Rational Solution Generator for the first case

```
> restart;with(plots):with(orthopoly):with(linalg):alias(w=w(z)):
> P4:=(diff(w,z)^2/2/w+3/2*w^3+4*z*w^2+2*(z^2-alpha)*w+beta/w)-diff
  (w,z,z);
```

$$P4 := \frac{1}{2} \frac{\left(\frac{\partial}{\partial z} w\right)^2}{w} + \frac{3}{2} w^3 + 4 z w^2 + 2 \left(z^2 - \alpha\right) w + \frac{\beta}{w} - \left(\frac{\partial^2}{\partial z^2} w\right) \qquad (1)$$

```
> c:=N->product(1/2^j*1/factorial(j),j=1..N-1):
> h:=(M,N)->sort(c(N)*det(wronskian([seq(H(M+j,z),j=0..N-1)],z))):
> w:=(m,n)->convert(diff(ln((h(m+1,n))/(h(m,n))),z),parfrac,z):
> m:=2;n:=2;
```

$$m := 2$$
$$n := 2 \qquad (2)$$

```
> w(m,n):alpha:=2*m+n+1;beta:=-2*n^2:
```

$$\alpha := 7 \qquad (3)$$

```
> RootOf(h(m+1,n),z):J1:=evalf(allvalues(%)):RootOf(h(m,n),z):J2:=
  (allvalues(%)):
> A:=complexplot([J1],style=point,symbol=solidcircle,color=magenta,
  symbolsize=15):
> B:=complexplot([J2],style=point,symbol=solidcircle,color=blue,
  symbolsize=15):
> C:=plot(w(m,n),z=-5..5,y=-8..8,colour=black,thickness=2):display
  (A,B,C);
```
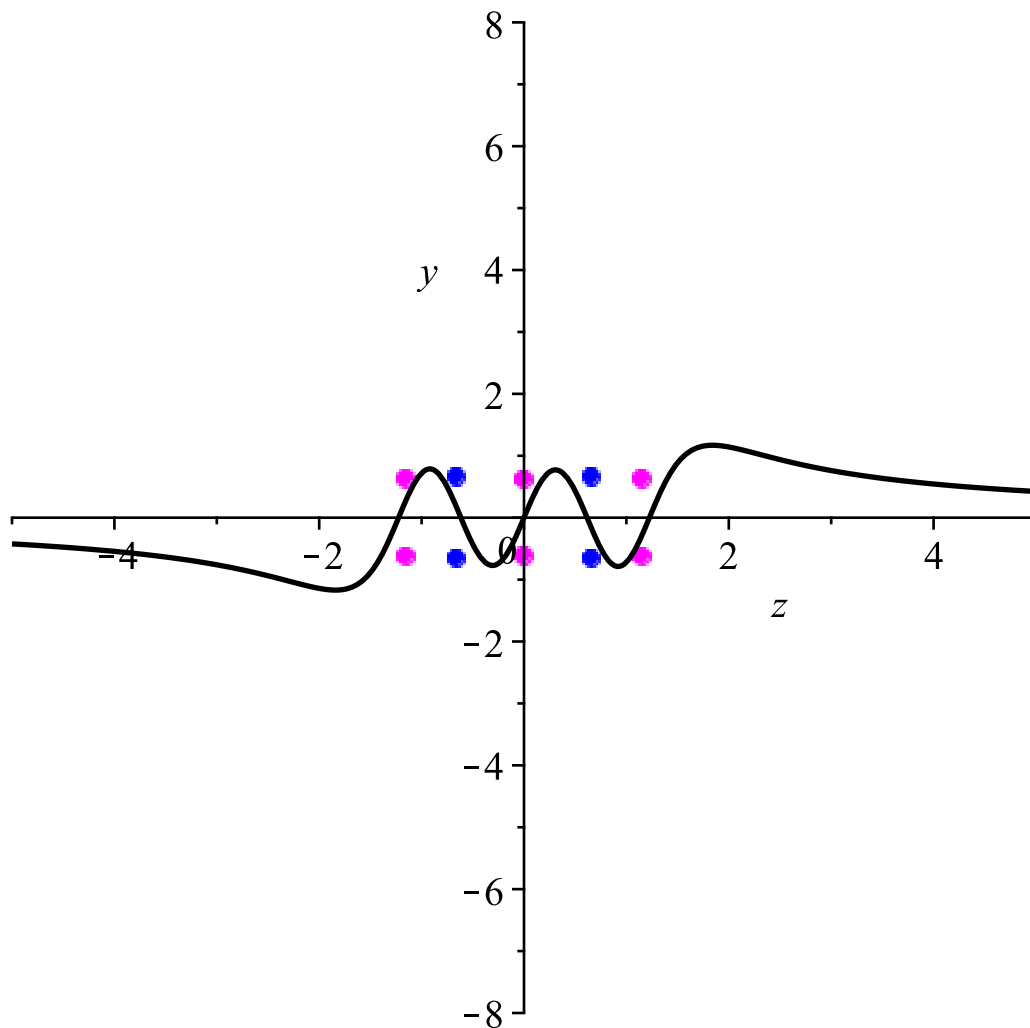
```
> w:=w(m,n):
> simplify(P4);
```
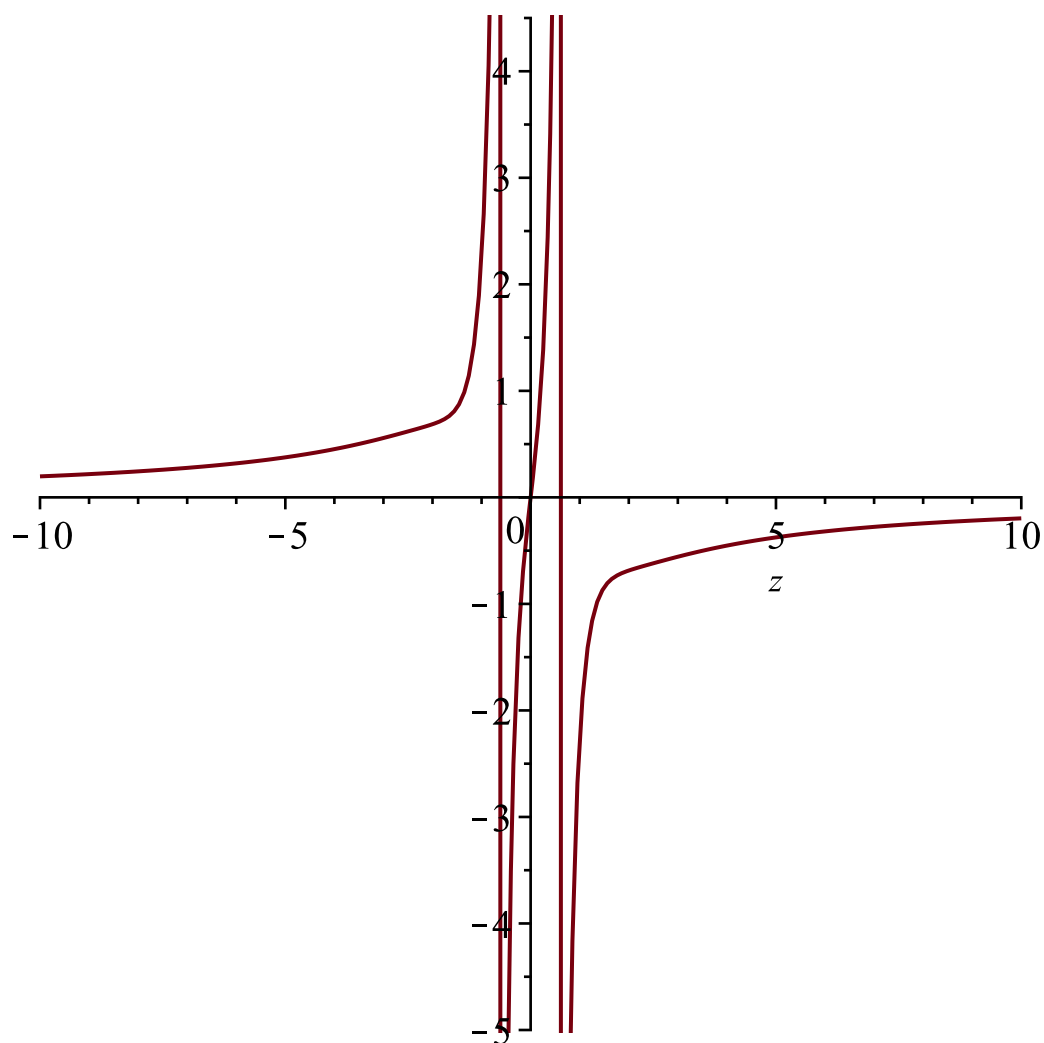$$0 \tag{4}$$

