Logistic growth

I. Logistic Equation

The logistic equation (sometimes called the Verhulst model or logistic growth curve) is a model of population growth first published by Pierre Verhulst (1845, 1847).

$$N'[t] == \frac{r\, N[t]\, (K - N[t])}{K}$$

where r is the Malthusian parameter (rate of maximum population growth) and K is the so-called carrying capacity (i.e., the maximum sustainable population).

$$\frac{N'[t]}{K} == \frac{r\, N[t] \left(1 - \frac{N[t]}{K}\right)}{K}$$

Letting $x = \frac{n}{k}$

x’(t) == r x(t)(1 - x(t))

Demonstration.

dN / dt = r N (1 – N / K)

Analytic Solution. The logistic equation can be solved by separation of variables:

$\int$(dN / (N (1 - N / K)) = $\int$r dt .

In order to evaluate the left hand side we write:

1 / (N (1 - N / K) = K / (N (K - N)) = 1 / N + 1 / (K - N) ,

hence

$\int$dN / N + $\int$dN / (K - N) = $\int$r dt ,

ln| N | - ln| K - N | = rt + C ,

ln| (K - N) / N | = -rt - C ,

| (K - N) / N | = e^(-rt - C) ,

(K - N) / N = A.e^(-rt)   (A = +/- e^(-C)).

From here we get :

N = K / (1 + A.e^(-rt))   where   A = (K - N_0) / N_0 ,

Dividing both sides by K and defining x ≡ N / K.

We finally obtain,

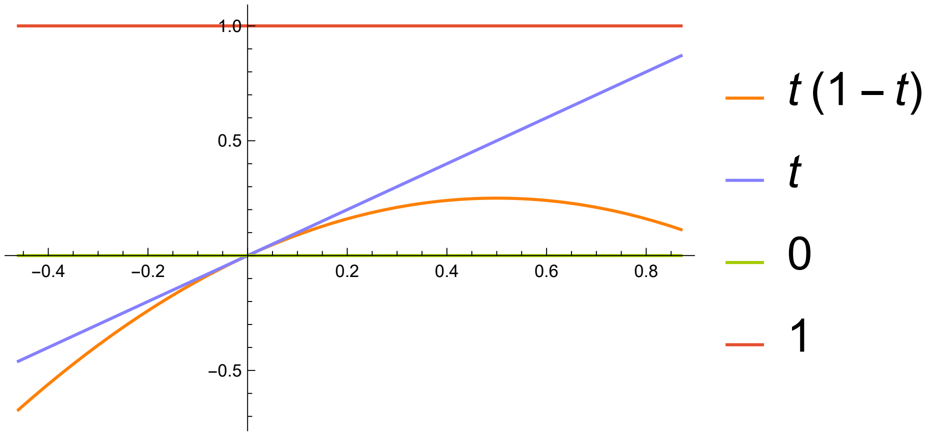
x(t) = 1 / (1 + B.e^(-rt))   where   B = (1 / N_0) - 1.

The function x(t) is sometimes known as the sigmoid function.

Plot[{1 t (1 – t), {t, 0, 1}]

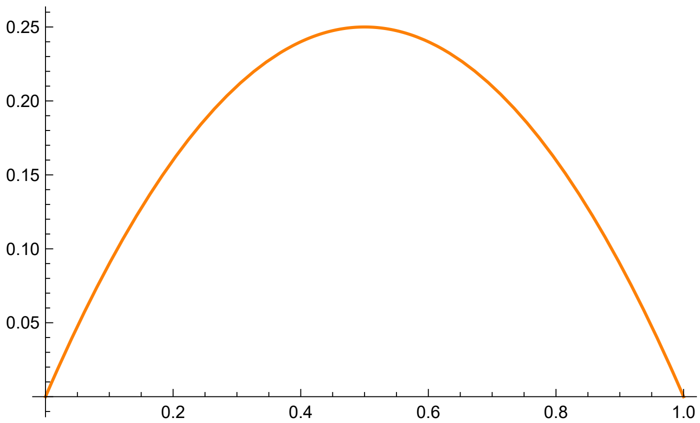
In[9]:=
```
Plot[{1 * t * (1 - t), {t, 0, 1}}, {t, -0.46, 0.87}]
```

