P4

```
> restart;with(ListTools):with(orthopoly):with(linalg):alias(sigma=
  sigma(z),psi=psi(z),phi=phi(z),w=w(z)):P4:=-diff(w,z,z)+(diff(w,
  z)^2/2/w+3/2*w^3+4*z*w^2+2*(z^2-A)*w+B/w);
```

$$P4 := -\left(\frac{\partial^2}{\partial z^2}\, w\right) + \frac{1}{2}\,\frac{\left(\frac{\partial}{\partial z}\, w\right)^2}{w} + \frac{3}{2}\, w^3 + 4\, z\, w^2 + 2\left(z^2 - A\right) w + \frac{B}{w} \tag{1}$$

```
> n:=2;epsilon:=1;
```

$$n := 2$$

$$\epsilon := 1 \tag{2}$$

The following shows that d/dz\psi_\nu=\psi_{\nu-1}

```
> diff(psi, z, z) = -2*nu*psi*epsilon+2*z*epsilon*(diff(psi, z));
```

$$\frac{\partial^2}{\partial z^2}\,\psi = -2\,\nu\,\psi + 2\, z\left(\frac{\partial}{\partial z}\,\psi\right) \tag{3}$$

```
> diff(%,z);collect(%,diff,factor);subs(diff(psi, z)=psi,%);
```

$$\frac{\partial^3}{\partial z^3}\,\psi = -2\,\nu\left(\frac{\partial}{\partial z}\,\psi\right) + 2\left(\frac{\partial}{\partial z}\,\psi\right) + 2\, z\left(\frac{\partial^2}{\partial z^2}\,\psi\right)$$

$$\frac{\partial^3}{\partial z^3}\,\psi = 2\, z\left(\frac{\partial^2}{\partial z^2}\,\psi\right) + \left(-2\,\nu + 2\right)\left(\frac{\partial}{\partial z}\,\psi\right)$$

$$\frac{\partial^2}{\partial z^2}\,\psi = 2\, z\left(\frac{\partial}{\partial z}\,\psi\right) + \left(-2\,\nu + 2\right)\psi \tag{4}$$

```
> diff(psi,z,z)=-epsilon*2*nu*psi+epsilon*2*z*diff(psi,z),seq(diff
  (psi,z$(j+2))=diff(-epsilon*2*nu*psi+epsilon*2*z*diff(psi,z),
  z$j),j=1..n):S1:=Reverse([%]):
> tau[nu]:=(n)->det(wronskian([psi,seq(diff(psi,z$j),j=1..n-1)],z))
  ;tau[nu-1]:=(n)->det(wronskian([diff(psi,z),seq(diff(psi,z$j),j=
  2..n)],z));
```

$$\tau_\nu := n \to linalg{:}\text{-}det\left(linalg{:}\text{-}wronskian\left(\left[\psi, seq\left(\frac{\partial^j}{\partial z^j}\,\psi, j = 1\,..n-1\right)\right], z\right)\right)$$

$$\tau_{\nu-1} := n \to linalg{:}\text{-}det\left(linalg{:}\text{-}wronskian\left(\left[\frac{\partial}{\partial z}\,\psi, seq\left(\frac{\partial^j}{\partial z^j}\,\psi, j = 2\,..n\right)\right], z\right)\right) \tag{5}$$

```
> tau[nu-1](0):=1;tau[nu](0):=1;
```

$$\tau_{\nu-1}(0) := 1$$

$$\tau_\nu(0) := 1 \tag{6}$$

```
> expand(-2*z+epsilon*diff(ln(tau[nu](n+1)/tau[nu](n)),z)):simplify
  (subs(S1,%)):simplify(subs(S1,%)):simplify(subs(S1,%)):simplify
  (subs(S1,%)):w:=convert(%,parfrac,diff(psi,z)):W:=convert
  (simplify(subs(diff(psi,z)=Phi*psi,%)),parfrac,Phi);
```

$$W := -\frac{2\left(\nu - 1\right)}{z} + \frac{2\,\Phi}{\Phi^2 - 2\,\Phi\, z + 2\,\nu} \tag{7}$$

