

Kinematics and Dynamics of the double pendulum

Legged Robots Course

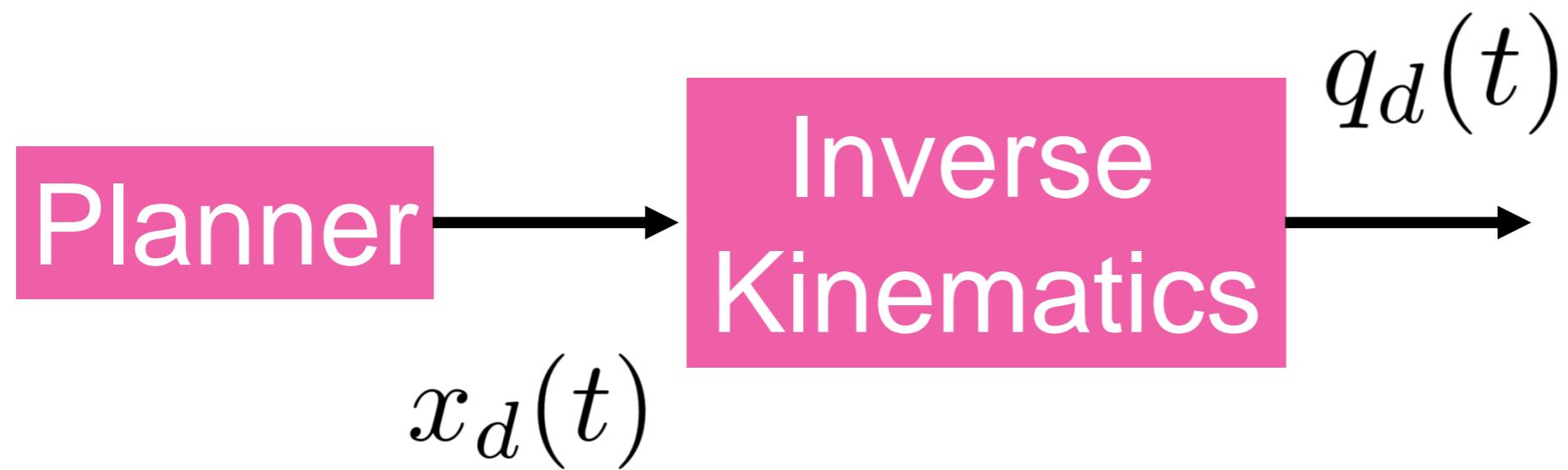
Control Design Pipeline

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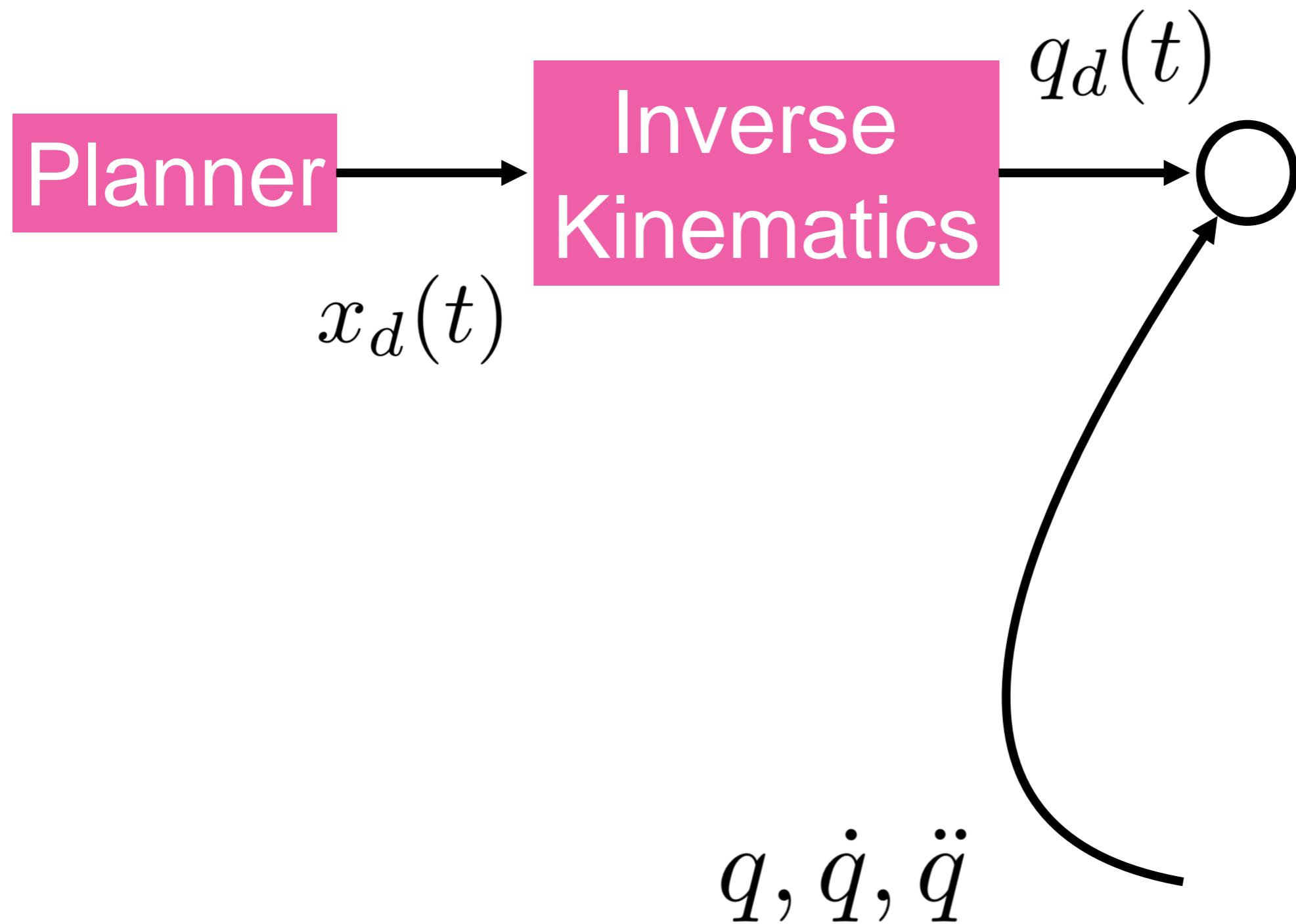


