# Hausdorff Measure and Fractal Geometry

James Guo

April 15, 2025

## I Motivation

### I.1 Preliminaries

A question that many mathematicians (as well as *engineers*) concern is "*How much stuffs do we have?*"

As many of you are *mathematicians*, we will try to establishing the following example of which set is "*larger*?"

**Example I.1.1. Comparing which set is larger.**

Let $\mathbb{R}$ be defined the set of real numbers, consider the following sets:

- $\mathbb{Z}$, denoting the set of integers.
- $[0]_{\mathbb{Z}/2}$, denoting the set of even integers.
- $(0,1)$, denoting all real numbers between 0 and 1, not inclusively.

In fact, a very typical thing that people say about cardinalities examples is that there are "*as many*" integers as even integers, because there exists a bijection function $f : \mathbb{Z} \to [0]_{\mathbb{Z}/2}$ such that:

$$f(x) = 2x.$$

Also, in this example, you would know that $[0,1]$ is *countable*, whereas $\mathbb{Z}$ and $[0]_{\mathbb{Z}/2}$ are countable, so one would claim that there are "*more*" stuffs in $[0,1]$ compared to $\mathbb{Z}$ or $[0]_{\mathbb{Z}/2}$. ⌟

But, the question is, can we extend this "bijection" construction to more sets within $\mathbb{R}$?

**Example I.1.2. Is that a valid way of comparison?**

Some readers might think the bijection argument is good for comparing the "size" of two sets, and a set would be "smaller" than the other if it has a injective map to another set, but no surjective maps, think about the following sets:

- $(0,1)$, denoting all real numbers between 0 and 1, not inclusively.
- $(0,2)$, denoting all real numbers between 0 and 2, not inclusively.

- $\mathbb{R}_{>0}$, denoting all positive real numbers.

It is clear that there exists bijections between all three of these sets, and in that regard, they are of the same "size." ⌟

Here, we can even have a standard argument from **complex analysis**.

**Theorem I.1.3. Riemann Mapping Theory.**

Let $U \subset \mathbb{C}$ be non-empty, simply connected, and open subset, then there exists a **biholomorphic mapping** (bijective holomorphic map whose inverse is also holomorphic) $f$ from $U$ onto the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

If we are to visualize this theorem, there exists a biholomorphic mapping between all the following subsets of $\mathbb{C}$, so we can classify them as the "same size."

1. The unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
2. The "unit square" $(-1, 1) \times \mathrm{i}(-1, 1)$.
3. A "special" shape.
4. The upper-half-plane $\mathbb{H}$.
5. The open strip $\big(\mathbb{R} \times \mathrm{i}(-\pi/2, \pi/2)\big)$.
6. The punctured slit plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

