CS-6210: HW 4

James Brissette

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1 Chapter 6

6.2 a) Using the composite trapezoidal rule with four subintervals we find that I_T can be calculated as follows:

$$
\begin{aligned}
I_T &= h\Big(\frac{1}{2}f_1 + f_2 + f_3 + f_4 + \frac{1}{2}f_5\Big) \\
&= h\Big(\frac{1}{2}e^2 + e^1 + e^0 + e^{-1} + \frac{1}{2}e^{-2}\Big) \\
&\approx 3.9242
\end{aligned}
$$

In order to evaluate the error on this approximation, we need to find $||f''||_\infty$:

$$
\begin{aligned}
\frac{d}{dx}\Big[e^{-2x}\Big] &= -2e^{-2x} \\
\frac{d^2}{dx^2}\Big[e^{-2x}\Big] &= 4e^{-2x}
\end{aligned}
$$

$$
\begin{aligned}
||f''||_\infty &= max_{-1\leq x\leq 1}|4e^{-2x}| \\
||f''||_\infty &= 4e^2
\end{aligned}
$$

Approximating our error we see:

$$
\begin{aligned}
\left|\int_{-1}^{1} e^{-2x}dx - I_T\right| &\leq \frac{2}{12}h^2(4e^2) \\
&\leq 1.2315
\end{aligned}
$$

b) We are able to use Simpson's rule in this case because the number of subintervals n is even, and I_S can be calculated as follows:

$$
\begin{aligned}
I_S &= \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + 4f_4 + f_5\Big) \\
&= \frac{1}{6}\Big(e^2 + 4e^1 + 2 + 4e^-1 + e^-2\Big) \\
&\approx 3.6448
\end{aligned}
$$

In order to evaluate the error on this approximation, we need to find $||f''''||_\infty$:

$$
\begin{aligned}
\frac{d^3}{dx^3}\Big[e^{-2x}\Big] &= -8e^{-2x} \\
\frac{d^4}{dx^4}\Big[e^{-2x}\Big] &= 16e^{-2x}
\end{aligned}
$$

$$||f''''||_\infty = max_{-1eqx\leq 1}|16e^{-2x}|$$
$$||f''''||_\infty = 16e^2$$

Approximating our error we see:

$$\left|\int_{-1}^{1} e^{-2x}dx - I_S\right| \leq \frac{2}{90}h^4(16e^2)$$
$$\leq 0.1642$$

c) Using the composite Hermite rule (or corrected trapezoidal rule) with four subintervals we find that I_H can be calculated as follows:

$$\begin{aligned} I_H &= h\Big(\frac{1}{2}f_1 + f_2 + f_3 + f_4 + \frac{1}{2}f_5\Big) + \frac{1}{12}h^2(f'_1 - f'_{n+1}) \\ &= \frac{1}{2}\Big(\frac{1}{2}e^2 + e^1 + e^0 + e^{-1} + \frac{1}{2}e^{-2}\Big) + \frac{1}{48}(-2e^2 + 2e^{-2}) \\ &\approx 3.6219 \end{aligned}$$

In order to evaluate the error on this approximation, we need to find $||f''''||_\infty$ which was calculated previously:

$$||f''''||_\infty = 16e^2$$

Approximating our error we see:

$$\left|\int_{-1}^{1} e^{-2x}dx - I_H\right| \leq \frac{2}{720}h^4(16e^2)$$
$$\leq 0.0205$$

d) From Theorem 6.2, using the trapezoidal rule we use can substitute in the definition of our step h, $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{12}h^2(4e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}}$$

Solving this inequality yields $n \geq 4,439$

e) From Theorem 6.3, using Simpson's rule we use can substitute in the definition of our step $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{90}h^4(16e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}}$$

