AS.110.407: Honors Complex Analysis
FINAL PROJECT

Fourier Series in Complex Analysis

James Guo

December 15, 2024

Introduction:

This project investigates Fourier Series in the context of complex analysis, especially on forming the Fourier Series on a holomorphic periodic function.

For the first part, the project will use Fourier series in real analysis as a motivation of writing functions in terms of series, whereas a proof on orthonormal basis utilizes conclusions from complex analysis. This part will mainly be following [1], with some differences in the definition of a Hilbert space.

Then, we will use the next two parts to investigate the periodic holomorphic function over complex analysis following [4]. We will first develop the foundations in the first half using factorization theorem, and eventually construct the Fourier series for periodic holomorphic functions with some worked examples.

Contents

I Fourier Series from Real Analysis **1**
I.1 $L^2([-\pi, \pi])$ Space . . . 1

II Period Holomorphic Functions **5**
II.1 Strips and Annuli . . . 6
II.2 Periodic Holomorphic Functions in Strips . . . 8

III Fourier Series in Stripes **11**
III.1 Fourier Development in Stripes . . . 11
III.2 Examples of Fourier Series for Periodic Functions . . . 12

I Fourier Series from Real Analysis

Out of the most common cases, we investigate the **Fourier Series** for functions that take in a real input and give a real output. In particular, we want to some "good" function f with input from $[-\pi, \pi]$, so that we may represent f in terms of a series of exponentials:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \exp(inx).$$

The first chapter aims to give a foundation of **Fourier series** in Real analysis context. Here, we will be following the information from [1].

I.1 $L^2([-\pi, \pi])$ Space

Here, we think of the space of the functions, just like a **vector space**. In particular, we consider a special type of vector space, the **Hilbert Space**.

Definition I.1.1. Hilbert Space.

A set $\mathcal{H}$ is a **Hilbert Space** if it satisfies the following properties:

(i) $\mathcal{H}$ is a vector space over $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$).

(ii) $\mathcal{H}$ is an inner product space, *i.e.*, having inner product $\langle \bullet, \bullet \rangle$ such that:

- Positivity: $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}$.
- Definiteness: $\langle f, f \rangle = 0$ if and only if $f = 0$.
- Linearity in first slot: $\langle \lambda f_1 + f_2, g \rangle = \lambda \langle f_1, g \rangle + \langle f_2, g \rangle$ for all $f_1, f_2, g \in \mathcal{H}$ and $\lambda \in \mathbb{F}$.
- Conjugate symmetry: $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all $f, g \in \mathcal{H}$.

Thus, we have the norm induced by $\|f\| = \langle f, f \rangle^{1/2}$ for $f \in \mathcal{H}$.

(iii) $\mathcal{H}$ is complete with respect the the metric $d(f, g) = \|f - g\|$ for $f, g \in \mathcal{H}$. ⌟

Note that in [1], $\mathcal{H}$ needs to be separable. However, we will lift this requirement for the sake of the project and give the proofs without this requirement.

As an inner product space, we also have **orthogonality** for granted, that is:
For $f, g \in \mathcal{H}$, f and g are orthogonal if $\langle f, g \rangle = 0$.

Moreover, the Hilbert space has many good properties (the verification is omitted), such as:

- **Triangular inequality** for the norm induced by inner product:
  For any $f, g, h \in \mathcal{H}$ $\|f - h\| \leq \|f - g\| + \|g - h\|$.

- **Cauchy-Schwarz inequality**:
  For $f, g \in \mathcal{H}$, $|\langle f, g \rangle|^2 \leq \|u\|^2 \cdot \|v\|^2$.

Definition I.1.2. Orthonormal Basis.

A set β consist of elements in $\mathcal{H}$ is the orthonormal basis of $\mathcal{H}$ if $\overline{\text{span}(\beta)} = \mathcal{H}$, every element is normal (for $x \in \mathcal{H}$, $\|x\| = 1$), and any two elements in β are orthogonal. ⌟

Another very important property of Hilbert Spaces is as follows.

Theorem I.1.3. Hilbert Spaces have Orthonormal Basis.

Any Hilbert space $\mathcal{H}$ has an orthonormal basis.

Since we did not assume **separability**, so we use another proof using **Zorn's Lemma**, from [2].

Proof. Let $\mathcal{O}$ be the set of all orthonormal sets of $\mathcal{H}$. Without loss of generality, we consider $\mathcal{H}$ being nontrivial, so there exists some $\{x\} \in \mathcal{O}$ such that $x \in \mathcal{H}$ and $\|x\| = 1$.
Then, we may establish the partial order by set inclusion. By **Zorn's Lemma**, there exists a maximal element $B \in \mathcal{O}$.
Since B is orthonormal, we just need to check that the span of B is dense in $\mathcal{H}$. Suppose $\overline{\text{span}(B)} \neq \mathcal{H}$, then we can decompose $\mathcal{H}$ as:

$$\mathcal{H} = \overline{\text{span}(B)} \oplus (\overline{\text{span}(B)})^{\perp},$$

where $(\overline{\text{span}(B)})^{\perp}$ is the subspace perpendicular to $\overline{\text{span}(B)}$, and since $(\overline{\text{span}(B)})^{\perp}$ is nonempty, we pick $y \in (\overline{\text{span}(B)})^{\perp}$ such that $\|y\| = 1$, while $B \sqcup \{y\}$ is in $\mathcal{O}$ and is greater than B, which is a contradiction. Hence B is the orthonormal basis. □