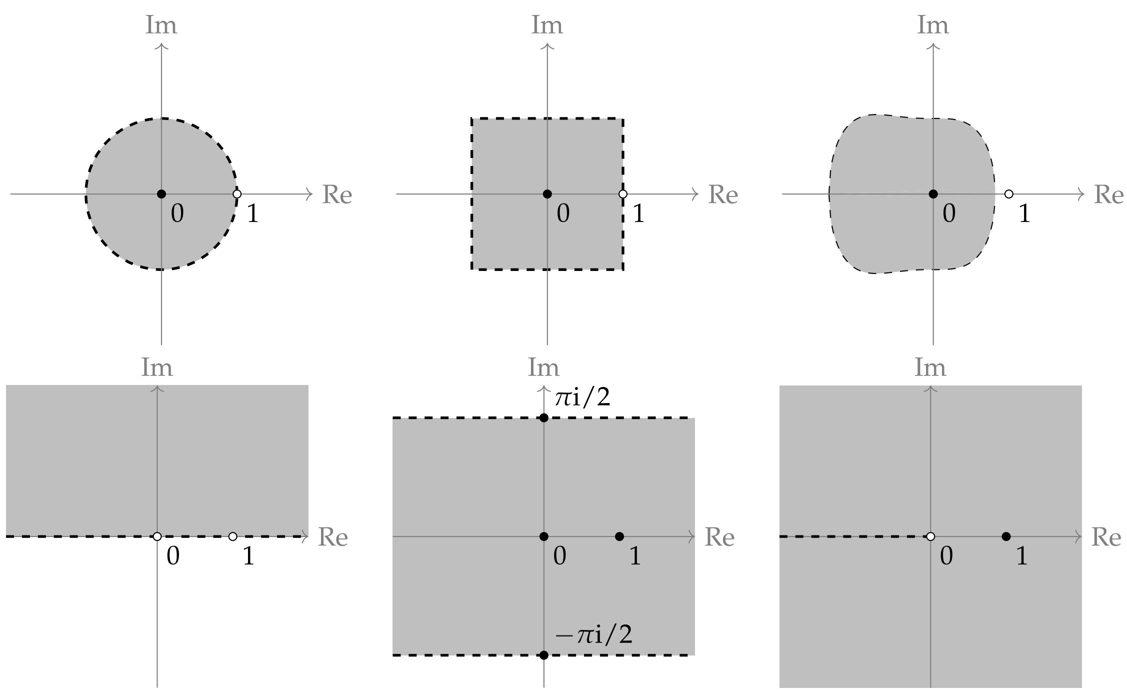


*Figure I.1. Examples of nonempty, simply connected, open subsets in $\mathbb{C}$.*

*Intuitively* speaking, they are not of the same size, so we need some other ways of measuring size.

To discuss about this size, let's recall how we compute area of a shape in a 2-d plane in *elementary school*.

**Example I.1.4. Area of pizzas.**

For the sake of this example, let's suppose that pizzas a triangular after being cut into slices, that is you do note eat the cornicione of the pizza (don't do this). Say there are 3 pieces left:

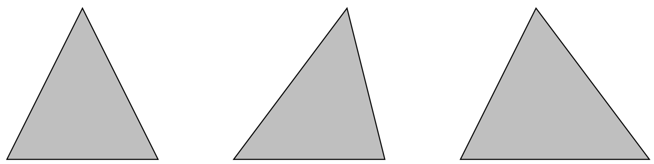

*Figure I.2. Left pieces of pizzas.*

It is clear that the rightmost piece is the largest, and you should pick this if you are hungry. Formally, you think about this as:

$$\text{Area} = \frac{1}{2}\text{base} \times \text{height}.$$

This can be thought of as a very basic motivation to area or measure. However, (some) mathematicians are always frantic, and they rather cut the pizza in the following way.

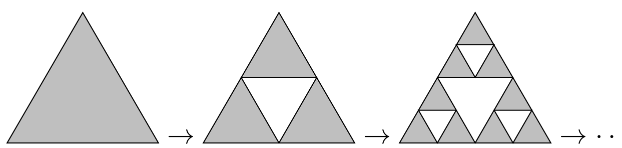

*Figure I.3. Step construction of Sierpiński triangle.*

Here, we want to ask some questions here:

- With this construction, assume the mathematician cuts the pizza in infinitely many stages, does it still have area?
- For these specially constructed shapes, is there still a way to compare if one of them is larger or not?

For the first question, if you have had some exposure from basic calculus and elementary school geometry, we can see that the areas of each stage forms a "sequence" (assuming side length of 1), of:

$$\left\{ \sqrt{3}, \frac{3}{4}\sqrt{3}, \frac{9}{16}\sqrt{3}, \cdots \right\},$$

so you hopefully agree that this sequence *converges* to 0.

Then, the problem becomes, are there someways of representing this set of its measure, given it is not empty nor *countable*, and we will explore this later on.

## I.2 Measure Theory

The key component of this is about **measure theory**, which concerns about measuring how much stuffs do we have.

Now, lets' very quickly define a very common measure, known as the Lebesgue (*outer*) measure.

**Remark I.2.1.** When having this measure, our aim is to have it consist with our current *intuitions* with length/area/volume in the Euclidean spaces. ⌟

**Definition I.2.2. Lebesgue Outer Measure.**

Let $E \subset \mathbb{R}^n$, and let $\{C_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ be any sequence of *rectangular cuboid* that covers $E$, *i.e.*, $\bigcup_{k=1}^{\infty} C_k \supset E$. We define the **Lebesgue outer measure** as:

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \mathrm{vol}(C_k) \right\},$$

where the volume is calculated as the product of the side lengths for the cubes. ⌟

It would be good to think about some examples in the Euclidean space, such as a normal cube, or balls.

