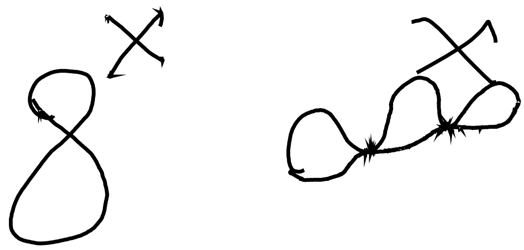
Def^n

A simple closed path is a closed path that does not intersect or touch itself.

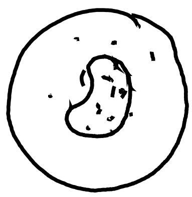
simple closed path



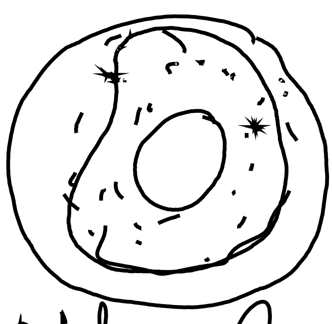
Def^n

Simply connected domain

It is a domain such that every simple closed path in D encloses only points of D.



simply connected



Not simply connected.

Cauchy integral theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0$$

Exp

analytic

$$\oint_C z\,dz \qquad C : |z| = r$$

$r(t) = re^{it}$ $\quad 0 \le t \le 2\pi$

$r'(t) = ire^{it}$

$$= \int_0^{2\pi} f(r(t))\, r'(t)\,dt$$

$$= \int_0^{2\pi} r(t)\, r'(t)\,dt$$

$$= \int_0^{2\pi} re^{it}\, ire^{it}\,dt = ir^2 \int_0^{2\pi} e^{2it}\,dt$$

$$= ir^2 \left[\frac{e^{2it}}{2i}\right]_0^{2\pi} = \frac{r^2}{2}\left[e^{4\pi i} - e^0\right]$$

$$= 0$$

Ex

$$\oint_{C:\,|z|=r} \cos z\, dz = 0$$

$$\oint_{C:\,|z|=r} e^{z}\, dz = 0$$

$$\oint_{C:\,|z|=r} (az^2+bz+c)\, dz = 0$$

For nonanalytic functions $\quad r(t) = re^{it}$

$f(z) = \bar{z}$ not analytic

$$\oint_{C:\,|z|=r} \bar{z}\, dz = \int_0^{2\pi} \overline{r(t)}\, r'(t)\, dt$$

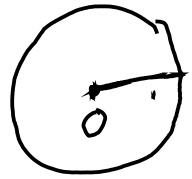
$$= \int_0^{2\pi} re^{-it}\, ire^{it}\, dt$$

$$= ir^2 \int_0^{2\pi} dt = \boxed{2\pi i r^2}$$

$\neq 0$

Analyticity is sufficient but not necessary for the integral to be 0.

$$\oint_{|z|=1} \frac{1}{z^2}\, dz$$



$|z| = 1$

$\gamma(t) = e^{it}$

$\gamma'(t) = i e^{it}$

$$\int_0^{2\pi} f(\gamma(t))\,\gamma'(t)\,dt$$

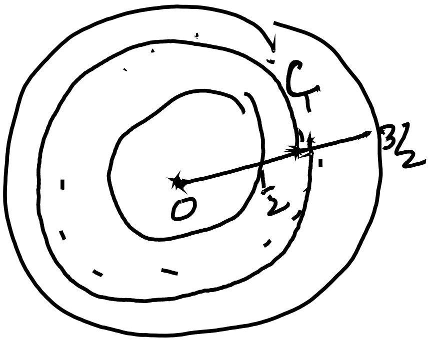
$$= \int_0^{2\pi} \frac{1}{e^{2it}}\, i e^{it}\, dt$$

$$= i\int_0^{2\pi} e^{-it}\, dt = i\left[\frac{e^{-it}}{-i}\right]_0^{2\pi}$$

$$= -\left[e^{-2\pi i} - e^{0}\right] = -[1-1] = 0$$

Simply connectedness is essential

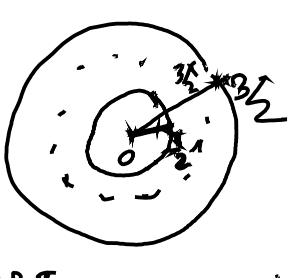
$D : \frac{1}{2} < |z| < \frac{3}{2}$



$$\oint \frac{1}{z}\, dz$$

$C : |z| = 1 \quad \gamma(t) = e^{it}$

$$= \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i$$



$$\int_C \frac{1}{z^2+2z+1}\,dz$$

D:

$C : |z| = 1$

Cauchy integral formula

If $f(z)$ is analytic in a simply connected domain D. Then for any point $z_0 \in D$ and any simple closed path C in D that encloses z_0

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

Proof $f(z) = f(z_0) + f(z) - f(z_0)$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \frac{f(z_0)}{(z-z_0)}\,dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

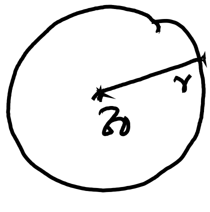
$$= f(z_0) \oint_C$$

$$I = \underbrace{f(z_0) \oint_C \frac{1}{z - z_0}\, dz}_{I_1} + \underbrace{\oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz}_{I_2}$$

$C : z_0 + re^{it}$

$0 \le t \le 2\pi$

$$I_1 = \oint_C \frac{1}{z - z_0}\, dz$$



$$= \int_0^{2\pi} \frac{1}{\cancel{z_0} + re^{it} - \cancel{z_0}}\, ire^{it}\, dt$$

$$= i\int_0^{2\pi} dt = 2\pi i$$

$$I_2 = \oint \frac{f(z) - f(z_0)}{z - z_0}\, dz$$

$f(z)$ is analytic $\Rightarrow f(z)$ continuous

$|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$.

Here $\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\epsilon}{r}$

choose $r = \delta$

$|z - z_0| < r$

$\frac{1}{|z - z_0|} > \frac{1}{r}$

(2)

$$\left| \oint_C \frac{f(z)-f(z_0)}{z-z_0}\, dz \right|$$

$$\leq \oint_C \left| \frac{f(z)-f(z_0)}{z-z_0} \right| dz$$

C: $r(t) = z_0 + re^{it}$

$r'(t) = ire^{it}$

$$< \frac{\epsilon}{r} \oint_C dz$$

$$= \frac{\epsilon}{r} \int_0^{2\pi} ire^{it}\, dt$$

$$= i\epsilon \int_0^{2\pi} e^{it}\, dt = 0$$

$$\Rightarrow \oint_C \frac{f(z)-f(z_0)}{z-z_0}\, dz = 0$$

Hence $\boxed{I = \oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0)}$

General Cauchy integral formula

$f(z)$ analytic

$$\boxed{\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)}$$

$f^{(n)}$ denote the nth derivative.

$n:0$

$$\oint \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{2\pi i}{0!} f^{(0)}(z_0)$$

