Complex Variables

A complex number z is an ordered pair (x, y), $x, y \in \mathbb{R}$.

written as

$$z = (x, y) \quad \text{or} \quad z = x + iy$$

$x \to$ real part

$y \to$ imaginary part

with the following structural properties.

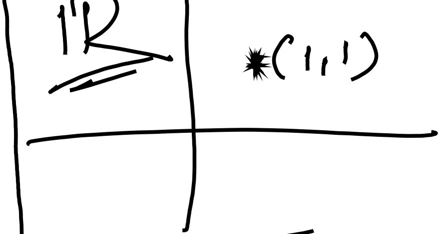
(1) Two complex numbers are equal iff their real parts are equal and their imaginary parts are equal.

(2) $z_1 \pm z_2$

$= (x_1 \pm x_2) + i(y_1 \pm y_2)$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$



$(x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

$(x_1, y_1)(x_2, y_2)$

$= x_1 x_2 + y_1 y_2$

(3) $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= \underbrace{(x_1 x_2 - y_1 y_2)}_{\text{Real part}} + i\,\underbrace{(x_1 y_2 + x_2 y_1)}_{\text{Imaginary}}$$

$$\boxed{\begin{aligned} &\mathbb{R}^2 \quad (x_1, y_1) \quad (x_2, y_2) \\ &= x_1 x_2 + y_1 y_2 \end{aligned}}$$

(4) $$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

$z_2 \neq 0$

$$= \frac{x_1 x_2 + y_1 y_2 + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$$

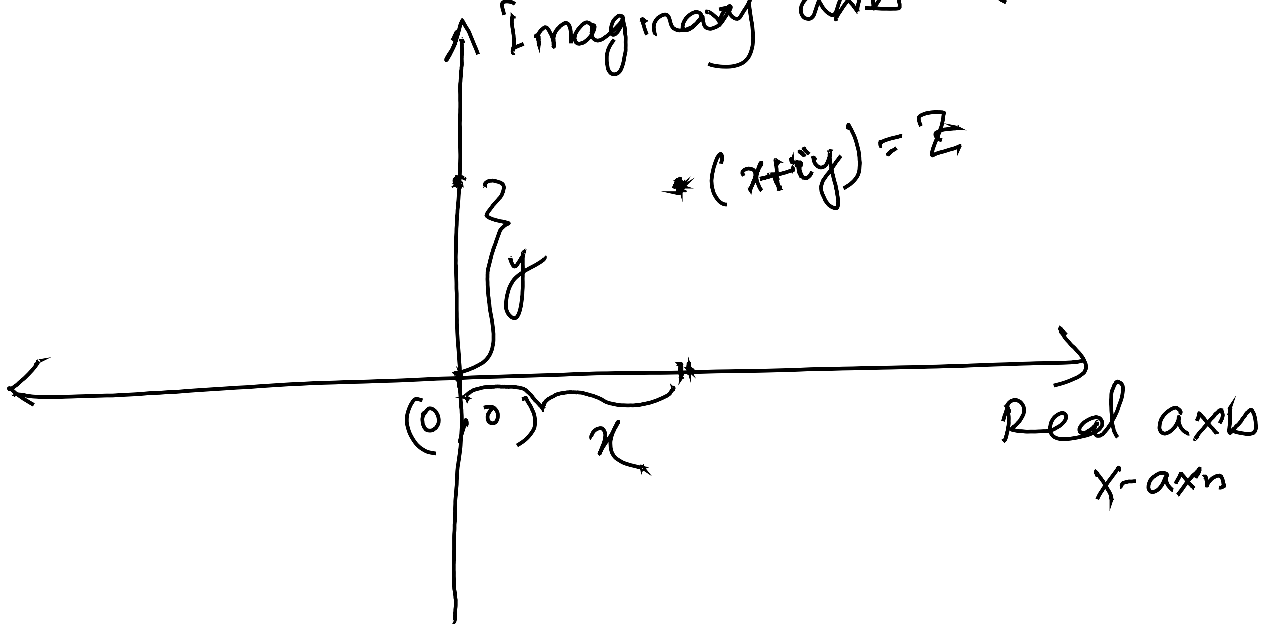
$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$$