**Remark I.2.3. Lebesgue Measure.**

The **Lebesgue measure** will be similarly defined a certain subsets of $\mathbb{R}^n$, and in the measure theory perspective, such sets are called a measurable set (or **Borel sets**). ⌟

Instead, many of us might wonder, are there sets that are not Borel, and we will give an interesting construction of a non-measurable set, the **Vitali Set**, using the **Axiom of choice**.

**Example I.2.4. Vitali Set.**

Consider the interval $[0,1] \subset \mathbb{R}$, we define an recurrence relationship that $a \sim_{\mathbb{Q}} b$ if $a - b \in \mathbb{Q}$. Hence, we effectively have $[0,1]/\mathbb{Q}$ as uncountably many equivalence classes, and by the **axiom of choice**, we can select exactly one element in each equivalent class, forming the **Vitali set**.
Here, we assume that the **Vitali set** is measurable and has measure $m(V)$, since the Lebesgue measure is translation invariant, we have $m(q+V) = m(V)$ for some $q \in \mathbb{Q} \cap [-1,1]$. Note that $\mathbb{Q} \cap [0,1]$ is countable, so we can have an enumeration, thus:

$$[0,1] \subset \bigcup_{k=1}^{\infty} q + V \subset [-1,2].$$

Then, note that the Lebesgue measure has monotonicity, we have:

$$1 = m(0,1) \leq m\left(\bigcup_{k=1}^{\infty} q + V\right) \leq m([-1,2]) = 3.$$

Recall translation invariant, we have the middle argument as an infinite product of $m(V)$, but there is no $m(V)$ such that $1 \leq \infty \cdot m(V) \leq 3$, hence it is not measurable. ⌟

**Remark I.2.5. Solovay's Theorem.**

**Axiom of choice** is essential for the construction of a non-measurable set. If we do not assume **axiom of choice**, then all subsets of $\mathbb{R}$ are Borel. ⌟

Then, we will look into a good old friend of mathematician in 1-D, so it cannot really be a pizza cutting thing, but the **Cantor set**.

**Example I.2.6. Cantor Set.**

The **Cantor set** can be constructed step-by-step, starting from $C_0 = [0, 1]$. For each step, we remove the middle $1/3$ proportion of all current segments, and we can consider $\mathcal{C} := \lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n$. ⌟

The first few level of **Cantor set** constructions can be demonstrated as follows:

*Figure I.4. First six steps of Cantor set construction.*

**Proposition I.2.7. Properties of Cantor Set.**

We will remark (but not prove) the following properties of the **Cantor set**:

- Cantor set is Borel.
- Cantor set is uncountable.
- Cantor set has Lebesgue measure $0$.

In fact, there are many variant of the Cantor set, such as removing a different proportion in the middle, but is there a way to compare between the *size* of them?

# II Hausdorff Measure and Dimension

## II.1 Hausdorff Measure

Now, you should notice that we need something else. There are many *uncountable* sets that are of Lebesgue measure zero. And if we are to think of them as the same *size*, this seems too arbitrary.

Here, mathematicians need another way to *measure* these null sets, and what about scale the volume of the cubes/balls?

**Definition II.1.1. Hausdorff Measure of Dimension $s$.**

Suppose $E \subset \mathbb{R}^n$ and $s$ is a non-negative number, for any $\delta > 0$, we define the **Hausdorff measure** of dimension $s$ as:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \left[ \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\} \right].$$

Here, the $\delta$-cover is a sequence of sets of diameter at most $\delta$ that cover $E$, *i.e.*, $E \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \leq |U_i| \leq \delta$ for all $i$. ⌟

Note that if we just have the $\delta$-covers, as $\delta \to 0$, by monotonicity and non-negativity of smaller $\delta$, the limit is guaranteed to exist, so we would not worry about the existence of the limit here.

### II.2 Hausdorff Dimension

## III Iterated Function System and Fractals

### III.1 Self Similar Sets

### III.2 Contraction Fractals

### III.3 Examples of Fractals

## IV Extensions

### IV.1 Random Fractals

### IV.2 Fractal Graphs