CS-6210: HW 4

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1 Chapter 6

6.2 a) Using the composite trapezoidal rule with four subintervals we find that I_T can be calculated as follows:

$$\begin{aligned} I_T &= h\Big(\frac{1}{2}f_1 + f_2 + f_3 + f_4 + \frac{1}{2}f_5\Big) \\ &= h\Big(\frac{1}{2}e^2 + e^1 + e^0 + e^{-1} + \frac{1}{2}e^{-2}\Big) \\ &\approx 3.9242 \end{aligned}$$

In order to evaluate the error on this approximation, we need to find $||f''||_\infty$:

$$\begin{aligned} \frac{d}{dx}\Big[e^{-2x}\Big] &= -2e^{-2x} \\ \frac{d^2}{dx^2}\Big[e^{-2x}\Big] &= 4e^{-2x} \end{aligned}$$

$$\begin{aligned} ||f''||_\infty &= max_{-1\leq x\leq 1}|4e^{-2x}| \\ ||f''||_\infty &= 4e^2 \end{aligned}$$

Approximating our error we see:

$$\begin{aligned} \left|\int_{-1}^{1} e^{-2x}dx - I_T\right| &\leq \frac{2}{12}h^2(4e^2) \\ &\leq 1.2315 \end{aligned}$$

b) We are able to use Simpson's rule in this case because the number of subintervals n is even, and I_S can be calculated as follows:

$$\begin{aligned} I_S &= \frac{h}{3}\Big(f_1 + 4f_2 + 2f_3 + 4f_4 + f_5\Big) \\ &= \frac{1}{6}\Big(e^2 + 4e^1 + 2 + 4e^-1 + e^-2\Big) \\ &\approx 3.6448 \end{aligned}$$

In order to evaluate the error on this approximation, we need to find $||f''''||_\infty$:

$$\begin{aligned} \frac{d^3}{dx^3}\Big[e^{-2x}\Big] &= -8e^{-2x} \\ \frac{d^4}{dx^4}\Big[e^{-2x}\Big] &= 16e^{-2x} \end{aligned}$$

$$||f''''||_\infty = max_{-1eqx\leq 1}|16e^{-2x}|$$
$$||f''''||_\infty = 16e^2$$

Approximating our error we see:

$$\left|\int_{-1}^{1} e^{-2x}dx - I_S\right| \leq \frac{2}{90}h^4(16e^2)$$
$$\leq 0.1642$$

c) Using the composite Hermite rule (or corrected trapezoidal rule) with four subintervals we find that I_H can be calculated as follows:

$$\begin{aligned} I_H &= h\Big(\frac{1}{2}f_1 + f_2 + f_3 + f_4 + \frac{1}{2}f_5\Big) + \frac{1}{12}h^2(f_1' - f_{n+1}') \\ &= \frac{1}{2}\Big(\frac{1}{2}e^2 + e^1 + e^0 + e^{-1} + \frac{1}{2}e^{-2}\Big) + \frac{1}{48}(-2e^2 + 2e^{-2}) \\ &\approx 3.6219 \end{aligned}$$

In order to evaluate the error on this approximation, we need to find $||f''''||_\infty$ which was calculated previously:

$$||f''''||_\infty = 16e^2$$

Approximating our error we see:

$$\left|\int_{-1}^{1} e^{-2x}dx - I_H\right| \leq \frac{2}{720}h^4(16e^2)$$
$$\leq 0.0205$$

d) From Theorem 6.2, using the trapezoidal rule we use can substitute in the definition of our step h, $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{12}h^2(4e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{4e^2}\frac{12}{2}\Big)^{1/2}}$$

Solving this inequality yields $n \geq 4,439$

e) From Theorem 6.3, using Simpson's rule we use can substitute in the definition of our step $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{90}h^4(16e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{16e^2}\frac{90}{2}\Big)^{1/4}}$$

Solving this inequality yields $n \geq 81$

f) From Theorem 6.4, using Simpson's rule we use can substitute in the definition of our step $h = \frac{b-a}{n}$ in to the following equation for error:

$$\frac{2}{720}h^4(16e^2) \leq 10^{-6}$$
$$h \leq \Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}$$
$$\frac{2}{n} \leq \Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}$$
$$n \geq \frac{2}{\Big(\frac{10^{-6}}{16e^2}\frac{720}{2}\Big)^{1/4}}$$

