```
> restart;
> alpha[0]+alpha[1]+2*alpha[2]+alpha[3]+alpha[4]=1;alpha[2]:=solve
  (%,alpha[2]);
```

$$\alpha_0 + \alpha_1 + 2\,\alpha_2 + \alpha_3 + \alpha_4 = 1$$

$$\alpha_2 := \frac{1}{2} - \frac{1}{2}\,\alpha_0 - \frac{1}{2}\,\alpha_1 - \frac{1}{2}\,\alpha_3 - \frac{1}{2}\,\alpha_4 \tag{1}$$

```
> nu[1]=-1/2*(alpha[3]+alpha[4]),nu[2]=1/2*(alpha[4]-alpha[3]),nu
  [3]=-1/2*(alpha[0]+alpha[1]-1),nu[4]=1/2*(alpha[0]-alpha[1]-1);
```

$$\nu_1 = -\frac{1}{2}\,\alpha_3 - \frac{1}{2}\,\alpha_4,\ \nu_2 = \frac{1}{2}\,\alpha_4 - \frac{1}{2}\,\alpha_3,\ \nu_3 = -\frac{1}{2}\,\alpha_0 - \frac{1}{2}\,\alpha_1 + \frac{1}{2},\ \nu_4 = \frac{1}{2}\,\alpha_0 - \frac{1}{2}\,\alpha_1 - \frac{1}{2} \tag{2}$$

```
> solve({%},{alpha[0],alpha[1],alpha[3],alpha[4]});alpha[0]:= nu[3]
  +nu[4]+1; alpha[1]:= -nu[4]+nu[3];alpha[3]:= nu[1]-nu[2];alpha[4]
  := nu[2]+nu[1];
```

$$\{\alpha_0 = -\nu_3 + \nu_4 + 1,\ \alpha_1 = -\nu_4 - \nu_3,\ \alpha_3 = -\nu_1 - \nu_2,\ \alpha_4 = \nu_2 - \nu_1\}$$

$$\alpha_0 := \nu_3 + \nu_4 + 1$$

$$\alpha_1 := -\nu_4 + \nu_3$$

$$\alpha_3 := \nu_1 - \nu_2$$

$$\alpha_4 := \nu_2 + \nu_1 \tag{3}$$

```
> H:=q*(q-1)*(q-x)*p^2-(alpha[4]*(q-1)*(q-x)+alpha[3]*q*(q-x)+
  (alpha[0]-1)*q*(q-1))*p+alpha[2]*(alpha[1]+alpha[2])*(q-x);
```

$$H := q\,(q-1)\,(q-x)\,p^2 - \left(\left(\nu_2 + \nu_1\right)(q-1)\,(q-x) + \left(\nu_1 - \nu_2\right) q\,(q-x) + \left(\nu_3 + \nu_4\right) q\,(q-1)\right) p + \left(-\nu_3 - \nu_1\right)\left(-\nu_4 - \nu_1\right)(q-x) \tag{4}$$

```
> collect(simplify(diff(H,p)),[q,p],factor);collect(simplify(diff
  (H,q)),[q,p],factor);
```

$$2\,p\,q^3 + \left((-2\,x-2)\,p - 2\,\nu_1 - \nu_3 - \nu_4\right) q^2 + \left(2\,p\,x + 2\,x\,\nu_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4\right) q - x\left(\nu_2 + \nu_1\right)$$

$$3\,q^2\,p^2 + \left((-2\,x-2)\,p^2 + \left(-4\,\nu_1 - 2\,\nu_3 - 2\,\nu_4\right) p\right) q + x\,p^2 + \left(2\,x\,\nu_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4\right) p + \left(\nu_1 + \nu_4\right)\left(\nu_1 + \nu_3\right) \tag{5}$$

```
> alias(q=q(x),p=p(x),sigma=sigma(x));
```

