CS-6210: HW 2

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1 Chapter 4

4.22 a) I calculated the eigenvalues by hand to be $\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$. Using the orthogonal iteration proceedure found in Table 4.9 of the text and with the modified Gram-Schmidt described on page 144, I constructed the following program used for compare my calculated eigenvalues above and consequently for part c below:

```
%Orthogonal Iteration Algorithm
function [out] = ch4q22(n)

%Value used to shift
omega = 4;
%Setup matrix A based on input n
A =  zeros(n);
for k=2:n
    A(k-1,k) = (k-1) / sqrt((2*(k-1) - 1)*(2*(k-1) + 1));
    A(k,k-1) = A(k-1,k);
end
As = A - omega*eye(n);

%Randomly select B and calculate preliminary eigenvalues
B = rand(n,n);
Q = GS(B);
eigenvals = zeros(n,1);

eigenvec = Q(:,1);
eigenval = dot(eigenvec, A*eigenvec) / dot(eigenvec, eigenvec);

%Run orthogonal iteration until condition satisfied
iterativeErr = 10000;
while abs(iterativeErr) > 1e-12
    B = As*Q;
    Q = GS(B);

    e = eigenval;
    eigenvec = Q(:,1);
    eigenval = dot(eigenvec, As*eigenvec) / dot(eigenvec, eigenvec);

    iterativeErr = e - eigenval;
end

%Compute eigenvalues
for k=1:n
    eigenvec = Q(:,k);
    eigenval = dot(eigenvec, As*eigenvec) / dot(eigenvec, eigenvec);
    eigenvals(k) = eigenval + omega;
```

```
end

out = eigenvals;
end

%Gram-Schmidt Algorithm
function [Q] = GS(A)
[m,n] = size(A);
e = zeros(m,n);
for j=1:n
    e(:,j) = A(:,j);
    for k=1:j-1
        r = dot(e(:,j),e(:,k)) / dot(e(:,k),e(:,k));
        e(:,j) = e(:,j) - r*e(:,k);
    end
    e(:,j) = e(:,j)/norm(e(:,j));
end

Q = e;
end
```

---

b) My approximations of 1 were off by 1.718e-06 in either direction. The exact values were:

$$1 - 2q_{11}^2 = 1.717584309335329e - 06$$

$$1 - 2q_{21}^2 = -1.717584309446352e - 06$$

c) Below are the returned eigenvalues of A computed using orthogonal iteration. You'll notice the the values are symmetric, which is a good sign considering the matrix A is also symmetric. Additional confidence comes from knowing the relationship between $tr(A)$ and the sum of the eigenvalues, namely that they should be equal, and in this case, the diagonals are 0 and the eigenvalues symmetric, hinting that our results appear to be valid.

$$\lambda_i' s = \begin{pmatrix} -0.9739 \\ -0.8651 \\ -0.6794 \\ -0.4334 \\ -0.1489 \\ 0.1489 \\ 0.4334 \\ 0.6794 \\ 0.8651 \\ 0.9739 \end{pmatrix}$$

d) Calculating the $\overline{w}_j$'s for each eigenvector resulted in the below vector w where

$$w_i = 2q_{j1}^2$$

$$w = \begin{pmatrix} 0.0667 \\ 0.3355 \\ 0.0405 \\ 0.3759 \\ 0.2300 \\ 0.2045 \\ 0.3612 \\ 0.1445 \\ 0.2854 \\ 0.0196 \end{pmatrix}$$

4.31 a) Given A, λ_i, and x_i as below:

$$A = \begin{pmatrix} a & b & & & & \\ b & a & c & & & \\ & b & a & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & b & a & c \\ & & & & b & a \end{pmatrix}$$

$$\lambda_i = a + 2|b|cos(i\theta) \quad for \quad i = 1, 2, \ldots, n \quad where \quad \theta = \frac{\pi}{n+1}$$

$$x_i = (sin(\phi), sin(2\phi), \ldots, sin(n\phi)) \quad for \quad \phi = i\theta$$

If x_i is and eigenvector for the eigenvalue λ_i, then the following should hold:

$$(A - \lambda_i I)x_i = 0$$

Solving for $A - \lambda_i I$ we see that the diagonals a become:

$$a - (a + 2|b|cos(i\theta)) = -2|b|cos(i\theta)$$

and we solve:

$$\begin{pmatrix} -2|b|cos(i\theta) & b & & & \\ b & -2|b|cos(i\theta) & c & & \\ & \ddots & \ddots & \ddots & \\ & & b & -2|b|cos(i\theta) & \end{pmatrix} \begin{pmatrix} sin(i\theta) \\ sin(2i\theta) \\ sin(3i\theta) \\ \vdots \\ sin(ni\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

