Hausdorff Measure and Fractal Geometry

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I Motivation

I.1 Preliminaries

A question that many mathematicians (as well as *engineers*) concern is "*How much stuffs do we have?*"

As many of you are *mathematicians*, we will try to establishing the following example of which set is "*larger*?"

Example I.1.1. Comparing which set is larger.

Let $\mathbb{R}$ be defined the set of real numbers, consider the following sets:

- $\mathbb{Z}$, denoting the set of integers.
- $[0]_{\mathbb{Z}/2}$, denoting the set of even integers.
- $(0, 1)$, denoting all real numbers between 0 and 1, not inclusively.

In fact, a very typical thing that people say about cardinalities examples is that there are "*as many*" integers as even integers, because there exists a bijection function $f : \mathbb{Z} \to [0]_{\mathbb{Z}/2}$ such that:

$$f(x) = 2x.$$

Also, in this example, you would know that $[0, 1]$ is *countable*, whereas $\mathbb{Z}$ and $[0]_{\mathbb{Z}/2}$ are countable, so one would claim that there are "*more*" stuffs in $[0, 1]$ compared to $\mathbb{Z}$ or $[0]_{\mathbb{Z}/2}$. ⌟

But, the question is, can we extend this "bijection" construction to more sets within $\mathbb{R}$?

Example I.1.2. Is that a valid way of comparison?

Some readers might think the bijection argument is good for comparing the "size" of two sets, and a set would be "smaller" than the other if it has a injective map to another set, but no surjective maps, think about the following sets:

- $(0, 1)$, denoting all real numbers between 0 and 1, not inclusively.
- $(0, 2)$, denoting all real numbers between 0 and 2, not inclusively.

- $\mathbb{R}_{>0}$, denoting all positive real numbers.

It is clear that there exists bijections between all three of these sets, and in that regard, they are of the same "size." ⌟

Here, we can even have a standard argument from **complex analysis**.

Theorem I.1.3. Riemann Mapping Theory.

Let $U \subset \mathbb{C}$ be non-empty, simply connected, and open subset, then there exists a **biholomorphic mapping** (bijective holomorphic map whose inverse is also holomorphic) f from U onto the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

If we are to visualize this theorem, there exists a biholomorphic mapping between all the following subsets of $\mathbb{C}$, so we can classify them as the "same size."

1. The unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
2. The "unit square" $(-1, 1) \times \mathrm{i}(-1, 1)$.
3. A "special" shape.
4. The upper-half-plane $\mathbb{H}$.
5. The open strip $\big(\mathbb{R} \times \mathrm{i}(-\pi/2, \pi/2)\big)$.
6. The punctured slit plane $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