As a side note, if we use the conventions in [1] such that $\mathcal{H}$ is separable, it is a immediate consequence through the **Gram-Schmidt process**, which happened to be an important technique in inner product space to construct an orthonormal basis. In particular, we start with a nonzero $f_1 \in \mathcal{H}$, and normalize it into $e_1 = f_1 / \|f_1\|$. When we have a set of orthonormal basis $\{e_1, \cdots, e_k\}$, for some $f_{k+1} \in \mathcal{H}$ that is nonzero, we have:

$$e'_{k+1} = f_{k+1} - \sum_{j=1}^{k} \langle f_{k+1}, e_j \rangle,$$

and normalize it into $e_{k+1} = e'_{k+1} / \|e'_{k+1}\|$ for e'_{k+1} being nonzero, which *inductively* forms an orthonormal basis.

On the other hand, the reason that we discuss about Hilbert space is that our function space, that is the $L^2([-\pi, \pi])$ space.

Definition I.1.4. $L^2([0,1])$ **Space.**

We define a function $f \in L^2([0,1]$ if:

$$\int_0^1 |f(x)|^2 dx < +\infty,$$

with inner product of $f, g \in L^2([0,1])$ as:

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

Here, we claim that $L^2([0,1])$ space is a Hilbert Space, one just need to check all properties in Definition I.1.1. As a result, it must have a set of **orthonormal basis**.

Proposition I.1.5. Exponentials are an Orthonormal Basis of $L^2([0,1])$ Space.

The set $\{\exp(2\pi i n x)\}_{n=-\infty}^{\infty}$ is an orthonormal basis of $L^2([0,1])$ space.

Here, we will prove the theorem using a few consequences from **Complex Analysis**, according to [3].

Proof. First, we want to show orthogonality, note that:

$$\begin{aligned}\langle \exp(2\pi i n x), \exp(2\pi i m x)\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(inx)\overline{\exp(imx)}dx \\ &= \int_0^1 \exp\left(2\pi i(n-m)x\right)dx = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m,\end{cases}\end{aligned}$$

so we can show that $\{\exp(2\pi i n x)\}_{n=-\infty}^{\infty}$ is an orthonormal set.
Then, we suppose that $f \in L^2([0,1])$ such that:

$$\int_0^1 f(x)\exp(-2\pi i n x)dx = 0 \text{ for all } n \in \mathbb{Z},$$

and we want to show that $f(x) \equiv 0$.
Here, we define an auxiliary function:

$$F(\lambda) = \frac{1}{e^{2\pi i\lambda} - 1}\int_0^1 e^{2\pi i \lambda t} f(t)dt.$$

Note that the above function F is holomorphic over $\mathbb{C} \setminus \mathbb{Z}$, and we consider the singularities at $\mathbb{Z}$. We note that for any $n \in \mathbb{Z}$:

$$\lim_{\lambda\to n}(\lambda - n)\frac{\int_0^1 e^{2\pi i\lambda t}f(t)dt}{e^{2\pi i\lambda}-1} = \lim_{\lambda\to n}\frac{\int_0^1 e^{2\pi i\lambda t}f(t)dt}{1 + 2\pi i(\lambda - n) + \{\text{higher order terms of } (\lambda - n)\} - 1} = 0.$$

Hence, by the **Riemann Removable Singularity theorem**, $f(x)$ has a removable singularity for all are removable. Hence, it extends to an entire function w.r.t λ.
Then, we consider the (boundary of) squares centered at the origin as S_n (being a contour):

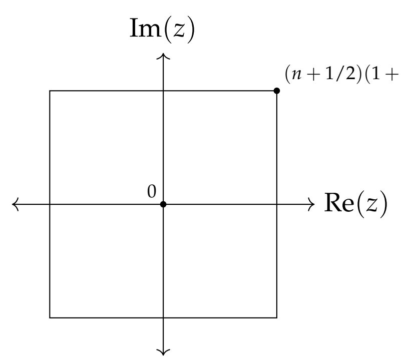

Figure I.1. Squares S_n with corresponding vertices.

Specifically, the vertices are:

$$\left(n+\frac{1}{2}\right)(1+\mathrm{i}), \left(n+\frac{1}{2}\right)(1-\mathrm{i}), \left(n+\frac{1}{2}\right)(-1-\mathrm{i}), \text{ and } \left(n+\frac{1}{2}\right)(-1+\mathrm{i}),$$

for $n \in \mathbb{Z}$.

Here, we want to show that $F(\lambda)$ is uniformly bounded for $\{S_n\}_{n\in\mathbb{N}}$. Here, we consider the Cauchy-Schwarz inequality as:

$$\begin{aligned} |F(\lambda)| &= \frac{1}{|e^{2\pi\mathrm{i}\lambda}-1|}\left|\int_0^1 e^{2\pi\mathrm{i}\lambda t} f(t)dt\right| \\ &\leq \frac{1}{|e^{2\pi\mathrm{i}\lambda}-1|}\sqrt{\int_0^1 |e^{2\pi\mathrm{i}\lambda t}|^2 dt}\,\underbrace{\sqrt{\int_0^1 |f(t)|^2 dt}}_{=K} \\ &= K\frac{1}{|e^{2\pi\mathrm{i}\lambda}-1|}\sqrt{\int_0^1 |e^{2\pi\mathrm{i}\lambda t}|^2 dt}. \end{aligned}$$

