$\boxed{\text{Example}}$ $3x^2 + 8x\log x = \Theta(x^2)$

D

$$0 \le 8x\log x \le 8x^2$$

$$\therefore \quad 3x^2 + 8x\log x \le 11x^2 \quad \text{for } x > 1$$

$$\therefore \quad 3x^2 + 8x\log x = O(x^2)$$

$$x^2 = O(3x^2 + 8x\log x)$$

$$3x^2 + 8x\log x = \Theta(x^2)$$

COMPLEXITY OF ALGO

computational complexity $\begin{cases} \to \text{time complexity} \\ \to \text{space } '' \end{cases}$

TIME COMPLEXITY # of operations used by algo when the input has a particular size

$\boxed{\text{Example}}$ Time complexity of finding the max element in a $<\infty$ seq.

☐ Comparisions are basic operation used

~~- One comparison max = a_1~~

— For each i $(1 \le i \le n-1)$ 2 comparison → to determine the end of lists; → updating the term? or not?
(2 to n^{th} elements)

— One comparison to exit the loop

$$\#\ \text{comparism} = 2(n-1)+1 = \Theta(n).$$

$\boxed{\text{Example}}$ linear search algo worst case complexity is $O(n)$

☐ – At each step of the loop in the algo → one to see end has reached; → compare x to the term

$\therefore$ one comparison is made outside of the loop
If $x = a_i$, $2i+1$ comparisons

When x is not in the list most comparisions $2n+2$ are required

$2n$ for x not in the list

one is outside the loop: one to exit the loop

$\therefore$ Worst-case $\cancel{O(2n+2)}$ $= O(n)$
$2n+2$

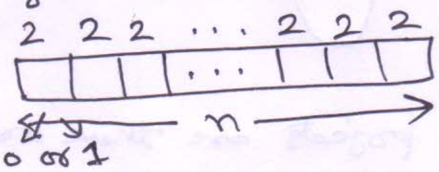
Feb 24, 2020

Counting

The Basics of Counting

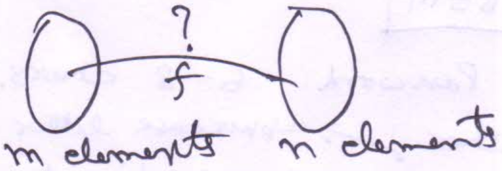
X Rule If a procedure can be broken down into seq. of two tasks and task one can be done in n_1 ways & for each of these ways task two the can be done in n_2 ways then procedure can be done in $n_1 n_2$ ways.

Ex-1 How many binary strings of length n?



2^n ways.

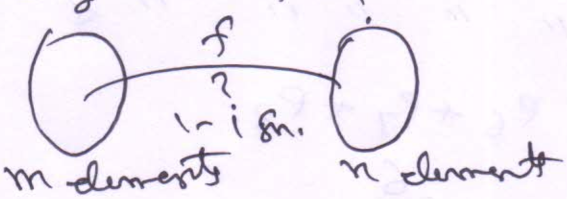
Ex-2 How many fns are there from a set with m elements to a set with n elements?



A fn. corresponds to a choice of one of the n element for each m elements of the domain

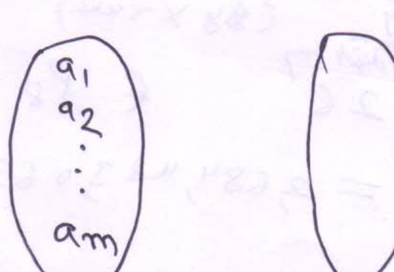
$\therefore$ by $\times$ rule there are $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m \text{ times}} = n^m$ functions.

Ex-3 How many 1-1 fns?



Case ① If $m > n$ There is no 1-1 fn. (it will be many to 1)

Case ② If $m \le n$



As the fn. is 1-1 there are n ways to choose the value of the fn. at a_1, a_2 can be chosen $n-1$ ways (as the value used for a_1 can not be used again). In general a_k can be chosen in $n-k+1$ ways $\therefore$ by $\times$ rule

$a_1, a_2, \ldots, a_m$ can be choosen in $n(n-1)(n-2)\cdots(n-m+1)$ ways

These many 1-1 fns.

$\boxed{+ \text{ Rule}}$ If a task can be done either in one of n_1 ways or in one of n_2 ways (where none of the set of n_1 ways is same as any of the set of n_2 ways) then there are (n_1+n_2) ways to do the task.