Rational Solution Generator for the second case
```
> w:=(m,n)->convert(diff(ln((h(m,n))/(h(m,n+1))),z),parfrac,z):
> w(m,n):alpha:=-(m+2*n+1);beta:=-2*m^2;
```
$$\alpha := -7$$
$$\beta := -8 \tag{5}$$
```
> w:=w(m,n):plot(w);
```

```
> simplify(P4);
```

$$0 \tag{6}$$

Rational Solution Generator for the third case

```
> w:=(m,n)->-2*z+convert(diff(ln((h(m,n+1))/(h(m+1,n))),z),parfrac,
  z):
> w(m,n):alpha:=n-m;beta:=-2*(m+n+1)^2;
```
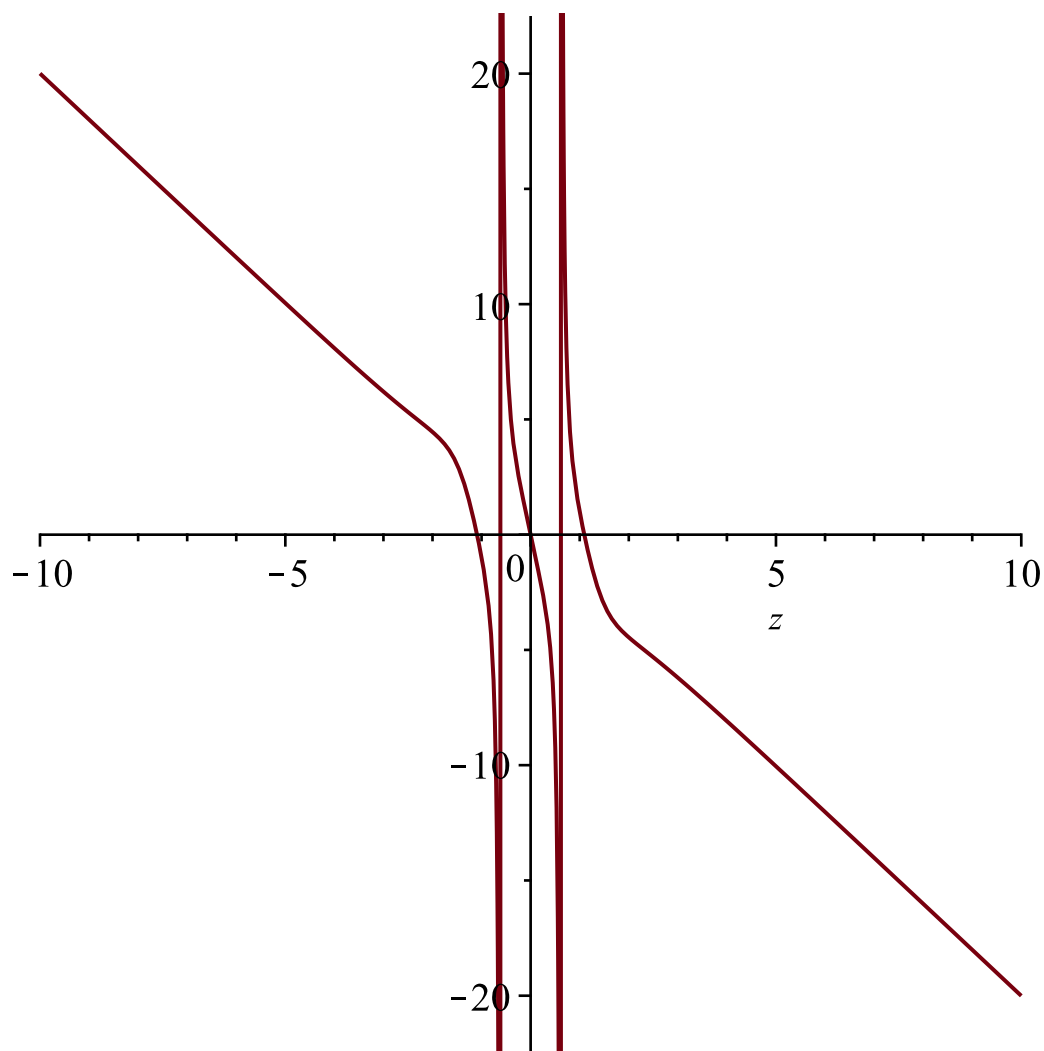
$$\alpha := 0$$

