Integration

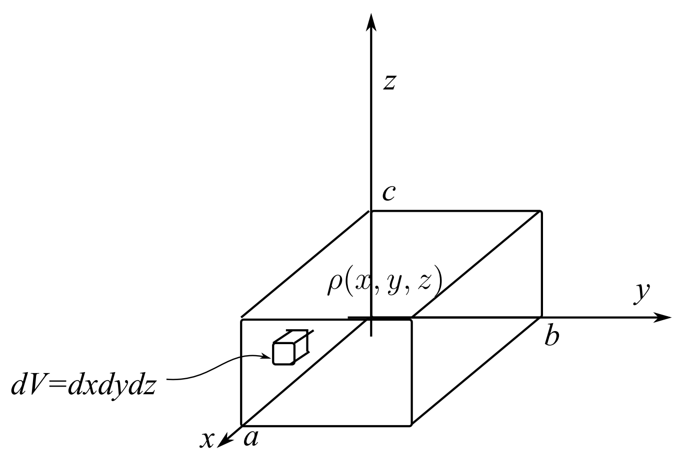
Integration of scalar and vector functions
We can discuss integrals of scalar and vector functions depending upon the infinitesimal elements, line element, surface element and volume element.

1 Volume Integral

1.1 Scalar function

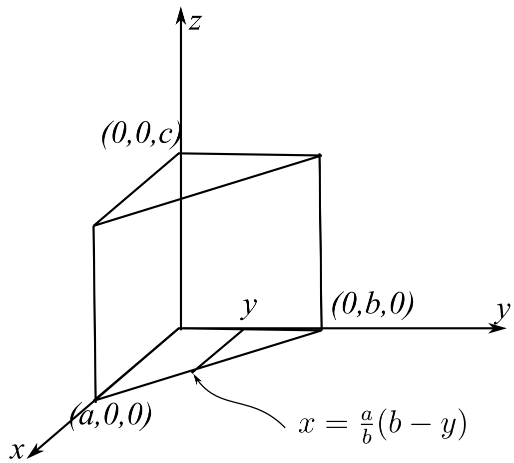
Here the integration is over a given volume. The function to be integrated can be a scalar or a vector function. Consider a scalar function, like density, $\rho(x, y, z)$. Integrating this over a specified volume bounded by certain boundaries will give the total mass contained within the volume.

Eg. $\rho(x, y, z) = xyz$
Let the volume V be the volume bounded by the planes $x = 0,\ x = a,\ y = 0,\ y = b, z = 0,\ z = c$.



$$\begin{aligned}
\int_V \rho dV &= \int_0^c \int_0^b \int_0^a (xyz) dxdydz \\
&= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c \\
&= \frac{a^2 b^2 c^2}{8}
\end{aligned}$$

If the volume over which we integrate is more complicated we have to work out proper limits of the variables. Consider the volume shown in the following figure. For a particular value



of y and z, x goes from 0 to $\frac{a}{b}(b-y)$. Thus we do the x integral first using this upper limit for x. Here the limits of y is independent of z. Hence y goes from 0 to b. Then z goes from 0 to c.

$$
\begin{aligned}
\int_V \rho dxdydz &= \int_0^c \int_0^b \int_0^{\frac{a}{b}(b-y)} (xyz)dxdydz \\
&= \int_0^c \int_0^b \frac{1}{2}\left[\frac{a}{b}(b-y)\right]^2 yzdydz \\
&= \left[\int_0^b \frac{1}{2}\frac{a^2}{b^2}(b^2-2yb+y^2)ydy\right]\left[\int_0^c zdz\right] \\
&= \frac{a^2}{2b^2}\left(\frac{b^4}{2}-\frac{2b^4}{3}+\frac{b^4}{4}\right)\frac{c^2}{2} \\
&= \frac{a^2b^2c^2}{48}
\end{aligned}
$$

1.2 Vector function

Consider integrating a vector function over a volume. For e.g. integrating infinitesimal dipole moments to calculate the total dipole moment or integrating forces over infinitesimal elements over a volume to find the total force. If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ then

