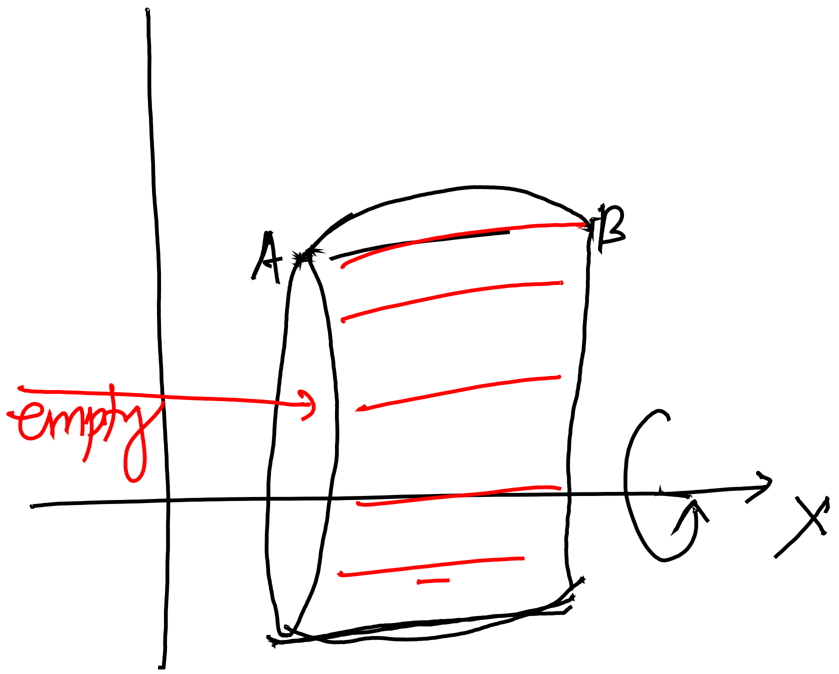
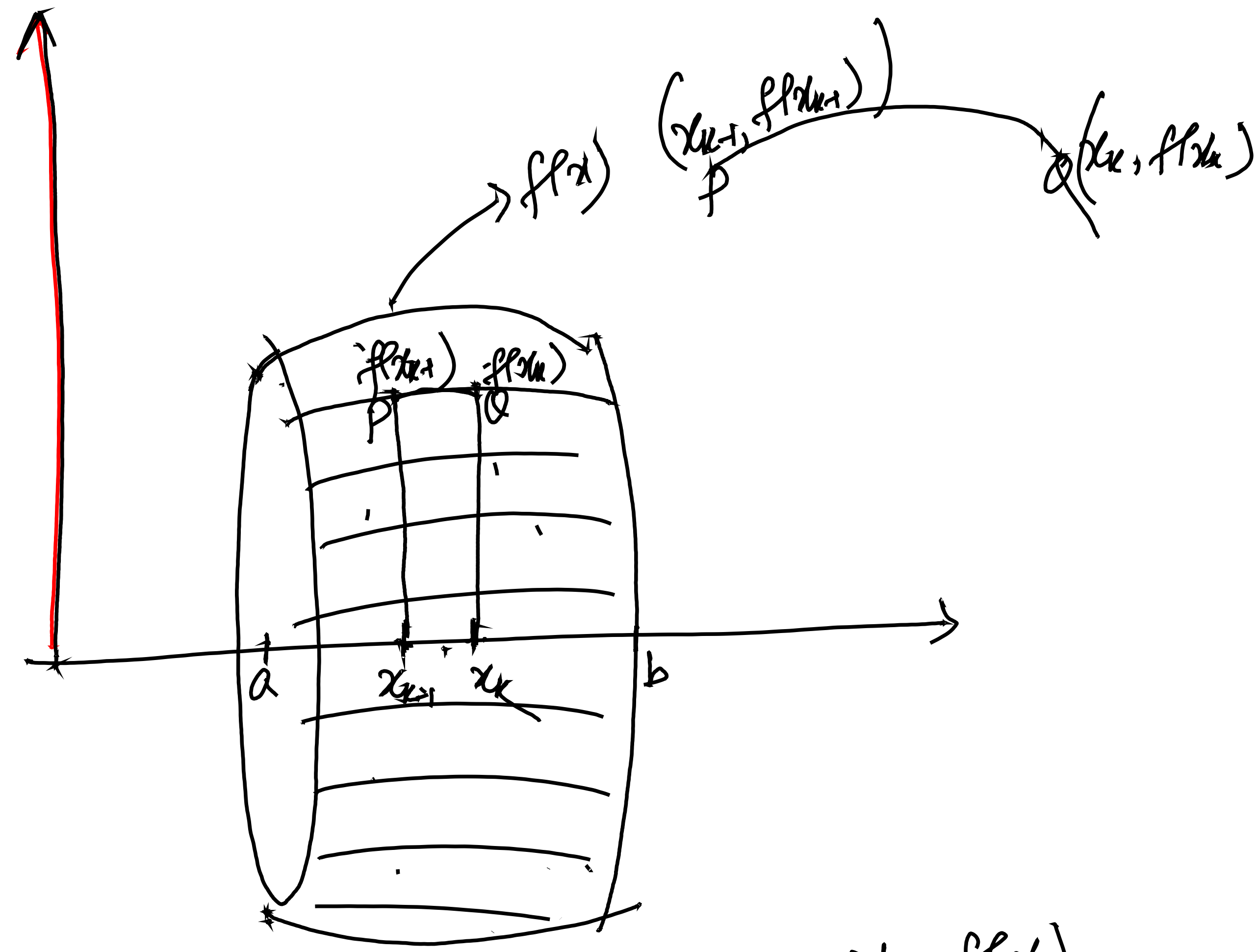
Areas of surfaces of revolution

