CS-6210: HW 4

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1 Chapter 6

6.2 a) Using the composite trapezoidal rule with four subintervals we find that I_T can be calculated as follows:

$$\begin{aligned}I_T &= h\Big(\frac{1}{2}f_1 + f_2 + f_3 + f_4 + \frac{1}{2}f_5\Big)\\ &= h\Big(\frac{1}{2}e^2 + e^1 + e^0 + e^{-1} + \frac{1}{2}e^{-2}\Big)\\ &\approx 3.9242\end{aligned}$$

In order to evaluate the error on this approximation, we need to find $||f''||_\infty$:

$$\begin{aligned}\frac{d}{dx}\Big[e^{-2x}\Big] &= -2e^{-2x}\\ \frac{d^2}{dx^2}\Big[e^{-2x}\Big] &= 4e^{-2x}\end{aligned}$$

$$\begin{aligned}||f''||_\infty &= max_{-1\leq x\leq 1}|4e^{-2x}|\\ ||f''||_\infty &= 4e^2\end{aligned}$$

Approximating our error we see:

$$\begin{aligned}\left|\int_{-1}^{1} e^{-2x}dx - I_T\right| &\leq \frac{2}{12}h^2(4e^2)\\ &\leq 1.2315\end{aligned}$$

b) We are able to use Simpson's rule in this case because the number of subintervals n is even, and I_S can be calculated as follows:

$$\begin{aligned}I_S &= \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + 4f_4 + f_5\Big)\\ &= \frac{1}{6}\Big(e^2 + 4e^1 + 2 + 4e^-1 + e^-2\Big)\\ &\approx 3.6448\end{aligned}$$

In order to evaluate the error on this approximation, we need to find $||f''''||_\infty$:

$$\begin{aligned}\frac{d^3}{dx^3}\Big[e^{-2x}\Big] &= -8e^{-2x}\\ \frac{d^4}{dx^4}\Big[e^{-2x}\Big] &= 16e^{-2x}\end{aligned}$$

$$||f''''||_\infty = max_{-1eqx\leq 1}|16e^{-2x}|$$
$$||f''''||_\infty = 16e^2$$

Approximating our error we see:

$$\left|\int_{-1}^{1} e^{-2x}dx - I_S\right| \leq \frac{2}{90}h^4(16e^2)$$
$$\leq 0.1642$$

c) Using the composite Hermite rule (or corrected trapezoidal rule) with four subintervals we find that I_H can be calculated as follows:

$$I_H = h\Big(\frac{1}{2}f_1 + f_2 + f_3 + f_4 + \frac{1}{2}f_5\Big) + \frac{1}{12}h^2(f_1' - f_{n+1}')$$
$$= \frac{1}{2}\Big(\frac{1}{2}e^2 + e^1 + e^0 + e^{-1} + \frac{1}{2}e^{-2}\Big) + \frac{1}{48}(-2e^2 + 2e^{-2})$$
$$\approx 3.6219$$

In order to evaluate the error on this approximation, we need to find $||f''''||_\infty$ which was calculated previously:

$$||f''''||_\infty = 16e^2$$

Approximating our error we see:

$$\left|\int_{-1}^{1} e^{-2x}dx - I_H\right| \leq \frac{2}{720}h^4(16e^2)$$
$$\leq 0.0205$$

d) From Theorem 6.2, using the trapezoidal rule we use can substitute in the definition of our step h, $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{12}h^2(4e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}}$$

Solving this inequality yields $n \geq 4,439$

e) From Theorem 6.3, using Simpson's rule we use can substitute in the definition of our step $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{90}h^4(16e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}}$$

Solving this inequality yields $n \geq 81$

f) From Theorem 6.4, using Simpson's rule we use can substitute in the definition of our step $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{720}h^4(16e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}}$$