Real path

Imaginary part

Complex Plane

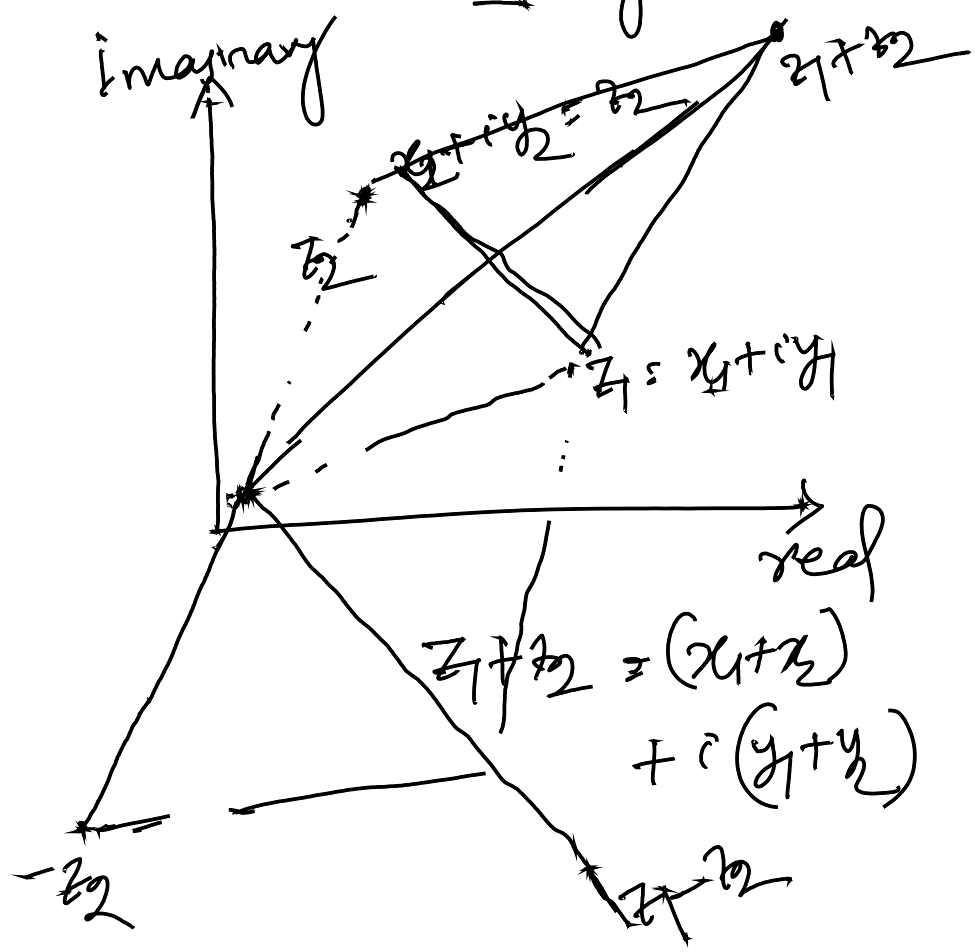
We represent all complex numbers in a plane.