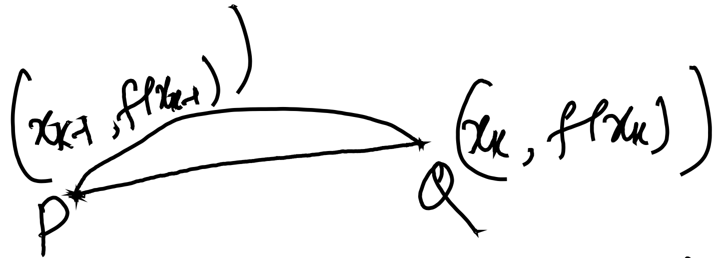
If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution.

But if you revolve only the boundary curve of the region, it does not sweep out any interior volume but rather a surface that sorrounds the solid.





$y = f(x)$

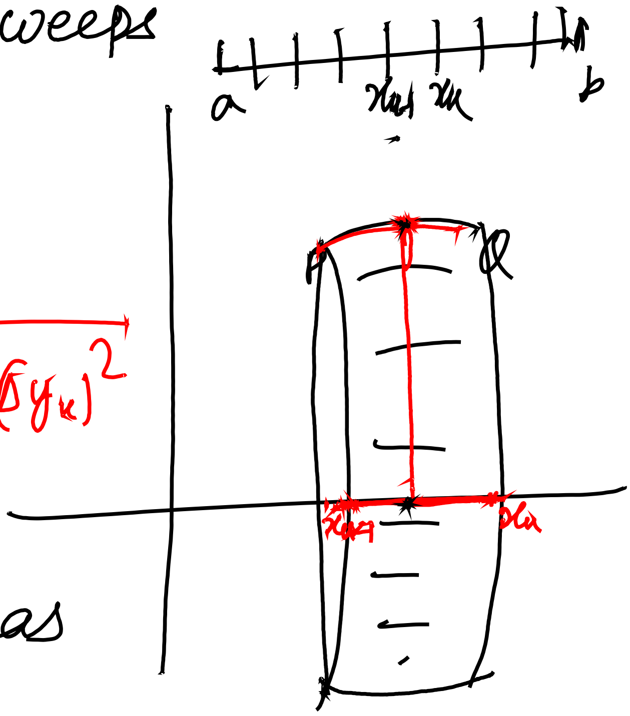


Length of the chord PQ

$$= \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

$$= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

After rotation consider the strip which PQ sweeps

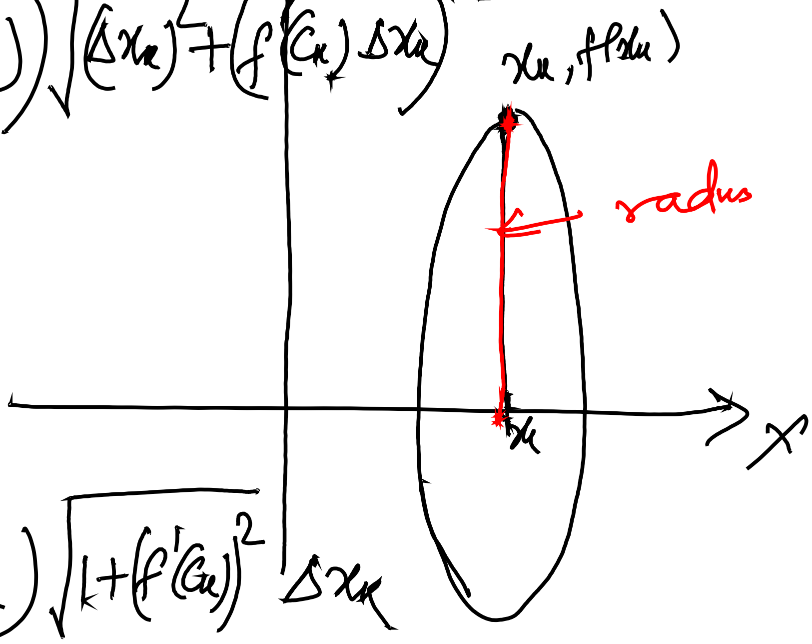