There are three cases to show this holds true: the two end cases $b + (a - \lambda_i)$ and $(a - \lambda_i) + c$, and the general case in between where $b + (a - \lambda_i) + c$

$(a - \lambda_i) + c$:

Here we use the trigonometric identity $sin(2x) = sin(x)cos(x)$ and substitute b for c since $b = c$

$$-2|b|cos(i\theta)sin(i\theta) + bsin(2i\theta) = 0$$

$$-|b|(2sin(i\theta)cos(i\theta)) + bsin(2i\theta) = 0$$

$$-b(sin(2i\theta)) + bsin(2i\theta) = 0$$

$$0 = 0$$

$b + (a - \lambda_i) + c$:

Here we use the trigonometric identity $sin(\alpha \pm \beta) = sin(\alpha)cos(\beta) \pm sin(\beta)cos(\alpha)$

$$bsin(i\theta) - 2|b|cos(i\theta)sin(2i\theta) + bsin(3i\theta) = 0$$

$$bsin(i\theta) - 2|b|cos(i\theta)sin(2i\theta) + bsin(2i\theta + i\theta) = 0$$

$$bsin(i\theta) - 2|b|cos(i\theta)sin(2i\theta) + b[sin(2i\theta)cos(i\theta) + sin(i\theta)cos(2i\theta)] = 0$$

$$bsin(i\theta) - bcos(i\theta)sin(2i\theta) + bsin(i\theta)cos(2i\theta) = 0$$

$$bsin(i\theta) - b(cos(i\theta)sin(2i\theta) - sin(i\theta)cos(2i\theta)) = 0$$

$$bsin(i\theta) - bsin(2i\theta - i\theta) = 0$$

$$0 = 0$$

$b + (a - \lambda_i)$:

Here we use the trigonometric identity $sin(\alpha \pm \beta) = sin(\alpha)cos(\beta) \pm sin(\beta)cos(\alpha)$

$$bsin((n-1)i\theta) - 2|b|cos(i\theta)sin(ni\theta) = 0$$

$$bsin(ni\theta - i\theta) - 2|b|cos(i\theta)sin(ni\theta) = 0$$

$$b[sin(ni\theta)cos(i\theta) - sin(i\theta)cos(ni\theta)] - 2|b|cos(i\theta)sin(ni\theta) = 0$$

$$-b(sin(ni\theta)cos(i\theta) - bcos(ni\theta)sin(i\theta) = 0$$

$$-b[sin(ni\theta)cos(i\theta) + cos(ni\theta)sin(i\theta)] = 0$$

$$-bsin((n+1)i\theta) = 0$$

$$-bsin(\frac{(n+1)i\pi}{n+1}) = 0$$

$$-bsin(i\pi) = 0$$

$$0 = 0$$

b) Because $cos(i\theta)$ simply oscillates between -1 and 1, we have in the worst case that $cos(i\theta) = -1$. Because the problem posits that $\lambda_i = a + 2|b|cos(i\theta)$ and we are given that $a \geq 2|b|$, we see that in the worst case that $cos(i\theta) = -1$ and a is no bigger than $2|b|$:

$$a + 2|b|(-1)$$

$$a - 2|b|$$

$$2|b| - 2|b| = 0$$

This does two things. First, it precludes the matrix A from being classified as *strictly diagonally dominant* since it is not strictly greater than the sum of the other elements in it's row (the first method I tried of proving positive definiteness), so we can't prove it that way. Second, it shows that by Theorem 3.1 we fail altogether in proving positive definiteness in that we can't prove that A has only positive eigenvalues. Without the constraint the $a \geq 2|b|$ tightened to $a > 2|b|$ we can only prove semi-positive definiteness.

