Magnetostatics

Certain materials attract iron. These materials, when they are brought close to each other, exert forces that can be attractive or repulsive depending upon their orientation. These forces are called magnetic forces, and the materials which are sources of magnetic forces are called magnetic materials.

Experiments with magnetic forces reveal that conducting wires carrying current experience a force in the vicinity of a magnetic material. These forces are proportional to the magnitude of the current through the wire. So when the current is 0 there is no force on the wire. the force on the wire is also found to be perpendicular to the wire. When the current is reversed, the force reverses its direction. Ince a current is understood to be a flow of charges through a conductor, the force exerted by the magnetic material appear to be influencing the charges. Since the force ceases when the current is 0, it appears to be related to the velocity of the charge.

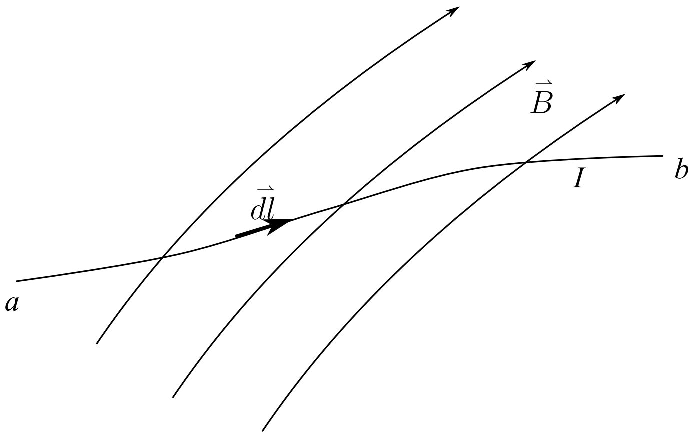
1 Magnetic Field

Just like in Electrostatics, the cause of the magnetic forces exerted on a current carrying wire or a moving charge is understood to be the presence of a magnetic field in the vicinity of a magnetic material. The field is denoted by the vector $\vec{B}$. The force on a charge q moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by the Lorentz force law

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Not only does a current carrying wire experience a force in a magnetic field but a wire carrying a current produces a magnetic field around it. This is seen from the effect of a current carrying wire on other current carrying wires and other moving charges. In fact every magnetic material or the sources of magnetic field is understood to be made of current carrying elements. So if we want to express $\vec{B}$ in terms of its sources we must be able to find out the magnetic field created due to a current carrying wire. This is given by the Biot-Savart Law, which we will see later.

2 Force on a current carrying wire



Consider a wire carrying a current I placed in a magnetic field. Let the charge per unit length in the wire be λ. If $\vec{v}$ is the velocity of the charge in the wire then $I = \lambda v$. Consider an element $\vec{dl}$ of the wire. Then the amount of charge in this element is

$$dq = \lambda dl$$

The magnetic force on this element is

$$\vec{df} = dq(\vec{v} \times \vec{B}) = \lambda dl(\vec{v} \times \vec{B})$$

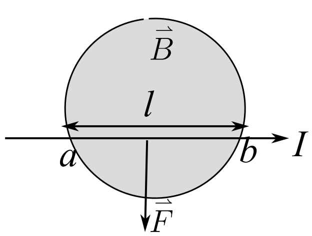
Since in a wire $\vec{v}$ and $\vec{dl}$ are in the same direction we have

$$\vec{df} = \lambda v(\vec{dl} \times \vec{B}) = I(\vec{dl} \times \vec{B})$$

In magnetostatics we will be dealing with currents which is same throughout the wire. These are called steady currents. So in this situation I is same throughout the wire and we have the total force

$$\vec{F} = I \int_a^b \vec{dl} \times \vec{B}$$

Eg:
Let us calculate the force on astraight wire, a length l of which lies perpendicular to a uniform magnetic field $\vec{B}$. The current in the wire is I. The magnetic field is perpendicular to the plane of the paper, coming out. The force on the wire is



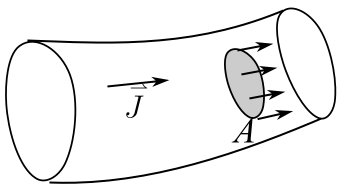
$$\vec{F} = I \int_a^b \vec{dl} \times \vec{B}$$

Since $\vec{B}$ is constant through the length l of the wire,

$$\vec{F} = I \int_a^b \vec{dl} \times \vec{B} = I\vec{l} \times \vec{B}$$

Since $\vec{l}$ is perpendicular to $\vec{B}$, we get $\vec{F} = BIL$ downward.