$\boxed{\text{Ex}}$ A student can choose a project from one of the 3 lists

No project is on more than one list

How many possible projects are there to choose from?

☐ By + Rule $23+15+19$ ways to choose the project. ▨

$\boxed{\text{COMPLEX COUNTING PROBEM}}$

Ex. ~~Each use~~ User ☺ Password 6-8 chars. long

char. → uppercase letter or a digit

- Each p.w. must contain at least one digit. How many possible passwords are there?

☐ Let $P :=$ Total # of possible passwords

$P_6 :=$ # of possible p.w. of length 6

$P_7 :=$ " " " " " " 7

$P_8 :=$ " " " " " " 8

By + rule $P = P_6 + P_7 + P_8$

Now $P_6 = \underbrace{36^6}_{\substack{\text{\# of strings of length 6 of upper case letter and digit} \\ \text{including those with no digit} \\ \text{(By × rule)}}} - \underbrace{26^6}_{\substack{\text{\# of string with no digit} \\ \text{(By × rule)}}}$

Similarly, $P_7 = 36^7 - 26^7$ & $P_8 = 36^8 - 26^8$

$P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

The Inclusion – Exclusion Principle

No + rule directly

— Principle

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$|A_1 \cup A_2|$ = # of ways to select an element from A_1 or A_2

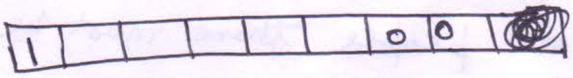
$|A_1|$ = ways to select element from A_1

$|A_2|$ = " " " A_2

$|A_1 \cap A_2|$ = # of ways to select an element from both A_1 & A_2

Example How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?

☐



— How many strings possible with 1st ~~string~~ bit is 1

$2^7 = 128$ ways ($\times$ rule)

— How many strings possible with last two bits 00

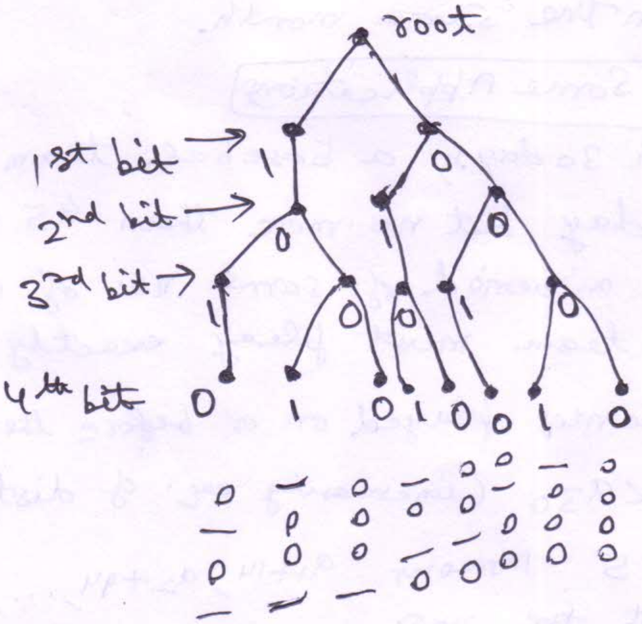
$2^6 = 64$ ways ($\times$ rule)

— Some of the ways to construct a bit string of length 8 are same as ways to const a string starting with 1 & ending in 00.

$\# \text{ ways} = 2^5 = 32$

$\therefore$ Total # of strings $= 128 + 64 - 32 = 160$ ▨

Tree Diagrams How many bit strings of length 4 do not have two consecutive 1's?



Total 8 strings

The Pigeonhole principle

If ($K>0$) integer & $K+1$ or more objects are placed into K boxes, then there is at least one box containing two or more of the objects.

Corollary 1 A fn. f from a set with $K+1$ or more elements to a set with K elements is not 1-1.

Ex. Among any gp. of 367 people there must be at least 2 with the same birthday.

The Generalized Pigeonhole principle

If N objects are placed into K objects, then there is at least one box containing at least $\lceil \frac{N}{K} \rceil$ objects

$\square$ Suppose none of the boxes contains more than $\lceil \frac{N}{K} \rceil - 1$ objects. Then the total no. of objects is at most

$$K\left(\left\lceil \frac{N}{K} \right\rceil - 1\right) < K\left(\left(\frac{N}{K}+1\right)-1\right) = N \quad \# \text{ contradiction}$$

$$\left(\because \left\lceil \frac{N}{K} \right\rceil < \frac{N}{K}+1\right)$$

Ex. Among 100 people there are at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who were born in the same month.