$$q, p, \sigma \tag{6}$$

```
> HM=collect(q*(q-1)*(q-x)*p^2-(alpha[4]*(q-1)*(q-x)+alpha[3]*q*(q-
  x)+(alpha[0]-1)*q*(q-1))*p+(1/2-(1/2)*alpha[0]-(1/2)*alpha[1]-
  (1/2)*alpha[3]-(1/2)*alpha[4])*((1/2)*alpha[1]+1/2-(1/2)*alpha[0]
  -(1/2)*alpha[3]-(1/2)*alpha[4])*(q-x),[p,q],factor);
```

$$HM = \left(q^3 + (-1-x)\,q^2 + x\,q\right) p^2 + \left(\left(-2\,\nu_1 - \nu_3 - \nu_4\right) q^2 + \left(2\,x\,\nu_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4\right) q - x\left(\nu_2 + \nu_1\right)\right) p + \left(\nu_1 + \nu_4\right)\left(\nu_1 + \nu_3\right) q - \left(\nu_1 + \nu_3\right)\left(\nu_1 + \nu_4\right) x \tag{7}$$

```
> HM := (q^3+(-1-x)*q^2+x*q)*p^2+((-2*nu[1]-nu[3]-nu[4])*q^2+(2*x*
  nu[1]+e[1])*q-x*(nu[2]+nu[1]))*p+(nu[1]+nu[4])*(nu[1]+nu[3])*q-
  (nu[1]+nu[3])*(nu[1]+nu[4])*x;
```

(8)

$$HM := \left(q^3 + (-1-x)\,q^2 + x\,q\right)p^2 + \left(\left(-2\,\nu_1 - \nu_3 - \nu_4\right)q^2 + \left(2\,x\,\nu_1 + e_1\right)q - x\left(\nu_2 + \nu_1\right)\right)p + \left(\nu_1 + \nu_4\right)\left(\nu_1 + \nu_3\right)q - \left(\nu_1 + \nu_3\right)\left(\nu_1 + \nu_4\right)x \tag{8}$$

```
> A := 2*p*q^3+((-2*x-2)*p-2*nu[1]-nu[3]-nu[4])*q^2+(2*p*x+2*x*nu
  [1]+e[1])*q-x*(nu[2]+nu[1]);
```

$$A := 2\,p\,q^3 + \left(\left(-2\,x-2\right)p - 2\,\nu_1 - \nu_3 - \nu_4\right)q^2 + \left(2\,x\,p + 2\,\nu_1\,x + e_1\right)q - x\left(\nu_2 + \nu_1\right) \tag{9}$$

```
> B := 3*q^2*p^2+((-2*x-2)*p^2+(-4*nu[1]-2*nu[3]-2*nu[4])*p)*q+x*
  p^2+(2*x*nu[1]+e[1])*p+(nu[1]+nu[4])*(nu[1]+nu[3]);
```

$$B := 3\,q^2\,p^2 + \left(\left(-2\,x-2\right)p^2 + \left(-4\,\nu_1 - 2\,\nu_3 - 2\,\nu_4\right)p\right)q + x\,p^2 + \left(2\,x\,\nu_1 + e_1\right)p + \left(\nu_1 + \nu_4\right)\left(\nu_1 + \nu_3\right) \tag{10}$$

```
> S6:=diff(sigma,x)*(x*(x-1)*diff(sigma,x,x))^2+(diff(sigma,x)*(2*
  sigma-(2*x-1)*diff(sigma,x))+nu[1]*nu[2]*nu[3]*nu[4])^2-product
  (diff(sigma,x)+nu[k]^2,k=1..4);
```

$$S6 := \left(\frac{\partial}{\partial x}\,\sigma\right)x^2\,(x-1)^2\left(\frac{\partial^2}{\partial x^2}\,\sigma\right)^2 + \left(\left(\frac{\partial}{\partial x}\,\sigma\right)\left(2\,\sigma - (2\,x-1)\left(\frac{\partial}{\partial x}\,\sigma\right)\right) + \nu_1\,\nu_2\,\nu_3\,\nu_4\right)^2 - \left(\frac{\partial}{\partial x}\,\sigma + \nu_1^2\right)\left(\frac{\partial}{\partial x}\,\sigma + \nu_2^2\right)\left(\frac{\partial}{\partial x}\,\sigma + \nu_3^2\right)\left(\frac{\partial}{\partial x}\,\sigma + \nu_4^2\right) \tag{11}$$

```
> H1:=diff(q,x)=(A)/(x*(x-1)):collect(%,[x,p,q],factor);
```