c) Because A is symmetric, according to Theorem 4.1, if x_i and x_j are eigenvectors for different eigenvalues (as we proved in part a), then: $x_i \bullet x_j = 0$
Additionally, to show $x_i \bullet x_i = \frac{n+1}{2}$ we use the trigonometric identity $sin^2(x) = \frac{1}{2}(1 - cos(2x))$

$$x_i \bullet x_i = sin^2(i\theta) + sin^2(2i\theta) + \cdots + sin^2(ni\theta)$$

$$= \frac{1}{2}(1 - cos(2i\theta)) + \frac{1}{2}(1 - cos(4i\theta)) + \cdots + \frac{1}{2}(1 - cos(2nix))$$

$$= \frac{(1 - cos(2i\theta)) + (1 - cos(4i\theta)) + \cdots + (1 - cos(2nix))}{2}$$

$$= \frac{n - \Sigma_{k=1}^{n} cos(\frac{2ki\pi}{n+1})}{2}$$

As it turns out because cosine has a period of 2π that $\forall n \geq 3 \quad \Sigma_{k=1}^{n} cos(\frac{2ki\pi}{n+1}) = -1$

$$x_i \bullet x_i = \frac{n - \Sigma_{k=1}^{n} cos(\frac{2ki\pi}{n+1})}{2} = \frac{n+1}{2}$$

d) Taking a look at the new λ_i we see in interesting coincidence stemming from the symmetry of the matrix, namely that $b = c$ so in substituting back into

$$\lambda_i = a + 2\sqrt{bc}cos(i\theta) = a + 2\sqrt{b^2}cos(i\theta) = a + 2bcos(i\theta)$$

From part a we determined $\lambda_i = a + 2bcos(i\theta)$ assuming $b > 0$ and in part c we solved $\lambda_i = a + 2bcos(i\theta)$ for $b = c$.
Turning our attention to the additional constraint $\kappa = \sqrt{\frac{b}{c}}$ again we see that due to the symmetric nature of A, $b = c$ and $\kappa = \sqrt{\frac{b}{b}} = \sqrt{1} = 1$. Substituting this back into x_i we see the structure has not changed from that of part a in that:

$$x_i = (\kappa sin(\phi), \kappa^2 sin(2\phi), \ldots, \kappa^n sin(n\phi)$$

$$x_i = (1sin(\phi), 1^2 sin(2\phi), \ldots, 1^n sin(n\phi)$$

$$x_i = (sin(\phi), sin(2\phi), \ldots, sin(n\phi))$$

And in part a we proved that x_i is an eigenvector for λ_i if the same conditions held true.

2 Chapter 9

9.2 a) The result of the normalization using $\overline{X} = 2.8215$, $\overline{Y} = 2.4224$, and scaling X by 7.1118 and Y by 2.0859 is:

p	q
−1.0000	0.5963
−0.8151	0.6407
−0.6096	0.2867
−0.3397	0.5540
−0.1761	0.4480
0.2006	−0.1704
0.2508	0.0572
0.4053	−0.0117
0.5854	−0.5213
0.6720	−0.8795
0.8265	−1.0000

b) The α calculated using the normalized table in part a:

$$\alpha = -0.9281$$

c) Since we have $G(X) = V_1 + V_2X$ where $G(X)$ is a linear function with slope V_2, it follows that $m = V_2 = v_2$ and for the analogous function Y,

$$Y = \overline{Y} + m(X - \overline{X})$$

Rearranging Y we group the constants and get:

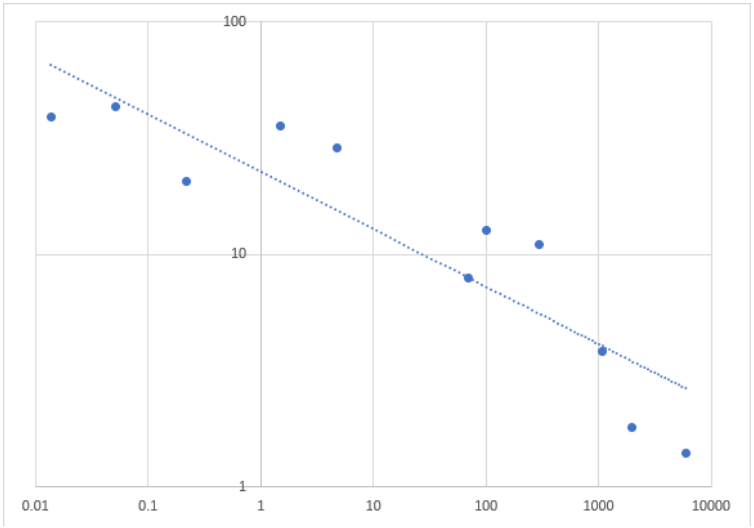
$$Y = (\overline{Y} - m\overline{X}) + mX$$

Since $V_1 = log(v_1)$, we can solve for v_1 as:

$$V_1 = log(v_1) = \overline{Y} - m\overline{X}$$

$$v_1 = 10^{\overline{Y} - m\overline{X}}$$

d)



e) Based on the data, and a thorough Google search of T-Rex massses, they tend to be in the range of 13e03 to 32e03 kg. Estimating the speed of a slightly heavier T-Rex with a mass of $\approx 20,000kg$ we get a speed of about 10.79814949 (converting back to standard metric).