Then, we consider $\lambda = a + \mathrm{i}b$, where $a, b \in \mathbb{R}$, and we consider it as:

$$\frac{|F(\lambda)|}{K} \leq \frac{1}{|e^{2\pi\mathrm{i}\lambda}-1|}\sqrt{\int_0^1 |e^{2\pi\mathrm{i}\lambda t}|^2 dt}.$$

Here, we make the claim that:

$$|e^{2\pi\mathrm{i}\lambda}-1|^2 \geq \left|\int_0^1 |e^{2\pi\mathrm{i}\lambda t}|^2 dt\right| \text{ on } S_n.$$

By replacing λ, we have that:

$$\begin{aligned} |e^{2\pi\mathrm{i}\lambda}-1|^2 &= |e^{2\pi\mathrm{i}a}e^{-2\pi b}-1|^2 \\ &= (e^{2\pi\mathrm{i}a}e^{-2\pi b}-1)(e^{-2\pi\mathrm{i}a}e^{-2\pi b}-1) \\ &= 1 - e^{-2\pi b}2\mathrm{Re}(e^{2\pi\mathrm{i}a}) + e^{-4\pi b} = 1 - e^{-2\pi b}2\cos(2\pi a) + e^{-4\pi b}. \end{aligned}$$

Then, we consider the other equation as:

$$\int_0^1 |e^{2\pi\mathrm{i}\lambda t}|^2 dt = \int_0^1 |e^{-4\pi bt}|dt = \frac{e^{-4\pi b}-1}{-4\pi b} = \frac{1-e^{-4\pi b}}{4\pi b}.$$

Consider on S_n, when $a = N + 1/2$, then $\cos(2\pi a) = 0$, so we can verify that:

$$e^{-4\pi b} + 1 > \frac{1-e^{-4\pi b}}{4\pi b}.$$

When $b = N + 1/2$, since we have $\cos(-)$ between -1 and 1 for real input, with second inequality valid for $N = -1$ or 0, to verify that:

$$1 - e^{-2\pi b}\cos(2\pi a) + e^{-4\pi b} > |e^{-4\pi b} - 1| > \frac{|1-e^{-4\pi b}|}{2\pi} \geq \frac{1-e^{-4\pi b}}{4\pi b}.$$

Hence, we have necessarily verified our claim, so $|F(\lambda)|$ is uniformly bounded on all S_n, say $|F(\lambda)| < M$ on all S_i.

Now, we consider that for any $\lambda \in \mathbb{C}$ such that $|\lambda| < N$, we use **Cauchy integral formula** to obtain:

$$F(\lambda) = \frac{1}{2\pi\mathrm{i}}\int_{S_n}\frac{F(z)}{z-\lambda}dz, \text{ for any } n \geq N,$$

and since $F(z)$ is bounded on S_n, so we have F bounded on $\mathbb{C}$. Recall it is entire, so by **Liouville's theorem**, we have $F \equiv C$, where $C \in \mathbb{C}$ is a constant. Thus, we have that:

$$\int_{-\pi}^{\pi} e^{\mathrm{i}\lambda t} f(t)dt = C(e^{2\pi\mathrm{i}\lambda} - 1) \text{ for all } \lambda \in \mathbb{C}.$$

Here, we consider the left hand side tending to 0 as $\lambda \to \infty$ on the positive imaginary axis, so we must have $C = 0$, and correspondingly:

$$H(\lambda) := \int_{-\pi}^{\pi} e^{\mathrm{i}\lambda t} f(t)dt \equiv 0.$$

As we take derivatives of $H(\lambda)$, we have:

$$0 = \frac{dH}{d\lambda}(\lambda) = \int_{-\pi}^{\pi} (\mathrm{i}t)e^{\mathrm{i}\lambda t} f(t)dt = \mathrm{i}\int_{-\pi}^{\pi} te^{\mathrm{i}\lambda t} f(t)dt.$$

By taking derivatives again, we have:

$$0 = \frac{d^2H}{d^2\lambda}(\lambda) = \mathrm{i}\int_{-\pi}^{\pi} (\mathrm{i}t^2)e^{\mathrm{i}\lambda t} f(t)dt = -\int_{-\pi}^{\pi} t^2e^{\mathrm{i}\lambda t} f(t)dt,$$

By taking the derivatives inductively, we have:

$$\int_{-\pi}^{\pi} t^n e^{\mathrm{i}\lambda t} f(t)dt = 0 \text{ for all } n \in \mathbb{N}.$$

Recall that polynomials are dense by **Stone-Weierstrass** in $L^2([-\pi,\pi])$, hence $e^{\mathrm{i}\lambda t} f(t) \equiv 0$, and so $f(t) = 0$. Therefore, we have shown that the span of $\{\exp(\mathrm{i}nx)\}_{n=-\infty}^{\infty}$ is dense in $L^2([-\pi,\pi])$. □

By the above verification, we know that $\exp(2\pi\mathrm{i}nx)$ is an orthonormal basis, hence for any $f \in L^2([0,1])$, we can write it as:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi\mathrm{i}nx),$$

where the coefficients are:

$$a_n = \int_0^1 f(x) \underbrace{\exp(-2\pi\mathrm{i}nx)}_{\overline{\exp(2\pi\mathrm{i}nx)}} dx.$$

Remark I.1.6. Fourier Series in Sine and Cosine.

