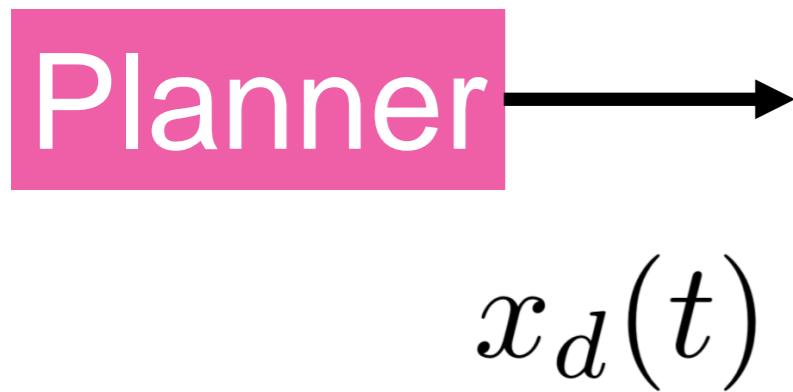


Kinematics and Dynamics of the double pendulum

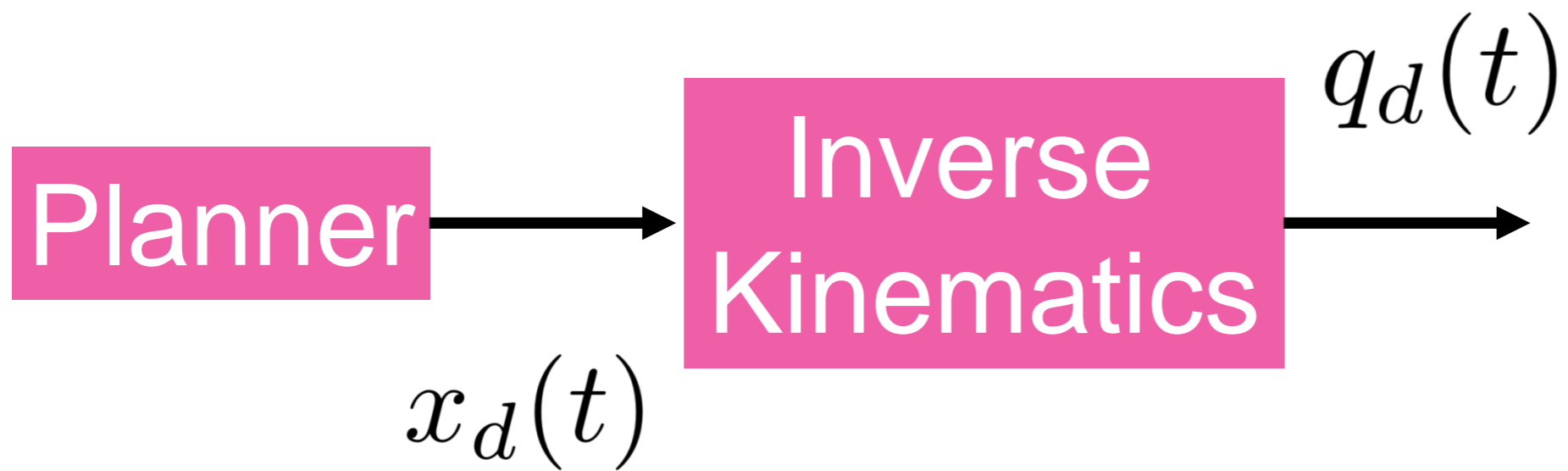
Legged Robots Course

