Partial derivatives

$z = f(x,y)$

The partial derivative of $f(x,y)$ with respect to x at (x_0, y_0)

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}}$$

provided the limit exists.

$$= \frac{d}{dx}\left(f(x, y_0)\right)_{x=x_0}$$

f_x, $\frac{\partial f}{\partial x}$, z_x, $\frac{\partial z}{\partial x}$

Similarly the partial derivative of $f(x,y)$ with respect to y at the point (x_0, y_0) is

$$\boxed{\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}}$$

$$= f_y$$

Exp

$$\boxed{f(x,y) = \frac{2y}{y+\cos x}}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)$$

$$= \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)\times 0 - 2y(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2\cos x}{(y+\cos x)^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f(x,y) = \begin{cases} \dfrac{x}{y}, & xy \neq 0 \quad \cancel{(x,y) \neq (0,0)} \\ 0, & xy = 0 \quad \cancel{(x,y) = (0,0)} \end{cases}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0) = (x_0, y_0)}$$

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\boxed{\text{function is not continuous at } (0,0) \text{ but the partial derivative exists at } (0,0)}$

Exp $\boxed{yz - \ln z = x + y}$

where $z(x,y)$, Find $\frac{\partial z}{\partial x}$

Sol^n

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = \frac{\partial}{\partial x}(x+y)$$

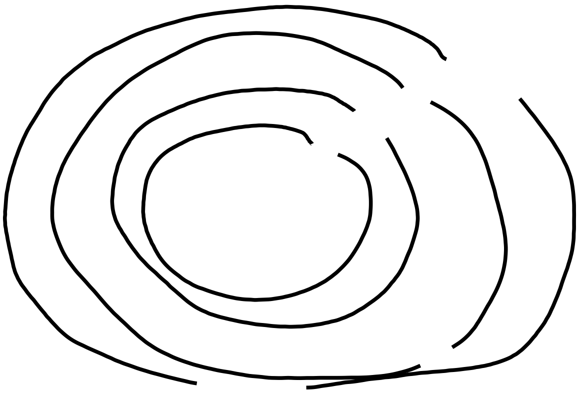
$$\Rightarrow y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1+0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz-1}$$

Exp

Find the slope of a surface $z = x^2+y^2$ in the y-direction at the point $(1,2)$.

$$\cancel{\partial f} \quad \frac{\partial z}{\partial y}\Big|_{(1,2)} = 4$$



$$z = f(x_1, x_2, \ldots, x_n)$$

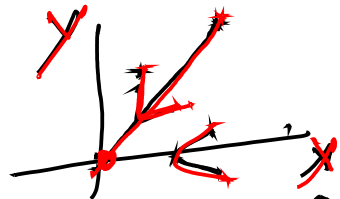
The partial derivatives similarly defined.

$$\frac{\partial z}{\partial x_1}\Big|_{(x_1^0, x_2^0, \ldots, x_n^0)} = \lim_{h \to 0} \frac{f(x_1^0+h, x_2^0, \ldots, x_n^0) - f(x_1^0, x_2^0, \ldots}{h}$$

Exp

$$f(x,y)=\begin{cases}0, & xy\neq 0\\ 1, & xy=0\end{cases}$$

whether $f(x,y)$ is continuous at $(0,0)$



Along the path $y=x$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0$$

Along the path $y=0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 1 = 1$$

So limit does not exist.

So $f(x,y)$ is not continuous.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{1-1}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0}\frac{1-1}{k} = 0$$

$$f(x,y) = \begin{cases} 1 & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

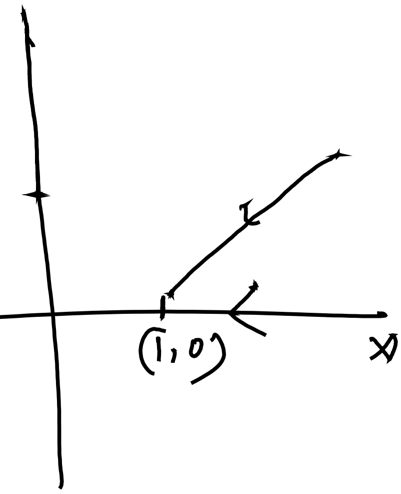
$$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = \lim_{h\to 0} \frac{f(1+h,0) - f(1,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,0)} = \lim_{K\to 0} \frac{f(1,0+K) - f(1,0)}{K}$$

$$= \lim_{K\to 0} \frac{1-0}{K}$$ does not exist

$$\lim_{(x,y)\to(1,0)} f(x,y)$$



Along x-direction $y=0$

$$\lim_{(x,y)\to(1,0)} f(x,y) = \lim_{(x,y)\to(1,0)} 0 = \underline{\underline{0}}$$

Along $\underline{\underline{y = x\cancel{+1}}}$

$$\lim_{(x,y)\to(1,0)} f(x,y) = \lim_{(x,y)\to(1,0)} 1 = \underline{1}$$

So $f(x,y)$ is not continuous.