Solving this inequality yields $n \geq 48$

6.4 a) Because we are given that $T = E\frac{du}{dx}$, we know that $\frac{du}{dx} = \frac{1}{E}T$. If we wanted to evaluate the the integral of $\frac{du}{dx}$ in the interval $[0, 1]$ we could write that as:

$$\int_0^x u'(x)\frac{d}{dx} = \frac{1}{E}\int_0^x T(s)ds$$
$$\frac{1}{E}\int_0^x T(s)ds = u(x)\Big|_0^x$$

or equivalently

$$u(x) - u(0) = \frac{1}{E}\int_0^x T(s)ds$$
$$u(x) = u(0) + \frac{1}{E}\int_0^x T(s)ds$$

b) To calculate $u(\frac{1}{4})$ using the trapezoidal rule and the data provided in table 6.10 we use the result from part a and set up the following:

$$u(1/4) = u(0) + \frac{1}{E}\int_0^{\frac{1}{4}} T(s)ds \quad where \quad u(0) = 0, E = 4$$
$$u(1/4) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + \frac{1}{2}(-1)\Big)$$
$$u(1/4) = 0$$

We use the trapezoidal rule similarly to calculate $u(1/2)$, $u(3/4)$, and $u(1)$ using the same values of $u(0)$ and E:

$$u(1/2) = 0 + \frac{1}{4}\int_0^{\frac{1}{2}} T(s)ds$$
$$u(1/2) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + \frac{1}{2}(2)\Big)$$
$$u(1/2) = \frac{1}{32}$$

$$u(3/4) = 0 + \frac{1}{4}\int_0^{\frac{3}{4}} T(s)ds$$
$$u(3/4) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + (2) + \frac{1}{2}(3)\Big)$$
$$u(3/4) = \frac{3}{16}$$

$$u(1) = 0 + \frac{1}{4}\int_0^{1} T(s)ds$$
$$u(1) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + (2) + (3) + \frac{1}{2}(4)\Big)$$
$$u(1) = \frac{13}{32}$$

c) In order to use the composite midpoint rule to evaluate $u(1)$ we need the values of the midpoints of each step. Since we're not give the value of the function at $x = \frac{1}{8}$, we must increase our step size from $\frac{1}{4}$ to $\frac{1}{2}$ so we are able to use the data that is provided as our midpoints:

$$u(1) = 0 + \frac{1}{4}\int_0^{1} T(s)ds$$
$$u(1) = 0 + \frac{1}{4}I_M$$
$$u(1) = 0 + \frac{1}{4} * \frac{1}{2}\Big((-1) + (3)\Big)$$
$$u(1) = \frac{1}{4}$$

d) We can use Simpson's rule to evaluate $u(1)$ since the number of intervals n is even:

$$u(1) = 0 + \frac{1}{4}\int_0^{1} T(s)ds$$
$$u(1) = 0 + \frac{1}{4}I_S$$
$$u(1) = 0 + \frac{1}{4} * \frac{1}{12}\Big((1) + 4(-1) + 2(2) + 4(3) + (4)\Big)$$
$$u(1) = \frac{17}{48}$$

e) If we use our result from part b we see $I_T = \frac{13}{8}$ and we use this in place of $I_T(2n)$. Following the theorem in the book we find that the Romberg Integration using the trapezoidal rule takes the form:

$$\int_a^b f(x)dx = \frac{4}{3}I_T(2n) - \frac{1}{3}I_T(n) + O(h^3)$$

Solving for $I_T(n)$ we get:

$$I_T(n) = \frac{1}{2}\Big(\frac{1}{2}(1) + (2) + \frac{1}{2}(4)\Big)$$
$$I_T(n) = \frac{9}{4}$$

Putting it all together we see:

$$u(1) = \frac{4}{3}(\frac{13}{8}) - \frac{1}{3}(\frac{9}{4})$$
$$u(1) = \frac{17}{12}$$

