Complexity of Algorithms

$f(x) = a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0$

$O(x^n)$

How to measure the efficiency of algorithms?

Q. Linear Search

$i = 1$

while ($i \le n$ and $x \ne a_i$)

$i = i + 1$

if $i \le n$ then location $= i$

else location $= 0$

$a_1 \; a_2 \; \ldots \; a_n$

$2 + 2 \ldots + 2 = 2n$

$1 \qquad 2n+2$

$O(n)$

Binary search

$i = 1$
$j = n$

while $i < j$
begin

$$m = \left\lfloor \frac{i+j}{2} \right\rfloor$$

if $x > a_m$ **then** $i = m+1$
else $j = m$

end
if $x = a_i$ **then** location $= i$
else location $= 0$

$a_1 \quad a_2 \quad \cdots \quad a_m \quad a_{m+1} \quad \cdots \quad a_n$

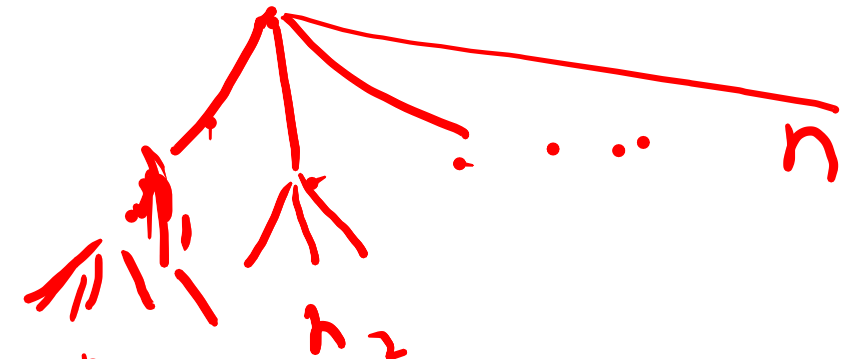
$k = \log_2 n \qquad \boxed{n = 2^k}$

$$\left.\begin{array}{cc} 2^k & 2 \\ 2^{k-1} & 2 \\ \vdots & \\ 2^2 & 2 \\ 2^1 & 2 \\ 1 & \end{array}\right\} \quad \boxed{2k+2} = 2\log n + 2$$

$$O(\log n)$$

Counting

The product Rule



a procedure is broken down into two task.
If there are n_1 ways to do the first task and each of these ways of doing the first task there are n_2 ways to do the second task. then there are $n_1 n_2$ ways to do the task.

Exp How many bit strings of length 7 are there?

_ _ _ _ _ _ _

0 0 0

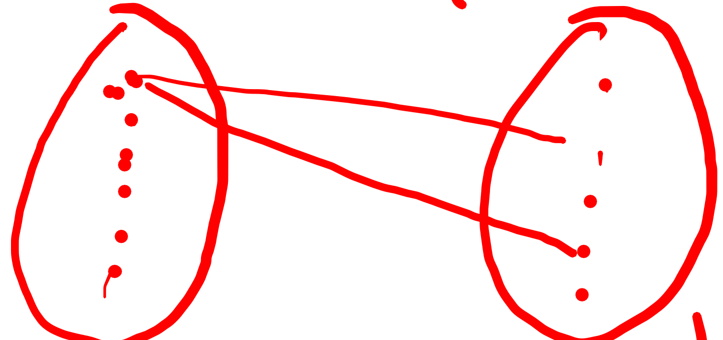
1 1 1

7

2

$$2 \times 2 \times \cdots \times 2 = 2^7$$

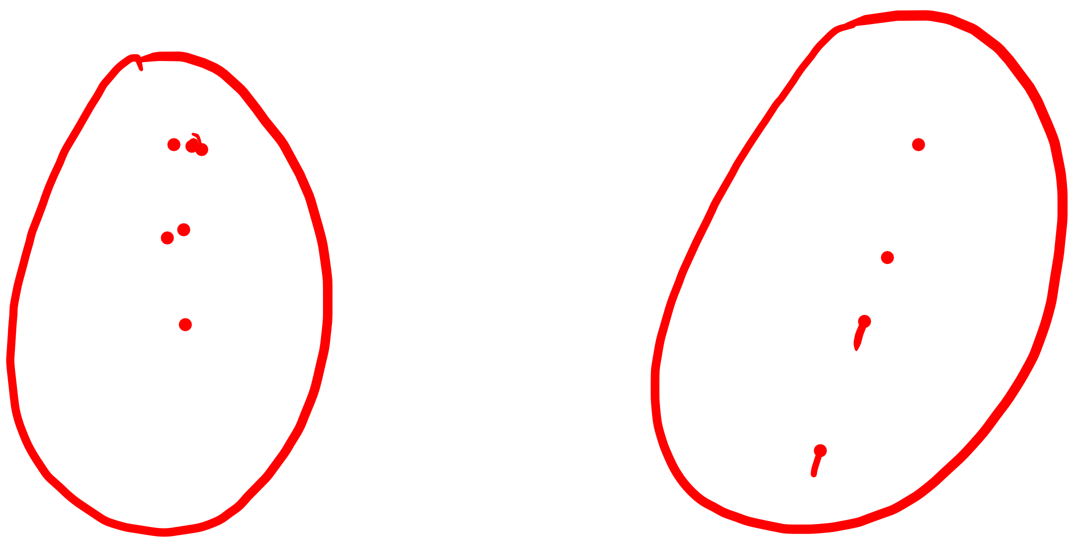
Exp How many functions are there from a set with m elements to a set with n elements?