$$+\left(2\,\Phi^2\,\nu\, z^2 - 4\,\Phi\,\nu\, z^3 - 4\,\Phi^2\,\nu^2 + 8\,\Phi\,\nu^2\, z + 4\,\nu^2\, z^2 + 6\,\Phi^2\,\nu - 10\,\Phi\,\nu\, z - 8\,\nu^3\right.$$

$$-2\,\Phi^2+8\,\nu^2\Big)\Big/\Big(\big(\Phi^3 z-2\,\Phi^2 z^2-2\,\Phi^2\nu+6\,\Phi\,\nu\,z+\Phi^2-4\,\nu^2\big)\,z\Big)$$

```
> A:=epsilon*(2*n-nu);B:=-2*(nu+1)^2;simplify(P4):collect(numer
  (expand(subs(S1,subs(S1,expand(%))))),diff,factor);
```

$$A := 4-\nu$$

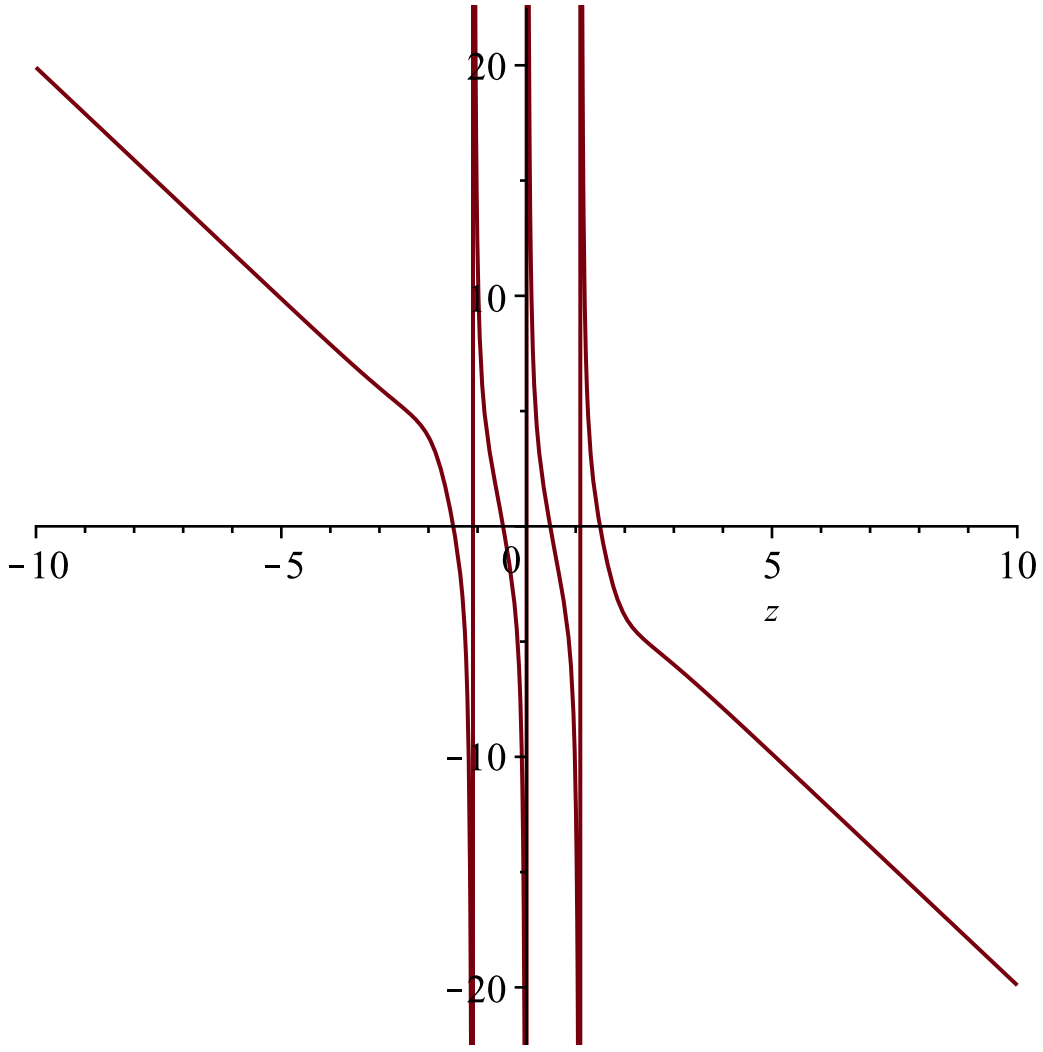
$$B := -2\,(\nu+1)^2$$

$$0 \tag{8}$$

```
> mu:=5;Digits:=300;plot(subs(Phi=diff(ln(exp(epsilon*z^2/2)*
  (CylinderD(mu,sqrt(2)*z))+0*CylinderD(mu,-sqrt(2)*z)),z),nu=mu,
  W),z);
```

$$\mu := 5$$

$$Digits := 300$$




```