Surface area

$$= 2\pi \frac{f(x_{k-1})+f(x_k)}{2} \times \sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$$

Sum of surface areas of all these small strips

$$\sum_{k=1}^{n} \pi \left(f(x_{k-1})+f(x_k)\right)\sqrt{(\Delta x_k)^2+(f'(c_k)\Delta x_k)^2}$$



$$= \sum_{k=1}^{n} \pi \left(f(x_{k-1})+f(x_k)\right)\sqrt{1+(f'(c_k))^2}\,\Delta x_k$$

When the partition is very finer or as $n \to \infty$

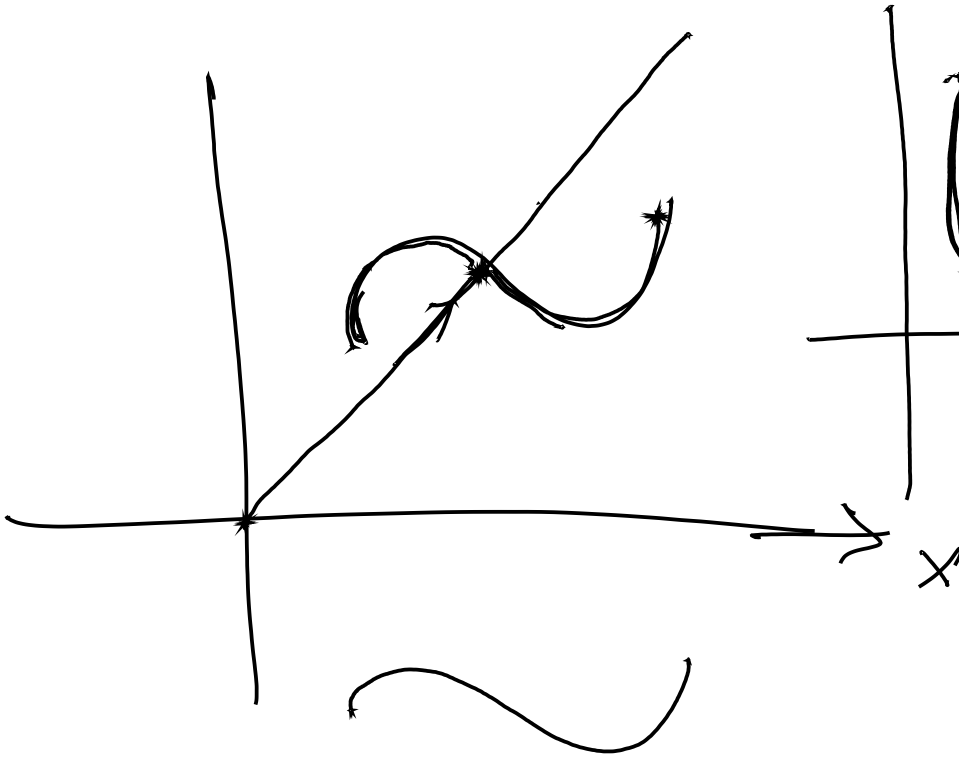
$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx}$$

