```
> restart;
> alpha:=2;beta:=2;
```

$$\alpha := 2$$

$$\beta := 2 \qquad (1)$$

```
> mu:=(k)->simplify(expand(int(x^k*(1-x)^alpha*(1+x)^beta*exp(-z*x),x=-1..1)));
```

$$\mu := k \rightarrow simplify\left(expand\left(\int_{-1}^{1} x^k (1-x)^{\alpha} (1+x)^{\beta} e^{-z x} \, dx\right)\right) \qquad (2)$$

```
> MU:=(k)->simplify(2^(alpha+beta+1)*int((2*u-1)^k*(1-u)^alpha*u^beta*exp(-z*(2*u-1)),u=0..1));
```

$$MU := k \rightarrow simplify\left(2^{\alpha+\beta+1}\left(\int_{0}^{1} (2u-1)^k (1-u)^{\alpha} u^{\beta} e^{-z(2u-1)} \, du\right)\right) \qquad (3)$$

```
> MU:=(k)->simplify(expand(2^(alpha+beta+1)*exp(z)*sum(binomial(k,r)*(2)^(r)*(-1)^(k-r)*int((1-u)^alpha*u^(beta+r)*exp(-z*2*u),u=0..1),r=0..k)));
```

$$MU := k \rightarrow simplify\left(expand\left(2^{\alpha+\beta+1} e^{z}\left(\sum_{r=0}^{k} \mathrm{binomial}(k, r)\, 2^r\, (-1)^{k-r}\left(\int_{0}^{1} (1-u)^{\alpha} u^{\beta+r} e^{-2zu} \, du\right)\right)\right)\right) \qquad (4)$$

```
> simplify(mu(2)-MU(2));
```

$$0 \qquad (5)$$

```
> MU:=(k)->simplify(expand(2^(alpha+beta+1)*GAMMA(alpha+1)*exp(z)*sum(binomial(k,r)*(2)^(r)*(-1)^(k-r)*GAMMA(beta+r+1)/GAMMA(alpha+beta+r+2)*KummerM(beta+r+1,alpha+beta+r+2,-2*z),r=0..k)));
```

$$MU := k \rightarrow simplify\left(expand\left(2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\, e^{z}\left(\sum_{r=0}^{k} \frac{\mathrm{binomial}(k, r)\, 2^r\, (-1)^{k-r}\,\Gamma(\beta+r+1)\,\mathrm{KummerM}(\beta+r+1, \alpha+\beta+r+2, -2z)}{\Gamma(\alpha+\beta+r+2)}\right)\right)\right) \qquad (6)$$

```
> MU:=(k)->simplify(expand(2^(alpha+beta+1)*sum(exp(z)*binomial(k,r)*(2)^(r)*(-1)^(k-r)*GAMMA(alpha+1)*GAMMA(beta+r+1)/GAMMA(alpha+beta+r+2)*KummerM(beta+r+1,alpha+beta+r+2,-2*z),r=0..k)));
```

$$MU := k \rightarrow simplify\left(expand\left(2^{\alpha+\beta+1}\left(\sum_{r=0}^{k} \frac{1}{\Gamma(\alpha+\beta+r+2)}\left(e^{z}\,\mathrm{binomial}(k, r)\, 2^r\, (-1)^{k-r}\,\Gamma(\alpha+1)\,\Gamma(\beta+r+1)\,\mathrm{KummerM}(\beta+r+1, \alpha+\beta+r+2, -2z)\right)\right)\right)\right) \qquad (7)$$

```
> MUdif:=(k)->simplify(expand(2^(alpha+beta+1)*GAMMA(alpha+1)*exp(z)*sum(binomial(k,r)*(2)^(r)*(-1)^(k-r)*GAMMA(beta+r+1)/GAMMA(alpha+beta+r+2)*KummerM(beta+r+1,alpha+beta+r+2,-2*z)-binomial
```

```
(k,r)*(2)^(r+1)*(-1)^(k-r)*GAMMA(beta+r+2)/GAMMA(alpha+beta+r+3)*
KummerM(beta+r+2,alpha+beta+r+3,-2*z),r=0..k)));
```