Current density: In general the current carrying conductor may not be in the form of a wire with infinitesimal width. In such a case we define a vector called current density at every point in the material



Once $\vec{J}$ is given, the current through any surface A is given by the surface integral

$$I_A = \int_A \vec{J} \cdot \hat{n} da$$

So the current flowing through the area A is the flux of the current density $\vec{J}$ through A. If ρ is the density of moving charges in the conductor and $\vec{v}$ is the velocity of the charges at a point then the current density $\vec{J}$ at the point is given as

$$\vec{J} = \rho\vec{v}$$

This is because the amount of charge passing a unit area of cross section per unit time is $\rho\vec{v}$. If this conductor is kept in a magnetic field $\vec{B}$ then the total force on the conductor is

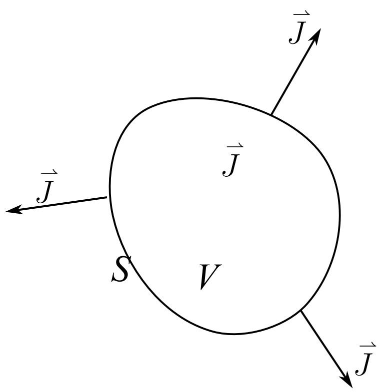
$$\begin{aligned} \vec{F} &= \int_{\mathcal{V}} dq(\vec{v} \times \vec{B}) \\ &= \int_{\mathcal{V}} (\rho d\tau(\vec{\nabla} \times \vec{B}) = \int_{\mathcal{V}} (\vec{J} \times \vec{B}) d\tau \end{aligned}$$

where $\mathcal{V}$ is the volume of the conductor in the magnetic field.
Often the current is not distributed over the body of the conductor but only over the surface. In such cases we define the surface current density $\vec{K}$. So if σ is the surface density of the moving charges in a conductor and $\vec{v}$ is the velocity at a point, then $\vec{K} = \sigma\vec{v}$. The force on the surface in a magnetic field is

$$\vec{F} = \int_S \vec{K} \times \vec{B} da$$

Note: $\vec{J}$ is current per unit cross sectional area
$\vec{K}$ is current per unit cross sectional length.

3 Steady current



Consider a volume $\mathcal{V}$ enclosed by a surface S. Let $\vec{J}$ be the current density at every point in the volume.
By divergence theorem

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{J} d\tau = \oint_S \vec{J} \cdot \hat{n} da$$

The surface integral on the right hand side is the total current flowing across the surface S. If Q is the amount of charge enclosed by the surface S at a time t then

$$\oint_S \vec{J} \cdot \hat{n} da = -\frac{dQ}{dt}$$

This equation means that the total current is the amount of charge flowing across the surface S per unit time. The negative sign is put because if the flux is positive, i.e, the current flows outwards, then the total charge Q within the surface decreases. If ρ is the charge density then

$$\begin{aligned}
\frac{dQ}{dt} &= \frac{d}{dt}\left(\int_{\mathcal{V}} \rho d\tau\right) = \int_{\mathcal{V}} \frac{\partial \rho V}{t} d\tau \\
\therefore \quad \oint_S \vec{J} \cdot \hat{n} da &= -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau \\
\therefore \quad \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{J} d\tau &= -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau
\end{aligned}$$

Since this is true for any arbitrary volume

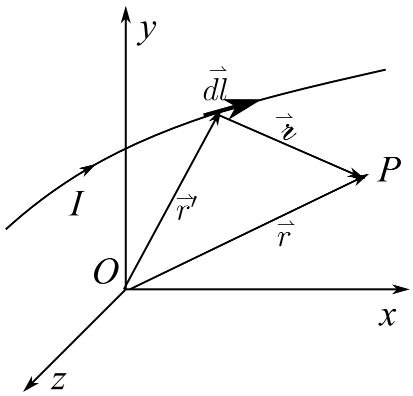
$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

