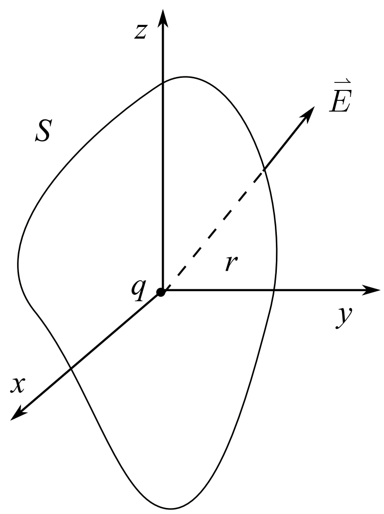
Gauss' Law



Consider a point charge q at the origin. Consider a volume V surrounded by a surface S. The flux of $\vec{E}$ over S is given as

$$\oint_S \vec{E} \cdot \hat{n} da$$

If we increase the charge q then the electric field will increase in the same proportion. So we may say that the flux of the electric field across the closed surface is proportional to the charge q at the center. It is clear that we need not keep the charge q at the origin. It can be kept anywhere within the surface S and the total flux of $\vec{E}$ over the surface S will be proportional to the charge q.

What is surprising is that that the constant of proportionality also doesn't change with the location of the charge within the surface. This constant is found to be $1/\epsilon_0$. So irrespective of the position of the charge q within the surface S

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{q}{\epsilon_0}$$

We can have several charges $q_1, q_2, ..., q_n$ within the surface S. The total electric field due to all these charges is $\vec{E}_1 + \vec{E}_2 + ... + \vec{E}_n = \vec{E}$. It is clear that

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{1}{\epsilon_0}(q_1 + q_2 + ... + q_n) = \frac{\text{total charge}}{\epsilon_0}$$

This is the statement of the Gauss' law.

If we have a continuous charge distribution $\rho(\vec{r})$ then the Gauss' law takes the form

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{1}{\epsilon_0} \int_V \rho(\vec{r}) dV$$

By divergence theorem

$$\oint_S \vec{E} \cdot \hat{n} da = \int_V \vec{\nabla} \cdot \vec{E} dV$$

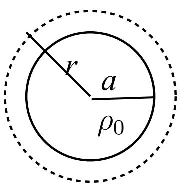
$$\therefore \quad \int_V \vec{\nabla} \cdot \vec{E} dV = \frac{1}{\epsilon_0} \int_V \rho(\vec{r}) dV$$

This is true for charge distribution over any closed volume V. So the integrands can be equated.

$$\therefore \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

This is the differential form of the Gauss' law.

Eg.:
Find the electric field inside and outside a uniformly charged sphere of radius a.



Let the uniform charge density be ρ_0.

$$\begin{aligned} \rho(\vec{r}) &= \rho_0 : \quad 0 \leq r \leq a \\ &= 0 : \quad r > a \end{aligned}$$

To find the electric field outside the sphere , $r > a$, consider a spherical surface of radius r. Due to spherical symmetry of the problem the magnitude of the electric field $\vec{E}$ is same over the surface of the sphere and directed radially outward. Let $E(r)$ be the magnitude of this electric field. Then the flux of this field over the sphere of radius r is

$$\oint_S \vec{E} \cdot \hat{n} da = E(r) \times 4\pi r^2$$

According to Gauss' law the flux is equal to $\frac{q}{\epsilon_0}$ where q is the charge enclosed by the sphere (we call this the imaginary sphere, the Gausian sphere).