6.8 a) Using the trapezoidal rule, we know our error terms looks like the following (evaluated at $erf(2)$):

$$\left|\frac{2}{\sqrt{\pi}}\int_0^2 e^{-s^2}ds\right| \leq \frac{2}{12}h^2||(4x^2-2)e^{-x^2}||_\infty$$

Taking the norm of $||f''||_\infty$ to be .8925 (solved using MATLAB), we solve for h:

$$\frac{2}{12}h^2(.8925) \leq 10^{-6}$$
$$h \leq \sqrt{\frac{10^{-6}}{.8925} * 6}$$
$$h \leq 2.592815e-03$$

b) Issue with this MATLAB code. Not particularly close to the true value..

```
function [output] = ch6q8()
% Solves for h using the trapezoidal rule
x = 2;
IT = 0;

erf2 = 0.995322265;

error = 1;
n = 200000;
while abs(error) > 10e-7
% while n < 10
h = x/n;
%     fprintf('N = %d; Step size is %d; ',n, h);
IT = (.5*exp(0) + .5*(1/exp((2)^2)));
%     fprintf('Intermediate eval at ');
for i = 1:n-1
IT = IT + (1/ exp((i*h)^2));
%         fprintf('x = %d; ',(i)*h);
end
IT = IT * h;
error = erf2 - IT
n = n+1;
end
end
```

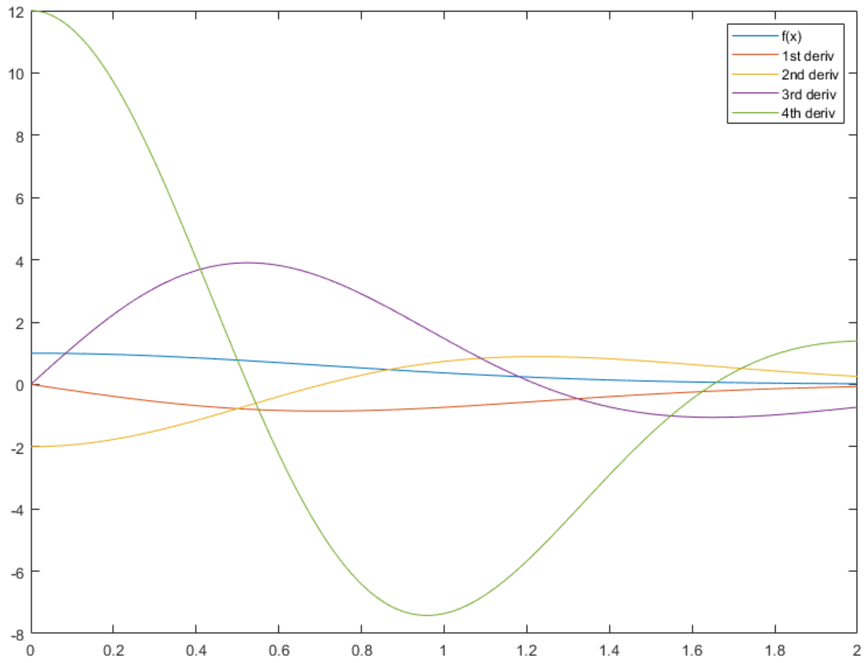
c) Using Simpson's rule, we know our error terms looks like the following (evaluated at $erf(2)$):

$$\left| \frac{2}{\sqrt{\pi}} \int_0^2 e^{-s^2} ds \right| \leq \frac{2}{90} h^4 ||f''''||_\infty$$

In this case, finding f'''' is not as trivial, so we include the process:

$$\begin{aligned}
\frac{d^2}{dx^2} f(x) &= (4x^2 - 2)e^{-x^2} \\
\frac{d^3}{dx^3} f(x) &= -2x(4x^2 - 2)e^{-x^2} + 8xe^{-x^2} \\
&= (-8x^3 + 12x)e^{-x^2} \\
\frac{d^4}{dx^4} f(x) &= (-24x^2 + 12)e^{-x^2} + -2x(-8x^3 + 12x)e^{-x^2} \\
&= (-16x^4 - 48x^2 + 12)e^{-x^2}
\end{aligned}$$

