1. Let $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$. Then verify that $\vec{\nabla}' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}} = -\vec{\nabla} \frac{1}{\mathscr{r}}$ where $\vec{\nabla}'$ is the gradient with respect to the primed coordinates.
soln:

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$\begin{aligned}
\vec{\nabla}\frac{1}{\mathscr{r}} &= \hat{i}\frac{\partial}{\partial x}\frac{1}{\mathscr{r}} + \hat{j}\frac{\partial}{\partial y}\frac{1}{\mathscr{r}} + \hat{k}\frac{\partial}{\partial z}\frac{1}{\mathscr{r}} \\
&= -\hat{i}\frac{x-x'}{\mathscr{r}^3} - \hat{j}\frac{y-y'}{\mathscr{r}^3} - \hat{k}\frac{z-z'}{\mathscr{r}^3} \\
&= -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \\
\vec{\nabla}'\frac{1}{\mathscr{r}} &= \hat{i}\frac{\partial}{\partial x'}\frac{1}{\mathscr{r}} + \hat{j}\frac{\partial}{\partial y'}\frac{1}{\mathscr{r}} + \hat{k}\frac{\partial}{\partial z'}\frac{1}{\mathscr{r}} \\
&= -\hat{i}\frac{(x-x')(-1)}{\mathscr{r}^3} - \hat{j}\frac{(y-y')(-1)}{\mathscr{r}^3} - \hat{k}\frac{(z-z')(-1)}{\mathscr{r}^3} \\
&= \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}
\end{aligned}$$

$$\therefore \quad \vec{\nabla}\frac{1}{\mathscr{r}} = -\vec{\nabla}'\frac{1}{\mathscr{r}}$$

2. A sphere of radius R, centered at the origin, carries charge density $\rho(r,\theta) = k(R-r)\cos\theta$. Find the approximate potential on the z axis far from the sphere.

soln:
Let us denote the charge distribution as $\rho(\vec{r'})$ and the observation point as $\vec{r}$. In this notation the multipole expansion of the potential at $\vec{r}$ is given as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int (r')^l P_i(\cos\theta')\rho(\vec{r'})d\tau'$$

The monopole term is the $l=0$ term. This is given as

$$\begin{aligned}
V_{mon}(z) &= \frac{1}{4\pi\epsilon_0 z}\int \rho(\vec{r'})d\tau' \\
&= \frac{2\pi}{4\pi\epsilon_0 z}\int_0^R k(R-r')r'^2 dr' \int_0^\pi \sin\theta'\cos\theta' d\theta' \\
&= 0 \qquad (1)
\end{aligned}$$

as the angular integral is 0. So we go to the next significant term

$$\begin{aligned}
V_{dip}(z) &= \frac{1}{4\pi\epsilon_0 z^2}\int_0^R\int_0^\pi r'\cos\theta' k(R-r')\cos\theta' r'^2\sin\theta' dr'd\theta'd\phi' \\
&= \frac{1}{2\epsilon_0 z^2}\int_0^R r'^3 k(R-r')dr'\int_0^\pi \cos^2(\theta')\sin\theta' d\theta' \\
&= \frac{1}{2\epsilon_0 z^2}\frac{kR^5}{20}\frac{2}{3} \\
&= \frac{1}{60\epsilon_0 z^2}kR^5
\end{aligned}$$

This is the most significnt term in the potential far away on the z axis. The charge distribution has an azimuthal symmetry. So at a general point (r,θ) the potential is

$$V_{dip}(r,\theta) = \frac{1}{60\epsilon_0 r^2}kR^5\cos\theta$$

3. A dipole $\vec{\mathrm{p}}$ is at a distance r from a point charge q and oriented so that $\vec{\mathrm{p}}$ makes an angle θ with the vector $\vec{\mathrm{r}}$ from q to $\vec{\mathrm{p}}$.

   (a) What is the force on $\vec{\mathrm{p}}$?

   (b) What is the force on q ?

   soln
   In both the parts it is easier if we take the dipole along $\hat{z}$.

   (a) Due to q at the origin the force on the dipole $\vec{p}$ is

   $$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} = p\frac{\partial\vec{E}}{\partial z}$$

   where $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$

   $$\begin{aligned}
   \therefore\quad \vec{F}_p &= \frac{qp}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left(\frac{\vec{r}}{r^3}\right) \\
   &= \frac{qp}{4\pi\epsilon_0}\left[\vec{r}\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\frac{\partial\vec{r}}{\partial z}\right] \\
   &= \frac{qp}{4\pi\epsilon_0}\left[-\frac{3z\vec{r}}{r^5} + \frac{\hat{z}}{r^3}\right] \\
   &= \frac{q}{4\pi\epsilon_0}\left[-\frac{(3\vec{p}\cdot\vec{r})\vec{r}}{r^5} + \frac{\vec{p}}{r^3}\right] \\
   &= -\frac{q}{4\pi\epsilon_0 r^3}\left[(3\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]
   \end{aligned}$$

   (b) For this part we place the dipole at the origin.