$$x_d(t)$$

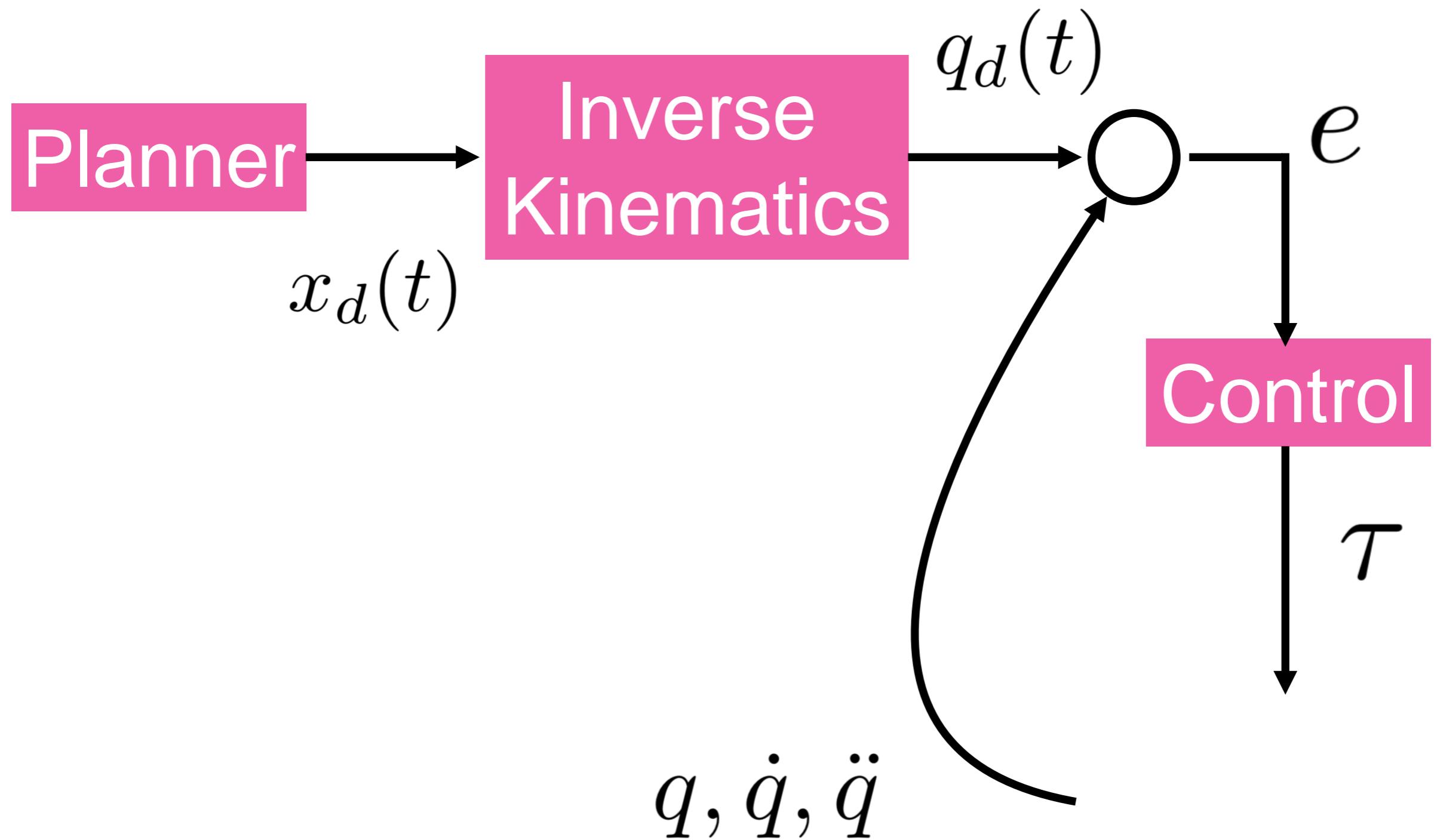
Control Design Pipeline



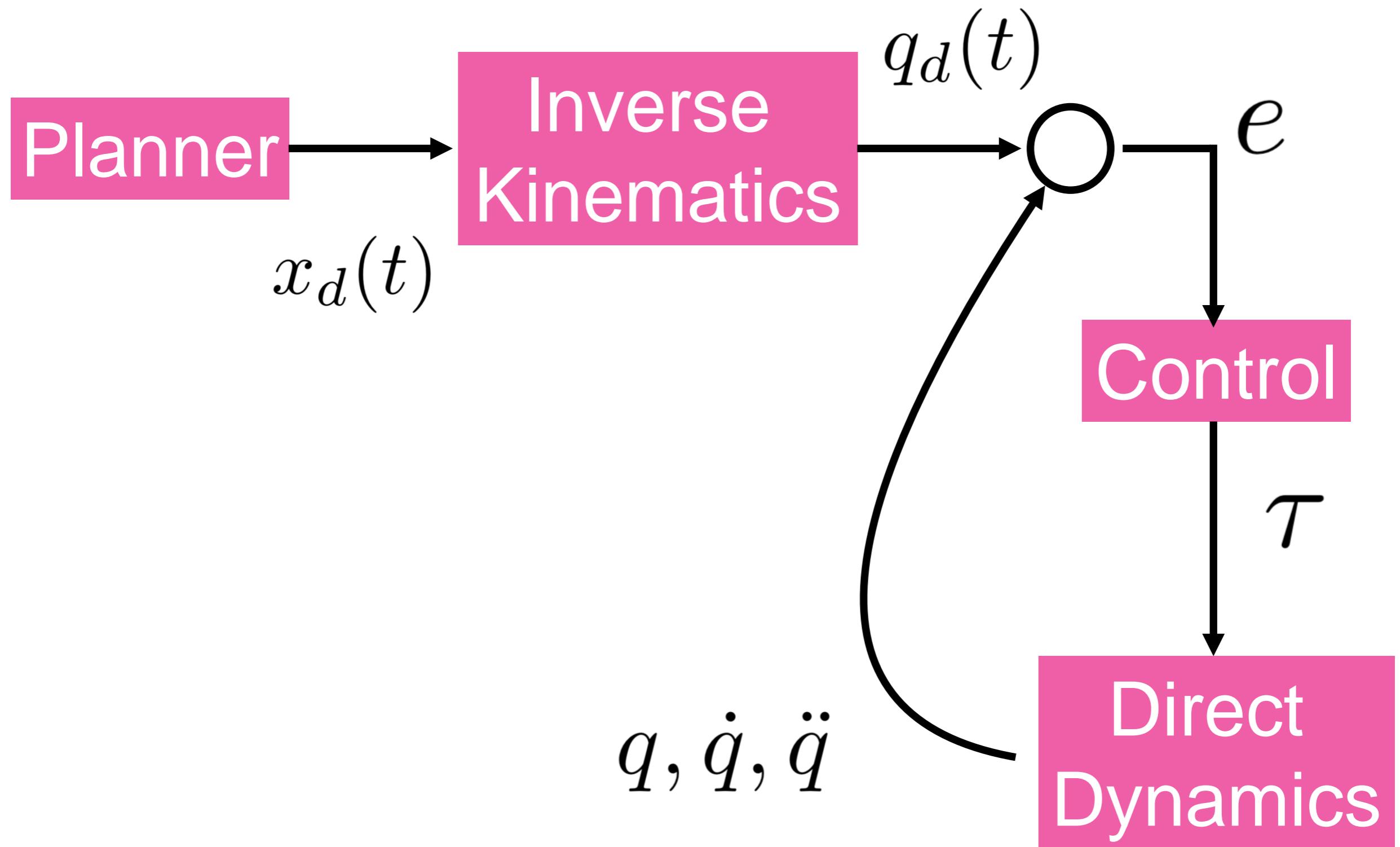