$z = x + iy$

$z_1 + z_2$

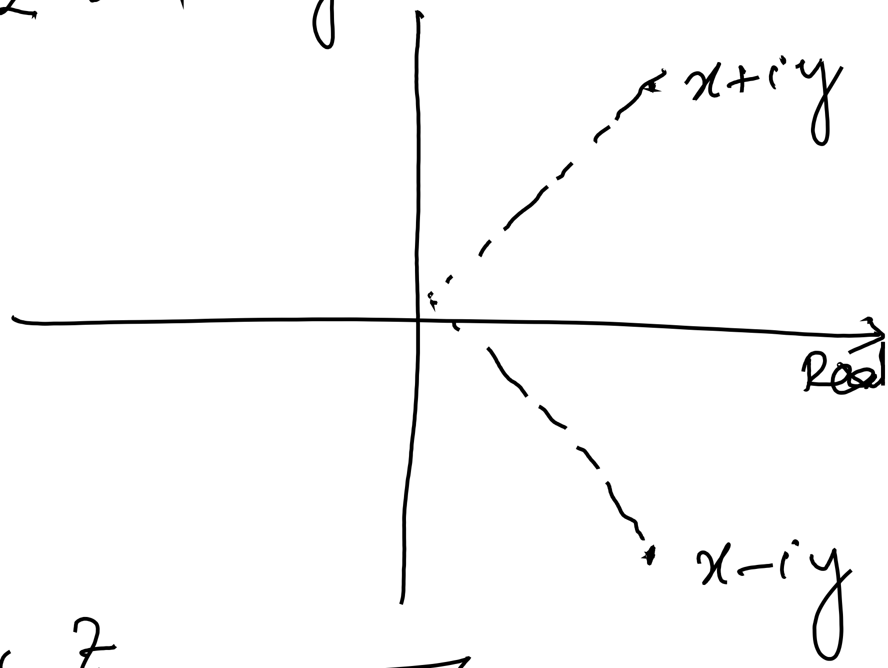
$z_1 - z_2$



Complex conjugate

$z = x + iy$

Conjugate $\bar{z} = x - iy$



Real part of z

$$= \boxed{x = \frac{z + \bar{z}}{2}}$$

Imaginary part of z

$$\boxed{y = \frac{z - \bar{z}}{2i}}$$

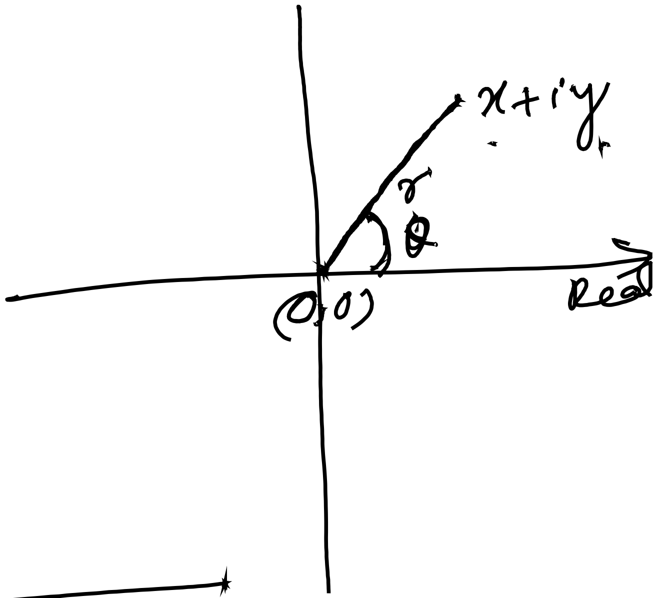
① $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ ③ $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

② $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ ④ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Polar form of complex numbers

$z = x + iy$

$x = r\cos\theta \quad , \quad y = r\sin\theta$



$z = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= r e^{i\theta}$

$|z| = |x+iy| = \sqrt{x^2+y^2}$

$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= r$

$|z| = |re^{i\theta}| = r$

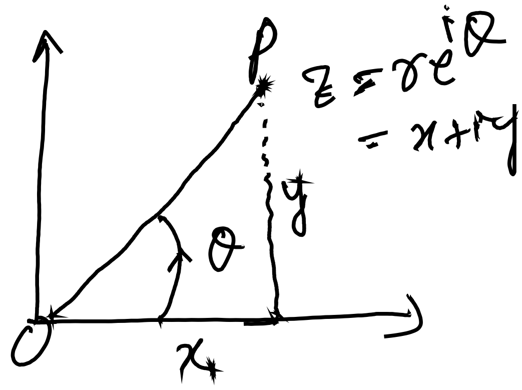
$|z| \rightarrow$ distance of the complex number z from the origin.

$|z_1 - z_2| \rightarrow$ distance between two complex numbers z_1 and z_2.

Argument of a complex number

$$z = x + iy = re^{i\theta}$$

θ is called the argument of z denoted by $\arg(z)$



$$\theta = \arg(z)$$

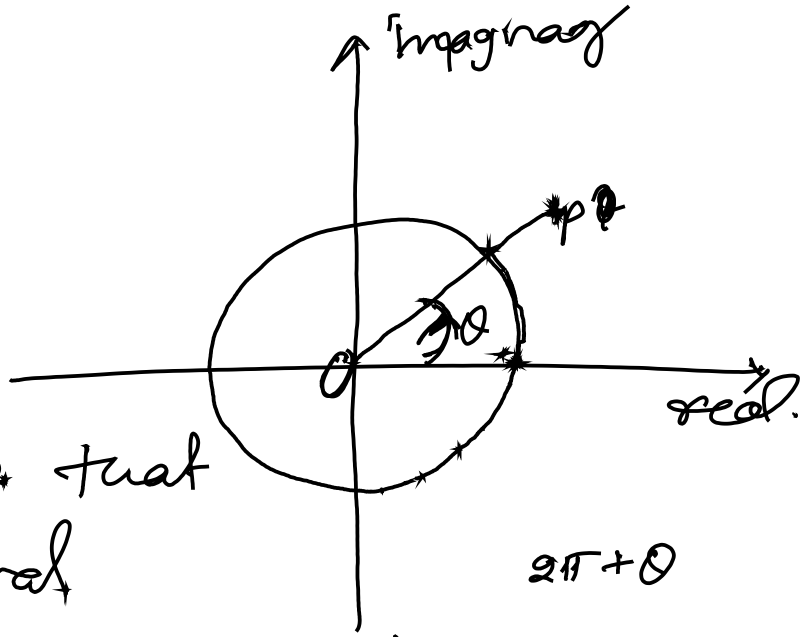
$$= \tan^{-1}\frac{y}{x}$$

$$= \arctan\frac{y}{x}$$

θ is the directed angle from the positive x-axis to the OP.