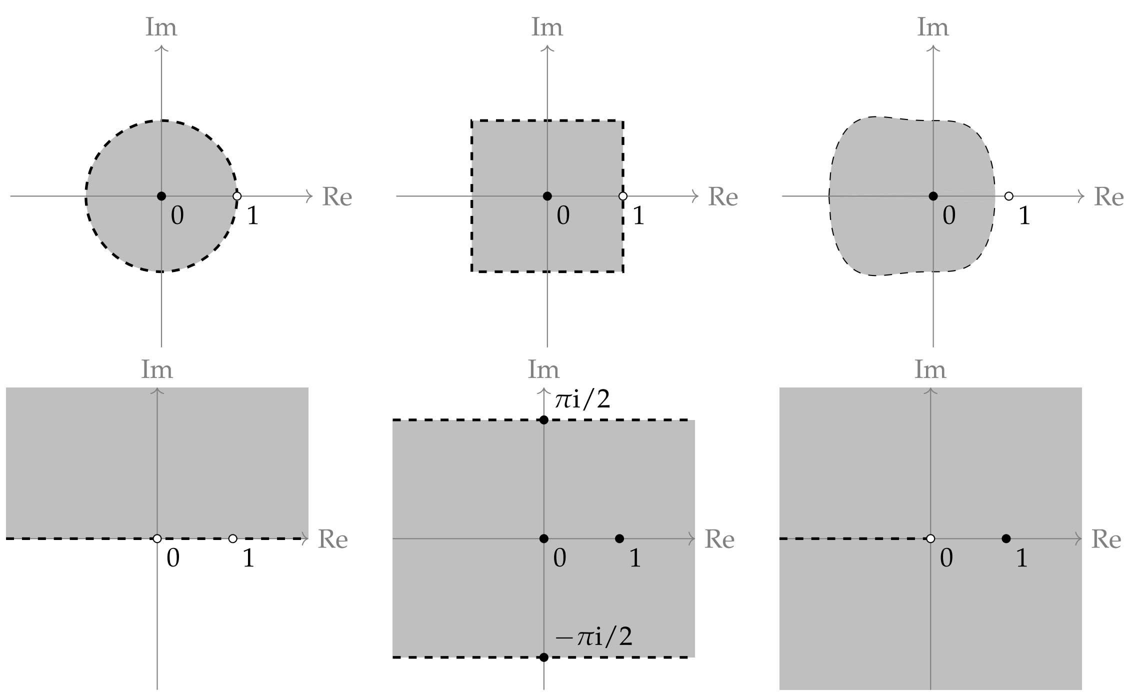

Figure I.1. Examples of nonempty, simply connected, open subsets in $\mathbb{C}$.

Intuitively speaking, they are not of the same size, so we need some other ways of measuring size.

To discuss about this size, let's recall how we compute area of a shape in a 2-d plane in *elementary school*.

Example I.1.4. Area of pizzas.

For the sake of this example, let's suppose that pizzas a triangular after being cut into slices, that is you do note eat the cornicione of the pizza (don't do this). Say there are 3 pieces left:

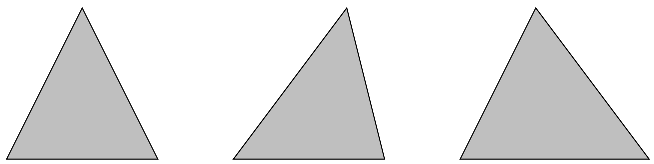

Figure I.2. Left pieces of pizzas.

It is clear that the rightmost piece is the largest, and you should pick this if you are hungry. Formally, you think about this as:

$$\text{Area} = \frac{1}{2}\text{base} \times \text{height}.$$

This can be thought of as a very basic motivation to area or measure. However, (some) mathematicians are always frantic, and they rather cut the pizza in the following way.

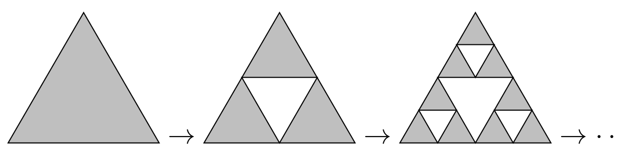

Figure I.3. Step construction of Sierpiński triangle.

Here, we want to ask some questions here:

- With this construction, assume the mathematician cuts the pizza in infinitely many stages, does it still have area?
- For these specially constructed shapes, is there still a way to compare if one of them is larger or not?

For the first question, if you have had some exposure from basic calculus and elementary school geometry, we can see that the areas of each stage forms a "sequence" (assuming side length of 1), of:

$$\left\{\sqrt{3}, \frac{3}{4}\sqrt{3}, \frac{9}{16}\sqrt{3}, \cdots\right\},$$

so you hopefully agree that this sequence *converges* to 0.

Then, the problem becomes, are there someways of representing this set of its measure, given it is not empty nor *countable*, and we will explore this later on.

I.2 Measure Theory

The key component of this is about **measure theory**, which concerns about measuring how much stuffs do we have.

Now, lets' very quickly define a very common measure, known as the Lebesgue (*outer*) measure.

