First order linear differential equation

The general form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{--- (1)}$$

(Linear differential equation means it is linear in dependent variable and there is no product of dependent variable and its derivatives)

Solution method

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{--- (1)}$$

We multiply a positive function $v(x)$ to (1) that transforms the LHS of (1) into derivative of the product $v(x) \cdot y$.

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x)$$

$$\frac{d}{dx}\left(v(x)\cdot y\right) = v(x)Q(x) \quad \text{—— (2)}$$

$v(x)$ is chosen to make

$$v(x)\frac{dy}{dx} + P(x)v(x)y = \frac{d}{dx}\left(v(x)\cdot y\right)$$

Integrating (2) w.r.t. x

$$v(x)\,y = \int v(x)Q(x)\,dx + C$$

(after integration)

$$\cancel{\Rightarrow y = \frac{1}{v(x)}\int v(x)Q(x)\,dx}$$

general solution

$v(x)$ is called the integrating factor because its presence makes the equation (1) integrable.

$$\frac{d}{dx}\left(v(x)\, y(x)\right) = v\frac{dy}{dx} + y\frac{dv}{dx} \quad \text{--- (3)}$$

$$v\frac{dy}{dx} + pvy = \frac{d}{dx}(vy) \quad \text{--- (4)}$$

Comparing (3) & (4)

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + pvy$$

$$\Rightarrow \quad y\frac{dv}{dx} = pvy$$

$$\Rightarrow \quad \frac{dv}{dx} = pv$$

$$\Rightarrow \quad \frac{dv}{v} = p\,dx$$

$$\log v = \int p\,dx$$

$$\Rightarrow \quad \boxed{v(x) = e^{\int p(x)\,dx}}$$

Exp solve

$$x\frac{dy}{dx} = x^2+3y \quad , \quad x>0 \quad \text{(1)}$$

Sol^n

$$\frac{dy}{dx} - \frac{3y}{x} = x \quad \text{(2)}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = -\frac{3}{x}, \quad Q(x) = x$$

$$\text{I.F.} \quad V(x) = e^{\int P(x)\,dx}$$

$$= e^{\int -\frac{3}{x}dx} = e^{-3\log x} = \frac{1}{x^3}$$

multiply (2) by $\frac{1}{x^3}$

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y(x)\right) = \frac{1}{x^2}$$

Integrating $\quad \frac{1}{x^3}y(x) = -\frac{1}{x} + c$

$$y(x) = -x^2 + cx^3$$ general solution.

Exp Find the particular solution of $3xy' - y = \log x + 1$, $x > 0$ satisfying $y(1) = -2$.

Sol^n

$$\frac{dy}{dx} - \frac{1}{3x} y = \frac{1}{3x}(\log x + 1)$$

$$P(x) = \frac{-1}{3x} \qquad Q(x) = \frac{1}{3x}(\log x + 1)$$

I.F. $e^{\int P(x)\,dx} = e^{\int -\frac{1}{3x} dx} = e^{-\frac{1}{3}\log x}$

$$= x^{-1/3}$$

Solution is

$$x^{-1/3} y(x) = \int x^{-1/3} \frac{1 + \log x}{3x} dx$$

$$= \frac{1}{3} \int (1 + \log x) x^{-4/3} dx$$

$$= -x^{-1/3}(1 + \log x) - 3x^{-1/3} + C$$

$$y(x) = -(1+\log x) - 3 + c\,x^{1/3}$$

general solution

$$y(1) = -2$$

$$-2 = -(1+\log 1) - 3 + c$$

$$\Rightarrow -2 = -4 + c \Rightarrow \boxed{c = 2}$$

The particular solution is

$$\boxed{y(x) = -(1+\log x) - 3 + 2x^{1/3}}$$

Equations reducible to linear equation

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n, \quad n \neq 1 \quad \text{——— (1)}$$

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \quad \text{——— (2)}$$

Put $y^{1-n} = z$

Differentiating w.r.t. x

$$\Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

putting in (2)

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)\,z = Q(x)$$

$$\Rightarrow \frac{dz}{dx} + \underline{(1-n)\,P(x)}\,z = Q(x)\,(1-n) \quad \text{——— (3)}$$

Linear equation in z and x.

$$\text{I.F.} = e^{\int (1-n)\,P(x)\,dx}$$

Sol^n

$$z \times \text{I.F.} = \int Q(x)\,(1-n) \times \text{I.F.}\,dx$$

Exp Solve

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}\log x \quad \text{---} (1)$$

Sol^n

$$y^{-2}\frac{dy}{dx} + \frac{y^{-1}}{x} = \frac{\log x}{x} \quad \text{---} (2)$$

put $y^{-1} = z$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$$

put in (2)

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x}$$

$$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$$

Linear equation.

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$$

Solution

$$z \cdot \frac{1}{x} = \int \frac{1}{x} \cdot -\frac{\log x}{x}\, dx = \frac{\log x}{x} + \frac{1}{x} + c$$

$=$

$$z = \log x + cx + 1$$

$$\Rightarrow \frac{1}{y} = \log x + cx + 1$$

$$\Rightarrow y(x) = \frac{1}{1 + \log x + cx}$$

Ex Solve

(1) $$x\frac{dy}{dx} + y = x^4 y^3$$

(2) $$\frac{dy}{dx}\left(x^2y^3 + xy\right) = 1$$

Second order differential equation

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = G(x) \quad \text{--- (1)}$$

This a general linear second order differential equation.

Assumption! $P(x), Q(x), R(x), G(x)$ are continuous functions.

If $G(x) = 0$ for all x, then

$$P(x)\, y''(x) + Q(x)\, y'(x) + R(x)\, y(x) = 0 \quad \text{--- (2)}$$

is said to be a homogeneous linear second order differential equation.

We also assume that $P(x) \neq 0$ for any x.

Result-1

If $y_1(x)$ and $y_2(x)$ are two solutions of linear homogeneous equation (2), then for any constants c_1 and c_2, the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of (2).

(Principle of superposition)

Result-2

If $P(x)$, $Q(x)$ and $R(x)$ are continuous functions and $P(x) \neq 0$ (for any x)

then the linear homogeneous equation (2) has two linearly independent solutions $y_1(x)$ and $y_2(x)$.

Moreover, if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (2) then the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

[c_1, c_2 arbitrary constants]

Homogeneous linear equations with constant coefficients

The general form is

$$ay''(x) + by'(x) + cy(x) = 0 \quad (3)$$

a, b, c are constants.

To solve equation (3) we seek a function which when multiplied by a constant and added to a constant time its 1st derivative plus a constant time its 2nd derivative, sums identically to zero.

One such function is exponential function.

We seek a solution of the form $y = e^{mx}$.

How to check ~~linearly~~ ~~dep~~ independent ?

$y_1(x)$ $y_2(x)$

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Wronskian

If $\neq 0$

then L. I.

linearly independent

If $= 0$ then L. D.

linearly dependent

$y_1(x) = x$, $y_2(x) = 2x$

(L. D.)

$$\begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix} = 2x - 2x = 0$$

$y_1(x) = x$, $y_2(x) = x^2$ non zero function.

L. I.

$$\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 \neq 0$$

L.