Solving this inequality yields $n \geq 81$

f) From Theorem 6.4, using Simpson's rule we use can substitute in the definition of our step $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{720}h^4(16e^2) \leq 10^{-6}$$

$$h \leq \Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}$$

$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}$$

$$n \geq \frac{2}{\Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}}$$

Solving this inequality yields $n \geq 48$

6.4 a) Because we are given that $T = E\frac{du}{dx}$, we know that $\frac{du}{dx} = \frac{1}{E}T$. If we wanted to evaluate the the integral of $\frac{du}{dx}$ in the interval $[0, 1]$ we could write that as:

$$\int_0^x u'(x)\frac{d}{dx} = \frac{1}{E}\int_0^x T(s)ds$$

$$\frac{1}{E}\int_0^x T(s)ds = u(x)\Big|_0^x$$

or equivalently

$$u(x) - u(0) = \frac{1}{E}\int_0^x T(s)ds$$

$$u(x) = u(0) + \frac{1}{E}\int_0^x T(s)ds$$

b) To calculate $u(\frac{1}{4})$ using the trapezoidal rule and the data provided in table 6.10 we use the result from part a and set up the following:

$$u(1/4) = u(0) + \frac{1}{E}\int_0^{\frac{1}{4}} T(s)ds \quad where \quad u(0) = 0, E = 4$$

$$u(1/4) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + \frac{1}{2}(-1)\Big)$$

$$u(1/4) = 0$$

We use the trapezoidal rule similarly to calculate $u(1/2)$, $u(3/4)$, and $u(1)$ using the same values of $u(0)$ and E:

$$u(1/2) = 0 + \frac{1}{4}\int_0^{\frac{1}{2}} T(s)ds$$
$$u(1/2) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + \frac{1}{2}(2)\Big)$$
$$u(1/2) = \frac{1}{32}$$

$$u(3/4) = 0 + \frac{1}{4}\int_0^{\frac{3}{4}} T(s)ds$$
$$u(3/4) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + (2) + \frac{1}{2}(3)\Big)$$
$$u(3/4) = \frac{3}{16}$$

$$u(1) = 0 + \frac{1}{4}\int_0^{1} T(s)ds$$
$$u(1) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + (2) + (3) + \frac{1}{2}(4)\Big)$$
$$u(1) = \frac{13}{32}$$

c) In order to use the composite midpoint rule to evaluate $u(1)$ we need the values of the midpoints of each step. Since we're not give the value of the function at $x = \frac{1}{8}$, we must increase our step size from $\frac{1}{4}$ to $\frac{1}{2}$ so we are able to use the data that is provided as our midpoints:

$$u(1) = 0 + \frac{1}{4}\int_0^{1} T(s)ds$$
$$u(1) = 0 + \frac{1}{4}I_M$$
$$u(1) = 0 + \frac{1}{4} * \frac{1}{2}\Big((-1) + (3)\Big)$$
$$u(1) = \frac{1}{4}$$

d) We can use Simpson's rule to evaluate $u(1)$ since the number of intervals n is even:

$$u(1) = 0 + \frac{1}{4}\int_0^{1} T(s)ds$$
$$u(1) = 0 + \frac{1}{4}I_S$$
$$u(1) = 0 + \frac{1}{4} * \frac{1}{12}\Big((1) + 4(-1) + 2(2) + 4(3) + (4)\Big)$$
$$u(1) = \frac{17}{48}$$

e) If we use our result from part b we see $I_T = \frac{13}{8}$ and we use this in place of $I_T(2n)$. Following the theorem in the book we find that the Romberg Integration using the trapezoidal rule takes the form:

$$\int_a^b f(x)dx = \frac{4}{3}I_T(2n) - \frac{1}{3}I_T(n) + O(h^3)$$

Solving for $I_T(n)$ we get:

$$I_T(n) = \frac{1}{2}\Big(\frac{1}{2}(1) + (2) + \frac{1}{2}(4)\Big)$$
$$I_T(n) = \frac{9}{4}$$

Putting it all together we see:

$$u(1) = \frac{4}{3}(\frac{13}{8}) - \frac{1}{3}(\frac{9}{4})$$
$$u(1) = \frac{17}{12}$$

6.8 a) Using the trapezoidal rule, we know our error terms looks like the following (evaluated at $erf(2)$):

$$\left|\frac{2}{\sqrt{\pi}}\int_0^2 e^{-s^2}ds\right| \leq \frac{2}{12}h^2\Big|\Big|\frac{(8x^2-4)e^{-x^2}}{\sqrt{\pi}}\Big|\Big|_\infty$$

Taking the norm of $||f''||_\infty$ to be 2.256758 (solved using MATLAB), we solve for h:

$$\frac{2}{12}h^2(2) \leq 10^{-6}$$
$$h \leq \sqrt{\frac{10^{-6}}{2.256758} * 6}$$
$$h \leq 1.6305468e-03$$
$$h \leq 0.0016$$