In many texts, the Fourier Series is decomposed into sum of $\sin(2\pi nx)$ and $\cos(2\pi nx)$ for $n \in \mathbb{Z}^+$. This is a consequence of the Euler's Formula that:

$$\exp(2\pi\mathrm{i}nx) = \cos(2\pi nx) + \mathrm{i}\sin(2\pi nx).$$

Readers can check that $\sin(2\pi nx)$ and $\cos(2\pi nx)$ are orthogonal by taking the inner products and the trigonometric identities.
Hence, we could have an orthonormal basis as $\{1\} \cup \{\sin(2\pi nx), \cos(2\pi nx)\}_{n=1}^{\infty}$ for $L^2([0,1])$. ⌟

Fourier series turns out to be widely applied to many fields given the real functions. However, we would also want to apply it to complex function.

II Period Holomorphic Functions

Now, we want to extend the **Fourier Series** to complex analysis. In the discussion of Fourier Series in $\mathbb{C}$, we want to consider **holomorphic functions** that has a **periodicity** $\omega \in \mathbb{C}$, that is $f \in \mathcal{O}(\mathbb{C})$ such that:

$$f(z+\omega) = f(z) \text{ for all } z \in \mathbb{C}.$$

By convention, we consider $\omega \neq 0$ as the **period** of a function.

Throughout the next chapters, we will be following the procedures in [4].

Example II.0.1. Periodic Holomorphic Functions.

Consider the function $f(z) = \exp\left(\frac{2\pi \mathrm{i}}{\omega}z\right)$, where $\omega \in \mathbb{C}^\times$. Since it is the pre-composition of $\exp(-)$ with a simple scale multiplication function, $f \in \mathcal{O}(\mathbb{C})$, we note that:

$$f(z+\omega) = \exp\left(\frac{2\pi \mathrm{i}}{\omega}(z+\omega)\right) = \exp\left(\frac{2\pi \mathrm{i}}{\omega}z + 2\pi \mathrm{i}\right) = \exp\left(\frac{2\pi \mathrm{i}}{\omega}z\right) \cdot \underbrace{\exp(2\pi \mathrm{i})}_{1} = \exp\left(\frac{2\pi \mathrm{i}}{\omega}z\right) = f(z).$$

Hence $f(z) = \exp\left(\frac{2\pi \mathrm{i}}{\omega}z\right)$ is periodic and holomorphic, with period ω. ⌟

Meanwhile, by considering **Euler's equation** that:

$$\exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta \qquad \text{and} \qquad \exp(-\mathrm{i}\theta) = \cos\theta - \mathrm{i}\sin\theta,$$

and through some algebraic deductions, we can obtain:

$$\sin\theta = \frac{\exp(\mathrm{i}\theta) - \exp(-\mathrm{i}\theta)}{2\mathrm{i}} \qquad \text{and} \qquad \cos\theta = \frac{\exp(\mathrm{i}\theta) + \exp(-\mathrm{i}\theta)}{2}.$$

Hence, we deduce the following holomorphic functions over $\mathbb{C}$ that have periodicity of ω:

$$\sin\left(\frac{2\pi}{\omega}z\right) = \frac{\exp\left(\frac{2\pi \mathrm{i}}{\omega}z\right) - \exp\left(\frac{-2\pi \mathrm{i}}{\omega}z\right)}{2\mathrm{i}},$$

$$\cos\left(\frac{2\pi}{\omega}z\right) = \frac{\exp\left(\frac{2\pi \mathrm{i}}{\omega}z\right) + \exp\left(\frac{-2\pi \mathrm{i}}{\omega}z\right)}{2}.$$

Just like the development of Fourier Series for real function, our goal here is to write the **periodic holomorphic functions** as a series of the form:

$$\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi \mathrm{i}}{\omega}nz\right).$$

II.1 Strips and Annuli

Again, considering the periodic function, we want to have functions being the same when adding or subtracting various amount of $\omega \in \mathbb{C}^\times$, which give rise to the concept of a **invariant region**.

Definition II.1.1. Invariant Region.

A open set $\Omega \subset \mathbb{C}$ is called ω-invariant if $z \pm \omega \in \Omega$ for all $z \in \Omega$. ⌟

Algebraically, we have a invariant region when every **translation** $\varphi_n : z \mapsto z + n\omega$ for $n \in \mathbb{Z}$ is an **automorphism** in the invariant region Ω.

Definition II.1.2. Stripe.

Given a pair $a, b \in \mathbb{R}$ such that $a < b$, we define the **stripe** as the set:

$$T_\omega(a,b) := \left\{ z \in \mathbb{C} : a < \operatorname{Im}\left(\frac{2\pi}{\omega}z\right) < b \right\}.$$ ⌟

Here, we can consider the **stripe** to be determined by information of ω, a, and b. Here, we may observe the following property.

Proposition II.1.3. The Boundary of Stripe is Linear.

With the same setup as above, $\mathrm{Im}\left(\frac{2\pi}{\omega}z\right) = c$, where $c \in \mathbb{R}$ is a constant, is a line on the complex plane.