Control Design Pipeline

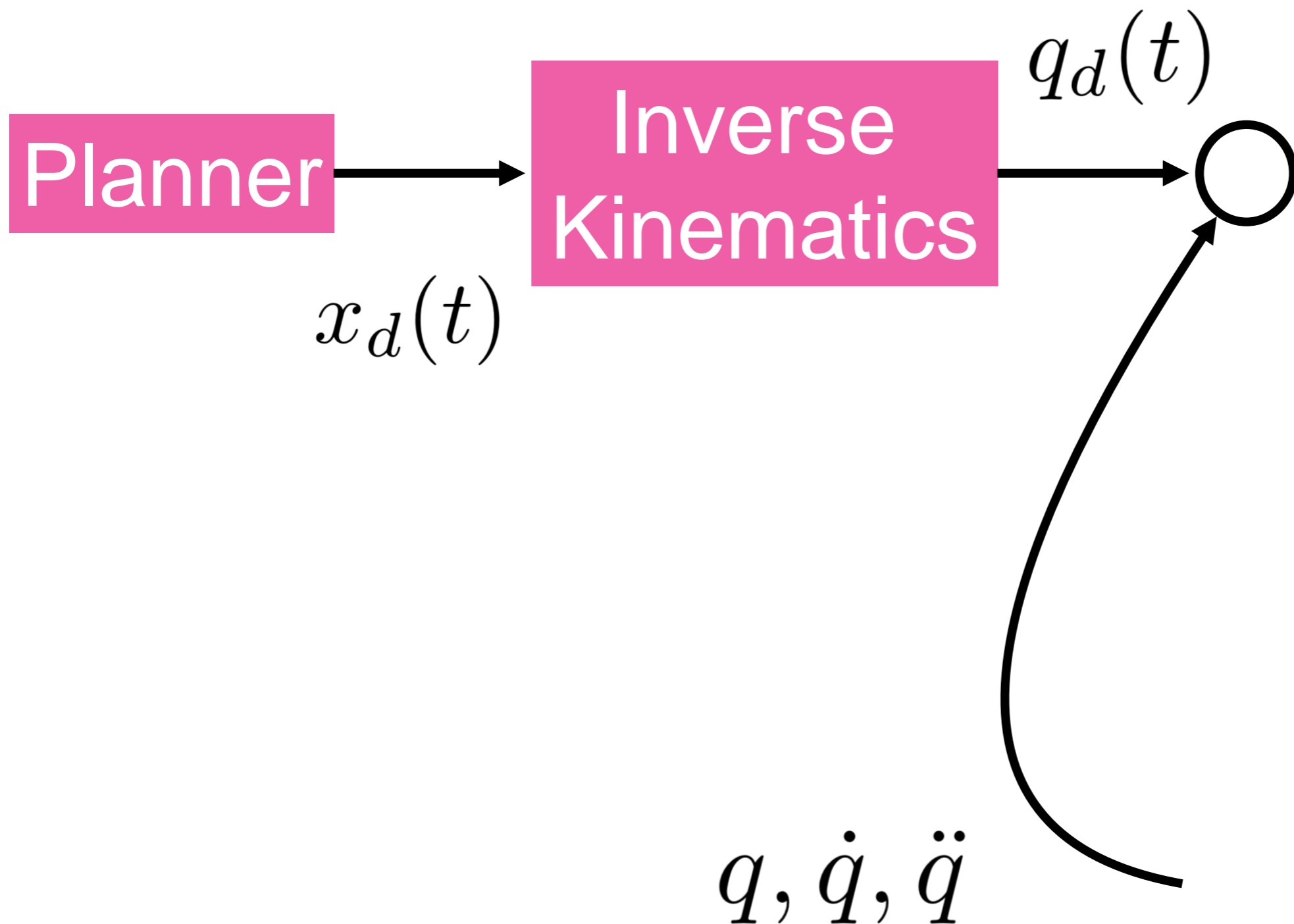
Control Design Pipeline



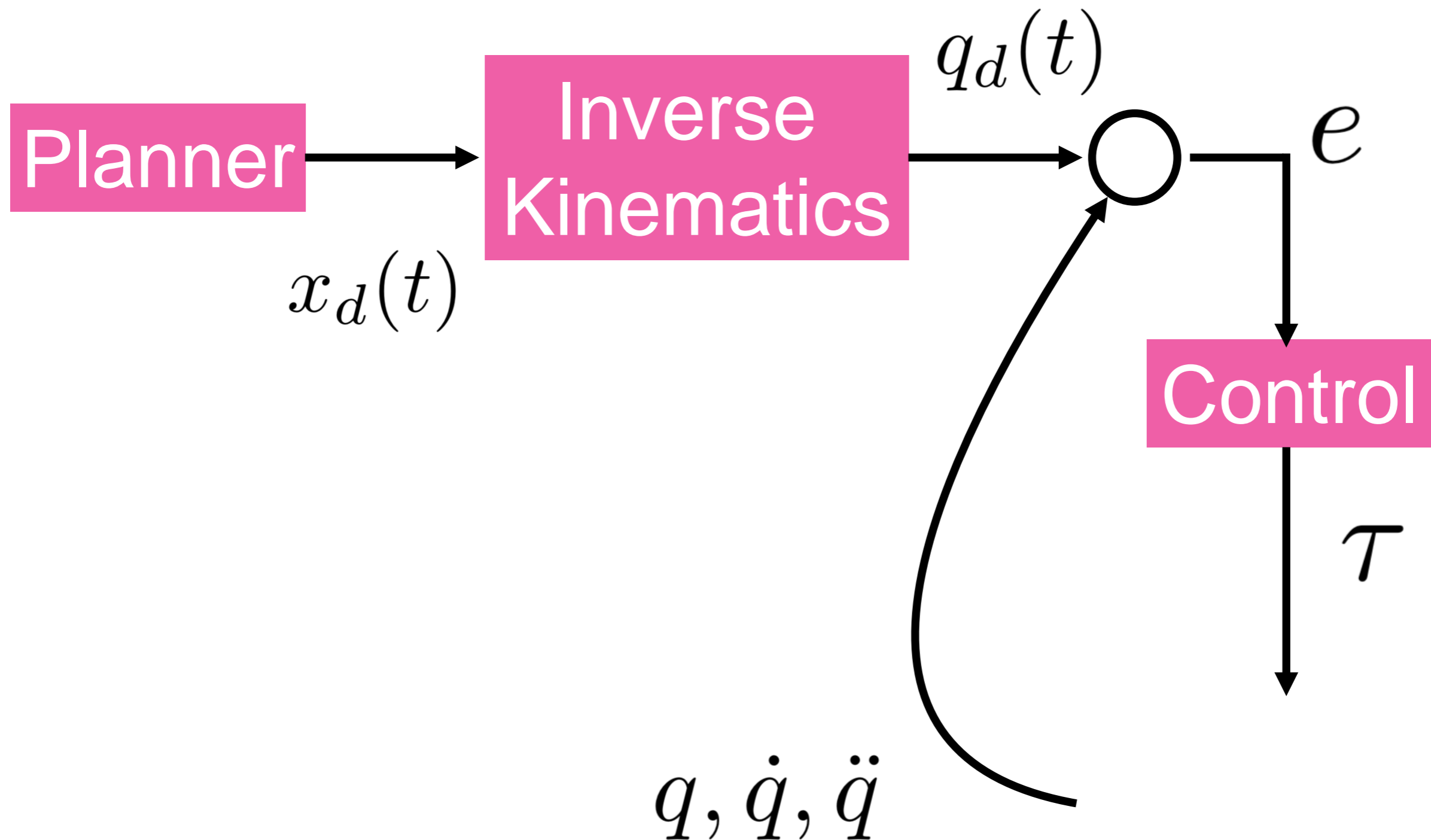
Control Design Pipeline



