Reinforcement Learning

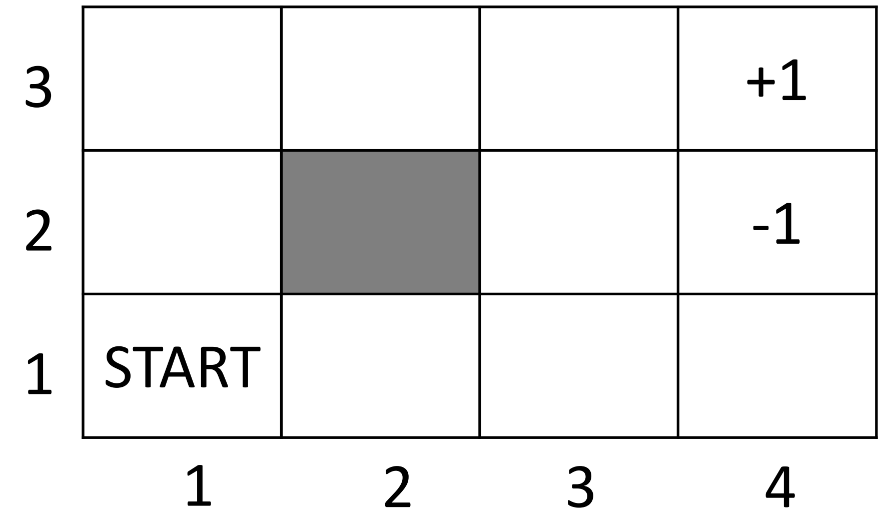
CSE 4309 – Machine Learning
Vassilis Athitsos
Computer Science and Engineering Department
University of Texas at Arlington

Markov Decision Processes (MDPs)

3				+1
2				-1
1	START			
	1	2	3	4

- In Markov Decision Processes, we made two assumptions:
- Assumption 1: we have a Markovian transition model, $p(s' \mid s, a)$.
  – This model gives us the probability of the next state being s', if the agent is at state s and takes action a.
- Assumption 2: **Discounted Additive Rewards**.
  – The utility U_h of a state sequence is: $U_h(s_0, s_1, \ldots, s_T) = \sum_{t=0}^{T} \gamma^t R(s_t)$
- In an MDP, both the transition model $p(s' \mid s, a)$ and the reward function $R(s)$ are <u>**known to the agent in advance**</u>.
- What we do not know, and we need to compute using specific algorithms, is the optimal policy and the utility of each state.

Reinforcement Learning



- Compared to MDPs, **<u>Reinforcement Learning</u>** (RL) is a more difficult version of sequential decision problems.
- As in MDPs, we want to compute:
  - The optimal policy (or, at least, a good policy).
  - The utility of each state
- However, in contrast to MDPs, in RL we **<u>do not know</u>**:
  - The transition model $p(s' \mid s, a)$.
  - The reward function $R(s)$.
- Figuring out what policy to follow is more complicated, with an unknown transition model and an unknown reward function.

Reinforcement Learning

	1	2	3	4
3	-0.04	-0.04	-0.04	+1
2	-0.04		-0.04	-1
1	-0.04	-0.04	-0.04	-0.04

- The only way the agent can learn anything in RL is by taking actions and observing results.
- The observed outcomes of an action are:
  - The resulting state, which can be used as data for estimating $p(s' \mid s, a)$.
  - The reward $R(s')$ at the resulting state s', which can simply be memorized.
- As the agent moves around, and observes state transitions and rewards, the agent can learn about the environment, by:
  - Memorizing the reward $R(s)$ at each state that the agent visits.
  - Learning a model of state transitions $p(s' \mid s, a)$.
  - Estimating the utility $U(s)$ of each state.
  - Estimating a good (or optimal) policy to follow.

Reinforcement Learning

3	-0.04	-0.04	-0.04	+1
2	-0.04		-0.04	-1
1	-0.04	-0.04	-0.04	-0.04
	1	2	3	4

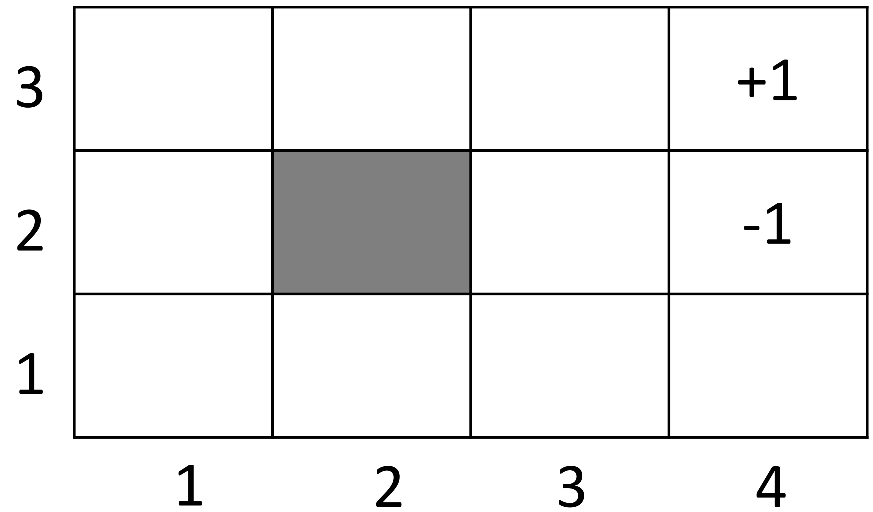
- For example:
  - The agent is at state $(1{,}3)$.
  - The agent takes action "go up".
  - The result of the action is to actually move left, to state $(1{,}2)$.
  - The agent gets a reward of -0.04 for visiting state $(1{,}2)$.
- Based on that experience, the agent learns the following:
  - $R(1{,}2) = -0.04$. This can be memorized, it will never change.
  - The result of action "go up" from state $(1{,}3)$, in a single attempt, was state $(1{,}2)$.

Reinforcement Learning

3	-0.04	-0.04	-0.04	+1
2	-0.04		-0.04	-1
1	-0.04	-0.04	-0.04	-0.04
	1	2	3	4

- How can the agent estimate probability $p\big((1,2) \mid (1,3), \text{"go up"}\big)$ from this experience?
- Frequentist approach: estimated $p\big((1,2) \mid (1,3), \text{"go up"}\big) = 1$.
  - Too aggressive!
  - It can be updated to more accurate estimates in the future, whenever the agent revisits state $(1,3)$ again and tries action "go up" again.
- Bayesian approach: start with some priors on the probability that $p\big((1,2) \mid (1,3), \text{"go up"}\big) = \theta$, and update the estimate after each observation.
  - Similar to the sunrise example we saw at the beginning of the course.

Reinforcement Learning



- Some times, rewards are only observed when the agent gets to specific states.
- Without loss of generality, we can assume that the rest of the states have zero rewards.
- Example: chess.
  - The agent plays games of chess, against itself or against others.
  - During each game, the agent makes different moves, without feedback as to whether each move was good or bad.
  - At some point, the game is over, and the reward is +1 for winning, 0 for a tie, -1 for losing.

Reinforcement Learning vs. Supervised Learning

- Learning to play chess can be approached as a reinforcement learning problem, or as a supervised learning problem.
- Reinforcement learning approach:
  - The agent plays games of chess, against itself or against others.
  - During each game, the agent makes different moves, without feedback as to whether each move was good or bad.
  - At some point, the game is over, and the reward is +1 for winning, 0 for a tie, -1 for losing.
- Supervised learning approach:
  - The agent plays games of chess, against itself or against others.
  - During each game, the agent makes different moves.
  - For every move, an expert provides an evaluation of that move.
- Pros and cons of each approach?