Out[9]=



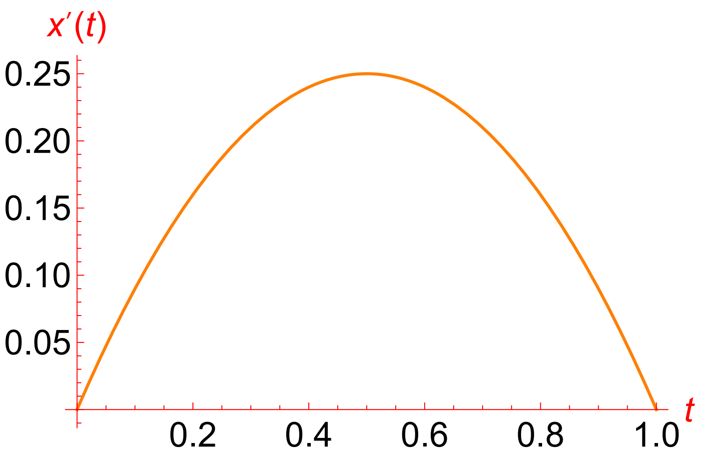
In[10]:=
```
logit = Plot[{(1 * t * (1 - t))}, {t, 0, 1}]
```

Out[10]=



In[11]:=
```
Show[logit, AxesLabel → {HoldForm[t], HoldForm[x′[t]]},
 PlotLabel → None, LabelStyle → {GrayLevel[0]},
 AxesStyle → {Directive[Red, 18], Directive[Red, 18]}]
```

Out[11]=



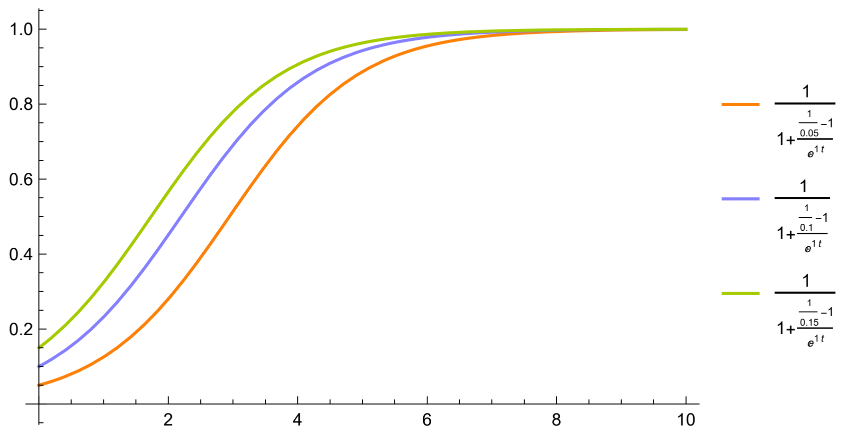
II. Sigmoid function

The logistic equation has a solution known has the sigmoid function.

Plot[{1/(1 + (1/0.05 – 1)/E^(1*t)), 1/(1 + (1/0.1 – 1)/E^(1*t)), 1/(1 + (1/0.15 – 1)/E^(1*t))}, {t, 0, 10}]