9.3 a) Given $ZZ^T = \frac{1}{n}\Sigma_{i=1}^{n} s_i(s_i)^T$ and $s_i^* = Z^{-1}s_i$ we can plug into 9.30 and get back the original ZZ^T as follows:

$$\frac{1}{n}\Sigma_{i=1}^{n} s_i^*(s_i^*)^T = I \tag{1}$$

$$\frac{1}{n}\Sigma_{i=1}^{n} Z^{-1}s_i(Z^{-1}s_i)^T = I \tag{2}$$

$$\frac{1}{n}\Sigma_{i=1}^{n} Z^{-1}s_i(s_i)^T(Z^{-1})^T = I \tag{3}$$

$$Z^{-1}(\frac{1}{n}\Sigma_{i=1}^{n} s_i(s_i)^T)(Z^{-1})^T = I \tag{4}$$

$$ZZ^{-1}(\frac{1}{n}\Sigma_{i=1}^{n} s_i(s_i)^T)(Z^{-1})^T Z^T = ZIZ^T \tag{5}$$

$$\frac{1}{n}\Sigma_{i=1}^{n} s_i(s_i)^T = ZZ^T \tag{6}$$

b) For this problem we will show that given $\frac{1}{n}\Sigma_{i=1}^{n} s_i (s_i)^T = U\Sigma U^T$ and $s_i^* = \Sigma^{-\frac{1}{2}} U^T s_i$ we can plug s_i^* into 9.30 and get back the identity matrix I:

$$\frac{1}{n}\Sigma_{i=1}^{n} s_i^* (s_i^*)^T = I \tag{1}$$

$$\frac{1}{n}\Sigma_{i=1}^{n} (\Sigma^{-\frac{1}{2}} U^T s_i)(\Sigma^{-\frac{1}{2}} U^T s_i)^T = I \tag{2}$$

$$\frac{1}{n}\Sigma_{i=1}^{n} (\Sigma^{-\frac{1}{2}} U^T s_i)(s_i^T U^{T^T} \Sigma^{-\frac{1}{2}T}) = I \tag{3}$$

$$\Sigma^{-\frac{1}{2}} U^T (\frac{1}{n}\Sigma_{i=1}^{n} s_i (s_i^T)) U \Sigma^{-\frac{1}{2}T} = I \tag{4}$$

$$\Sigma^{-\frac{1}{2}} U^T (U\Sigma U^T) U \Sigma^{-\frac{1}{2}T} = I \tag{5}$$

$$\Sigma^{-\frac{1}{2}} (\Sigma) \Sigma^{-\frac{1}{2}T} = I \tag{6}$$

And since we know Σ is a diagonal matrix:

$$I = I \tag{7}$$

c) For the first matrix, after centering the data, we find that plugging it into the summation gives us a new matrix:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

If we select Z to be equal to the below:

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

then we satisfy the condition

$$ZZ^T = \frac{1}{n}\Sigma_{i=1}^{n} s_i (s_i)^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

However, this Z is not invertible and so we can't use it to whiten the data. Exploring other options we see that there's no other immediately obvious answer given that Z is not unique (per the text).

9.5 a) Since we have that $p = \alpha_1 v_1$ we can decompose the vector p into its two parts, namely p_1 and p_2 which correspond to the first and second entries of v_1 times some scaling factor α_1:

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$$

$$p_1 = \alpha_1 v_{11} \quad p_2 = \alpha_1 v_{21}$$

Solving for α_1 in the first equation we get that $\alpha_1 = \frac{p_1}{v_{11}}$ and substituting into the equation for p_2 we see:

$$p_2 = \alpha_1 v_{21} = \frac{v_{21}}{v_{11}} p_1 \quad where \quad \alpha_1 = \frac{v_{21}}{v_{11}}$$

b) Since in part a we saw that $p_2 = \alpha p_1$ we can use the fact that the variables p_1 and p_2 are the normalized versions of q_1 and q_2 (namely that they are centered and scaled) to derive the equation that relates them:

$$p_1 = \frac{1}{S_1}(q_1 - M_1), \quad p_2 = \frac{1}{S_2}(q_2 - M_2)$$

$$p_2 = \frac{1}{S_2}(q_2 - M_2) = \alpha p_1 = \alpha \frac{1}{S_1}(q_1 - M_1)$$
$$\frac{1}{S_2}(q_2 - M_2) = \alpha \frac{1}{S_1}(q_1 - M_1)$$
$$q_2 - M_2 = \frac{\alpha S_2}{S_1}(q_1 - M_1) \quad where \quad a = \frac{\alpha S_2}{S_1}$$
$$q_2 = M_2 + a(q_1 - M_1)$$