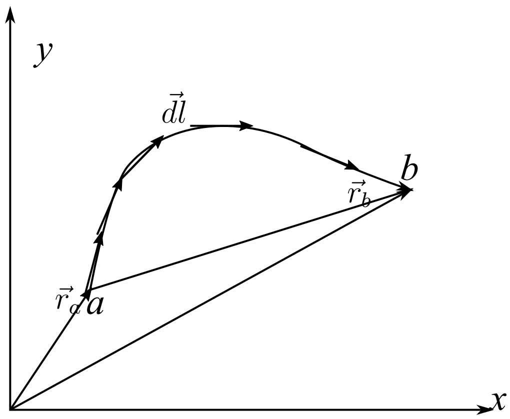
$$
\int_V \vec{A}dV = \hat{\imath}\int_V A_xdV + \hat{\jmath}\int_V A_ydV + \hat{k}\int_V A_zdV
$$

Now the individual integrals can be done as discussed above.

2 Line Integral

Here we integrate scalar or vector functions along a curve over infinitesimal line elements.

2.1 Scalar functions



$$\int_a^b \rho(x,y)\vec{dl}$$

The result of this integral is a vector quantity.
If $\rho(x,y)$ is a constant function then

$$\int_a^b \rho(x,y)\vec{dl} = \rho\int_a^b \vec{dl} = \rho(\vec{r}_b - \vec{r}_a$$

2.2 Vector function

Consider the following line integral of a vector function

$$\int_a^b \vec{A}\cdot\vec{dl}$$

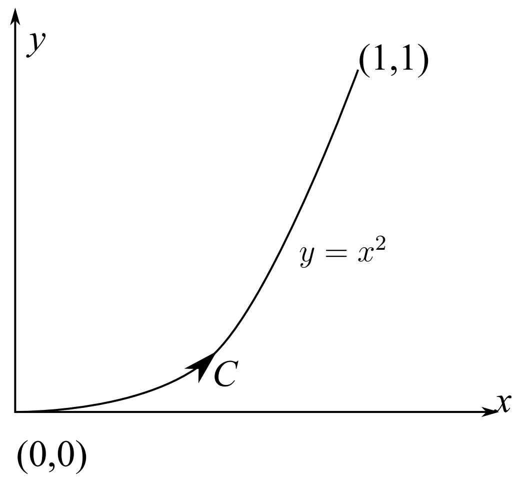
Here $\vec{dl} = \hat{i}dx + \hat{j}dy + \hat{k}dz$.
This kind of line integral have to be evaluated to compute the work done by a force in moving a particle from point a to point b.
Eg. 1
Let $\vec{A} = x\hat{i} + y\hat{j}$. Let us integrate along a curve $y = x^2$ from $a(0,0)$ t0 $b(1,1)$

$$\int_a^b \vec{A}\cdot\vec{dl} = \int_a^b xdx + ydy$$

Along the curve $y = x^2$, $\quad dy = 2xdx$.

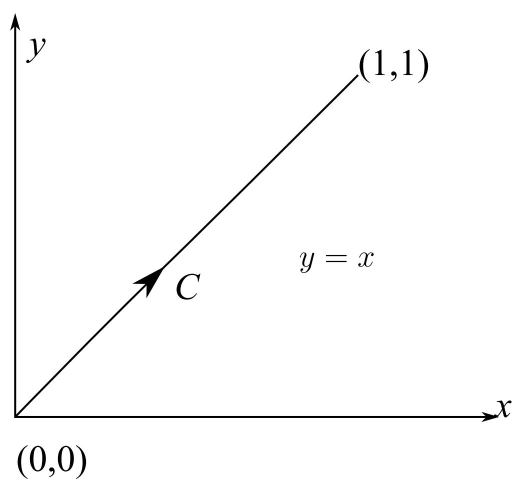
So the line integral gets converted into a single integral , in this case, only over x.

