CS-6210: HW 3

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1 Chapter 5

5.10 a)

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b)

c)

d)

5.11 a) For $g(x)$ to be a cubic spline, the following basic conditions must be met:

$$g_1(x_1) = y1 \quad g_1(x_2) = y2$$
$$g_2(x_2) = y2 \quad g_2(x_3) = y3$$

$$g_1'(x_2) = g_2'(x_2)$$
$$g_1''(x_2) = g_2''(x_2)$$

In an attempt to find the relationship between α and β, and in order to deduce the values that make $g_1'(x_2) = g_2'(x_2)$ true, we equate the derivatives of g_1 and g_2 and we see that @ $x = 0$ this is true for all values of α and β:

$$6x + 3\alpha x^2 = 2\beta x + 3x^2 \rightarrow \quad true \quad \forall \alpha, \beta \quad @x = 0$$

Taking the second derivatives we see that for $6 + 6\alpha x = 2\beta + 6x$ that at the point where the two curves meet ($x = 0$), we get $2\beta = 6$ or $\beta = 3$. Again here we see that the we're not constrained to a specific value of α. As an aside, interestingly enough if you plug $\beta = 3$ back into the first derivatives and set them equal to each as we did before, you find that $\alpha = -1$ for every value of x where $x \neq 0$ and plugging that α and β into g_1 and g_2 you see that they're equivalent for all values of x:

$$g_1 = 2 + 3x^2 - x^3 \quad @\alpha = -1$$

$$g_2 = 2 + 3x^2 - x^3 \quad @\beta = 3$$

b) Taking $\beta = 3$ we have points $(1, 4)$,$(0, 2)$ and if we assume $\alpha = -1$, (-1,6). If we don't make that assumption, we have $(-1, 5 - \alpha)$

c) We can find the value for which $g(x)$ is a natural cubic spline by setting the second derivatives of g_1 and g_2 at the endpoints to 0 and solving for α and β:

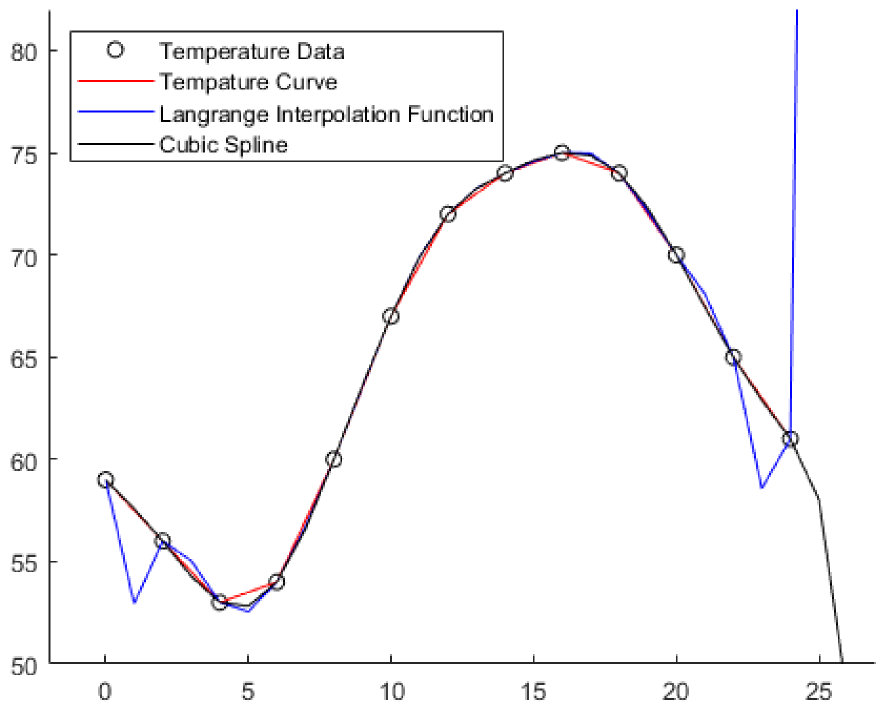
$$\begin{array}{cc} s_1''(-1) = 0 & s_2''(+1) = 0 \\ 6 + 6\alpha x = 0 & 2\beta - 6x = 0 \\ 6\alpha = 6 & \beta = 3 \\ \alpha = 1 & \end{array}$$

d) We can find the value for which $g(x)$ is a clamped cubic spline by setting the first derivatives of g_1 and g_2 equal to y' at the endpoints and solving for α and β. However, since we don't have a distinct value for α from part a, we can't find an exact solution:

$$\begin{array}{rcccc} s_1'(-1) & = & 6x + 3\alpha x^2 & = & y'_{-1} \\ s_1'(-1) & = & -6 + 3\alpha & = & y'_{-1} \\ & & \alpha & = & (y'_{-1} + 6)/3 \end{array}$$

$$\begin{array}{rcccc} s_2'(1) & = & 2\beta x - 3x^2 & = & y'_{+1} \\ s_2'(1) & = & 2\beta - 3 & = & y'_{+1} \\ & & 3 & = & y'_{+1} \end{array}$$

