CS-6210: HW 1

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1 Chapter 3: Questions 6,12,16,24,25

3.6 a) Here I solve for A^{-1} the old school way:

$$\underset{R3-R1\to R3}{\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&1&0&1&0\\1&0&1&0&0&1\end{array}\right)} \rightarrow \quad \underset{R1-R2\to R1}{\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&1&0&1&0\\0&-1&1&-1&0&1\end{array}\right)} \rightarrow$$

$$\underset{R3+R2\to R3}{\left(\begin{array}{ccc|ccc}1&0&-1&1&-1&0\\0&1&1&0&1&0\\0&-1&1&-1&0&1\end{array}\right)} \rightarrow \quad \underset{R3/2\to R3}{\left(\begin{array}{ccc|ccc}1&0&-1&1&-1&0\\0&1&1&0&1&0\\0&0&2&-1&1&1\end{array}\right)} \rightarrow$$

$$\underset{Solve}{\left(\begin{array}{ccc|ccc}1&0&-1&1&-1&0\\0&1&1&0&1&0\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right)} \rightarrow \quad \left(\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&1&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right)$$

b)

$$||A|| = \max(2,2,2) = 2$$
$$||A^{-1}|| = \max(\frac{3}{2},\frac{3}{2},\frac{3}{2}) = \frac{3}{2}$$
$$k_\infty(A) = ||A|| * ||A^{-1}|| = 3$$

c)

$$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\l_{12}&1&0\\l_{13}&l_{23}&1\end{pmatrix}\begin{pmatrix}u_{11}&u_{21}&u_{31}\\0&u_{22}&u_{32}\\0&0&u_{33}\end{pmatrix}$$

$$= \begin{pmatrix}u_{11}&u_{21}&u_{31}\\u_{11}l_{12}&u_{21}l_{12}+u_{22}&u_{31}l_{12}+u_{32}\\u_{11}l_{13}&u_{21}l_{13}+u_{22}l_{23}&u_{31}l_{13}+u_{32}l_{23}+u_{33}\end{pmatrix}$$

Solving for each unknown we get:

$$\begin{array}{lll}u_{11}=1 & u_{21}=1 & u_{31}=0\\ l_{12}=0 & u_{22}=1 & u_{32}=1\\ l_{13}=1 & l_{23}=-1 & u_{33}=2\end{array}$$

And the Doolittle factorization becomes:

$$\begin{pmatrix}1&0&0\\0&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&0&2\end{pmatrix}$$

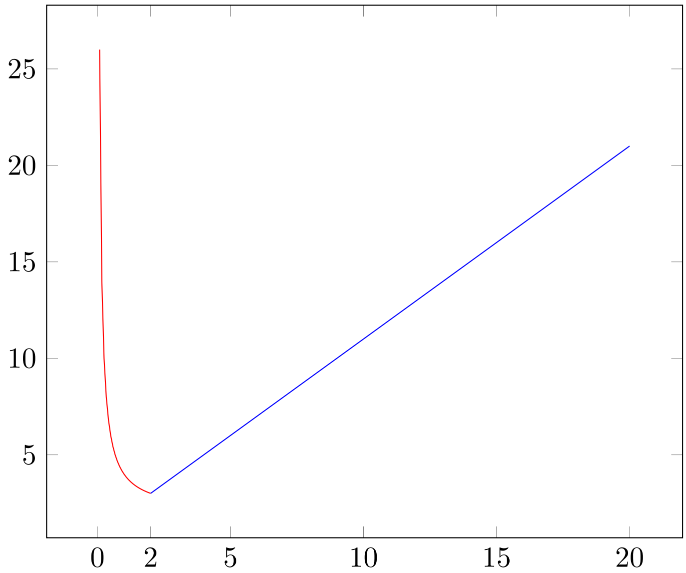
3.12 a) To determine the values α that make A *ill-conditioned*, we can compute the condition number k using the infinity norm, denoted k_∞:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & \alpha \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -1 & 1/\alpha \\ 0 & 1/\alpha \end{pmatrix}$$
$$||A|| = \max(2, \alpha) \qquad ||A^{-1}|| = 1 + \tfrac{1}{\alpha}$$

The condition number can then be computed as:

$$k_\infty(A) = ||A|| * ||A^{-1}||$$

For $0 < \alpha \leq 2$, k takes the values: $2 * (\frac{1+\alpha}{\alpha})$; and for $2 < \alpha$, k can be calculated as: $1 + \alpha$. This is most easily seen as we graph the curve for both domains of α:



The matrix A will be ill-conditioned for very small values of α ($0 < \alpha \ll 1$) and for very large values of α (k increases linearly with α).