Some Applications

Ex. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some no. of consecutive days during which the team must play exactly 14 games.

$\square$ Let $a_j :=$ # of games played on or before the j^{th} day of the month

Then $a_1 < a_2 < \dots < a_{30}$ (increasing seq. of distinct +ve no.s) with $1 \le a_j \le 45$. Moreover $a_1+14, a_2+14, \dots, a_{30}+14$ is also incr. seq. of distinct +ve nos.

The 60 nos. $a_1, a_2, a_3, \dots, a_{30}, a_1+14, a_2+14, \dots, a_{30}+14$ are all less ≤ 59

$\therefore$ By pigeonhole principle two of them must be equal $\because a_j\ (1 \le j \le 30)$ are all distinct & $a_j+14\ (1 \le j \le 30)$ are all distinct

$\exists$ indices i & j with $a_i = a_j + 14$

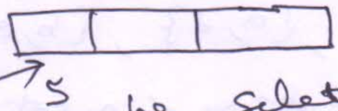
$\Rightarrow$ exactly 14 games were played from day $j+1$ to day i. ∎

Permutation & Combination

Permutations

In how many ways can we select 3 students from a gp. of 5 students to stand in line for a picture?

□ first student $\rightarrow$ select in 5 ways



2nd student can be selected in 4 ways

3rd " " " " 3 ways.

$\#$ of ways $= 5 \times 4 \times 3 = 60$ ways.

Def: Permutation : A permutation of a set of distinct objects in an ordered arrangement of these objects

— An ordered arrangements of r-elements of a set with n-elements is called r-permutation.

Ex $\quad S = \{1,2,3\}$ $\quad$ 3, 1, 2 is a permutation

3, 2 is a 2-perm.

$$\boxed{P(n,r) = n(n-1)(n-2)\cdots(n-r+1)} \quad 1 \le r \le n$$

r-perm. of a set of n distinct elements. $\quad$ $\boxed{n > 0 \text{ integer}}$

$P(n,0) = 1$

$$P(n,r) = \frac{n!}{(n-r)!} \quad 0 \le r \le n$$

$\boxed{r\text{-combination}}$ An r-combination of elements of a set is an unordered selection of r-elements from the set
∴ r-combination simply a subset of the set with r elements.

- The # of r-combinations of a set with n distinct elements is denoted by $C(n,r)$ or $\binom{n}{r}$ → binomial coeff.

<u>Ex</u> - $C(4,2)$ or $\binom{4}{2} = 6$

As 2 combinations of $\{a,b,c,d\}$ are the 6 subsets $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$ & $\{c,d\}$.

<u>Th.</u> The # of r-combinations of a set with n elements

$$\binom{n}{r} \text{ or } C(n,r) = \frac{n!}{r!\,(n-r)!} \qquad 0 \le r \le n$$

□ The r-perm. of the set can be obtained by forming the $C(n,r)$ r-combinations of the set and then ordering the elements in each r-combination (can be done in $P(r,r)$ ways)

$$P(n,r) = C(n,r) \cdot P(r,r)$$

$$\Rightarrow C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!\,/\,(n-r)!}{r!\,/\,(r-r)!} = \frac{n!}{r!\,(n-r)!}$$

$$C(n,r) = \frac{n(n-1)\ldots(n-r+1)}{r!}$$

<u>Cor.</u> $r \le n$ $\quad C(n,r) = C(n,n-r)$

□ $$C(n,r) = \frac{n!}{r!\,(n-r)!} \quad \& \quad C(n,n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!}$$

$$\Rightarrow C(n,r) = C(n,n-r)$$

Binomial coefficients

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

Th. $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j$

$\square$ Terms are $x^{n-j}y^j \; (j=0,1,2,\dots,n)$

To count the no. of terms of the form $x^{n-j}y^j$ choose $(n-j)$ x's from the n sum (so the other j terms in the $\times$ are y's)

$\therefore$ coeff of $x^{n-j}y^j = \binom{n}{n-j} = \binom{n}{j}$ ▨

Cor. ① $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ In B.T. Put $x=1$ & $y=1$ ▨

Cor ② $n>0$ integer $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Put $x=-1$ & $y=1$ $0=0^n=(-1+1)^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k}$

$\rightarrow \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Remark: $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\dots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\dots$

Cor ③ $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$ Put $x=1$ & $y=2$

$3^n = (1+2)^n = \sum_{k=0}^{n}\binom{n}{k} 1^{n-k}2^k = \sum_{k=0}^{n}\binom{n}{k}2^k$ ▨

Pascal's Identity & Δ

Let n & k be integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

□ Suppose T set, $|T| = n+1$ elements

Let $a \in T$ & let $S = T - \{a\}$

Note there are $\binom{n+1}{k}$ subsets of T containing k elements.

