Monte Carlo Method for Option Valuation

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1 Introduction

This piece of work is motivated by Chapter 15 from the book "An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation" [1], and the theory I have covered around the subject of Mathematical Finance in my final year studying for my Master's in Mathematics at Durham University.

2 Background of Option Valuation

Consider a European-style option with payoff that is some function $\Phi(\cdot)$ of the asset price at expiry. Our model for the asset price is:

$$S_t = S_0 e^{(\mu-\sigma^2)t+\sigma\sqrt{t}Z}, \text{ where } Z \sim \mathcal{N}(0,1).$$

We interpret $r > 0$ as the interest rate, μ as the expected return per unit time, and σ as the historic volatility. We treat these parameters as constant, which admittedly may lead to shortfalls, due to the fact that stock volatility is not constant over time. Nevertheless, over short time periods any prices we calculate will be sufficiently accurate so long as the volatility does not change too dramatically.

In order to have a model for the asset price at expiry, we consider the above equation at time $t = T$. Further, we can utilise the risk neutrality approach in order to compute the time-zero option value as follows.

Under the real world measure $\mathbb{P}$ the price dynamics of the Black-Scholes market is given by

$$dB_t = rB_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $(W_t)_{t\geq 0}$ is a Brownian motion and $(S_t)_{t\geq 0}$, $(B_t)_{t\geq 0}$ are risky and risk-free assets, respectively.

We find a new measure $\mathbb{Q}$, the *risk-neutral measure*, defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-(\frac{\mu-r}{\sigma})W_T - \frac{1}{2}(\frac{\mu-r}{\sigma})^2 T}.$$

Under this new measure, the process $W'_t = W_t + \frac{\mu-r}{\sigma}t, t \geq 0$ is a Brownian motion.

We therefore have that

$$dS_t = \mu S_t dt + \sigma S_t dW'_t,$$

so that $S'_t = \frac{S_t}{B_t}$ is a martingale under $\mathbb{Q}$.

Theorem 1 *Risk-neutral valuation formula*

For a contingent claim X on the Black-Scholes markets, with expiry time T, then the no-arbitrage price $\Pi_t(X)$ at time $t \leq T$ satisfies

$$\Pi_t(X) = e^{-r(T-t)\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]}$$

Where $\mathbb{Q}$ is the risk-neutral or martingale measure and $\mathcal{F}_t, t \geq 0$ is the Brownian filtration. Note that this is merely an overview of the topic that is necessary for our application with Monte Carlo methods in the next section. Notably, several proofs have been omitted.

3 Monte Carlo Methods

We consider the case of a random variable X, which has expected value $\mathrm{E}(X) = a$ and variance $\mathrm{Var}(X) = b^2$ which are unknown. Suppose we wish to approximate a and b^2 and further that we can take independent samples of X using a pseudo-random number generator. Since we will need to take expectations in our calculations, it will be useful for us to consider unbiased estimators for a and b^2, where an unbiased estimator Y is such that $\mathrm{E}(Y) = Y$. We shall let $X_1, \ldots, X_M$ denote independent random variables with the same distribution as X. We note that the following are unbiased estimators for a, b^2

$$a_M = \frac{1}{M}\sum_{i=1}^{M} X_i,$$

$$b_M^2 = \frac{1}{M-1}\sum_{i=1}^{M}(X_i - a_M)^2 .$$

Since

$$\mathrm{E}(a_M) = \frac{1}{M}\sum_{i=1}^{M}\mathrm{E}(X_i) = \frac{M}{M}a_M = a_M$$

and

$$
\begin{aligned}
\mathrm{E}(b_M^2) &= \frac{1}{M-1}\mathrm{E}(\sum_{i=1}^{M}(X_i - a_M)^2) \\
&= \frac{1}{M-1}\sum_{i=1}^{M}\mathrm{E}(X_i^2 - 2a_M X_i + a_M^2) \\
&= \frac{1}{M-1}\left(\sum_{i=1}^{M}\mathrm{E}(X_i^2) - 2\mathrm{E}(a_M\sum_{i=1}^{M}(X_i)) + \sum_{i=1}^{M}\mathrm{E}(a_M^2)\right) \\
&= \frac{1}{M-1}\left(\sum_{i=1}^{M}\mathrm{E}(X_i^2) - M\mathrm{E}(a_M^2)\right) \\
&= \frac{1}{M-1}(M-1)b_M^2 \\
&= b_M^2.
\end{aligned}
$$

Where we have used linearity of expectation, and the simple rule that $\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2$.