9.8 a) Included for reference below is the code used to generate the below times for each part of the question.
Below is one reference table for the calculated times for each test and each run time:

	Size	*Times*
PartA	2000	$3.607285975884664e-03$
	20000	$3.317933575600404e-02$
	200000	$3.302640852477376e-01$
PartB	2000	$3.758302269764120e-04$
	20000	$3.413897520502541e-03$
	200000	$3.592276018280914e-02$
PartC	2000	$3.826568492037185e-04$
	20000	$3.482651358648984e-03$
	200000	$3.866706231818633e-02$
PartD	2000	$4.195206092311732e-04$
	20000	$3.526829356777124e-03$
	200000	$4.035226277658426e-02$

b) Right off the bat I noticed that for each order of magnitude increase in the size, there was a proportional increase in the time to compute. Parts B, C, and D were all faster than part A, however, they appeared to slow down as the tests progressed (e.g. D was slower than C was slower than B). This may be due to my implementation, or may be a system nuance. In any case, the difference in computing time was not super large and may not be as pronounced without a significantly larger dataset.

```
function [out] = ch9q8()

len1 = 2000;
len2 = 20000;
len3 = 200000;

x1 = rand(2,len1);
x2 = rand(2,len2);
x3 = rand(2,len3);

%n=2000
x1a = @() summation(x1,len1);
time_x1a = timeit(x1a);

x1b = @() centersum(x1,len1);
time_x1b = timeit(x1b);
```

```
x1c = @() cstars(x1,len1);
time_x1c = timeit(x1c);

x1d = @() callCov(x1,len1);
time_x1d = timeit(x1d);


%n=20000
x2a = @() summation(x2,len2);
time_x2a = timeit(x2a);

x2b = @() centersum(x2,len2);
time_x2b = timeit(x2b);

x2c = @() cstars(x2,len2);
time_x2c = timeit(x2c);

x2d = @() callCov(x2,len2);
time_x2d = timeit(x2d);


%n=200000
x3a = @() summation(x3,len3);
time_x3a = timeit(x3a);

x3b = @() centersum(x3,len3);
time_x3b = timeit(x3b);

x3c = @() cstars(x3,len3);
time_x3c = timeit(x3c);

x3d = @() callCov(x3,len3);
time_x3d = timeit(x3d);

disp('Times:')
disp('Part A, n=2000')
disp(time_x1a)
disp('Part B, n=2000')
disp(time_x1b)
disp('Part C, n=2000')
disp(time_x1c)
disp('Part D, n=2000')
disp(time_x1d)
disp('%%%%%%%%%%%%%%%%%%%%%%%%')
disp('Part A, n=20000')
disp(time_x2a)
disp('Part B, n=20000')
disp(time_x2b)
disp('Part C, n=20000')
disp(time_x2c)
disp('Part D, n=20000')
disp(time_x2d)
disp('%%%%%%%%%%%%%%%%%%%%%%%%')
disp('Part A, n=200000')
disp(time_x3a)
disp('Part B, n=200000')
disp(time_x3b)
disp('Part C, n=200000')
```

```
disp(time_x3c)
disp('Part D, n=200000')
disp(time_x3d)
end

function [out] = calculateCenter(in)
    [m,n] = size(in);
    out = in;
    avg = mean(out);
    for k=1:n
        out(:,k) = out(:,k) - avg(k);
    end
end

function summation(x,n)

xc = calculateCenter(x);
B = zeros(2,2);
for k=1:n
    vec = xc(:,k);
    B = B + (vec*vec');
end
B = B / n;
end

function centersum(x,n)

xc = calculateCenter(x);
B = (xc*xc')/n;

end

function cstars(x,n)

xc = calculateCenter(x);
cstar1 = xc(1,:)';
cstar2 = xc(2,:)';

B = dot(cstar1,cstar2)/n;

end

function callCov(x,n)

xc = calculateCenter(x);
B = cov(xc');

end
```