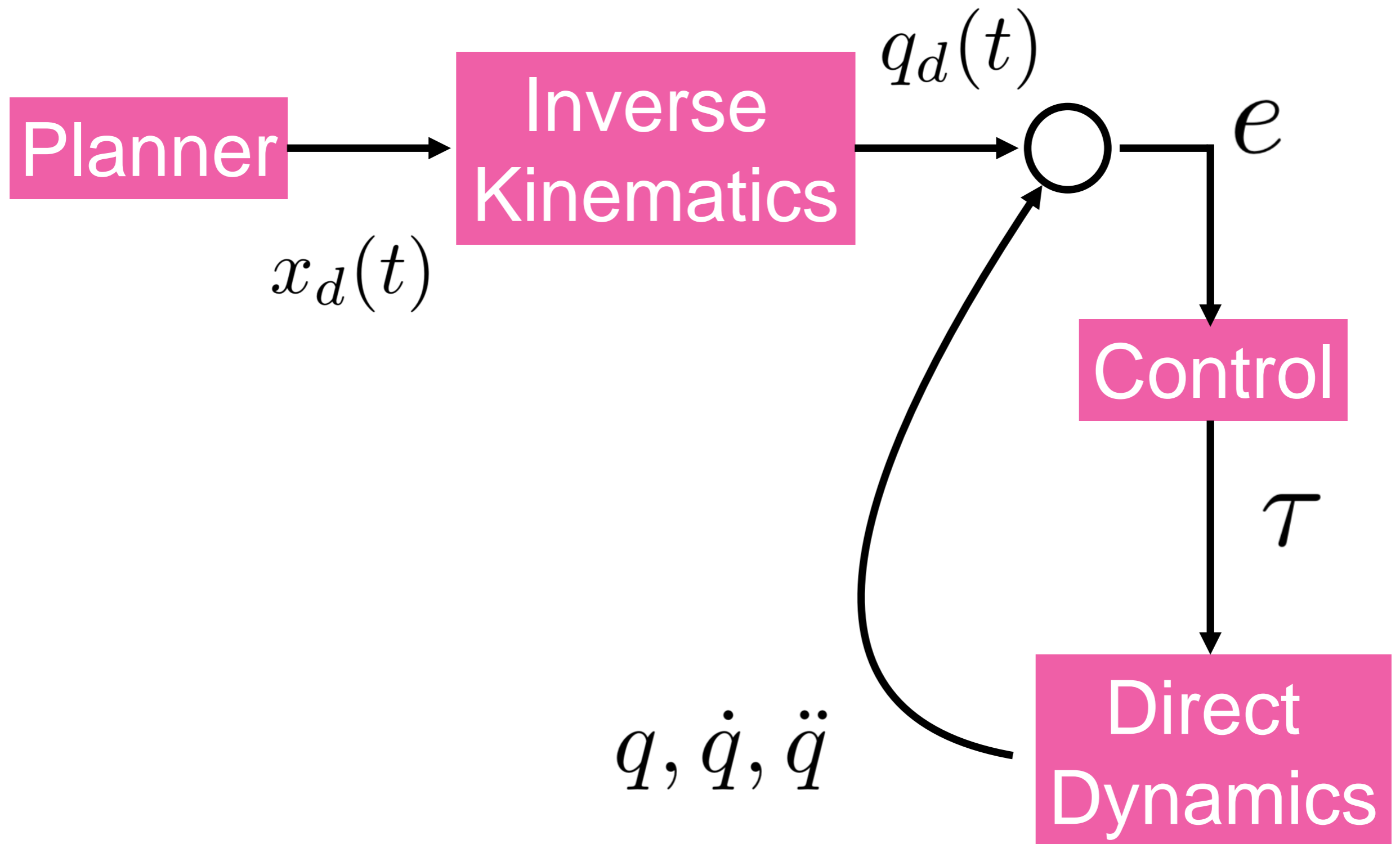
Control Design Pipeline



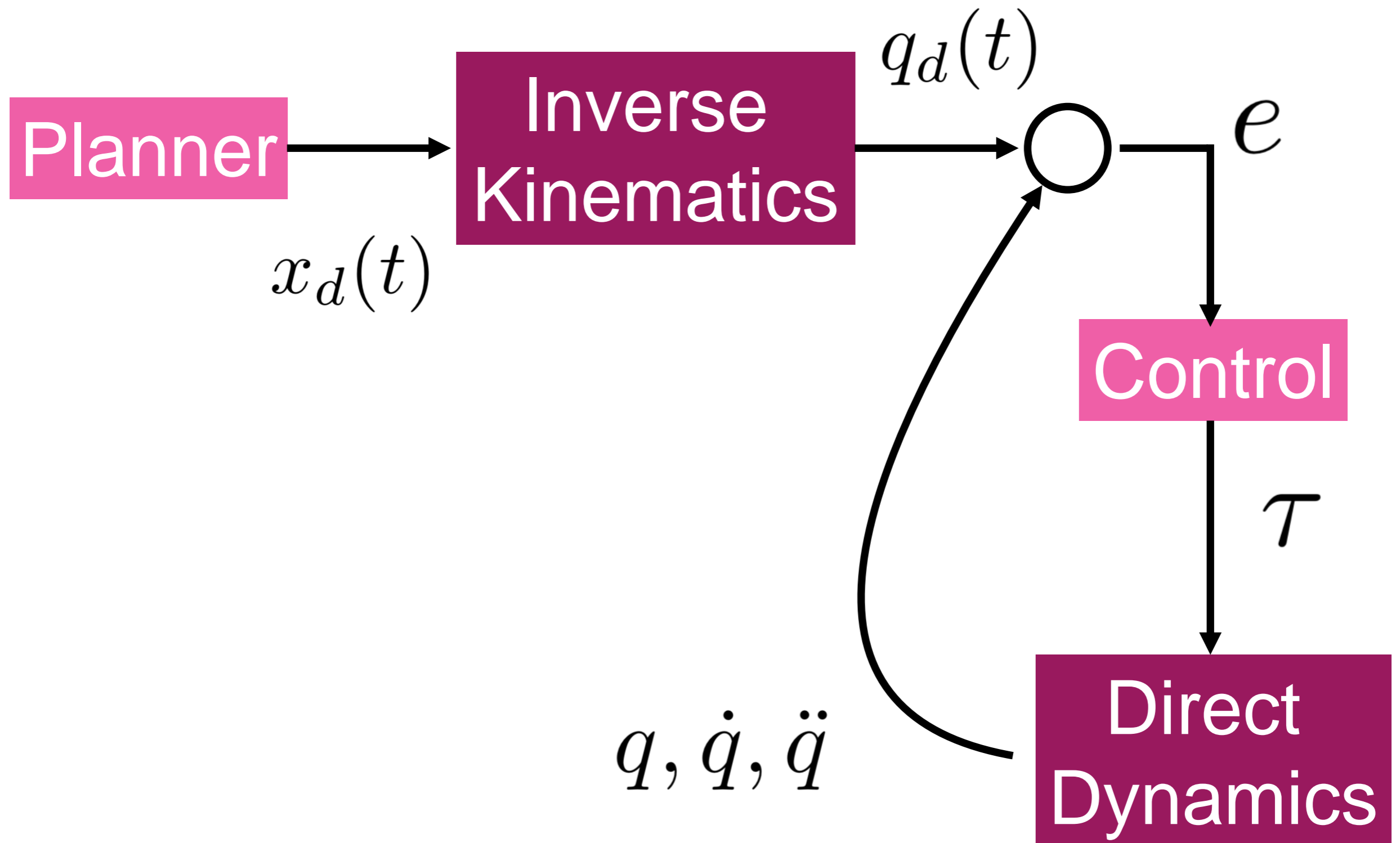
Control Design Pipeline