Control Design Pipeline



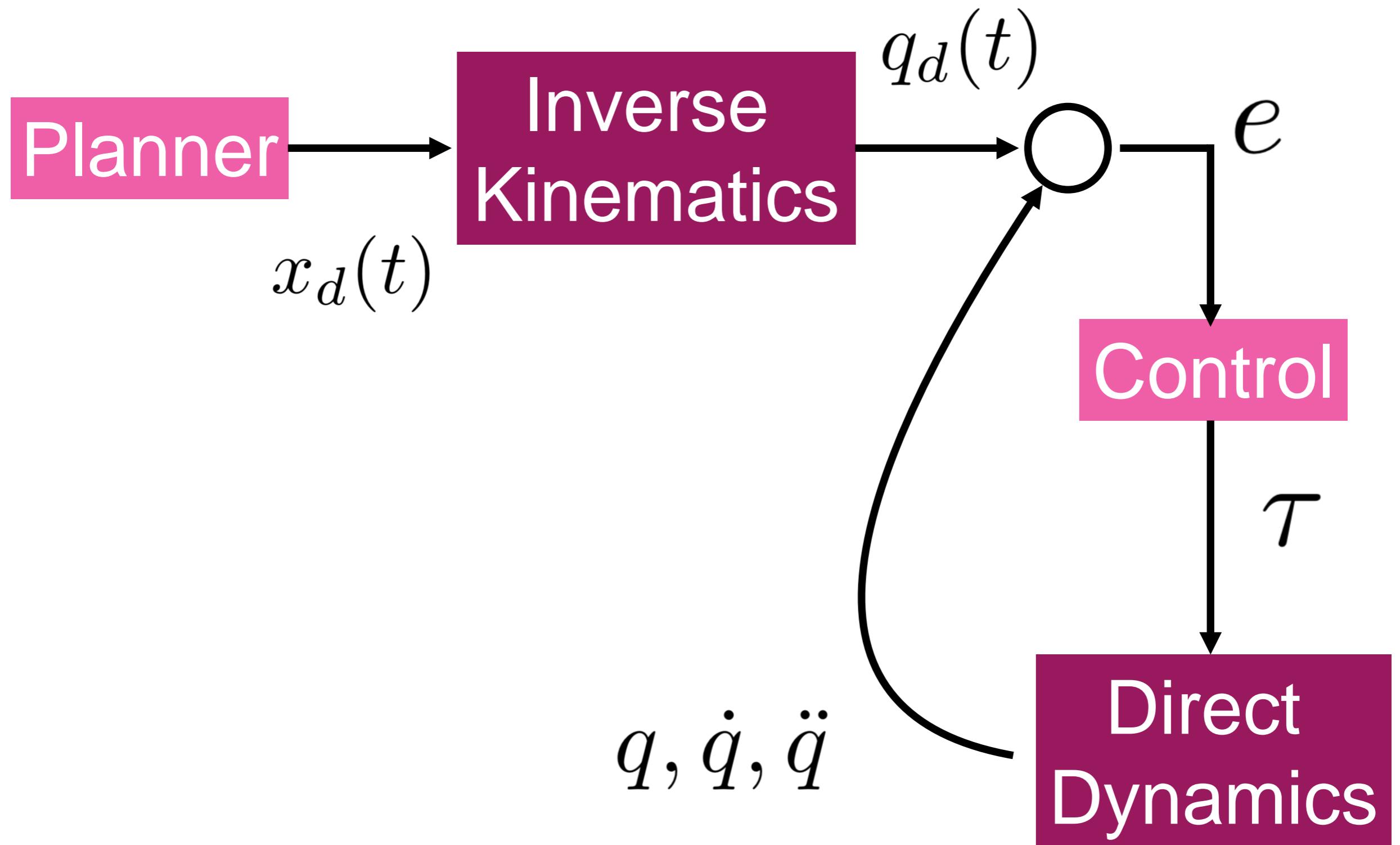
Control Design Pipeline



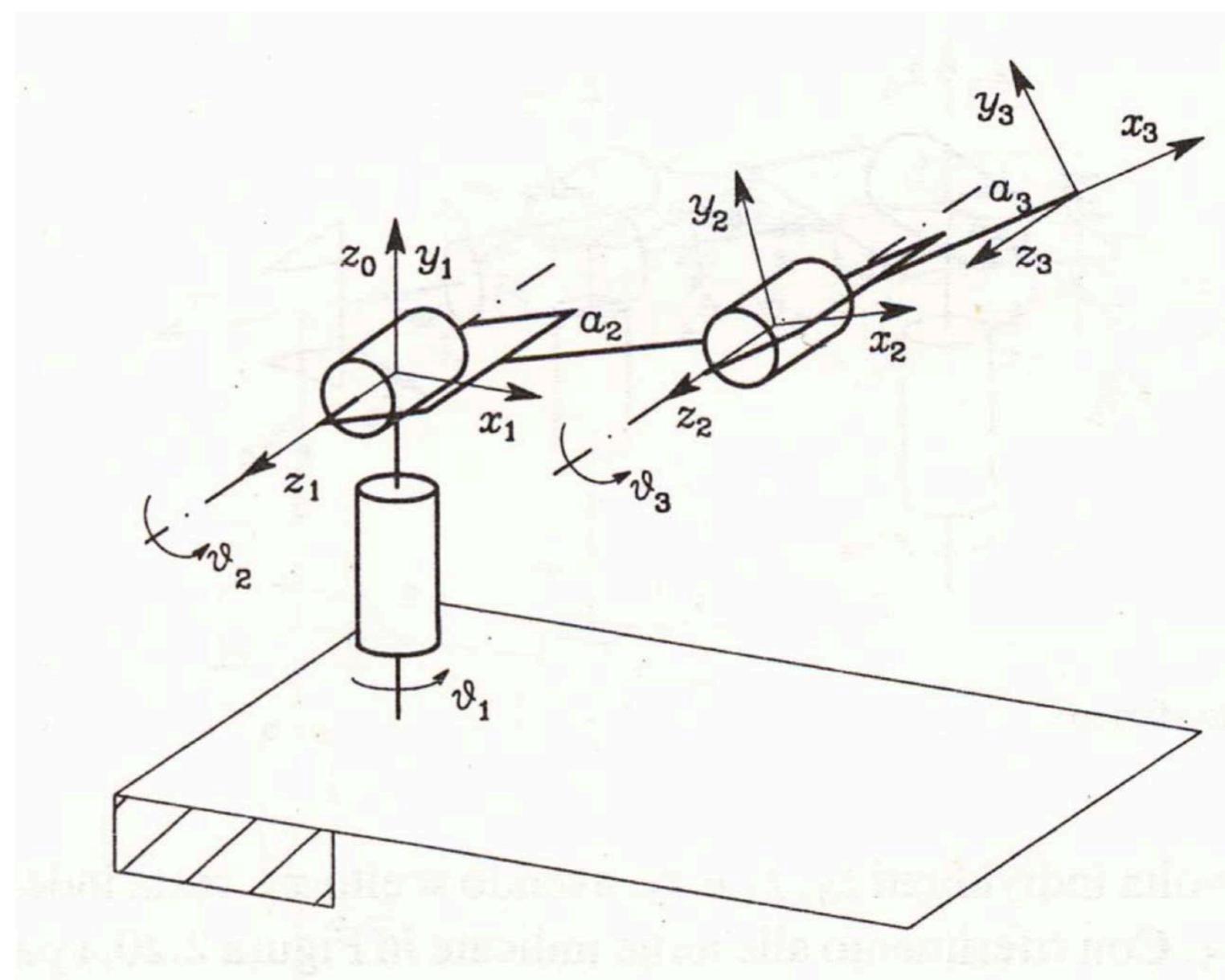
