Tangent plane to $f(x,y,z)=C$

at P_0 is $P_0(x_0, y_0, z_0)$

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0)+\left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0)+\left.\frac{\partial f}{\partial z}\right|_{P_0}(z-z_0)=0$$

Normal line to $f(x,y,z)=C$ at P_0

is
$$x = x_0+\left.\frac{\partial f}{\partial x}\right|_{P_0} t$$

$$y = y_0+\left.\frac{\partial f}{\partial y}\right|_{P_0} t$$

$$z = z_0+\left.\frac{\partial f}{\partial z}\right|_{P_0} t$$

Taylor's Theorem

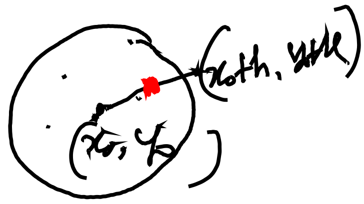
Single variable:

$$f(x)=f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots+\frac{h^n}{n!}f^{(n)}(x_0)+R_n$$

$$R_n=\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x_0+\theta h),\quad 0<\theta<1$$

Taylor's Theorem (two variable case)

Let a function $f(x,y)$ be defined in some domain $D \subset \mathbb{R}^2$ and have continuous partial derivatives upto $(n+1)$th order in some neighbourhood of (x_0, y_0) in D. Then



$$f(x_0+h, y_0+k)$$

$$= f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$$

$$h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x \partial y}$$

$$+ \cdots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0) + R_n$$

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k)$$

$$0 < \theta < 1$$

Ex

$f(x,y) = \sin xy$

Express in Taylor series about $(0,0)$.

Extreme values of functions of multivariables

$f : G \to \mathbb{R}$ $\quad G \subset \mathbb{R}^2$ ~~open region~~.

$f(x,y)$ has a local maximum at (a,b) if there is an open disk D centering (a,b) such that

$$f(x,y) \le f(a,b) \quad \forall (x,y) \in D.$$

$f(x,y)$ has a local minimum at (a,b) if there is an open disk D centering (a,b) such that

$$f(x,y) \ge f(a,b) \quad \forall (x,y) \in D.$$

$f(x,y)$ has global maximum or maximum at (a,b) if

$$f(x,y) \le f(a,b) \quad \forall (x,y) \in G$$

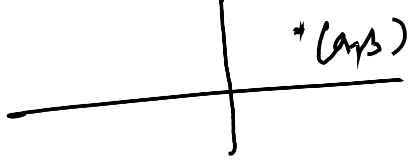
$f(x,y)$ has global minimum or minimum at (a,b) if

$$f(x,y) \ge f(a,b) \quad \forall (x,y) \in G.$$

Th^m

If f is continuous on a closed bounded region G, then f has maximum and minimum in G.

Th^m If f has continuous first partial derivatives and if $f(x,y)$ has a local maximum or local minimum at (a,b), then



$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

(a,b) interior point.

$\underline{\text{Def}^n}$ A pair (a, b) is called a critical point of $f(x,y)$ if either

① $\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$

or ② $\frac{\partial f}{\partial x}(a,b)$ or $\frac{\partial f}{\partial y}(a,b)$ does not exist.

$\underline{\text{Def}^n}$ Let $f(x,y)$ has continuous second partial derivatives.

The discriminant D of $f(x,y)$ is given by

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$$

$$= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Theorem

Let $f(x,y)$ has continuous second partial derivatives throughout an ~~open~~ region R, containing (a,b), where (a,b) interior point.

If $f_x(a,b) = 0 = f_y(a,b)$ and

if $\boxed{f_{xx} f_{yy} - (f_{xy})^2 \Big|_{(a,b)} > 0}$, then

① (a,b) is a local maximum if $f_{xx}(a,b) < 0$

② (a,b) is a local minimum if $f_{xx}(a,b) > 0$.

If $f_{xx} f_{yy} - (f_{xy})^2 \Big|_{(a,b)} < 0$

then the critical point (a,b) is neither a local maximum nor a local minimum.
(saddle point).

Exp

If $f(x,y) = x^2 - 4xy + y^3 + 4y$

Find the local extrema and saddle points of f.

Sol^n $f_x = \frac{\partial f}{\partial x} = 2x - 4y = 0 \Rightarrow x = 2y$

$f_y = \frac{\partial f}{\partial y} = -4x + 3y^2 + 4 = 0$

$3y^2 - 8y + 4 = 0$

$\Rightarrow y = 2, \quad y = \frac{2}{3}$

$x = 4, \quad x = \frac{4}{3}$

The critical points are

$(4, 2)$ and $\left(\frac{4}{3}, \frac{2}{3}\right)$

At $(4,2)$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2$ $\qquad f_{xx}(4,2) = 2$

$f_{yy} = 6y$ $\qquad f_{yy}(4,2) = 12$

$f_{xy} = -4$ $\qquad f_{xy}(4,2) = -4$

At $(4,2)$

$$D(4,2) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \text{ at } (4,2)$$

$$= \begin{vmatrix} 2 & -4 \\ -4 & 12 \end{vmatrix} = 24-16 = 8 > 0$$

$f_{xx}(4,2) = 2 > 0$

$f(x,y)$ has local minimum at $(4,2)$.