If we plot each these curves we can see at f'''' the maximum value is obtained when $x = 0$ and $||f''''||_\infty = 12$:



Taking the norm of $||f''''||_\infty$ to be 12 (solved using MATLAB), we solve for h:

$$\begin{aligned}
\frac{2}{90} h^4 (12) &\leq 10^{-6} \\
h &\leq \sqrt[4]{\frac{10^{-6}}{12} * \frac{90}{2}} \\
h &\leq 4.4005587e - 02
\end{aligned}$$

d) Using the same code from part B adapted for Simpson's rule...

6.15 a) Since we're given that $v(0) = 0$, and for any interval from 0 to t we get:

$$\int_0^t a(r)dr = v(t) - v(0) = v(t)$$

and we know the trapezoidal rule gives us

$$\int_0^t a(r)dr = h\Big(\frac{1}{2}a_0 + \frac{1}{2}a_t\Big)$$

we can combine the results to give us the following for the subinterval $t_i \leq t \leq t_{i+1}$

$$\begin{aligned}
\int_i^{i+1} a(r)dr &= v(i+1) - v(i) \\
\int_i^{i+1} a(r)dr &= h\Big(\frac{1}{2}a_i + \frac{1}{2}a_{i+1}\Big) \\
v(i+1) - v(i) &= h\Big(\frac{1}{2}a_i + \frac{1}{2}a_{i+1}\Big) \\
v(i+1) &= v(i) + \frac{1}{2}h\Big(a_i + a_{i+1}\Big)
\end{aligned}$$

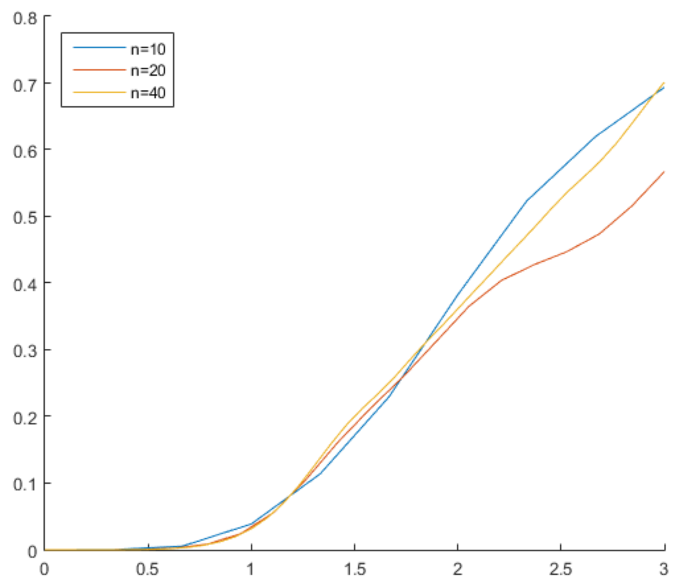
The same logic applies to solving for y_{i+1} since we know $y(0) = 0$, and for any interval from 0 to t we get:

$$\int_0^t v(r)dr = y(t) - y(0) = y(t)$$

On the interval from $t_i \leq t \leq t_{i+1}$ we get:

$$\begin{aligned}
y(i+1) - y(i) &= h\Big(\frac{1}{2}v_i + \frac{1}{2}v_{i+1}\Big) \\
y(i+1) &= y(i) + \frac{1}{2}h\Big(v_i + v_{i+1}\Big)
\end{aligned}$$

b) Plotting $y(t)$ for $n = 10$, 20, and 40 yields the following:



c) For each value of n, the computed value of $y(t)$ was:

n	$computed$	$difference$
10	0.69359564141	$3.3727e-02$
20	0.56714220363	$1.6018e-01$
40	0.70120254590	$2.6120e-02$