Solving for n this gives us approximately 1227 sub intervals necessary to reach a error $\leq 10^{-6}$

b) Issue with this MATLAB code. Not particularly close to the true value..

```
function [error] = ch6q8(n)
% Solves for h using the trapezoidal rule
a = 0;
b = 2;
x = linspace(a,b,n+1);
h = x(3)-x(2);

trueError = 0.995322265;

IT = f(x);
IT(1) = .5*IT(1);
IT(n+1) = .5*IT(n+1);
IT = IT*h;

error = abs(trueError - sum(IT));

end

function y=f(x)
    y = 2/sqrt(pi).*exp(-x.*x);
end
```

Using the above code we are able to find an error of $9.807542619477694e-06$ with as little as $n = 53$ sub intervals. This perhaps highlights that h is our worst case scenario, when in practice, rarely are we dealing with worst case.

c) Using Simpson's rule, we know our error terms looks like the following (evaluated at $erf(2)$):

$$\left| \frac{2}{\sqrt{\pi}} \int_0^2 e^{-s^2} ds \right| \leq \frac{2}{90} h^4 ||f''''||_\infty$$

In this case, finding f'''' is not as trivial, so we include the process:

$$\frac{d^2}{dx^2} f(x) = \frac{(8x^2 - 4)e^{-x^2}}{\sqrt{\pi}}$$

$$\frac{d^3}{dx^3} f(x) = \frac{-2x(8x^2-4)}{\sqrt{\pi}} e^{-x^2} + \frac{(16x)}{\sqrt{\pi}} e^{-x^2}$$

$$= \frac{e^{-x^2}}{\sqrt{\pi}} \big(16x - 16x^3 + 8x\big)$$

$$= \frac{e^{-x^2}}{\sqrt{\pi}} \big(- 16x^3 + 24x\big)$$

$$\frac{d^4}{dx^4} f(x) = \Big(\frac{16}{\sqrt{\pi}} e^{-x^2} + \frac{-32x^2}{\sqrt{\pi}} e^{-x^2}\Big) + \Big(\frac{-48x^2}{\sqrt{\pi}} e^{-x^2} + \frac{32x^4}{\sqrt{\pi}} e^{-x^2}\Big) + \Big(\frac{8}{\sqrt{\pi}} e^{-x^2} - \frac{16x^2}{\sqrt{\pi}} e^{-x^2}\Big)$$

$$= \frac{e^{-x^2}\big(32x^4 - 96x^2 + 24\big)}{\sqrt{\pi}}$$

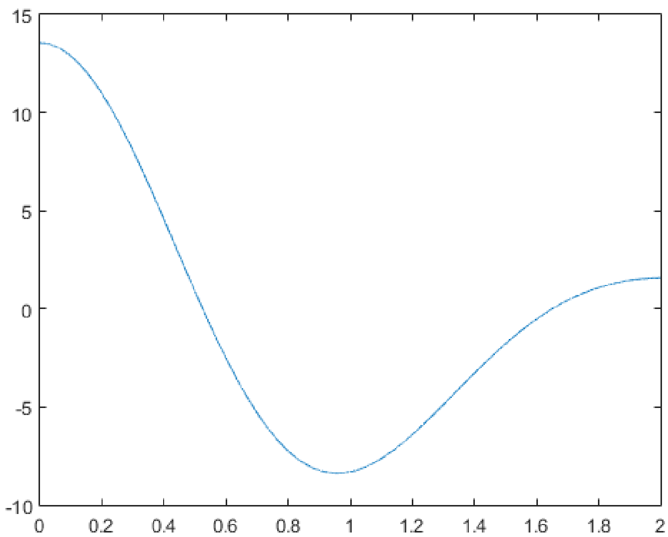
Taking the norm of $||f''''||_\infty$ to be 13.54055 or $\frac{24}{\sqrt{pi}}$ (solved using MATLAB and observing the plot below), we solve for h:

$$\frac{2}{90} h^4 (13.54055) \leq 10^{-6}$$

$$h \leq \sqrt[4]{\frac{10^{-6}}{13.54055} * \frac{90}{2}}$$

$$h \leq 4.269667464679832e - 02$$

Plotting the curve we see:



d) Using the same code from part B adapted for Simpson's rule, we have that we need 48 sub intervals for an error less than 10^{-6}. At $n = 48$ we have an error of $1.3824814e - 08$

```
function [error] = ch6q8d(n)
% Solves for h using Simpson's rule
a = 0;
b = 2;
x = linspace(a,b,n+1);
h = x(3)-x(2);

trueError = 0.995322265;
c = ones(1,n+1);

for i=2:n
    if mod(i,2)==0
        c(i) = 4;
    else
        c(i) = 2;
    end
end

IT = f(x);
IT = c.*IT;
IT = IT*h/3;

error = abs(trueError - sum(IT));

end

function y=f(x)
    y = 2/sqrt(pi).*exp(-x.*x);
end
```

6.15 a) Since we're given that $v(0) = 0$, and for any interval from 0 to t we get:

$$\int_0^t a(r)dr = v(t) - v(0) = v(t)$$

and we know the trapezoidal rule gives us

$$\int_0^t a(r)dr = h\Big(\frac{1}{2}a_0 + \frac{1}{2}a_t\Big)$$

we can combine the results to give us the following for the subinterval $t_i \leq t \leq t_{i+1}$

$$\begin{aligned}
\int_i^{i+1} a(r)dr &= v(i+1) - v(i) \\
\int_i^{i+1} a(r)dr &= h\Big(\frac{1}{2}a_i + \frac{1}{2}a_{i+1}\Big) \\
v(i+1) - v(i) &= h\Big(\frac{1}{2}a_i + \frac{1}{2}a_{i+1}\Big) \\
v(i+1) &= v(i) + \frac{1}{2}h\Big(a_i + a_{i+1}\Big)
\end{aligned}$$

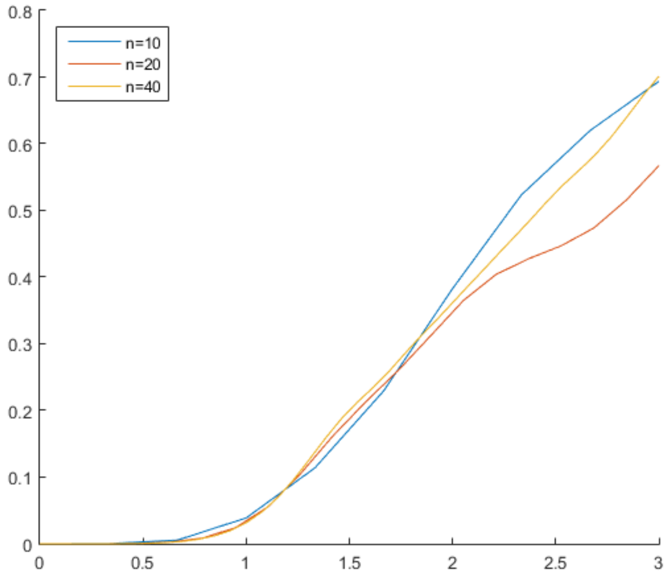
The same logic applies to solving for y_{i+1} since we know $y(0) = 0$, and for any interval from 0 to t we get:

$$\int_0^t v(r)dr = y(t) - y(0) = y(t)$$

On the interval from $t_i \le t \le t_{i+1}$ we get:

$$y(i+1) - y(i) = h\Big(\frac{1}{2}v_i + \frac{1}{2}v_{i+1}\Big)$$
$$y(i+1) = y(i) + \frac{1}{2}h\Big(v_i + v_{i+1}\Big)$$

b) Plotting $y(t)$ for $n = 10$, 20, and 40 yields the following:



c) For each value of n, the computed value of $y(t)$ was:

n	$computed$	$difference$
10	0.69359564141	$3.3727e-02$
20	0.56714220363	$1.6018e-01$
40	0.70120254590	$2.6120e-02$

In order to for our error to be less than 10e-8 we solve for n using the trapezoidal error. We also use Matlab to calculate $||_\infty \le 10^{-8}$ as :

$$|f''||_\infty \approx 1.147687e+04$$

Which allows us to solve for n as follows:

$$\frac{3}{12}h * 2||f''||_\infty \le 10^-8$$
$$h \le \sqrt{\frac{10^{-8}}{11477} * 4} \quad where \quad h = \frac{3}{n}$$
$$n \ge \frac{3}{\sqrt{\frac{10^{-8}}{11477} * 4}}$$
$$n \ge 160,696$$

