SC223 - Linear Algebra

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Lecture 3

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Linear Combination of vectors

- **Linear combination** of a_{*i} and a_{*j} with real numbers x_i and x_j is defined as

$$x_i \cdot \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} + x_j \cdot \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} x_i a_{1i} + x_j a_{1j} \\ x_i a_{2i} + x_j a_{2j} \\ \vdots \\ x_i a_{mi} + x_j a_{mj} \end{bmatrix}$$

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- Linear combination of n vectors:

$$x_1 \cdot \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{a_{*1} \in \mathbb{R}^n} + x_2 \cdot \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{a_{*2} \in \mathbb{R}^n} + \ldots + x_n \cdot \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{a_{*n} \in \mathbb{R}^n} = \begin{bmatrix} \sum_{j=1}^n x_j a_{1j} \\ \sum_{j=1}^n x_j a_{2j} \\ \vdots \\ \sum_{j=1}^n x_j a_{mj} \end{bmatrix}$$

- In general, for an $m \times n$ system of linear equations $A_{m\times n}x = b$,

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- If $\exists z \in N(A), z \neq \mathbf{0}_n$, then $Ax = b$ will have infinitely many solutions, if one exists!

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- **Beware of the notation:** a_{i*} denotes the i^{th} row of A *written as a column matrix*.

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- Since this is the same as Nullspace of the matrix A^T, left nullspace is denoted by $N(A^T)$.

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$$\Rightarrow v^T v = 0 \Rightarrow v = A^T y = x = \mathbf{0}_n.$$

Thus $C(A^T) \cap N(A) = \mathbf{0}_n$.

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- Let $x \in C(A^T) \cap N(A)$. Since $x \in C(A^T), \exists y \in \mathbb{R}^m$ such that $x = A^T y$, and since $x \in N(A), Ax = \mathbf{0}_m$.
- Thus,

$$Ax = AA^T y = \mathbf{0}_m$$

$$y^T A \underbrace{A^T y}_{v \in \mathbb{R}^n} = 0$$

$$\Rightarrow v^T v = 0 \Rightarrow v = A^T y = x = \mathbf{0}_n.$$

Thus $C(A^T) \cap N(A) = \mathbf{0}_n$.
- Similarly, $C(A) \cap N(A^T) = \mathbf{0}_m$.

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$$\begin{bmatrix} a_{*1} & \dots & a_{*r} & a_{*r+1} & \dots & a_{*n} \end{bmatrix}_{m\times n}$$
$$= \begin{bmatrix} a_{*1} & \dots & a_{*r} \end{bmatrix}_{m\times r} \underbrace{\begin{bmatrix} I_{r\times r} & b_{*r+1} & \dots & b_{*n} \end{bmatrix}_{r\times n}}_{C}$$

$$\begin{bmatrix} a_{1*}^T \\ a_{2*}^T \\ \vdots \\ a_{m*}^T \end{bmatrix} = \begin{bmatrix} a_{*1} & \dots & a_{*r} \end{bmatrix}_{m\times r} \begin{bmatrix} c_{1*}^T \\ c_{2*}^T \\ \vdots \\ c_{r*}^T \end{bmatrix}_{r\times n}$$