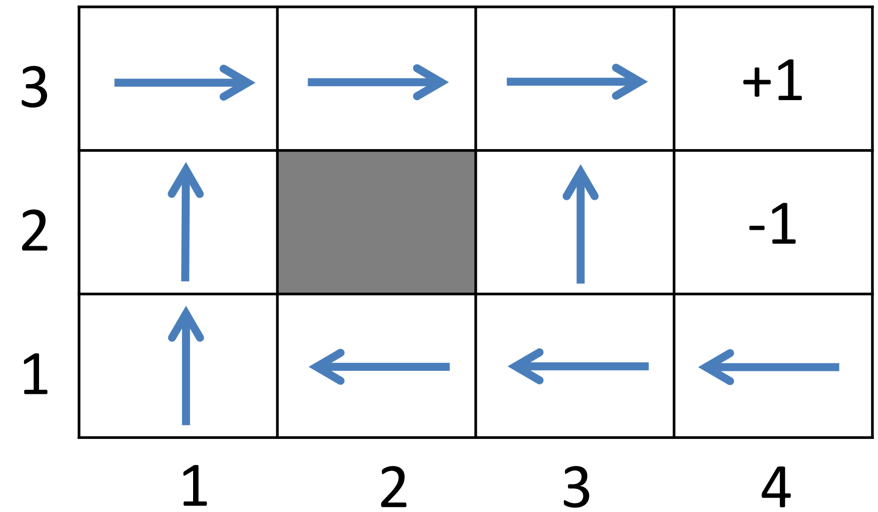
Reinforcement Learning vs. Supervised Learning

- In the chess example, the big advantage of reinforcement learning is that no effort is required from human experts to evaluate moves.
  - Lots of training data can be generated by having the agent play against itself or other artificial agents.
  - No human time is spent.
- Supervised learning requires significant human effort.
  - If that effort can be spared, supervised learning has more information, and thus should learn a better strategy.
  - However, in many sequential decision problems, the state space is so large, that it is infeasible for humans to evaluate a sufficiently large number of states.

Passive and Active RL

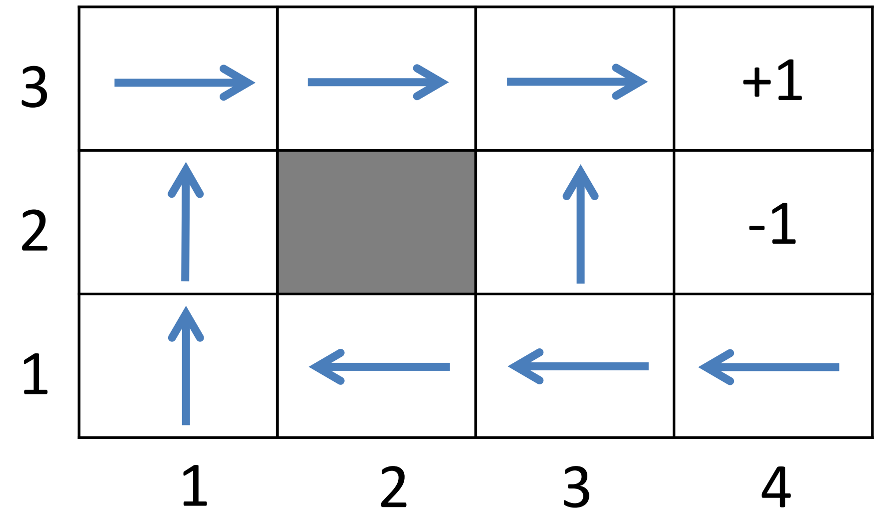
- In typical RL problems, the agent proceeds step-by-step, where every step involves:
  - Deciding what action to take, based on its current policy.
  - Taking the action, observing the outcome, and modifying accordingly its current policy.
- This problem is called **active reinforcement learning**, and we will look at some methods for solving it.
- However, first, we will study an easier RL problem, called **passive reinforcement learning**.
- In this easier version:
  - The policy is fixed.
  - The transition model and reward function are still unknown.
  - The goal is simply to compute the utility value of each state.

Passive Reinforcement Learning



- In the figure we see an example of a specific environment and a policy.
  - Suppose that $R(s) = -0.04$ if s is non-terminal.
  - Suppose that $\gamma = 1$.
  - Suppose that the transition model $p(s' \mid s, a)$ is the same as before:
    - 80% chance of moving in the direction the agent wanted.
    - 20% chance of moving perpendicular to the intended direction.
    - If the agent hits a wall, the agent remains in the current state.

Passive Reinforcement Learning



- This problem has similarities with the policy evaluation problem in MDPs.
- The main differences:
  - Here, the reward function $R(s)$ is not known in advance.
  - Here, the transition model $p(s' \mid s, a)$ is not known in advance.
- The **adaptive dynamic programming** (ADP) algorithm address these issues, by doing the following at each step:
  - Update $R(s)$ and $p(s' \mid s, a)$ based on current observations (reward of current state, result of last action).
  - Using the updated $R(s)$ and $p(s' \mid s, a)$, run the **PolicyEvaluation** function to estimate the utility of each state.

Adaptive Dynamic Programming

- The ADP algorithm takes many arguments.
- There are **input** arguments, whose value is fixed:
  - (s, a): the previous state s and the previous action a that led from s to s'.
  - (s', r'): the current state s', and the reward r' received at state s'.
  - π: the policy mapping each state s to an action $\pi(s)$.
  - γ: the discount factor.
  - K: parameter, that is passed to **PolicyEvaluation**, and specifies how many rounds of updates to perform in that function.
- There are also **input/output** arguments, whose values can change.
  - $\mathbb{S}$: the set of states, which **is not known in advance**.
  - R, U: the reward and utility tables, storing rewards and utilities of states.
  - p: a 3D table of transitions, storing values $p(s' \mid s, a)$.
  - N_{sa}: a table, where $N_{sa}[s, a]$ counts all times a was performed from state s.
  - N_{sat}: a table, where $N_{sat}[s, a, t]$ counts all times a led from s to t.

Adaptive Dynamic Programming

- **Input** arguments:
  - (s, a): previous state and action.
  - (s', r'): current state and reward.
  - π: the policy.
  - γ: the discount factor.
  - K: used in **PolicyEvaluation**.
- **Input/output** arguments:
  - $\mathbb{S}$: the set of states.
  - R, U : reward and utility tables.
  - p: table of values $p(s' \mid s, a)$.
  - N_{sa}: 2D table, counts times action a was taken in state s.
  - N_{sat}: 3D table, counts number of times a led from s to t.

- The task of the ADP algorithm is to process a single step:
  - Process the result of the last action a, which led from state s to state s'.
    - Update values $p(t \mid s, a)$.
  - Record the observed reward at state s'.
  - Call **PolicyEvaluation** to update the utility table.
- The agent runs the ADP algorithm **after each move**.