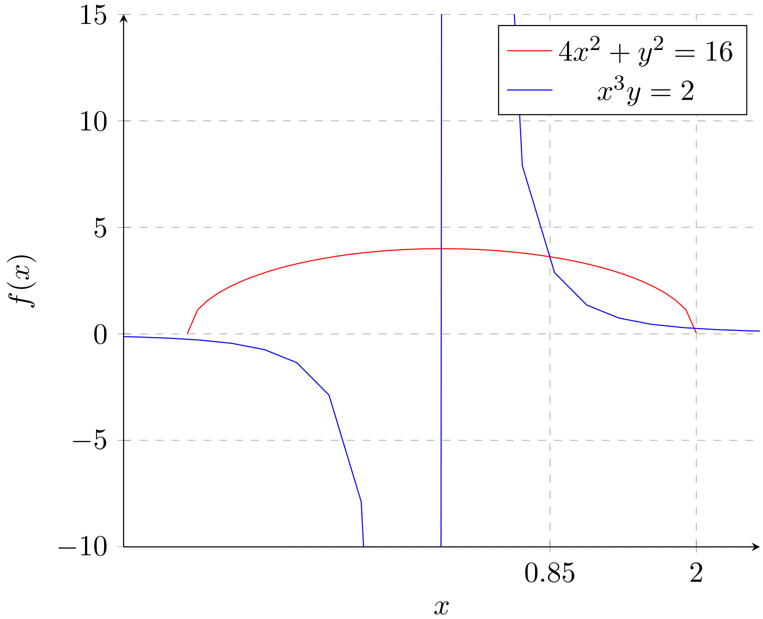
b) Error can also be defined as $e = A^{-1}r$, which is convenient considering α is an element of the matrix A. In that case $e_1 = -r_1 + \frac{r_2}{\alpha}$ and $e_2 = \frac{r_2}{\alpha}$. It quickly becomes apparent that a matrix A which contains very small ($\alpha \ll 1$) values of α could take a small residual and create a large error.

c) Conversely, residual can be written as $r = Ae$, which allows us to see the effect of α on r. Here we see $r_1 = -e_1 + e_2$ and $r_2 = \alpha e_2$. The effect of α on error is opposite, where r can be large given a small e if $\alpha \gg 1$.

3.16 1. Need to solve in MATLAB

3.24

a) Based solely on the sketch, the solutions are located (approximately) at ($x_1 = .85$) and ($x_2 = 2$):



b) Calculating the Jacobian matrix J of the nonlinear system of equations involves taking the partial derivatives of each equation with respect to x and y:

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 4x^2 + y^2 - 16 \\ x^3y - 2 \end{bmatrix}$$

$$J = \begin{bmatrix} 8x & 2y \\ 3x^2y & x^3 \end{bmatrix} \textit{ and } f = \begin{bmatrix} 4x^2 + y^2 - 16 \\ x^3y - 2 \end{bmatrix}$$

c) Good initial starting values for the solutions, again going based off the sketch, would be anywhere in the general range of the expected solution. A convention that appears to be common (I could be mistaken) is to choose an easy to calculate point like $x = 1$. I would guess, given the curves, that wouldn't be a bad place to begin here either. After 2 or 3 iterations, you would have a good approximation of either solution depending on which curve you plugged x into initially. In fact, any initial starting value of x where $x > 0$ seems like it would work if you began with the blue curve, and any point $0 < x < 2$ if starting with the red curve.

