Monte Carlo Error Scaling for Value-at-Risk Estimation

1 Problem Setting

Let L be a real-valued random variable representing portfolio loss over a fixed time horizon (e.g. one day), with cumulative distribution function (CDF) F_L and probability density function (PDF) f_L.

For a confidence level $\alpha \in (0, 1)$, the Value-at-Risk (VaR) is defined as the α-quantile of the loss distribution:

$$\mathrm{VaR}_\alpha = q_\alpha := \inf\{x \in \mathbb{R} \mid F_L(x) \geq \alpha\}.$$

In Monte Carlo estimation, we generate N independent samples

$$L_1, L_2, \ldots, L_N \sim L$$

and estimate VaR_α using the empirical quantile $\hat{q}_\alpha$.

The central question is:

> Why does reducing the Monte Carlo estimation error by a factor of 2 require approximately 4 times as many samples?

2 Monte Carlo Error for Sample Means

We begin with the classical case of estimating an expectation. Let Y be a random variable with

$$\mathbb{E}[Y] = \mu, \quad \mathrm{Var}(Y) = \sigma^2.$$

The Monte Carlo estimator of μ using N samples is

$$\bar{Y}_N = \frac{1}{N}\sum_{i=1}^{N} Y_i.$$

Because the samples are independent,

$$\mathrm{Var}(\bar{Y}_N) = \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}(Y_i) = \frac{\sigma^2}{N}.$$

Therefore, the standard deviation (typical error magnitude) is

$$\mathrm{SD}(\bar{Y}_N) = \frac{\sigma}{\sqrt{N}}.$$

This $\mathcal{O}(N^{-1/2})$ scaling is fundamental and arises from the additive nature of variance.

3 Sample Complexity Consequence

Suppose we desire a Monte Carlo error no larger than ε. Since

$$\mathrm{SD}(\bar{Y}_N) \approx \frac{C}{\sqrt{N}},$$

achieving $\mathrm{SD}(\bar{Y}_N) \leq \varepsilon$ requires

$$N \geq \left(\frac{C}{\varepsilon}\right)^2.$$

Thus, halving the error tolerance ($\varepsilon \mapsto \varepsilon/2$) requires quadrupling the number of samples:

$$N_{\text{new}} = 4N_{\text{old}}.$$

4 Monte Carlo Estimation of Quantiles

VaR estimation differs from expectation estimation because it involves a quantile rather than a mean. Nevertheless, sample quantiles exhibit similar asymptotic behavior.

Let $\hat{q}_\alpha$ be the empirical α-quantile of the Monte Carlo samples. Under mild regularity conditions (specifically, $f_L(q_\alpha) > 0$), the asymptotic distribution of $\hat{q}_\alpha$ is

$$\sqrt{N}(\hat{q}_\alpha - q_\alpha) \xrightarrow{d} \mathcal{N}\left(0, \frac{\alpha(1-\alpha)}{f_L(q_\alpha)^2}\right).$$

Equivalently,

$$\mathrm{Var}(\hat{q}_\alpha) \approx \frac{\alpha(1-\alpha)}{N\, f_L(q_\alpha)^2}.$$

Taking the square root yields the standard error:

$$\mathrm{SD}(\hat{q}_\alpha) \approx \frac{\sqrt{\alpha(1-\alpha)}}{f_L(q_\alpha)} \cdot \frac{1}{\sqrt{N}}.$$

5 Interpretation

The $1/\sqrt{N}$ scaling persists for VaR because:

- The number of samples falling below the true VaR behaves like a binomial random variable with variance proportional to N.
- Translating uncertainty in sample counts into uncertainty in the quantile value requires dividing by the local density $f_L(q_\alpha)$.
- The resulting estimator variance is therefore proportional to $1/N$.

As with expectations, reducing VaR estimation error by a factor of 2 requires increasing the number of Monte Carlo samples by a factor of 4.

6 Practical Implications for VaR

While the convergence rate is universal, the constant factor

$$\frac{1}{f_L(q_\alpha)}$$

can be large for:

- high confidence levels (e.g. $\alpha = 0.99$ or 0.999),
- heavy-tailed loss distributions,
- flat densities near the quantile.

As a result, VaR estimation can require very large sample sizes even though the asymptotic scaling remains $\mathcal{O}(N^{-1/2})$.

7 Summary

- Monte Carlo estimators typically exhibit error proportional to $1/\sqrt{N}$.
- This scaling arises from variance additivity and averaging.
- Sample quantiles, including VaR estimators, obey the same asymptotic law.
- Consequently, reducing error by a factor of k requires k^2 times as many samples.