ADP Pseudocode

function ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if this is the first time we visit s':

$\mathrm{R}[\mathrm{s}'] = \mathrm{r}'$

$\mathrm{U}[\mathrm{s}'] = \mathrm{r}'$

$\mathbb{S} = \mathbb{S} \cup \{s'\}$ // add s' to set of known states.

if s is not **null**: // s is null if no states have been visited before

if $N_{sa}[s, a]$ exists: $N_{sa}[s, a] \mathrel{+}= 1$ **else:** $N_{sa}[s, a] = 1$

if $N_{sat}[s, a, t]$ exists: $N_{sat}[s, a, t] \mathrel{+}= 1$ **else:** $N_{sat}[s, a, t] = 1$

for each state t such that $N_{sat}[s, a, t] \neq 0$:

$$p(t \mid s, a) = \frac{N_{sat}[s,a,t]}{N_{sa}[s,a]}$$

$\mathrm{U} =$ **PolicyEvaluation**$(\mathbb{S}, \mathrm{p}, \mathrm{R}, \gamma, \pi, K, \mathrm{U})$

ADP Pseudocode

Step 1: If we never saw s' before, add s' to set of states, and to tables R and U.

function ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if this is the first time we visit s':

$\mathrm{R[s'] = r'}$

$\mathrm{U[s'] = r'}$

$\mathbb{S} = \mathbb{S} \cup \{s'\}$ // add s' to set of known states.

if s is not **null**: // s is null if no states have been visited before

if $N_{sa}[s, a]$ exists: $N_{sa}[s, a] \mathrel{+}= 1$ **else:** $N_{sa}[s, a] = 1$

if $N_{sat}[s, a, t]$ exists: $N_{sat}[s, a, t] \mathrel{+}= 1$ **else:** $N_{sat}[s, a, t] = 1$

for each state t such that $N_{sat}[s, a, t] \neq 0$:

$$p(t \mid s, a) = \frac{N_{sat}[s,a,t]}{N_{sa}[s,a]}$$

$\mathrm{U} =$ **PolicyEvaluation**$(\mathbb{S}, \mathrm{p}, \mathrm{R}, \gamma, \pi, K, \mathrm{U})$

ADP Pseudocode

Step 2: update transition model $p(t \mid s, a)$.

function ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if this is the first time we visit s':

$\mathrm{R}[\mathrm{s}'] = \mathrm{r}'$

$\mathrm{U}[\mathrm{s}'] = \mathrm{r}'$

$\mathbb{S} = \mathbb{S} \cup \{s'\}$ // add s' to set of known states.

if s is not **null**: // s is null if no states have been visited before

if $N_{sa}[s, a]$ exists: $N_{sa}[s, a] \mathrel{+}= 1$ **else:** $N_{sa}[s, a] = 1$

if $N_{sat}[s, a, t]$ exists: $N_{sat}[s, a, t] \mathrel{+}= 1$ **else:** $N_{sat}[s, a, t] = 1$

for each state t such that $N_{sat}[s, a, t] \neq 0$:

$$p(t \mid s, a) = \frac{N_{sat}[s,a,t]}{N_{sa}[s,a]}$$

$\mathrm{U} =$ **PolicyEvaluation**$(\mathbb{S}, \mathrm{p}, \mathrm{R}, \gamma, \pi, K, \mathrm{U})$

ADP Pseudocode

Step 3: Call **PolicyEvaluation** to update utility table U.

function ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if this is the first time we visit s':

$\mathrm{R}[\mathrm{s}'] = \mathrm{r}'$

$\mathrm{U}[\mathrm{s}'] = \mathrm{r}'$

$\mathbb{S} = \mathbb{S} \cup \{s'\}$ // add s' to set of known states.

if s is not **null**: // s is null if no states have been visited before

if $N_{sa}[s, a]$ exists: $N_{sa}[s, a] \mathrel{+}= 1$ **else:** $N_{sa}[s, a] = 1$

if $N_{sat}[s, a, t]$ exists: $N_{sat}[s, a, t] \mathrel{+}= 1$ **else:** $N_{sat}[s, a, t] = 1$

for each state t such that $N_{sat}[s, a, t] \neq 0$:

$$p(t \mid s, a) = \frac{N_{sat}[s,a,t]}{N_{sa}[s,a]}$$

$\mathrm{U} =$ **PolicyEvaluation**$(\mathbb{S}, \mathrm{p}, \mathrm{R}, \gamma, \pi, K, \mathrm{U})$

The Policy Evaluation Function

- We have seen this function before, as a module in the Policy Iteration algorithm for Markov Decision Processes.
- Approach: iteratively update the utility of each state based on the utilities of the neighboring states.
  - K rounds of updates. Each round updates once the utility of each state.
- Parameter K allows us to trade speed for accuracy.

function PolicyEvaluation$(\mathbb{S}, \mathrm{p}, \mathrm{R}, \gamma, \pi_i, K, \mathrm{U})$
    $\mathrm{U}_0 =$ copy of U
    for $\boldsymbol{k = 1}$ **to** $\boldsymbol{K}$**:**
        for each state s in $\mathbb{S}$:
            $U_k(s) = R(s) + \gamma \sum_{s'} [p(s' \mid s, \pi_i(s)) U_{k-1}(s')]$
    return U_k

ADP Implementation Notes

- The algorithm uses several tables:
  - $\mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat}$.
- In MDPs, we could implement these tables as arrays, because we knew the dimensions. For example:
  - The size of R and U was the number of states N.
  - The size of p was $N \times N \times \text{number\_of\_actions}$.
- In the ADP algorithm, **we do not know the number of states in advance**.
  - Thus, tables $\mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat}$ should be stored using data structures that allow new elements to be added, like:
    - Dictionaries in Python.
    - HashMaps in Java.

ADP Implementation Notes

- For the state transition model p:
  - In the pseudocode, we denote entries of p as $p(t \mid s, a)$.
  - In an actual implementation, p can just be a three-dimensional hash map (or any other appropriate structure) mapping values t, s, a to the current estimate of $p(t \mid s, a)$.

ADP Implementation Notes

if $N_{sa}[s,a]$ exists: $N_{sa}[s,a] \mathrel{+}= 1$ **else:** $N_{sa}[s,a] = 1$

if $N_{sat}[s,a,t]$ exists: $N_{sat}[s,a,t] \mathrel{+}= 1$ **else:** $N_{sat}[s,a,t] = 1$

- When updating entries $N_{sa}[s,a]$ and $N_{sat}[s,a,t]$, we need to check if those entries already exist.
  - If this is the first time we take action a from state s, then $N_{sa}[s,a]$ has not been initialized yet.
  - If this is the first time that state t was the outcome of action a from state s, then $N_{sat}[s,a,t]$ has not been initialized yet.
  - If any of those entries has not been initialized, we initialize it to a value of 1.
  - If any of those entries has been initialized, we increment it.

End-to-End Estimation of Utilities

- ADP processes a single move of the agent at a time.
- How would we model an agent's behavior over time, if the agent uses ADP after each move?
- **Input**: values π, γ, K must be specified in advance.
- **Initialization**: initialize all variables appropriately.
- **Main loop** (running for ever):
  – Start from a legal initial state.
  – Repeat:
    - Use sensors to sense the current state and current reward.
    - Call **ADP** to update variables appropriately.
    - If the current state is non-terminal, pick and execute the next action.
  – Until the current state is a terminal state.

Agent Model According to ADP

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variable s to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**()

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Agent Model According to ADP

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variable s to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

First, we initialize all variables appropriately.

Agent Model According to ADP

Main loop: Execute mission after mission, forever.

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variable s to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Agent Model According to ADP

To start the mission, move to a legal initial state.

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variable s to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Agent Model According to ADP

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variable s to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

The inner loop processes a single mission, from beginning to end.

Agent Model According to ADP

function AgentModelADP(π, γ, K)

Use the sensors to sense current state and reward.

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variable s to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Agent Model According to ADP

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variables s, a to **null** **// no previous state**

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while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Call **ADP** to update the model.

Agent Model According to ADP

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variables s, a to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

If we reached a terminal state, we are done with this mission.

Agent Model According to ADP

Pick and execute the next action.