Proof. Consider $z = x + \mathrm{i}y$ and $\omega = a + \mathrm{i}b$, we have:

$$\begin{aligned}\mathrm{Im}\left(\frac{2\pi}{\omega}z\right) &= \mathrm{Im}\left(\frac{2\pi}{a+\mathrm{i}b}(x+\mathrm{i}y)\right) = \mathrm{Im}\left(\frac{2\pi}{(a+\mathrm{i}b)(a-\mathrm{i}b)}(x+\mathrm{i}y)(a-\mathrm{i}b)\right)\\ &= \mathrm{Im}\left(\frac{2\pi}{a^2+b^2}\big((ax+by)+\mathrm{i}(ay-bx)\big)\right) = \frac{2\pi}{a^2+b^2}(ay-bx).\end{aligned}$$

When constraint to a real constant c, we have:

$$\frac{2\pi}{a^2+b^2}(ay-bx) = c,$$

which is a linear function on the complex plane. □

Note that in our proof, we have shown that the slope on the complex plane is b/a, which is exactly the slope if we were to plot ω on the complex plane[1], so this $T_\omega(a,b)$ is ***ω*-invariant**, and we will denote the ω-invariant region as T_ω. In particular, we say that for any $z \in T_\omega(a,b)$, we have $z + \omega\mathbb{R} \subset T_\omega(a,b)$. Moreover, since the boundary is linear, it is **convex**.

Here, we consider a particular example of the stripe.

Example II.1.4. A Stripe $T_{1+\mathrm{i}}(0,2)$.

Consider $T_{1+\mathrm{i}}(0,2)$, from the previous proof, we note that the linear equations of the boundaries are:

$$\pi(y-x) = 0 \qquad \text{and} \qquad \pi(y-x) = 2,$$

which can be graphically represented by:

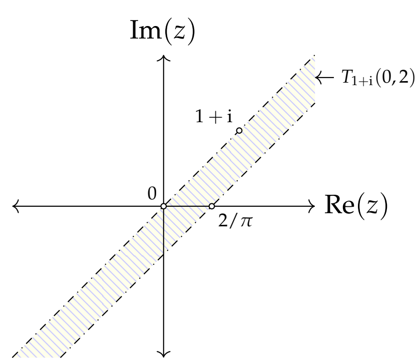

Figure II.1. The Stripe $T_{1+\mathrm{i}}(0,2)$.

[1]Note that we can also say that the angle is the argument of ω.

Moreover, we can consider $T_\omega(-\infty, b)$ for $b \in \mathbb{R}$ as the **open half-plane**. Moreover, $T_\omega(-\infty, \infty)$ will be the **whole complex plane**.

Then, we want to expand our arguments to **biholomorphic** relationships between the stripes and other sets on the complex plane.

Proposition II.1.5. Holomorphic Maps of the Stripe.

Let $\omega \in \mathbb{C}^\times$ and $a, b \in \mathbb{R}$:

- $T_\omega(a, b)$ is **biholomorphic** to the unit period stripe $T_1(a, b)$.
- $T_1(a, b)$ can be **holomorphically** mapped to $\mathbb{A}_{e^{-b}, e^{-a}}(0)$, *i.e.*, annulus with inner radius e^{-b} and outter radius e^{-a}.

Proof.

- For the first **biholomorphic** relationship, we can define:
$$z \mapsto z/\omega,$$
which is well-defined, holomorphic, and bijective.
- For the second **holomorphic** map, we define:
$$z \mapsto \exp(2\pi \mathrm{i} z).$$
This map is the composition of a constant multiple map and an exponential function, which is holomorphic, and each straight line $\operatorname{Im}(2\pi z) = \ell$ is mapped onto the circle $|\exp(2\pi \mathrm{i} z)| = e^{-\ell}$ for all $s \in \mathbb{R}$, hence it is mapped to the annulus $\mathbb{A}_{e^{-b}, e^{-a}}(0)$, respectively for the right and left boundary into the inner and outer circle. □

Also, we can extend $a, b \in \mathbb{R}$ into $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, where we induct $e^{-\infty} = 0$ and $e^{\infty} = \infty$, which leads to some special cases.

Remark II.1.6. Special Case of Punctured Disk.

Following the second **holomorphic map**, we map $T_1(a, \infty)$ into the puncture disc $\mathbb{D}_{e^{-a}}(0)^\times$.
Also, consider $\mathbb{C} = T_1(-\infty, \infty)$, it is mapped to the punctured complex plane, *i.e.*, $\mathbb{C}^\times$. ⌟

II.2 Periodic Holomorphic Functions in Strips

In this section, we first make a formal definition of a **periodic holomorphic function**.

Definition II.2.1. Periodic Holomorphic Function.

f is **holomorphic** over a ω-invariant region Ω if $f(z + n\omega)$ is well defined for all $z \in \Omega$ and $n \in \mathbb{Z}$. f is **periodic** in Ω if:
$$f(z + \omega) = f(z) \text{ for all } z \in \Omega.$$

For this case, we denote set of all ω-periodic over invariant region Ω holomorphic function as $\mathcal{O}_\omega(\Omega)$. ⌟

Proposition II.2.2. Extended Period Behavior.

Suppose f is **holomorphic periodic** over ω-invariant Ω, then:

$$f(z+n\omega) = f(z) \text{ for all } n \in \mathbb{Z}.$$

Proof. This can be proven recursively. For positive integer n:

$$f(z+n\omega) = f\big(z+(n-1)\omega\big) = \cdots = f\big(z+\omega\big) = f(z),$$

where as for negative integer $-n$:

$$f(z-n\omega) = f(z-n\omega+\omega) = \cdots = f(z-n\omega+n\omega) = f(z),$$

as desired. □

It is very natural to consider $\mathcal{O}_\omega(\Omega)$ as a subset of $\mathcal{O}(\Omega)$, which is the set of all holomorphic function. However, there turns out to be a stronger conclusion.

Proposition II.2.3. ω-invariant Region is a Subalgebra.