Solving this inequality yields $n \geq 48$

6.4 a) Because we are given that $T = E\frac{du}{dx}$, we know that $\frac{du}{dx} = \frac{1}{E}T$. If we wanted to evaluate the the integral of $\frac{du}{dx}$ in the interval $[0, 1]$ we could write that as:

$$\int_0^x u'(x)\frac{d}{dx} = \frac{1}{E}\int_0^x T(s)ds$$
$$\frac{1}{E}\int_0^x T(s)ds = u(x)\Big|_0^x$$

or equivalently

$$u(x) - u(0) = \frac{1}{E}\int_0^x T(s)ds$$
$$u(x) = u(0) + \frac{1}{E}\int_0^x T(s)ds$$

b) To calculate $u(\frac{1}{4})$ using the trapezoidal rule and the data provided in table 6.10 we use the result from part a and set up the following:

$$u(1/4) = u(0) + \frac{1}{E}\int_0^{\frac{1}{4}} T(s)ds \quad where \quad u(0) = 0, E = 4$$
$$u(1/4) = 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + \frac{1}{2}(-1)\Big)$$
$$u(1/4) = 0$$

We use the trapezoidal rule similarly to calculate $u(1/2)$, $u(3/4)$, and $u(1)$ using the same values of $u(0)$ and E:

$$
\begin{aligned}
u(1/2) &= 0 + \frac{1}{4}\int_0^{\frac{1}{2}} T(s)ds \\
u(1/2) &= 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + \frac{1}{2}(2)\Big) \\
u(1/2) &= \frac{1}{32}
\end{aligned}
$$

$$
\begin{aligned}
u(3/4) &= 0 + \frac{1}{4}\int_0^{\frac{3}{4}} T(s)ds \\
u(3/4) &= 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + (2) + \frac{1}{2}(3)\Big) \\
u(3/4) &= \frac{3}{16}
\end{aligned}
$$

$$
\begin{aligned}
u(1) &= 0 + \frac{1}{4}\int_0^{1} T(s)ds \\
u(1) &= 0 + \frac{1}{4} * \frac{1}{4}\Big(\frac{1}{2}(1) + (-1) + (2) + (3) + \frac{1}{2}(4)\Big) \\
u(1) &= \frac{13}{32}
\end{aligned}
$$

c) In order to use the composite midpoint rule to evaluate $u(1)$ we need the values of the midpoints of each step. Since we're not give the value of the function at $x = \frac{1}{8}$, we must increase our step size from $\frac{1}{4}$ to $\frac{1}{2}$ so we are able to use the data that is provided as our midpoints:

$$
\begin{aligned}
u(1) &= 0 + \frac{1}{4}\int_0^{1} T(s)ds \\
u(1) &= 0 + \frac{1}{4} I_M \\
u(1) &= 0 + \frac{1}{4} * \frac{1}{2}\Big((-1) + (3)\Big) \\
u(1) &= \frac{1}{4}
\end{aligned}
$$

d) We can use Simpson's rule to evaluate $u(1)$ since the number of intervals n is even:

$$
\begin{aligned}
u(1) &= 0 + \frac{1}{4}\int_0^{1} T(s)ds \\
u(1) &= 0 + \frac{1}{4} I_S \\
u(1) &= 0 + \frac{1}{4} * \frac{1}{12}\Big((1) + 4(-1) + 2(2) + 4(3) + (4)\Big) \\
u(1) &= \frac{17}{48}
\end{aligned}
$$

e) If we use our result from part b we see $I_T = \frac{13}{8}$ and we use this in place of $I_T(2n)$. Following the theorem in the book we find that the Romberg Integration using the trapezoidal rule takes the form:

$$
\int_a^b f(x)dx = \frac{4}{3} I_T(2n) - \frac{1}{3} I_T(n) + O(h^3)
$$

Solving for $I_T(n)$ we get:

$$I_T(n) = \frac{1}{2}\Big(\frac{1}{2}(1) + (2) + \frac{1}{2}(4)\Big)$$
$$I_T(n) = \frac{9}{4}$$

Putting it all together we see:

$$u(1) = \frac{4}{3}(\frac{13}{8}) - \frac{1}{3}(\frac{9}{4})$$
$$u(1) = \frac{17}{12}$$

6.8 a) Using the trapezoidal rule, we know our error terms looks like the following (evaluated at $erf(2)$):

$$\left|\frac{2}{\sqrt{\pi}}\int_0^2 e^{-s^2}ds\right| \leq \frac{2}{12}h^2||(4x^2-2)e^{-x^2}||_\infty$$

