CS-6190: Homework 1

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1 Warm Up

1. To get $p(x_1)$ from $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we need to use a fair bit of wizardry, and assume $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$, so $\Sigma^{-1} = V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$. Furthermore, we know that $x^T Ax = \sum_i \sum_j x_i A_{ij} x_j$, so the term in our multivariate gaussian exponent becomes:

$$\Big(\frac{1}{2\pi}\Big)^{\frac{k}{2}} \frac{1}{|\Sigma|^{\frac{1}{2}}} exp\big[\frac{1}{2} x^T \Sigma^{-1} x\big]$$
$$= x_1^T V_{11} x_1 + x_1^T V_{12} x_2 + x_2^T V_{21} x_1 + x_2^T V_{22} x_2$$

When we go through and tediously complete the square we get something that looks like the following:

$$x^T \Sigma^{-1} x = (x_2 + V_{22}^{-1} V_{21} x_1)^T V_{22} (X_2 + V_{22}^{-1} V_{21} x_1) + x_1^T (V_{11} - V_{21}^T V_{22}^{-1} V_{21}) x_1$$

and with some guidance from various sources that illustrate that $f(x) = f(x_2|x_1)f(x_1)$, we can simplify this and recover the marginal distribution of x_1 as follows:

$$f(x_2|x_1) \propto exp(x_2 + V_{22}^{-1} V_{21} x_1)^T V_{22} (X_2 + V_{22}^{-1} V_{21} x_1)$$
$$f(x_1) \propto exp((x_1^T - \mu)^T (V_{11} - V_{21}^T V_{22}^{-1} V_{21})(x_1 - \mu))$$
$$X_1 \sim \mathcal{N}(\mu, (V_{11} - V_{21}^T V_{22}^{-1} V_{21}))$$

Which turns out to be exactly $X_1 \sim \mathcal{N}(\mu, \Sigma_{11})$

2.

3.

4. We know that $KL(q||p) = \int \big[log(q(x)) - log(p(x))\big] p(x)dx$. If we expand out each term we get:

$$\int log\Big[\frac{1}{2|\Lambda|} exp(\frac{1}{2}(x-m)^T \Lambda^{-1}(x-m))\Big] - log\Big[\frac{1}{2|\Sigma|} exp(\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))\Big] p(x)dx$$
$$= \int \Big[\frac{1}{2} log\frac{|\Lambda|}{|\Sigma|} - \frac{1}{2}(x-m)^T \Lambda^{-1}(x-m) + \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\Big] p(x)dx$$
$$= \frac{1}{2} log\frac{|\Lambda|}{|\Sigma|} - tr\big[E[(x-\mu)(x-\mu)^T]\Sigma^{-1}\big] + \frac{1}{2} E[(x-m)^T \Lambda^{-1}(x-m)]$$

and we know from the matrix cookbook that $E(x^T Ax) = tr(A\Sigma) + m^T Am$:

$$= \frac{1}{2} log\frac{|\Lambda|}{|\Sigma|} - tr\big[I_d\big] + \frac{1}{2}(m-\mu) + \frac{1}{2} tr(^{-1}\Sigma))$$
$$= \frac{1}{2}\big[log\frac{|\Lambda|}{|\Sigma|} - d + tr(\Lambda^{-1}\Sigma) + (m-\mu)^T \Lambda^{-1}(m-\mu)\big]$$

5. So if we use a fair bit of what is in the lecture notes, this isn't too bad. We know that $\nabla log(Z(\eta)) = \frac{\nabla Z(\eta)}{Z(\eta)}$, that $Z(\eta) = \int h(x)exp(u(x)^T\eta)dx$ and that $\nabla Z(\eta) = \int h(x)exp(u(x)^T\eta)u(x)dx$.

   If we combine terms to simplify the gradient by taking $\frac{1}{Z(\eta)}$ to be $exp(-log(Z(\eta)))$ we can combine terms and calculate the second derivative as follows:

$$\nabla log(Z(\eta)) = \int u(x)h(x)exp(u(x)^T\eta - log(Z(\eta)))$$
$$\nabla^2 \frac{log(Z(\eta))}{d\eta} = \int u(x)h(x)exp(u(x)^T\eta - log(Z(\eta)))(u(x) - \nabla log(Z(\eta))$$

   And we can simplify $u(x)exp(u(x)^T\eta - log(Z(\eta))) = \mathbb{E}[u(x)]$ which gives us:

$$\begin{aligned}
&= \int u(x)^2 exp(u(x)^T\eta - log(Z(\eta)) - u(x)\mathbb{E}[u(x)]exp(u(x)^T\eta - log(Z(\eta))) \\
&= \mathbb{E}[u(x)^2] - \mathbb{E}[u(x)]^2 \\
&= \mathbb{V}[u(x)]
\end{aligned}$$

6. I've never heard of negative Variance, so the covariance matrix must all be positive, meaning it's positive semi-definite, meaning it's convex.

7. In order to calculate the mutual information in terms of the entropy of two random variables, we'll start with the definition for Mutual Information:

$$\begin{aligned}
I &= -\int\int p(x,y)log\frac{p(x)p(y)}{p(x,y)} \\
&= -\int\int p(x,y)\Big[log\frac{p(y)}{p(x,y)} + log(p(x))\Big] \\
&= -\int\int p(x,y) * log\frac{p(y)}{p(x,y)} + \int\int p(x,y)log(p(x)) \\
&= -\int\int p(y)p(x|y) * log(p(x|y)) + \int\int p(x,y)log(p(x)) \\
&= -\int p(y)\int p(x|y) * log(p(x|y)) + \int log(p(x))\int p(x,y) \\
&= -\int p(y) * H(x|y) + \int log(p(x)) * p(x) \\
&= -\int p(y) * H(x|y) \quad + H(x) \\
&= -H(x|y) + H(x) \\
&= H(x) - H(x|y)
\end{aligned}$$

8. (a)
   (b)
   (c)

9.

10. Yes.

11. (a)
    (b)
    (c)
    (d)