5.15 a) Fitting the data with a global polynomial using Lagrange interpolation, and a natural cubic spline:



The code was all self-written and is located at the end of this file.

b) At 11 AM the two interpolation functions give the following:

$Lagrange$	$NaturalCubicSpline$
$6.9913129e+01$	$6.9881177e+01$
$69.9°$	$69.8°$

c) At 1 AM the next day the two interpolation functions give the following:

$Lagrange$	$NaturalCubicSpline$
$1.5204891e+02$	$5.8029609e+01$
$152.05°$	$58.03°$

d) At 9 AM the next day the sun hides it face for shame at the blazing temperature Lagrange has unleashed upon the world:

$Lagrange$	$NaturalCubicSpline$
$4.5226229e+05$	0
$452,262.29°$	$0°$

Fun fact, this is about 1.8x hotter than the dead star at the center of the Red Spider Nebula ($\approx 250,000°$) which itself is 25 times hotter than the surface of the sun.

5.22 a) Looking first at Lagrange interpolation, we know that the polynomial $p_n(x)$ is a summation as follows:

$$p_n(x) = \Sigma_{i=1}^{n+1} yi\ell_i(x), \quad where \quad \ell_i = \prod_{j=1, i\neq j}^{n+1} \frac{x - x_j}{x_i - x_j}$$

In the formation of the Lagrange coefficient we see two subtractions and one division (3 FLOPS), which are done n times ($n + 1$ - 1 (when $i = j$)). The calculated coefficient is then multiplied by the value y_i (add 1) $n+1$ times (multiply by $n+1$). From this we have $(3n + 1)(n + 1)$. Lastly, each of the those terms is the summed, for a total of n additions. This gives us $(3n^2 + 3n + n + 1) + (n) = 3n^2 + 5n + 1$.

Looking next at the piecewise linear interpolation, we have:

$$G_i(x) = G(\frac{x - x_i}{h}) \quad where \quad G(x) = \begin{cases} 1 - |x| & if \quad |x| \leq 1 \\ 0 & if \quad 1 \leq |x| \end{cases}$$

and we are solving for

$$g(x) = \Sigma_{i=1}^{n+1} y_i G_i(x)$$

As seen above, calculating $y_i G_i(x)$ takes 4 FLOPS (2 to evaluate $\frac{x-x_i}{h}$, 1 to evaluate $1 - |x|$, and 1 to multiply $y_i G_i(x)$), and this is done $n + 1$ times, after which each term is summed for a total of n additions. We have $4(n+1)+n = 5n+4$ total FLOPS.

Lastly, looking at the cubic spline interpolation, we have several parts to consider. The equation to solve takes the form:

$$s(x) = \Sigma_{i=0}^{n+2} \ a_i B_i(x)$$

If we start by looking at the B_i's we see that we need at most 9 FLOPS where each $B_i(x)$ is solved as follows:

$$B_i(x) = B(\frac{x - x_i}{h}) \quad where \quad B(x) = \begin{cases} \frac{2}{3} - x^2(1 - \frac{1}{2}|x|) & if \quad |x| \leq 1 \\ \frac{1}{6}(2 - |x|)^3 & if \quad 1 \leq |x| \leq 2 \\ 0 & if \quad 2 \leq |x| \end{cases}$$

We count 2 to solve for $\frac{x-x_i}{h}$, and 7 to solve for $\frac{2}{3} - x^2(1 - \frac{1}{2}|x|)$). We use this one instead of $\frac{1}{6}(2 - |x|)^3$ because we see that when we evaluate a x the highest FLOP count will come from the first statement.
Now that we have the the B_i's, we solve for the a_i's which requires considerably more steps. To solve for the $n + 3$ coefficients necessary, we need an $n - 1$ by $n - 1$ tridiagonal matrix A, we need and $n - 1$ vector z and we need to evaluate a few a_i's (a_0 and a_{n+2}) by hand. First, a_0 and a_{n+2} require 2 FLOPS each to calculate

$a_0 = 2a_1 - a_2$, and $a_{n+2} = 2a_{n+1} - a_n$ for 4 total. Next, z is formed by making $n-1$ multiplications (every element multiplied by 6) and two subtractions at the first and last elements for $n+1$ FLOPS. Finally, to form A, we assume that each entry in the matrix is added to an initial matrix of zeros, in which case there are $n-1$ entries along the main diagonal and an additional $2(n-2)$ for the sub/super diagonals giving a total of $3n-5$ entries to add.

