Note-2

Properties of limits

① If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.

Then $\lim_{x \to c} \left(f(x) \pm g(x) \right) = L \pm M$

② If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. and $f(x) \le g(x)$ for all x in an open interval containing c (except possibly c itself), then $L \le M$.

③ $\lim_{x \to c} (\alpha f)(x) = \alpha \lim_{x \to c} f(x)$

④ If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

Then $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = LM$

⑤ $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$.

⑥ $\lim_{x \to c} \left(f(x) \right)^n = L^n$ n is a positive integer.

⑦ $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, n is positive.

EXP For the $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$.

That is find a $\delta > 0$ such that for all x

$0 < |x-5| < \delta \Rightarrow |\sqrt{x-1} - 2| < 1$

Sol^n

Step-1 Solve the inequality $|\sqrt{x-1}-2|<1$ to find an interval containing $x_0=5$ on which the inequality holds for all $x \neq x_0 = 5$

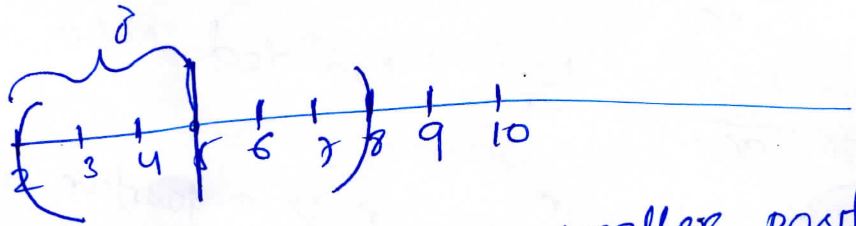
$|\sqrt{x-1}-2|<1$

$\Rightarrow -1 < \sqrt{x-1}-2 < 1$

$\Rightarrow 1 < \sqrt{x-1} < 3 \Rightarrow 1 < x-1 < 9 \Rightarrow 2 < x < 10$

The inequality $|\sqrt{x-1}-2|<1$ holds true for all x in the interval $(2,10)$.

Step-2 Find a value of $\delta > 0$ to place the centered interval $(5-\delta, 5+\delta)$ inside the interval $(2,10)$



If we take $\delta = 3$ or any smaller positive number, then the inequality $0<|x-5|<\delta$ will automatically place x between 2 and 10 to make

$|\sqrt{x-1}-2|<1$.

ie $0<|x-5|<3 \Rightarrow |\sqrt{x-1}-2|<1$.

One sided limit

To have a limit L as x approaches x_0, the function f must be defined on both sides of x_0 and its values $f(x)$ must approach L as $x \to x_0$ from either side.

Right hand limit

$$\lim_{x \to x_0^+} f(x) = L$$

If for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that for all x in $(x_0, x_0+\delta)$,

$|f(x) - L| < \epsilon$.

Left hand limit

$$\lim_{x \to x_0^-} f(x) = L$$

If for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all x in $(x_0-\delta, x_0)$, $|f(x)-L| < \epsilon$.

Exp $\lim_{x \to 0^+} \sqrt{x} = 0$

Solⁿ Let $\epsilon > 0$ be given.

Here $x_0 = 0$, $L = 0$, We want to find $\delta > 0$ such that for all x in $0 < x < \delta$, $|\sqrt{x} - 0| < \epsilon$.

So for $|\sqrt{x} - 0| < \epsilon$

ie $\sqrt{x} < \epsilon$ as $x \to 0^+$

ie $x < \epsilon^2$

So if we choose $\delta = \epsilon^2$ ie on the interval $(0, \epsilon^2)$

it holds true that $|\sqrt{x} - 0| < \epsilon$.

$\Rightarrow \lim_{x \to 0^+} \sqrt{x} = 0$.

~~$f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$~~

Continuity

A function $f(x)$ is continuous at an interior point x_0 of its domain if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Precise definition

~~$f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$~~

f is continuous at x_0 if for every $\epsilon>0$ there exists $\delta>0$ such that for all x in

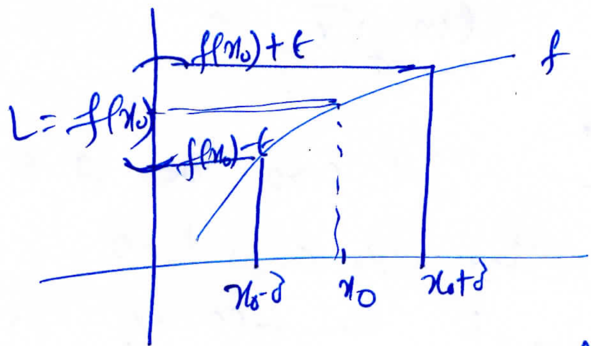
$$|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\epsilon .$$

Difference between the definitions of limit and continuity

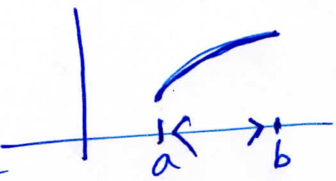
In case of limit $\quad 0<|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\epsilon$

In case of continuity $\quad |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\epsilon .$

So in case of continuity function has to be defined at x_0 and it should be equal to the limiting value of $f(x)$ at x_0.