> expand(epsilon*diff(ln(tau[nu](n)/tau[nu-1](n)),z)):simplify(subs
  (S1,%)):simplify(subs(S1,%)):simplify(subs(S1,%)):simplify(subs
  (S1,%)):w:=convert(%,parfrac,diff(psi,z)):W:=convert(simplify
  (subs(diff(psi,z)=Phi*psi,%)),parfrac,Phi);
```

$$W := \frac{2\,z}{\nu-1}-\frac{2\,\nu\,\left(-2\,\Phi\,z^2+\Phi\,\nu+2\,\nu\,z-\Phi\right)}{\left(\Phi^2\nu-2\,\Phi\,\nu\,z-\Phi^2+2\,\nu^2\right)\,(\nu-1)}-\frac{2\,\Phi}{\Phi^2-2\,\Phi\,z+2\,\nu} \tag{9}$$

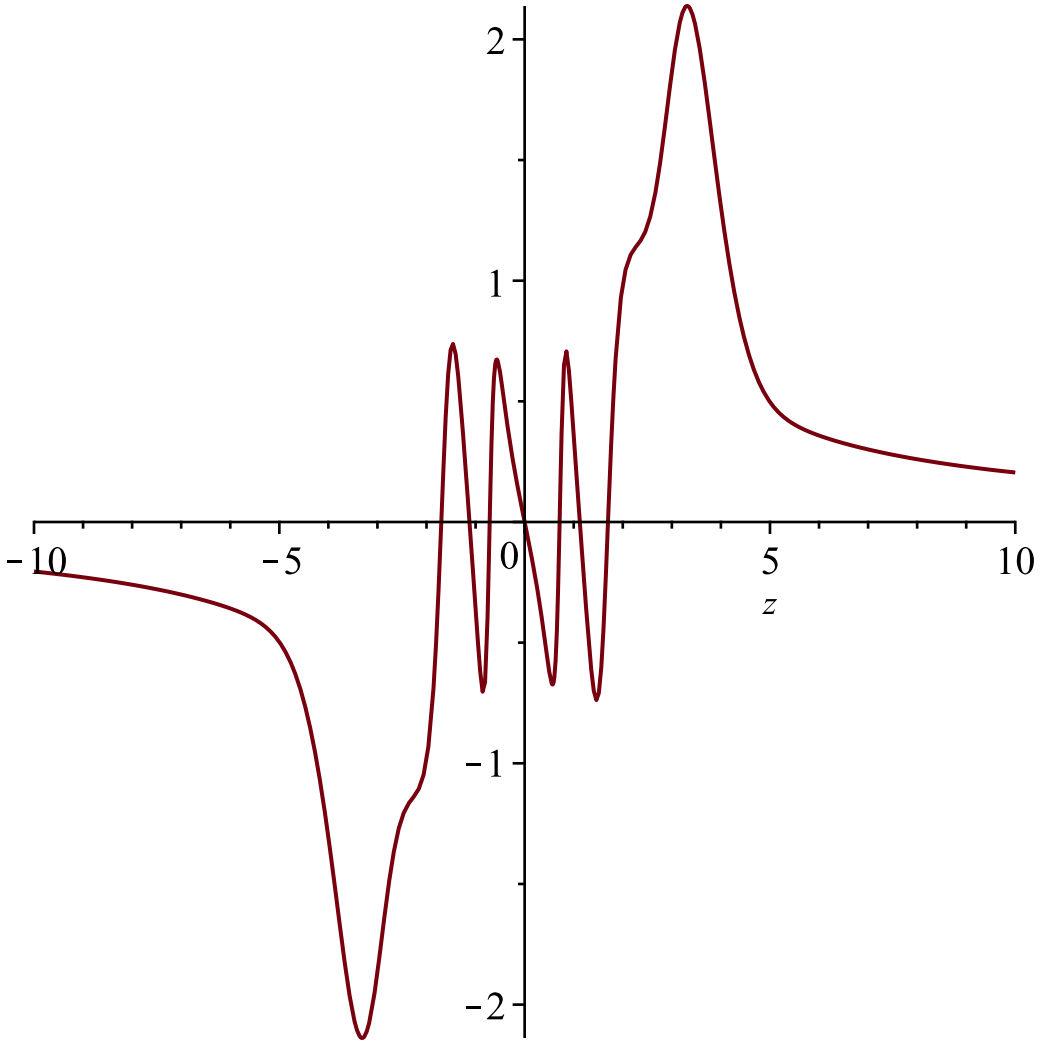
```
> A:=epsilon*(2*nu-n+1);B:=-2*(n)^2;simplify(P4):collect(numer
```

```
(expand(subs(S1,subs(S1,expand(%)))))),diff,factor);plot(subs(Phi=
diff(ln(exp(epsilon*z^2/2)*(CylinderD(mu,sqrt(2)*z))+CylinderD
(mu,-sqrt(2)*z)),z),nu=mu,W),z);
```