$$\frac{\partial}{\partial x}\,q = \frac{1}{x\,(x-1)}\left(\left(\left(-2\,q^2 + 2\,q\right)p + 2\,\nu_1\,q - \nu_1 - \nu_2\right)x + \left(2\,q^3 - 2\,q^2\right)p + \left(-2\,\nu_1 - \nu_3 - \nu_4\right)q^2 + e_1\,q\right) \tag{12}$$

```
> H2:=diff(p,x)=-(B)/(x*(x-1)):collect(%,[x,p,q],factor);
```

$$\frac{\partial}{\partial x}\,p = -\frac{1}{x\,(x-1)}\left(\left(\left(-2\,q+1\right)p^2 + 2\,\nu_1\,p\right)x + \left(3\,q^2 - 2\,q\right)p^2 + \left(e_1 + \left(-4\,\nu_1 - 2\,\nu_3 - 2\,\nu_4\right)q\right)p + \left(\nu_1 + \nu_4\right)\left(\nu_1 + \nu_3\right)\right) \tag{13}$$

```
> `e'`[1]:=nu[1]+nu[3]+nu[4];e[1]:=nu[1]+nu[2]+nu[3]+nu[4];
```

$$e'_1 := \nu_1 + \nu_3 + \nu_4$$

$$e_1 := \nu_1 + \nu_2 + \nu_3 + \nu_4 \tag{14}$$

```
> `e'`[2]:=nu[1]*nu[3]+nu[1]*nu[4]+nu[3]*nu[4];e[2]:=nu[4]*(nu[3]+
  nu[2]+nu[1])+nu[3]*(nu[2]+nu[1])+nu[2]*(nu[1]);
```

$$e'_2 := \nu_1\,\nu_3 + \nu_1\,\nu_4 + \nu_3\,\nu_4$$

$$e_2 := \nu_4\left(\nu_3 + \nu_2 + \nu_1\right) + \nu_3\left(\nu_2 + \nu_1\right) + \nu_2\,\nu_1 \tag{15}$$

```
> `e'`[3]:=nu[1]*nu[3]*nu[4];e[3]:=nu[4]*(nu[1]*nu[2]+nu[1]*nu[3]+
  nu[2]*nu[3])+nu[3]*(nu[1]*nu[2]);
```

$$e'_3 := \nu_1\,\nu_3\,\nu_4$$

$$e_3 := \nu_4\left(\nu_1\,\nu_2 + \nu_1\,\nu_3 + \nu_2\,\nu_3\right) + \nu_3\,\nu_2\,\nu_1 \tag{16}$$

```
> S:=sigma=HM+`e'`[2]*x-1/2*(e[2]);
```

$$S := \sigma = \left(q^3 + (-1-x)\,q^2 + x\,q\right)p^2 + \left(\left(-2\,\nu_1 - \nu_3 - \nu_4\right)q^2 + \left(2\,x\,\nu_1 + \nu_1 + \nu_2 + \nu_3 + \nu_4\right)q - x\left(\nu_2 + \nu_1\right)\right)p + \left(\nu_1 + \nu_4\right)\left(\nu_1 + \nu_3\right)q - \left(\nu_1 + \nu_3\right)\left(\nu_1 + \nu_4\right)x + \left(\nu_1\,\nu_3 + \nu_1\,\nu_4 + \nu_3\,\nu_4\right)x - \frac{1}{2}\,\nu_4\left(\nu_3 + \nu_2 + \nu_1\right) - \frac{1}{2}\,\nu_3\left(\nu_2 + \nu_1\right) - \frac{1}{2}\,\nu_2\,\nu_1 \tag{17}$$

```
> s1:=collect(simplify(subs(H1,H2,diff(S,x))),[p,nu[1]],factor);
```