In order to for our error to be less than 10e-8 we solve for n using the trapezoidal error. We also use Matlab to calculate $||_\infty \leq 10^{-8}$ as :

$$|f''||_\infty \approx 1.147687e+04$$

Which allows us to solve for n as follows:

$$\frac{3}{12}h * 2||f''||_\infty \leq 10^-8$$
$$h \leq \sqrt{\frac{10^{-8}}{11477} * 4} \quad where \quad h = \frac{3}{n}$$
$$n \geq \frac{3}{\sqrt{\frac{10^{-8}}{11477} * 4}}$$
$$n \geq 160,696$$

6.18 a) If we look first at I_M we see that the function is evaluated at the midpoints between our known function values. Since Simpson's rule does not use midpoints, we need a way to convert this, and we find that by doubling the step size, our midpoint rule gives us every other function value (e.g. midpoint between f_1 and f_3, f_3 and f_5, etc.):

$$I_M(n) = h\Big(f_{1+\frac{1}{2}} + f_{2+\frac{1}{2}} + \cdots + f_{n+\frac{1}{2}}\Big)$$
$$I_M(n/2) = 2h\Big(f_2 + f_4 + \dots\Big)$$

Combining this result with the trapezoidal rule we show fairly easily that in the proportions outlined in the question, we arrive back at Simpson's rule:

$$I_S(n) = \frac{2}{3}I_T(n) + \frac{1}{3}I_M\Big(\frac{n}{2}\Big)$$
$$I_S(n) = \frac{2}{3}h\Big(\frac{1}{2}f_1 + f_2 + \cdots + \frac{1}{2}f_{n+1}\Big) + \frac{1}{3}2h\Big(f_2 + f_4 + \dots\Big)$$
$$I_S(n) = \frac{h}{3}\Big(f_1 + 2f_2 + 2f_3 + \cdots + f_{n+1}\Big) + \frac{h}{3}\Big(2f_2 + 2f_4 + \dots\Big)$$
$$I_S(n) = \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + \cdots + f_{n+1}\Big)$$

b) When we look at $I_T\Big(\frac{n}{2}\Big)$ we see a similar trend occur as in part a, where since our number of subdivisions is halved over the same interval, our step subsequently doubles, yielding:

$$I_T\Big(\frac{n}{2}\Big) = 2h\Big(\frac{1}{2}f_1 + f_3 + f_5 + \cdots + \frac{1}{2}f_{n+1}\Big)$$
$$I_T\Big(\frac{n}{2}\Big) = h\Big(f_1 + 2f_3 + 2f_5 + \cdots + f_{n+1}\Big)$$

If we plug this back into the equation given in the question, we see that again for these proportions, again, we get back Simpson's rule:

$$
\begin{aligned}
I_S(n) &= \frac{4}{3}I_T(n) - \frac{1}{3}I_T\Big(\frac{n}{2}\Big) \\
I_S(n) &= \frac{4}{3}h\Big(\frac{1}{2}f_1 + f_2 + \cdots + \frac{1}{2}f_{n+1}\Big) - \frac{1}{3}h\Big(f_1 + 2f_3 + 2f_5 + \cdots + f_{n+1}\Big) \\
I_S(n) &= \frac{h}{3}\Big(2f_1 + 4f_2 + 4f_3 + \cdots + 2f_{n+1}\Big) - \frac{h}{3}\Big(f_1 + 2f_3 + 2f_5 + \cdots + f_{n+1}\Big) \\
I_S(n) &= \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + \cdots + f_{n+1}\Big)
\end{aligned}
$$

6.19 a)

b)

c)