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = f_{yx}$$

Mixed derivative theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a,b) and all are continuous at (a,b).

then $f_{xy}\Big|_{(a,b)} = f_{yx}\Big|_{(a,b)}$.

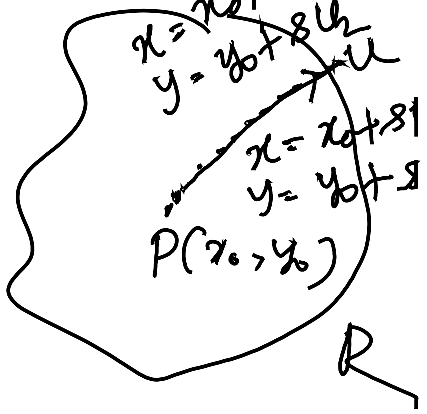
Directional derivative

If we want to investigate the rate of change of a surface $z = f(x,y)$ along any ~~other~~ direction other than x-direction or y-direction

$z = f(x,y)$ is defined throughout a region R in the xy-plane

$P(x_0, y_0)$ is a point in R.

$u = u_1\hat{i} + u_2\hat{j}$ is a unit vector

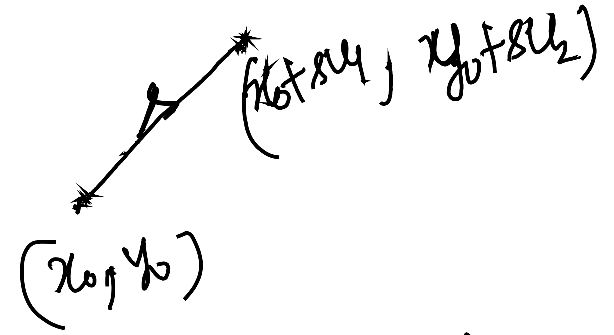


then the equations

$$x = x_0 + su_1 \quad , \quad y = y_0 + su_2$$

parametrize the line through P_0 in the direction of u.

$u = u_1\hat{i} + u_2\hat{j}$



<u>Def^n</u> The directional derivative of f at $P(x_0, y_0)$ in the direction of the unit vector $u = u_1\hat{i} + u_2\hat{j}$ is denoted by

$$\left(\frac{df}{ds}\right)_{u,P} = \cancel{\lim_{s\to 0} f(x_0+su_1, y_0+su_2) - f(x_0}$$

$$= \lim_{s\to 0} \frac{f(x_0+su_1,\ y_0+su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

<u>If $u = \hat{i}$</u> along x-direction

$$\left.\frac{df}{ds}\right|_{\hat{i},P} = \left.\frac{\partial f}{\partial x}\right|_P$$

If $u = \hat{j}$ along y-direction

$$\left.\frac{df}{ds}\right|_{\hat{j},P} = \left.\frac{\partial f}{\partial y}\right|_P$$

Exp

Find the derivative of $f(x,y) = x^2 + xy$ at $(1,2)$ in the direction of $u = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Sol^n

$$\left(\frac{df}{ds}\right)_{u,P} = \underline{\underline{(D_u f)_{P_0}}}$$

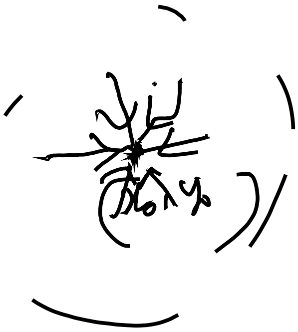
$$= \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0, y_0)}{s}$$

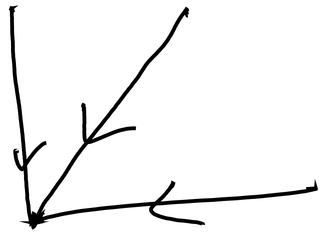
$$= \lim_{s\to 0} \frac{(x_0+su_1)^2 + (x_0+su_1)(y_0+su_2) - (x_0^2 + x_0y_0)}{s}$$

$$= \lim_{s\to 0} \frac{\left[\left(1+\frac{s}{\sqrt{2}}\right)^2 + \left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right) - (1^2 + 1\times 2)\right]}{s}$$

$$= \lim_{s\to 0} \frac{\left[1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2} + 2 + \frac{s}{\sqrt{2}} + \frac{2s}{\sqrt{2}} + \frac{s^2}{2} - 3\right]}{s}$$

$$= \lim_{s\to 0} \frac{s^2 + \frac{5s}{\sqrt{2}}}{s} = \lim_{s\to 0}\left(s + \frac{5}{\sqrt{2}}\right) = \boxed{\frac{5}{\sqrt{2}}}$$





$\lim_{x \to x_0}$