Remark I.2.1. When having this measure, our aim is to have it consist with our current *intuitions* with length/area/volume in the Euclidean spaces. ⌟

Definition I.2.2. Lebesgue Outer Measure.

Let $E \subset \mathbb{R}^n$, and let $\{C_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ be any sequence of *rectangular cuboid* that covers E, *i.e.*, $\bigcup_{k=1}^{\infty} C_k \supset E$. We define the **Lebesgue outer measure** as:

$$m^*(E) = \inf\left\{\sum_{k=1}^{\infty} \mathrm{vol}(C_k)\right\},$$

where the volume is calculated as the product of the side lengths for the cubes. ⌟

It would be good to think about some examples in the Euclidean space, such as a normal cube, or balls.

Remark I.2.3. Lebesgue Measure.

The **Lebesgue measure** will be similarly defined a certain subsets of $\mathbb{R}^n$, and in the measure theory perspective, such sets are called a measurable set (or **Borel sets**). ⌟

Instead, many of us might wonder, are there sets that are not Borel, and we will give an interesting construction of a non-measurable set, the **Vitali Set**, using the **Axiom of choice**.

Example I.2.4. Vitali Set.

Consider the interval $[0,1] \subset \mathbb{R}$, we define an recurrence relationship that $a \sim_{\mathbb{Q}} b$ if $a - b \in \mathbb{Q}$. Hence, we effectively have $[0,1]/\mathbb{Q}$ as uncountably many equivalence classes, and by the **axiom of choice**, we can select exactly one element in each equivalent class, forming the **Vitali set**.
Here, we assume that the **Vitali set** is measurable and has measure $m(V)$, since the Lebesgue measure is translation invariant, we have $m(q+V) = m(V)$ for some $q \in \mathbb{Q} \cap [-1,1]$. Note that $\mathbb{Q} \cap [0,1]$ is countable, so we can have an enumeration, thus:

$$[0,1] \subset \bigcup_{k=1}^{\infty} q + V \subset [-1,2].$$

Then, note that the Lebesgue measure has monotonicity, we have:

$$1 = m(0,1) \leq m\left(\bigcup_{k=1}^{\infty} q + V\right) \leq m([-1,2]) = 3.$$

Recall translation invariant, we have the middle argument as an infinite product of $m(V)$, but there is no $m(V)$ such that $1 \leq \infty \cdot m(V) \leq 3$, hence it is not measurable. ⌟

Remark I.2.5. Solovay's Theorem.

Without the **Axiom of choice** we can construct a model such that all subsets of $\mathbb{R}^n$ are Borel. ⌟

Then, we will look into a good old friend of mathematician in 1-D, so it cannot really be a pizza cutting thing, but the **Cantor set**.

Example I.2.6. Cantor Set.

The **Cantor set** can be constructed step-by-step, starting from $C_0 = [0, 1]$. For each step, we remove the middle $1/3$ proportion of all current segments, and we can consider $\mathcal{C} := \lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n$. ⌟

The first few level of **Cantor set** constructions can be demonstrated as follows:

Figure I.4. First six steps of Cantor set construction.

Proposition I.2.7. Properties of Cantor Set.

We will remark (but not prove) the following properties of the **Cantor set**:

- Cantor set is Borel.
- Cantor set is uncountable.
- Cantor set has Lebesgue measure 0.

In fact, there are many variant of the Cantor set, such as removing a different proportion in the middle, but is there a way to compare between the *size* of them?

II Hausdorff Measure and Dimension

II.1 Hausdorff Measure

Now, you should notice that we need something else. There are many *uncountable* sets that are of Lebesgue measure zero. And if we are to think of them as the same *size*, this seems too arbitrary.

Here, mathematicians need another way to *measure* these null sets, and what about scale the volume of the cubes/balls?

Definition II.1.1. Hausdorff Measure of Dimension s.