6.18 a) If we look first at I_M we see that the function is evaluated at the midpoints between our known function values. Since Simpson's rule does not use midpoints, we need a way to convert this, and we find that by doubling the step size, our midpoint rule gives us every other function value (e.g. midpoint between f_1 and f_3, f_3 and f_5, etc.):

$$
\begin{aligned}
I_M(n) &= h\Big(f_{1+\frac{1}{2}} + f_{2+\frac{1}{2}} + \dots + f_{n+\frac{1}{2}}\Big) \\
I_M(n/2) &= 2h\Big(f_2 + f_4 + \dots\Big)
\end{aligned}
$$

Combining this result with the trapezoidal rule we show fairly easily that in the proportions outlined in the question, we arrive back at Simpson's rule:

$$
\begin{aligned}
I_S(n) &= \frac{2}{3}I_T(n) + \frac{1}{3}I_M\Big(\frac{n}{2}\Big) \\
I_S(n) &= \frac{2}{3}h\Big(\frac{1}{2}f_1 + f_2 + \dots + \frac{1}{2}f_{n+1}\Big) + \frac{1}{3}2h\Big(f_2 + f_4 + \dots\Big) \\
I_S(n) &= \frac{h}{3}\Big(f_1 + 2f_2 + 2f_3 + \dots + f_{n+1}\Big) + \frac{h}{3}\Big(2f_2 + 2f_4 + \dots\Big) \\
I_S(n) &= \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + \dots + f_{n+1}\Big)
\end{aligned}
$$

b) When we look at $I_T\Big(\frac{n}{2}\Big)$ we see a similar trend occur as in part a, where since our number of subdivisions is halved over the same interval, our step subsequently doubles, yielding:

$$
\begin{aligned}
I_T\Big(\frac{n}{2}\Big) &= 2h\Big(\frac{1}{2}f_1 + f_3 + f_5 + \dots + \frac{1}{2}f_{n+1}\Big) \\
I_T\Big(\frac{n}{2}\Big) &= h\Big(f_1 + 2f_3 + 2f_5 + \dots + f_{n+1}\Big)
\end{aligned}
$$

If we plug this back into the equation given in the question, we see that again for these proportions, again, we get back Simpson's rule:

$$
\begin{aligned}
I_S(n) &= \frac{4}{3}I_T(n) - \frac{1}{3}I_T\Big(\frac{n}{2}\Big) \\
I_S(n) &= \frac{4}{3}h\Big(\frac{1}{2}f_1 + f_2 + \dots + \frac{1}{2}f_{n+1}\Big) - \frac{1}{3}h\Big(f_1 + 2f_3 + 2f_5 + \dots + f_{n+1}\Big) \\
I_S(n) &= \frac{h}{3}\Big(2f_1 + 4f_2 + 4f_3 + \dots + 2f_{n+1}\Big) - \frac{h}{3}\Big(f_1 + 2f_3 + 2f_5 + \dots + f_{n+1}\Big) \\
I_S(n) &= \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + \dots + f_{n+1}\Big)
\end{aligned}
$$