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

Initialize $\mathrm{U}, p, N_{sa}, N_{sat}$ to empty tables

while (true): **// Main loop: Execute mission after mission**

Initialize variables s, a to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**()

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Agent Model According to ADP

function AgentModelADP(π, γ, K)

// Initialization

Initialize $\mathbb{S}$ to an empty set

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(s', r') = **SenseStateAndReward**$()$

ADP$(s, a, s', r', \pi, \gamma, K, \mathbb{S}, \mathrm{R}, \mathrm{U}, p, N_{sa}, N_{sat})$

if s' is terminal: **break** **// Done with this mission, move on to next**

$a = \pi(s')$

ExecuteAction(a)

$s = s'$

Store in s the value of the previous state.

Agent Model According to ADP

- The **AgentModelADP** algorithm models an agent that behaves according to a **fixed** policy π, and that updates its utility estimates using **ADP**.
- The agent interacts with the environment using two functions:
  - **SenseStateAndReward**(), for getting the current state and reward from sensors.
  - **ExecuteAction**(a), for executing a specific action.
- These functions can be simulated, if we want to simply test the algorithm in software (for example, for a homework assignment).

Temporal-Difference Learning

- **Temporal-Difference Learning (TDL)** is an alternative to ADP for solving the passive reinforcement learning problem.
- Key difference: complicated vs. simple update at each step.
- ADP does a more complicated update, where the **PolicyEvaluation** function is called to update the utilities of all known states.
- TDL does a very simple update, it only changes the utility value of the previous state.

Temporal-Difference Learning

- The TDL algorithm takes mostly the same arguments as ADP.
- There are **input** arguments, whose value is fixed:
  - (s, r, a): the previous state s, the reward r obtained at state s, the previous action a that led from s to s'.
  - (s', r'): the current state s', and the reward r' received at state s'.
  - π: the policy mapping each state s to an action $\pi(s)$.
  - γ: the discount factor.
  - $\boldsymbol{\eta}$: a **<u>function</u>**, specifying **a learning rate that decreases over time**.
- There are also **input/output** arguments, whose values can change.
  - R, U : the reward and utility tables, storing rewards and utilities of states.
  - N_s: a table, where $N_s[s]$ counts all times state s has been visited.

TDL Pseudocode

function TDL$(s, r, a, s', r', \pi, \gamma, \eta, \mathrm{R}, \mathrm{U}, N_s)$

  if this is the first time we visit s':

    $\mathrm{R[s']} = \mathrm{r'}$

    $\mathrm{U[s']} = \mathrm{r'}$

  if s is not **null**: // s is null if no states have been visited before

    if $N_s[s]$ exists: $N_s[s] \mathrel{+}= 1$ **else:** $N_s[s] = 1$

    $c = \eta(N_s[s])$ // we call function η with input $N_s[s]$

    $\mathrm{U}[s] = (1 - c)\mathrm{U[s]} + c(\mathrm{R}[s] + \gamma \mathrm{U[s']})$

- Step 1:
  - If we never saw s' before, add values for s' to tables R and U.

TDL Pseudocode

function TDL$(s, r, a, s', r', \pi, \gamma, \eta, \mathrm{R}, \mathrm{U}, N_s)$

 if this is the first time we visit s':

  $\mathrm{R[s'] = r'}$

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  $\mathrm{U}[s] = (1 - c)\mathrm{U[s]} + c(\mathrm{R}[s] + \gamma \mathrm{U[s']})$

- Step 2:
  - Increment $N_s[s]$ to indicate that we have visited state s one more time.

TDL Pseudocode

function TDL$(s, r, a, s', r', \pi, \gamma, \eta, \mathrm{R}, \mathrm{U}, N_s)$
    if this is the first time we visit s':
        $\mathrm{R[s'] = r'}$
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    if s is not **null**:   // s is null if no states have been visited before
        if $N_s[s]$ exists: $N_s[s] \mathrel{+}= 1$ **else:** $N_s[s] = 1$
        $c = \eta(N_s[s])$ // we call function η with input $N_s[s]$
        $\mathrm{U}[s] = (1 - c)\mathrm{U[s]} + c(\mathrm{R}[s] + \gamma \mathrm{U[s']})$

- Step 3:
  - Compute the learning rate to be used at the current step.
  - **Do not make the mistake to read** $\eta(N_s[s])$ **as multiplication.**
  - $\eta(N_s[s])$ is a function call. We call function η, with argument $N_s[s]$.
  - Function η must be chosen appropriately to guarantee that **TDL** updates converge eventually to the right utility values.

TDL Pseudocode

function TDL$(s, r, a, s', r', \pi, \gamma, \eta, \mathrm{R}, \mathrm{U}, N_s)$
    if this is the first time we visit s':
        $\mathrm{R[s'] = r'}$
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        if $N_s[s]$ exists: $N_s[s] \mathrel{+}= 1$ **else:** $N_s[s] = 1$
        $c = \eta(N_s[s])$ // we call function η with input $N_s[s]$
        $\mathrm{U}[s] = (1 - c)\mathrm{U[s]} + c(\mathrm{R}[s] + \gamma \mathrm{U[s']})$

- Step 3:
  - We will not prove it, but it is sufficient if function η decays as fast as $\Theta\left(\frac{1}{n}\right)$.
  - For the 3×4 grid example, the Russell & Norvig textbook defines η as:

$$\eta(n) = \frac{60}{59 + n}$$

TDL Pseudocode

function TDL$(s, r, a, s', r', \pi, \gamma, \eta, \mathrm{R}, \mathrm{U}, N_s)$

    if this is the first time we visit s':

        $\mathrm{R}[\mathrm{s}'] = \mathrm{r}'$

        $\mathrm{U}[\mathrm{s}'] = \mathrm{r}'$

    if s is not **null**:   // s is null if no states have been visited before

        if $N_s[s]$ exists: $N_s[s] \mathrel{+}= 1$ **else:** $N_s[s] = 1$

        $c = \eta(N_s[s])$ // we call function η with input $N_s[s]$

        $\mathrm{U}[s] = (1 - c)\mathrm{U}[\mathrm{s}] + c(\mathrm{R}[s] + \gamma \mathrm{U}[\mathrm{s}'])$

- Step 4: Adjust $\mathrm{U}[s]$, by taking a **weighted average** of:
  - The previous estimate for $\mathrm{U}[s]$, with weight $(1 - \mathrm{c}) = (1 - \eta(N_s[s]))$.
  - Estimate $\mathrm{R}[s] + \gamma \mathrm{U}[\mathrm{s}']$, which would be true if **the successor of *s* were always *s*′**, with weight $\mathrm{c} = \eta(N_s[s])$.
- Even though this update is very simple, values $\mathrm{U}[s]$ converge (over multiple calls to **TDL**) to the correct value.

A Closer Look at the TDL Update

$$\mathrm{U}[s] = (1 - c)\mathrm{U}[\mathrm{s}] + c(\mathrm{R}[s] + \gamma \mathrm{U}[\mathrm{s}'])$$

- The TDL update equation is shown above.
  - s is the previous state.
  - s' is the current state, produced by performing some action a.
- Sometimes, unlikely transitions occur.
  - It may be that s' was a low-probability outcome for action a and state s.
- In that case, $\mathrm{U}[s]$ may become less accurate after the update.
  - If this is the first time that the agent visited state s, then the $\mathrm{U}[s]$ value may end up being far from the correct value.
- However: as long as the learning rate decreases appropriately over time, $\mathrm{U}[s]$ will converge to the right value.
  - Unlikely transitions happen rarely, so, over time, they affect the $\mathrm{U}[s]$ value only a small fraction of the times.

Agent Model According to TDL

End-to-end model of an agent that uses TDL.