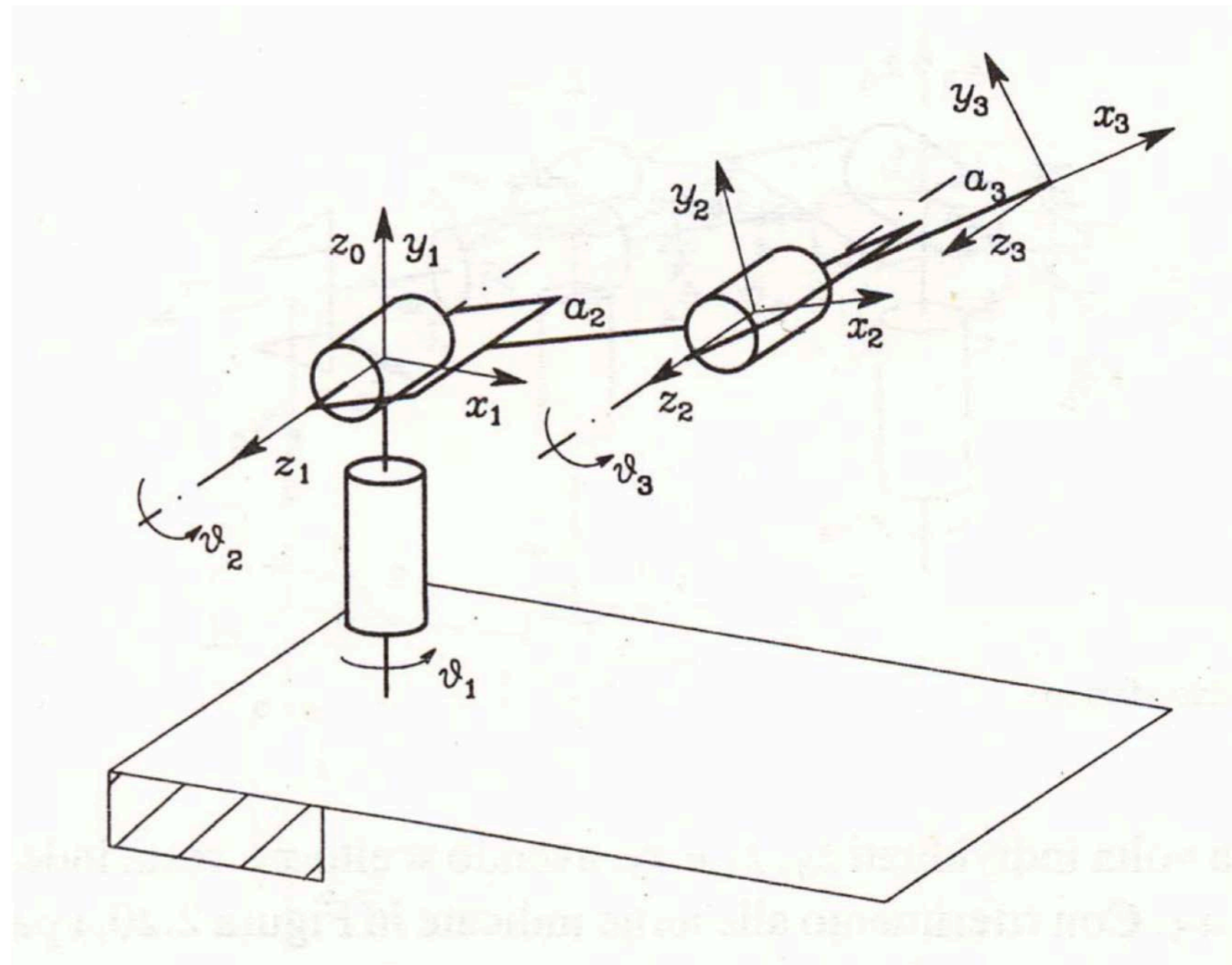
Control Design Pipeline



Control Design Pipeline

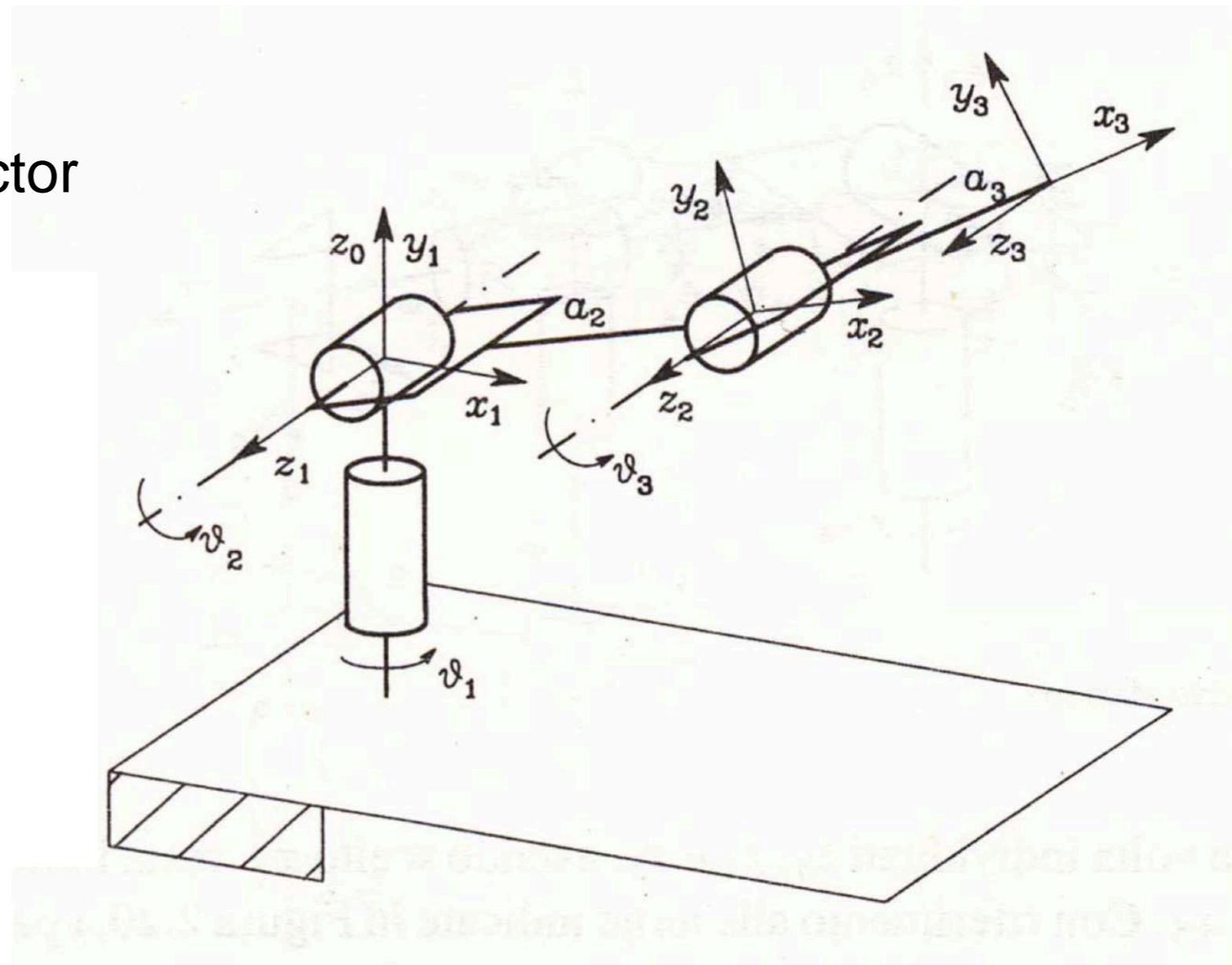


Kinematics