$$\begin{aligned}\therefore \int_a^b \vec{A}\cdot\vec{dl} &= \int_0^1 xdx + \int_0^1 x^2\cdot 2xdx \\ &= \int_0^1 (2x^3 + x)dx \\ &= \left[2\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$



Instead of the curve $y = x^2$ if we consider the curve $y = x$ from $(0,0)$ to $(1,1)$ then $dy = dx$.
We will have

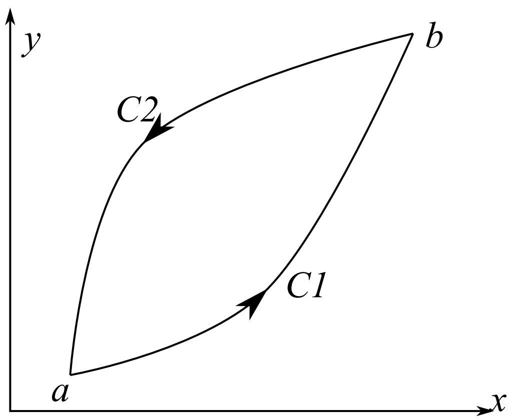
$$\begin{aligned} \int_a^b \vec{A} \cdot \vec{dl} &= \int_0^1 x dx + \int_0^1 x dx \\ &= 2 \left[\frac{x^2}{2} \right]_0^1 = 2 \times \frac{1}{2} = 1 \end{aligned}$$



The line integral is same along both the paths. This is not a coincidence. For the given function the line integral is independent of the path.
Now

$$\int_{a(alongC)}^b \vec{A} \cdot \vec{dl} = -\int_{b(alongC)}^a \vec{A} \cdot \vec{dl}$$

This is true for any curve C.
So if we have a closed curve, the line integral will be zero.
Such integrals are denoted as $\oint \vec{A} \cdot \vec{dl}$.



$$\begin{aligned}
\oint \vec{A}\cdot\vec{dl} &= \int_{a\ C1}^{b} \vec{A}\cdot\vec{dl} + \int_{b\ C2}^{a} \vec{A}\cdot\vec{dl} \\
&= \int_{a\ C1}^{b} \vec{A}\cdot\vec{dl} - \int_{a\ C2}^{b} \vec{A}\cdot\vec{dl} = 0
\end{aligned}$$

Eg. 2
Let $\vec{A} = y\hat{i} - x\hat{j}$. Then line integral with this function is dependent on the path taken. Along the path $y = x^2$ we will have

$$\int_a^b \vec{A}\cdot\vec{dl} = \int_a^b ydx - xdy = \int_a^b x^2dx - xdy$$

Along the given path $y = x^2$, $\;dy = 2xdx$. Substituting these in the above integral we have

$$\int_a^b \vec{A}\cdot\vec{dl} = \int_0^1 (x^2dx - 2x^2dx) = \int_0^1 -x^2dx = -\frac{1}{3}$$

Along the path $y = x$, $\;dy = dx$. So we have

$$\int_a^b \vec{A}\cdot\vec{dl} = \int_0^1 xdx - xdx = 0$$

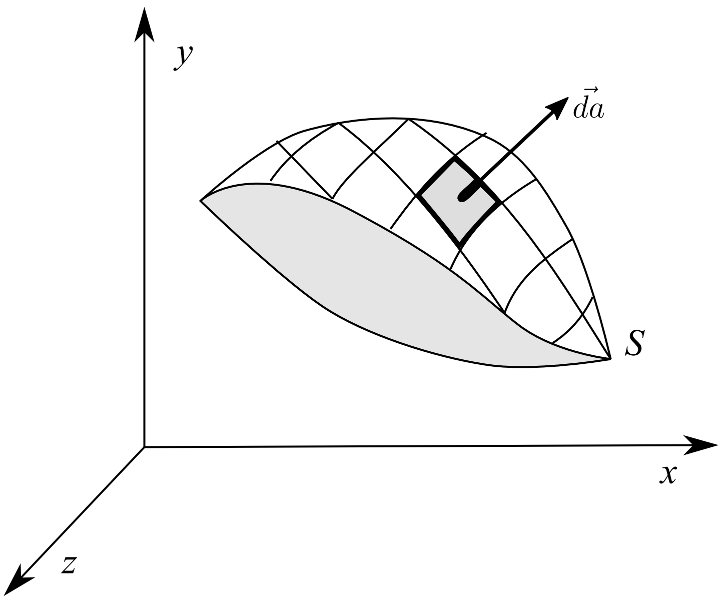
So this integral is path dependent.
Occasionally we do have line integral of the type $\int_a^b \vec{A}\times\vec{dl}$ along a curve C from point a to point b. The result of this integral is evidently a vector quantity. In the cartesian coordinate system we can write down this integral as