This equation is called the continuity equation. It relates the divergence of the current density at a point to the rate of change of charge density at the point. If the flow of charge in a region is such that ρ doesn't change with time then we will have

$$\vec{\nabla} \cdot \vec{J} = 0$$

A current distribution which satisfies this condition everywhere is called a steady current distribution. It says the total flux of the current entering a volume is same as the total flux exiting the volume. In such a case there is no accumulation of charge or decay of charge at any point. This is the condition under which Krchhoff's law is applicable.
Magnetostatics is the study of magnetic fields created by steady currents. In electrostatics we start with electrostatic field of a point charge at rest. In magnetostatics we start with the magnetostatic field of a wire carrying a steady current I. Since the current is steady there is no accumulation of charge at any point in the wire. For such a case the magnetostatic field at a point is given by the Bio-Savart law.

4 Bio-Savart law



Consider an element of a wire of length dl' at the location r'. The current through the lement is I. The magnetic field due to the small element of current at the point P is given as

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \frac{\vec{dl}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$. This is equivalent to the Coulomb's law which gives us the electric field at a point due to a point charge at $\vec{r}'$. Since we cannot have a point charge, the basic form of Biot-Savart law have to be for a wire, which is obtained by integrating over the current elements along the wire. So the magnetic field at the point $P(\vec{r})$ is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

μ_0 is a fundamental constant like ϵ_0 in electrostatics. It is called the permeability of free space. If the current is measured in amperes $A = coulombs/sec$ then

$$\mu_0 = 4\pi^{-7}\mathrm{N/A^2}$$

Since the force on a current carrying wire of length l placed perpendicular to a magnetic field $\vec{B}$ is given by $F = BIL$, the unit of magnetic field in the standard unit is N/Am. This gives the unit of μ_0 in Bio Savart law.
1N/Am is called the Tesla denoted as T.
In C.G.S units the Bio Savart law is given as

$$\vec{B} = \frac{1}{c} I \int \frac{\vec{dl}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

where c is the velocity of light in vacuum, which is 3^{10}cm/s. In this system the unit of $\vec{B}$ is Gauss.

$$1Tesla = 10^4 \text{ Gauss}$$

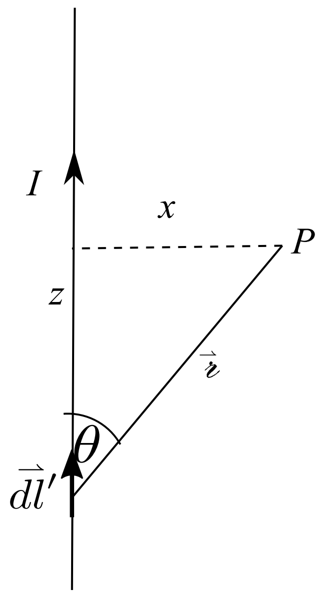
For a conductor carrying a volume current density $\vec{J}(\vec{r})$ we have

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \tag{1}$$

For a specified surface current density $\vec{K}(\vec{r})$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} da'$$

5 Magnetic field due to a long wire



Comsider a long straight wire alongthe z axis carrying a current I along the +ve z direction. Let us find the magnetic field $\vec{B}$ at a point P whose distance from the straight wire is x.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl'} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

All along the wire $\vec{dl} \times \hat{\mathscr{r}}$ is along $\hat{y}$, $\mathscr{r}^2 = x^2 + z^2$, $\quad dl = dz$

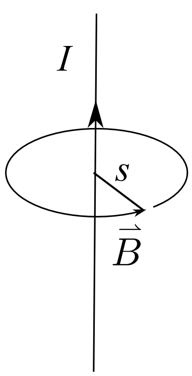
$$\therefore \quad \vec{B} = \frac{\mu_0}{4\pi} I \hat{y} \int_{\infty}^{\infty} \frac{dz \sin\theta}{x^2 + z^2}$$

$z = x \cot\theta$. The integral gives

$$\vec{B} = \frac{\mu_0 I}{2\pi x} \hat{y}$$

So this magnetic field due to an infinite long wire varies as $\frac{1}{s}$ where s is the distance from the wire. This field curls around the wire as shown inthe figure. This expresion is equivalent to the field due to a point charge in electrodynamics. Any surface or volume current densities can be understood to be made up of a number of thin wires carrying current.