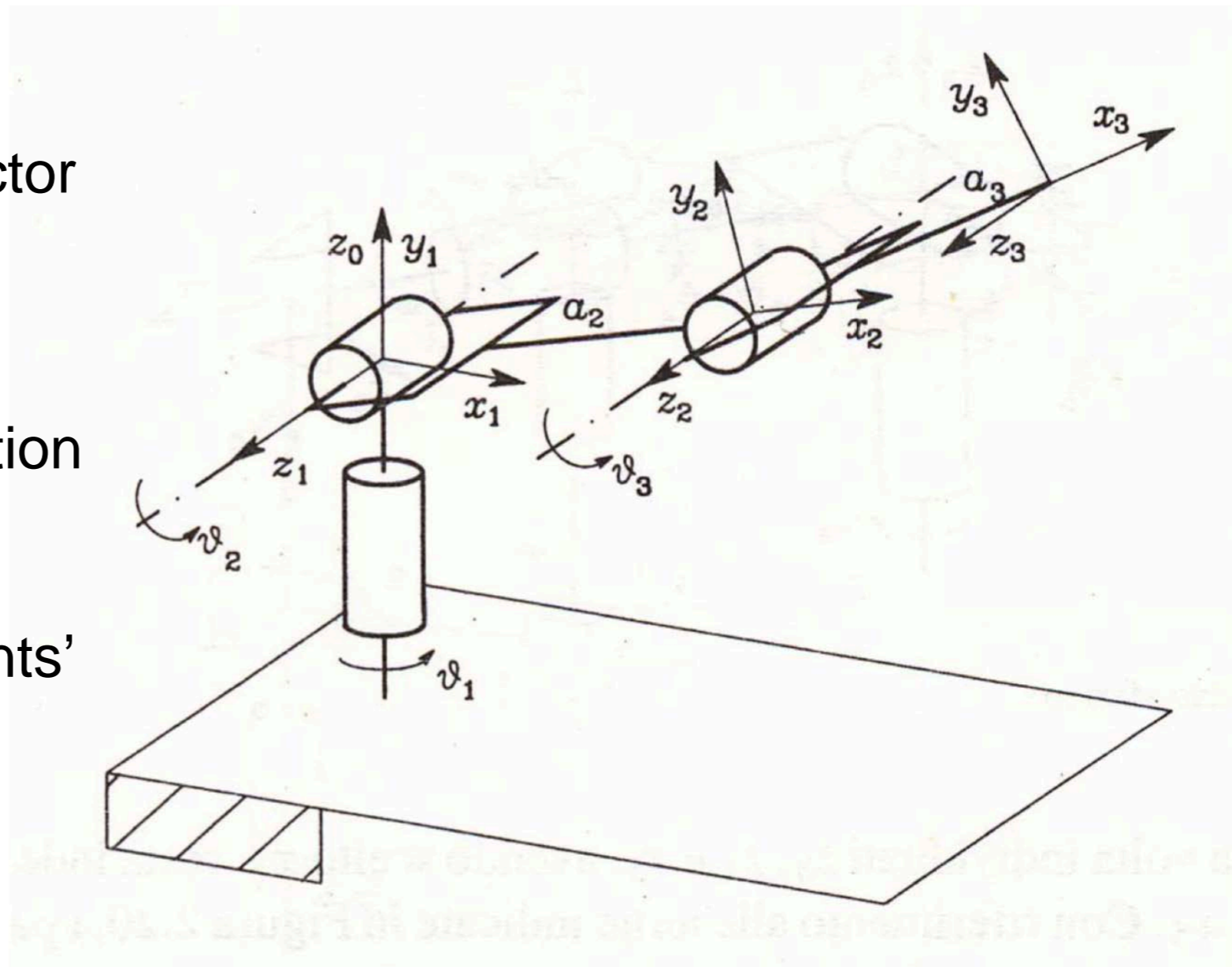
Kinematics

- **Direct Kinematics:**
 - Input: joints' angles
 - Output: position and orientation of the end effector

