```
> restart;Digits:=30;
```

$$Digits := 30 \tag{1}$$

```
> with(PDEtools):with(linalg):with(LinearAlgebra):with(plots):alias
  (w=w(z),phi=phi(t),psi=psi(t)):d:=-1/2:epsilon[3]:=1;
```

$$\varepsilon_3 := 1 \tag{2}$$

```
> P5:=(diff(w, z, z))-((1/(2*w)+1/(w-1))*(diff(w,z)^2)-1/z*diff(w,
  z)+(w-1)^2/z^2*(A*w+B/w)+C*w/z+d*w*(w+1)/(w-1));
```

$$P5 := \frac{\partial^2}{\partial z^2} w - \left(\frac{1}{2\,w} + \frac{1}{w-1} \right) \left(\frac{\partial}{\partial z} w \right)^2 + \frac{\frac{\partial}{\partial z} w}{z} - \frac{(w-1)^2 \left(A\,w + \frac{B}{w} \right)}{z^2} - \frac{C\,w}{z} + \frac{1}{2} \frac{w\,(w+1)}{w-1} \tag{3}$$

```
> a:=-6;b:=-30;phi:=simplify(KummerU(a,b,z));#phi:=simplify
  (LaguerreL(-a,b-1,z)):
```

$$a := -6$$

$$b := -30$$

$$\phi := z^6 + 150\,z^5 + 9750\,z^4 + 351000\,z^3 + 7371000\,z^2 + 85503600\,z + 427518000 \tag{4}$$

```
> n:=3;
```

$$n := 3 \tag{5}$$

```
> diff(exp(-z)*phi,z):for K from 1 to n+1 do;l[K]:=diff(%,z)*z;
  od:wronskian([diff(exp(-z)*phi,z),seq(l[k],k=1..n)],z):for K from
  1 to n+1 do;h[K]:=Row(%,1);row(%%,2);wronskian(%*z,z):od:simplify
  (subs(<seq(simplify(h[k]),k=1..n+1)>)):tau[n+1]:=det(%):
> exp(-z)*phi:for K from 1 to n do;l[K]:=diff(%,z)*z;od:wronskian(
  [exp(-z)*phi,seq(l[k],k=1..n-1)],z):for K from 1 to n do;h[K]:=
  Row(%,1);row(%%,2);wronskian(%*z,z):od:simplify(<seq(simplify(h
  [k]),k=1..n)>):tau[n]:=det(%):
> w:=convert(simplify(expand(1+1/(a-b-n)*(b+z+z*diff(ln((tau[n+1])/
  (tau[n])),z)))),parfrac,z):
> H:={A=1/2*(b-a+n)^2,B=-1/2*a^2,C=1+n-b};
```

$$H := \left\{ A = \frac{441}{2}, B = -18, C = 34 \right\} \tag{6}$$

```
> numer(subs(H,P5));
```

$$0 \tag{7}$$

```
> epsilon[3]:=-1;
```

$$\varepsilon_3 := -1 \tag{8}$$

```
> diff(phi,z):for K from 1 to n+1 do;l[K]:=diff(%,z)*z;od:wronskian
  ([diff(phi,z),seq(l[k],k=1..n)],z):for K from 1 to n+1 do;h[K]:=
  Row(%,1);row(%%,2);wronskian(%*z,z):od:simplify(subs(<seq
  (simplify(h[k]),k=1..n+1)>)):tau[n+1]:=det(%):
> phi:for K from 1 to n do;l[K]:=diff(%,z)*z;od:wronskian([phi,seq
  (l[k],k=1..n-1)],z):for K from 1 to n do;h[K]:=Row(%,1);row(%%,2)
  ;wronskian(%*z,z):od:simplify(<seq(simplify(h[k]),k=1..n)>):tau
  [n]:=det(%):
> w:=convert(simplify(expand(1+1/(a+n)*(z-b-z*diff(ln((tau[n+1])/
  (tau[n])),z)))),parfrac,z):
> H:={A=(a+n)^2/2,B=-(b-a)^2/2,C=(b-n-1)};
```