6.20 a) It's given that the error involved with Simpson's rule takes the form

$$I_S(n) + \alpha h^4 + \beta h^6 + \gamma h^8 + \ldots$$

and so substituting in the known error for $I_S(n)$ and $I_S(n/2)$ yields the following:

$$
\begin{aligned}
I_R &= \frac{1}{15}\Big[16I_S(2n) - I_S\Big] \\
I_R &= \frac{1}{15}\Big[16(\alpha\frac{h}{2}^4 + \beta\frac{h}{2}^6 + \gamma\frac{h}{2}^8 + \ldots) - (\alpha h^4 + \beta h^6 + \gamma h^8 + \ldots)\Big] \\
I_R &= \frac{1}{15}\Big[\alpha\frac{16h^4}{16} + \beta\frac{16h^6}{64} + \gamma\frac{h}{2}^8 + \ldots) - (\alpha h^4 + \beta h^6 + \gamma h^8 + \ldots)\Big] \\
I_R &= \frac{1}{15}\Big[\alpha h^4 + \beta\frac{h^6}{4} + \gamma\frac{h}{2}^8 + \ldots) - (\alpha h^4 + \beta h^6 + \gamma h^8 + \ldots)\Big] \\
I_R &= \frac{1}{15}\Big[\beta\frac{h^6}{4} + \gamma\frac{h}{2}^8 + \ldots) - (\beta h^6 + \gamma h^8 + \ldots)\Big]
\end{aligned}
$$

We assume here that since the number of sub intervals is doubled, the step size is necessarily halved. Also, we see the α error term is eliminated, leaving the dominant term here as h^6 meaning $I_R = O(h^6)$

b)

c)

6.21 a) Since we're solving for three unknowns, we need at least 3 equations. These are given in the book in Table 6.5 as:

k	$f(x)$	$\int_{x_i}^{x_{i+1}} f(x)dx)$
0	1	h
1	x	$h\Big(x_i + \frac{1}{2}h\Big)$
2	x^2	$h\Big(x_i^2 + hx_i + \frac{1}{3}h^2\Big)$

First taking $k = 0$ we solve for w_1:

$$
\begin{aligned}
h &= w_1 + w_2 \\
w_1 &= h - w_2
\end{aligned}
$$

Using our second formula we solve for w_2 remembering that in the problem, z is given as $x_i + \alpha h$ where we are to solve for α:

$$\begin{aligned}
h\Big(x_i + \frac{1}{2}h\Big) &= w_1 x_i + w_2 z \\
&= (h - w_2)x_i + w_2 x \\
hx_i + \frac{1}{2}h^2 &= hx_i - w_2 x_i + w2_z \\
\frac{1}{2}h^2 &= -w_2 x_i + w_2 z \\
\frac{h^2}{2} &= w_2(z - x_i) \\
\frac{h^2}{2} &= w_2(x_i + \alpha h - x_i) \\
\frac{h^2}{2} &= w_2(\alpha h) \\
w_2 &= \frac{h}{2\alpha}
\end{aligned}$$

Using our third formula and having equations in place for w_1 and w_2 we can solve for α as follows:

$$\begin{aligned}
h\Big(x_i^2 + hx_i + \frac{1}{3}h^2\Big) &= w_1 x_i^2 + w_2 z^2 \\
&= (h - w_2)x_i^2 + w_2 z^2 \\
&= hx_i^2 - w_2 x_i^2 + w_2(x_i + \alpha h)^2 \\
&= hx_i^2 - w_2 x_i^2 + w_2(x_i^2 + 2x_i\alpha h + \alpha^2 h^2) \\
&= hx_i^2 + w_2(2x_i\alpha h + \alpha^2 h^2) \\
&= hx_i^2 + \frac{h}{2\alpha}(2x_i\alpha h + \alpha^2 h^2) \\
hx_i^2 + h^2 x_i + \frac{h^3}{3} &= hx_i^2 + h^2 x_i + \frac{h^3\alpha}{2} \\
\frac{h^3}{3} &= \frac{h^3\alpha}{2} \\
\alpha &= \frac{2}{3}
\end{aligned}$$

b)