Control Design Pipeline



Control Design Pipeline

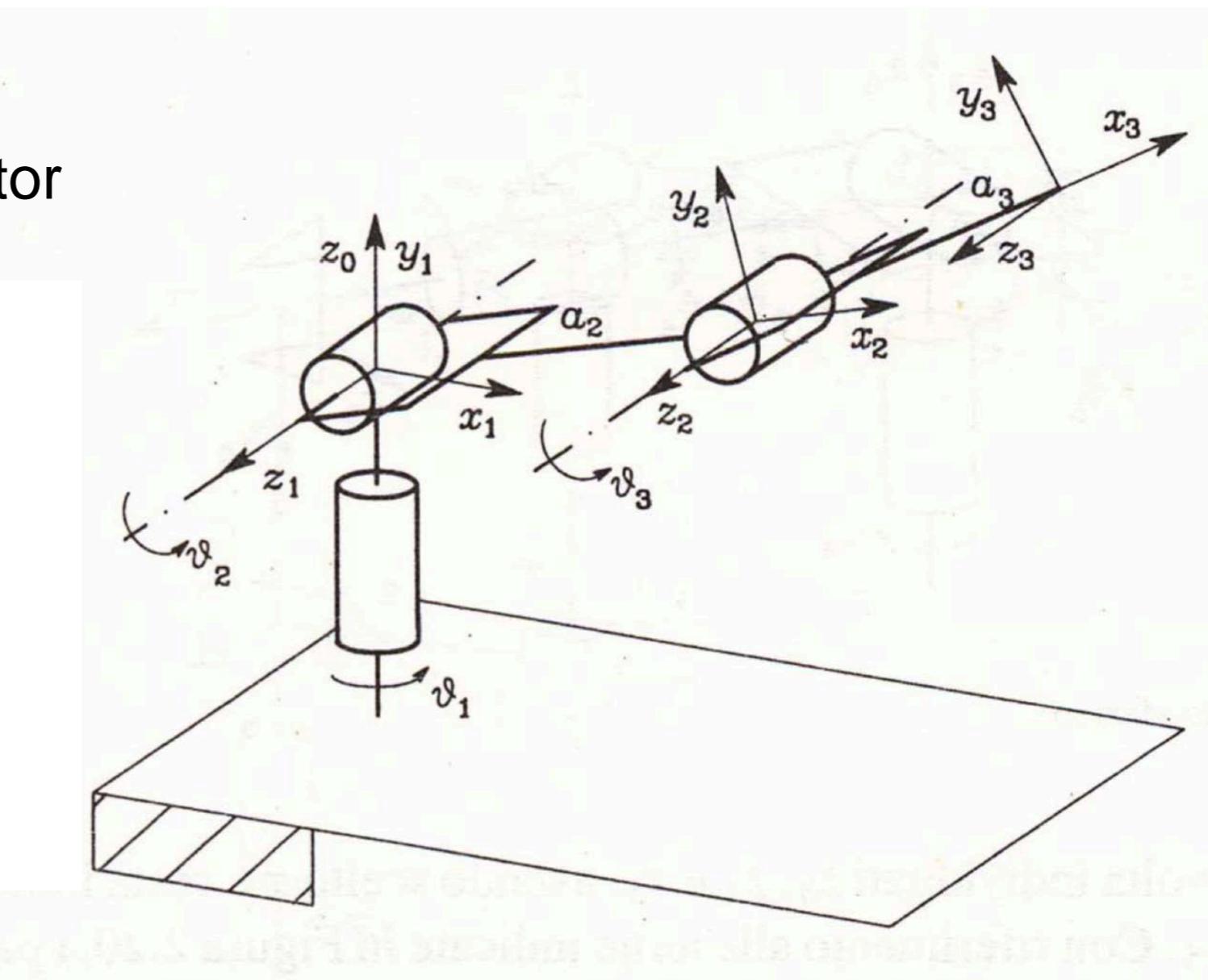


Kinematics



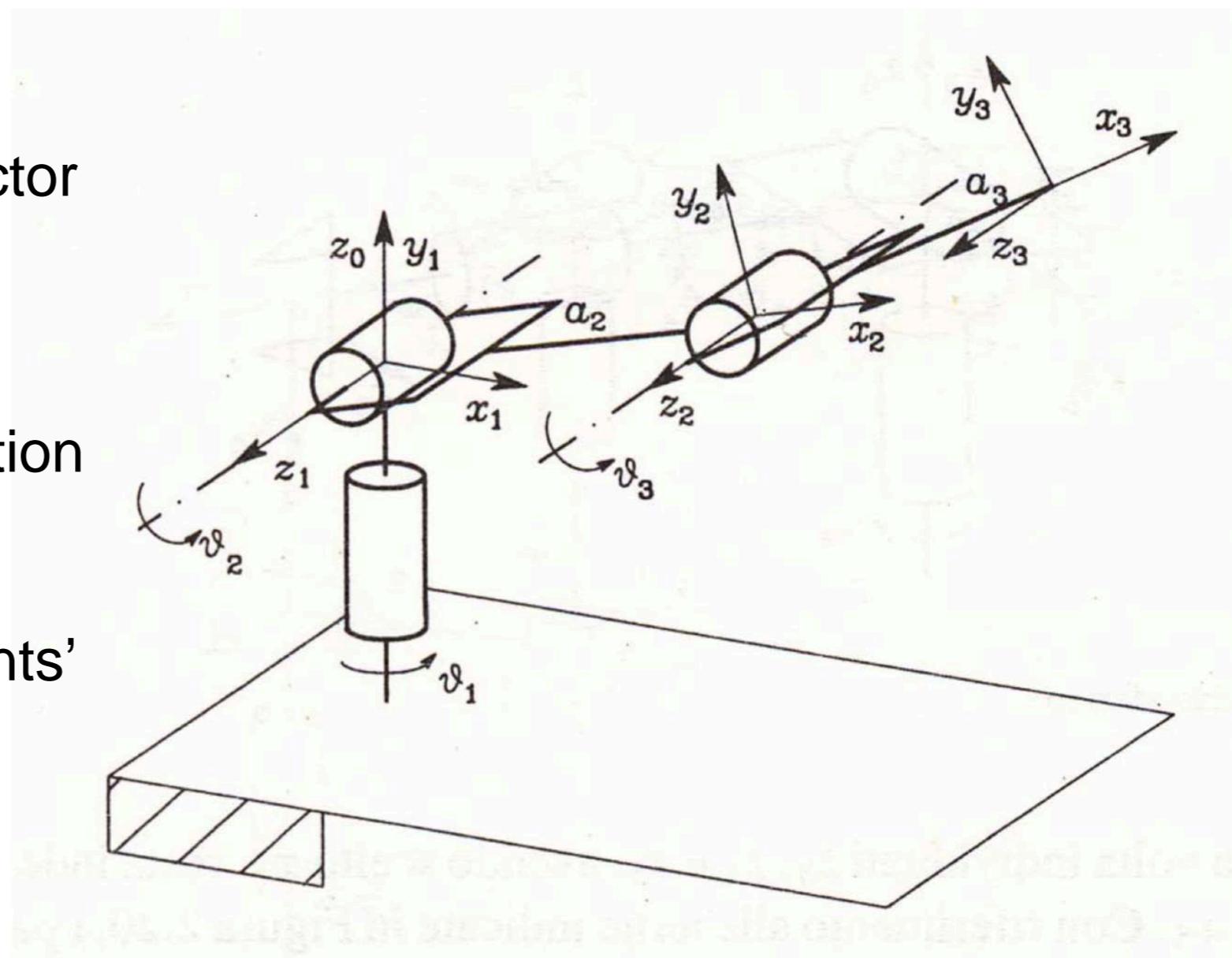
Kinematics

- Direct Kinematics:
 - Input: joints' angles
 - Output: position and orientation of the end effector