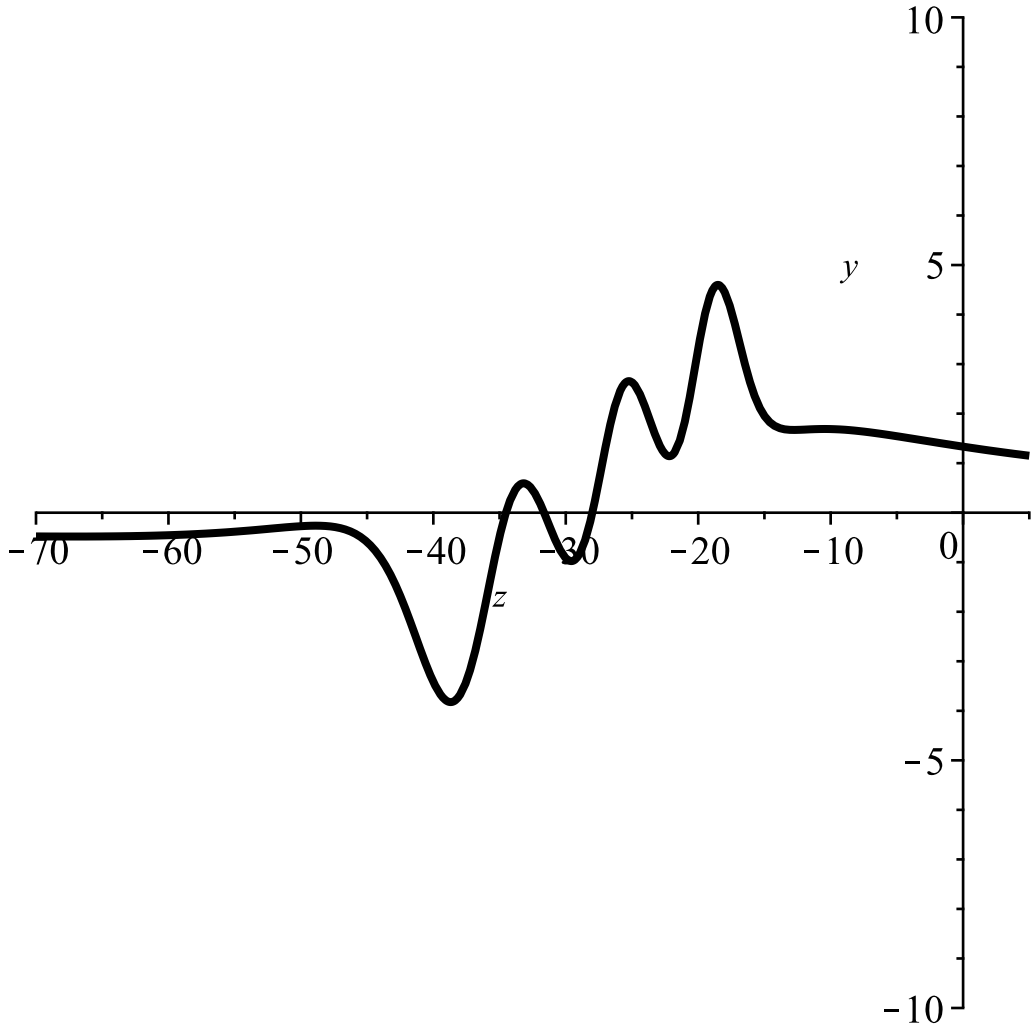
(9)

$$H := \left\{A = \frac{9}{2}, B = -288, C = -34\right\} \quad (9)$$

```
> collect(numer(subs(H,P5)),z,factor);
```

$$0 \quad (10)$$

```
> RootOf(tau[n+1]*(z^((n+1)/2*(1-(n+1)))),z):J1:=evalf(allvalues(%)
  ):RootOf(tau[n]*(z^(n/2*(1-n))),z):J2:=(allvalues(%)):
> #A:=complexplot([J1],style=point,symbol=solidcircle,color=
  magenta,symbolsize=25):
> #B:=complexplot([J2],style=point,symbol=solidcircle,color=blue,
  symbolsize=25):
> C:=plot(w-((z+3*n+2*a+1-b)/(a+n)),z=-70..5,y=-10..10,colour=
  black,thickness=3):display(C);
```




```
> restart;Digits:=100:with(PDEtools):with(linalg):with
  (LinearAlgebra):with(plots):alias(S[n]=S[n](z),sigma=sigma(z),
  phi=phi(z));
```

$$S_n, \sigma, \phi \quad (11)$$

```
> S5:=z^2*(diff(sigma, z, z))^2-(2*(diff(sigma, z))^2-z*(diff
  (sigma, z))+sigma)^2+(4*(diff(sigma, z)+k[0]))*(diff(sigma, z)+k
  [1])*(diff(sigma, z)+k[2])*(diff(sigma, z)+k[3]);
```