6.19 a) We know from the trapezoidal rule that we have

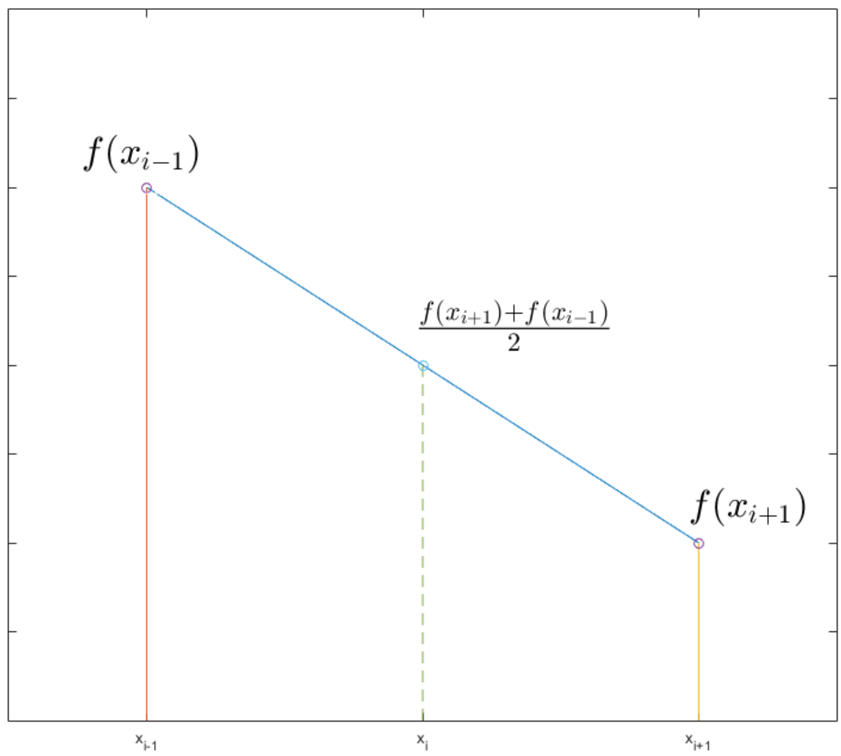
$$
I_T = h\Big(\frac{1}{2}f_{i-1} + f_i + \frac{1}{2}f_{i+1}\Big)
$$

However, since we don't know the value of f at x_i we need to find a way to exclude it during out calculation. We do this by increasing the step size h so we evaluate only at the end points x_{i-1} and x_{i+1}:

$$
\begin{aligned}
I_T &= 2h\Big(\frac{1}{2}f_{i-1} + \frac{1}{2}f_{i+1}\Big) \\
&= h\Big(f_{i-1} + f_{i+1}\Big)
\end{aligned}
$$

b) If we use linear interpolation to find a value for f_i we see from the plot below that f_i takes the value:

$$f_i = \frac{f_{i+1}+f_{i-1}}{2}$$



If we take this value of f_i and substitute it into Simpson's rule we get, by simplifying, the following:

$$\begin{aligned}
I_S &= \frac{h}{3}\Big(f_{i-1} + 4f_i + f_{i+1}\Big) \\
&= \frac{h}{3}\Big(f_{i-1} + 4(\frac{f_{i+1} + f_{i-1}}{2}) + f_{i+1}\Big) \\
&= \frac{h}{3}\Big(f_{i-1} + 2\big(f_{i+1} + f_{i-1}\big) + f_{i+1}\Big) \\
&= \frac{h}{3}\Big(3f_{i-1} + 3f_{i+1}\Big) \\
&= h\Big(f_{i-1} + f_{i+1}\Big)
\end{aligned}$$

We see this is the same result we reached in part (a)

c) If we assume there are constants A and B, we can plug these into Simpson's rule to solve for the weights that maximize precision as follows:

$$\begin{aligned}
I_S &= \frac{h}{3}\Big(f_{i-1} + 4f_i + f_{i+1}\Big) \\
&= \frac{h}{3}\Big(f_{i-1} + 4\big(Af_{i-1} + Bf_{i+1}\big) + f_{i+1}\Big) \\
&= \frac{h}{3}\Big((4A+1)f_{i-1} + (4B+1)f_{i+1}\Big) \\
&= \frac{(4A+1)h}{3}f_{i-1} + \frac{(4B+1)h}{3}f_{i+1}
\end{aligned}$$

where

$$w_1 = \frac{(4A+1)h}{3}, \quad w_2 = \frac{(4B+1)h}{3}$$

Here we also note that w_1 and w_2 appear to be symmetric, differing only by their respective values of A and B, which are likely to be symmetric around some point.

6.20 a) It's given that the error involved with Simpson's rule takes the form

$$I_S(n) + \alpha h^4 + \beta h^6 + \gamma h^8 + \dots$$

and so substituting in the known error for $I_S(n)$ and $I_S(n/2)$ yields the following:

$$\begin{aligned}
I_R &= \frac{1}{15}\Big[16 I_S(2n) - I_S\Big] \\
I_R &= \frac{1}{15}\Big[16(\alpha\frac{h}{2}^4 + \beta\frac{h}{2}^6 + \gamma\frac{h}{2}^8 + \dots) - (\alpha h^4 + \beta h^6 + \gamma h^8 + \dots)\Big] \\
I_R &= \frac{1}{15}\Big[\alpha\frac{16h^4}{16} + \beta\frac{16h^6}{64} + \gamma\frac{h}{2}^8 + \dots) - (\alpha h^4 + \beta h^6 + \gamma h^8 + \dots)\Big] \\
I_R &= \frac{1}{15}\Big[\alpha h^4 + \beta\frac{h^6}{4} + \gamma\frac{h}{2}^8 + \dots) - (\alpha h^4 + \beta h^6 + \gamma h^8 + \dots)\Big] \\
I_R &= \frac{1}{15}\Big[\beta\frac{h^6}{4} + \gamma\frac{h}{2}^8 + \dots) - (\beta h^6 + \gamma h^8 + \dots)\Big]
\end{aligned}$$

