Divide-and-Conquer

$7\ 2\ |\ 9\ 4\ \rightarrow\ 2\ 4\ 7\ 9$

$7\ |\ 2\ \rightarrow\ 2\ 7$

$9\ |\ 4\ \rightarrow\ 4\ 9$

$7\ \rightarrow\ 7$

$2\ \rightarrow\ 2$

$9\ \rightarrow\ 9$

$4\ \rightarrow\ 4$
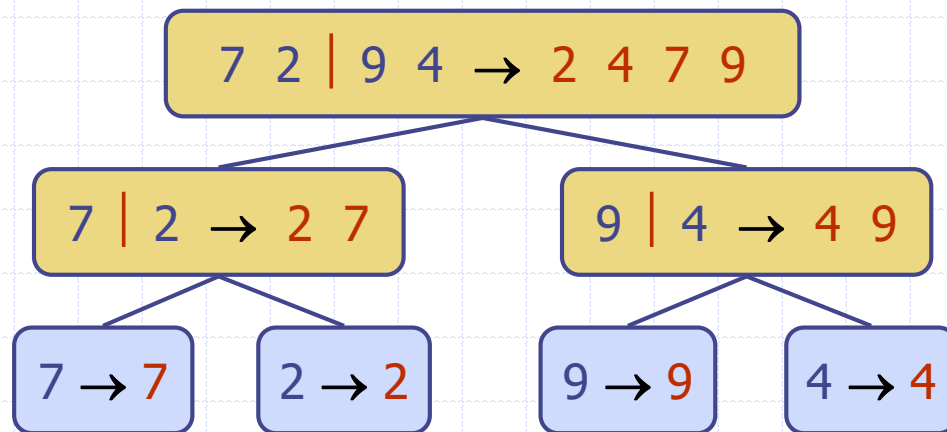
# Outline and Reading

- Divide-and-conquer paradigm
- Review Merge-sort
- Recurrence Equations
  - Iterative substitution
  - Recursion trees
  - Guess-and-test
  - The master method

Divide-and-Conquer

◈ **Divide-and conquer** is a general algorithm design paradigm:

- Divide: divide the input data S in two or more disjoint subsets S_1, S_2, …
- Recur: solve the subproblems recursively
- Conquer: combine the solutions for S_1, S_2, …, into a solution for S

◈ The base case for the recursion are subproblems of constant size

◈ Analysis can be done using **recurrence equations**
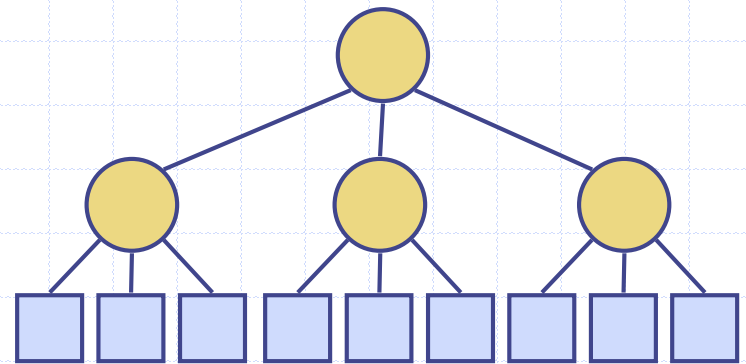
Merge-Sort Review

◆ Merge-sort on an input sequence S with n elements consists of three steps:

- Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
- Recur: recursively sort S_1 and S_2
- Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S)

   Input sequence S with n elements

   Output sequence S sorted

   if $S.size() > 1$

      $(S_1, S_2) \leftarrow partition(S, n/2)$

      $mergeSort(S_1,)$

 $mergeSort(S_2)$

      $S \leftarrow merge(S_1, S_2)$

Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b.
- Likewise, the basis case ($n < 2$) will take at b most steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- We can therefore analyze the running time of merge-sort by finding a **closed form solution** to the above equation.
  - That is, a solution that has $T(n)$ only on the left-hand side.
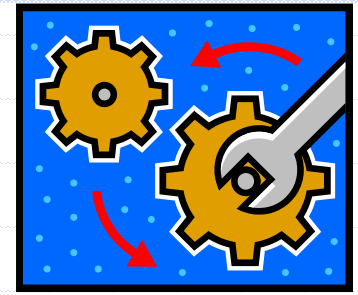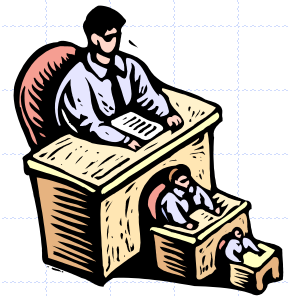
Iterative Substitution