$$\beta := -50 \tag{7}$$

```
> w:=w(m,n):plot(w);
```

```
> simplify(P4);
```

$$0 \tag{8}$$

S4 Rational Solution Generator for the first case

```
> restart;with(plots):with(orthopoly):with(linalg):alias(w=w(z),
  sigma=sigma(z)):
> S4:=diff(sigma,z$2)^2-4*(z*diff(sigma,z)-sigma)^2+4*diff(sigma,z)
  *(diff(sigma,z)+2*theta[0])*(diff(sigma,z)+2*theta[infinity]);
```

$$S4 := \left(\frac{\partial^2}{\partial z^2}\, \sigma \right)^2 - 4 \left(z \left(\frac{\partial}{\partial z}\, \sigma \right) - \sigma \right)^2 + 4 \left(\frac{\partial}{\partial z}\, \sigma \right) \left(\frac{\partial}{\partial z}\, \sigma + 2\,\theta_0 \right) \left(\frac{\partial}{\partial z}\, \sigma + 2\,\theta_\infty \right) \tag{9}$$

```
> c:=N->product(1/2^j*1/factorial(j),j=1..N-1):
> h:=(M,N)->simplify(c(N)*det(wronskian([seq(HermiteH(M+j,z),j=0..
  N-1)],z))):
> tau:=(m,n)->simplify(diff(ln(h(m,n)),z)):
> m:=4;n:=2;
```

$$m := 4$$
$$n := 2 \tag{10}$$

```
> sigma1:=tau(m,n);theta[0]:=-n;theta[infinity]:=m;simplify(expand
  (subs(sigma=sigma1,S4)));
```

$$\sigma 1 := \frac{32\,z^3\left(4\,z^4 - 12\,z^2 + 15\right)}{16\,z^8 - 64\,z^6 + 120\,z^4 + 45}$$

$$\theta_0 := -2$$

$$\theta_\infty := 4$$

$$0 \tag{11}$$

```
> sigma2:=-2*n*z+tau(m,n);theta[0]:=n;theta[infinity]:=m+n;simplify
  (subs(sigma=sigma2,S4));
```

$$\sigma 2 := -4\,z + \frac{32\,z^3\left(4\,z^4 - 12\,z^2 + 15\right)}{16\,z^8 - 64\,z^6 + 120\,z^4 + 45}$$

$$\theta_0 := 2$$

$$\theta_\infty := 6$$

$$0 \tag{12}$$

```
> sigma3:=2*m*z+tau(m,n);theta[0]:=-m;theta[infinity]:=-m-n;
  simplify(subs(sigma=sigma3,S4));
```

$$\sigma 3 := 8\,z + \frac{32\,z^3\left(4\,z^4 - 12\,z^2 + 15\right)}{16\,z^8 - 64\,z^6 + 120\,z^4 + 45}$$

$$\theta_0 := -4$$

$$\theta_\infty := -6$$

$$0 \tag{13}$$