Variance Discrete Uniform Distribution

The aim is to replicate the demonstration than can be found here :
https://en.wikibooks.org/wiki/Statistics/Distributions/Discrete_Uniform

In[1]:= $\text{EXU} = \sum_{i=0}^{n-1} \left(\frac{1}{n} (a + i)\right)$

Out[1]= $\frac{1}{2} (-1 + 2 a + n)$

$$Var(X) = E\left[(X - E[X])^2\right] = \sum_{x \in S} f(x) (x - E[X])^2 = \sum_{i=0}^{n-1} \left(\frac{1}{n}\left((a+i) - \frac{a+b}{2}\right)^2\right)$$

In[2]:= $\text{VXU} = \sum_{i=0}^{n-1} \left(\frac{1}{n} ((a + i) - \text{EXU})^2\right)$

Out[2]= $\frac{1}{12} (-1 + n^2)$

$$Var(X) = \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{a + 2 i - b}{2}\right)^2$$

In[6]:= $\frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{a + 2 i - b}{2}\right)^2$

Out[6]= $\frac{1}{12} (2 - 6 a + 3 a^2 + 6 b - 6 a b + 3 b^2 - 6 n + 6 a n - 6 b n + 4 n^2)$

$$Var(X) = \frac{1}{4 n} \sum_{i=0}^{n-1} (a^2 + 4 a i - 2 a b + 4 i^2 - 4 i b + b^2)$$

In[9]:= $\frac{1}{4 n} \sum_{i=0}^{n-1} (a^2 + 4 a \times i - 2 a \times b + 4 i^2 - 4 i \times b + b^2)$

Out[9]= $\frac{1}{12} (2 - 6 a + 3 a^2 + 6 b - 6 a b + 3 b^2 - 6 n + 6 a n - 6 b n + 4 n^2)$

$$Var(X) = \frac{1}{4 n} \left[\sum_{i=0}^{n-1} (a^2 - 2 a b + b^2) + \sum_{i=0}^{n-1} (4 a i - 4 i b) + \sum_{i=0}^{n-1} 4 i^2\right]$$

In[10]:= $\text{Simplify}\left[\frac{1}{4 n} \left(\sum_{i=0}^{n-1} (a^2 - 2 a \times b + b^2) + \sum_{i=0}^{n-1} (4 a \times i - 4 i \times b) + \sum_{i=0}^{n-1} (4 i^2)\right)\right]$

Out[10]=

$\frac{1}{12} (2 + 3 a^2 + 3 b^2 - 6 a (1 + b - n) - 6 b (-1 + n) - 6 n + 4 n^2)$

$$Var(X) = \frac{1}{4 n} \left[n (a^2 - a b + b^2) + 4 (a - b) \sum_{i=0}^{n-1} i + 4 \sum_{i=0}^{n-1} i^2\right]$$

In[11]:= $\text{Simplify}\left[\frac{1}{4\,n}\left(n\left(a^2 - 2\,a\times b + b^2\right) + 4\,(a-b)\sum_{i=0}^{n-1}(i) + 4\sum_{i=0}^{n-1}\left(i^2\right)\right)\right]$

Out[11]=

$$\frac{1}{12}\left(2 + 3\,a^2 + 3\,b^2 - 6\,a\,(1+b-n) - 6\,b\,(-1+n) - 6\,n + 4\,n^2\right)$$

In[12]:=
```
(1 / 12) (2 + 3 a^2 + 3 b^2 - 6 a (1 + b - n) - 6 b (-1 + n) - 6 n + 4 n^2) ==
  (1 / 12) (2 - 6 a + 3 a^2 + 6 b - 6 a b + 3 b^2 - 6 n + 6 a n - 6 b n + 4 n^2)
```

Out[12]=

$$\frac{1}{12}\left(2 + 3\,a^2 + 3\,b^2 - 6\,a\,(1+b-n) - 6\,b\,(-1+n) - 6\,n + 4\,n^2\right) ==$$
$$\frac{1}{12}\left(2 - 6\,a + 3\,a^2 + 6\,b - 6\,a\,b + 3\,b^2 - 6\,n + 6\,a\,n - 6\,b\,n + 4\,n^2\right)$$

In[13]:= **Expand[%]**

Out[13]=

True

Remember that $\sum_{i=0}^{m}\left(i^2\right) = [m\,(m+1)\,(2\,m+1)]\,/\,6$:

In[14]:= $\sum_{i=0}^{m}\left(i^2\right)$

Out[14]=

$$\frac{1}{6}\,m\,(1+m)\,(1+2\,m)$$

$$Var(X) = \frac{1}{4\,n}\left[n\,(b-a)^2 + 4\,(a-b)\,[(n-1)\,n/2] + 4\,[(n-1)\,n\,(2\,n-1)/6]\right]$$

In[16]:= $\text{Simplify}\left[\frac{1}{4\,n}\left(n\,(b-a)^2 + 4\,(a-b)\left(\frac{(n-1)\,n}{2}\right) + 4\left(\frac{(n-1)\,n\,(2\,n-1)}{6}\right)\right)\right]$