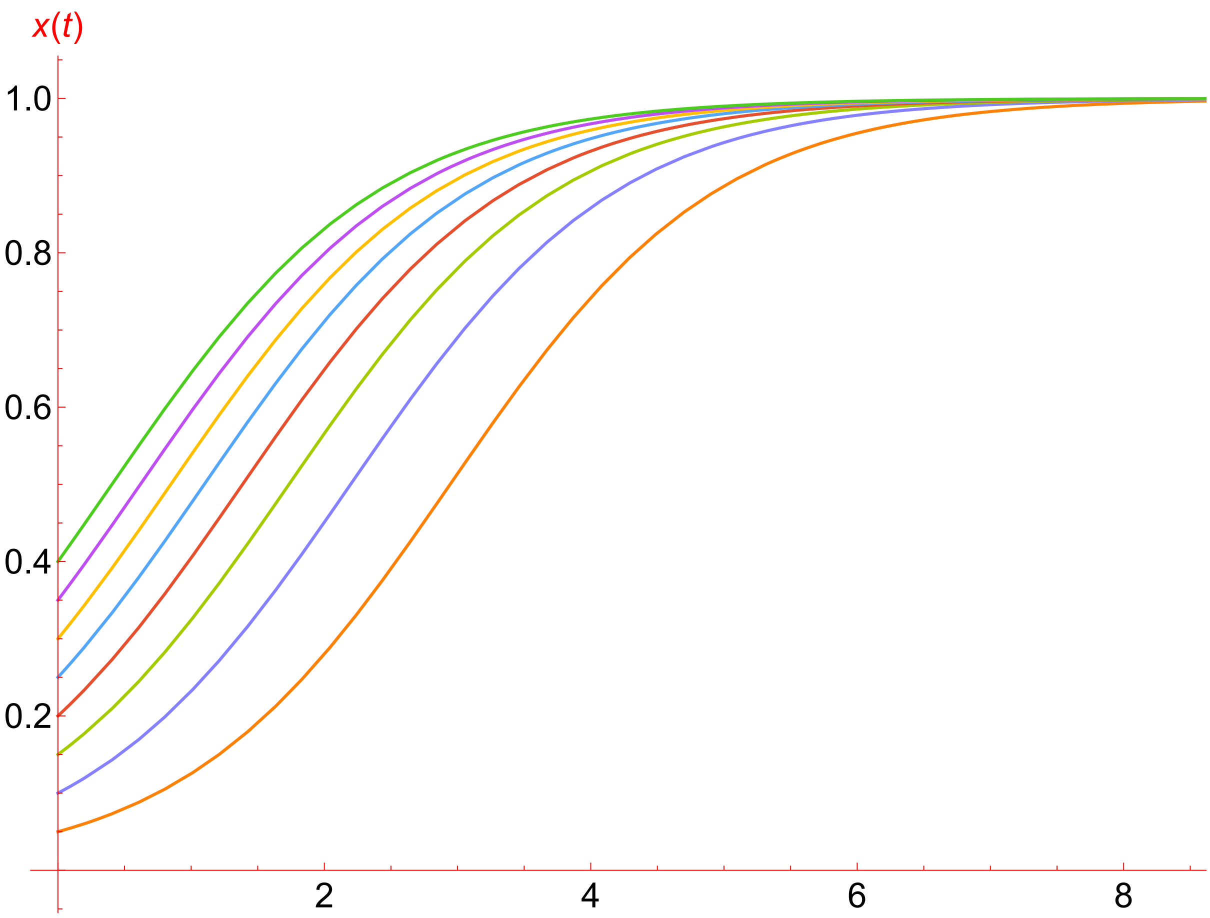
In[12]:=
```
Plot[{1/(1 + (1/0.05 - 1)/E^(1*t)),
  1/(1 + (1/0.1 - 1)/E^(1*t)), 1/(1 + (1/0.15 - 1)/E^(1*t))}, {t, 0, 10}]
```

Out[12]=



In[13]:=
```
sig = Plot[{1 / (1 + (1 / 0.05 - 1) / E^(1 * t)),
   1 / (1 + (1 / 0.1 - 1) / E^(1 * t)), 1 / (1 + (1 / 0.15 - 1) / E^(1 * t)),
   1 / (1 + (1 / 0.2 - 1) / E^(1 * t)), 1 / (1 + (1 / 0.25 - 1) / E^(1 * t)),
  1 / (1 + (1 / 0.3 - 1) / E^(1 * t)), 1 / (1 + (1 / 0.35 - 1) / E^(1 * t)),
   1 / (1 + (1 / 0.4 - 1) / E^(1 * t))}, {t, 0, 10},
  ImageSize -> 750, AxesLabel → {HoldForm[t], HoldForm[x[t]]},
  PlotLabel → None, LabelStyle → {GrayLevel[0]},
  AxesStyle → {Directive[Red, 18], Directive[Red, 18]}]
```

Out[13]=



References

Weisstein, Eric W. “Logistic Equation.” From MathWorld--A Wolfram Web Resource. https://-mathworld.wolfram.com/LogisticEquation.html. Accessed 11 April 2020.

Lerma, Miguel A. “Notes on Calculus II.” https://sites.math.northwestern.edu/~mlerma/courses/math214-2-04f/notes/c2-logist.pdf. Accessed 11 April 2020.