By the Central Limit Theorem, $\sum_{i=1}^{M} X_i \approx \mathcal{N}(Ma, Mb^2)$. This implies therefore that $a_M - a \approx \mathcal{N}(0, \frac{b^2}{M})$. Note that if we had an equality instead of an approximation here, then

$$
\mathbb{P}\left(a - \frac{1.96b}{\sqrt{M}} \leq a_M \leq a + \frac{1.96b}{\sqrt{M}}\right) = 0.95.
$$

This follows from the symmetry of the standard normal distribution and $\mathrm{N}(1.96) \approx 0.975$. We can re-arrange the above to read

$$
\mathbb{P}\left(a_M - \frac{1.96b}{\sqrt{M}} \leq a \leq a_M + \frac{1.96b}{\sqrt{M}}\right) = 0.95.
$$

Replacing the unknown b with our approximation b_M we see that the unknown value a lies in the interval

$$
\left[a_M - \frac{1.96b_M}{\sqrt{M}}, a_M + \frac{1.96b_M}{\sqrt{M}}\right].
$$

Thus we have found our basic Monte Carlo method for approximating a. We compute M independent samples and form a_M. We can use our approximation b_M^2 to compute a confidence interval, which obviously does not necessarily have to be 95% as shown above.

4 Applications of Monte Carlo

4.1 Option Valuation Example

Considering a European style option and using the risk neutrality approach as discussed previously, we wish to find the expected value of the following random variable

$$e^{rT}\Phi\left(S_0\exp\left[\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z\right]\right), \text{where } Z\sim\mathcal{N}(0,1).$$

Therefore the Monte Carlo algorithm we wish to use is as follows:

Algorithm 1 Option Valuation

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procedure
    for i = 1 to M do
        compute an N(0,1) sample ψ_i
        set S_i = S_0 e^{(r - ½σ²)T + σ√T ψ_i}
        set V_i = e^{-rT} Φ(S_i)
    end

        a_M ← (1/M) Σ_{i=1}^{M} V_i
        b_M^2 ← (1/(M-1)) Σ_{i=1}^{M} (V_i - a_M)^2

```

We use this Monte Carlo method to value a European call option, so that $\Phi(X)=\max\{X-K,0\}$. We can use Black-Scholes to compute the exact value for the option and compare it to the Monte Carlo computation.

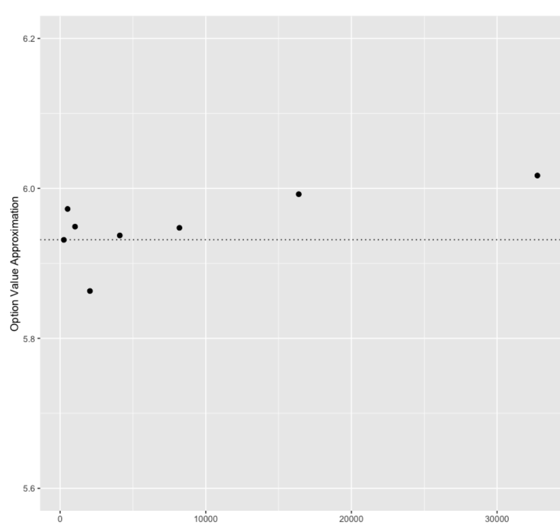

Figure 1: Monte Carlo method to value a European call option

Figure 1 above shows the Monte Carlo method for a range of sample sizes for the following values of our parameters: $S_0 = 32$, $K = 28$, $\sigma = 0.2$, $r = 0.05$, $T = 1$. The horizontal line at 5.931494 is the exact Black-Scholes value which we can compare with. We see that the Monte Carlo method provides a rather strong approximation.

5 Conclusions and Future Work

This project has only just touched the surface of the capability of Monte Carlo in this area of mathematical finance. Note that since we were able to compute an exact value for our European call option using Black-Scholes, we did not necessarily need to use Monte Carlo here. However, it provided a way to demonstrate the efficacy of the method, which we now know we can reliably apply to a range of different scenarios. For example, we could apply the method to compute the value of more exotic or complex options, or even Greeks, by applying Monte Carlo to a finite difference approximation to the derivative.

References

[1] Higham, Desmond J., *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation*, (Cambridge University Press, 2004).

Computer Programs

R version 3.6.2

For a repository of R code used for this project, please see:
https://github.com/Jamaloney/Mathfinance