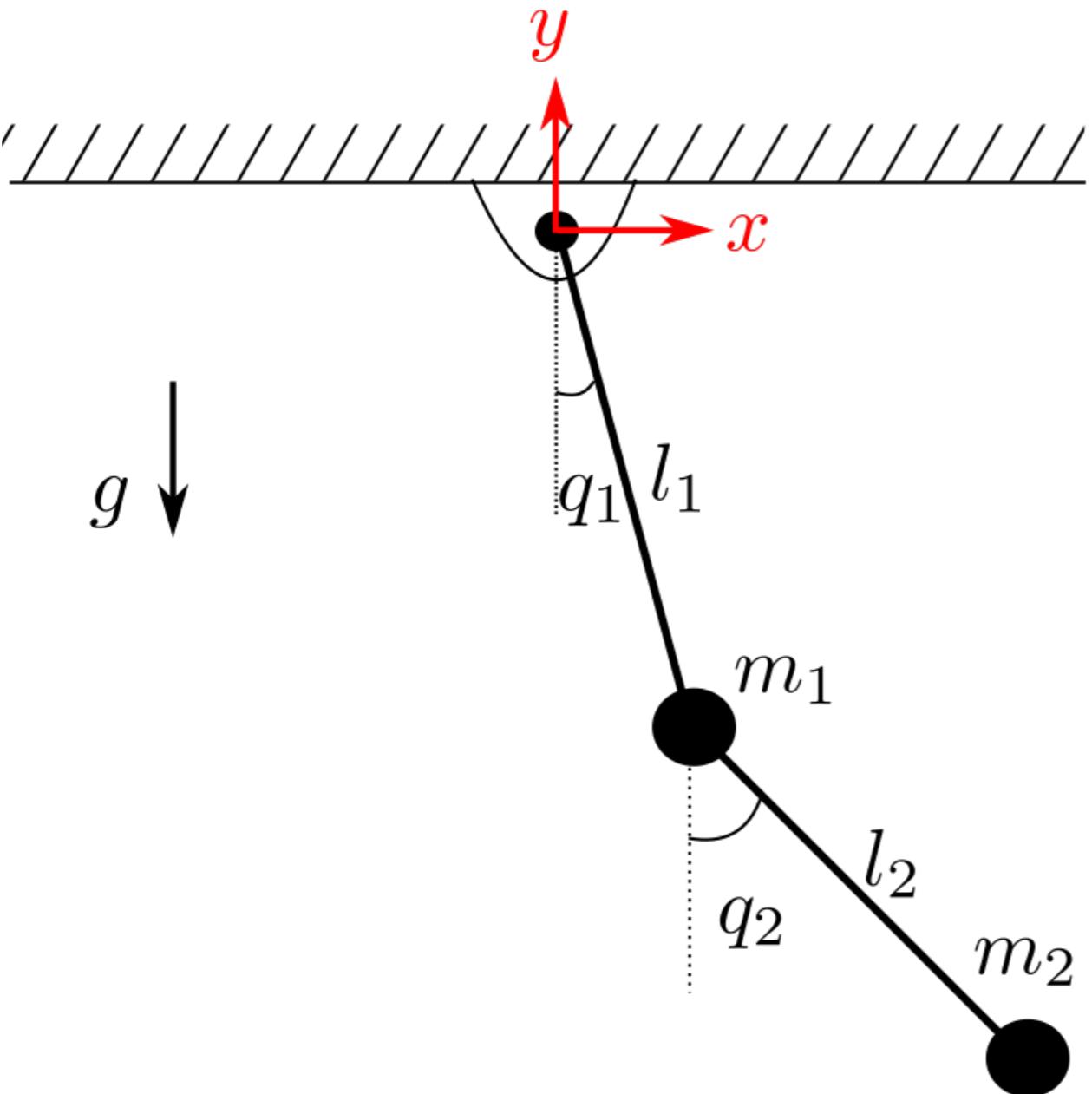


Kinematics

- **Direct Kinematics:**
 - Input: joints' angles
 - Output: position and orientation of the end effector
- **Inverse Kinematics:**
 - Input: position and orientation of the end effector
 - Output: all the possible joints' angles combinations



Kinematics



$$x_1 = l_1 \sin(\theta_1)$$

$$y_1 = -l_1 \cos(\theta_1)$$

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$$

$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2)$$

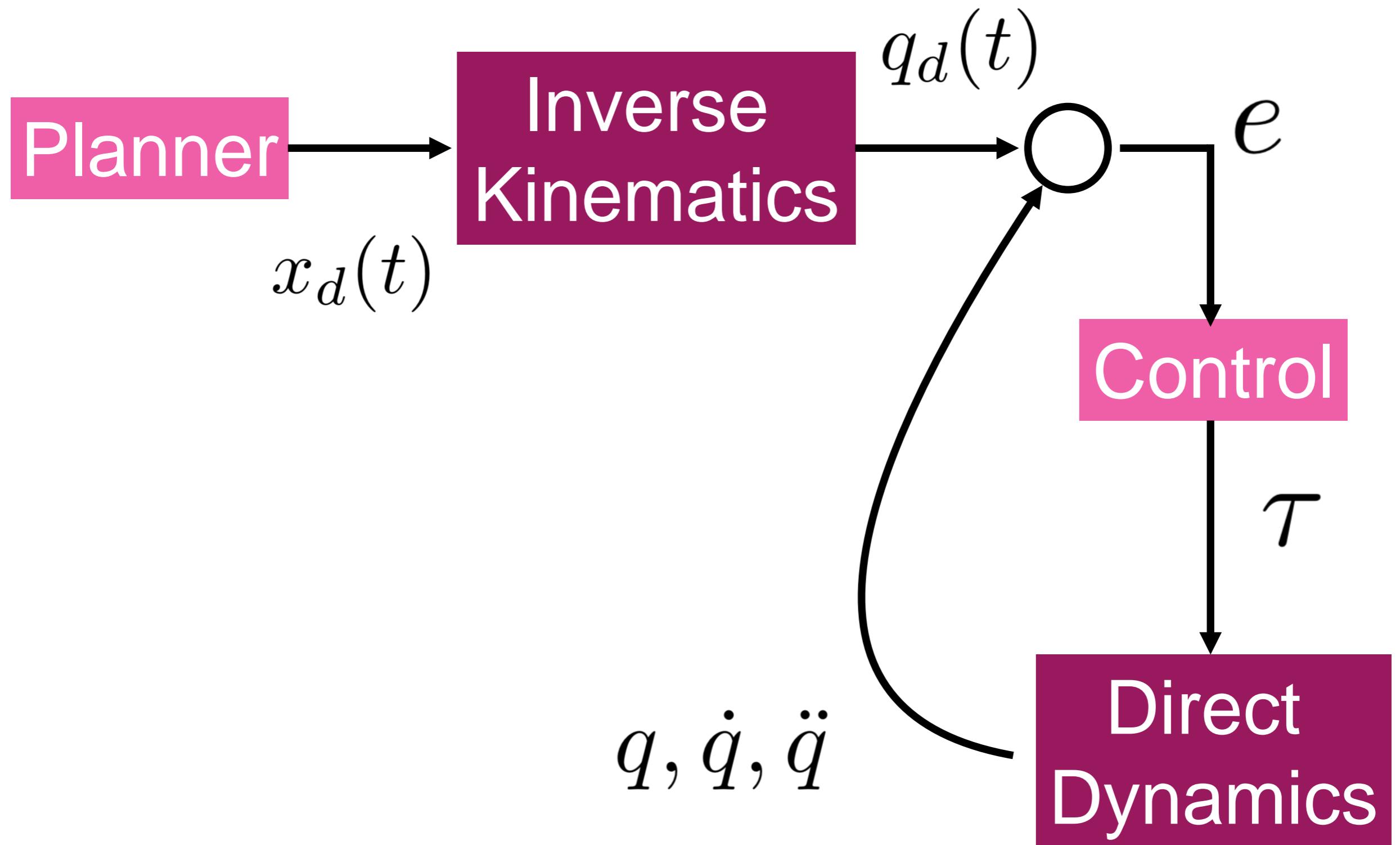
$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos(\theta_1)$$

