If $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some nbd of a point $z = x+iy$ and analytic at z.

Then at that point the 1st order partial derivatives of u & v exist and they satisfy CR equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$\underline{f(z) = z^2} = (x+iy)^2$

$= x^2 - y^2 + i\,2xy$

$u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

C.R equations are satisfying.

$$f(z) = \bar{z}$$
$$= x - iy \qquad u(x,y) = x$$
$$v(x,y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$f(z) = \bar{z}$ is not analytic.

Theorem

If two real valued continuous functions $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives and satisfy the C-R equations in some ~~or~~ domain D.

Then the function

$f(z) = u(x,y) + i v(x,y)$ is analytic in D.

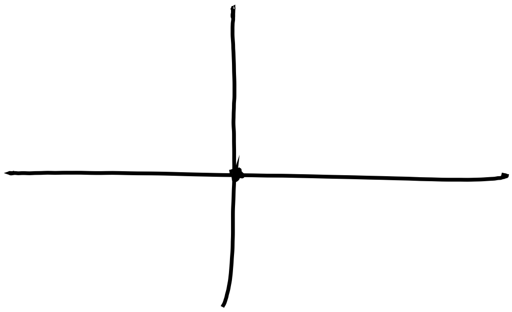
$f(z) = |z|^2$ $\qquad$ $z = x+iy$

$$= \underbrace{x^2+y^2}_{u(x,y)} \qquad v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$$

$(0,0)$



The function $f(z) = |z|^2$ is differentiable at $(0,0)$ but not analytic at $(0,0)$.

Q Let $f(z)$ be analytic and whose real part is constant. Then the function is constant function.

$$\boxed{\begin{array}{l} a+ie^{y} \quad \times \\ \underset{u}{a}+i\underset{v}{y} \\ \frac{\partial u}{\partial x}=0, \quad \frac{\partial v}{\partial y}=1 \end{array}}$$

Proof

$f(z) = u+iv$ analytic

$u = C$ constant

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$0 = \frac{\partial v}{\partial y} \Rightarrow v \stackrel{*}{=} C$$

$\Rightarrow f$ is constant function.

Q

$f(z)$ analytic function whose ~~of~~ modulus is constant. Then $f(z)$ is constant function.

Sol^n

$$f(z) = u + iv$$

$$|f(z)| = \sqrt{u^2+v^2} = K \quad \text{constant.}$$

$$\Rightarrow u^2+v^2 = K^2$$

Take partial derivatives w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{——} (1)$$

Similarly take partial derivatives w.r.t y.

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \text{——} (2)$$

Since $f(z)$ is analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

& $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

From (1) $\quad u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{—(3)}$

From (2) $\quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0 \quad \text{—(4)}$

Multiplying (3) coeffn u and
(4) coeffn v.

$$u^2\frac{\partial u}{\partial x} - uv\cancel{\frac{\partial u}{\partial y}} = 0$$

$$+ \quad uv\cancel{\frac{\partial u}{\partial y}} + v^2\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \underline{(u^2+v^2)\frac{\partial u}{\partial x} = 0}$$

Either $\underline{u=0 \text{ and } v=0}$ or $\frac{\partial u}{\partial x} = 0$

then $f(z)=0$ which is constant.

If $\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$

$\Rightarrow u = c_1, \; v = c_2$

$\Rightarrow f(z) = c_1 + ic_2$ is constant.

$$\boxed{\begin{aligned} f(z) &= r\cos\theta + i\, r\sin\theta = r e^{i\theta} \\ |f(z)| &= r \end{aligned}}$$

CR equation in polar form

$f(z) = f(r, \theta)$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}}$$

If $f(z) = u(x,y) + i v(x,y)$ is analytic in a domain D.

Then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ Laplace equation

u is harmonic

v is also harmonic

analytic

$$f(z) = \underline{\underline{z^2}} = \overline{(x+iy)^2}$$

$$= x^2 - y^2 + i\,2xy$$

$$u(x,y) = x^2 - y^2$$

$$v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

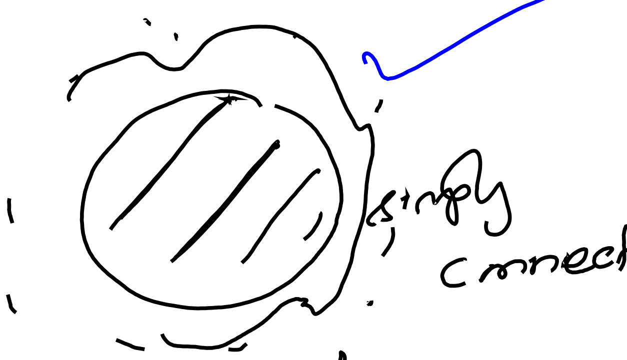
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial^2 v}{\partial x^2} = 0$$

$$\frac{\partial v}{\partial y} = 2x \qquad \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

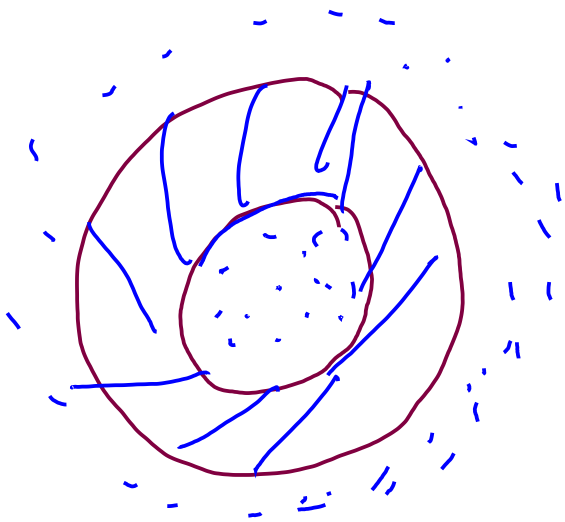
Simply connected domain



simply connected

Not simply connected

It's complement is connected with respect to extended complex plane



Not simply connected

$u(x,y) + i\,v(x,y)$

Result

Every harmonic function $u(x,y)$ in a simply connected domain has a harmonic conjugate $v(x,y)$ and

$$f(z) = u(x,y) + i\,v(x,y) \text{ is analytic.}$$

Exp $u(x,y) = \boxed{x^2 - y^2}$ $+ \, i \, V$?

u is harmonic

Find the harmonic conjugate

that is find v such that

$u + iv$ is analytic function.

Sol^n

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$ (C-R equation)

$$\Rightarrow \frac{\partial v}{\partial y} = 2x$$

$$\Rightarrow v = 2xy + \phi(x)$$

$\phi(x)$ to be determined.

$$\frac{\partial v}{\partial x} = 2y + \phi'(x)$$

By C-R equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$-2y = -2y - \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \phi(x) = K$$

$$\Rightarrow \boxed{V = 2xy + K}$$ Answer

Laplace eqn in polar form

$$\boxed{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$