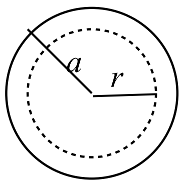
$$q = \int_V \rho(\vec{r}) dV = \rho_0 \frac{4}{3} \pi a^3$$

$$\therefore \quad E(r) 4\pi r^2 = \frac{1}{\epsilon_0} \frac{4}{3} \pi a^3 \rho_0$$

$$\therefore \quad E(r) = \frac{\rho_0}{3\epsilon_0} \frac{a^3}{r^2}$$

So at a point $\vec{r}$ the electric field outside the sphere is

$$\vec{E}(\vec{r}) = \frac{\rho_0 a^3}{3\epsilon_0 r^2} \hat{r}$$



To calculate the electric field inside the sphere we consider a Gausian surface as a sphere with $r < a$. Then the total charge inside the sphere is $q = \frac{4}{3}\pi r^3 \rho_0$.
By Gauss' Law,

$$E(r) \times 4\pi r^2 = \frac{\rho_0}{\epsilon_0}\frac{4}{3}\pi r^3$$

$$\therefore \quad E(r) = \frac{\rho_0}{3\epsilon_0} r$$

$$\therefore \quad \vec{E}(\vec{r}) = \frac{\rho_0}{3\epsilon_0} r\hat{r} = \frac{\rho_0}{3\epsilon_0}\vec{r}$$

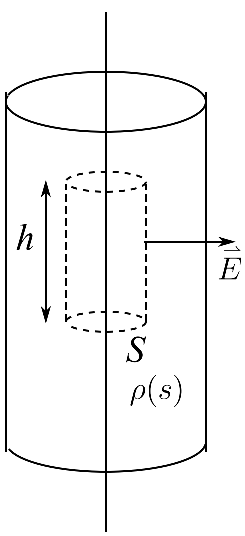
Now we verify the differential form of the Gauss' law.
For outside the charge configuration

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho_0 a^3}{3\epsilon_0}\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right)$$

We have seen that this divergence is 0 for $r > 0$.
Hence $\vec{\nabla} \cdot \vec{E}(\vec{r}) = 0$ for $r > a$.
Inside the sphere

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho_0}{3\epsilon_0}\vec{\nabla} \cdot \vec{r} = \frac{\rho_0}{s\epsilon_0} \times 3 = \frac{\rho_0}{\epsilon_0}$$

This is consistent with the differential form of the Gauss' law inside as well as outside the sphere.
Eg.1:
A cylindrically symmetric charge distribution is given with $\rho(s) = ks, \quad 0 \leq s \leq R$. Find the electric field inside the cylinder of radius R and outside it.



By cylindrical symmetry $\vec{E}$ is along $\hat{s}$ everywhere. Inside the charge distribution

$$\oint_S \vec{E} \cdot \hat{n} da = E_s(s) \cdot 2\pi s h = \frac{q_{\text{enc}}}{\epsilon_0}$$

The flux through the upper and lower flat surfaces of the Gausian cylinder is 0 since $\vec{E}$ is perpendicular to the normal to this surfaces.

$$\begin{aligned} q_{enc} &= \int_V \rho(s) dV \\ &= \int_0^h \int_0^{2\pi} \int_0^s ks \cdot s d\theta ds dz \\ &= kh \cdot 2\pi \int_0^s s^2 ds \\ &= \frac{2\pi k h s^3}{3} \end{aligned}$$

$$\begin{aligned}
\therefore \quad E_s(s)\cdot 2\pi sh &= \frac{1}{\epsilon_0}\frac{2\pi khs^3}{3} \\
\therefore \quad E_s(s) &= \frac{ks^2}{3\epsilon_0} \\
\therefore \quad \vec{E}_{in} &= \frac{ks^2}{3\epsilon_0}\hat{s} \quad \text{(proportional to} s^2\text{)}
\end{aligned}$$

Now we calculate electric field outside.

$$\begin{aligned}
E_s \times 2\pi sh &= \frac{1}{\epsilon_0}\frac{2\pi khR^3}{3} \quad \text{The charge densidy exists only upto } s = R \\
\therefore \quad E_s &= \frac{1}{3\epsilon_0}\frac{kR^3}{s} \\
\therefore \quad \vec{E}_{\text{out}} &= \frac{kR^3}{3\epsilon_0}\frac{1}{s}\hat{s} \quad \text{(Inversely proportional to} s\text{)}
\end{aligned}$$