3.25

a) The algorithm works only because of the combination of the simplified and generalizable LU factorization of tri-diagonal matricies and the convenience of Cholesky Factorization. Factorizations of tri-diagonal matrices take the form:

$$\begin{pmatrix} l_{11} & & & & \\ l_{12} & l_{22} & & & \\ & l_{23} & l_{33} & & \\ & & \ddots & \ddots & \\ & & & & \\ & & & l_{i-1,i} & l_{ii} \end{pmatrix} \begin{pmatrix} u_{11} & u_{21} & & & & \\ & u_{22} & u_{32} & & & \\ & & u_{33} & u_{43} & & \\ & & & \ddots & \ddots & \\ & & & & & u_{i,i-1} \\ & & & & & u_{ii} \end{pmatrix}$$

And Cholesky allows us to simplify the process by assuming $A = U^TU$ rather than

$A = LU$ by setting $L = U^T$. This allows our factorization to become:

$$\begin{pmatrix} u_{11} & & & & & \\ u_{21} & u_{22} & & & & \\ & u_{32} & u_{33} & & & \\ & & \ddots & \ddots & & \\ & & & & & \\ & & & & u_{i,i-1} & u_{ii} \end{pmatrix} \begin{pmatrix} u_{11} & u_{21} & & & & \\ & u_{22} & u_{32} & & & \\ & & u_{33} & u_{43} & & \\ & & & \ddots & \ddots & \\ & & & & & u_{i,i-1} \\ & & & & & u_{ii} \end{pmatrix}$$

Where the super/sub diagonals have the same values. Combining the matrices, you come to the general form:

$$\begin{pmatrix} u_{11}^2 & u_{11}u_{21} & & & & \\ u_{11}u_{21} & u_{21}^2 u_{22}^2 & u_{22}u_{32} & & & \\ & u_{22}u_{32} & u_{32}^2 u_{33}^2 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & u_{ij}u_{i+1,j} \\ & & & & u_{ij}u_{i+1,j} & u_{i,j-1}^2 u_{ij}^2 \end{pmatrix}$$

It then becomes apparent that the algorithm solves this factorization. The upper-leftmost value u_{11}^2 (or d_1^2 in the algorithm) equals the upper-leftmost value in the original tridiagonal matrix, a_1. And solving for d_1 we see $d_1 = \sqrt{a_1}$. The rest is equally trivial to connect if we remember that $L = U^T$ and so $u_{i,i+1} = u_{i+1,i}$ for every entry along the super/sub diagonal, and to fill in the blanks in the algorithm, $v_{i-1} = \frac{b_{i-1}}{d_{i-1}}$, where b_{i-1} is the subdiagonal entry b at index i in the original matrix A; d_i is then calculated by solving:

$$v_{i-1}^2 d_i^2 = a_i$$

$$d_i = \sqrt{a_i}/v_{i-1}$$

b) The equation needs to be solved in two parts: First, we solve $U^T y = z$. Second, using our answer for y, we calculate $Ux = y$. The first part of the algorithm solves for y, by first calculating the trivial case where $d_1 y_1 = z_1$ (because the upper diagonal of U^T is zero) $\rightarrow y_1 = z_1/d_1$.
The remaining y's can be solved by calculating $v_{i-1}y_{i-1} + d_i y_i = z_i$. For $i = 2 : n$, $y_i = (z_i - y_{i-1}v_{i-1})/d_i$.
Once y has been calculated, we solve for x via the equation $Ux = y$. It's important to remember the matrix U is upper diagonal with values on the main diagonal, super diagonal and zeros everywhere else. The trivial case here is no longer located at d_1, but at d_n where $d_n x_n = y_n$. The algorithm takes this into account by first solving $x_n = y_n/d_n$ and then backsolving the remaining x's:

$$d_n x_n + d_{n-1}x_{n-1} = y_{n-1}$$

$$x_{n-1} = (y_{n-1} - d_n x_n)/d_{n-1}$$

Or for the arbitrary case i for $i = n-1 : 1 \rightarrow x_i = (y_i - d_{n+1}x_{i+1})/d_i$

c) Need MATLAB

d)

2 Chapter 4: Questions 10,22,31

4.10

4.22

4.31