Similar in logic to **AgentModelADP**.

function AgentModelTDL(π, γ, η)

// **Initialization**

Initialize $\mathrm{R}, \mathrm{U}, N_s$ to empty tables

while (true): // **Main loop**

Initialize variables s, r, a to **null**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): // **Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

TDL$(s, r, a, s', r', \pi, \gamma, \eta, \mathrm{R}, \mathrm{U}, N_s)$

if s' is terminal: **break** // **Done with this mission**

$a = \pi(s')$

ExecuteAction(a)

$(s, r) = (s', \mathrm{r}')$

ADP vs. TDL

- ADP spends more time and effort in its updates.
  - It calls **PolicyEvaluation** to update utilities as much as possible using the new information.
- TDL does a rather minimal update.
  - It just updates the utility of the previous state, taking a weighted average of:
    - the previous estimate for the utility of the previous state
    - the estimated utility that was obtained as a result of the last action.
- The pros and cons are rather obvious:
  - ADP takes more time to process a single step, and estimates converge after fewer steps.
  - TDL is faster to execute for a single step, but estimates need more steps to converge, compared to ADP.

Active Reinforcement Learning

- Active Reinforcement Learning is the problem of actually figuring out what to do.
  - The policy is not given.
  - Rewards are not known in advance.
  - The transition model is not known in advance.
- MDPs and Passive Reinforcement Learning solved easier problems.
  - In MDPs, rewards and transitions are known.
  - In passive reinforcement learning, the policy is given and the agent just wants to estimate the utility of each state.

A Greedy Approach

- Main loop: at each step
  - Use sensors to sense the current state s' and current reward r'.
  - Call **ADP** to update the utility estimates $\mathrm{U}[s]$.
  - Choose the next action using the same equation that we used in MDPs:

$$a = \underset{a \in A(s')}{\operatorname{argmax}} \left\{ \sum_t \left[p(t | s', a) U[t] \right] \right\}$$

- The key idea is that, at each step, the agent picks the best action it can find, according to its current model.
  - Since the current model may not be correct, the action is not necessarily the truly best action.
  - However, the model keeps getting updated at each step.
- Does this approach eventually converge to the optimal policy?

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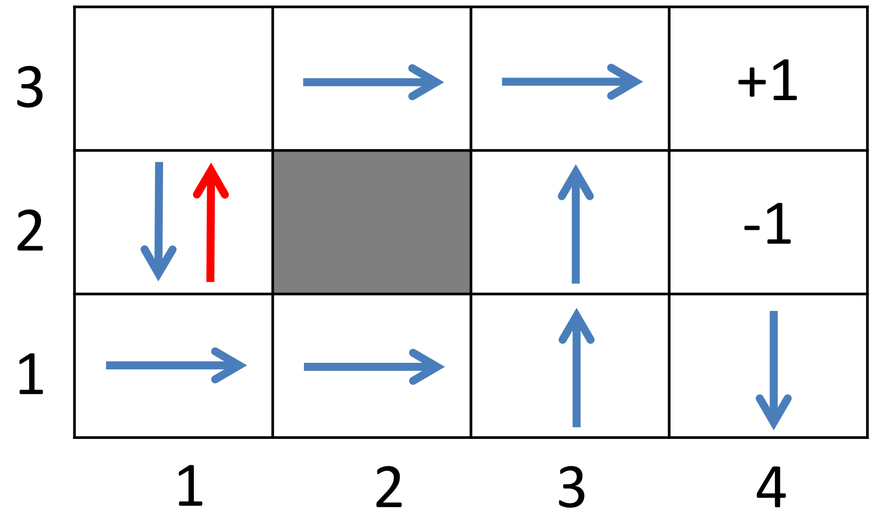
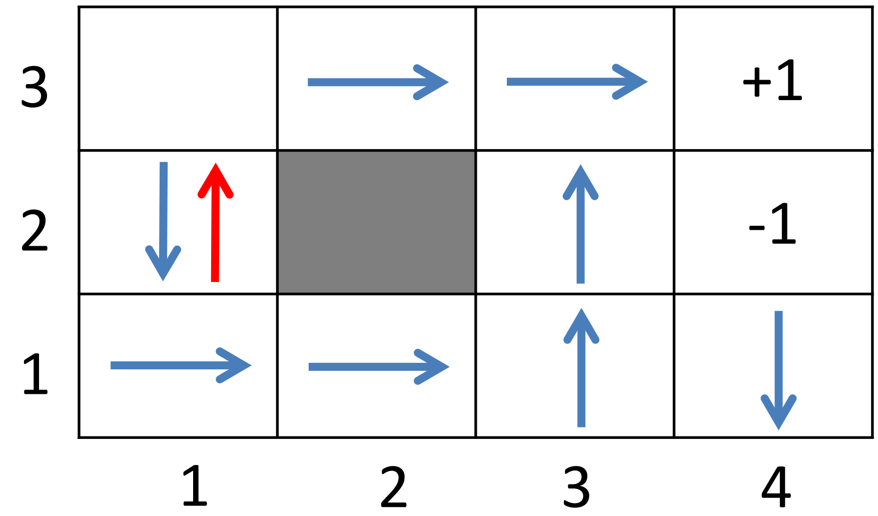
- The key idea is that, at each step, the agent picks the best action it can find, according to its current model.
  - Since the current model may not be correct, the action is not necessarily the truly best action.
  - However, the model keeps getting updated at each step.
- Does this approach eventually converge to the optimal policy?
  - Unfortunately, no.

Problem with the Greedy Approach



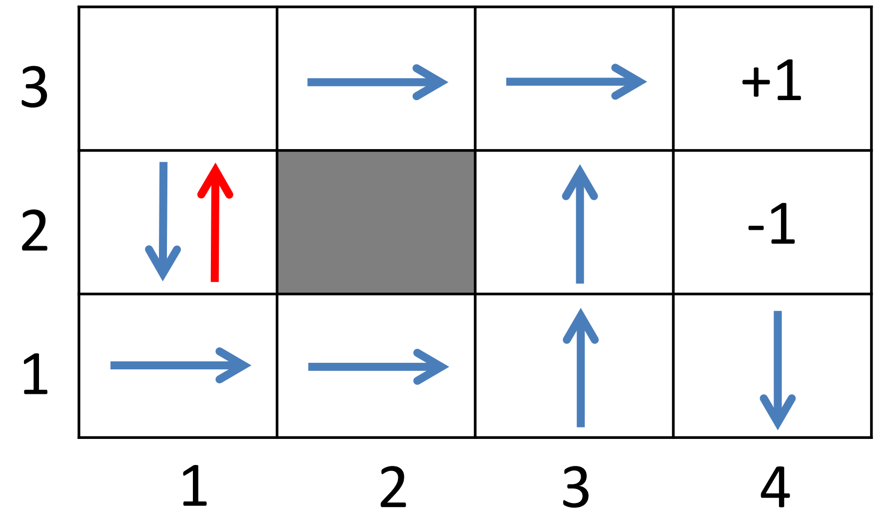
- This example illustrates the problem with the previous approach.
- In state $(2,1)$, according to the current (inaccurate) model, the optimal action is "go down".
  - State $(3,1)$ has never been visited, so its estimated utility $\mathrm{U}[(3,1)]$ is 0.
- The true optimal action is "go up" (shown with the red arrow).
- However, the agent will never visit state $(3,1)$, because none if the actions in the current policy can possibly lead to state $(3,1)$.
- As a result, estimated utility $\mathrm{U}[(3,1)]$ will never be updated.
- Thus, the agent will never figure out that "go up" is the best choice for state $(2,1)$.