$$A := 2\,\nu - 1$$

$$B := -8$$

$$0$$




```
> expand(epsilon*diff(ln(tau[nu-1](n)/tau[nu](n+1)),z)):simplify
(subs(S1,%)):simplify(subs(S1,%)):simplify(subs(S1,%)):simplify
(subs(S1,%)):w:=convert(%,parfrac,diff(psi,z)):W:=convert
(simplify(subs(diff(psi,z)=Phi*psi,%)),parfrac,Phi);
```

$$W := \frac{2\left(-\nu z^2+\nu^2-2\nu+1\right)}{(\nu-1)\,z}+\frac{2\nu\left(-2\,\Phi z^2+\Phi\nu+2\nu z-\Phi\right)}{\left(\Phi^2\nu-2\,\Phi\nu z-\Phi^2+2\nu^2\right)(\nu-1)} \tag{10}$$

$$+\left(-2\,\Phi^2\nu z^2+4\,\Phi\nu z^3+4\,\Phi^2\nu^2-8\,\Phi\nu^2 z-4\,\nu^2 z^2-6\,\Phi^2\nu+10\,\Phi\nu z+8\,\nu^3\right.$$

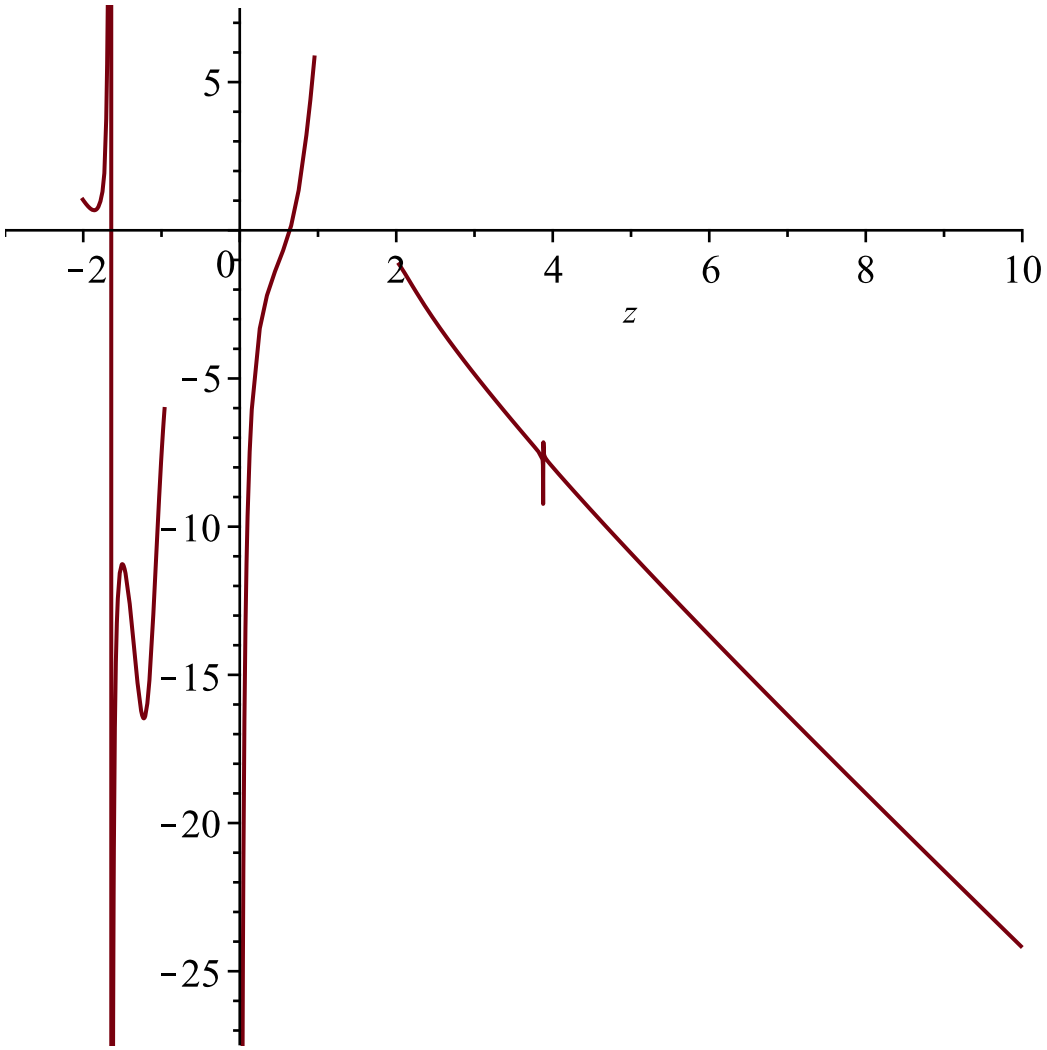
$$\left.+2\,\Phi^2-8\,\nu^2\right)\Big/\left(\left(\Phi^3 z-2\,\Phi^2 z^2-2\,\Phi^2\nu+6\,\Phi\nu z+\Phi^2-4\,\nu^2\right)z\right)$$

```
> A:=-epsilon*(n+nu+1);B:=-2*(nu-n)^2;simplify(P4):collect(numer
(expand(subs(S1,subs(S1,expand(%)))))),diff,factor);plot(subs(Phi=
diff(ln(CylinderD(mu,sqrt(2)*z)+0*CylinderD(mu,-sqrt(2)*z))*exp
(epsilon*z^2/2),z),nu=mu,W),z);
```

$$A := -3 - \nu$$

$$B := -2\,(\nu - 2)^2$$

$$0$$



S4

```
> S4 := (diff(sigma, z, z))^2-4*z^2*(diff(sigma, z))^2+8*z*(diff
  (sigma, z))*sigma-4*sigma^2+4*(diff(sigma, z))^3+8*(diff(sigma,
  z))^2*theta[infinity]+8*theta[0]*(diff(sigma, z))^2+16*(diff
  (sigma, z))*theta[0]*theta[infinity];
```