Once these values are found, the Thomas algorithm solves for the remaining a_i's in 8n-7 FLOPS. Putting it all together, we have (9)+(4)+(n+1)+(3n-5)+(8n-7) = $12n+2$ FLOPS and fortunately we only have to calculate the a_i's once. Returning to our equation for $s(x)$ we have that we calculate the value of the a_i's once for $12n+2$ FLOPS, we then have 9 FLOPS to calculate B_i and 1 multiplication for $a_iB_i(x)$. This is done $n+3$ times and then the values are summed for a total of $n+2$ additions:

$$(12n+2) + (n+3)(10) + (n+2) = 23n + 34$$

From greatest to least we have Lagrange, cubic, piecewise linear.

5.26 a)

5.27 a)

5.28 a)

2 Code:

```
function [ output ] = ch5q15()
%ch5q15
x = linspace(0,24,13);
[m,n] = size(x);
y = [59,56,53,54,60,67,72,74,75,74,70,65,61];

nx = 25+9; %Adding 9 to evaluate at 9am the next day
interpolationPoints = linspace(x(1),x(n)+9,nx);

%p is the output using the interpolation polynomial
p = zeros(nx,1);

%Compute the Lagrange Polynomial
for i = 1:nx
px = 0;
lagrangeCoefficient = ell(interpolationPoints(i),x);
for j = 1:n
px = px + y(j)*lagrangeCoefficient(j);
end
p(i) = px;
end

hold on

output = zeros(nx,2);
output(:,1) = p;
scatter(x,y,'ko');
plot(x,y,'r',interpolationPoints,p,'b');
%The CubicSpline was handled seperately
```

```
output(:,2) = CubicSpline(x,y,nx);
hold off
legend('Temperature Data','Tempature Curve','Langrange Interpolation
    Function','Cubic Spline','Location','northwest');
axis([-2,27,50,82]);

end
```

```
function [coefficients] = ell(interp, xs)
%Computes the Lagrange Coefficients
[m,n] = size(xs);
coefficients = zeros(n,1);

for i=1:n
val = 1;
for j=1:n;
if (i == j)
continue
end
val = val * (interp - xs(j)) / (xs(i)-xs(j));
end
coefficients(i) = val;
end
end
```

```
function [ output ] = CubicSpline(x,y,nx)

h = x(2)-x(1);
[m,n] = size(x);

%s is the output using the cubic spline
s = zeros(1,nx);
interpolationPoints = linspace(x(1),x(n)+9,nx);

n = n-1;
B = zeros(1,n+3);

%%%%Compute the Natural Cubic Spline
% Calculate ai's
A = tridiag(4,1,n-1);
z = zeros(n-1,1);
z(1) = 6*y(2)-y(1);
z(n-1) = 6*y(n)-y(n+1);
for k = 2:n-2
z(k) = 6*y(k+1);
end

%a is incremented to reflect 1-indexing, not zero
a = zeros(1,n+3);
% a(1) & a(n+1)
a(2) = y(1);
a(n+2) = y(n+1);

%  Solve for middle ai's using Thomas Algorithm
a(3:n+1) = Thomas(A,z);
```

```
%  a(0) & a(n+2)
a(1) = 2*a(2)-a(3);
a(n+3) = 2*a(n+2)-a(n+1);

%new vector xx is the vector x with one additonal point at each end
xx = zeros(n+3,1);
xx(2:n+2) = x;
xx(1) = xx(2)-h;
xx(n+3) = xx(n+2)+h;

for i = 1:nx
val = 0;
for j = 1:n+3
%Calculate Bi
cx = (interpolationPoints(i)-xx(j))/h;

if (abs(cx) < 1)
B(j) = (2/3)-(cx^2)*(1-(.5*abs(cx)));
elseif (abs(cx) < 2)
B(j) = ((2-abs(cx))^3)/6;
else
B(j) = 0;
end

val = val + a(j)*B(j);
end
s(i) = val;
end

output = s';
plot(interpolationPoints,s,'k')
hold off
end
```

```
function [x] = Thomas(A,z)
%Thomas Algorithm
%   For solving tridiagonal matrices as given in Table 3.6
[m,n] = size(A);
x = zeros(n,1);
v = zeros(n,1);
w = A(1,1);
x(1) = z(1)/w;

for i = 2:n
v(i) = A(i-1,i)/w;
w = A(i,i)-A(i,i-1)*v(i);
x(i) = (z(i)-A(i,i-1)*x(i-1))/w;
end

for j = (n-1):-1:1
x(j) = x(j)-v(j+1)*x(j+1);
end

end
```

```
function [ output ] = tridiag(a,b,n)
```

```
%Creates an nxn tridiagonal matrix with 'a' along the main diagonal

output = zeros(n);
output(1,1) = a;
for i = 2:n
output(i,i) = a;
output(i-1,i) = b;
output(i,i-1) = b;
end

end
```