Exploration and Exploitation



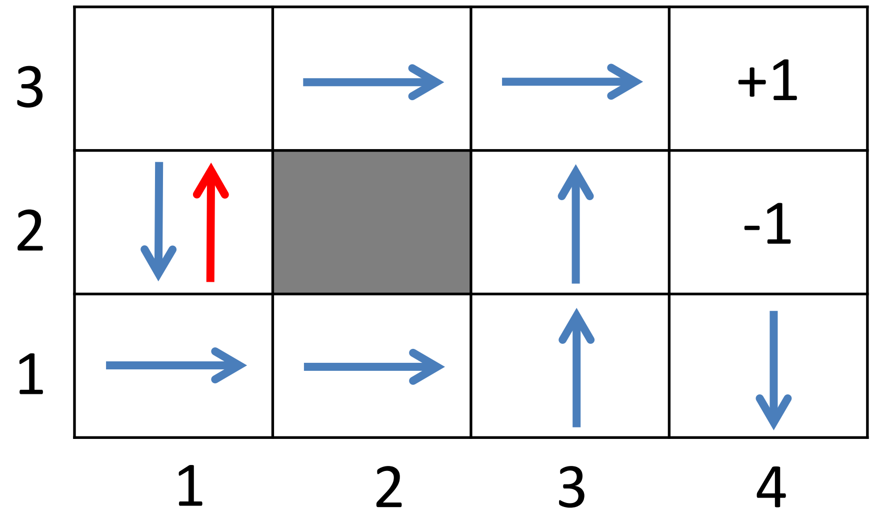
- The greedy approach, where the agent always chooses what seems to be the best action, that approach is called **exploitation**.
  - In that case, the agent may never figure out what the best action is.
- The only way to solve this problem is to allow some **exploration**.
  - Every now and then, the agent should take actions that, according to its current model, are not the best actions to take.
  - This way the agent can, eventually, identify better choices that it was not aware of at first.

How Much Exploration?



- Allowing zero exploration makes it possible that the agent will never figure out an optimal (or a good) policy.
- Doing too much exploration means that the agent, while learning a lot, keeps making bad choices and accumulating poor rewards.
- One approach is to choose, at each step t, a random action with probability $\frac{1}{t}$, so that this probability decreases over time.
- This way, in our example, action "go up" will eventually be chosen from state $(2,1)$, state $(3,1)$ will eventually be visited, and the policy will eventually converge to the optimal policy.

How Much Exploration?



- Allowing zero exploration makes it possible that the agent will never figure out an optimal (or a good) policy.
- Doing too much exploration means that the agent, while learning a lot, keeps making bad choices and accumulating poor rewards.
- One approach is to choose, at each step t, a random action with probability $\frac{1}{t}$, so that this probability decreases over time.
- If we follow this approach, it can be proven (we skip the proof) that the agent's policy eventually converges to the optimal policy.
- However, convergence is relatively slow.

Encouraging Exploration

- Another approach is to artificially bump up the utility values of states that have not been visited much.
- We denote these "bumped-up" utilities as $U^{+}[s]$:

$$U^{+}(s) = R(s) + \gamma \max_{a \in A(s)} f\left(\sum_{s'} [p(s' | s, a) U^{+}(s')], N_{sa}[s, a] \right)$$

- In the above equation, note the use of function $f(u, n)$:
  - u is the estimated utility value of taking action a at state s.
  - n is the number of times the agent has taken action a at state s.
- The job of f is to "bump up" utility values of (state, action) pairs that have not been tried many times.

Encouraging Exploration

$$U^+(s) = R(s) + \gamma \max_{a \in A(s)} f\left(\sum_{s'} [p(s' | s, a) U^+(s')], N_{sa}[s, a]\right)$$

- Function f can be defined in different ways. For example:

$$f(u, n) = \begin{cases} R^+ & \text{if } n < N_e \\ u & \text{otherwise} \end{cases}$$

- In the above definition:
  - R^+ is the maximum possible reward in the environment.
  - N_e is a hard-coded paramter, provided as input to the algorithm.
- With this definition:
  - If the agent has not tried action a from state s many times, then $N_{sa}[s, a] < N_e$, and function f returns R^+ which is the max possible reward.
  - Otherwise, function f returns the more realistic utility estimate u.

Q-Learning

- Q-Learning is a method for active reinforcement learning.
- It is based on temporal-difference learning.
- Q-Learning learns a function $Q(s,a)$, which is defined as the utility obtained by performing action a from state s:

$$\mathrm{Q}(s,a) = R(s) + \gamma \sum_{s'} [p(s' |\, s,a) U(s')]$$

Q-Learning

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- Q-Learning learns a function $Q(s,a)$, which is defined as the utility obtained by performing action a from state s:

$$\mathrm{Q}(s,a) = \boxed{R(s)} + \gamma \boxed{\sum_{s'} [p(s' | s,a) U(s')]}$$

Reward at state s

Expected utility of outcomes of action a

Q-Learning

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- It is based on temporal-difference learning.
- Q-Learning learns a function $Q(s, a)$, which is defined as the utility obtained by performing action a from state s:

$$\mathrm{Q}(s, a) = R(s) + \gamma \sum_{s'} [p(s' | s, a) U(s')]$$

- Q-values are obviously related to state utilities:

$$U(s) = \max_{a} Q(s, a)$$

Q-Learning

- Definition of the Q function:

$$Q(s,a) = R(s) + \gamma \sum_{s'} [p(s' \mid s,a) U(s')]$$

- Relation between Q values and utilities of states:

$$U(s) = \max_a Q(s,a)$$

- By substituting $\max_{a'} Q(s',a')$ for $U(s')$, we get

$$Q(s,a) = R(s) + \gamma \sum_{s'} [p(s' \mid s,a) \max_{a'} Q(s',a')]$$

Q-Learning Update Step

- The Q-Learning method updates Q values after each step.
- There are **input** arguments, whose value is fixed:
  - (s, r, a): the previous state s, the reward r obtained at state s, the previous action a that led from s to s'.
  - (s', r'): the current state s', and the reward r' received at state s'.
  - γ: the discount factor.
  - $\boldsymbol{\eta}$: a **function**, specifying **a learning rate that decreases over time**.
- There are also **input/output** arguments, whose values can change.
  - Q : the table of Q-values, storing utilities of (state, action) pairs.
  - N_{sa}: a table, where $N_{sa}[s, a]$ counts all times that the agent was at state s AND chose action a as its next action.

Q-Learning Update: Pseudocode

function Q_Learning_Update$(s, r, a, s', r', \gamma, \eta, \mathrm{Q}, N_{sa})$
    if s' is a terminal state:
        $\mathrm{Q}[s', \mathrm{None}] = r'$
    if s is not **null**: **// s is null if no states have been visited before**
        if $N_{sa}[s, a]$ exists: $N_{sa}[s, a] \mathrel{+}= 1$
        else: $N_{sa}[s, a] = 1$
        $c = \eta(N_{sa}[s, a])$ // we call function η with input $N_{sa}[s, a]$
        $\mathrm{Q}[s, a] = (1 - c)\mathrm{Q}[s, a] + c\left(r + \gamma \max_{a'} Q(s', a')\right)$

- The update step, similar to TD-Learning, computes the weighted average of:
  - The previous estimate for $\mathrm{Q}[s, a]$.
  - $r + \gamma \max_{a'} Q(s', a')$, which is the expected utility given that the successor of s is s'.

Agent Model for Q-Learning

function AgentModel_Q_Learning(γ, η)

// Initialization

Initialize Q, N_{sa} to empty tables,

while (true) **// Main loop: Execute mission after mission**

Initialize variables s, r, a to **null** **// no previous state**

s' = initial state (chosen randomly, if multiple initial states exist)

while (true): **// Execute one mission, from start to end**

(s', r') = **SenseStateAndReward**$()$

Q_Learning_Update$(s, r, a, s', r', \gamma, \eta, \mathrm{Q}, N_{sa})$

if s' is terminal: **break** **// Done with this mission**

$$a = \operatorname*{argmax}_{a' \in A(s')} f(Q[s', a'], N_{sa}[s', a'])$$

ExecuteAction(a)

$(s, r) = (s', r')$

The line in red makes this version of RL **active** as opposed to **passive**.

