Separation of variables

This is one method to convert a partial differential equation like the Laplace's Equation into a set of ordinary differential equations. We will study this method for the various coordinate system though the spirit of the method in every system is the same.

1 Cartesian System

In the cartesian system the Laplace's equation is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

where $V(x, y, z)$ is the potential in a chargeless region. We try a solution to this equation of the form

$$V(x, y, z) = X(x)Y(y)Z(z)$$

where $X(x)$ is a function of only x, $Y(y)$ is a function of only y, and $Z(z)$ is a function of only z. Substituting this form in the Laplace's equation gives

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0$$

Multiplying this equation by $\frac{1}{V}$ gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

There are three terms in the above equation. The term $\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$ is purely a function of x. Likewise the other two terms are purely functions of y and z respectively. So we have

$$f(x) + g(y) + h(z) = 0$$

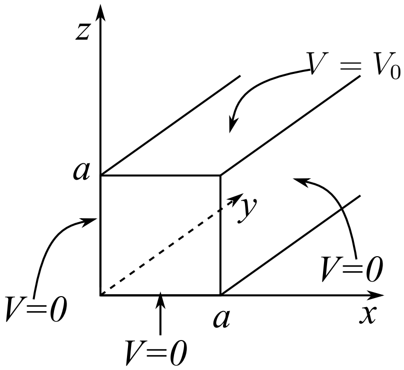
The only way we can satisfy this equation is when each of the functions $f(x), g(y)$ and $h(z)$ are constants.

$$f(x) = c_1, g(y) = c_2, h(z) = c_3$$

So

$$\frac{d^2 X}{dx^2} = c_1 X, \quad \frac{d^2 Y}{dy^2} = c_2 Y, \quad \frac{d^2 Z}{dz^2} = c_3 Z$$

Each of the above three differential equations are ordinary differential equations and easy to solve.
The kind of values that c_1, c_2 and c_3 can take depend upon the type of boundary conditions in the problem. So now we consider an example.



A metal tube with square cross section has three sides grounded while the fourth surface at $z = a$ is maintained at potential V_0. We have to find the potential at all the points inside the tube.

The potential will only vary with x and z. It is independent of y. After separation of variables in cartesian co-ordinates the Laplace's equation becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Each term is a constant. Since the sum of these constants is 0, one of them is positive while the other is negative.

Let $\quad \frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \quad$ and $\quad \frac{1}{X}\frac{d^X}{dx^2} = -k^2$

$$\begin{aligned} \therefore \quad X &= A\sin kx + B\cos kx \\ Z &= Ce^{kz} + De^{-kz} \end{aligned} \tag{1}$$

At $x = 0, V = 0 \quad \Longrightarrow \quad B = 0$
At $z = 0, V = 0 \quad \Longrightarrow \quad C + D = 0 \quad \Longrightarrow \quad D = -C$
At $x = a, V = 0 \quad \Longrightarrow \quad A\sin(ka) = 0 \quad \Longrightarrow \quad k = \frac{n\pi}{a},\ n = 1, 2, ...$

$$\begin{aligned} \therefore \quad V &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) C_n \left(e^{\frac{n\pi z}{a}} - e^{-\frac{n\pi z}{a}}\right) \\ &= \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right) 2\sinh\left(\frac{n\pi z}{a}\right) \quad \text{where} K_n = A_n C_n \end{aligned}$$

At $z = a \quad V = V_0$

$$\therefore \quad V_0 = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right) 2\sinh(n\pi) \tag{2}$$

To obtain K_n we use the following integral

$$\int_0^a \sin\frac{n\pi x}{a}dx \sin\frac{n'\pi x}{a}dx = \frac{a}{2}\delta_{nn'}$$

$$\begin{aligned} \int_0^a \sin\frac{n\pi x}{a}dx &= \frac{2a}{n\pi} && \text{for } n = 1, 3, 5, ... \\ &= 0 && \text{for } n = 2, 4, 6, ... \end{aligned}$$

Multiplying both sides of Eq.2 by $\sin \frac{n'\pi x}{a}$ and integrating from 0 to a we get

$$V_0 \frac{2a}{n'\pi} = K_{n'} a \sinh(n'\pi) \text{for} n' = 1, 3, 5, ...$$

$$\begin{aligned} \therefore \quad K_{n'} &= \frac{2V_0}{n'\pi \sinh(n'\pi)} && \text{for} \quad n' = 1, 3, 5, ... \\ &= 0 && \text{for} \quad n' = 2, 4, 6, ... \end{aligned}$$

$$\therefore \quad V(x, y, z) = \sum_{n=1,3,5,...} \frac{4V_0}{n\pi \sinh(n\pi)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi z}{a}\right)$$

This is the appropriate potential for all points inside the tube satisfying the given boundary conditions.