Suppose $E \subset \mathbb{R}^n$ and s is a non-negative number, for any $\delta > 0$, we define the **Hausdorff measure** of dimension s as:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \left[\inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\} \right].$$

Here, the δ-cover is a sequence of sets of diameter at most δ that cover E, *i.e.*, $E \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \le |U_i| \le \delta$ for all i. ⌟

Note that if we just have the δ-covers, as $\delta \to 0$, by monotonicity and non-negativity of smaller δ, the limit is guaranteed to exist, so we would not worry about the existence of the limit here.

Proposition II.1.2. Hausdorff and Lebesgue Measure.

Suppose $F \subset \mathbb{R}^n$ is a Borel set, then the n-dimensional Hausdorff measure is a constant multiple of Lebesgue measure, *i.e.*:

$$\mathcal{H}^n(F) = c_n^{-1} m_{\mathbb{R}^n}(F),$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$.

This is a direct consequence of the definition of how to measure the "volume" of small objects, whether by the n-th power of the diameter for balls or the the actual volume.

Another important property is that Hausdorff measure is also a metric outer measure, just like the Lebesgue measure.

Proposition II.1.3. Hausdorff Measure is Metric Outer Measure.

Let $E, F \subset \mathbb{R}^n$ be such that $d(E,F) := \inf\{d(x,y) : x \in E, y \in F\} > 0$, we have:

$$\mathcal{H}^s(E \sqcup F) = \mathcal{H}^s(E) + \mathcal{H}^s(F).$$

Remark II.1.4. Natural Measurable Set.

Note that the **Borel sets** satisfy a σ-algebra and the Hausdorff measure is metric outer measure, we can restrict to Borel sets $\mathcal{B}$ to be the set of (Carathéodory) measurable set. ⌟

There are also some other properties, such as how the Hausdorff measure behaves under maps of scaling maps, maps satisfying Hölder condition, or Lipschitz maps.

Example II.1.5. Some special cases of Hausdorff measures.

Here are some specific example of sets corresponding to certian dimensions of Hausdorff measure:

1. For hyper-planes $F \subset \mathbb{R}^n$, $\mathcal{H}^0(F)$ is the cardinality of F.
2. Let $F \subset \mathbb{R}^n$ be a smooth curve, $\mathcal{H}^1(F)$ gives its length.
3. For $F \subset \mathbb{R}^n$ being a smooth m-dimensional manifold, then $c_m \mathcal{H}^m(F)$ gives its volume, where c_m is the volume of unit ball in $\mathbb{R}^m$.

⌟

II.2 Hausdorff Dimension

Now, we kind of see that the *Hausdorff measure* is capable of measuring the measure so that Lebesgue measure cannot measure. Here, we will speak about an additional feature of Hausdorff measure leading to an extension to dimension.

Proposition II.2.1. Dimension s of Finite Measure $\Longrightarrow$ Higher Dimensions have Zero Measure.

Suppose $F \subset \mathbb{R}^n$ and $\mathcal{H}^s(F) < \infty$. If $t > s$, then $\mathcal{H}^t(F) = 0$.

Hence, the graph of $\mathcal{H}^s(F)$ against s has some "jump" discontinuity from ∞ to 0, as there could only be at most one s with non-zero and finite Hausdorff measure. If there are more than one point whose measure is within $(0, \infty)$, the proposition incurs a contradiction that the smaller s has to be zero. Therefore, the graph could only have one "jump" discontinuity from ∞ to 0.

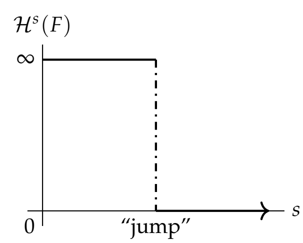

Figure II.1. Graph of $\mathcal{H}^s(F)$ against F with jump at its dimension.

Hence, this leads to the definition of the Hausdorff dimension.

