SC223 - Linear Algebra
Autumn 2022
Tutorial 2: Linear Independence and Rank of Matrices

Definitions:
A set of vectors in R^n are **linearly independent** if none of them can be generates as a linear combination of the others.

The **row rank** of an $m \times n$ dimensional matrix A is the maximum number of linearly independent rows of A.

Clearly this number cannot be more than m, the number of rows of A.

The **column rank** is the maximum number of linearly independent columns of A.

As above, this number cannot exceed n.

Given an $n \times m$ matrix A with real numbered entries, the set of all dimension m vectors that can be generated as linear combination of columns of A are called, in aggregate, its **column space**. The vectors that cannot be generated, thus, as well as the all 0 vector are called its **left-null space**.

This definition is, rather, an indirect one. The direct definition is the **column space** is all vectors b that are solutions to

$$Ax = b$$

Here A is $m \times n$ and the other two are automatically defined. Here, The vector of all zeroes always belongs.

The **left null space** is the solution to

$$A^T y = 0$$

Here, too, the vector of all zeroes always belongs.

Here, A^T, refers to the transpose matrix.

The **row space** is the solutions to $A^T y = b$, and the **null space** is the solutions to $Ax = 0$.

A well known result, is that the row-rank of a matrix is the same as the column-rank of that same matrix. Thus it is often just referred to as the **rank** of a matrix.

1. Find the rank of the following matrices.

   (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \\ 1 & n \end{bmatrix} \in \mathbb{R}^{n \times 2}$

   (b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n \times 2}$

   (c) $A = \begin{bmatrix} 1 & 5 & -2 \\ 4 & -2 & 14 \\ 4 & 6 & 6 \end{bmatrix}$

(d) $A \in \mathbb{R}^{n \times n}, A(p,q) = (-1)^{p+q}, 1 \leq p,q \leq n$. $A(p,q)$ denotes the entry at row p, column q.

(e) $A \in \mathbb{R}^{5\times 6}$. Every solution to $Ax = \mathbf{0}_5$ is a scalar multiple of $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \\ 2 \\ 4 \end{bmatrix}$.

2. For any matrix $A \in \mathbb{R}^{m\times n}$, find the intersection of its row space and null space, i.e., $C(A^T) \cap N(A)$. Similarly, find $C(A) \cap N(A^T)$.

3. Let $A \in \mathbb{R}^{m\times n}$ be an arbitrary matrix. The matrix can be used to define a function $A : \mathbb{R}^n \to \mathbb{R}^m$ by matrix multiplication. What is the rank of matrix A if the function A is injective?

4. Suppose we are allowed to change the entry at a position of a matrix to any value of our choice. If the rank of the matrix A is r, then what can be the value of the new matrix A' in terms of r, where A' differs from A in just one position?

5. If we wish to change the rank of a matrix by k, then what is the minimum number of entries to be altered? What is the condition on the position of these entries within the matrix?