Generating functions

To solve

Counting problems, ~~&~~ recurrence relations,

Combinatorial identities

Def^n (a_n) is a seq^n.

The generating function for the sequen $a_0, a_1, a_2 \ldots a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Ex $(a_k) = 3$ $\quad \sum_{k=0}^{\infty} 3x^k$

$a_k = k+1$

$$\sum_{k=0}^{\infty} (k+1) x^k$$

finite sequence $a_0, a_1, \ldots a_n$

$a_{n+1} = 0$

Generating function

$a_{n+2} = 0$

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$\vdots$

So on

Exp generating function for $1, 1, 1, 1, 1, 1$

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{x^6 - 1}{x - 1}$$

$$\left[a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} \quad r \neq 1 \right]$$

Important Facts

Ex) $f(x) = \dfrac{1}{1-x}$

$1, 1, 1, \ldots$

$1 + x + x^2 + x^3 \ldots = \sum_{n=0}^{\infty} x^n = \dfrac{x^{n}-1}{x-1}$

$|x|<1$

$\dfrac{x^n - 1}{x-1}$

$|x|<1$

$x^n \to 0$

$\dfrac{1}{1-x}$ $\dfrac{1}{1-x}$

Exp $f(x) = \dfrac{1}{1-ax}$

$\underbrace{1, a, a^2, a^3, a^4 \ldots}$

$\underbrace{1 + ax + a^2x^2 + a^3x^3 \ldots}$

$1 + ax + (ax)^2 + (ax)^3 + \ldots$

$\dfrac{1}{1-ax}$

$a_n = 1$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} 1 + x + x^2 + \cdots$$

$a_n = (a^k)$

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$

Th^m

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ (a_n)

$g(x) = \sum_{k=0}^{\infty} b_k x^k$ (b_n)

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x)\, g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Exp

$$\frac{1}{1-x}$$

$1, 1, 1, \ldots$

$$\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2}$$

Find the coeffs $a_0, a_1, a_2, \ldots$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1) x^k$$

Using Generating Functions to solve Recurrence Relations

Ex) Valid code word : String of digits with **even** no. of 0 digits.

0, 1, 2, ..., 9.

257030 valid

283590 invalid.

Let a_n be the no. of n digit valid codewords

Find a recurrence relation for a_n.

Sol^n

a_n = no. of valid n-digit codewords

$a_1 = 9$

0, 1 . . 9

$a_n =$

a_{n-1} = no. of valid n-1 digits codewords

first n-1

$9 a_{n-1}$ ~~ic~~ $+ \left(10^{n-1} - a_{n-1}\right)$

$10^{n-1} - a_{n-1}$

$a_n = 8 a_{n-1} + 10^{n-1}$

$a_n = 8a_{n-1} + 10^{n-1}$

$a_1 = 8 \cdot 1 + 10^0 = 8 + 1 = 9$

$a_1 = 9$

$a_0 = 1$

consistent.

$a_n x^n = 8a_{n-1} x^n + 10^{n-1} x^n$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$ (a_n)

$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n$

$$G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 8 a_{n-1} x^n + 10^{n-1} x^n$$

$$= 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$$

$$= 8x \underbrace{\sum_{n=0}^{\infty} a_n x^n} + x \underbrace{\sum_{n=0}^{\infty} 10^n x^n}$$

$$= 8x\, G(x) + \frac{x}{1-10x}$$

solve for $G(x)$

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

$$\underline{G(x)} = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$\underline{\sum_{n=0}^{\infty} a_n x^n} = \sum_{n=0}^{\infty} \frac{1}{2}\left(8^n + 10^n\right) x^n$$

$$\underline{a_n = \frac{1}{2}\left(8^n + 10^n\right)}$$

Proving Identities via Generating Functions

Exp Use generating functions to show that

$$\sum_{k=0}^{n} \underbrace{C(n,k)^2} = \underbrace{C(2n,n)}$$

Coefficient of $\underline{x^n}$ in $\underbrace{(1+x)^{2n}}$

$$(1+x)^{2n} = \left[(1+x)^n\right]^2 = \left[C(n,0) + C(n,1)\underline{x} + C(n,2)x^2 + \cdots \right.$$
$$\left. \cdots + C(n,n)\underline{x^n}\right]^2$$

Coefficient of x^n is

$$C(n,0)\,C(n,n) + C(n,1)\,C(n,n-1) + C(n,2)\,C(n,n-2)$$

$$\cdots + C(n,n)\cdot C(n,0) = \sum_{k=0}^{n} C(n,k)^2$$

$$\text{As } C(n,n-k) = C(n,k)$$