2 Spherical Polar system

In the spherical polar co-ordinate system, the Laplace's Equation can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Generally in this system we encounter problems that have azimuthal symmetry i.e symmetry about the zenith axis. So the potential is independent of ϕ. So $V(r, \theta, \phi)$ can be written as $V(r, \theta)$. For variable separable technique we assume

$$V(r, \theta) = R(r)\Theta(\theta)$$

Substituting this into the Laplace's equation we get

$$\Theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial R(r)}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) = 0$$

Dividing by $V(r, \theta)$ we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = 0$$

$$\therefore \quad f(r) + g(\theta) = 0$$

f is a function of only r, and g is a function of only θ. So both must be constants. It proves convenient to take these constants as

$$f(r) = l(l+1) \quad \text{and} \quad g(\theta) = -l(l+1)$$

So we have the separated ordinary differential Equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

and $$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\Theta$$

The most general solution to the radial Equation is

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

The acceptable (physically meaningful) solutions to the θ equation demands l to be a non-negative integer. They are denoted as

$$\Theta = CP_l(\cos\theta)$$

$P_l(\cos\theta)$ is a polynomial in $\cos\theta$ of degree l. They are called the Legendre polynomials.

$$\begin{aligned} P_0(\cos\theta) &= 1 \\ P_1(\cos\theta) &= \cos\theta \\ P_2(\cos\theta) &= \frac{1}{2}(3\cos^2\theta - 1) \\ P_3(\cos\theta) &= \frac{1}{2}(5\cos^3\theta - 3\cos\theta)... \end{aligned}$$

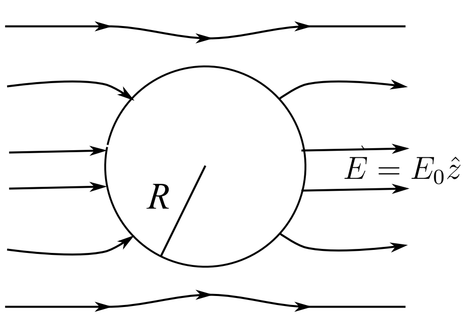
$P_l(\cos\theta)$ is even in $\cos\theta$ if l is even and odd in $\cos\theta$ if l is odd.

$$\therefore \quad V(r,\theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

The most general solution to the Laplace's Equation with Azimuthal symmetry is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Eg:
A metal sphereof radius R is placed in a region having uniform electric field $\vec{E} = E_0\hat{z}$. Find the potential at all points outside the sphere.



Far away from the sphere the electric field is uniform. $\vec{E} = E_0\hat{z}$. The field distorts only near the sphere as shown. By symmetry along the z axis it is convenient to keep the plane $z = 0$ at 0 potential. Hence the whole sphere will be at 0 potential.

Since the problem have azimuthal symmetry we have

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\vec{E} = -\vec{\nabla} V = -\left[\sum_{l=1}^{\infty} \left(l A_l r^{l-1} - (l+1)\frac{B_l}{r^{l+2}} \right) P_l(\cos\theta) - \frac{B_0}{r^2} \right] \hat{r}$$
$$+ \left[\sum_{l=1}^{\infty} \left(A_l r^{l-1} + \frac{B_l}{r^{l+2}} \right) \frac{d}{d\theta}(P_l(\cos\theta)) \right] \hat{\theta}$$

As $r \to \infty$, $\vec{E} = E_0\hat{z} = E_0 \cos\theta \hat{r} - E_0 \sin\theta \hat{\theta}$
Comparing the $\hat{r}$ component in th is limit

$$-\sum_{l=1}^{\infty} l A_l r^{l-1} P_l(\cos\theta) = E_0 \cos\theta = E_0 P_1(\cos\theta) \tag{3}$$

The legendre's polynomial $P_l(\cos\theta)$ are linearly independent of each other and they satisfy the following orthogonality conditions

$$\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$$

So multiplying Equation 3 both sides by $P_{l'}(\cos\theta)$ and integrating from 0 to π we get

$$-l' A_{l'} r^{l'-1} \frac{2}{2l'+1} = E_0 \frac{2}{2l'+1} \delta 1l'$$

$$\therefore \quad -A_1 = E_0 \quad \implies \quad A_1 = -E_0$$
$$A_2 = A_3 = ... = 0$$

The potential at $r = R$ is 0

$$\therefore \quad \sum_{l=0\infty} \left(A_l R^l + \frac{B_l}{R^{l+1}} \right) P_l(\cos\theta) = 0$$

Since $P_l(\cos\theta)$ are all linearly independent

$$B_l = -A_l R^{2l+1} \;\; ; \;\; B_0 = -A_0 R$$
$$\therefore \;\; B_2 = B_3 = ... = 0$$
$$B_1 + -A_1 R^3 = E_0 R^3$$

$$\therefore \;\; V = A_0 \left(1 - \frac{R}{r} \right) + \left(-E_0 r + \frac{E_0 R^3}{r^2} \right) \cos\theta$$

The electric field at $r = R$ is

$$\vec{E}(R) = \left(-\frac{A_0}{R} + 3E_0\cos\theta\right)\hat{r}$$

Since the sphere is chargeless

$$\oint_{sphere} \vec{E}\cdot\hat{n}da = 0 \quad \Longrightarrow \quad -\frac{A_0}{R}\times 4\pi R^2 = 0 \quad \Longrightarrow \quad A_0 = 0$$

$$\therefore \quad V = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$