$|A| = m$ $\qquad$ $|B| = n$

$$\underbrace{n \times n \times \cdots \times n}_{n^m}$$

Exp How many one-one functions from a set with m elements to a set with n elements



$|A| = m$ $\quad$ $|B| = n$

$m > n$ X

$m \le n$

$n(n-1)\cdot(n-2)\cdots(n-m+1)$

Sum Rule

If a task can be done either in n_1 ways or in n_2 ways when none of the set of n_1 ways is same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task.

Exp

Three list of project

A	B	C
23	15	19

No project is on more than one list

How many possible projects are there to choose from.

$$23 + 15 + 19 = 57$$

Complex Counting Problem

Exp A password ~~is~~ is six to eight characters long. where each character is an upper case letter or digit. Each password must contain at least one digit. How many possible passwords are there?

P = total no. of passwords

Sol^n $P = P_6 + P_7 + P_8$

P_6 = no. of passwords of length 6

P_7 = " " " " 7

P_8 = " " " " 8

$P = P_6 + P_7 + P_8$

$P_6 = 36^6 - 26^6$

$P_7 = 36^7 - 26^7$

$P_8 = 36^8 - 26^8$

The Inclusion-Exclusion Principle

Suppose that a task can be done in n_1 or in n_2 ways. but some of the set of n_1 ways to do the task are same as some of the n_2 ways to do the task.

- Let A_1, A_2 two set.

$|A_1|$ ways to select an element from A_1

$|A_2|$ " " " " " " A_2

- Then the no. of ways to select an element from A_1 or A_2 is $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Ex) How many bit strings of length 8 either start with a 1 bit or end with two bits 00 ?

$$\underset{2}{\underline{1}}\ \underline{\ \ }\ \underline{\ \ }\ \underset{2}{\underline{\ \ }}\ \underset{2}{\underline{\ \ }}\ \underset{2}{\underline{\ \ }}\ \underset{1}{\underline{0}}\ \underset{1}{\underline{0}}$$

length 8

$|A_1| = 2^7$ no. of bit strings start with $1 = 2^7$

$|A_2| =$ no. of bit strings end with $00 = 2^6$

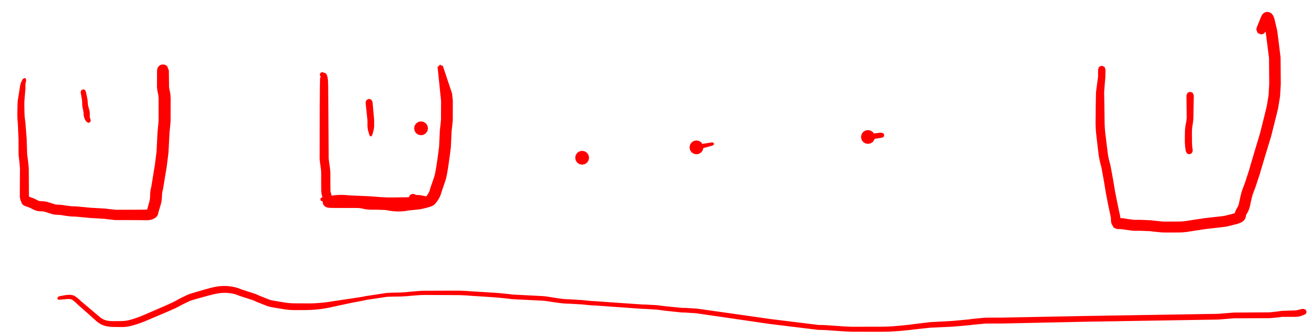
$|A_1 \cap A_2| = 2^5$

$|A_1 \cup A_2| = 2^7 + 2^6 - 2^5$

The Pigeonhole Principle

Thm If k is a positive integer and $k+1$ or more objects are placed into k boxes then there is at least one box containg two or more object.

$k+1$ objects



k boxs

Exp: 367 people

There must be at least two with same birthday

there are 366 possible birthdays

The Generalized Pigeonhole Principle

Th^m If N objects are placed into k boxes then there is at least one box containg at least $\left\lceil \frac{N}{k} \right\rceil$ objects

Exp Among 100 people there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who were born in the same month.

Exp During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some no. of consecutive days during when the team must play exactly 14 games

$a_{30} = a_1$ 1st + ... + 30th

Sol^n $1 \le a_j \le 45$ $\underset{1}{-}$ $\underset{2}{-}$ . . . $\underset{30}{-}$

Let a_j be the no. of games played on or before the jth day of the month.

$a_1, a_2 \ldots a_{30}$ is an increasing seq^n, distinct

a_1 = 1st

a_2 = 1st + 2nd

a_3 : 1st + 2nd + 3rd

$a_1 + 14, \quad a_2 + 14, \quad \ldots \quad a_{30} + 14$

$1 + 14 \le a_i + 14 \le 45 + 15$

~~increasing seq, distinct~~

$1 \le a_i \le 45$

$15 \le a_i + 14 \le 59$

$a_1 \quad a_2 \quad \ldots \quad a_{30}$ increasing seq, distinct

$1 \le a_i \le 45$

$60 \quad a_i\text{'s}$

$1 \le a_i, a_i + 14 \le 59$

$a_i = a_j + 14 \qquad a_j$