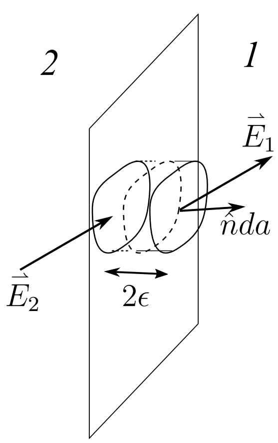
Boundary Conditions

Once we solve Poisson's equation we will get the potential due to a given charge distribution but along with a number of arbitrary constants. The values of these constants have to be determined by a number of specified conditions on the fields in the region of interest. Generally these conditions are in the form of the value of potential or electric fields over certain bounding surfaces of the region. The boundary conditions can also be certain specified surface charge densities or linear charge densities.

1 Boundary condition on the Electric fields:



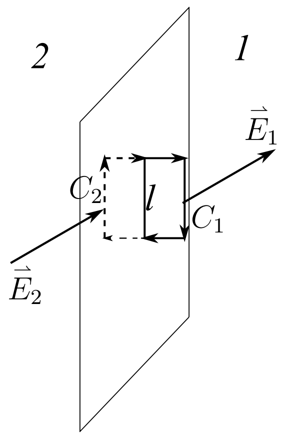
A surface divides a region in two parts, say 1 and 2. Let $\vec{E}_1$ be the electric field near the surface in region 1 and $\vec{E}_2$ in region 2. Consider a Gaussian surface of area da and thickness 2ϵ where $\epsilon \to 0$. The total flux of the electric field from this Gaussian surface is

$$\vec{E}_1 \cdot \hat{n}da - \vec{E}_2 \cdot \hat{n}da \tag{1}$$

It must be noted that the flux through the curved cylindrical surface is not 0 since the electric field is not necessarily perpendicular to the surface. Secondly the $-$ sign in Eq. 1 is because the normal $\hat{n}$ on the two sides of the surface are oppositely directed. Now if the surface has a surface charge density σ then the total charge enclosed by the Gaussian surface is σda. Then by Gauss's law

$$\begin{aligned}
\vec{E}_1 \cdot \hat{n}da - \vec{E}_2 \cdot \hat{n}da &= \frac{\sigma da}{\epsilon_0} \\
\therefore \quad \vec{E}_1 \cdot \hat{n} - \vec{E}_2 \cdot \hat{n} &= \frac{\sigma}{\epsilon_0} \\
\therefore \quad E_{1\perp} - E_{2\perp} &= \frac{\sigma}{\epsilon_0} \qquad (2)
\end{aligned}$$

Eq. 2 specifies the boundary condition on the perpendicular component of the electric field on the two sides. If the surface charge density at a place is 0 then $E_{1\perp} = E_{2\perp}$.



The condition on the tangential component of electric field comes from the property that $\vec{\nabla} \times \vec{E} = 0$. Consider a rectangular loop as shown in the figure. The tangential elements are of length l and very near the surface. The parts of the loop that pierces the surface are negligibly small. So the integral becomes

$$\begin{aligned} \oint \vec{E} \cdot \vec{dl} &= \int (\vec{\nabla} \times \vec{E}) \cdot \hat{n} da = 0 \\ \therefore \int_{C_1} \vec{E}_1 \cdot \vec{dl} &+ \int_{C_2} \vec{E}_2 \cdot \vec{dl} = 0 \end{aligned}$$

The curves C_1 and C_2 are tangential to the surface and since they lie close to the surface we have $C_2 = -C_1$

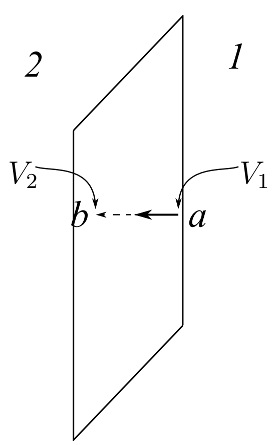
$$\begin{aligned} \therefore \quad \int_{C_1} \vec{E}_1 \cdot \vec{dl} + \int_{-C_1} \vec{E}_2 \cdot \vec{dl} &= 0 \\ \therefore \quad \int_{C_1} \vec{E}_1 \cdot \vec{dl} - \int_{C_1} \vec{E}_2 \cdot \vec{dl} &= 0 \\ \therefore \quad \int_{C_1} \vec{E}_1 \cdot \vec{dl} = \int_{C_1} \vec{E}_2 \cdot \vec{dl} & \end{aligned}$$

Since this is true for any arbitrary loop we select and hence any arbitrary curve C_1 we have

$$E_{1\parallel} = E_{2\parallel} \tag{3}$$

where $E_{1\parallel}$ and $E_{2\parallel}$ are parallel or tangential components of the electric fields to the surface. We see that the boundary condition 3 is independent of the surface charge density.

2 Boundary condition on the electric potential



Let V_1 be the potential at a point very near to the surface in region 1 and V_2 in region 2.

$$V_2 - V_1 = \int_a^b \vec{E} \cdot \vec{dl}$$

Since $\vec{E}$ is finite in the two regions, though it may be discontinuous , the integral in the R.H.S $\to$ 0 as $a \to b$. So $V_2 \to V_1$ as a and b approach the surface.

Eg:

A sphere has a uniform charge density σ over its surface. Find the potential inside and outside the sphere.

Here $\rho = 0$ both, inside and outside.

$$\begin{aligned}
\nabla^2 V_{out} &= 0 \\
\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_{out}}{\partial r}\right) &= 0 \\
\therefore \quad V_{out} &= -\frac{c_1}{r} + c_2
\end{aligned}$$

Similarly $\nabla^2 V_{in} = 0 \implies \quad V_{in} = -\frac{d_1}{r} + d_2$

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{d_1}{r^2}\hat{r}$$

Applying Gauss's law over a spherical surface inside we get

$$-\frac{d_1}{r^2} \times 4\pi r^2 = 0 \implies d_1 = 0$$

If we demand that the potential at ∞ be 0 then $c_2 = 0$

$$\begin{aligned}
\therefore \quad V_{out} &= -\frac{c_1}{r} \\
\therefore \quad \vec{E}_{out} &= -\vec{\nabla} V_{out} = -\frac{c_1}{r^2}\hat{r}
\end{aligned}$$

At $r = R$ we have the boundary condition

$$\begin{aligned}
E_{out\perp} - E_{in\perp} &= \frac{\sigma}{\epsilon_0} \quad \implies \quad -\frac{c_1}{R^2} = \frac{\sigma}{\epsilon_0} \\
\therefore \quad c_1 &= -\frac{\sigma R^2}{\epsilon_0} \\
\therefore \quad V_{out} &= \frac{\sigma R^2}{r\epsilon_0} \\
V_{in} &= d_2
\end{aligned}$$

At the surface of the sphere we have

$$\begin{aligned}
V_{out}|_R &= V_{in}|_R \\
\therefore \quad \frac{\sigma R^2}{R\epsilon_0} &= d_2 \quad \implies d_2 = \frac{\sigma R}{\epsilon_0} \\
\therefore \quad V_{in} &= \frac{\sigma R}{\epsilon_0}
\end{aligned}$$

So we get the potential everywhere. From this we can obtain the electric field

$$\begin{aligned}
\vec{E}_{out} &= -\vec{\nabla} V_{out} = \frac{\sigma R^2}{\epsilon_0}\frac{1}{r^2}\hat{r} \\
\vec{E}_{in} &= 0
\end{aligned}$$

Verify that this is indeed the right electric field obtained by other methods.