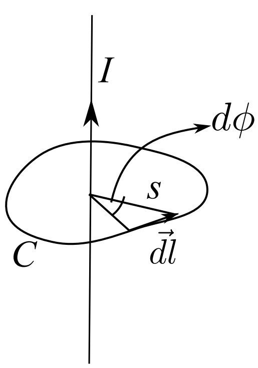
In cylindrical co-ordinates, when the current carrying wire is along the z axis



$$\vec{B}(s, \phi) = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

$$\therefore \quad B_s = 0, \quad B_\phi = \frac{\mu_0 I}{2\pi s}, \quad B_z = 0$$

$$\therefore \quad \vec{\nabla} \times \vec{B} = \hat{z} \frac{1}{s} \frac{\partial}{\partial s} (s B_\phi) = 0, \quad \text{if} \quad s > 0$$

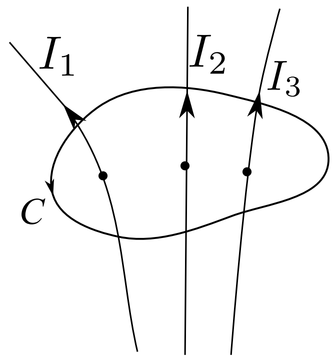


To find $\vec{\nabla} \times \vec{B}$ at $s = 0$ consider a loop around the wire. We consider $\oint_C \vec{B} \cdot \vec{dl}$ along the curve C.
Along the loop $\vec{dl} = s d\phi \hat{\phi} + ds \hat{s}$

$$\begin{aligned} \therefore \quad \vec{B} \cdot \vec{dl} &= \frac{\mu_0}{2\pi} d\phi \\ \therefore \quad \oint_C \vec{B} \cdot \vec{dl} &= \int_0^{2\pi} \frac{\mu_0 I}{2\pi} d\phi = \mu_0 I \end{aligned}$$

$\therefore \quad \vec{\nabla} \times \vec{B}$ is a two dimensional delta function given by

$$\vec{\nabla} \times \vec{B} = \mu_0 I \delta^{(2)}(x, y)$$



Now if we have a number of wires passing through a loop carrying currents $I_1, I_2, ..., I_n$ then

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 (I_1 + I_2 + ... + I_n)$$

Let S be a surface enclosed by the curve C. Then by Stoke's theorem we have

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot \hat{n} da = \oint_C \vec{B} \cdot \vec{dl} = \mu_0 (I_1 + I_2 + ... + I_n)$$

Now instead of thin wires we have a current density $\vec{J}$ in the region enclosed by the loop. Then

$$\oint_S \vec{\nabla} \times \vec{B} \cdot \hat{n} da = \mu_0 \int_S \vec{J} \cdot \hat{n} da$$

This implies

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

This is equivalent to the Gauss' law in electrostatics, $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. Static charge density ρ is the cause of the electric field $\vec{E}$ and steady current density $\vec{J}$ is the cause of static magnetic field $\vec{B}$. From the Biot-Savart law for a current density $\vec{J}$ [1] $\vec{\nabla} \cdot \vec{B}$ is obtained as

$$\vec{\nabla} \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$\vec{J}$ is only a function of $\vec{r}'$. So we have

$$\vec{\nabla} \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = -\vec{J} \cdot \left(\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 0 \qquad \text{since} \qquad \vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$$

$$\therefore \quad \vec{\nabla} \cdot \vec{B} = 0$$

So irrespective of the current density $\vec{\nabla} \cdot \vec{B} = 0$.
It may appear that we have proved $\vec{\nabla} \cdot \vec{B} = 0$. But this is not true. It is an experimental observation that $\vec{\nabla} \cdot \vec{B} = 0$. It is important to note that in electrostatics, $\vec{\nabla} \cdot \vec{E}$ gives the charge density at a point. Everything starts from the existence of point charges which is an experimental fact. In magnetostatics we don't have any magnetic charges equivalent to electric charges. We only have magnetic dipoles (North and south poles). The dipoles are understood to be formed by current carrying loops. These fields can be evaluated using the Bio-Savart law. So the experimental fact is the Biot-Savart law which is equivalent to the Coulomb's law in electrostatics. This experimental law lead to the consequence that $\vec{\nabla} \cdot \vec{B} = 0$. So the possibility of the existence of magnetic monopoles is not ruled out theoretically.

