Electrostatic Energy

1 Work and Energy

Let $V(\vec{r})$ be the potential in a region due to a charge distribution. Now if we move a charge q from ∞ to a point $\vec{a}$ then the work done on the charge is $W = qV(\vec{a})$. If we are given acharge distribution consisting of point charges $q_1, q_2, ..., q_n$ situated at positions $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$, then how much work is spent in making this configuration ?.

We start with q_1 at $\vec{r}_1$ and bring q_2 from ∞ and place it at $\vec{r}_2$. This will need an energy

$$W_2 = \frac{q_2}{4\pi\epsilon_0}\frac{q_1}{r_{12}}$$

where $r_{12} = |\vec{r}_2 - \vec{r}_1|$. Now we bring q_3 from ∞ to $\vec{r}_3$. The work done for this is

$$W_3 = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

So the energy required to accumulate these three charges is

$$\begin{aligned} W_2 + W_3 &= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right] \\ &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\sum_{j=i+1}^{3}\frac{q_i q_j}{r_{ij}} \end{aligned}$$

When we have accumulated n charges this becomes

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{q_i q_j}{r_{ij}}$$

If we want to treat the indices symmetrically then we can write the summation as

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{q_i q_j}{r_{ij}} \tag{1}$$

This can be written as

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i) \tag{2}$$

where $V(\vec{r}_i) = \frac{1}{4\pi\epsilon_0} \sum j \neq i^n \frac{q_j}{r_{ij}}$ is the potential at the point $\vec{r}_i$ due to the other $n-1$ charges. The energy stored in a charge distribution as expressed in the form 2 can be extended to a continuous charge distribution. as follows:

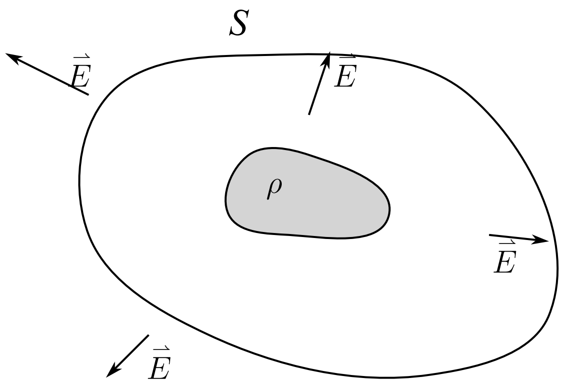
$$W = \frac{1}{2}\int_\tau \rho(\vec{r})V(\vec{r})d\tau \tag{3}$$

where τ is the region within which the charge density $\rho(\vec{r})$ exist.
By Gauss's Law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$. So we have

$$W = \frac{1}{2}\int_\tau \epsilon_0(\vec{\nabla}\cdot\vec{E}V(\vec{r})d\tau$$

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{E}V) &= (\vec{\nabla}\cdot\vec{E})V + \vec{E}\cdot\vec{\nabla}V \\
\therefore\ (\vec{\nabla}\cdot\vec{E})V &= \vec{\nabla}\cdot(\vec{E}V) - \vec{E}\cdot\vec{\nabla}V \\
&= \vec{\nabla}\cdot(\vec{E}V) + E^2 \\
\therefore\ W &= \frac{\epsilon_0}{2}\left[\int_\tau E^2 d\tau + \int_\tau \vec{\nabla}\cdot(\vec{E}V)d\tau\right] \\
&= \frac{\epsilon_0}{2}\left[\int_\tau E^2 d\tau + \int_S \vec{E}V\cdot\hat{n}d\tau\right] \qquad (4)
\end{aligned}$$

where S is the surface enclosing τ.
We started with the energy stored in a charge configurarion within a volume τ in Eq. 3. In Eq. 4 we see that this is equal to just integral over a volume and a surface foe electric fields and electric potentials $\vec{E}$ and V.



In Eq. 4 the integral over τ extends over a region where there may not be any charge density, but we may have non zero electric field $\vec{E}$. We can push the surface S to infinity and make the integral over volume to be over the whole space. If the charge configuration is confined in space then $\vec{E}$ and V goes to 0 at infinity. In this case the surface integral tends to 0 and the energy of the charge configuration is

$$W = \frac{1}{2}\int_{\text{whole space}} E^2 d\tau \tag{5}$$

Eq. 5 says that if we have electric field $\vec{E}$ in a region then we have an energy density $\frac{\epsilon_0}{2}E^2$ in the region. This implies that electric field is not just a mathematical convenience, but a physical entity carrying energy. In fact we can have electric field without charges which can carry energy as in electromagnetic waves.

When we started calculating the energy of a charge configuration we started accumulating point charges. We didn't bother what is the energy needed to make these point charges. Eq. 5 gives a way to calculate the electrostatic energy of a point charge.
Since $E = \frac{q}{4\pi\epsilon_0 r^2}$

$$\begin{aligned} W &= \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}\int_{\text{whole space}}\frac{1}{r^4}r^2\sin\theta d\theta d\phi dr \\ &= \frac{q^2}{32\pi^2\epsilon_0}\left(\int_0^\infty \frac{1}{r^2}dr\right)4\pi \rightarrow \infty \end{aligned}$$

We can also calculate energy of a point charge by other methods and convince ourselves that it is indeed ∞.

Point charge is a fiction that demands and infinite energy to be created. Typically we ignore this infinite background reference energy and live our humble life considering only the variation around it.

Superposition

Suppose we have two charge configurations in a region. Due to one configuration the electric field is $\vec{E}_1$. Due to the other the electric field is $\vec{E}_2$. So due to the combined charge distribution of the two configuration we have

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

The electrostatic energies of the individual configuration are

$$W_1 = \int E_1^2 d\tau \quad \text{and} \quad W_2 = \int E_2^2 d\tau$$

Now due to the combined configuration $\vec{E}_1 + \vec{E}_2$ the total energy is

$$\begin{aligned} W_{tot} &= \frac{\epsilon_0}{2}\int(\vec{E}_1 + \vec{E}_2)^2 d\tau \\ &= \frac{\epsilon_0}{2}\left[\int E_1^2 d\tau + \int E_2^2 d\tau + 2\int(\vec{E}_1\cdot\vec{E}_2)d\tau\right] \\ &= W_1 + W_2 + \epsilon_0\int(\vec{E}_1\cdot\vec{E}_2)d\tau \end{aligned}$$

In the spaecial case when the individual configuration are point charges q_1 and q_2, W_1 and W_2 are infinities and the energy of the configuration W_{tot} is considered to be only the term

$$W_{tot} = \epsilon_0\int(\vec{E}_1\cdot\vec{E}_2)d\tau$$

We can show that this is equal to $\frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$.