Johns Hopkins Mathematics Competition 2025

High School - Power Round

SOLUTIONS

April 26, 2025

Do NOT reveal this solution manual to any candidates before the end of competition.

I Preliminaries

This power round explores measure theory and fractals in mathematics. As we delve into such exploration, we will be going through various preliminary definitions that support the topic.

The first concept that we are about to explore is "countability." However, prior to defining this concept, let's take a look of a special class of function.

Definition I.1. Injective Function.

Let $f : X \to Y$, that is f maps each element $x \in X$ (where X is a set, or collection of elements) to some $y \in Y$. f is **injective** if for any $x, x' \in X$ such that $f(x) = f(x')$, this implies that $x = x'$. ⌟

Remark. For the convention of this power round, 0 is a natural number. Notation-wise, $\mathbb{N}$ represents the set of *natural numbers*, $\mathbb{Z}$ represents the set of *integers*, $\mathbb{Q}$ represents the set of *rational numbers*, $\mathbb{R}$ represents the set of *real numbers*, and $\mathbb{C}$ represents the set of *complex numbers*. ♢

Example. Let $f : \mathbb{R} \to \mathbb{R}$, in which $f(x) = x + 1$. This is **injective**, since for any $x, x' \in \mathbb{R}$ such that $x + 1 = f(x) = f(x') = x' + 1$, we can conclude that $x = x'$.
Let $g : \mathbb{R} \to \mathbb{R}$, in which $g(x) = x^2$. This is **not injective**, since we have $g(1) = 1 = g(-1)$, but $1 \neq -1$. However, if we restrict $\tilde{g} : \mathbb{N} \to \mathbb{N}$, so that $\tilde{g} := g\big|_{\mathbb{N}} = x^2$, it is **injective**, since for any $x, x' \in \mathbb{N}$ such that $x^2 = (x')^2$, since x and x' are nonnegative, we must have $x = x'$. ♢

Problem 1. [8 marks] Identify if the following functions are injective. Give a counterexample if the function is not injective.

- $f : \mathbb{R} \to \mathbb{R}$, in which $f(x) = x^3$.
- $g : \mathbb{N} \to \mathbb{N}$, in which $g(x) = (x - 1)^2$.
- $h : \mathbb{N} \to \mathbb{Z}$, in which $h(x) = h(x - 1) + h(x - 2)$ for $x \geq 2$ and $h(0) = -1$ and $h(1) = 1$.
- $i : \mathbb{R} \to \mathbb{C}$, in which $i(x) = \cos x + \mathrm{i} \sin x$, where i is the imaginary unit.

Solution.

- f is injective.
- g is not injective, since $g(0) = g(2)$.
- h is not injective, since $h(1) = h(3)$.
- i is not injective, since $i(0) = i(2\pi)$.

Definition I.2. Countable Sets.

A set is **countable** if there exists an *injective* function from it to the natural numbers. ⌟

A set is **finite** if there are finitely many elements in it. If it has infinitely many elements, it is **countable** as defined above, otherwise it is **uncountable**.

In fact, we can easily show that $\mathbb{Z}$ is countable.

Proof. We construct the injective function $f : \mathbb{Z} \to \mathbb{N}$ as:

$$f(x) := \begin{cases} 2x, & \text{when } x \geq 0, \\ -2x - 1, & \text{otherwise.} \end{cases}$$

Note that the function f is well defined since it maps any integer to a nonnegative integer, *i.e.*, natural number. Also, since $2x$ and $-2x-1$ are injective, and all nonnegative numbers are mapped to even integers and all negative numbers are mapped to odd integers, the function f is injective. □

Proposition I.3. $\mathbb{Q}$ is Countable.

The set of rational numbers, $\mathbb{Q}$, is countable. ⌟

Here is a "proof" on the above proposition.

"Proof." First, we recall the definition of rational numbers:

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0, \text{ and } \gcd(p, q) = 1 \right\}.$$

Here, we want to construct an injective function to $\mathbb{N}$, note that in the previous proof, we can propose an injective function to $\mathbb{Z}$ first, and compose the maps necessary. Hence, we can first consider the case for all positive rational numbers, or $\mathbb{Q}^+ \to \mathbb{Z}^+$. Now, we may write the rational numbers as a table:

$$\begin{array}{cccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Figure 1. Tabular representation of $\mathbb{Q}^+$.

Hence, we can enumerate the numbers through diagonals, namely through:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \dots.$$

Note that there will be repetitive terms, but as long as we remove the terms such that the fraction is not coprime, we can end up with an *injective* from $\mathbb{Q}^+ \to \mathbb{N}^+$, then we can enforce:

$$g(x) = \begin{cases} 0, & \text{when } x = 0, \\ \text{enumeration such that } x \text{ has coprime numerator and denominator}, & \text{when } x > 0, \\ -f(-x), & \text{when } x < 0. \end{cases}$$

Hence, we have constructed an injective map $g : \mathbb{Q} \to \mathbb{Z}$ since every positive rational is mapped to a unique order that is well defined, so do the negative rationals as negative integer and zero to zero, and if we want to have an *injective* map from $\mathbb{Q}$ to $\mathbb{N}$, just apply $f \circ g$, where f is given in the previous proof.