$$S4 := \left(\frac{\partial^2}{\partial z^2}\,\sigma\right)^2 - 4\,z^2\left(\frac{\partial}{\partial z}\,\sigma\right)^2 + 8\,z\left(\frac{\partial}{\partial z}\,\sigma\right)\sigma - 4\,\sigma^2 + 4\left(\frac{\partial}{\partial z}\,\sigma\right)^3 + 8\left(\frac{\partial}{\partial z}\,\sigma\right)^2\theta_\infty \qquad (11)$$

$$+ 8\,\theta_0\left(\frac{\partial}{\partial z}\,\sigma\right)^2 + 16\left(\frac{\partial}{\partial z}\,\sigma\right)\theta_0\,\theta_\infty$$

```
> expand(diff(ln(tau[nu](n)),z)):simplify(subs(S1,%)):simplify(subs
  (S1,%)):simplify(subs(S1,%)):simplify(subs(S1,%)):sigma:=convert
  (%,parfrac,diff(psi,z));S:=convert(simplify(subs(diff(psi,z)=Phi*
  psi,%)),parfrac,Phi);
```

$$\sigma := 2\,z - \frac{2\,\psi\left(\frac{\partial}{\partial z}\,\psi\right)}{-2\left(\frac{\partial}{\partial z}\,\psi\right)\psi\,z + 2\,\psi^{2}\,\nu + \left(\frac{\partial}{\partial z}\,\psi\right)^{2}}$$

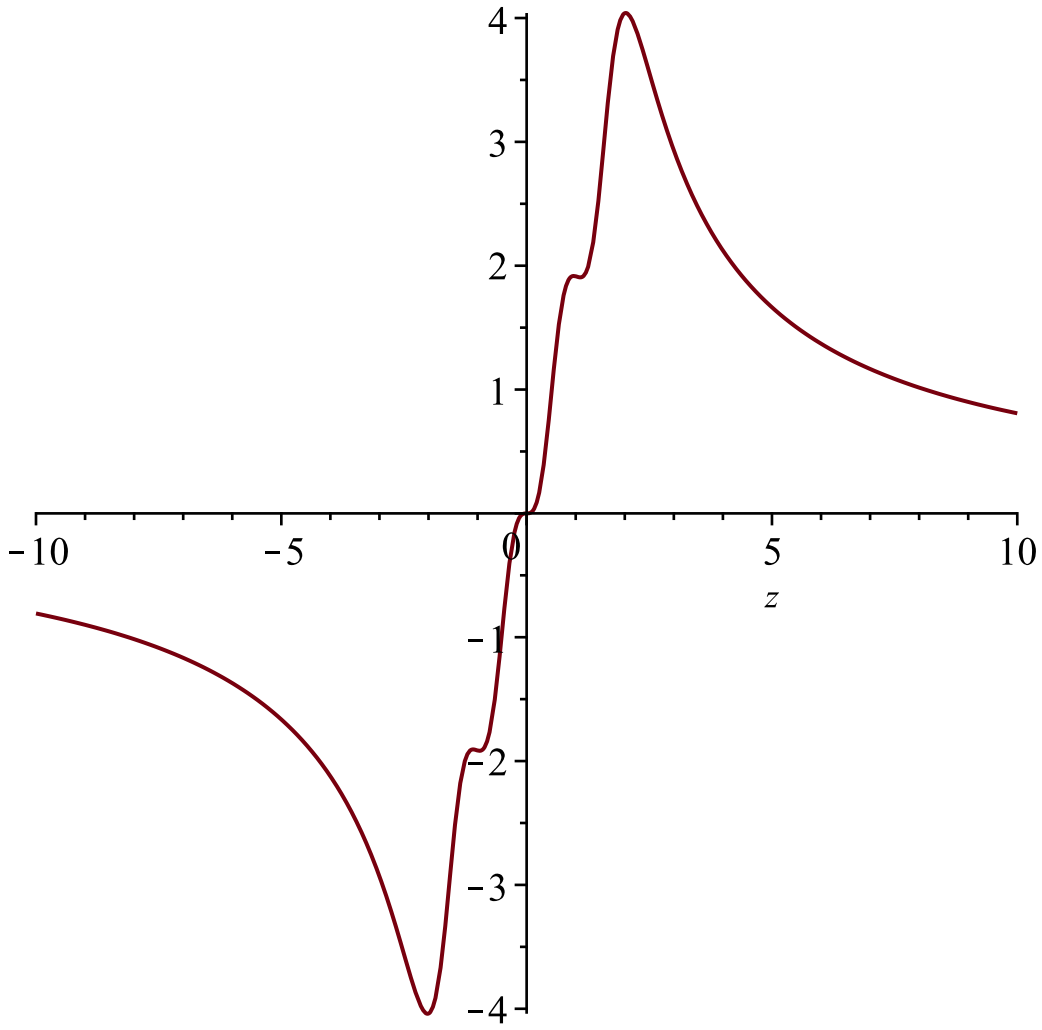
$$S := 2\,z - \frac{2\,\Phi}{\Phi^{2} - 2\,\Phi\,z + 2\,\nu} \qquad \textbf{(12)}$$

```
> theta[0]:=epsilon*(nu-n+1);theta[infinity]:=-epsilon*n;simplify
  (S4):collect(numer(expand(subs(S1,subs(S1,expand(%))))),diff,
  factor);plot(subs(Phi=diff(ln(exp(epsilon*z^2/2)*CylinderD(mu,
  sqrt(2)*z)),z),nu=mu,S),z);
```