$$\dot{y}_1 = l_1 \dot{\theta}_1 \sin(\theta_1)$$

$$\dot{x}_2 = l_1 \dot{\theta}_1 \cos(\theta_1) + l_2 \dot{\theta}_2 \cos(\theta_2)$$

$$\dot{y}_2 = l_1 \dot{\theta}_1 \sin(\theta_1) + l_2 \dot{\theta}_2 \sin(\theta_2)$$

Control Design Pipeline



Dynamics

- Dynamics studies the relation between the **joint actuator torques** and the **resulting motion**

Dynamics

- Dynamics studies the relation between the **joint actuator torques** and the **resulting motion**
- **Inverse Dynamics:**

$$\tau = B(q)\ddot{q} + C(q, \dot{q}) + g(q) + F_v\dot{q} + F_s sign(\dot{q})$$

- **Direct Dynamics**

$$\ddot{q} = B^{-1}(q) \cdot [\tau - C\dot{q} - g - F_v\dot{q} - F_s sign(\dot{q})]$$

Dynamics

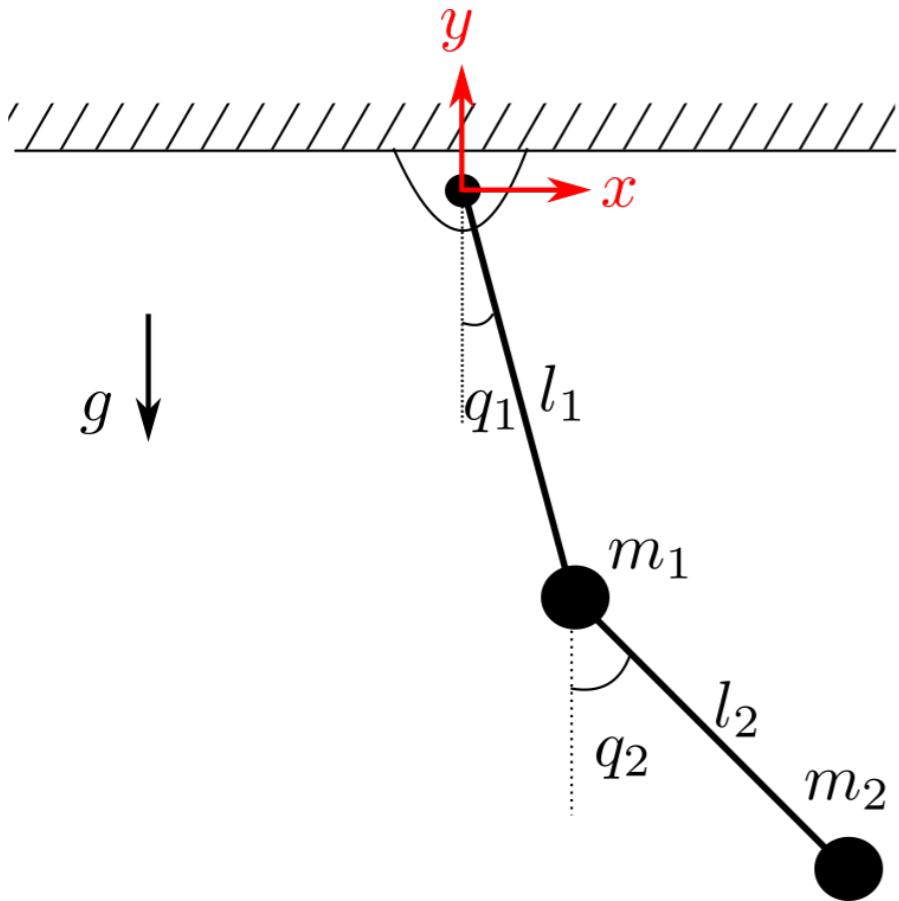
- **Lagrangian Mechanics Method:**
 - Variational approach based on kinetics and potential energy

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

- **Newton-Euler formulation**
 - Relies on $F=ma$ applied to each individual link of the robot