6 Ampere's law

The equation

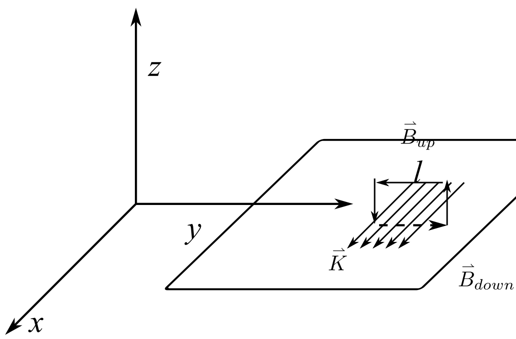
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

is called the Ampere's law in differential form. In integral form this equation is

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{enclosed}}$$

where I_{enclosed} is the total current passing through a surface enclosed by the loop C.
Like Gauss' law, Ampere's law is useful in evaluating magnetic fields when there are certain symmetries in the problem.

Eg.1
Let $\vec{K}$ be the surface current density (current perunit length) over an infinite plane. Find the magnetic field produced by this current.



Consider a loop (amperian loop) as shown. The loop has length l parallel to the plane and has tiny elements piercing the plane. Let us traverse the loop anticlockwise and evaluate $\oint_C \vec{B} \cdot \vec{dl}$

$$\oint_C \vec{B} \cdot \vec{dl} = B_{up} l + B_{down} l = \mu_0 K l$$

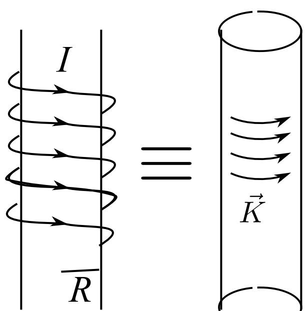
B_{up} and B_{down} are the magnitude of the magnetic field above and below the plane respectively. Above, the field is along $-\hat{y}$, and below, the field is along $\hat{y}$. Now at the same distance above and below,

$$\begin{aligned}
&B_{up} = B_{down} = B \\
&\therefore \quad 2Bl = \mu_0 K l \\
&\therefore \quad \frac{\mu_0}{2} K
\end{aligned}$$

$\therefore$ Above the plane $\vec{B} = -\frac{\mu_0}{2} K \hat{y}$
Below the plane $\vec{B} = \frac{\mu_0}{2} K \hat{y}$
This field is independent of the distance from the plane. This situation is similar to an infinite plane with uniform surface charge density in electrostatics where $E = \frac{\sigma}{2\epsilon_0}$.

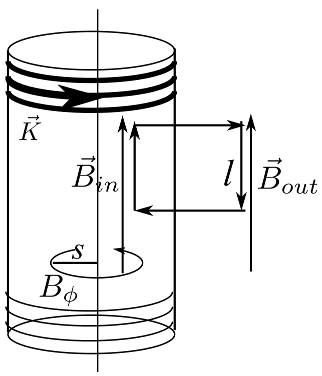
Eg. 2

As a second example let us find the magnetic field due to a solenoid having n turns per unit length, radius R and a very long length. The current in the solenoid is I. We assume that n is very large so that the windings of the solenoid wire is nearly in a plane perpendicular to the axis of the solenoid.