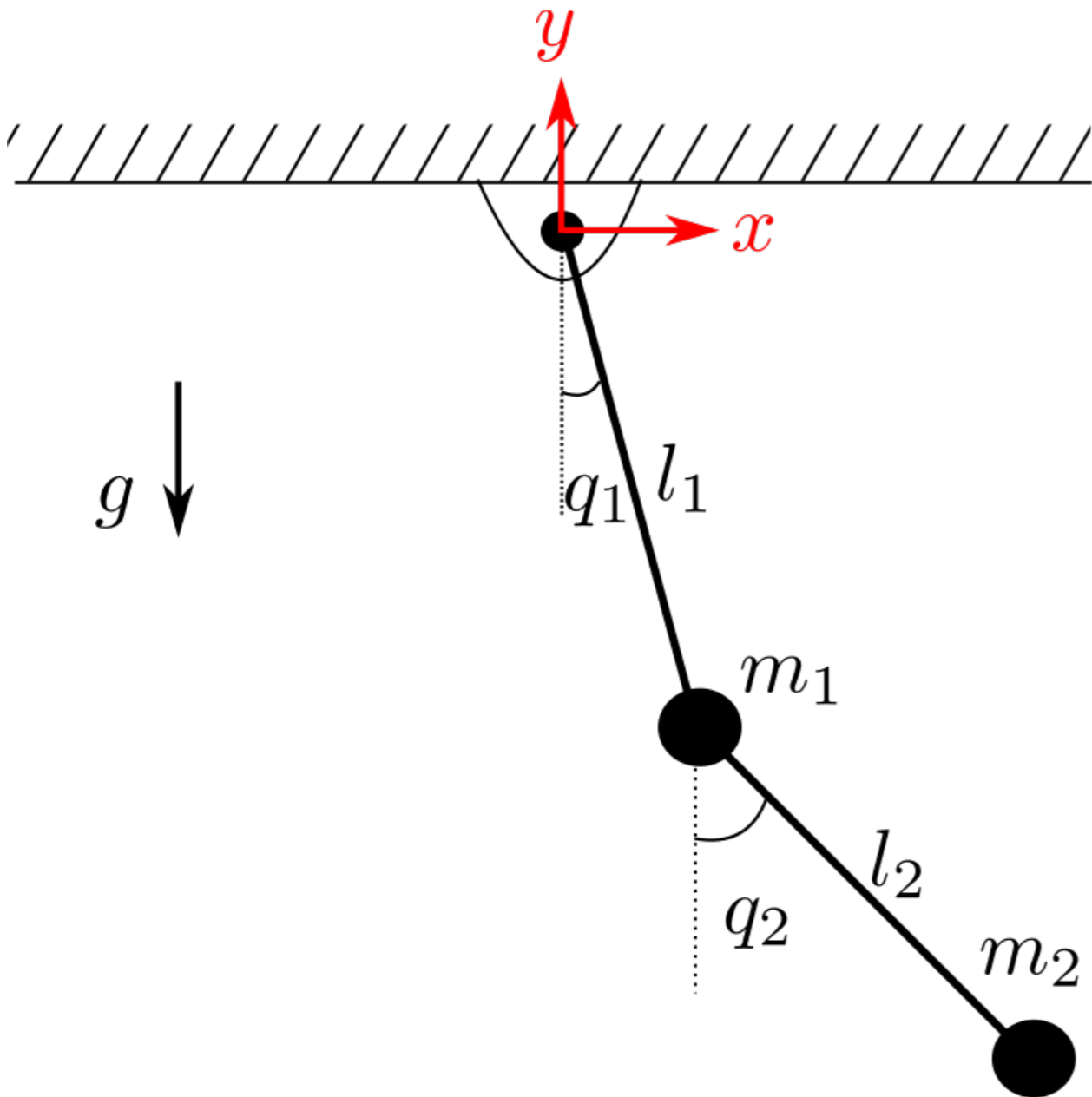


Kinematics

- **Direct Kinematics:**
 - Input: joints' angles
 - Output: position and orientation of the end effector
- **Inverse Kinematics:**
 - Input: position and orientation of the end effector
 - Output: all the possible joints' angles combinations



Kinematics



$$x_1 = l_1 \sin(\theta_1)$$

$$y_1 = -l_1 \cos(\theta_1)$$

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$$

$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2)$$

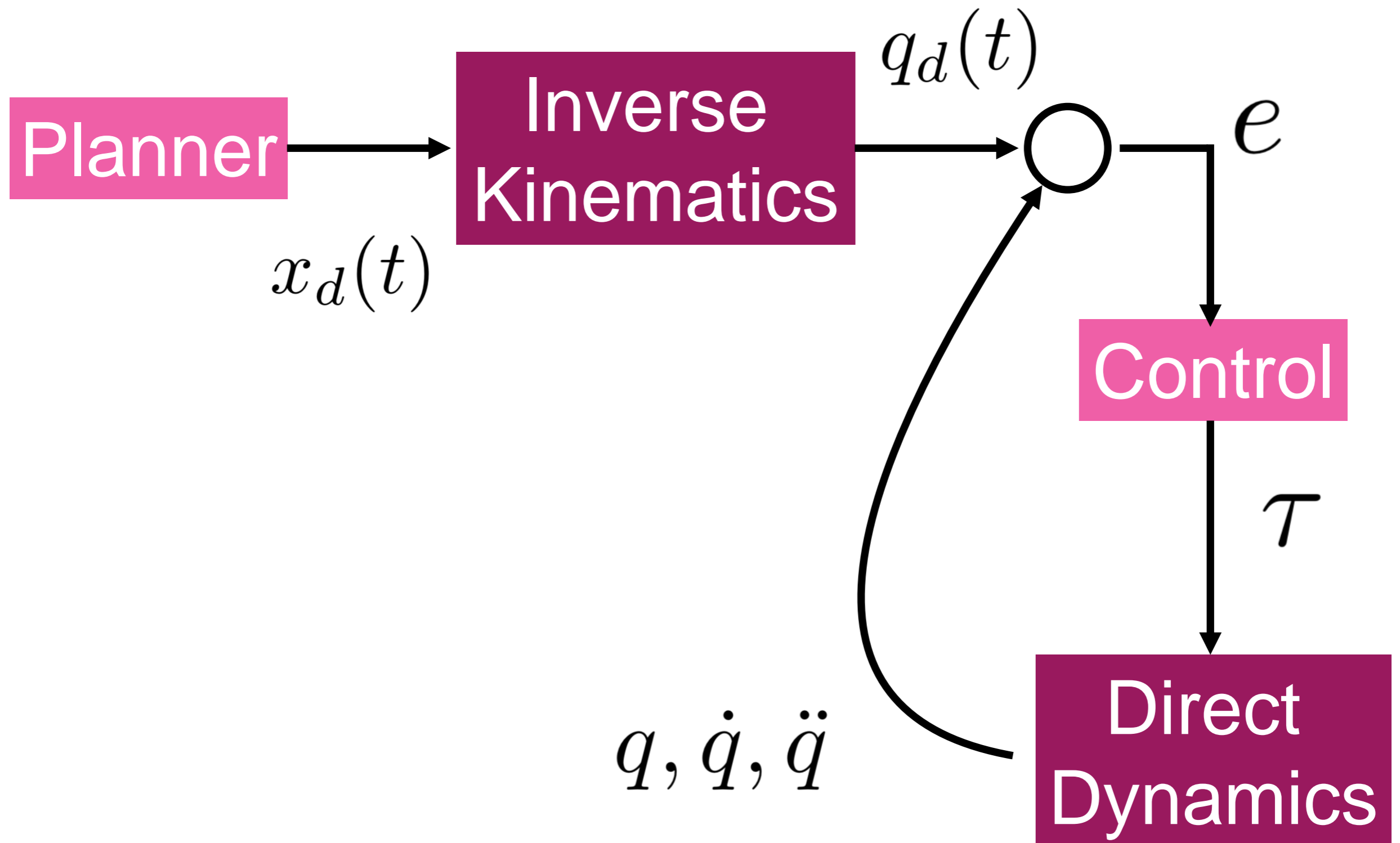
$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos(\theta_1)$$