However, a subset of T with k elements either contains a together with $k-1$ elements of S or contains k elements of S & does not contain a.

$\therefore$ there are $\cancel{\binom{n}{k}}$ $\binom{n}{k-1}$ ~~subsets of k elements~~ subsets of $k-1$ elements of S

there are $\binom{n}{k-1}$ subsets of k elements of T that contain a

& there are $\binom{n}{k}$ subsets of k elements of T that do not contain a.

$$\therefore \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$ ▨

$\cancel{\binom{n}{0} = \binom{n}{n} = 1}$

or

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

□ $\binom{n}{k} :=$ # of subsets with k elements from a set of n elements.

let $a \in T$ (set with n elements)

- <u>To construct a subset of k elements containing 'a'</u>, choose 'a' & $k-1$ elements from the remaining $n-1$ elements in the set there are $\binom{n-1}{k-1}$ such subsets.
- <u>To construct a subset of k elements not containing 'a'</u>, choose k elements from the remaining $n-1$ elements of the set. There are $\binom{n-1}{k}$ such subsets.
- Every subset of k elements either contains 'a' or not

$\therefore$ - The total # of subsets with k elements in a set of n elements is the sum of the # of subsets containing 'a' & the # of subsets that do not contain 'a'. i.e. $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ ▨

Vandeermonde's Identity

Let m, n & $r \geq 0$ integers r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

$\square$ Suppose there are m elements in one set $= T$
n elements " other set $= S$

total # of ways to pick r elements from the $T \cup S$ is $\binom{m+n}{r}$. Another way to pick r elements from the $T \cup S$ is to pick k element from set T & then $r-k$ elements from set S $(0 \leq k \leq r)$

$\therefore$ This can be done in $\binom{m}{k} \cdot \binom{n}{r-k}$ by $\times$ rule

$\therefore$ total # of ways to pick r elements from $T \cup S$ equals

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} \quad ▨$$

Cor $n \geq 0$ integer

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

$\because \binom{n}{k} = \binom{n}{n-k}$

$\square$ Put $m = r = n$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Th. $n, r \geq 0$ integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

$\boxed{\text{Advanced Counting}}$

Ex # of bacteria doubles every hour. If a colony begins with 5 bacterias, how many will be ∃ in n hours?

□ Let $a_n :=$ # of bacteria at the end of n hours.

$$\Rightarrow \quad \underbrace{a_n = 2a_{n-1}}_{\boxed{\text{Recurrence relation}}} \quad \forall n>0 \ \& \ a_0 = 5$$

Ex. $a_n = 2a_{n-1} - a_{n-2}, \ n = 2, 3, 4, \ldots$

□ Soln. is $a_n = 3n$

as $2a_{n-1} - a_{n-2} = 2\cdot 3(n-1) - 3(n-2)$

$= 3n = a_n$ ∴ its a soln. ▨

– If $a_n = 2^n$ $\quad 2a_{n-1} - a_{n-2} = 2\cdot 2^{n-1} - 2^{n-2} = 2^n - 2^{n-1}$

$\neq 2^n$ not a soln.

– If $a_n = 5$ $\quad 2a_{n-1} - a_{n-2} = 2\cdot 5 - 5 = 5 = a_n$ is a soln. ▨

Ex. Hemachandra Numbers:

$$f_n = f_{n-1} + f_{n-2}, \quad f_1 = 1 \ \& \ f_2 = 1$$

– A linear homogeneous recurrence relation of degree k with const. coff. is a rec. relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$\forall c_i \in \mathbb{R} \ \& \ c_k \neq 0$$

– Linear

– Homo. (no term occur that is not multiple of a_j)

– deg k → a_n is expressed in previous k terms

Hemachana rec. relation is linear homo. of deg 2.

We seek solns of the form $a_n = r^n$ (r const.)

