Large deviations in the stock market

james.pallister

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1 Introduction

We consider the methods of large deviation theory [1] and geometric optics [2] as applied to the stock market. We are interested in the probability a stock wildly deviates from its equilibrium price.

2 Mapping between finance and physics

The dynamics of the stock market are modelled using the stochastic differential equation appropriate to the Black-Scholes model

$$\frac{ds}{s} = v(s,t)dt + \sigma dW_t. \tag{1}$$

s is the stock price, $v(s,t)$ the drift, σ (a constant) the volatility and W_t is the Weiner process, i.e.

$$\frac{dW_t}{dt} = \frac{\eta(t)}{\sigma}, \tag{2}$$

where the noise, $\eta(t)$, is Gaussian distributed,

$$P(\eta) \sim \exp\left[-\frac{1}{2\sigma^2}\int dt \eta^2(t)\right]. \tag{3}$$

Defining $dx = ds/s$ we have

$$\frac{dx}{dt} - v(x,t) - \eta(t) = 0. \tag{4}$$

The probability distribution of x is given by the Martin-Siggia-Rose method as the path integral

$$P = \int \mathcal{D}\eta P(\eta)\delta(\partial_t x - v(x,t) - \eta), \tag{5}$$

where P is the probability of a particular 'trajectory of the stock' $x(t)$ occurring. Representing the delta function as a further integral over a new variable $p(t)$ we have

$$P = \int \mathcal{D}[x(t), p(t)]\mathcal{D}\eta e^{-\int dtp(t)(\partial_t x - v(x,t) - \eta(t)) - \frac{1}{2\sigma^2}\int dt\eta^2(t)}. \tag{6}$$

Performing the integral over η we find

$$P = \int \mathcal{D}[x(t), p(t)]e^{-\int dtp(t)(\partial_t x - v(x,t)) + \frac{\sigma^2}{2}\int p^2(t)dt}. \tag{7}$$

This is just like the path-integral for a quantum particle when written

$$P = \int \mathcal{D}[x(t), p(t)]e^{-\int dtp\dot{x} + \int dtH(p,x)}, \tag{8}$$

i.e. we have a quantum particle with a Hamiltonian $H = \frac{\sigma^2 p^2}{2} + pv(x,t)$. For future convenience we introduce the diffusion coefficient $D = \sigma^2/2$.

3 Large deviations in the stock market

3.1 Herding behaviour

In a recent article [3], this exact model was considered with the choice $v(x,t) = -\partial_x V$ where $V(x) = \alpha|x-a|$. They wanted to look at large deviations; that is the probability that the stock starts at some price $x(0) = x = a + \log s_0$ that is close to its equilibrium a and then deviates wildly to some price infinitely far away $x(t) = y \to \infty$. The potential they introduced acts as a restoring force that tries to drive the stock back to equilibrium which represents a 'herding' effect where stocks cluster around their long term average a. This force means it is unlikely *but not impossible* that the stock will make a large deviation. In [3] they dealt with the behaviour of P directly by finding the PDE it satisfied here we will use the Martin-Siggia-Rose formalism.

Consider sending $t \to \infty$ (i.e. we are waiting a large time such that the large deviation -which is unlikely to happen normally- will occur). In order for the exponential in Eq. (8) to be well-defined it cannot diverge, therefore we must have $H = 0$. This implies $p = -2\partial_x V/\sigma^2$. Substituting this into the exponential of Eq. (8) leads to

$$P \sim \int \mathcal{D}xe^{-\int dtp\dot{x}} = e^{\frac{1}{D}\int \partial_x V dx}. \tag{9}$$

The upper and lower integration limits are $x(\infty) = y = \infty, x(0) = a + \log(s_0)$. Formally, then, it looks like the result diverges, however, implementing proper normalisation is crucial. We want it such that

$$A\int_{-\infty}^{\infty} Pdx = 1, \tag{10}$$

for some constant A. We find A diverges in precisely the way Eq. (9) does and thus the two cancel. We are then left with

$$P \sim (s_0)^{-\frac{\alpha}{D}} , \tag{11}$$

recovering the result of [3]. During times of uncertainty, like a recession, the drive to equilibrium is slow (α is small) and the volatility is high, D is large. Therefore the exponent is small and P is high meaning there is a great deal of herding.

References

[1] Hugo Touchette. The large deviation approach to statistical mechanics. *Physics Reports*, 478(1–3):1–69, July 2009.

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[3] Kwangwon Ahn, Linxiao Cong, Hanwool Jang, and Daniel Kim. Business cycle and herding behavior in stock returns: theory and evidence. *Financial Innovation*, 10, 01 2024.