- In the iterative substitution, or "plug-and-chug," technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern:

$$T(n) = 2T(n/2) + bn$$

$$= 2(2T(n/2^2)) + b(n/2)) + bn$$

$$= 2^2 T(n/2^2) + 2bn$$

$$= 2^3 T(n/2^3) + 3bn$$

$$= 2^4 T(n/2^4) + 4bn$$

$$= \dots$$

$$= 2^i T(n/2^i) + ibn$$

- Note that base, $T(n)=b$, case occurs when $2^i=n$. That is, $i = \log n$.
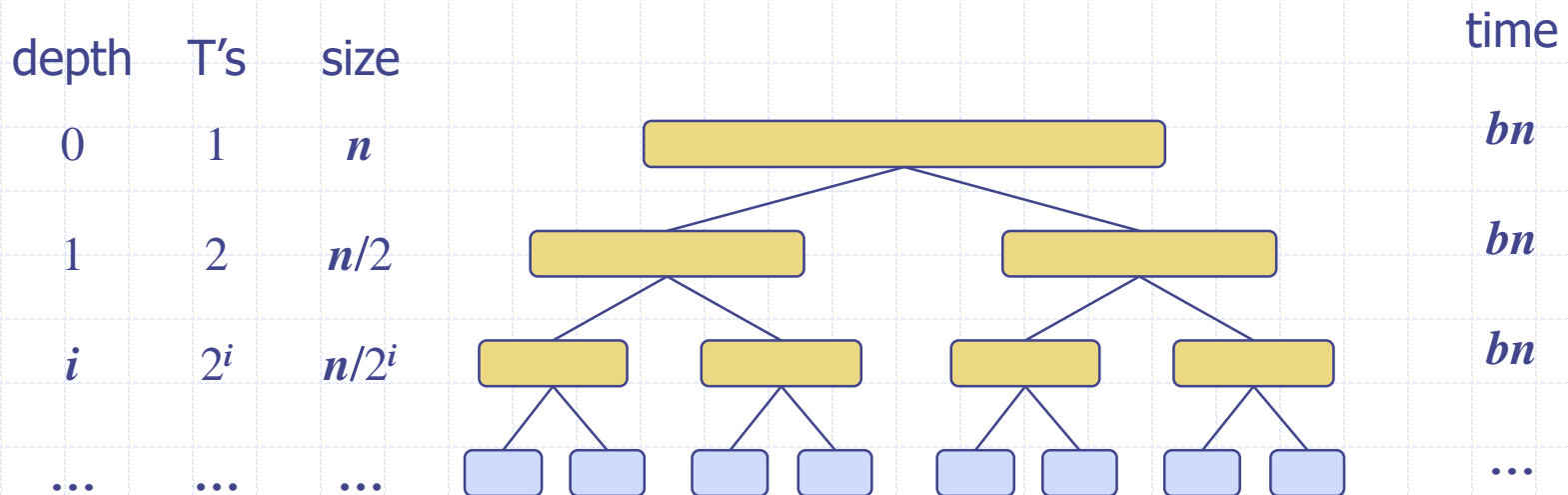- So,

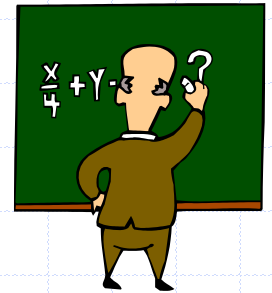$$T(n) = bn + bn \log n$$

- Thus, $T(n)$ is $O(n \log n)$.

The Recursion Tree

♦ Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

depth	T's	size		time
0	1	n		bn
1	2	$n/2$		bn
i	2^i	$n/2^i$		bn
...	...	...		...

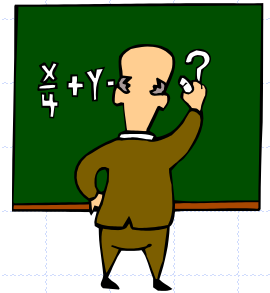Total time $= bn + bn \log n$

(last level plus all previous levels)

Guess-and-Test Method

- In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

- Guess: T(n) < cn log n.