$a_n = r^n$ is a soln. iff

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$\Rightarrow \boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0}$$

Char. eq^n

roots

$\boxed{\text{Th-1}}$ Let $c_1, c_2 \in \mathbb{R}$. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 & r_2. Then the seq. $\{a_n\}$ is a soln. of the rec. relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, \ldots, \ \alpha_1, \alpha_2 \text{ const.}$$

□ Claim ① If $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ then $\{a_n\}$ is a soln. of the rec. relation.

Δ If r_1, r_2 roots of $r^2 - c_1 r - c_2 = 0$

$$\Rightarrow r_1^2 = c_1 r_1 + c_2 \ \& \ r_2^2 = c_2 r_2 + c_2$$

$$\Rightarrow c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$

$$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$

$$= \alpha_1 r_1^{n-2} \cdot r_1^2 + \alpha_2 r_2^{n-2} \cdot r_2^2$$

$$= \alpha_1 r_1^n + \alpha_2 r_2^n = a_n$$

Claim ② Every soln. $\{a_n\}$ of the rec. relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ $n = 0, 1, 2, \ldots$ for some const. α_1 & α_2

Δ Suppose $\{a_n\}$ is a soln. of the rec. relation. & $a_0 = c_0$ & $a_1 = c_1$ hold. ∃ const. α_1 & α_2 s.t. the seq. $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies with initial condition.

$$\left.\begin{aligned} a_0 &= c_0 = \alpha_1 + \alpha_2 \\ a_1 &= c_1 = \alpha_1 r_1 + \alpha_2 r_2 \end{aligned}\right\} \text{Solve for } \alpha_1 \ \& \ \alpha_2$$

$$\Rightarrow c_1 = \alpha_1 r_1 + (c_0 - \alpha_1) r_2 = \alpha_1(r_1 - r_2) + c_0 r_2$$

$$\Rightarrow \alpha_1 = \frac{c_1 - c_0 r_2}{r_1 - r_2} \ \& \ \alpha_2 = c_0 - \alpha_1 = \frac{c_0 r_1 - c_1}{r_1 - r_2} \quad \text{provided } r_1 \neq r_2$$

$\{a_n\}$ satisfies initial condition.

Ex Solve $a_n = a_{n-1} + 2a_{n-2}$

$a_0 = 2$ & $a_1 = 7$

□ Ch. eqn $r^2 - r - 2 = 0 \Rightarrow r = 2$ & $r = -1$

$\{a_n\}$ is soln. iff $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ for some constt α_1 & α_2

Using initial conditions

$$\left.\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 \\ a_1 &= 7 = \alpha_1 . 2 + \alpha_2 (-1) \end{aligned}\right\} \alpha_1 = 3 \ \& \ \alpha_2 = -1$$

$$\Rightarrow \quad a_n = 3 . 2^n - (-1)^n$$ ▨

Ex. Find explicit formula for Hemachandra number.

□ $$f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0 \ \& \ f_1 = 1$$

Roots of $r^2 - r - 1 = 0$ are $r_1 = \frac{(1+\sqrt{5})}{2}$ & $r_2 = \frac{(1-\sqrt{5})}{2}$

$\Rightarrow$ Using Th-1,

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constt. α_1 & α_2

Initial conditions $f_0 = 0$ & $f_1 = 1$ yields

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\alpha_1 = \frac{1}{\sqrt{5}} \quad \& \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

$$\therefore f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$ ▨

Th-1 does not work when one root is of multiplicity 2.

Th-2 : Let $c_1, c_2 \in \mathbb{R}$, $(c_2 \neq 0)$ Suppose

$r^2 - c_1 r - c_2 = 0$ has only one root

r_0. A seq. $\{a_n\}$ is a soln. of the rec. relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$

$n = 0, 1, 3, \ldots$

α_1, α_2 are constt.

□ Pick up a cup of tea. ▨

Ex. Solve $a_n = 6 a_{n-1} - 9 a_{n-2}$

$a_0 = 1$ & $a_1 = 6$

□ $r^2 - 6r + 9 = 0 \Rightarrow r = 3$

$\therefore$ $a_n = \alpha_1 3^n + \alpha_2 \cdot n \cdot 3^n$

Using initial conditions

$a_0 = 1 = \alpha_1$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$

$\Rightarrow \alpha_1 = 1$ & $\alpha_2 = 1$

$\Rightarrow a_n = 3^n + n \cdot 3^n$ ▨

$\boxed{\text{Th-3 (deg } k)}$ Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$ suppose ch. eqn

$r^k - c_1 r^{k-1} \ldots - c_k = 0$ has k distinct roots

$r_1, r_2, \ldots, r_k$. Then a seq. $\{a_n\}$ is a soln. of the

rec. relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$

iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$

for $n = 0, 1, 2, \ldots$ α_i are constt.