We can calculate this even using the differential form of Gauss' law.
By symmetry of the problem, only E_s component of the electric field is non zero. Using expression for divergence in cylindrical coordinates we get

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho \\
\therefore \quad \frac{1}{s}\frac{\partial}{\partial s}(sE_s) &= \frac{1}{\epsilon_0}ks \quad \text{(inside the radius } R\text{)} \\
\therefore \quad E_s &= \frac{ks^2}{3\epsilon_0} \quad \Longrightarrow \vec{E} = \frac{ks^2}{3\epsilon_0}\hat{s} \qquad (1)
\end{aligned}$$

See footnote[1]

Outside the radius R

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 0 \\
\therefore \quad \frac{1}{s}\frac{\partial}{\partial s}(sE_s) &= 0 \quad \Longrightarrow \quad sE_s = c \quad \Longrightarrow \quad E_s = \frac{c}{s}
\end{aligned}$$

where c is some constant.
We demand $\vec{E}$ to be continuous at $s = R$ (not always true as we will see later).

$$\begin{aligned}
\therefore \quad \frac{c}{R} &= \frac{kR^2}{3\epsilon_0} \quad \Longrightarrow \quad c = \frac{kR^3}{3\epsilon_0} \\
\therefore \quad \vec{E}_{\text{out}} &= \frac{kR^3}{3\epsilon_0}\frac{1}{s}\hat{s}
\end{aligned}$$

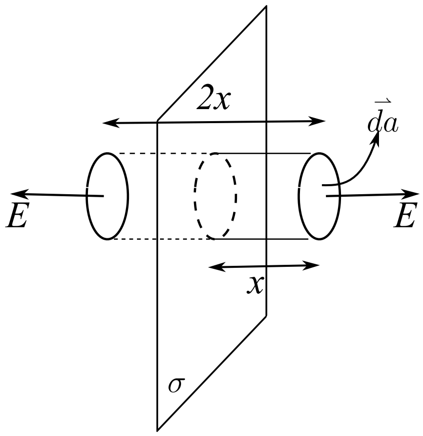
[1] In fact when we solve the differential equation in s (Eq.1) we will get $E_s = \frac{ks^2}{3\epsilon_0} + \frac{c_1}{s}$ where c_1 is an arbitrary constant. Let us check by calculating the divergence of this field at $s = 0$. This has to be done carefully.

$$\vec{\nabla}\cdot\vec{E}\Big|_{s=0} = \lim_{s\to 0}\frac{1}{\pi s^2 h}(E_s \cdot 2\pi sh) = \lim_{s\to 0}\left(\frac{2ks}{3\epsilon_0} + \frac{2c_1}{s^2}\right)$$

If $c_1 \neq 0$ then $\vec{\nabla}\cdot\vec{E} \to \infty$ as $s \to 0$ which is not consistent with the given charge density at $s = 0$

Eg. 2:

Electric field due to an infinite plane of charge with uniform surface charge density σ.



Let us find the $\vec{E}$ at a distance x from the plane. Due to symmetry $\vec{E}$ is directed perpendicular to the plane. The Gausian surface we consider is a cylinder as shown in the figure whose length is $2x$ and extends symmetrically on both sides of the charged plane.

There is no flux from the side walls of the cylinder since $\vec{E}$ is orthogonal to the normal to the surface. However the flux from the 'lid' of the cylinder is

$$\vec{E}\cdot\vec{da}+\vec{E}\cdot\vec{da}=2Eda$$

By Gauss' Law $2Edx=\sigma/\epsilon_0$.

$$\therefore \quad E=\frac{\sigma}{\epsilon_0}$$

$$\therefore \quad \vec{E}=\frac{\sigma}{2\epsilon_0}\hat{n}$$

where $\hat{n}$ is the normal to the plane.

Note that $\hat{n}$ is directed opposite on the two sides of the plane. This electric field is independent of distance x and extends upto ∞. Ofcourse this is practically not possible. It holds true only when x is much smaller than the lengths of the size of the finite plane.