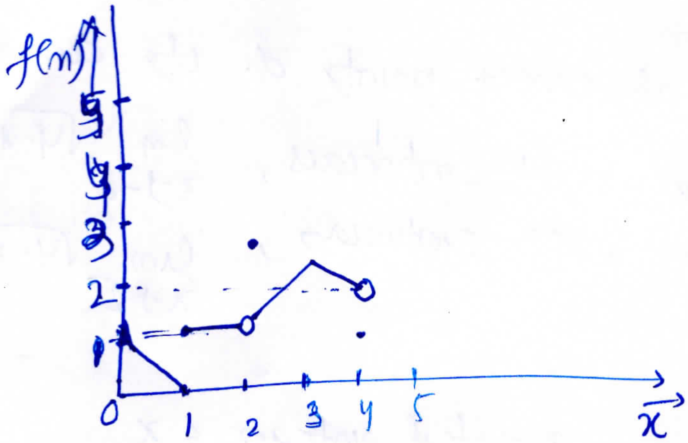


Defn A function $f(x)$ is continuous at a left end point a if $\lim_{x\to a^+} f(x) = f(a)$



Defn A function $f(x)$ is continuous at right end point b if $\lim_{x\to b^-} f(x) = f(b)$

XP



Continuous

At $x=0$, $\lim_{x\to 0^+} f(x) = f(0)$

At $x=3$, $\lim_{x\to 3} f(x) = f(3)$

At $0<c<4$, $c\neq 1,2$, $\lim_{x\to c} f(x) = f(c)$

Discontinuous

At $x=1$, $\lim_{x\to 1} f(x)$ does not exist.

At $x=2$, $\lim_{x\to 2} f(x) = 1$ but $f(2) \neq 1$

At $x=4$, $\lim_{x\to 4^-} f(x) = 2$ but $f(4)\neq 1$

At $c<0$ or $c>4$, ~~the~~ points are not in the domain of f.

Right continuous

f is said to be right continuous at c if $\lim_{x\to c^+} f(x) = f(c)$

Left continuous

f is said to be left continuous at c if $\lim_{x\to c^-} f(x) = f(c)$.

EXP

$f(x) = \sqrt{4-x^2}$

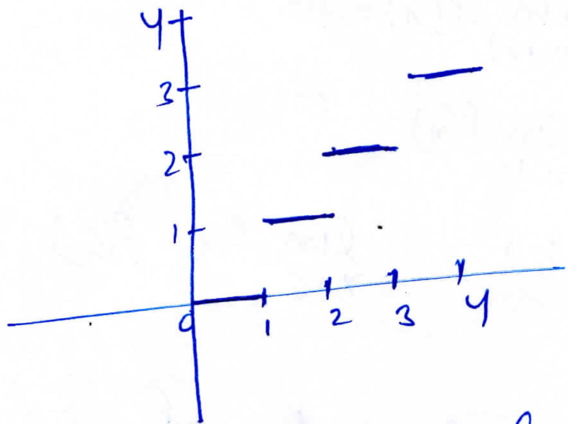
is continuous at every point of its domain $[-2, 2]$

At $x = -2$, f is right continuous, $\lim_{x \to -2^+} \sqrt{4-x^2} = 0 = f(-2)$

At $x = 2$, f is left continuous, $\lim_{x \to 2^-} \sqrt{4-x^2} = 0 = f(2)$.

EXP

$f(x) = \lfloor x \rfloor$ greatest integer $\leq x$



$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} \lfloor x \rfloor = n-1$

$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} \lfloor x \rfloor = n$

$\lfloor n \rfloor = n$

So $\lfloor x \rfloor$ is right continuous ~~but~~ every integer n. But $\lfloor x \rfloor$ is not left continuous at n. Therefore f is discontinuous at every integer point

This type of discontinuity ~~points~~ ~~are~~ is called Jump discontinuity. $f(x)$ has jump discontinuity at every integer point.

Similarly for $f(x) = \lceil x \rceil$ smallest integer $\geq x$.

Removable discontinuity

x_0 is said to be a removable discontinuity point of f if $\lim_{x\to x_0} f(x)$ exists but not equal to $f(x_0)$. or $f(x_0)$ does not exist.

Exp $\quad f(x) = \frac{x^2-1}{x-1}$

At $x=1$, $\lim_{x\to 1} f(x) = 2$ exists.

But $f(1)$ is undefined.

So if we assign $f(1) = 2$

then $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

Thus $f(x)$ is continuous.

Therefore $f(x) = \frac{x^2-1}{x-1}$ has a removable discontinuity point $x=1$.

Exp $\quad f(x) = x \sin\frac{1}{x}$

At $x=0$, $\lim_{x\to 0} x\sin\frac{1}{x} = \lim_{x\to 0} x \left(\lim_{x\to 0} \sin\frac{1}{x}\right)$ finite

$= 0$

$\lim_{x\to 0} f(x) = 0$ exists

but $f(0)$ is undefined.

So if we define $f(x) = \begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Then $f(x)$ is continuous.

So $f(x) = x\sin\frac{1}{x}$ has a removable discontinuity at $x=0$.