$$\dot{y}_1 = l_1 \dot{\theta}_1 \sin(\theta_1)$$

$$\dot{x}_2 = l_1 \dot{\theta}_1 \cos(\theta_1) + l_2 \dot{\theta}_2 \cos(\theta_2)$$

$$\dot{y}_2 = l_1 \dot{\theta}_1 \sin(\theta_1) + l_2 \dot{\theta}_2 \sin(\theta_2)$$

Control Design Pipeline



Dynamics

- Dynamics studies the relation between the **joint actuator torques** and the **resulting motion**

Dynamics

- Dynamics studies the relation between the **joint actuator torques** and the **resulting motion**
- **Inverse Dynamics:**

$$\tau = B(q)\ddot{q} + C(q, \dot{q}) + g(q) + F_v\dot{q} + F_s\text{sign}(\dot{q})$$

- **Direct Dynamics**

$$\ddot{q} = B^{-1}(q) \cdot [\tau - C\dot{q} - g - F_v\dot{q} - F_s\text{sign}(\dot{q})]$$

Dynamics

- **Lagrangian Mechanics Method:**

- Variational approach based on kinetics and potential energy

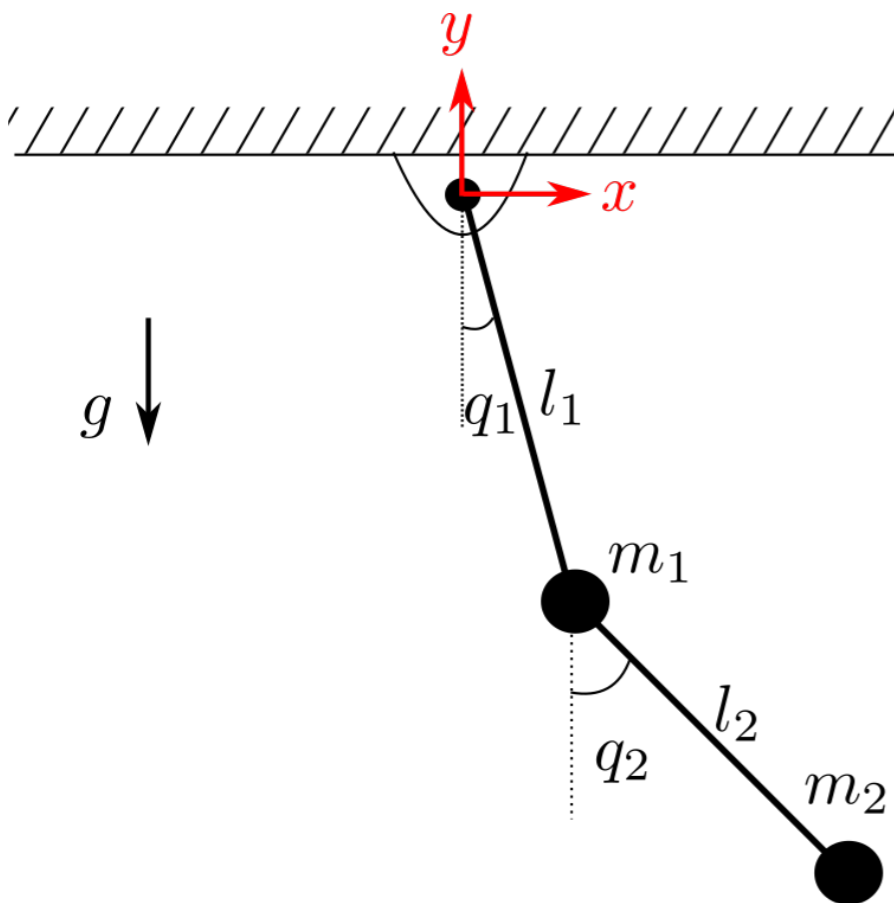
$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