Choosing Actions with Q-Learning

- In the **AgentModelADP** and **AgentModelTDL** pseudocode, the next action was chosen using this line:

$$a = \pi(s')$$

  - In those cases, the next action is provided by a **fixed** policy π.
- In the **AgentModel_Q_Learning** pseudocode, the action is chosen using this line:

$$a = \underset{a' \in A(s')}{\operatorname{argmax}} f(Q[s', a'], N_{sa}[s', a'])$$

  - The next action is chosen taking into account the updated $Q[s', a']$ values.
  - Thus, the policy that the agent follows is not fixed. As values $Q[s', a']$ change, the winning action for each state can also change.

Choosing Actions with Q-Learning

- In the **AgentModel_Q_Learning** pseudocode, the action is chosen using this line:

$$a = \underset{a' \in A(s')}{\operatorname{argmax}} f(Q[s', a'], N_{sa}[s', a'])$$

- For the next action, we don't just choose the action that maximizes the estimated utility $Q[s', a']$.
  – Doing that would allow for zero exploration.
- Instead, we choose the action that maximizes $f(Q[s', a'], N_{sa}[s', a'])$.
- As we saw before, the role of function f is to encourage exploration, by giving higher values to (s', a') pairs that have not been explored a sufficient number of times.

Generalization in RL

- The Q-Learning method is a solution to the active reinforcement learning.
- The complexity of that solution is at least linear to the number of states.
  - We cannot expect utility estimates to be accurate for most states, unless the agent visits most states.
- In some cases, linear time complexity is prohibitive.
  - The number of states for backgammon is estimated at 10^{20}.
  - The number of states for chess is estimated at 10^{40}.
- We need a way to estimate utilities of states that have never been visited.

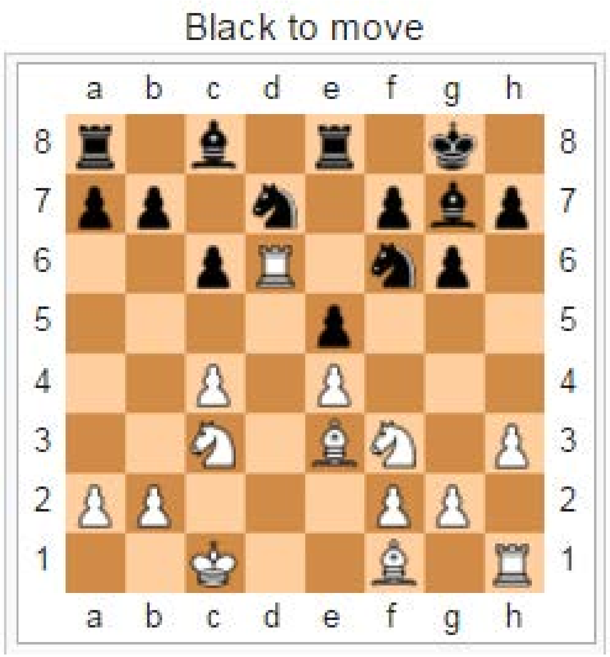
Function Approximation

- We can approximate a utility function, or a Q-function, using a parametric function.
- For example: we can approximate the Q-function using a function $\hat{Q}_\theta$ that is a linear combination of a set of features:

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

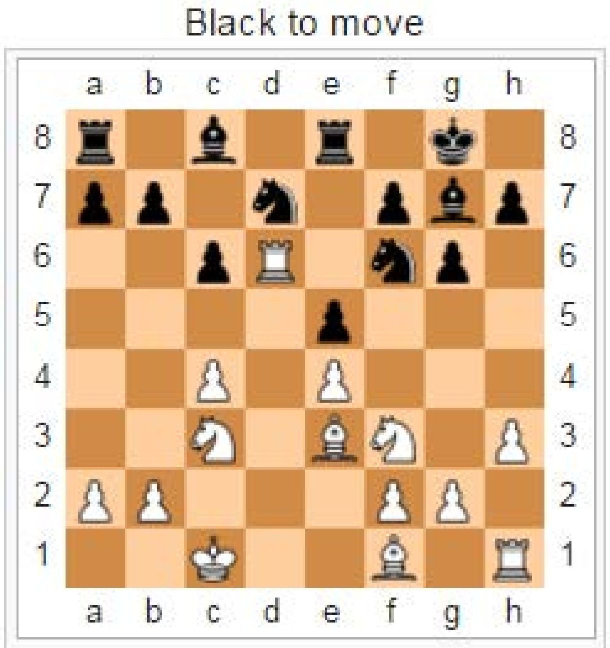
- The parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ is a vector of real numbers, that get optimized during learning.
- The f_i functions are basis functions, extracting features from a state-action pair.

Basis Functions for Chess



- As an example, we can look at one way to define basis functions for chess.
- A state s is defined by:
  - The position of pieces on the board.
  - A number indicating whose turn it is.
    - e.g., 1 for white, -1 for black.
- An action a is a move that can be played at state s.
  - We denote by $G(s, a)$ the resulting state if we play move a at state s.

Basis Functions for Chess



- We can define some basis functions $f_i(s, a)$ as follows:
  - $f_1(s, a) =$ number of white queens in s.
  - $f_2(s, a) =$ number of black queens in s.
  - $f_3(s, a) =$ number of white bishops in s.
  - $f_4(s, a) =$ number of black bishops in s.
  - $f_5(s, a) =$ number of white knights in s.
  - $f_6(s, a) =$ number of black knights in s.
  - $f_7(s, a) =$ number of white rooks in s.
  - $f_8(s, a) =$ number of black rooks in s.
  - $f_9(s, a) =$ number of white pawns in s.
  - $f_{10}(s, a) =$ number of black pawns in s.

Basis functions f_1 to f_{10} extract features from the state s, but tell us nothing about the action a.

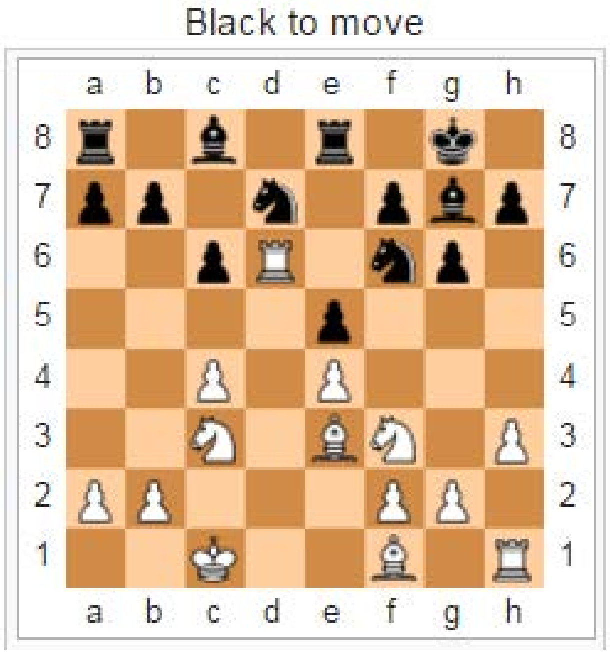
Basis Functions for Chess

- We can define some basis functions $f_i(s,a)$ as follows:
  - $f_{11}(s,a) =$ # of white queens in $\mathrm{G}(s,a)$.
  - $f_{12}(s,a) =$ # of black queens in $\mathrm{G}(s,a)$.
  - $f_{13}(s,a) =$ # of white bishops in $\mathrm{G}(s,a)$.
  - $f_{14}(s,a) =$ # of black bishops in $\mathrm{G}(s,a)$.
  - $f_{15}(s,a) =$ # of white knights in $\mathrm{G}(s,a)$.
  - $f_{16}(s,a) =$ # of black knights in $\mathrm{G}(s,a)$.
  - $f_{17}(s,a) =$ # of white rooks in $\mathrm{G}(s,a)$.
  - $f_{18}(s,a) =$ # of black rooks in $\mathrm{G}(s,a)$.
  - $f_{19}(s,a) =$ # of white pawns in $\mathrm{G}(s,a)$.
  - $f_{20}(s,a) =$ # of black pawns in $\mathrm{G}(s,a)$.