$$s1 := \frac{\partial}{\partial x}\,\sigma = -q\,(q-1)\,p^2 + \left((2\,q-1)\,\nu_1 - \nu_2\right)p - \nu_1^2 \tag{18}$$

```
> collect(S-x*s1,[p,nu[1]],factor);algsubs(-q*(q-1)*p^2+((2*q-1)*nu
  [1]-nu[2])*p-nu[1]^2=diff(sigma, x),%):subs(p=Q/q/(q-1),%)
  :collect(expand(%),q,factor):P[1]:=subs(Q=q*(q-1)*p,collect(%,[q,
  Q],factor));P[2]:=collect(%%,[q,Q],factor);
```

$$-x\left(\frac{\partial}{\partial x}\,\sigma\right) + \sigma = q^2\,(q-1)\,p^2 + \left(-q\,(2\,q-1)\,\nu_1 - q\left(q\,\nu_3 + q\,\nu_4 - \nu_2 - \nu_3 - \nu_4\right)\right)p + q\,\nu_1^2 + \left(q\,\nu_3 + q\,\nu_4 - \frac{1}{2}\,\nu_2 - \frac{1}{2}\,\nu_4 - \frac{1}{2}\,\nu_3\right)\nu_1 + \nu_3\,\nu_4\,q - \frac{1}{2}\,\nu_4\,\nu_2 - \frac{1}{2}\,\nu_3\,\nu_4 - \frac{1}{2}\,\nu_3\,\nu_2$$

$$P_1 := -x\left(\frac{\partial}{\partial x}\,\sigma\right) + \sigma = \left(\nu_3\,\nu_1 + \nu_1\,\nu_4 + \nu_3\,\nu_4 - \left(\frac{\partial}{\partial x}\,\sigma\right)\right)q + \left(-\nu_4 - \nu_3\right)q\,(q-1)\,p - \frac{1}{2}\,\nu_1\,\nu_4 - \frac{1}{2}\,\nu_3\,\nu_2 - \frac{1}{2}\,\nu_2\,\nu_1 - \frac{1}{2}\,\nu_3\,\nu_1 - \frac{1}{2}\,\nu_4\,\nu_2 - \frac{1}{2}\,\nu_3\,\nu_4$$

$$P_2 := -x\left(\frac{\partial}{\partial x}\,\sigma\right) + \sigma = \left(\nu_3\,\nu_1 + \nu_1\,\nu_4 + \nu_3\,\nu_4 - \left(\frac{\partial}{\partial x}\,\sigma\right)\right)q + \left(-\nu_4 - \nu_3\right)Q - \frac{1}{2}\,\nu_1\,\nu_4 - \frac{1}{2}\,\nu_3\,\nu_2 - \frac{1}{2}\,\nu_2\,\nu_1 - \frac{1}{2}\,\nu_3\,\nu_1 - \frac{1}{2}\,\nu_4\,\nu_2 - \frac{1}{2}\,\nu_3\,\nu_4 \tag{19}$$

```
> x*(x-1)*diff(sigma,x,x)=2*q*(`e'`[1]*diff(sigma,x)-`e'`[3])-2*Q*
  (diff(sigma,x)-nu[3]*nu[4])-e[1]*diff(sigma,x)+e[3];
```

$$x\,(x-1)\left(\frac{\partial^2}{\partial x^2}\,\sigma\right) = 2\,q\left(\left(\nu_1 + \nu_3 + \nu_4\right)\left(\frac{\partial}{\partial x}\,\sigma\right) - \nu_1\,\nu_3\,\nu_4\right) - 2\,Q\left(\frac{\partial}{\partial x}\,\sigma - \nu_3\,\nu_4\right) - \left(\nu_1 + \nu_2 + \nu_3 + \nu_4\right)\left(\frac{\partial}{\partial x}\,\sigma\right) + \nu_4\left(\nu_1\,\nu_2 + \nu_1\,\nu_3 + \nu_2\,\nu_3\right) + \nu_3\,\nu_2\,\nu_1 \tag{20}$$

```
> collect(simplify(subs(H1,H2,x*(x-1)*diff(s1,x))),[p,q],factor)
  :collect(algsubs(-q*(q-1)*p^2+((2*q-1)*nu[1]-nu[2])*p-nu[1]^2=
  diff(sigma, x),%),[p],factor):expand(%):collect(algsubs(-q*(q-1)*
  p^2+((2*q-1)*nu[1]-nu[2])*p-nu[1]^2=diff(sigma, x),%),[p,diff],
  factor):subs(p=Q/q/(q-1),%):collect(expand(%),q,factor):R[1]:=
  subs(Q=q*(q-1)*p,collect(%,[q,Q],factor));R[2]:=collect(%%,[q,Q,
  diff],factor);
```

