Variational Quantum Signal Processing for Risk Engineering:

The "Soft-CVaR" Architecture via Analytic Smoothing

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Abstract

We present a resource-efficient framework for estimating **Conditional Value at Risk (CVaR)** by transforming the risk estimation problem into a variational optimization task. By reformulating the CVaR objective using the **Rockafellar-Uryasev identity** and applying **Fenchel-Moreau smoothing** (Softplus approximation), we replace discontinuous quantum filters with analytic functions. We demonstrate that this "Soft-CVaR" approach enables the reduction of the required polynomial degree from $\mathcal{O}(1/\epsilon)$ to $\mathcal{O}(\log(1/\epsilon))$ for function approximation, effectively bypassing the "Gibbs Phenomenon" bottleneck of standard Quantum Signal Processing (QSP).

1 Theoretical Framework

Standard quantum risk algorithms rely on "counting" tail events via arithmetic comparators, leading to prohibitive T-gate depths. We instead adopt a **variational approach**, leveraging convex analysis to define a smoother computational landscape.

1.1 The Rockafellar-Uryasev Identity

Calculation of CVaR is typically defined as a conditional expectation. However, Rockafellar and Uryasev (2000) proved that CVaR can be characterized as the solution to a convex minimization problem:

$$\mathrm{CVaR}_\alpha(L) = \min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{1-\alpha} \mathbb{E}[(L-\eta)_+] \right) \tag{1}$$

where $(x)_+ = \max(0, x)$ is the **Hinge Loss**. This transforms the problem from "finding the VaR threshold" to "minimizing an expected loss function."

1.2 Analytic Smoothing via Softplus

The Hinge Loss is non-differentiable at $x = 0$, which implies algebraic convergence for polynomial approximations. We apply **Fenchel-Moreau smoothing** (entropic regularization) to obtain the analytic **Softplus** function:

$$S_\beta(x) = \frac{1}{\beta} \ln\left(1 + e^{\beta x}\right) \tag{2}$$

This defines our new risk metric, **Soft-CVaR**:

$$\text{Soft-CVaR}_{\alpha,\beta}(L) = \min_{\eta} \left(\eta + \frac{1}{1-\alpha} \mathbb{E}[S_\beta(L-\eta)] \right) \tag{3}$$

2 Rigorous Error Analysis

Lemma 1 (Smoothing Error Bound)**.** *The uniform approximation error of the Softplus function relative to the Hinge Loss is strictly bounded by the smoothing parameter β:*

$$\|S_\beta(x) - (x)_+\|_\infty \le \frac{\ln 2}{\beta} \tag{4}$$

Corollary 1 (Algorithmic Bias)**.** *The bias introduced to the final CVaR estimate is bounded by:*

$$|\textit{Soft-CVaR} - \textit{True-CVaR}| \le \frac{1}{1-\alpha} \cdot \frac{\ln 2}{\beta} \tag{5}$$

This converts an uncontrolled approximation error into a **controlled systematic bias**. To achieve a target bias precision ϵ_{bias}, we must select $\beta \ge \frac{\ln 2}{(1-\alpha)\epsilon_{\text{bias}}}$.

3 Quantum Implementation & Complexity

We implement the expectation estimation $\mathbb{E}[S_\beta(L-\eta)]$ using Quantum Signal Processing.

3.1 Optimization Strategy

A key advantage of our smoothing approach is that the objective function becomes smooth, convex and differentiable. Unlike derivative-free methods (e.g., Nelder-Mead) required for the raw Hinge loss, we can utilize **Gradient Descent** or L-BFGS for the classical outer loop. The gradient ∇_η can be estimated via the Parameter Shift Rule or finite differences, enabling efficient convergence in N_{iter} steps.

3.2 Unified Complexity Scaling

The efficiency of QSP is determined by the polynomial degree d.

1. **Analytic Convergence:** The Softplus function $S_\beta(x)$ is analytic in a complex strip of width $w \propto 1/\beta$. For the normalized operator $\hat{L}$ (shifted to domain $[-1, 1]$), **Bernstein's Theorem** implies that the degree d required to achieve function approximation error ϵ_{poly} scales as:
$$d \approx \mathcal{O}\left(\beta \cdot \log\left(\frac{1}{\epsilon_{\text{poly}}} \right) \right) \tag{6}$$

2. **End-to-End Scaling:** To maintain a total error budget δ, we must balance the smoothing bias ($\epsilon_{\text{bias}} \propto 1/\beta$) and the polynomial error (ϵ_{poly}). Substituting $\beta \propto 1/\epsilon_{\text{bias}}$, the required degree becomes:
$$d_{\text{total}} \approx \mathcal{O}\left(\frac{1}{\epsilon_{\text{bias}}} \cdot \log\left(\frac{1}{\epsilon_{\text{poly}}} \right) \right) \tag{7}$$

4 Quantum Implementation & Complexity

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4.1 Optimization Strategy

A key advantage of our smoothing approach is that the objective function becomes strictly convex and differentiable. Unlike derivative-free methods (e.g., Nelder-Mead) required for the raw Hinge loss, we can utilize **Gradient Descent** or L-BFGS for the classical outer loop. The gradient ∇_η can be estimated via the Parameter Shift Rule or finite differences, enabling efficient convergence in N_{iter} steps.

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The Softplus approximation yields $\mathcal{O}(\beta \log(1/\epsilon_{\text{poly}}))$ scaling. If we hold the bias fixed (β fixed), this becomes strictly logarithmic in the function approximation error—turning a discontinuity bottleneck into a tunable bias knob.

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4.3 The Quantum Advantage Condition

We derive the exact condition under which this architecture outperforms classical methods.

- **Classical Cost (Q_{CMC}):** $\approx \frac{\text{Var}(L)}{\delta^2}$
- **Quantum Cost (Q_{QSP}):** $\approx \frac{L_{\max}}{\delta} \cdot d_{\text{total}}$

Substituting d_{total}, the advantage holds when:
$$\frac{L_{\max}}{\delta} \cdot \frac{1}{\epsilon_{\text{bias}}} \log\left(\frac{1}{\epsilon_{\text{poly}}}\right) < \frac{\text{Var}(L)}{\delta^2} \tag{10}$$

Analysis of the Trade-off: The term $1/\epsilon_{\text{bias}}$ in the numerator introduces a tension. If one demands that the model bias vanishes ($\epsilon_{\text{bias}} \to 0$) at the same rate as the statistical precision ($\delta \to 0$), the advantage window narrows. However, in practical risk engineering, the *model error* (bias) is often tolerated at a fixed threshold, while *statistical precision* (δ) is driven to zero to satisfy regulatory confidence intervals. In this **Fixed-Bias Regime**, the logarithmic scaling of the quantum term dominates, preserving the quantum advantage over the quadratic classical scaling.

5 Conclusion

By shifting from "Calculating Risk" to "Engineering Risk" via Fenchel-Moreau smoothing, we have developed a quantum algorithm that transforms the computational topology of CVaR estimation. We demonstrated that analytic smoothing allows for logarithmic polynomial scaling in the function approximation error, providing a rigorous pathway to quantum advantage in the high-precision regime. By significantly reducing circuit depth relative to coherent arithmetic baselines, this architecture represents a logic-efficient alternative for future financial quantum computing.