$$MUdif := k \to simplify\left(expand\left(2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,e^{z}\left(\sum_{r=0}^{k}\left(\frac{\mathrm{binomial}(k,r)\,2^{r}\,(-1)^{k-r}\,\Gamma(\beta+r+1)\,\mathrm{KummerM}(\beta+r+1,\alpha+\beta+r+2,-2z)}{\Gamma(\alpha+\beta+r+2)} - \frac{1}{\Gamma(\alpha+\beta+r+3)}\left(\mathrm{binomial}(k,r)\,2^{r+1}\,(-1)^{k-r}\,\Gamma(\beta+r+2)\,\mathrm{KummerM}(\beta+r+2,\alpha+\beta+r+3,-2z)\right)\right)\right)\right)\right) \quad (8)$$

```
> MUdif:=(k)->simplify(expand(2^(alpha+beta+1)*GAMMA(alpha+1)*exp
(z)*sum(binomial(k,r)*(2)^(r)*(-1)^(k-r)*GAMMA(beta+r+1)/GAMMA
(alpha+beta+r+2)*KummerM(beta+r+1,alpha+beta+r+2,-2*z)-binomial
(k,r)*(2)^(r+1)*(-1)^(k-r)*GAMMA(beta+r+2)/GAMMA(alpha+beta+r+3)*
KummerM(beta+r+2,alpha+beta+r+3,-2*z),r=0..k)));
```

$$MUdif := k \to simplify\left(expand\left(2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,e^{z}\left(\sum_{r=0}^{k}\left(\frac{\mathrm{binomial}(k,r)\,2^{r}\,(-1)^{k-r}\,\Gamma(\beta+r+1)\,\mathrm{KummerM}(\beta+r+1,\alpha+\beta+r+2,-2z)}{\Gamma(\alpha+\beta+r+2)} - \frac{1}{\Gamma(\alpha+\beta+r+3)}\left(\mathrm{binomial}(k,r)\,2^{r+1}\,(-1)^{k-r}\,\Gamma(\beta+r+2)\,\mathrm{KummerM}(\beta+r+2,\alpha+\beta+r+3,-2z)\right)\right)\right)\right)\right) \quad (9)$$

```
> expand(mu(1)+MUdif(0));
```

$$0 \quad (10)$$

```
> restart;alias(S[n]=S[n](z),sigma=sigma(z),u=u(Zeta),H[n]=H[n](z),
H[n]=H[n](t)):with(PDEtools):
> t^2*(diff(H[n], t, t))^2-4*(-n*(a+n)-H[n]+(1/2*(a+2*n+b+2*t))*
(diff(H[n], t)))^2-8*(diff(H[n], t))*(t*(diff(H[n], t))-H[n])*(-
b-(1/2)*(diff(H[n], t)));
```

$$t^2\left(\frac{\partial^2}{\partial t^2}H_n\right)^2 - 4\left(-n\,(a+n) - H_n + \frac{1}{2}\,(a+2n+b+2t)\left(\frac{\partial}{\partial t}H_n\right)\right)^2 - 8\left(\frac{\partial}{\partial t}H_n\right)\left(t\left(\frac{\partial}{\partial t}H_n\right) - H_n\right)\left(-b - \frac{1}{2}\,\frac{\partial}{\partial t}H_n\right) \quad (11)$$

```
> S5 := z^2*(diff(H[n], z, z))^2-(-n*(alpha+n)-H[n]+(alpha+2*n+
beta+z)*(diff(H[n], z)))^2-4*(diff(H[n], z))*(z*(diff(H[n], z))-H
[n])*(-beta-(diff(H[n], z)));
```

$$S5 := z^2\left(\frac{\partial^2}{\partial z^2}H_n\right)^2 - \left(-n\,(\alpha+n) - H_n + (\alpha+2n+\beta+z)\left(\frac{\partial}{\partial z}H_n\right)\right)^2 \quad (12)$$

$$-4\left(\frac{\partial}{\partial z}H_n\right)\left(z\left(\frac{\partial}{\partial z}H_n\right)-H_n\right)\left(-\beta-\left(\frac{\partial}{\partial z}H_n\right)\right)$$

```
> JMOeq:=(z*diff(sigma,z,z))^2-(2*diff(sigma,z)^2-z*diff(sigma,z)+sigma)^2+4*product(diff(sigma,z)+k[j],j=0..3);
```

$$JMOeq := z^2\left(\frac{\partial^2}{\partial z^2}\sigma\right)^2-\left(2\left(\frac{\partial}{\partial z}\sigma\right)^2-z\left(\frac{\partial}{\partial z}\sigma\right)+\sigma\right)^2+4\left(\frac{\partial}{\partial z}\sigma+k_0\right)\left(\frac{\partial}{\partial z}\sigma+k_1\right)\left(\frac{\partial}{\partial z}\sigma+k_2\right)\left(\frac{\partial}{\partial z}\sigma+k_3\right) \quad \textbf{(13)}$$

```
> eq1:=collect(expand(subs(H[n]=sigma+b*z+c,S5)-JMOeq),[diff,sigma,z],factor):
> coeff(eq1,diff(sigma,z)):bc:=solve({op(1,%),op(2,%)},{b,c}):
> H[n]=collect(subs(%,sigma+b*z+c),[z,n],factor);collect(%-sigma,[z],factor):collect(%,[z,n],factor);
```