We assume here that since the number of sub intervals is doubled, the step size is necessarily halved. Also, we see the α error term is eliminated, leaving the dominant term here as h^6 meaning $I_R = O(h^6)$

b) Here we have that the error associated with $f(x)$ is equal to

$$\int_a^b f(x)dx = I(n) + \alpha h^2 + \beta h^3 + \gamma h^4 + \dots$$

By adjusting the step size within the same interval to account for increased numbers of n sub intervals we solve for I_R as

$$\begin{aligned}
I_R &= \frac{1}{21}\Big[32I(4n) - 12I(2n) + I(n)\Big] \\
&= \frac{1}{21}\Big[32\Big(\alpha\big(\frac{h}{4}\big)^2 + \beta\big(\frac{h}{4}\big)^3 + \gamma\big(\frac{h}{4}\big)^4\Big) - 12\Big(\alpha\big(\frac{h}{2}\big)^2 + \beta\big(\frac{h}{2}\big)^3 + \gamma\big(\frac{h}{2}\big)^4\Big) + \Big(\alpha h^2 + \beta h^3 + \gamma h^4\Big)\Big] \\
&= \frac{1}{21}\Big[32\Big(\alpha\frac{h^2}{16} + \beta\frac{h^3}{64} + \gamma\frac{h^4}{256}\Big) - 12\Big(\alpha\frac{h^2}{4} + \beta\frac{h^3}{8} + \gamma\frac{h^4}{16}\Big) + \Big(\alpha h^2 + \beta h^3 + \gamma h^4\Big)\Big] \\
&= \frac{1}{21}\Big[\Big(2\alpha h^2 + \frac{1}{2}\beta h^3 + \frac{1}{8}\gamma h^4\Big) - \Big(3\alpha h^2 + \frac{3}{2}\beta h^3 + \frac{3}{4}\gamma h^4\Big) + \Big(\alpha h^2 + \beta h^3 + \gamma h^4\Big)\Big]
\end{aligned}$$

Combining like terms we reduce this to

$$\begin{aligned}
I_R &= \frac{1}{21}\Big[\Big(2\alpha h^2 - 3\alpha h^2 + \alpha h^2\Big) + \Big(\frac{1}{2}\beta h^3 - \frac{3}{2}\beta h^3 + \beta h^3\Big) + \Big(\frac{1}{8}\gamma h^4 - \frac{3}{4}\gamma h^4 + \gamma h^4\Big)\Big] \\
&= \frac{1}{21}\Big[\Big(\frac{3}{8}\gamma h^4\Big)\Big]
\end{aligned}$$

From this we see that the α and β terms cancel out and we are left with an error that is $O(h^4)$

c) Here we have that the error associated with $f(x)$ is equal to

$$\int_a^b f(x)dx = I(n) + \alpha h^2 + \beta h^3 + \gamma h^4 + \dots$$

Again, by adjusting the step size within the same interval to account for increased numbers of n sub intervals we solve for I_R as

$$\begin{aligned}
I_R &= \frac{1}{12}\Big[27I(3n) - 16I(2n) + I(n)\Big] \\
&= \frac{1}{12}\Big[27\Big(\alpha(\frac{h}{3})^2 + \beta(\frac{h}{3})^3 + \gamma(\frac{h}{3})^4\Big) - 16\Big(\alpha(\frac{h}{2})^2 + \beta(\frac{h}{2})^3 + \gamma(\frac{h}{2})^4\Big) + \Big(\alpha h^2 + \beta h^3 + \gamma h^4\Big)\Big] \\
&= \frac{1}{12}\Big[27\Big(\alpha\frac{h^2}{9} + \beta\frac{h^3}{27} + \gamma\frac{h^4}{81}\Big) - 16\Big(\alpha\frac{h^2}{4} + \beta\frac{h^3}{8} + \gamma\frac{h^4}{16}\Big) + \Big(\alpha h^2 + \beta h^3 + \gamma h^4\Big)\Big] \\
&= \frac{1}{12}\Big[\Big(3\alpha h^2 + \beta h^3 + \frac{1}{3}\gamma h^4\Big) - \Big(4\alpha h^2 + 2\beta h^3 + \gamma h^4\Big) + \Big(\alpha h^2 + \beta h^3 + \gamma h^4\Big)\Big]
\end{aligned}$$