The electric field at q due to $\vec{p}$ is

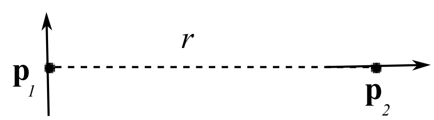
$$\vec{E} = \frac{1}{4\pi\epsilon_0 r'^3}\left[3(\vec{p}\cdot\hat{r}')\hat{r}' - \vec{p}\right]$$

Now $\vec{r}' = -\vec{r}$ (used in part (a)).
$\therefore\ \ \hat{r}' = -\hat{r}$.
$\therefore$ force on q is

$$\vec{F}_q = q\vec{E} = \frac{q}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

We see that the forces are equal and opposite.

4. $\vec{\mathrm{p}}_1$ and $\vec{\mathrm{p}}_2$ are perfect dipoles a distance r apart. $\vec{\mathrm{p}}_2$ is along $\vec{\mathrm{r}}$ while $\vec{\mathrm{p}}_1$ is orthogonal to $\vec{\mathrm{r}}$. Calculate the torque on the dipoles. Are they equal and opposite?



soln
To calculate torque on $\vec{p}_2$ we consider $\hat{z}$ along $\vec{p}_1$. So at $\vec{p}_2$ the electric field is

$$\begin{aligned}
\vec{E}_{p1} &= \frac{p_1}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \\
&= \frac{p_1}{4\pi\epsilon_0 r^3}\hat{\theta} \quad \text{since} \quad \theta = \frac{\pi}{2} \\
\therefore\quad \vec{\tau}_{p2} &= \vec{p}_2 \times \vec{E}_{p1} = \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{n})
\end{aligned}$$

where $\hat{n}$ is a normal to the paper outward.

To calculate the torque on $\vec{p}_1$ due to $\vec{p}_2$ we consider the origin at $\vec{p}_2$ with $\hat{z}$ along $\hat{p}_2$.

$$\begin{aligned}
\vec{E}_{p2} \ \text{ at } \ \vec{p}_1 &= \frac{p_2}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right) \\
\frac{p_2}{4\pi\epsilon_0 r^3}(-2\hat{r}) & \\
\therefore\quad \vec{\tau}_{p1} &= \vec{p}_1 \times \vec{E}_{p2} = \frac{2p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{n})
\end{aligned}$$

Note that the torques are not equal and opposite. Did you expect them to be so?

5. A sphere of radius R carries a polarization $\vec{\mathrm{P}}(\vec{\mathrm{r}}) = k\vec{\mathrm{r}}$
(a) Calculate the bound charges ρ_b and σ_b and the electric field caused due to them inside and outside the sphere.
(b) Find the electric field using the Gauss' law for the displacement vector $\vec{D}$ given as $\oint_S \vec{D}\cdot\hat{n}da = Q_{f(enc)}$.

soln:

(a)

The bound volume charge density is given as

$$\begin{aligned}\rho_b &= -\vec{\nabla}\cdot\vec{P} = -3k \\ \sigma_b &= \vec{P}\cdot\hat{n} = kR\end{aligned}$$

The given electrostatic configuration has a spherical symmetry. So by Gauss's law inside the sphere $r < R$ we have

$$E_{in}4\pi r^2 = -\frac{3k}{\epsilon_0}\frac{4}{3}\pi r^3$$

This gives $\vec{E}_{in} = -\frac{k\vec{r}}{\epsilon_0}$.

Outside the sphere $r > R$ we have

$$E_{out}4\pi r^2 = \frac{1}{\epsilon_0}\left[-3k\frac{4}{3}\pi R^3 + kR\times 4\pi R^2\right] = 0$$

This gives $\vec{E}_{out} = 0$.

(b)

Since there is no free charges anywhere we have $Q_{f(enc)} = 0$. So using the Gauss' law for the displacement vector we get $\vec{D} = 0$ everywhere.

Since $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ we have $\vec{E} = -\frac{\vec{P}}{\epsilon_0}$.

This directly gives

$$\vec{E}_{in} = -\frac{\vec{P}}{\epsilon_0} = -\frac{k\vec{r}}{\epsilon_0}, \quad \text{and} \quad \vec{E}_{out} = 0$$

6. A point charge q is imbedded at the center of a sphere of linear dielectric material with susceptibility χ_e and radius R. Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

soln

The problem has a spherical symmetry.