Black to move

Basis functions f_{11} to f_{20} extract features from state $G(s,a)$, which is the result of performing action a from state s.

Basis Functions for Chess



- The previous basis functions are far from exhaustive.
- We can define more basic functions, to capture other important aspects of a state, such as:
  - Which pieces are threatened.
  - Which pieces are protected by other pieces.
  - The number of legal moves available to each player.
  - …

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

- Once we have decided on the basis functions to use, we can use Q-Learning to learn the parameters θ_i.
- In the original version of Q-Learning, the update rule was:

$$\mathrm{Q}[s,a] = (1-c)\mathrm{Q}[s,a] + c\left(r + \gamma \max_{a'} Q(s',a')\right)$$

- Now that we are using parametric function $\hat{Q}_\theta$, we can optimize parameters θ_i using gradient descent:
  - Define an error function $E(s, a, \theta_1, \dots, \theta_n)$
  - Compute derivatives $\frac{\partial E}{\partial \theta_i}$.
  - Move each θ_i away from the direction of the gradient: $\theta_i = \theta_i - c\,\frac{\partial E}{\partial \theta_i}$.

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

- We define the error $E(s,a,\theta)$ as:

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\left(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)}}_{\text{observation}}\right)^2$$

- This is a standard definition of squared error, where:
- $\hat{Q}_\theta(s,a)$ is the **prediction** of the utility for pair (s,a), according to parameter vector $\boldsymbol{\theta}$.
- $R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')$ is the **observed utility** for pair (s,a), according to the last step, where action a led from state s' to state s'.

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

- We define the error $E(s,a,\theta)$ as:

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\left(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)}}_{\text{observation}} \right)^2$$

- In this error definition, if we were doing supervised learning, instead of the observation we would use the ground truth.
- In reinforcement learning, we do not have ground truth, so we treat each observation as ground truth.
- This may lead to a few inaccurate updates, but in the long term the estimate will converge to the best approximation.

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\Big(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\Big(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\Big)}}_{\text{observation}} \Big)^2$$

- The next step is to compute $\frac{\partial E}{\partial \theta_i}$. We use the chain rule.
- Define:
  - $g(x) = \frac{1}{2}x^2$
  - $h(s,a,\boldsymbol{\theta}) = \hat{Q}_\theta(s,a) - \Big(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\Big)$
- Then, $E(s,a,\boldsymbol{\theta}) = g\big(h(s,a,\boldsymbol{\theta})\big)$, and $\frac{\partial E}{\partial \theta_i} = \frac{\partial g}{\partial h}\frac{\partial h}{\partial \theta_i}$

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\left(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)}}_{\text{observation}}\right)^2$$

- $g(h) = \frac{1}{2}h^2$
- $h(s,a,\boldsymbol{\theta}) = \hat{Q}_\theta(s,a) - \left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)$
- $\frac{\partial g}{\partial h} = \frac{1}{2} 2h = h$
- So, $\frac{\partial E}{\partial \theta_i} = \frac{\partial g}{\partial h}\frac{\partial h}{\partial \theta_i} = h\frac{\partial h}{\partial \theta_i}$

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\Bigg(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\Big(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\Big)}}_{\text{observation}} \Bigg)^2$$

- $h(s,a,\boldsymbol{\theta}) = \hat{Q}_\theta(s,a) - \Big(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\Big)$
- To compute $\frac{\partial h}{\partial \theta_i}$, we treat the observation as independent of θ_i.
  - It is not really a constant, since θ_i affects $\hat{Q}_\theta(s',a')$.
  - However, treating the observation as independent of θ_i simplifies calculations, and does not hurt accuracy of the final Q-learning result.

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\Bigg(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)}}_{\text{observation}} \Bigg)^2$$

- $h(s,a,\boldsymbol{\theta}) = \hat{Q}_\theta(s,a) - \left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)$
- To compute $\frac{\partial h}{\partial \theta_i}$, we treat the observation as independent of θ_i.
- Thus, $\frac{\partial h}{\partial \theta_i} = \frac{\partial \hat{Q}_\theta(s,a)}{\partial \theta_i} = f_i(s,a)$.

Learning a Parametric Q-Function

$$\hat{Q}_\theta(s,a) = \theta_1 f_1(s,a) + \theta_2 f_2(s,a) + \cdots + \theta_n f_n(s,a)$$

$$E(s,a,\boldsymbol{\theta}) = \frac{1}{2}\left(\underbrace{\boxed{\hat{Q}_\theta(s,a)}}_{\text{prediction}} - \underbrace{\boxed{\left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)}}_{\text{observation}} \right)^2$$

- Combining our previous results we get:

$$\frac{\partial E}{\partial \theta_i} = \frac{\partial g}{\partial h}\frac{\partial h}{\partial \theta_i} = h\frac{\partial h}{\partial \theta_i} \Rightarrow$$

$$\frac{\partial E}{\partial \theta_i} = \left(\hat{Q}_\theta(s,a) - \left(R(s) + \gamma \max_{a'} \hat{Q}_\theta(s',a')\right)\right) f_i(s,a)$$

Learning a Parametric Q-Function

$$\frac{\partial E}{\partial \theta_i} = \left(\hat{Q}_\theta(s,a) \; - \; \left(R(s) + \; \gamma \max_{a'} \hat{Q}_\theta(s',a') \right) \right) f_i(s,a)$$

- So, for Q-Learning, the update rule becomes:

$\theta_i = \theta_i - c\,\frac{\partial E}{\partial \theta_i}$, or equivalently

$$\theta_i = \theta_i - c \left(\hat{Q}_\theta(s,a) \; - \; \left(R(s) + \; \gamma \max_{a'} \hat{Q}_\theta(s',a') \right) \right) f_i(s,a)$$

- As usual when using gradient descent, we are moving parameter vector $\boldsymbol{\theta}$ a small step away from the direction of the error gradient.
  - c is the size of that step.
  - c itself is calculated as $c = \eta(N_{sa}[s,a])$, and decreases as $N_{sa}[s,a]$ increases.

Reinforcement Learning - Recap

- The goal in reinforcement learning is to learn what action to take at each state, so as to maximize the expected utility of the agent.
  - The states themselves are unknown until they are observed.
  - Rewards of states are unknown until they are observed.
  - The state transition model, i.e., the probability $\mathrm{p}(s' \mid s, a)$ that taking action a at state s leads to state s', is unknown.

Reinforcement Learning - Recap

- To solve the reinforcement learning problem, we solved a sequence of easier problems:
  - The MDP problem: rewards and transition probabilities are known in advance.
  - Passive reinforcement learning: rewards and transition probabilities are unknown, but the policy is fixed.
    - The agent updates utility estimates after each step, and the estimates converge to the correct values eventually.
  - Q-Learning: learns optimal policy. Rewards and transition probabilities are unknown.
    - The agent updates utility estimates after each step, and picks the next action balancing those estimates and the need to explore.
  - Parametric Q-Learning: allows describing utilities in large state spaces with relatively few parameters.