Dynamics



Potential Energy

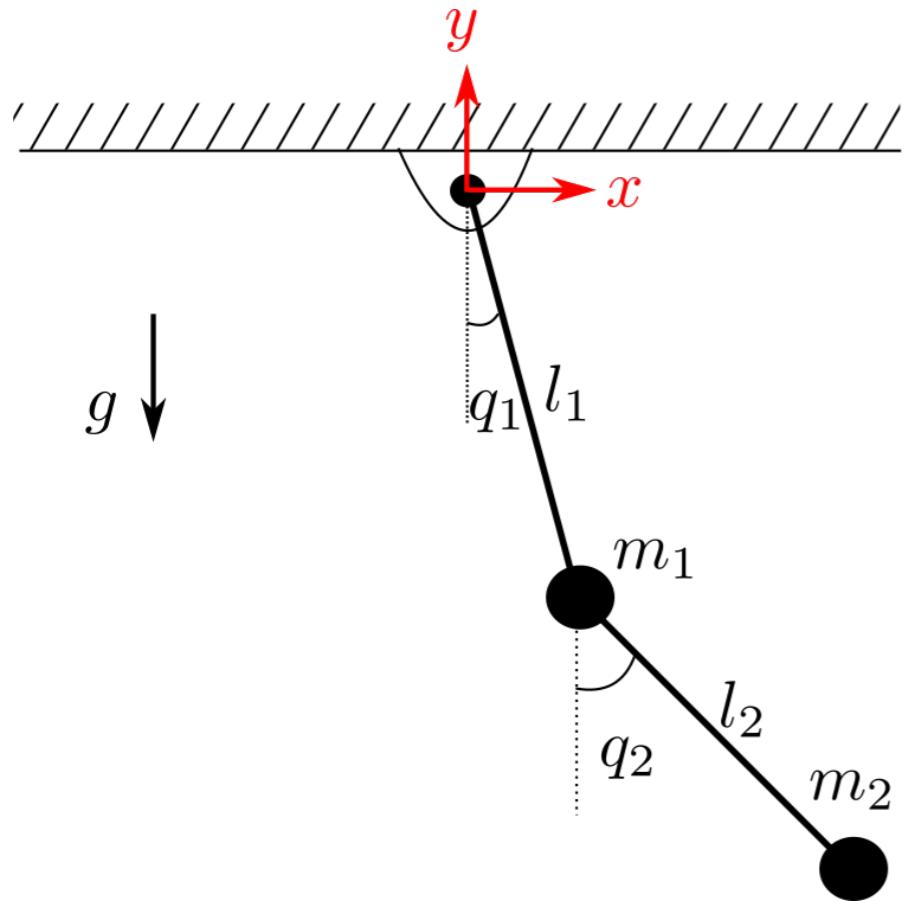
$$q = (\theta_1, \theta_2)$$

$$V = V_1 + V_2$$

$$V_1 = -m_1 y_1 g = -m_1 l_1 g \cos(\theta_1)$$

$$V_2 = -m_2 y_2 g = -m_2 g(l_1 \cos(\theta_1) + l_2 \cos(\theta_2))$$

Dynamics

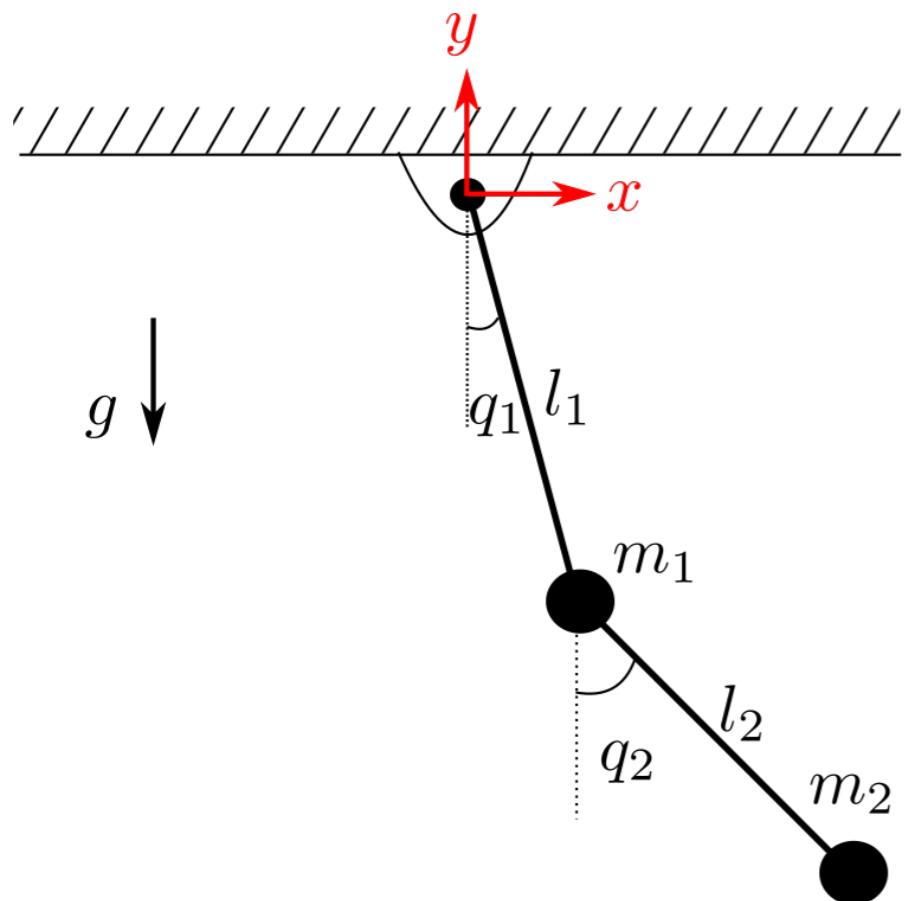


Kinetic Energy

$$\begin{aligned}T_1 &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) \\&= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2\end{aligned}$$

$$\begin{aligned}T_2 &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\&= \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 \\&\quad + l_1l_2m_2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)) \\&= \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + l_1l_2m_2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1 - \theta_2))\end{aligned}$$

Dynamics



Equation of Motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0$$



$$\frac{\partial T_1}{\partial \dot{\theta}_1}$$

$$\frac{\partial T_2}{\partial \dot{\theta}_1}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

Dynamics

$$\frac{\partial T_1}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1$$

$$\frac{\partial T_2}{\partial \dot{\theta}_1} = m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial T_1}{\partial \theta_1} = 0$$

$$\frac{\partial T_2}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial V_1}{\partial \theta_1} = m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial V_2}{\partial \theta_1} = m_2 g l_1 \sin(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = 0$$

Dynamics

$$\frac{\partial T_1}{\partial \dot{\theta}_2} = 0$$

$$\frac{\partial T_2}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial T_1}{\partial \theta_2} = 0$$

$$\frac{\partial T_2}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial V_1}{\partial \theta_2} = 0$$

$$\frac{\partial V_2}{\partial \theta_2} = m_2 g l_2 \sin(\theta_1)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$\begin{aligned} m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 & - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) & = 0 \end{aligned}$$

$$\begin{aligned} m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 & - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) & = 0 \end{aligned}$$

$$\tau = B(q)\ddot{q} + C(q, \dot{q}) + g(q)$$

Dynamics

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \\ - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) = 0$$

$$\tau = B(q) \ddot{q} + C(q, \dot{q}) + g(q)$$