Definition II.2.2. Hausdorff Dimension.

Let $F \subset \mathbb{R}^n$ be arbitrary, the Hausdorff dimension, denoted $\dim_{\mathrm{H}} F$, is:

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

For simplicity, the supremum of the empty set is 0. ⌟

With this definition of Hausdorff Dimension, we have:

$$\mathcal{H}^s(F) = \begin{cases} \infty, & \text{if } 0 \leq s < \dim_{\mathrm{H}} F, \\ 0, & \text{if } s > \dim_{\mathrm{H}} F. \end{cases}$$

Proposition II.2.3. Properties of Hausdorff Dimension.

Hausdorff dimension satisfies the following properties:

1. *Monotonicity*. If $E \subset F$, then $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$.
2. *Countably stability*. Let $\{F_i\}$ be a countable sequence of sets, then $\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty}\{\dim_{\mathrm{H}} F_i\}$.
3. *Countably sets*. If F is countable, then $\dim_{\mathrm{H}} F = 0$.
4. *Open sets*. If $G \subset \mathbb{R}^n$ is a non-empty, open set, then $\dim_{\mathrm{H}} F = n$.
5. *Smooth sets*. If $F \subset \mathbb{R}^n$ is a smooth m-dimensional manifold, then $\dim_{\mathrm{H}} F = m$.

Again, just like how we had measures relatively invariant over maps of scaling maps, maps satisfying Hölder condition, or Lipschitz maps, those naturally extends to the Hausdorff dimension.

Then, we will get back with our old friend, the **Cantor set** back. Recall that:

- It is not countable, so we cannot directly conclude that $\dim_{\mathrm{H}} \mathcal{C} = 0$.
- It had Lebesgue measure of 0, so we can conclude that $\dim_{\mathrm{H}} \mathcal{C} \leq 1$.

What should it be?

Example II.2.4. Hausdorff dimension of the Cantor set.

For the Cantor set, it splits into a left and right part, *i.e.*:

$$\mathcal{C}_L = \mathcal{C} \cap \left[0, \frac{1}{3}\right] \text{ and } \mathcal{C}_R = \mathcal{C} \cap \left[\frac{2}{3}, 0\right].$$

We note that both parts are geometrically similar to $\mathcal{C}$ but scaled by $1/3$. Moreover, they are disjoint and $\mathcal{C} = \mathcal{C}_L \sqcup \mathcal{C}_R$. Moreover, we have $d(\mathcal{C}_L, \mathcal{C}_R) \geq 1/3$, so they satisfies the metric outer measure. Therefore, by the scaling property, we have:

$$\mathcal{H}^s(\mathcal{C}) = \mathcal{H}^s(\mathcal{C}_L) + \mathcal{H}^s(\mathcal{C}_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(\mathcal{C}) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(\mathcal{C}).$$

Here, we *claim* that at $s = \dim_{\mathrm{H}} F$, $0 < \mathcal{H}^s(F) < \infty$. Hence, we get $1 = 2 \cdot \left(\frac{1}{3}\right)^s$ or $s = \log_3 2$. ⌟

This claim can be proven by iteratively exploiting the levels of the Cantor set, whose proof is long and tedious. However, we will have a stronger theorem later on.

III Iterated Function System and Fractals

III.1 Self Similar Sets

Many fractals are made with ways of iterated processes, the self-similarities, defined by iterated functions, can be helpful in finding the dimensions.

Definition III.1.1. Contraction Mapping.

Let $D \subset \mathbb{R}^n$ be closed, a mapping $S : D \to D$ is a contraction on D if there exists a number c with $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in F$. ⌟

Remark III.1.2. Remarks for Contractions.

Let S be a contraction, the following properties hold:

1. Any contraction is continuous, by the definition of continuity.
2. If the equality holds, *i.e.*, $|S(x) - S(y)| = c|x - y|$, then S transforms sets into geometrically similar sets, then S is a contracting similarity.