All angles are measured in radians and positive in counter clockwise direction.

Principal value of the argument of z



The value of θ that lies in the interval $-\pi < \theta \leq \pi$ is called the principal value of the argument of z. $(z \neq 0)$

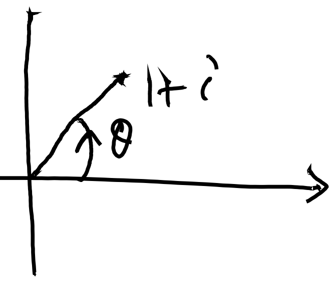
$\text{Arg}(z) \rightarrow$ Principal value of the argument of z.

$$-\pi < \text{Arg}(z) \leq \pi$$

<u>Note</u> We must be careful to the quadrant in which z lies.

$z = 1+i$

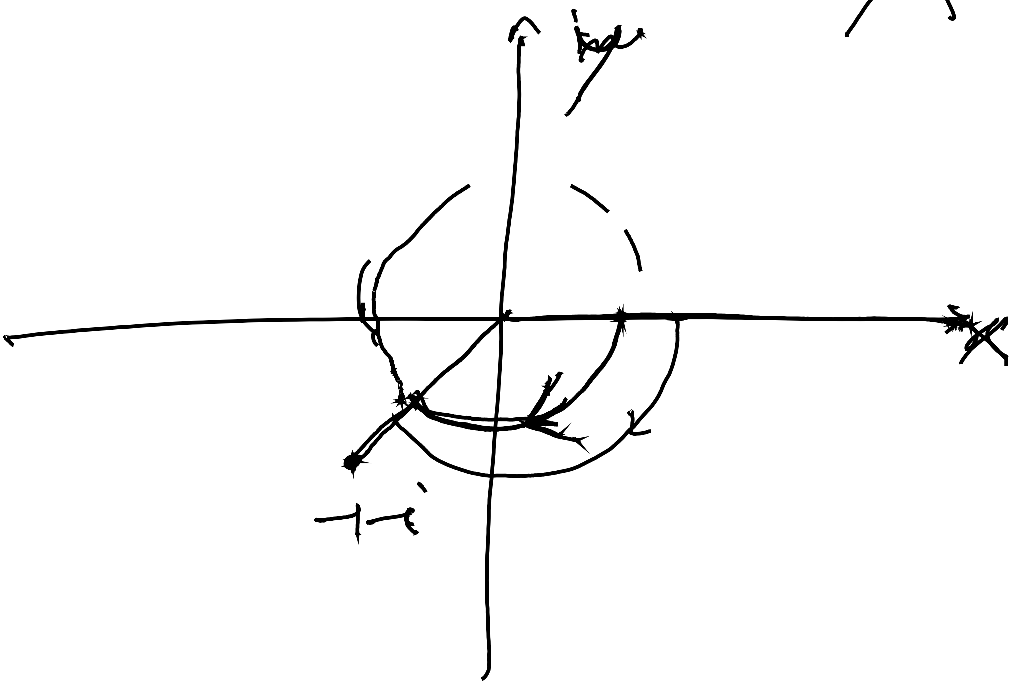
$\text{Arg}(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$



$z = -1 - i$

$$\text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{-1}\right) =$$

$$= \tan^{-1} 1 = \pi/4$$



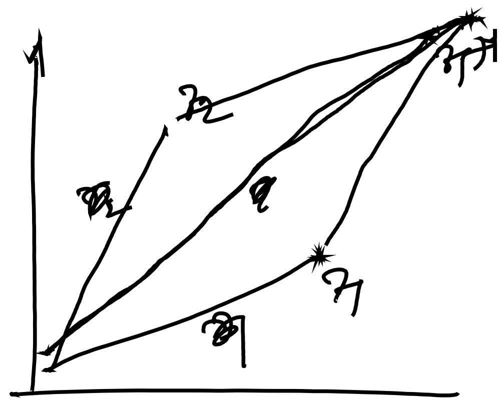
$$\text{Arg}(z) = -\frac{3\pi}{4}$$

$$\text{Arg}(z) = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } x > 0 \\ \pi + \tan^{-1}\frac{y}{x} & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi + \tan^{-1}\frac{y}{x} & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0,\ y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0,\ y < 0 \end{cases}$$