The set $\mathcal{O}_\omega(\Omega)$ of all ω-periodic holomorphic functions in the ω-invariant region Ω is a $\mathbb{C}$-subalgebra of $\mathcal{O}(\Omega)$.

Proof. Here, we consider any $f, g \in \mathcal{O}_\omega(\Omega)$ and $\lambda \in \mathbb{C}$, by closeness holomorphic functions, we have:

$$f+g \in \mathcal{O}(\Omega) \text{ and } \lambda f \in \mathcal{O}(\Omega),$$

so we just need to show that $f+g$ and $\lambda \cdot f$ has the same period ω, we first note that:

$$(f+g)(z+\omega) = f(z+\omega) + g(z+\omega) = f(z) + g(z) = (f+g)(z).$$

In terms of scalar multiplication, we also have:

$$(\lambda \cdot f)(z+\omega) = \lambda f(z+\omega) = \lambda f(z) = (\lambda \cdot f)(z).$$

Hence, we have $f+g, \lambda \cdot f \in \mathcal{O}_\omega(\Omega)$, as desired. □

Theorem II.2.4. Existence and Uniqueness of Periodic Function over Holomorphic Function.

For every 1-periodic holomorphic function f on Ω, there exists unique holomorphic function F on $\mathbb{A}$ such that:

$$f(z) = F\big(\exp(2\pi \mathrm{i} z)\big) \text{ for all } z \in \Omega.$$

To establish the proof of the above theorem, we would want to establish the following result first.

Theorem II.2.5. Factorization Theorem.

Let $g : G \to G'$ be a holomorphic mapping of a open set $G \subset \mathbb{C}$ onto a open, nonempty set G'. Let f be holomorphic function on G which is constant on each g-fiber $g^{-1}(w)$ for $w \in G'$. Then, there exists a unique holomorphic function h in G' such that $g^*(h) = f$, that is $h\big(g(z)\big) = f(z)$ for all $z \in G$.

This can be represented in terms of commutative diagram as:

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "f"] & G' \arrow[dl, dashed, bend left, "\exists ! h"] \\ f(G) & \end{tikzcd}$$

Figure II.2. Commutative diagram for factorization theorem.

Proof. Since g is onto and by the fact that f is uniquely defined on each g-fiber, we know that for all $w \in G'$, $g^{-1}(w)$ is nonempty and $f\big(g^{-1}(w)\big)$ is unique, so we can assign this value to $h(w)$, which is well defined such that $f = h \circ g = g^*(h)$.

Then, we are left to show that our definition of h is holomorphic, *i.e.*, $h \in \mathcal{O}(G')$. Since G' is not trivial and g is onto, g is *nonconstant*, so by the **open mapping theorem**, g is an open map.

Here, we suppose that $V \subset f(G)$ is an open set, then since f is holomorphic, it is continuous, so $f^{-1}(V)$ is open. Since g is open map, $g\big(f^{-1}(V)\big)$ is open. Hence, the preimage of H of an open set is open, so h is continuous on G'.

Now, we consider about the derivative of g, let $Z(g')$ denote the zeros of g'. Note that g is holomorphic and *nonconstant*, so g' is also holomorphic and nonzero, so its zeros are *discrete* since the zeros of nonzero holomorphic function are isolated. Since g' is continuous, the preimage of $\{0\}$ as a closed set is closed, hence $Z(g')$ is discrete and closed in G.

Then, we define the set M as:

$$M := \{b \in G' : g^{-1}(b) \subset Z(g')\}.$$

Here, we want to show that M is closed and discrete.

- For closeness, we want to show that:

  $$g\big(G \setminus Z(g')\big) = G' \setminus \{b \in G' : g^{-1}(b) \subset Z(g')\}.$$

  Specifically, if $y \in g\big(G \setminus Z(g')\big)$, then we have some $x \in G \setminus Z(g')$ so that $g^{-1}(y)$ contain an element that is not in $Z(g')$ so $y \in G' \setminus \{b \in G' : g^{-1}(b) \subset Z(g')\}$, so we have $g\big(G \setminus Z(g')\big) \subset G' \setminus \{b \in G' : g^{-1}(b) \subset Z(g')\}$.

  For the other inclusion, consider if $y \in G' \setminus \{b \in G' : g^{-1}(b) \subset Z(g')\}$, so there exists some $x \in G \setminus Z(g')$ such that $y = g(x)$, so this means that $y \in g\big(G \setminus Z(g)\big)$, hence, we have $g\big(G \setminus Z(g')\big) \supset G' \setminus \{b \in G' : g^{-1}(b) \subset Z(g')\}$, which completes the proof.

  Note that $G \setminus Z(g')$ is open, and since g is open map, so $g\big(G \setminus Z(g')\big) = G' \setminus \{b \in G' : g^{-1}(b) \subset Z(g')\}$ is open, and thus $M = \{b \in G' : g^{-1}(b) \subset Z(g')\}$ is closed.

- For discreteness, we note that for any $b \in M$, we have $g^{-1}(b) \subset N(g')$, so we have $M \subset g\big(N(g')\big)$, and it suffices to show that $g\big(N(g')\big)$ is discrete.

  Note that $N(g')$ is discrete, so for any $S \subset N(g')$, it is open in $N(g')$, and for any subset of $X \subset g\big(N(g')\big)$, there exists some $S \subset N(g')$ such that $X \subset g(S)$, and since g is open map, we have $g(S)$ open in G', so any subset of $g\big(N(g')\big)$ is open in $g\big(N(g')\big)$, thus M is discrete.