Guess-and-Test Method

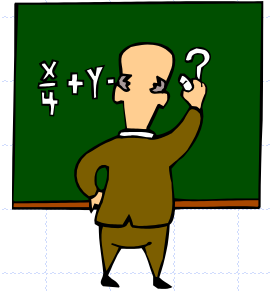◆ In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

◆ Guess: T(n) < cn log n.

$$T(n) = 2T(n/2) + bn \log n$$

$$\leq 2(c(n/2) \log(n/2)) + bn \log n$$

$$\leq cn(\log n - \log 2) + bn \log n$$

$$\leq cn \log n - cn + bn \log n$$

◆ The above guess is correct only if c.n is greater than equal to b.n.(log n). This is not possible.
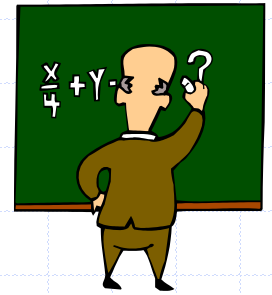
Guess-and-Test Method, Part 2

◆ Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

◆ Guess #2: T(n) < cn log^2 n.

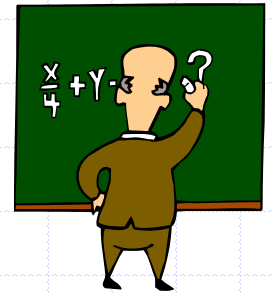# Guess-and-Test Method, Part 2

◈ Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

◈ Guess #2: T(n) < cn log^2 n.

$$T(n) = 2T(n/2) + bn \log n$$

$$\leq 2(c(n/2)\log^2(n/2)) + bn \log n$$

$$\leq cn(\log n - \log 2)^2 + bn \log n$$

$$\leq cn \log^2 n - 2cn \log n + cn + bn \log n$$

▪ Here the above guess is correct if c > b. This is very much possible.
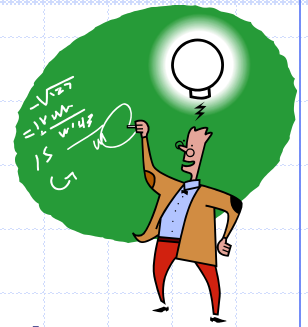
Guess-and-Test Method, Part 2

◆ Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

◆ Guess #2: T(n) < cn log² n.

$$T(n) = 2T(n/2) + bn \log n$$

$$\leq 2(c(n/2) \log^2 (n/2)) + bn \log n$$

$$\leq cn(\log n - \log 2)^2 + bn \log n$$

$$\leq cn \log^2 n - 2cn \log n + cn + bn \log n$$

■ Here the above guess is correct if c > b. This is very much possible. So, T(n) is O(n log² n). In general, to use this method, you need to have a good guess and you need to be good at induction proofs.
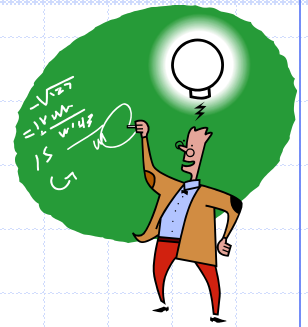
Master Method

- Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

- The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Master Method, Example 1

◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:
$$T(n) = 4T(n/2) + n$$

Solution: $\log_b a = 2$, so case 1 says T(n) is $\Theta(n^2)$.

Master Method, Example 2

◆ The form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
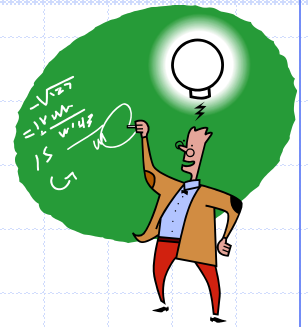
◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

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3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 2T(n/2) + n\log n$$

Solution: $\log_b a = 1$, so case 2 says $T(n)$ is $\Theta(n \log^2 n)$.

Master Method, Example 3

◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
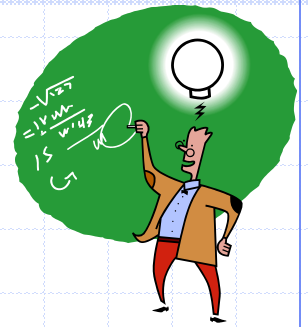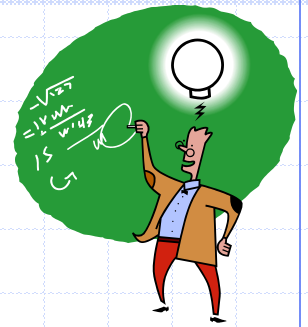
◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:
$$T(n) = T(n/3) + n \log n$$

Solution: $\log_b a = 0$, so case 3 says T(n) is Θ(n log n).