Combining like terms we reduce this to

$$\begin{aligned}
I_R &= \frac{1}{12}\Big[\Big(3\alpha h^2 - 4\alpha h^2 + \alpha h^2\Big) + \Big(\beta h^3 - 2\beta h^3 + \beta h^3\Big) + \Big(\frac{1}{3}\gamma h^4 - \gamma h^4 + \gamma h^4\Big)\Big] \\
&= \frac{1}{12}\Big[\Big(\frac{1}{3}\gamma h^4\Big)\Big]
\end{aligned}$$

From this we see that the α and β terms cancel out and we are left with an error that is $O(h^4)$

6.21 a) Since we're solving for three unknowns, we need at least 3 equations. These are given in the book in Table 6.5 as:

k	$f(x)$	$\int_{x_i}^{x_{i+1}} f(x)dx)$
0	1	h
1	x	$h\Big(x_i + \frac{1}{2}h\Big)$
2	x^2	$h\Big(x_i^2 + hx_i + \frac{1}{3}h^2\Big)$

First taking $k = 0$ we solve for w_1:

$$\begin{aligned}
h &= w_1 + w_2 \\
w_1 &= h - w_2
\end{aligned}$$

Using our second formula we solve for w_2 remembering that in the problem, z is given as $x_i + \alpha h$ where we are to solve for α:

$$\begin{aligned}
h\Big(x_i + \frac{1}{2}h\Big) &= w_1 x_i + w_2 z \\
&= (h - w_2)x_i + w_2 x \\
hx_i + \frac{1}{2}h^2 &= hx_i - w_2 x_i + w2_z \\
\frac{1}{2}h^2 &= -w_2 x_i + w_2 z \\
\frac{h^2}{2} &= w_2(z - x_i) \\
\frac{h^2}{2} &= w_2(x_i + \alpha h - x_i) \\
\frac{h^2}{2} &= w_2(\alpha h) \\
w_2 &= \frac{h}{2\alpha}
\end{aligned}$$

Using our third formula and having equations in place for w_1 and w_2 we can solve for α as follows:

$$\begin{aligned}
h\Big(x_i^2 + hx_i + \frac{1}{3}h^2\Big) &= w_1x_i^2 + w_2z^2 \\
&= (h - w_2)x_i^2 + w_2z^2 \\
&= hx_i^2 - w_2x_i^2 + w_2(x_i + \alpha h)^2 \\
&= hx_i^2 - w_2x_i^2 + w_2(x_i^2 + 2x_i\alpha h + \alpha^2h^2) \\
&= hx_i^2 + w_2(2x_i\alpha h + \alpha^2h^2) \\
&= hx_i^2 + \frac{h}{2\alpha}(2x_i\alpha h + \alpha^2h^2) \\
hx_i^2 + h^2x_i + \frac{h^3}{3} &= hx_i^2 + h^2x_i + \frac{h^3\alpha}{2} \\
\frac{h^3}{3} &= \frac{h^3\alpha}{2} \\
\alpha &= \frac{2}{3}
\end{aligned}$$

b) From Theorem 6.5 we have

$$E_G = Kh^{2\ell+1}f^{(2\ell)}\big(\eta\big)$$

where K is given to be

$$\frac{(\ell!)^4}{(2\ell+1)[(2\ell)!]^3}$$

Since we have 2-point Gaussian quadrature but one of the points is fixed, we are left with $\ell = 1$. Plugging this into our equation for K we see:

$$\begin{aligned}
K &= \frac{(1!)^4}{(2+1)[(2)!]^3} \\
&= \frac{1}{(3)(2)^3} \\
&= \frac{1}{24}
\end{aligned}$$