Triangle inequality

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$$

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$



$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Multiplication and division in polar form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$$

$$|z_1 z_2| = |r_1 e^{i\theta_1} \, r_2 e^{i\theta_2}|$$

$$= |r_1 r_2 e^{i(\theta_1+\theta_2)}|$$

$$= r_1 r_2 = |z_1||z_2|$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \left|\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}\right|$$

$$= \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\boxed{\begin{aligned}\arg(z_1 z_2) &= \theta_1+\theta_2 \\ &= \arg(z_1)+\arg(z_2)\end{aligned}}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\boxed{\begin{aligned}\arg\left(\frac{z_1}{z_2}\right) &= \theta_1-\theta_2 \\ &= \arg(z_1)-\arg(z_2)\end{aligned}}$$

De-Moivre's formula

$$z = r(\cos\theta + i\sin\theta)$$

$$z^n = z \cdot z \cdots z$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

When $|z|=1$. $\boxed{(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

Roots

If $z = w^n$ $\qquad f(z) = z^{1/n}$

$w = z^{1/n}$ $\qquad z \neq 0$

nth root of z.

multivalued function.

If we write $z = r(\cos\theta + i\sin\theta)$

and $w = R(\cos\phi + i\sin\phi)$

Then

$$w^n = R^n(\cos n\phi + i\sin n\phi)$$

$$= z = r(\cos\theta + i\sin\theta)$$

Equating $R^n = r \Rightarrow R = r^{1/n}$

$$n\phi = \theta + 2k\pi$$

$$\Rightarrow \phi = \frac{\theta + 2k\pi}{n}$$

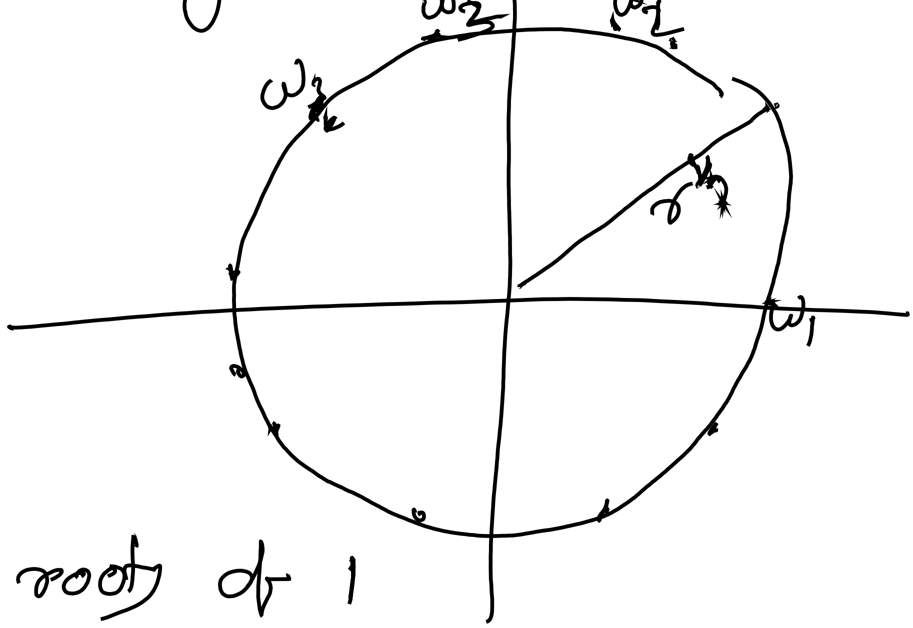
$k = 0, 1, 2, \ldots, n-1$

There n complex numbers which we get by putting the values of R and ϕ.

$\omega_1, \omega_2, \ldots, \omega_n$

are called the nth roots of Z.

The all lie on a circle of radius $r^{1/n}$ with center at the origin.



<u>Find nth roots of 1</u>

$Z = 1$ we have $|z| = r = 1$

$Arg\, z = 0 = \theta$

Then $\sqrt[n]{1} = 1^{1/n}\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$

$k = 0, 1, 2, \ldots, n-1$

There are nth roots of 1.

If ω corresponds to the complex number corresponds to $\underline{k=1}$

Then n values of $\sqrt[n]{1}$ can be written as $1, \omega, \omega^2, \dots, \omega^{n-1}$

$$\boxed{\begin{aligned} &\omega^n = 1 \\ &1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0 \end{aligned}}$$

$\cancel{f_2}(x_1, x_2, \dots x_n)$

$\cancel{f_1}(x_1, x_2, \dots x_n)$

f_1

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \dots \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \dots \\ \vdots & \vdots & \\ 1 & \frac{\partial f_m}{\partial x_1} & \dots \end{bmatrix}$$