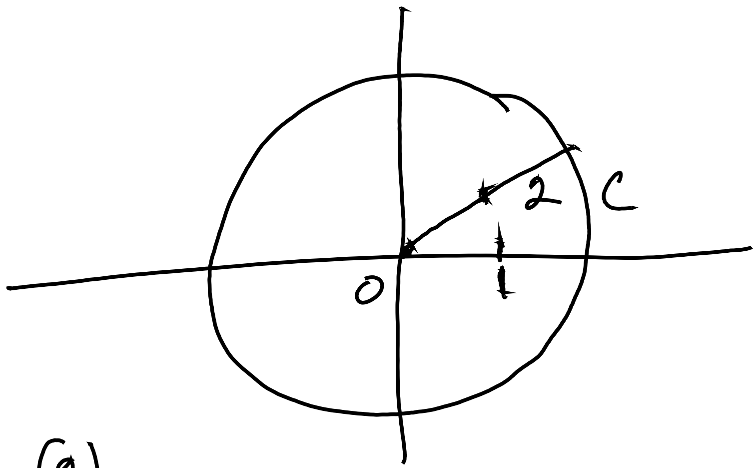
$$= 2\pi i\, f(z_0)$$

Ex: Evaluate

$$\oint \frac{\sin z}{(z-1)^3}\,dz$$

$C: |z| = 2$

$f(z) = \sin z$ is analytic



$$= \frac{2\pi i}{2!} f^{(2)}(1)$$

$$= \cancel{2} -\pi i \sin 1$$

$f(z) = \sin z$

$f'(z) = \cos z$

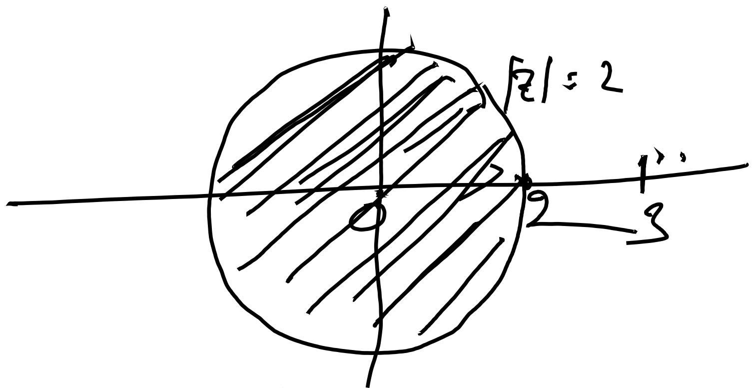
$f^{(2)}(z) = -\sin z$

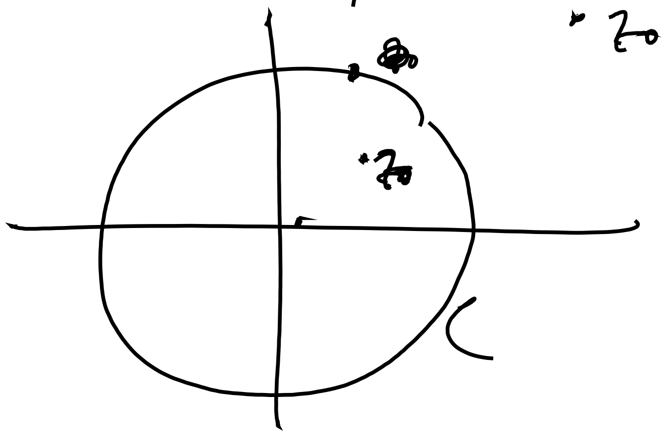
$f^{(2)}(1) = -\sin 1$

$$\oint_C \frac{\sin z}{(z-3)^5}\,dz$$

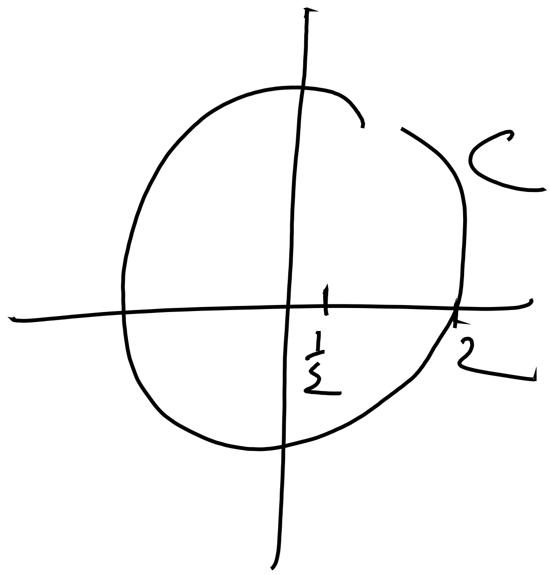
$C : |z| = 2$

$= 0$





$$\oint \frac{\sin z}{(z-\frac{1}{2})^5}\,dz$$