(12)

$$S5 := z^2 \left(\frac{\partial^2}{\partial z^2}\,\sigma\right)^2 - \left(2\left(\frac{\partial}{\partial z}\,\sigma\right)^2 - z\left(\frac{\partial}{\partial z}\,\sigma\right) + \sigma\right)^2 + 4\left(\frac{\partial}{\partial z}\,\sigma + k_0\right)\left(\frac{\partial}{\partial z}\,\sigma + k_1\right)\left(\frac{\partial}{\partial z}\,\sigma + k_2\right)\left(\frac{\partial}{\partial z}\,\sigma + k_3\right) \tag{12}$$

```
> A:=-5/8*(n+1)^2+1/4*(2*alpha+1+beta+3*z)*(n+1)-1/8*(-2*alpha-1+
  beta)*(-2*alpha-1-2*z+beta);k[0]:= 1/4*(2*alpha-beta+n+2);k[1]:=
  -1/4*(2*alpha+beta-n-2);k[2]:= 1/4*(2*alpha-beta-3*n-2);k[3]:=
  -1/4*(2*alpha-3*beta-n+2);
```

$$A := -\frac{5}{8}\,(n+1)^2 + \frac{1}{4}\,(2\,\alpha + 1 + \beta + 3\,z)\,(n+1) - \frac{1}{8}\,(-2\,\alpha - 1 + \beta)\,(-2\,\alpha - 1 - 2\,z + \beta)$$

$$k_0 := \frac{1}{2}\,\alpha - \frac{1}{4}\,\beta + \frac{1}{4}\,n + \frac{1}{2}$$

$$k_1 := -\frac{1}{2}\,\alpha - \frac{1}{4}\,\beta + \frac{1}{4}\,n + \frac{1}{2}$$

$$k_2 := \frac{1}{2}\,\alpha - \frac{1}{4}\,\beta - \frac{3}{4}\,n - \frac{1}{2}$$

$$k_3 := -\frac{1}{2}\,\alpha + \frac{3}{4}\,\beta + \frac{1}{4}\,n - \frac{1}{2} \tag{13}$$

```
> n:=2;alpha:=-12;beta:=20;phi:=simplify(KummerM(alpha,beta,z))
  :phi:=simplify(LaguerreL(-alpha,beta-1,z)):
```

$$n := 2$$

$$\alpha := -12$$

$$\beta := 20 \tag{14}$$

```
> exp(-z)*phi:for K from 1 to n+1 do;l[K]:=diff(%,z)*z;od:wronskian
  ([exp(-z)*phi,seq(l[j],j=1..n)],z):for j from 1 to n+1 do;h[j]:=
  Row(%,1);row(%%,2);wronskian(%*z,z):od:<seq(simplify(h[j]),j=1..
  n+1)>:W:=simplify(det(%)):
> sigma:=convert(simplify(z*diff(ln(W),z))+A,parfrac,z):
> simplify(S5):simplify(%);
```

$$0 \tag{15}$$

```
> A:=-5/8*n^2+1/4*(3*beta+2-2*alpha-3*z)*n-1/8*(2*alpha-beta)*(2*
  alpha+2*z-beta);k[0]:=-1/4*(2*alpha+beta+n);k[1]:=1/4*(3*n+2*
  alpha-beta);k[2]:= 1/4*(3*beta-n-2*alpha);k[3]:= 1/4*(2*alpha-
  beta-n);
```

$$A := -\frac{403}{2} + \frac{19}{2}\,z$$

$$k_0 := \frac{1}{2}$$

$$k_1 := -\frac{19}{2}$$

$$k_2 := \frac{41}{2}$$

$$k_3 := -\frac{23}{2} \tag{16}$$

```
> phi:=simplify(KummerM(alpha,beta,z)):phi:=simplify(LaguerreL(-
  alpha-1,beta,z)):
> phi:for K from 1 to n do;l[K]:=diff(%,z)*z;od:wronskian([phi,seq
  (l[j],j=1..n-1)],z):for j from 1 to n do;h[j]:=Row(%,1);row(%%,2)
  ;wronskian(%*z,z):od:<seq(simplify(h[j]),j=1..n)>:W:=factor(det
  (%)):
> sigma:=convert(simplify(z*diff(ln(W),z))+A,parfrac,z):
> simplify(S5):simplify(%);
```

$$0 \tag{17}$$

```
> RootOf(W/z^(n*(n-1)/2),z):J1:=evalf(allvalues(%)):
Warning,  computation interrupted
> A1:=complexplot([J1],style=point,symbol=solidcircle,color=blue,
  symbolsize=25):
> C:=plot(sigma-A-((3*n-1)*n/2),z=-10..70,y=-65..35,colour=black,
  thickness=3):display(A1,C);
```