LU Decomposition

August 26, 2022

We saw in the previous lecture how Guassian elimination combines a series of elementary row operations to render an arbitrary square matrix into an equivalent upper triangular matrix.

In this lecture we will see how each of these elementary row operations can be viewed as the left multiplication of the original matrix A (or an intermediate matrix, when looking beyond the first operation) by a matrix that directly encodes the elementary row operation. We will further see that all such matrices are lower triangular.

For the first type of elementary row operation (multiplying a row by a nonzero scalar), we simply left multiply the matrix by a matrix by a diagonal matrix of dimension n, where all entries are 1, exceot the one on the row we are applying the multiplying on. On that row the diagonal entry is equal to the factor by which we are multiplying.

For the second type of elementary row operation (subtracting a row from another row), we left multiply the matrix by a matrix which has a 1 on all the diagonal positions and a -1 at (j, i) where we are subtracting row j from i and $j < i$. The remaining entries are all 0.

We already established in the lecture that both these type of matrices are lower triangular. Thus, the Guassian elimination algorithm that we saw in terms of elementary row operations, in the previous lecture, is now recast in the framework of left multiplying by an appropriate lower triangular matrix.

Let us now assume it takes t elementary row operations to convert A into U, an upper triangular matrix. Let the lower triangular matrices corresponding to this sequence of elementary row operations be $E_1, \ldots, E_t$. Then we have:

$$E_t \times E_{t-1} \times \cdots \times E_1 \times A = U$$

By a simple chain of multiplying the inverses of these matrices, we get:

$$A = A = E_1^{-1} \times E_2^{-1} \times \cdots \times E_t^{-1} \times U$$

This set of lower triangular matrices all have determinant and henc inverses. It turns out that the inverse of the first type of matrix is the same matrix with just the entry on the diagonal, not equal to 1, being replaced by its reciprocal. The inverse of the second type of matrix is to replace the only occurrence of -1 by 1. It can be easily verified that these are the respective inverses. Thus after moving the multiplier matrices to the right hand side of the equation, each of them is still a lower triangular matrix. The product of several lower triangular matrices is known to be lower triangular.

Thus, we have established that these matrices can be expressed as the product of a lower triangular matrix with an upper triangular matrix.

That is

$$A = LU$$

We have thus provided an algorithm for the LU decomposition.