$$= \frac{2\pi i}{4!} f^{(4)}\left(\frac{1}{2}\right)$$

$$= \frac{2\pi i}{4!}\left(+\sin \tfrac{1}{2}\right)$$

Exp Integrate $g(z) = \frac{z^2+1}{z^2-1}$

counterclockwise around the

circle

(1) $|z-1| = 1$

(2) $|z-1| = \frac{1}{2}$

(3) $|z+1| = 1$

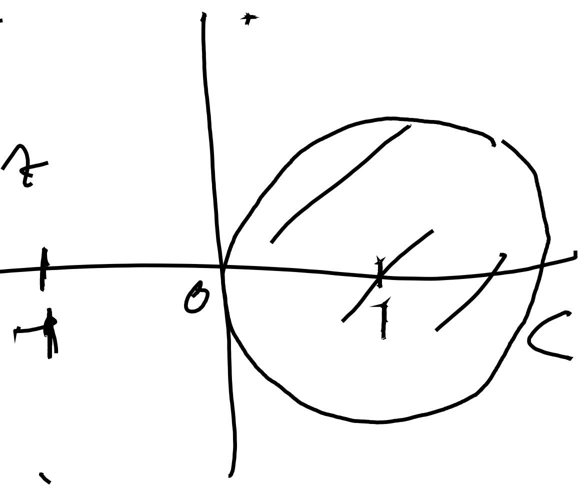
(4) $|z+1| = \frac{1}{2}$

(5) $|z| = 2$

(1) $C: |z-1| = 1$

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_C \frac{z^2+1}{(z-1)(z+1)}\,dz$$

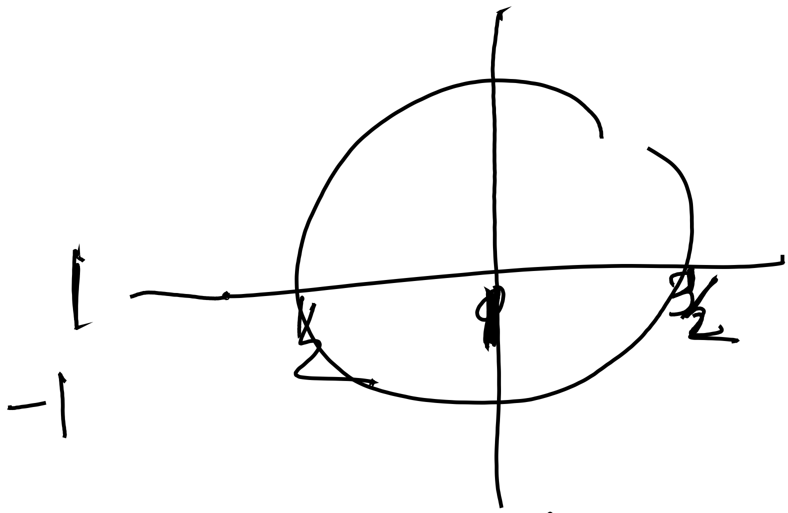
$$= \oint_C \frac{z}{z-\underset{z_0}{1}}\,dz + \oint_C \frac{-1}{z+1}\,dz$$