Ex Solve $a_n = 6 a_{n-1} - 11 a_{n-2} + 6 a_{n-3}$

$a_0 = 2$, $a_1 = 5$ & $a_2 = 15$

□ The char. eqn $r^3 - 6r^2 + 11r - 6 = 0$

Roots $r = 1$, $r = 2$ & $r = 3$

$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$

Solving $a_n = 1 - 2^n + 2 \cdot 3^n$

▨

Multiple root Th.

Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$ Suppose ch. eqn

$r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has t distinct roots

$r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ resp.

$(m_i \geq 1), i = 1, \ldots, t$ & $m_1 + m_2 + \cdots + m_t = k$

then a seq. $\{a_n\}$ is a soln. of the rec. relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$
$$+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

$\alpha_{i,j}$ are consts. $n = 0, 1, 2, \ldots$

$1 \leq i \leq t$

$0 \leq j \leq m_i - 1$.

Ex. Solve $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$a_0 = 1, a_1 = -2$ & $a_2 = -1$

□ ch. eqn: $r^3 + 3r^2 + 3r + 1 = 0 \Rightarrow (r+1)^3 = 0$

$\Rightarrow r = -1$ is a root of order 3.

$\Rightarrow a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1} \cdot n (-1)^n + \alpha_{1,2} n^2 (-1)^n$

Using initial conditions

$a_0 = 1 = \alpha_{1,0}$

$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$

$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$

$\Rightarrow \alpha_{1,0} = 1, \alpha_{1,1} = 3$ & $\alpha_{1,2} = -2$

$\therefore a_n = (1 + 3n - 2n^2)(-1)^n$

Linear Non-homo. Rec. Relation with constt. coff.

Ex. $a_n = 3a_{n-1} + 2n$

$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n) \quad \forall c_i \in \mathbb{R}$

$\rightarrow$ fn. $\not\equiv 0$

$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-1}$ is called Associated homo-rec. relation (AHRR)

Th If $\{a_n^{(p)}\}$ is a P.S. of non-homo linear rec. relation with constt. coff. $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n)$ then every soln. is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is the soln. of the AHRR.

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} .$$

☐ $\because \{a_n^{(p)}\}$ is a P.S.

$\Rightarrow a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-1}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n)$

If $\{b_n\}$ is second soln.

$b_n = c_1 b_{n-1} + \cdots + c_k b_{n-k} + F(n)$

$\Rightarrow b_n - a_n^{(p)} = c_1 (b_{n-1} - a_{n-1}^{(p)}) + \cdots + c_k (b_{n-k} - a_{n-k}^{(p)})$

$\Rightarrow \{b_n - a_n^{(p)}\}$ is a soln. of AHRR, say $\{a_n^{(h)}\}$

$\Rightarrow b_n = a_n^{(p)} + a_n^{(h)} \quad \forall n$ ▨

Ex. Solve $a_n = 3a_{n-1} + 2n$ What if $a_1 = 3$?

AHRR $a_n = 3a_{n-1}$ has soln. $a_n^{(h)} = \alpha \cdot 3^n$ α constt.

By trial we seek P.S.

$$a_n = 3a_{n-1} + 2n$$

Seek $p_n = cn + d$ is a soln.

$\Rightarrow cn + d = 3(c(n-1) + d) + 2n$

$\Rightarrow (2 + 2c)n + (2d - 3c) = 0$

$\Rightarrow cn + d$ is a soln. $\Leftrightarrow 2 + 2c = 0$ & $2d - 3c = 0$

$\Leftrightarrow c = -1$ & $d = -3/2$ $\therefore a_n^{(p)} = -n - \frac{3}{2}$

$\therefore a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n$

▨

Th-6 Suppose $\{a_n\}$ satisfies the linear non homo. rec. relation

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n)$$

where $c_i \in \mathbb{R}$ &

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

$b_i, s \in \mathbb{R}$

If s is not the root of the ch. eq^n of the AHRR there is P.S. of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$$

when s is a root of the ch. eq^n with multiplicity m

then P.S. is $n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n$