$$\theta_0 := \nu - 1$$

$$\theta_\infty := -2$$

$$0$$




```
> expand(diff(ln(tau[nu](n)),z)):simplify(subs(S1,%)):simplify(subs
  (S1,%)):simplify(subs(S1,%)):simplify(subs(S1,%)):sigma:=convert
  (%,parfrac,diff(psi,z))-2*epsilon*n*z;S:=convert(simplify(subs
  (diff(psi,z)=Phi*psi,%)),parfrac,Phi);
```

$$\sigma := -2\,z - \frac{2\,\psi\left(\frac{\partial}{\partial z}\,\psi\right)}{-2\left(\frac{\partial}{\partial z}\,\psi\right)\psi\,z + 2\,\psi^{2}\,\nu + \left(\frac{\partial}{\partial z}\,\psi\right)^{2}}$$

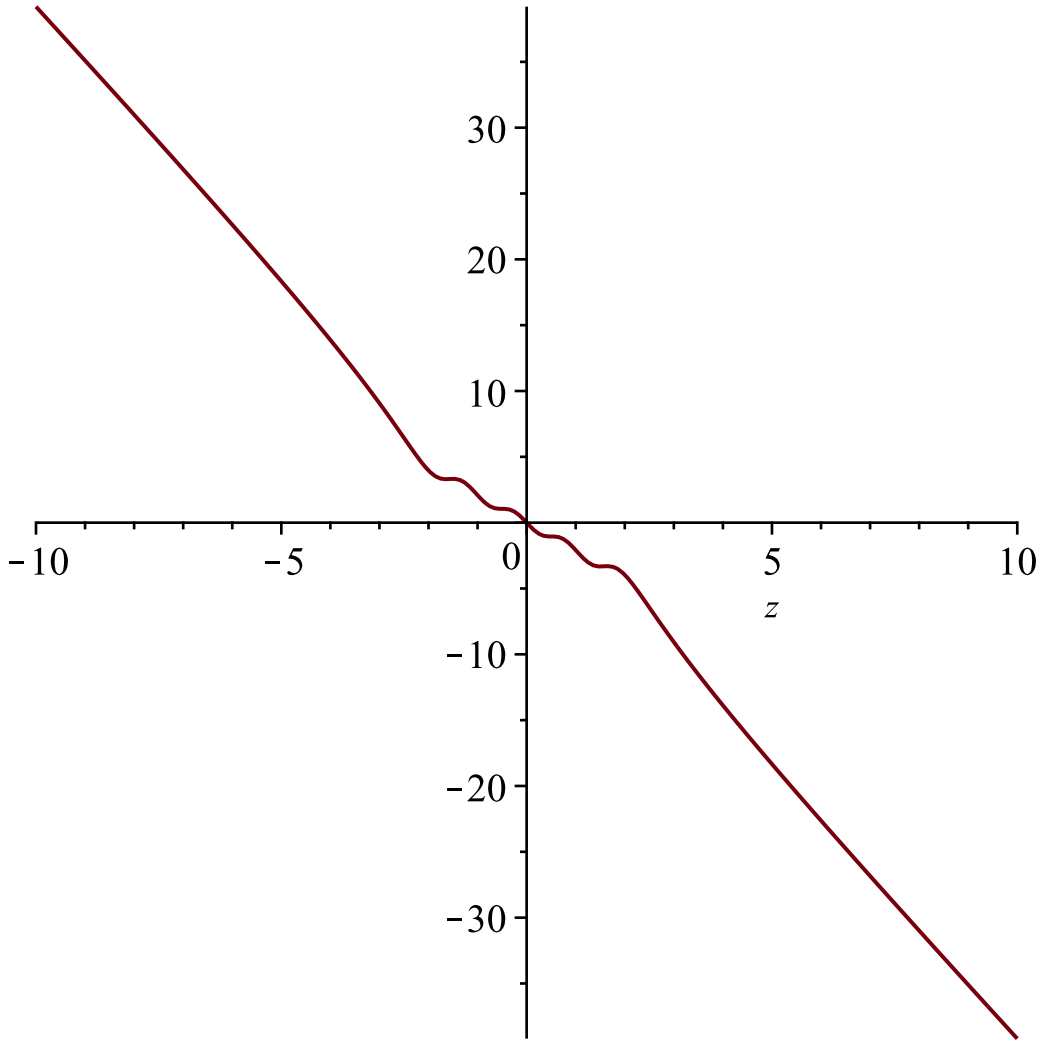
$$S := -2\,z - \frac{2\,\Phi}{\Phi^{2} - 2\,\Phi\,z + 2\,\nu} \qquad \textbf{(13)}$$

```
> theta[0]:=epsilon*n;theta[infinity]:=epsilon*(nu+1);simplify(S4)
  :collect(numer(expand(subs(S1,subs(S1,expand(%))))),diff,factor);
  plot(subs(Phi=diff(ln(exp(epsilon*z^2/2)*CylinderD(mu,sqrt(2)*z)
  ),z),nu=mu,S),z);
```

$$\theta_0 := 2$$