$= 2\pi i f(z_0) + 0$

$= 2\pi i \times 1 + 0 = 2\pi i$

② $C: |z-1| = \frac{1}{2}$

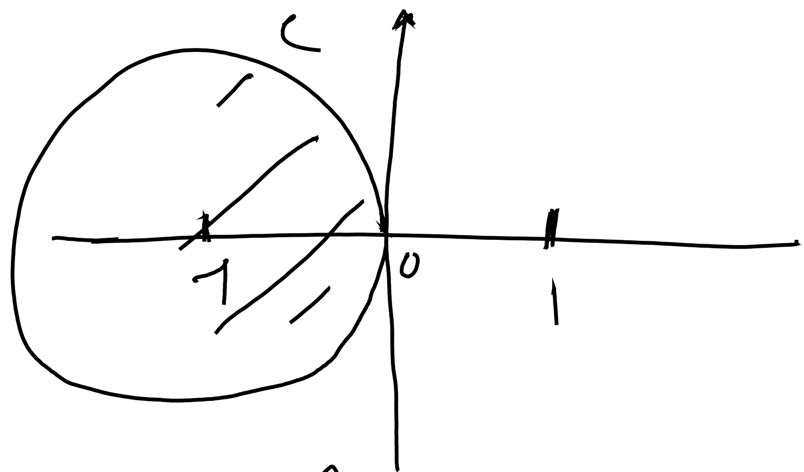


$$\int_C g(z)\,dz = \oint_C \frac{z}{z-1}\,dz + \oint_C \frac{-1}{z-1}\,dz$$

$$= 2\pi i + 0$$

$$= 2\pi i$$

③ $|z+1| = 1$

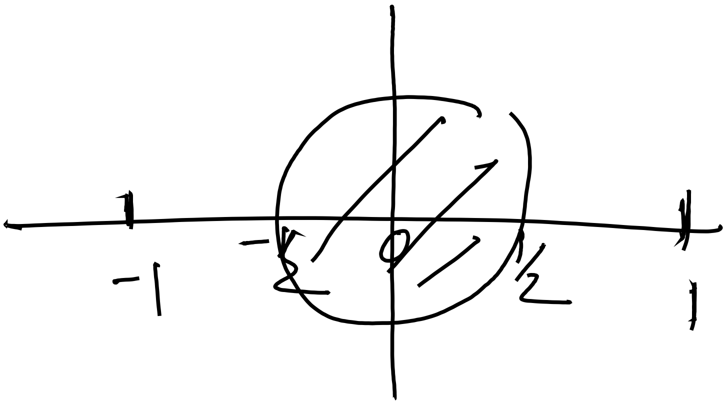


$$\int_C g(z)\,dz = \oint \frac{z}{z-1}\,dz + \oint_C \frac{-1}{z+1}\,dz$$

$$= 0 + 2\pi i\, f(-1) = \underline{-2\pi i}$$

$f(z) = -1$ $\quad f(-1) = -1$

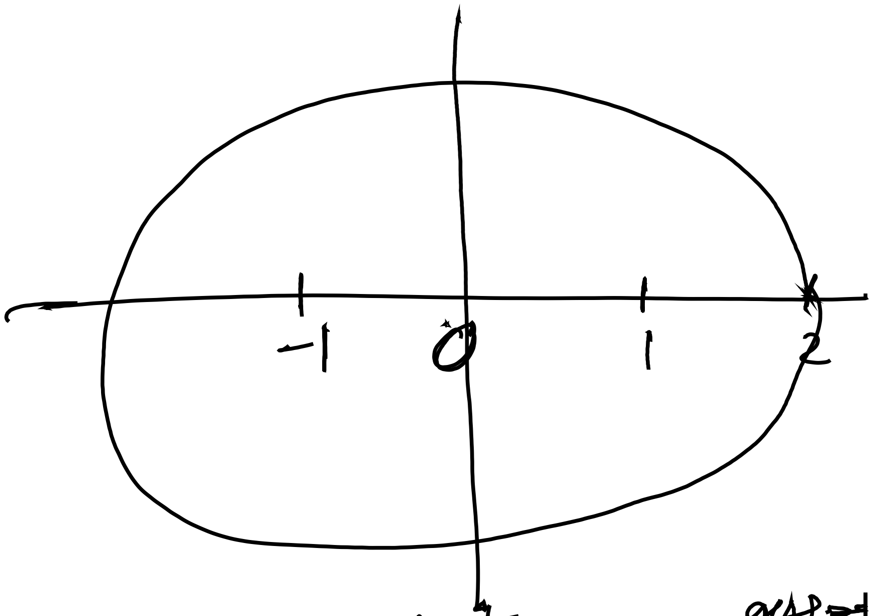
(1) $|z| = \frac{1}{2}$



$$\oint_{|z|=\frac{1}{2}} g(z)\, dz$$

$$= 0$$

(2) $|z| = 2$



$$\oint_C g(z)\, dz$$

$f(z) = z$ $\quad$ $g(z) = -1$

$$= \oint_C \frac{z^2+1}{z^2-1}\, dz = \oint_C \frac{z}{z-1}\, dz + \oint_C \frac{-1}{z+1}\, dz$$

$$= 2\pi i\, f(1) + 2\pi i\, g(-1)$$

$$= 2\pi i \times 1 + 2\pi i\,(-1)$$

$$= 0$$