- **Newton-Euler formulation**

- Relies on $F=ma$ applied to each individual link of the robot

Dynamics



Potential Energy

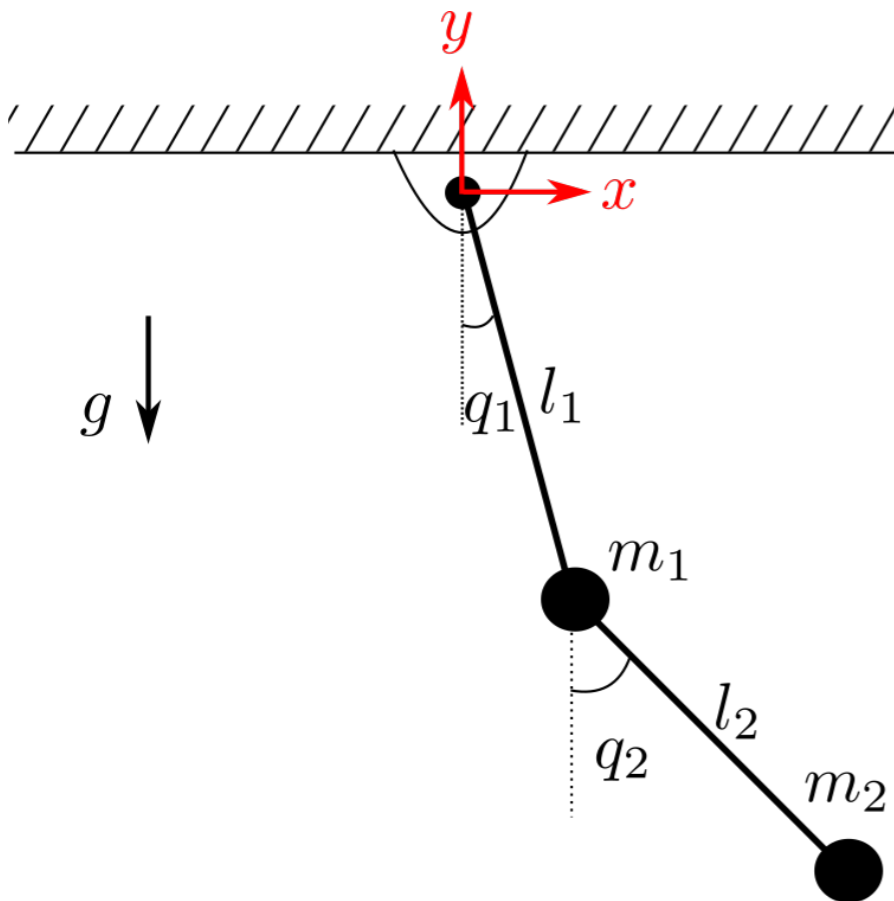
$$q = (\theta_1, \theta_2)$$

$$V = V_1 + V_2$$

$$V_1 = -m_1 y_1 g = -m_1 l_1 g \cos(\theta_1)$$

$$V_2 = -m_2 y_2 g = -m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_2))$$

Dynamics

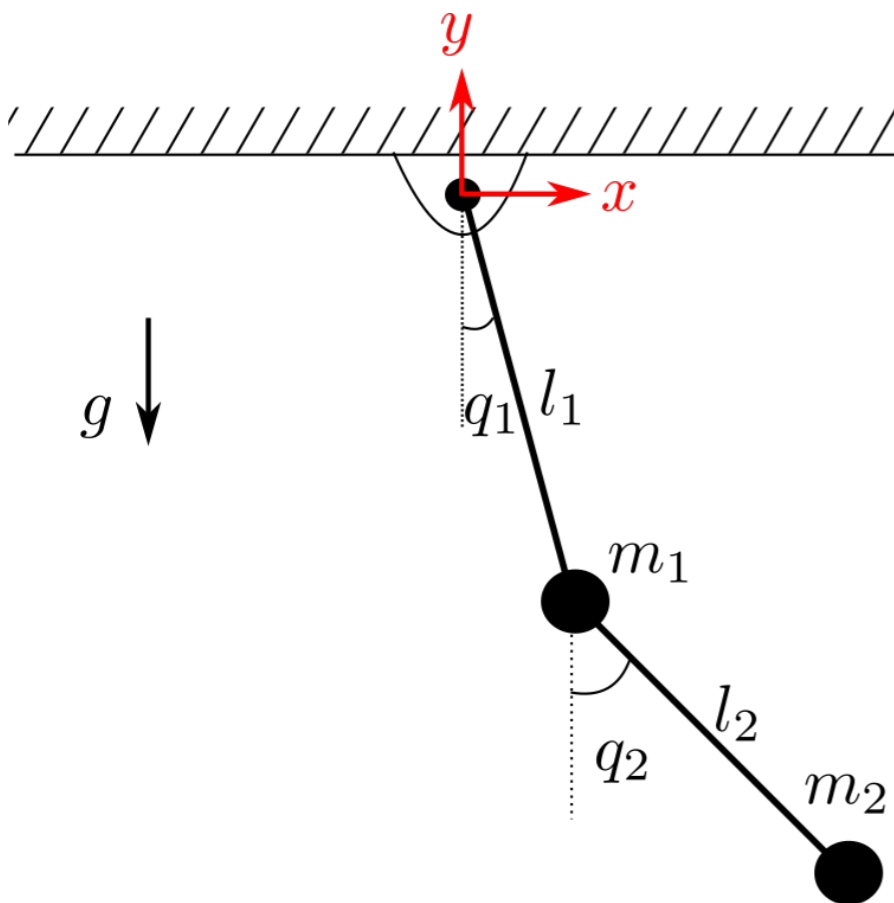


Kinetic Energy

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 \\ &\quad + l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 (\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2)) \\ &= \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 (\cos(\theta_1 - \theta_2)) \end{aligned}$$

Dynamics



Equation of Motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0$$



$$\frac{\partial T_1}{\partial \dot{\theta}_1}$$

$$\frac{\partial T_2}{\partial \dot{\theta}_1}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

Dynamics

$$\frac{\partial T_1}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1$$

$$\frac{\partial T_2}{\partial \dot{\theta}_1} = m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial T_1}{\partial \theta_1} = 0$$

$$\frac{\partial T_2}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial V_1}{\partial \theta_1} = m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial V_2}{\partial \theta_1} = m_2 g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

Dynamics

$$\frac{\partial T_1}{\partial \dot{\theta}_2} = 0$$

$$\frac{\partial T_2}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial T_1}{\partial \theta_2} = 0$$

$$\frac{\partial T_2}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial V_1}{\partial \theta_2} = 0$$

$$\frac{\partial V_2}{\partial \theta_2} = m_2 g l_2 \sin(\theta_1)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$\begin{aligned}
 & m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
 & + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0
 \end{aligned}$$

$$\begin{aligned}
 & m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\
 & - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0
 \end{aligned}$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$