Given

$D = \{x_1, x_2 \dots x_n\} \sim$ poisson

1) Prior = exponential, with mean β

and $\beta = \lambda_{MME} = \lambda$ — ①

exponential $(X) = \lambda e^{-\lambda x}$. $E[X] = 1/\lambda$ — ②

from ① and ②

$\lambda = 1/\beta$.

2) likelihood $= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x!} = \frac{e^{-n\lambda} \lambda^{\sum_i x_i}}{\prod_i x_i!}$

3) Posterior $\propto$ likelihood $\times$ prior

$$\frac{e^{-n\lambda} \lambda^{\sum_i x_i}}{\prod_{i=1}^{n} x_i!} \cdot \frac{e^{-\lambda/\beta}}{\beta}$$

$$= e^{-\lambda(n+1/\beta)} \cdot \lambda^{\sum_i x_i} \quad \text{— ③}$$

4) Gamma $(\alpha, \beta) = x^{\alpha-1} e^{-x/\beta}$ — (4)

Comparing (3) and (4) we have

$$\alpha - 1 = \sum_{i=1}^{n} x_i \implies \alpha = \sum_{i=1}^{n} x_i + 1$$

$$n + \frac{1}{\beta_p} = \frac{1}{\beta} \implies \beta = \frac{1}{\left(n + \frac{1}{\beta}\right)}$$

→ from posterior.