$$\hat{i}\int_a^b (A_y dz - A_z dy) + \hat{j}\int_a^b (A_z dx - A_x dz) + \hat{k}\int_a^b (A_x dy - A_y dx)$$

The individual integrals can be evaluated with the procedure stated above along a specified curve from point a to point b.

3 Surface Integral

Here a scalar or a vector function is integrated over infinitesimal surface elements. In three dimensions infinitesimal surface elements are vector quantities. Given an infinitesimal surface element we associate a vector whose magnitude is the area of the surface and which is perpendicular to the infinitesimal surface.



The surface integral that we generally encounter in Electrodynamics is given as

$$\int_S \vec{A} \cdot \vec{da}$$

where $\vec{A}$ is the vector field to be integrated over a given surface S. $\vec{da}$ is an infinitesimal element of the surface S. If the surface over we integrate is closed, i.e, encloses a volume then the surface integral is denoted as

$$\oint_S \vec{A} \cdot \vec{da}$$

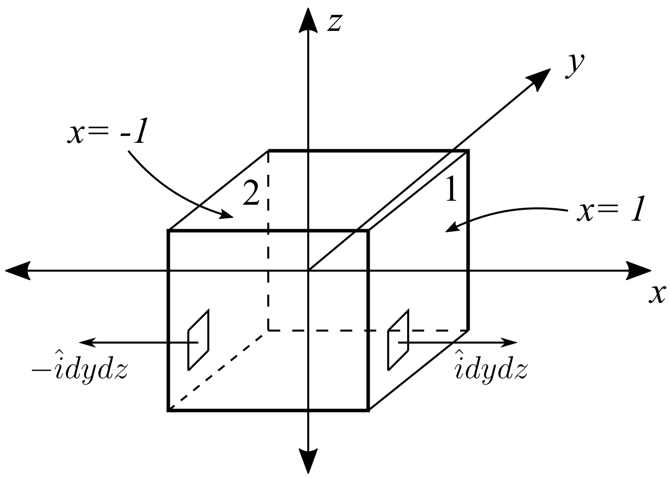
Eg.
Let $\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$. Let us integrate this over a closed surface of a cube formed by the planes $x = \pm 1,\ y = \pm 1,\ z = \pm 1$ as shown:

There are six surfaces enclosing a cubical volume. Consider the surface $x = 1$. At every point on this surface $\vec{da} = \hat{i}dydz$. Let us call this surface as surface 1. So

$$\int_1 \vec{A} \cdot \vec{da} = \int_{-1}^{1}\int_{-1}^{1} dydz = \int_{-1}^{1} dy \int_{-1}^{1} dz = 2 \times 2 = 4$$

On the opposite surface $x = -1$, call surface 2, we have $\vec{da} = -\hat{i}dydz$ and $x = -1$.
So

$$\begin{aligned}
\vec{A} \cdot \vec{da} &= (-\hat{i} + y\hat{j} + z\hat{k}) \cdot (-\hat{i}dydz) = dydz \\
\therefore \quad \int_2 \vec{A} \cdot \vec{da} &= \int_{-1}^{1}\int_{-1}^{1} dydz = 4
\end{aligned}$$



So the total contribution from this pair of surfaces is 4+4 =8. We have 3 such pairs and by symmetry of the function we have

$$\oint_S \vec{A} \cdot \vec{da} = 8 \times 3 = 24$$