Consider a Gaussian sphere of radius r. We have $D4\pi r^2 = q$.

$\therefore\quad D = q/4\pi r^2$.

$$D = \epsilon_0 E + P = \epsilon_0 E + \epsilon_0\chi_e E = \epsilon_0(1+\chi_e)E$$

$$\therefore\quad E = \frac{D}{\epsilon_0(1+\chi_e)} = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}$$

Polarization $\vec{P} = \epsilon_0\chi_e\vec{E}$.

$$\therefore\quad P = \frac{\chi_e}{1+\chi_e}\frac{q}{4\pi r^2}$$

$$\text{Hence}\quad \vec{\nabla}\cdot\vec{P} = 0 \;\text{ for }\; r > 0$$

$$\therefore\quad \rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

On the surface of the sphere

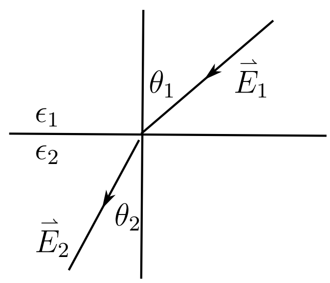
$$\sigma_b = \vec{P} \cdot \hat{r} = \frac{\chi_e}{(1+\chi_e)} \frac{q}{4piR^2}$$

Total bound charge on the surface of the sphere is $\frac{\chi_e}{1+\chi_e} q$.
Since the total bound charge has to be 0, the remaining bound charge is concentrated at the center surrounding the point charge. We can write

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{\chi_e}{1+\chi_e} q \delta^{(3)}(\vec{r})$$

Inside the dielectric the charge q is screened by ρ_b and reduces the electric field

7. At the interface between one linear dielectric and another the electric field lines bend. Show that $\tan\theta_2 / \tan\theta_1 = \epsilon_2/\epsilon_1$ assuming there is no free charge at the boundary. Refer to fig.1 below.



soln:

$$\vec{D}_1 = \epsilon_0 \epsilon_1 \vec{E}_1 \qquad \text{and} \qquad \vec{D}_2 = \epsilon_0 \epsilon_2 \vec{E}_2$$

Since there are no free charges at the interface

$$\begin{aligned} D_1^{\perp} &= D_2^{\perp} \\ \therefore \quad \epsilon_0 \epsilon_1 E_1 \cos\theta_1 &= \epsilon_0 \epsilon_2 E_2 \cos\theta_2 \\ \therefore \quad \epsilon_1 E_1 \cos\theta_1 &= \epsilon_2 E_2 \cos\theta_2 \end{aligned} \tag{2}$$

The parallel component of electric field must be equal.

$$\therefore \quad E_1 \sin\theta_1 = E_2 \sin\theta_2 \tag{3}$$

From 2 and 3 we have

$$\frac{\tan\theta_1}{\epsilon_1} = \frac{\tan\theta_2}{\epsilon_2} \qquad \Longrightarrow \qquad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_2}{\epsilon_1}$$

8. Suppose the field inside a large piece of dielectric is $\vec{\mathrm{E}}_0$, so that the electric displacement is $\vec{\mathrm{D}}_0 = \epsilon_0 \vec{\mathrm{E}}_0 + \vec{\mathrm{P}}$.

(a) If we have a narrow cylindrical(needle-like) cavity inside the material running parallel to $\vec{\mathrm{P}}$ find the field near the center of the cavity in terms of $\vec{\mathrm{E}}_0$ and $\vec{\mathrm{P}}$. Also find the displacement at the center of the cavity in terms of $\vec{\mathrm{D}}_0$ and $\vec{\mathrm{P}}$.

(b) Do the same for a thin wafer shaped cavity perpendicular to $\vec{\mathrm{P}}$.

soln:

(a) The tangential component of the electric field along the cylindrical walls of the cavity must be continuous.

$$\begin{aligned} \therefore\ \vec{E} &= \vec{E}_0 \\ \vec{D} &= \epsilon_0 \vec{E}_0 = \vec{D}_0 - \vec{P} \end{aligned}$$

(b) Here we use the boundary condition on the perpendicular component of $\vec{D}$ since there are no free charges.
Near the center of the cavity

$$\begin{aligned} \vec{D} &= \vec{D}_0 \\ \vec{E} &= \frac{1}{\epsilon_0}\vec{D}_0 = \vec{E}_0 + \frac{1}{\epsilon_0}\vec{P} \end{aligned}$$