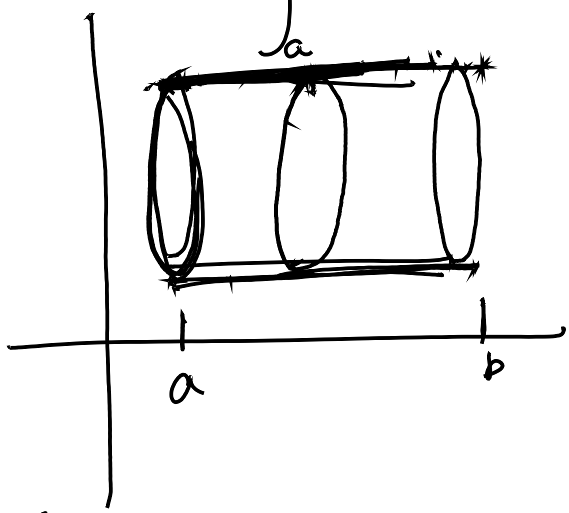
If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph $y = f(x)$ about the x-axis is

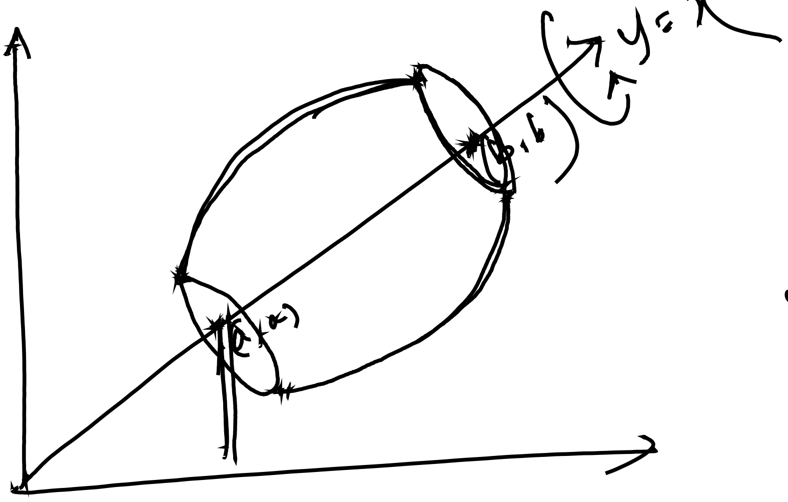
$y = f(x)$

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

$$\int_a^b 2\pi f(x)\, dx$$



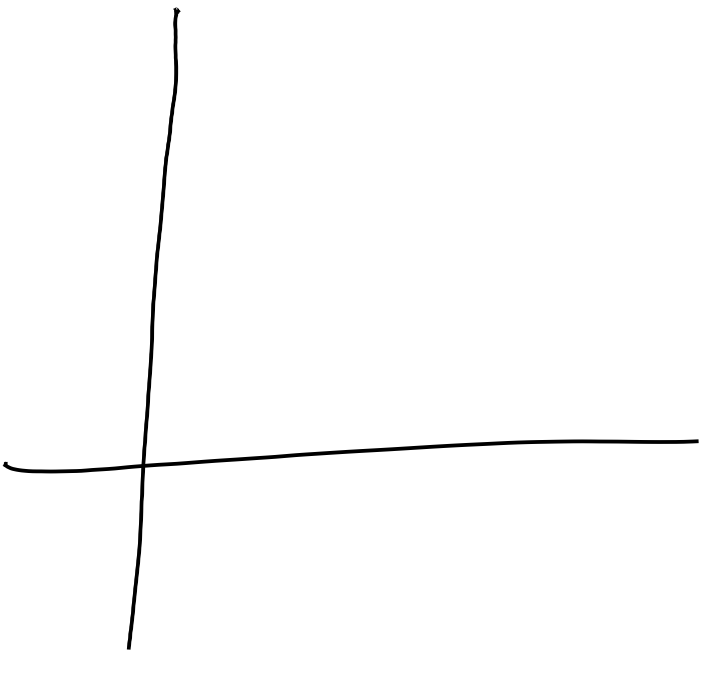


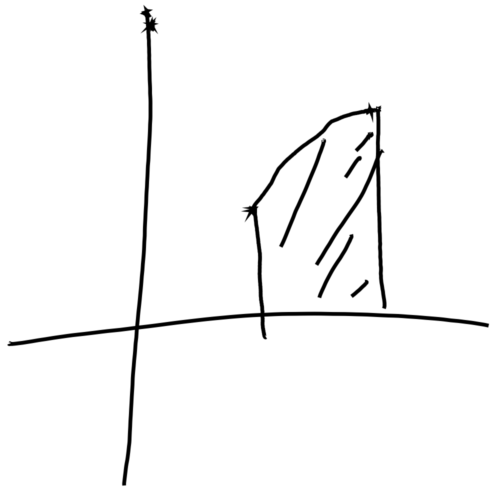


$$\int^{r}$$

$$\int_a^b 2\pi x^2\,dx$$

$$2\pi \int_a^b x^2\,dx$$





Revolution about the y-axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph $x = g(y)$ about y-axis is $\boxed{x = g(y)}$