With this, we consider that $g' \neq 0$ on $G \setminus M$, so it is locally biholomorphic on a open neighborhood V,[2] so there exists a biholomorphic inverse $\tilde{g} : V \to G$ such that:

$$h\mid_V = h \circ g \circ \tilde{g} = f \circ \tilde{g},$$

hence h is holomorphic over V, and so $h \in \mathcal{O}(G' \setminus M)$. Recall that M is closed and isolated, we can consider each point in M separately using **Riemann removable singularity theorem** such that since h is continuous, we consider it extends across continuously, so it equivalently extends across holomorphically. Therefore, our definition of h is holomorphic. □

With such consequence, we can prove Theorem II.2.4, as follows:

Proof of Theorem II.2.4. By Proposition II.1.5, without loss of generality, we can assume $\omega = 1$ and Ω is then the unit stripe T_1. Then, the holomorphic mapping $h : \Omega \to \mathbb{A}$ is from the stripe to the annulus, which happened to be defined as $z \mapsto \exp(2\pi \mathrm{i} z)$. Hence, we can create the commutative diagram for the case.

$$\begin{tikzcd} \Omega \arrow[r, "h"] \arrow[d, "f"] & \mathbb{A} \arrow[dl, dashed, bend left, "\exists! F"] \\ f(g) & \end{tikzcd}$$

Figure II.3. Commutative diagram for establishing the desired F map.

Here, we want to use factorization theorem, so we need to verify that f is constant on each h-fiber. Consider any $z \in \mathbb{A}$, we have its preimage as $\frac{1}{2\pi \mathrm{i}}\big(\log(z) + 2k\pi\mathrm{i}\big) = \frac{1}{2\pi\mathrm{i}}\log(z) + k$ for integer k. Since f has period 1, we have $f(z+k) = f(z)$ for all $k \in \mathbb{Z}$, then $f\big(\frac{1}{2\pi\mathrm{i}}\log(z)+k\big) = f\big(\frac{1}{2\pi\mathrm{i}}\log(z)\big)$, so f is constant on each h-fiber, and thus there exists a unique holomorphic function $F : \mathbb{A} \to f(G)$ such that the above diagram commutes. □

III Fourier Series in Stripes

As we have thought about the previous chapter, we have had a solid foundation to write the **periodic holomorphic** functions on the complex plane as a sum.

III.1 Fourier Development in Stripes

Now, we have established the existence and uniqueness of a holomorphic function $F : \mathbb{A} \to f(\Omega)$ for all 1-periodic $f : \Omega \to f(\Omega)$. We want to use this as a consequence to the following theorem.

Theorem III.1.1. Fourier Series for Holomorphic Functions over Stripes.

Let f be holomorphic and ω-periodic in the stripe $T_\omega(a, b)$, then f can be expanded into a unique Fourier series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi\mathrm{i}}{\omega} nz\right),$$

[2] This corresponds to the **Problem 6.3** in Homework, the biholomorphic inverse exists on a open neighborhood when the derivative at the point is nonzero.

which converges uniformly on every substripe $T_\omega(a', b')$ of $T_\omega(a, b)$ where $a < a' < b' < b$.
Moreover, the for any $z_0 \in T_\omega(a, b)$, we have the coefficients c_n for $n \in \mathbb{Z}$ as:

$$c_n = \frac{1}{\omega}\int_{z_0}^{z_0+\omega} f(\zeta)\exp\left(-\frac{2\pi \mathrm{i}}{\omega}n\zeta\right)d\zeta.$$

Proof. Again, without loss of generality, we can reduce the problem to $\omega = 1$ (by Proposition II.1.5).
Then, recall from Theorem II.2.4, there exists a unique holomorphic function F on the annulus $\mathbb{A} := \{w \in \mathbb{C} : e^{-b} < |w| < e^{-a}\}$ in which $f(z) = F\big(\exp(2\pi \mathrm{i}z)\big)$.
Then, we write the function F uniquely as a Laurent series in A, *i.e.*:

$$F(w) = \sum_{n=-\infty}^{\infty} c_n w^n \text{ where } c_n = \frac{1}{2\pi \mathrm{i}}\int_{\partial D_\epsilon(0)} \frac{F(\xi)}{\xi^{n+1}}d\xi,$$

in which $\partial D_\epsilon(0) \subset \mathbb{A}$ lies in the annulus.
Recall that when we proved the **Laurent series**, it is unique and converges uniformly inside the annulus.
Then, we think of a way to characterize the coefficients. Here, we consider $[z_0, z_0+1]$ as the parametrization of:

$$\zeta(t) := z_0 + \frac{1}{2\pi}t,$$

and we consider the circle as:

$$\xi(t) := re^{\mathrm{i}t},$$

where $r = \exp(2\pi \mathrm{i}z_0) \in A$, so we have the coefficients as:

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}}\int_{\partial D_\epsilon(0)} \frac{F(\xi)}{\xi^{n+1}}d\xi &= \frac{1}{2\pi}\int_{\partial D_\epsilon(0)} \frac{F\big(\exp(2\pi \mathrm{i}r)\exp(2\pi \mathrm{i}(\xi - r))\big)}{\xi^n}\frac{d\xi}{\mathrm{i}\xi} \\ &= \frac{1}{2\pi}\int_{\partial D_\epsilon(0)} \frac{F\big(\exp(2\pi \mathrm{i}r\zeta(t))\big)}{\xi^n}\frac{d\xi}{\mathrm{i}\xi} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{f\big(\zeta(t)\big)}{(re^{\mathrm{i}t})^n}dt = \int_{z_0}^{z_0+1} f(\zeta)\exp(-2\pi \mathrm{i}n\zeta)d\zeta.\end{aligned}$$

Here, we note that when ω is not 1, we simply make the upper bound of the integration to $z_0 + \omega$, and we then consider dividing the exponent into $-2\pi \mathrm{i}n\zeta/\omega$ correspondingly, which completes the proof. □

It is noteworthy that such calculations involve integral formulas, but there will be examples of function that we do not have use the integrations.

