \text{and } h_1(y) \le x \le h_2(y) \end{array}\right]$

with $h_1(y)$ and $h_2(y)$ continuous on $[c,d]$

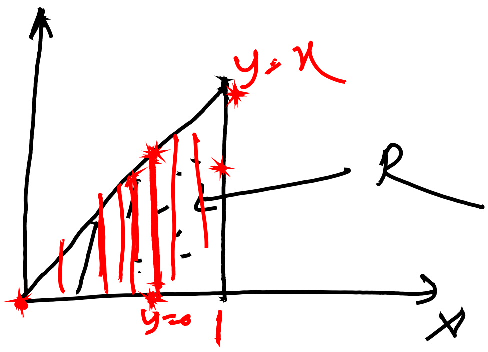
then

$$V = \iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Exp: Find the volume of the prism whose base is the triangle in the xy plane bounded by the x-axis and the lines $y=x$ and $x=1$ and whose top lies in the plane $f(x,y) = 3-x-y$.

Sol^n

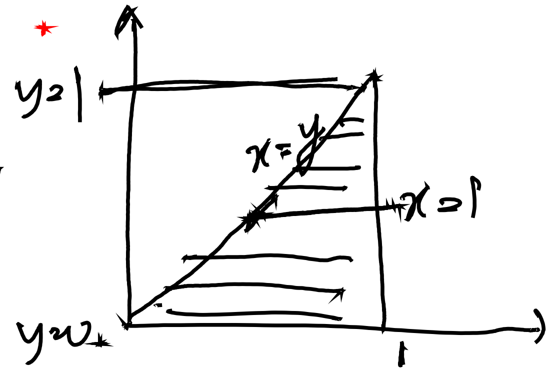
$$V = \int_{x=0}^{x=1}\int_{y=0}^{y=x} (3-x-y)\,dy\,dx$$



$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{3x^3}{6}\right]_0^1 = \frac{3}{2} - \frac{3}{6} = 1$$

$$V = \int_{y=0}^{1}\int_{x=y}^{x=1} (3-x-y)\,dx\,dy$$



$$= 1$$