$$\boxed{S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

Exp: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$ about the x-axis.

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 2\pi \, 2\cancel{\sqrt{x}} \, \frac{\sqrt{1+x}}{\cancel{\sqrt{x}}}\, dx$$

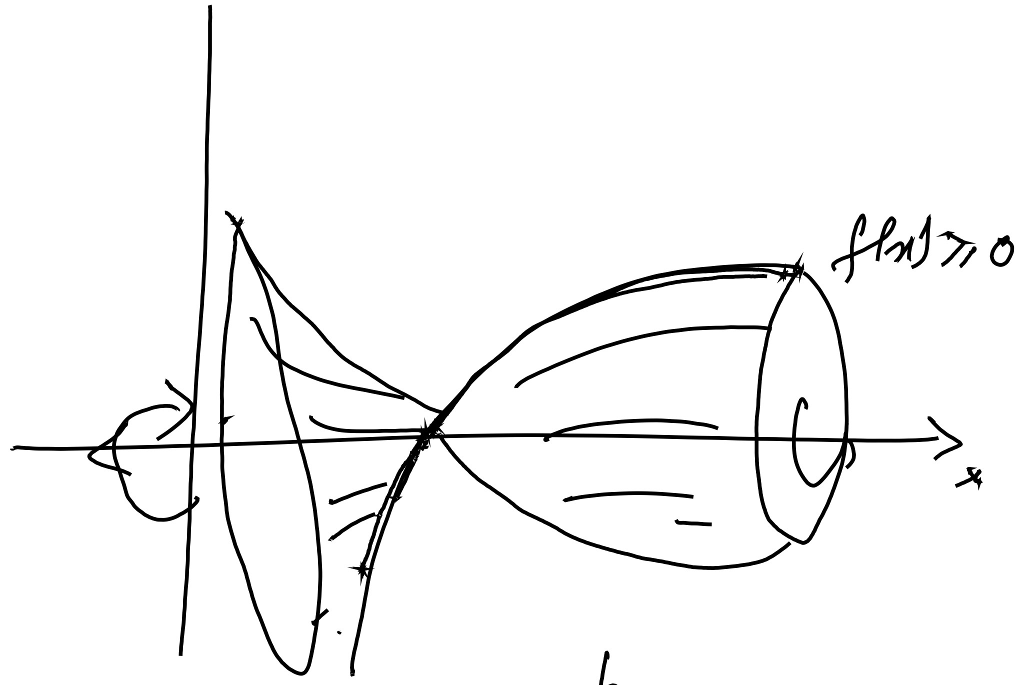
$$= 4\pi \int_1^2 \sqrt{1+x}\, dx$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x}} = \sqrt{\frac{x+1}{x}}$$