⌟

Definition III.1.3. Iterated Function System.

Let $\{S_1, S_2, \cdots, S_m\}$ be a *finite* family of contractions ($S_i : D \to D$) with $m \geq 2$, it is an iterated function system, *or* IFS. ⌟

Note that the we have the *IFS* defined for finite functions, not (countably) infinitely many functions.

Definition III.1.4. Attractor.

A compact subset $F \subset D$ is an attractor (*or* invariant set) for the IFS $\{S_i\}$, in which $S_i : D \to D$, if:

$$F = \bigcup_{i=1}^{m} S_i(F).$$

⌟

Conceptually thinking, when you have a map/image/graph so that you can zoom it in/out, there should be a point that has its position preserved, right? The set of all preserved points would be the attractor of the IFS.

For example, consider the Cantor set again.

Example III.1.5. Attractor as Cantor set.

If we define the iterated function system $\{S_1, S_2\}$ where $S_1, S_2 : D \to D$ is:

$$S_1(x) = \frac{1}{3}x \text{ and } S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Here, for $\mathcal{C}$, we have that $S_1(\mathcal{C}) = \mathcal{C}_L$ and $S_2(\mathcal{C}) = \mathcal{C}_R$, and hence we consider $\mathcal{C}$ as an attractor of $\{S_1, S_2\}$ as IFS, representing the self-similarities of $\mathcal{C}$. ⌟

Now, there is a formal theorem of its existence and uniqueness.

Theorem III.1.6. Existence and Uniqueness of Attractor.

Consider the iterated function system given by the contractions $\{S_1, \cdots S_m\}$ on $D \subset \mathbb{R}^n$, so that:

$$|S_i(x) - S_i(y)| \le c_i |x - y| \text{ for } x, y \in D.$$

with $c_i < 1$ for each i. Then there is a unique attractor F, *i.e.*, a non-empty compact set such that:

$$F = \bigcup_{i=1}^{m} S_i(F).$$

III.2 Self Similar Sets and Dimensions

To get the stronger conclusion, we consider an additional condition for the fractals, namely, the *open set condition*. Recall our concern with the proof of Cantor set's dimension, this condition can guarantee our claim in the previous section.

Definition III.2.1. Open Set Condition.

Let $S_i : D \to D$ be contractions in IFS $\{S_i\}_{i=1}^m$, then $\{S_i\}$ satisfy the open set condition if there exists a non-empty bounded open set V such that:

$$V \supset \bigcup_{i=1}^{m} S_i(V).$$

⌟

Theorem III.2.2. Open Set Condition $\Longrightarrow$ Conclusion of Dimensions.

Suppose that open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \le i \le m$. If F is the attractor of the IFS $\{S_1, \cdots, S_m\}$, *i.e.*:

$$F = \bigcup_{i=1}^{m} S_i(F),$$

then $\dim_{\mathrm{H}} F = s$, where s is given by:

$$\sum_{i=1}^{m} c_i^s = 1.$$

Moreover, for this value of s, $0 < \mathcal{H}^s(F) < \infty$.

Now, we just need to show that the Cantor set satisfies the **open set condition**, which is trivial. Say $(0,1)$ and consider the IFS:

$$S_1((0,1)) = \left(0, \frac{1}{3}\right) \text{ and } S_2((0,1)) = \left(\frac{2}{3}, 1\right),$$

and they are clearly subsets of $(0,1)$, so we have legitimately concluded that the Hausdorff dimension of the Cantor set is $\log_3(2)$.

Remark III.2.3. In fact, for the interest of the audiences, the $\log_3(2)$-dimensional Hausdorff measure of $\mathcal{C}$ is 1, which contrasts to the Lebesgue measure of 0. ⌟

III.3 Examples of Fractals

Recall the earlier pizza example? Let's compute what the dimension of that weird pizza.

Example III.3.1. Sierpiński Triangle.