$$R_1 := x\,(x-1)\left(\frac{\partial^2}{\partial x^2}\,\sigma\right) = \left(-2\,\nu_1\,\nu_3\,\nu_4 + 2\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_1 + 2\,\nu_3\left(\frac{\partial}{\partial x}\,\sigma\right) + 2\,\nu_4\left(\frac{\partial}{\partial x}\,\sigma\right)\right)q + \left(2\,\nu_3\,\nu_4 - 2\left(\frac{\partial}{\partial x}\,\sigma\right)\right)q\,(q-1)\,p + \nu_3\,\nu_2\,\nu_1 + \nu_1\,\nu_2\,\nu_4 + \nu_1\,\nu_3\,\nu_4 + \nu_2\,\nu_3\,\nu_4 - \left(\frac{\partial}{\partial x}\,\sigma\right)\nu_1 - \left(\frac{\partial}{\partial x}\,\sigma\right)\nu_2 - \nu_3\left(\frac{\partial}{\partial x}\,\sigma\right) - \nu_4\left(\frac{\partial}{\partial x}\,\sigma\right)$$

$$R_2 := x\,(x-1)\left(\frac{\partial^2}{\partial x^2}\,\sigma\right) = \left(\left(2\,\nu_1 + 2\,\nu_3 + 2\,\nu_4\right)\left(\frac{\partial}{\partial x}\,\sigma\right) - 2\,\nu_1\,\nu_3\,\nu_4\right)q + \left(2\,\nu_3\,\nu_4\right. \tag{21}$$

$$-2\left(\frac{\partial}{\partial x}\sigma\right)\Big)Q+\left(-\nu_1-\nu_2-\nu_3-\nu_4\right)\left(\frac{\partial}{\partial x}\sigma\right)+\nu_3\nu_2\nu_1+\nu_1\nu_2\nu_4+\nu_1\nu_3\nu_4$$
$$+\nu_2\nu_3\nu_4$$

```
> solve({R[2],P[2]},{Q,q}):collect(simplify(%),[diff,x],factor);
```

$$\left\{Q \right. \tag{22}$$

$$=\frac{1}{2}\,\frac{1}{\left(\frac{\partial}{\partial x}\sigma\right)^2+\left(\nu_3^2+\nu_4^2\right)\left(\frac{\partial}{\partial x}\sigma\right)+\nu_3^2\nu_4^2}\left(\left(\left(-x^2+x\right)\left(\frac{\partial}{\partial x}\sigma\right)+\left(\nu_1\nu_3\right.\right.\right.$$

$$\left.\left.\left.+\nu_1\nu_4+\nu_3\nu_4\right)x^2+\left(-\nu_1\nu_3-\nu_1\nu_4-\nu_3\nu_4\right)x\right)\left(\frac{\partial^2}{\partial x^2}\sigma\right)\right)$$

$$+\frac{1}{2}\,\frac{1}{\left(\frac{\partial}{\partial x}\sigma\right)^2+\left(\nu_3^2+\nu_4^2\right)\left(\frac{\partial}{\partial x}\sigma\right)+\nu_3^2\nu_4^2}\left(\left(\left(2\nu_1+2\nu_3+2\nu_4\right)x-\nu_1-\nu_2-\nu_3\right.\right.$$

$$\left.\left.-\nu_4\right)\left(\frac{\partial}{\partial x}\sigma\right)^2+\left(-2x\nu_1\nu_3\nu_4-\nu_1^2\nu_2+\nu_1\nu_3\nu_4-\nu_2\nu_3^2-\nu_2\nu_4^2-2\sigma\nu_1-2\sigma\nu_3\right.\right.$$