Master Method, Example 4

- The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

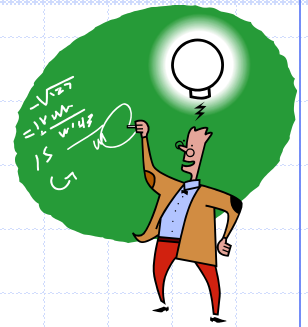- The Master Theorem:

    1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

    2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

    3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

    provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

- Example:
$$T(n) = 8T(n/2) + n^2$$

    Solution: $\log_b a = 3$, so case 1 says $T(n)$ is $\Theta(n^3)$.

Master Method, Example 5

◆ The form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
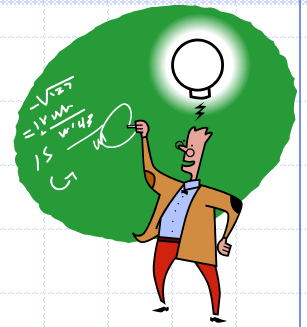
◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a = 2$, so case 3 says $T(n)$ is $\Theta(n^3)$.

Master Method, Example 6

◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
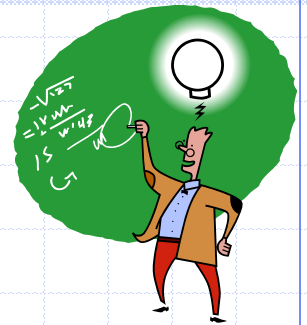
◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:
$$T(n) = T(n/2) + 1 \quad \text{(binary search)}$$

Solution: $\log_b a = 0$, so case 2 says T(n) is Θ(log n).

Master Method, Example 7

◆ The form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
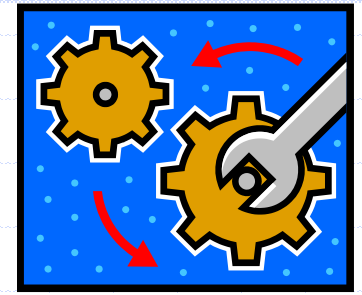
◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$

2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$

3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 2T(n/2) + \log n \quad \text{(heap construction)}$$

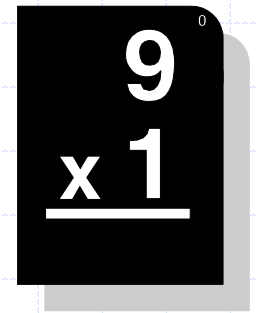Solution: $\log_b a = 1$, so case 1 says $T(n)$ is $\Theta(n)$.

Iterative "Proof" of the Master Theorem

- Using iterative substitution, let us see if we can find a pattern:

$$T(n) = aT(n/b) + f(n)$$

$$= a(aT(n/b^2)) + f(n/b)) + bn$$

$$= a^2 T(n/b^2) + af(n/b) + f(n)$$

$$= a^3 T(n/b^3) + a^2 f(n/b^2) + af(n/b) + f(n)$$

$$= \ldots$$

$$= a^{\log_b n} T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)$$

$$= n^{\log_b a} T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)$$

- We then distinguish the three cases as
  - The first term is dominant
  - Each part of the summation is equally dominant
  - The summation is a geometric series

Integer Multiplication

9
x 1

◆ Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

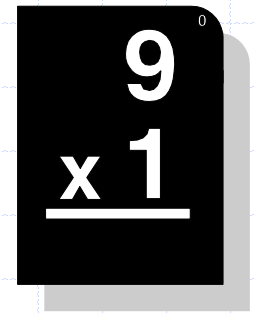$$J = J_h 2^{n/2} + J_l$$

- We can then define I*J by multiplying the parts and adding:

$$I * J = (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l)$$

$$= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l$$

- So, T(n) = 4T(n/2) + n, which implies T(n) is $O(n^2)$.
- But that is no better than the algorithm we learned in grade school.

An Improved Integer Multiplication Algorithm

9
x 1

◆ Algorithm: Multiply two n-bit integers I and J.

- ■ Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- ■ Observe that there is a different way to multiply parts:

$$I * J = I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l$$

$$= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l$$

$$I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l$$

- ■ So, T(n) = 3T(n/2) + n, which implies T(n) is $O(n^{\log_2 3})$, by the Master Theorem.
- ■ Thus, T(n) is $O(n^{1.585})$.