Taking the norm of $||f''||_\infty$ to be .8925 (solved using MATLAB), we solve for h:

$$\frac{2}{12}h^2(.8925) \leq 10^{-6}$$
$$h \leq \sqrt{\frac{10^{-6}}{.8925} * 6}$$
$$h \leq 2.592815e - 03$$

b) Issue with this MATLAB code. Not particularly close to the true value..

```
function [output] = ch6q8()
% Solves for h using the trapezoidal rule
x = 2;
IT = 0;

erf2 = 0.995322265;

error = 1;
n = 200000;
while abs(error) > 10e-7
% while n < 10
h = x/n;
%     fprintf('N = %d; Step size is %d; ',n, h);
IT = (.5*exp(0) + .5*(1/exp((2)^2)));
%     fprintf('Intermediate eval at ');
for i = 1:n-1
IT = IT + (1/ exp((i*h)^2));
%         fprintf('x = %d; ',(i)*h);
end
IT = IT * h;
error = erf2 - IT
n = n+1;
end
end
```

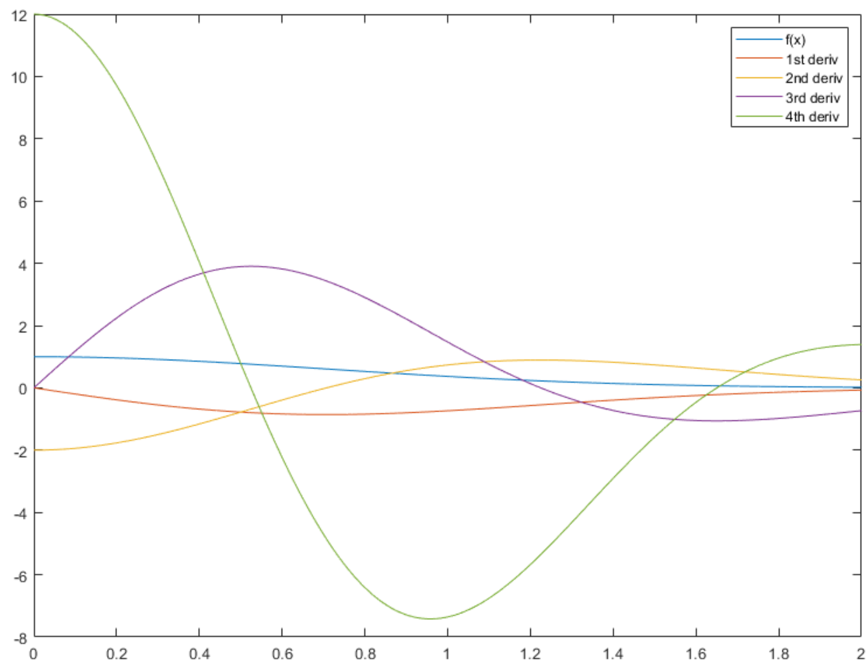
c) Using Simpson's rule, we know our error terms looks like the following (evaluated at $erf(2)$):

$$\left| \frac{2}{\sqrt{\pi}} \int_0^2 e^{-s^2} ds \right| \leq \frac{2}{90} h^4 ||f''''||_\infty$$

In this case, finding f'''' is not as trivial, so we include the process:

$$\begin{aligned}
\frac{d^2}{dx^2} f(x) &= (4x^2 - 2)e^{-x^2} \\
\frac{d^3}{dx^3} f(x) &= -2x(4x^2 - 2)e^{-x^2} + 8xe^{-x^2} \\
&= (-8x^3 + 12x)e^{-x^2} \\
\frac{d^4}{dx^4} f(x) &= (-24x^2 + 12)e^{-x^2} + -2x(-8x^3 + 12x)e^{-x^2} \\
&= (-16x^4 - 48x^2 + 12)e^{-x^2}
\end{aligned}$$

If we plot each these curves we can see at f'''' the maximum value is obtained when $x = 0$ and $||f''''||_\infty = 12$:



Taking the norm of $||f''''||_\infty$ to be 12 (solved using MATLAB), we solve for h:

$$\begin{aligned}
\frac{2}{90} h^4 (12) &\leq 10^{-6} \\
h &\leq \sqrt[4]{\frac{10^{-6}}{12} * \frac{90}{2}} \\
h &\leq 4.4005587e - 02
\end{aligned}$$

d) Using the same code from part B adapted for Simpson's rule...

6.15 a)

b)
c)
d)

6.18 a)
b)

6.19 a)
b)
c)

6.20 a)
b)
c)

6.21 a)
b)