$$\theta_\infty := \nu + 1$$

$$0$$




```
> expand(diff(ln(tau[nu](n)),z)):simplify(subs(S1,%)):simplify(subs
  (S1,%)):simplify(subs(S1,%)):simplify(subs(S1,%)):sigma:=convert
  (%,parfrac,diff(psi,z))+2*epsilon*(nu-n+1)*z;S:=convert(simplify
  (subs(diff(psi,z)=Phi*psi,%)),parfrac,Phi);
```

$$\sigma := 2\,z - \frac{2\,\psi\left(\frac{\partial}{\partial z}\,\psi\right)}{-2\left(\frac{\partial}{\partial z}\,\psi\right)\psi\,z + 2\,\psi^{2}\,\nu + \left(\frac{\partial}{\partial z}\,\psi\right)^{2}} + 2\,(\nu - 1)\,z$$

$$S := 2\,\nu\,z - \frac{2\,\Phi}{\Phi^{2} - 2\,\Phi\,z + 2\,\nu} \qquad \textbf{(14)}$$

```
> theta[0]:=-epsilon*(nu+1);theta[infinity]:=-epsilon*(nu-n+1);
  simplify(S4):collect(numer(expand(subs(S1,subs(S1,expand(%))))),
  diff,factor);plot(subs(Phi=diff(ln(exp(epsilon*z^2/2)*CylinderD
  (mu,sqrt(2)*z)),z),nu=mu,S),z);
```

$$\theta_0 := -\nu - 1$$

$$\theta_\infty := -\nu + 1$$

$$0$$