The Sierpiński triangle F is constructed from a equilateral triangle by removing the inverted equilateral triangle(s) of $1/4$ area, with such step iterated.

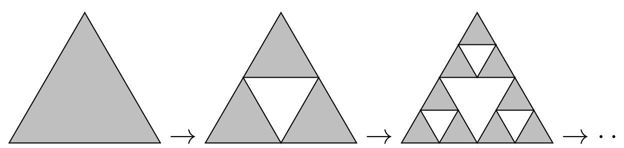

Figure III.1. Sierpiński Triangle.

Note that we can form the IFS as:

$$S_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right), \qquad S_2(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{1}{2}\right), \qquad S_3(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right).$$

Specifically, consider E_0° (the interior of the largest triangle) as an open set, we note that $S_1(E_0^\circ) \cup S_2(E_0^\circ) \cup S_3(E_0^\circ)$ can be covered by E_0°. Hence, F satisfies the open set conditions. Thus, by open set condition $\Longrightarrow$ conclusion of dimensions, we have the dimensions $s = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F$ that:

$$\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s = 1,$$

which implies that $s = \log_{1/2}(1/3) = \log_2(3)$. ⌟

Dimension-wise, you can say that the mathematician has left you something more than a single 1-dimensional line to eat. Then, let's explore different Cantor sets, as mathematicians can always find ways to do more.

Definition III.3.2. k-Cantor Set.

Recall in the definition of a **Cantor set**, it removes the middle $1/3$ component in each section. For $0 < k < 1$, we can just remove the middle k component in each section to construct the k-Cantor set. ⌟

It should be simple to use the IFS with each similar contraction scaled by $\frac{1-k}{2}$ and conclude on the dimension of $\log 2/\log(\frac{2}{1-k})$.

What about the products? Consider the Cantor dust of the products, such as $\mathcal{C}\times\mathcal{C}, \mathcal{C}\times\mathcal{C}\times\mathcal{C}, \cdots, \mathcal{C}^n$. The IFS would be composed of 2^n functions, and so the assume removing middle $1/3$, we have the Hausdorff dimension $\log_3(2^n) = n\log_3(2)$.

It should be noted that there is not yet computation of the s-dimensional Hausdorff measure for these Cantor dusts, and it is up to you to find the exact values of these measures.

Aside, consider the second product of cantor set, which is:

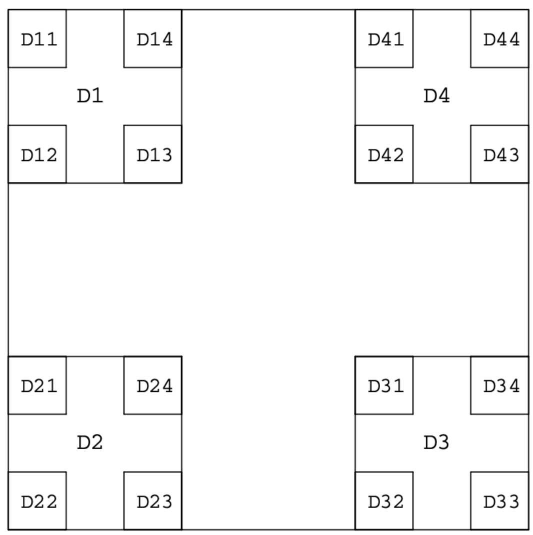

Figure III.2. The second level of $\mathcal{C}^2$.

You should realize that this set has Hausdorff dimension $2\log_3(2)$, in which we have:

$$2\log_3(2) < \log_2 3.$$

So for the Sierpiński Triangle and $\mathbb{C}^2$, we have Sierpiński Triangle having a larger dimension.

Hopefully, in the future, when a crazy mathematician hands you pizzas that are cut into fractals, and you are supper hungry (or you belong to some different dimension), you will be able to find the pizza with the highest dimension.