$$\left.\left.-2\sigma\nu_4\right)\left(\frac{\partial}{\partial x}\sigma\right)-\nu_1^2\nu_2\nu_3^2-\nu_1^2\nu_2\nu_3\nu_4-\nu_1^2\nu_2\nu_4^2-\nu_1\nu_2\nu_3^2\nu_4-\nu_1\nu_2\nu_3\nu_4^2-\nu_2\nu_3^2\nu_4^2\right.$$

$$\left.+2\sigma\nu_1\nu_3\nu_4\right),\ q=\frac{1}{2}\,\frac{\left(\left(\nu_3+\nu_4\right)x^2+\left(-\nu_4-\nu_3\right)x\right)\left(\frac{\partial^2}{\partial x^2}\sigma\right)}{\left(\frac{\partial}{\partial x}\sigma\right)^2+\left(\nu_3^2+\nu_4^2\right)\left(\frac{\partial}{\partial x}\sigma\right)+\nu_3^2\nu_4^2}$$

$$+\frac{1}{2}\,\frac{1}{\left(\frac{\partial}{\partial x}\sigma\right)^2+\left(\nu_3^2+\nu_4^2\right)\left(\frac{\partial}{\partial x}\sigma\right)+\nu_3^2\nu_4^2}\left(2\left(\frac{\partial}{\partial x}\sigma\right)^2x+\left(-2\nu_3\nu_4x-\nu_2\nu_1+\nu_3^2\right.\right.$$

$$\left.\left.+\nu_3\nu_4+\nu_4^2-2\sigma\right)\left(\frac{\partial}{\partial x}\sigma\right)-\nu_3^2\nu_2\nu_1-\nu_1\nu_2\nu_3\nu_4-\nu_1\nu_2\nu_4^2+\nu_3^2\nu_4^2+2\sigma\nu_3\nu_4\right)\Big\}$$

```
> factor(solve(subs(Q=q*(q-1)*p,%),{q,p})):factor(subs(%,S6));
  expand(%-S6);
```

$$\left(\frac{\partial}{\partial x}\,\sigma\right)\left(4\left(\frac{\partial}{\partial x}\,\sigma\right)^3 x^2-4\left(\frac{\partial}{\partial x}\,\sigma\right)^3 x+4\left(\frac{\partial}{\partial x}\,\sigma\right)^2\sigma+4\left(\frac{\partial}{\partial x}\,\sigma\right)\sigma^2-\left(\frac{\partial}{\partial x}\,\sigma\right)^2\nu_1^2\right.$$
$$-\left(\frac{\partial}{\partial x}\,\sigma\right)^2\nu_2^2-\left(\frac{\partial}{\partial x}\,\sigma\right)^2\nu_3^2-\left(\frac{\partial}{\partial x}\,\sigma\right)^2\nu_4^2+\left(\frac{\partial^2}{\partial x^2}\,\sigma\right)^2 x^4-2\left(\frac{\partial^2}{\partial x^2}\,\sigma\right)^2 x^3$$
$$+x^2\left(\frac{\partial^2}{\partial x^2}\,\sigma\right)^2-\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_2^2\nu_3^2-\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_2^2\nu_4^2-8\left(\frac{\partial}{\partial x}\,\sigma\right)^2\sigma\,x-\nu_1^2\nu_3^2\nu_4^2$$
$$-\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_3^2\nu_4^2-\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_1^2\nu_2^2-\nu_1^2\nu_2^2\nu_4^2-\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_1^2\nu_3^2-\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_1^2\nu_4^2-\nu_2^2\nu_3^2\nu_4^2$$
$$\left.-\nu_1^2\nu_2^2\nu_3^2+2\left(\frac{\partial}{\partial x}\,\sigma\right)\nu_1\nu_2\nu_3\nu_4+4\,\sigma\,\nu_1\nu_2\nu_3\nu_4-4\left(\frac{\partial}{\partial x}\,\sigma\right)x\,\nu_1\nu_2\nu_3\nu_4\right)$$
$$0 \qquad \textbf{(23)}$$