The solenoid can be approximated with a cylinder carrying a surface current with density $\vec{K} = nI\hat{\phi}$. Since the solenoid has n turns per unit length, the current flowing tangentially over the cylindrical surface per unit length is nI.
There can't be any magnetic field in the $\hat{\phi}$ direction. This is because if we consider an amperian loop which is circular and concentric with the cylinder, no current cuts the palne of the loop. By symmatry of the problem we have by Ampere's law

$$B_\phi \times 2\pi s = \mu_0 \times 0 \quad \Longrightarrow \quad B_\phi = 0$$

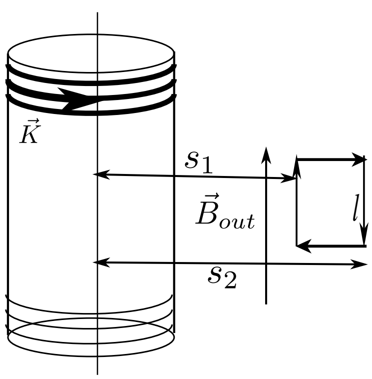


Consider an amperian loop as shown above traversed clockwise

$$\begin{aligned} (B_{in} - B_{out})l &= \mu_0 K l \\ \therefore \quad B_{in} - B_{out} &= \mu_0 K \end{aligned} \tag{2}$$

The contribution from the horizontal pieces of the loop is 0 since $\vec{B}$ is perpendicular to the loop element. Now we claim that B_{out} is 0. For this consider an amperian loop outside the solenoid. Then we have

$$\begin{aligned} [B(s_1) - B(s_2)]\, l &= 0 \\ \therefore \quad B(s_1) &= B(s_2) \end{aligned}$$



So if there is a magnetic field outside the solenoid it has to be constant everywhere. We have just seen that that an infinite sheet of current produces a magnetic field which is independent of distance from the sheet. The effect of the solenoid has to be less pronounced than this. Moreover, here we have the two sides of the solenoid creating opposite fields at any point outside if it exists. The only magnetic field that is consistent with all these arguments is $B_{out} = 0$ everywhere outside.
So from Eq. 2 we have

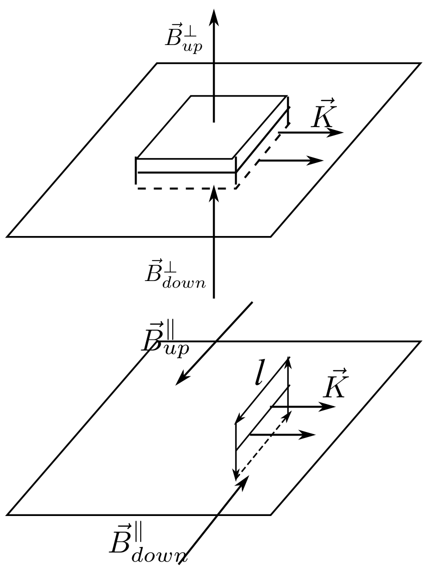
$$B_{in} = \mu_0 K = \mu_0 n I$$

independent of the distance from the center.
So a solenoid produces a uniform magnitic field inside along the axis of the solenoid.

$$\begin{aligned} \vec{B}_{in} &= \mu_0 K \hat{z} = \mu_0 n I \hat{z} \\ \vec{B}_{out} &= 0 \end{aligned}$$

7 Boundary Value Problems

When magnetic fields are calculated in regions free of current density we have to suitably match the fields at the boundary surfaces that may have currents flowing over them. We have conditions on the components of magnetic fields, perpendicular and parallel to the boundary surface.



Consider a pillbox parallel to the surface and lying on either side of the surface. Since $\vec{\nabla} \cdot \vec{B} = 0$ we have

$$\begin{aligned} &\oint_{pillbox} \vec{B} \cdot \hat{n} = 0 \\ \therefore \quad &\int_A (B^{\perp}_{up} - B^{\perp}_{down}) da = 0 \\ &B^{\perp}_{up} = B^{\perp}_{down} \end{aligned} \tag{3}$$

To get condition on the tangential (parallel) components we consider the line integral over a loop close to the surface

$$\begin{aligned} \oint \vec{B} \cdot \vec{dl} = (B^{\parallel}_{up} - B^{\parallel}_{down}) l &= \mu_0 K l \\ \therefore \quad B^{\parallel}_{up} - B^{\parallel}_{down} &= \mu_0 K \end{aligned} \tag{4}$$

The parallel components are perpendicular to the local direction of K.
The two boundary conditions can be combined into a single vector Equation

$$\vec{B}_{up} - \vec{B}_{down} = \mu_0 (\vec{K} \times \hat{n})$$

where $\hat{n}$ is normal to the surface on the 'up' side.