At $\left(\frac{4}{3}, \frac{2}{3}\right)$

$$f_{xx}\Big|_{\left(\frac{4}{3},\frac{2}{3}\right)} = 2 \qquad f_{yy}\Big|_{\left(\frac{4}{3},\frac{2}{3}\right)} = 4$$

$$f_{xy}\Big|_{\left(\frac{4}{3},\frac{2}{3}\right)} = -4$$

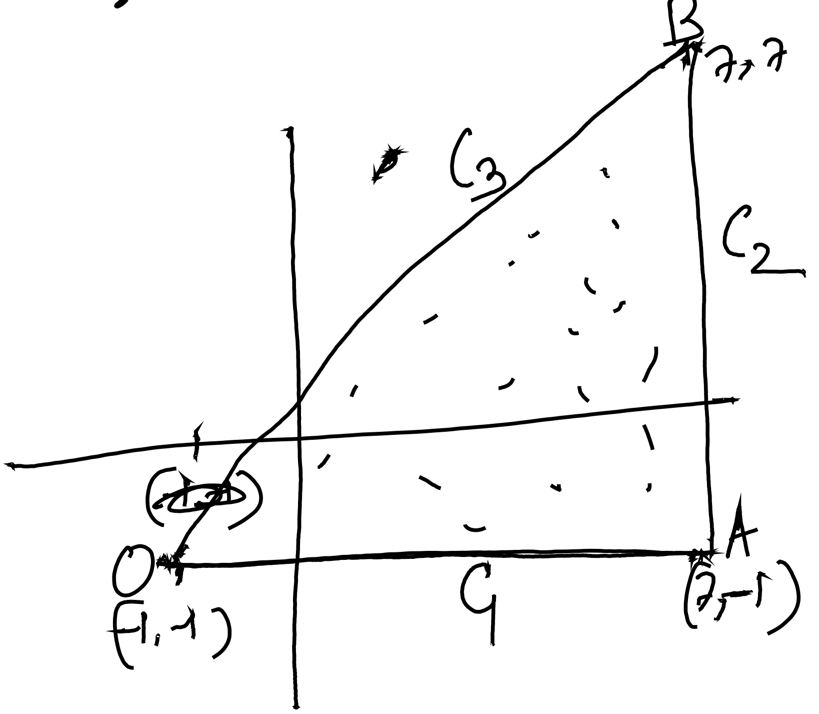
$$D\left(\frac{4}{3}, \frac{2}{3}\right) = \begin{vmatrix} 2 & -4 \\ -4 & 4 \end{vmatrix} = 8-16 = -8 < 0$$

So $\left(\frac{4}{3}, \frac{2}{3}\right)$ is a saddle point of $f(x,y)$.

Ex

$$f(x,y) = x^2 - 4xy + y^3 + 4y$$

Find the extreme values of f on the triangular region R that has vertices $(-1,-1)$, $(7,-1)$ and $(7,7)$.



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

at $(4,2)$, $\left(\frac{4}{3}, \frac{2}{3}\right)$

$(4,2)$ local minimum

$$\boxed{f(4,2) = 0}$$

$\left(\frac{4}{3}, \frac{2}{3}\right)$ saddle point

Along C_1 $\quad y = -1$

$$f(x,-1) = x^2 + 4x - 1 - 4 = x^2 + 4x - 5 = g(x)$$

$g'(x) = 0 \Rightarrow 2x + 4 = 0 \Rightarrow \underline{x = -2}$

$g''(x) = 2 > 0$

But $(-2, -1)$ is a point outside our region $\triangle OAB$.

Thus there is no local extreme values of $f(x,-1)$ inside C_1.

At the end point

$$\boxed{f(-1,-1) = -8}$$

$$\boxed{f(7,-1) = 72}$$

Along C_2

We have $x = 7$

$f(7,y) = 49 - 28y + y^3 + 4y$

$-1 \le y \le 7$

$= y^3 - 24y + 49 = h(y)$

$h'(y) = 3y^2 - 24 = 0 \Rightarrow y^2 = 8$

$\Rightarrow y = \pm 2\sqrt{2}$

But $y = -2\sqrt{2}$ is outside C_2

$y = 2\sqrt{2}$ is inside

$h''(y) = 6y \Big|_{2\sqrt{2}} = 12\sqrt{2} > 0$

$(7, 2\sqrt{2})$ is a local minimum on C_2

$$\boxed{f(7, 2\sqrt{2}) \approx 3.7}$$

At the end point of C_2

$$\boxed{\begin{aligned} f(7,-1) &= 72 \\ f(7,7) &= 224 \end{aligned}}$$

Along C_3 we have $y = x$

$$f(x,x) = x^2 - 4x^2 + x^3 + 4x$$

$$= x^3 - 3x^2 + 4x = k(x)$$

$k'(x) = 0 \Rightarrow 3x^2 - 6x + 4 = 0$

no real roots.

At the end point of C_3

$$\boxed{\begin{aligned} f(-1,-1) &= -8 \\ f(7,7) &= 224 \end{aligned}}$$

Collect all the values

global maximum $f(7,7) = 224$

global minimum $f(-1,-1) = -8$