Out[16]=

$$\frac{1}{12}\left(2 + 3\,(a-b)^2 + 6\,(a-b)\,(-1+n) - 6\,n + 4\,n^2\right)$$

In[17]:= $\% == \frac{1}{12}\left(2 - 6\,a + 3\,a^2 + 6\,b - 6\,a\,b + 3\,b^2 - 6\,n + 6\,a\,n - 6\,b\,n + 4\,n^2\right)$

Out[17]=

$$\frac{1}{12}\left(2 + 3\,(a-b)^2 + 6\,(a-b)\,(-1+n) - 6\,n + 4\,n^2\right) ==$$
$$\frac{1}{12}\left(2 - 6\,a + 3\,a^2 + 6\,b - 6\,a\,b + 3\,b^2 - 6\,n + 6\,a\,n - 6\,b\,n + 4\,n^2\right)$$

In[18]:= **Expand[%]**

Out[18]=

True

```
(* n=b-a+1 is the number of possibilities,so n-1=b-a  *)
```

In[19]:= $\text{Simplify}\left[\frac{1}{4\,n}\left(n\,(n-1)^2 - 2\,(n-1)\,(n-1)\,n + \left(\frac{2\,(n-1)\,n\,(2\,n-1)}{3}\right)\right)\right]$

Out[19]=

$$\frac{1}{12}\left(-1 + n^2\right)$$

```
(* n=b-a+1 *)
```

In[20]:= `n`

Out[20]=

n

In[21]:= `Remove[n]`

$$Var(X) = \frac{1}{4}\left[-(n-1)^2 + 2(n-1)(2n-1)/3\right]$$

In[22]:= $\text{Simplify}\left[\frac{1}{4}\left(-(-1+n)^2 + \frac{2}{3}(-1+n)(-1+2n)\right)\right]$

Out[22]=

$$\frac{1}{12}\left(-1+n^2\right)$$

$$Var(X) = \frac{1}{12}\left[-3(n-1)^2 + 2(n-1)(2n-1)\right]$$

In[23]:= $\text{Simplify}\left[\frac{1}{12}\left(-3(n-1)^2 + 2(n-1)(2n-1)\right)\right]$

Out[23]=

$$\frac{1}{12}\left(-1+n^2\right)$$

$$Var(X) = \frac{1}{12}\left[-3\left(n^2-2n+1\right) + 2\left(2n^2-3n+1\right)\right]$$

In[24]:= $\text{Simplify}\left[\frac{1}{12}\left(-3\left(n^2-2n+1\right) + 2\left(2n^2-3n+1\right)\right)\right]$

Out[24]=

$$\frac{1}{12}\left(-1+n^2\right)$$

Look at the two terms in the parenthesis,

In[30]:= $\text{si} = \text{Expand}\left[-3\left(n^2-2n+1\right)\right]$

Out[30]=

$-3+6n-3n^2$

In[31]:= $\text{sb} = \text{Expand}\left[2\left(2n^2-3n+1\right)\right]$

Out[31]=

$2-6n+4n^2$

In[33]:= `si + sb`

Out[33]=

$-1+n^2$

$$Var(X) = \frac{n^2-1}{12}$$

In[34]:= $\frac{1}{12}\left(-1+n^2\right) == \frac{n^2-1}{12}$

Out[34]=

True

In[35]:=
```
Expand[%]
```
Out[35]=
```
True
```

In[36]:=
```
n = b - a + 1
```
Out[36]=

$1 - a + b$

In[ ]:=
```
Variance[DiscreteUniformDistribution[{a, b}]]
```

In[37]:=

$$\frac{1}{12}\left(-1+(1-a+b)^2\right)$$

Out[37]=

$$\frac{1}{12}\left(-1+(1-a+b)^2\right)$$

In[38]:=

$$\frac{1}{12}\left(-1+(1-a+b)^2\right)$$

Out[38]=

$$\frac{1}{12}\left(-1+(1-a+b)^2\right)$$

In[39]:=
```
Remove [n]
```

In[40]:=

$$\frac{1}{12}\left(-1+n^2\right) == \frac{n^2-2}{12}$$

Out[40]=

$$\frac{1}{12}\left(-1+n^2\right) == \frac{1}{12}\left(-2+n^2\right)$$

In[41]:=
```
Expand[%]
```
Out[41]=

$$-\frac{1}{12}+\frac{n^2}{12} == -\frac{1}{6}+\frac{n^2}{12}$$

With variable, the Expand function indicates that the expression are not equivalent.

In[42]:=
```
n = b - a + 1
```
Out[42]=

$1 - a + b$

In[43]:=

$$\frac{1}{12}\left(-1+n^2\right) == \frac{n^2-2}{12}$$

Out[43]=

$$\frac{1}{12}\left(-1+(1-a+b)^2\right) == \frac{1}{12}\left(-2+(1-a+b)^2\right)$$

```
(* the previous equation does not include equivalent terms *)
```

In[45]:=

$$\frac{1}{12}\left(-1+n^2\right) == \frac{n^2-1}{12}$$

Out[45]=
```
True
```