III.2 Examples of Fourier Series for Periodic Functions

Example III.2.1. Sine and Cosine Curves.

Consider the sine and cosine curves as:

$$\sin z = \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}} \qquad \text{and} \qquad \cos z = \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2},$$

they can be directly written as:

$$\sin z = -\frac{1}{2\mathrm{i}}e^{-\mathrm{i}z} + \frac{1}{2\mathrm{i}}e^{\mathrm{i}z} \qquad \text{and} \qquad \cos z = \frac{1}{2}e^{-\mathrm{i}z} + \frac{1}{2}e^{\mathrm{i}z},$$

which is calculated from the Euler's formula. ⌟

Then, we can think of developing more examples of Fourier Series.

Example III.2.2. Inverse Sine and Cosine.

Note that $\frac{1}{\sin z}$ and $\frac{1}{\cos z}$ are holomorphic over the open upper half-plane and open lower half plane with period 2π, we can develop the Fourier series for them, respectively.

- For inverse sine, we have:
$$\frac{1}{\sin z} = \frac{2\mathrm{i}}{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)} = \frac{2\mathrm{i}\exp(-\mathrm{i}z)}{1 - \exp(-2\mathrm{i}z)} = 2\mathrm{i}\exp(-\mathrm{i}z)\frac{1}{1 - \exp(-2\mathrm{i}z)},$$
- Whereas for inverse cosine, we have:
$$\frac{1}{\cos z} = \frac{2}{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)} = \frac{2\exp(\mathrm{i}z)}{1 + \exp(2\mathrm{i}z)} = 2\exp(\mathrm{i}z)\frac{1}{1 + \exp(2\mathrm{i}z)}.$$

One thing that we should note is that for $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$, we can write:
$$\exp(2\mathrm{i}z) = \exp\big(2\mathrm{i}(x + \mathrm{i}y)\big) = \exp(2x\mathrm{i})\exp(-2y),$$
$$\exp(-2\mathrm{i}z) = \exp\big(-2\mathrm{i}(x + \mathrm{i}y)\big) = \exp(-2x\mathrm{i})\exp(2y).$$

Hence, we have $|\exp(2\mathrm{i}z)| < 1$ when $y > 0$ and $|\exp(-2\mathrm{i}z)| < 1$ when $y < 0$. So we can use geometric sequence, respectively.

Therefore, on the lower half plane, we can write inverse of sine as:
$$\begin{aligned}\frac{1}{\sin z} &= 2\mathrm{i}\exp(-\mathrm{i}z)\sum_{n=0}^{\infty}\big(\exp(-2\mathrm{i}z)\big)^n \\ &= 2\mathrm{i}\exp(-\mathrm{i}z)\sum_{n=0}^{\infty}\exp(-2\mathrm{i}nz) = \sum_{n=0}^{\infty}2\mathrm{i}\exp\big(-(2n+1)\mathrm{i}z\big).\end{aligned}$$

Similarly, on the upper half plane, we can write inverse of cosine as:
$$\begin{aligned}\frac{1}{\cos z} &= 2\exp(\mathrm{i}z)\sum_{n=0}^{\infty}(-1)^n\big(\exp(2\mathrm{i}z)\big)^n \\ &= 2\exp(\mathrm{i}z)\sum_{n=0}^{\infty}(-1)^n\exp(2\mathrm{i}nz) = \sum_{n=0}^{\infty}(-1)^n\cdot 2\exp\big((2n+1)\mathrm{i}z\big).\end{aligned}$$

Note that for the other domains, respectively, since we do not have the common ratio having absolute value less than 1, we cannot use geometric series to expand.

Also, we note that the worked example for $\frac{1}{\cos z}$ is not correct on [4]. ⌟

This completes the all the contents and worked examples for this project.

Special thanks to Professor Gregoric in assisting me through some contents of the proofs and having answered many of my questions through the semester.

References

[1] R. S. Elias M. Stein, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton University Press, November 2009.

[2] Asteroid, "every hilbert space has an orthonormal basis," Planetmath, March 2013. [Online]. Available: https://planetmath.org/everyhilbertspacehasanorthonormalbasis#:~:text=Proof%20%3A%20As%20could%20be%20expected,1%20%20x%20%20%3D%201%20.

[3] D. B. P. (https://math.stackexchange.com/users/112478/disintegrating-by parts), "Want to show that $\{e^{i2\pi nx} : n \in \mathbb{Z}\}$ form an orthonormal basis for 1-periodic l^2 functions." Mathematics Stack Exchange, uRL:https://math.stackexchange.com/q/787350 (version: 2014-05-09). [Online]. Available: https://math.stackexchange.com/q/787350

[4] R. B. Reinhold Remmert, *Theory of Complex Functions*. Springer, April 1990.