$$H_n=\left(-\frac{1}{4}\beta+\frac{1}{4}\alpha+\frac{1}{2}n\right)z-\frac{1}{2}n^2+\left(-\frac{1}{2}\alpha-\frac{1}{2}\beta\right)n+\sigma+\frac{1}{8}\alpha^2+\frac{1}{8}\beta^2-\frac{1}{4}\alpha\beta$$

$$H_n-\sigma=\left(-\frac{1}{4}\beta+\frac{1}{4}\alpha+\frac{1}{2}n\right)z-\frac{1}{2}n^2+\left(-\frac{1}{2}\alpha-\frac{1}{2}\beta\right)n+\frac{1}{8}(\alpha-\beta)^2 \quad \textbf{(14)}$$

```
> eq2:=collect(expand(subs(bc,eq1)),[diff,sigma,z],factor):
> solve({seq(coeff(eq2,diff(sigma,z),j),j=0..3)},{k[0],k[1],k[2],k[3]}): collect(%[1],factor);
```

$$\left\{k_0=-\frac{1}{4}\beta+\frac{1}{4}\alpha-\frac{1}{2}n,\ k_1=\frac{3}{4}\beta+\frac{1}{4}\alpha+\frac{1}{2}n,\ k_2=-\frac{1}{4}\beta+\frac{1}{4}\alpha+\frac{1}{2}n,\ k_3=-\frac{1}{4}\beta-\frac{3}{4}\alpha-\frac{1}{2}n\right\} \quad \textbf{(15)}$$

```
> restart;with(PDEtools):with(linalg):with(VectorCalculus):with(LinearAlgebra):alias(H[n]=H[n](t),sigma=sigma(t),phi=phi(t));
```

$$H_n, \sigma, \phi \quad \textbf{(16)}$$

```
> S5 := t^2*(diff(H[n], t, t))^2-(-n*(alpha+n)-H[n]+(alpha+2*n+beta+t)*(diff(H[n], t)))^2-4*(diff(H[n], t))*(t*(diff(H[n], t))-H[n])*(-beta-(diff(H[n], t)));
```

$$S5 := t^2\left(\frac{\partial^2}{\partial t^2}H_n\right)^2-\left(-n(\alpha+n)-H_n+(\alpha+2n+\beta+t)\left(\frac{\partial}{\partial t}H_n\right)\right)^2-4\left(\frac{\partial}{\partial t}H_n\right)\left(t\left(\frac{\partial}{\partial t}H_n\right)-H_n\right)\left(-\beta-\left(\frac{\partial}{\partial t}H_n\right)\right) \quad \textbf{(17)}$$

```
> K2 := diff(phi, t, t) = (alpha+1)*phi/t-(alpha+beta+2-t)*(diff(phi, t))/t:K3:=diff(K2,t):K4:=diff(K3,t):
> n:=2;
```

$$n := 2 \quad \textbf{(18)}$$

```
> tau[n]:=collect(subs(K3,K2,det(Wronskian([exp(-t/2)*phi,seq(diff(exp(-t/2)*phi,t$j),j=1..n-1)],t))),diff(phi,t),factor):
> H[n]:=convert(simplify(subs(K4,K3,K2,t*diff(ln(tau[n]),t)+n*t/2)),parfrac,diff(phi,t));
```

$$H_2(t) := -\alpha-\beta-2+t+\frac{\phi\left(-\left(\frac{\partial}{\partial t}\phi\right)\alpha-2\left(\frac{\partial}{\partial t}\phi\right)-\left(\frac{\partial}{\partial t}\phi\right)\beta+\phi\alpha+\phi\right)}{\left(\frac{\partial}{\partial t}\phi\right)^2 t+(-t\phi+2\phi+\phi\alpha+\beta\phi)\left(\frac{\partial}{\partial t}\phi\right)-\phi^2\alpha-\phi^2} \quad \textbf{(19)}$$

```
> subs(K3,K2,S5):collect(%,[diff(phi, t),t,phi],factor);
```

$$0 \tag{20}$$

```
> tr := {t=z*2};
```

$$tr := \{t = 2\,z\} \tag{21}$$

```
> H52:=dchange(tr,H5):alias(H[n]=H[n](z)):H52;
```