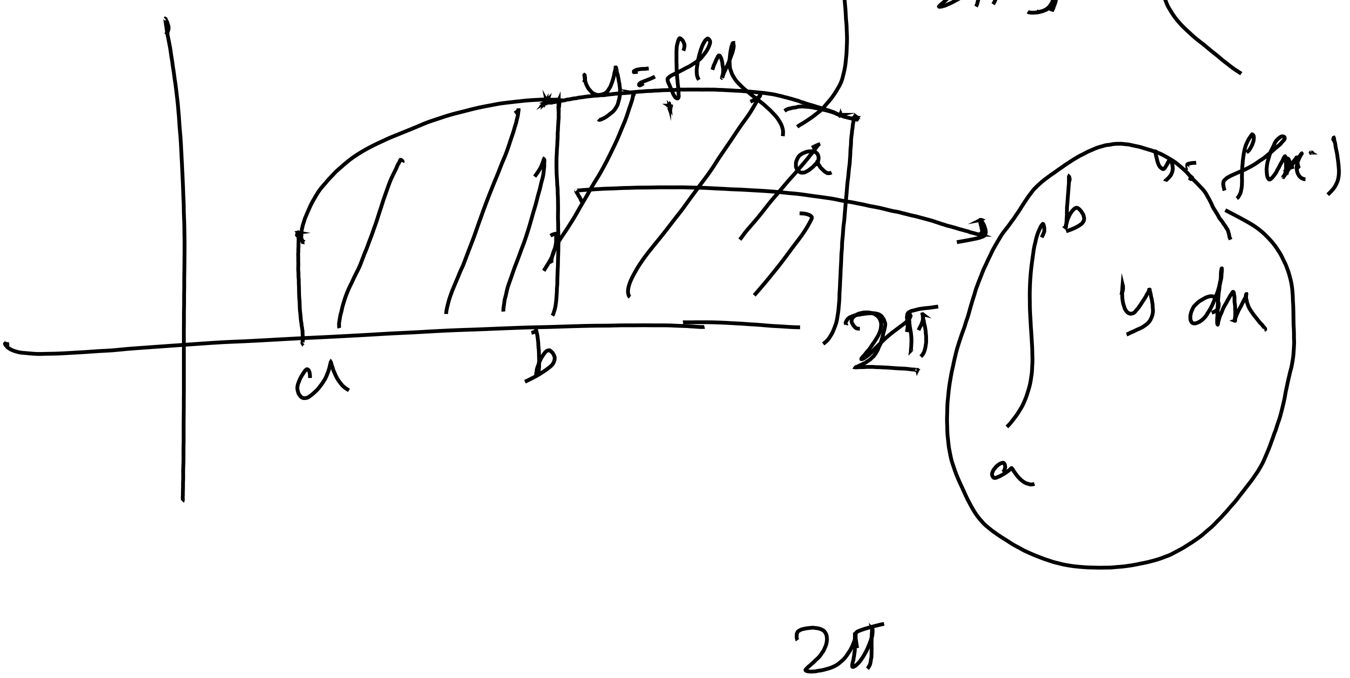
$$= \left[4\pi \, \frac{2}{3} \, (1+x)^{3/2}\right]_1^2$$

$$= \frac{8\pi}{3}\left(3^{3/2} - 2^{3/2}\right)$$

$$= \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$



$$\int^{b} 2\pi y \, dx$$



2π