End of "proof."

Problem 2. [6 marks] Is the above *"proof"* valid? If it is not valid, find the incorrect assumption or logic with the "proof."
Note: You do **not** have to fix the error in case the "proof" is not valid.

Solution.
The *proof* is valid, as $f \circ g$ is a injection by construction.

Theorem I.4. $\mathbb{R}$ is not Countable.

The set of real numbers, $\mathbb{R}$, is *not* countable. ⌟

A famous *proof* to that real numbers is not countable is by *Georg Cantor*, a German-Russian mathematician, who has an important role of establishing the **set theory** in mathematics. Here, we will provide you with the proof for **Theorem I.4** that is proposed by Cantor.

Proof. Here, we suppose that $\mathbb{R}$ is countable, then $(0, 1)$ is at most countable, and since it is infinite, $(0, 1)$ must be countable.
Hence, we may enumerate $(0, 1)$ in terms of:

$$\begin{aligned} x_1 &= 0.a_{1,1}a_{1,2}\cdots, \\ x_2 &= 0.a_{2,1}a_{2,2}\cdots, \\ x_3 &= 0.a_{3,1}a_{3,2}\cdots, \\ &\vdots \end{aligned}$$

In particular, $a_{i,j}$ can be any number from 0 to 9, then, we can construct a decimal $x \in (0, 1)$ as:

$$x = 0.b_1b_2b_3\cdots,$$

such that $b_1 \neq a_{1,1}$, $b_2 \neq a_{2,2}$, $b_3 \neq a_{3,3}$, so we consequently have $x \neq x_i$ for any $i = 1, 2, \cdots$, then x is not in this enumeration, hence this is a contradiction that $\mathbb{R}$ is countable. □

Problem 3. [5 marks] Is $\mathbb{C}$ countable? Justify your answer.

Solution.
Here, we recall the definition of complex numbers, $\mathbb{C}$ as:

$$\mathbb{C} = \{a + b\mathrm{i} : a, b \in \mathbb{R}\} \text{ where i is the imaginary unit.}$$

It is should be clear that $\mathbb{R} \subset \mathbb{C}$, so there exists an *injective* inclusion map $\iota : \mathbb{R} \to \mathbb{C}$ such that $\iota(x) = x$. Now, for the sake of contradiction, suppose that $\mathbb{C}$ is countable, then there exists an *injective map*

$\varphi : \mathbb{C} \to \mathbb{N}$, then we have $\varphi \circ \iota : \mathbb{R} \to \mathbb{N}$ that is a composition of injective functions, which is also injective. This is a contradiction to **Theorem I.4**, as $\mathbb{R}$ is not countable.
Thus, $\mathbb{C}$ must not be countable. □

Now, we will get to a specially constructed set, namely the **Cantor set**.

Definition I.5. Cantor Set.

The **Cantor set** can be constructed step-by-step, starting from:

$$C_0 = [0, 1].$$

Then, for $n \geq 1$, we define the construction, recursively:

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right).$$

Eventually, we define the **Cantor set**, denoted $\mathcal{C}$, as:

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n.$$

The first few level of **Cantor set** constructions can be demonstrated as follows:

Figure 2. First six steps of Cantor set construction.

For example, we can explicitly list the first few levels of the **Cantor set**, such as:

$$\begin{aligned}
C_0 &= [0, 1], \\
C_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \\
C_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \\
&\vdots
\end{aligned}$$

Problem 4. [6 marks]

(a) Write down C_4 explicitly.

(b) Prove or disprove each of the following statements:

- $3/54 \in \mathcal{C}$,
- $2/3^{12} \in \mathcal{C}$.

Solution.
(a) We first construct C_3 correspondingly:

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right].$$

Then, we can construct C_4 as:

$$\begin{aligned} C_4 = {} & \left[0, \frac{1}{81}\right] \cup \left[\frac{2}{81}, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{7}{81}\right] \cup \left[\frac{8}{81}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{19}{81}\right] \cup \left[\frac{20}{81}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{25}{81}\right] \cup \left[\frac{26}{81}, \frac{1}{3}\right] \cup \\ & \left[\frac{2}{3}, \frac{55}{81}\right] \cup \left[\frac{56}{81}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{61}{81}\right] \cup \left[\frac{62}{81}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{73}{81}\right] \cup \left[\frac{74}{81}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, \frac{79}{81}\right] \cup \left[\frac{80}{81}, 1\right]. \end{aligned}$$

(b) For $3/54$, it is not in $\mathcal{C}$, we note that $\frac{3}{54} \in \left(\frac{1}{27}, \frac{2}{27}\right)$, and it is removed in set C_3 already.
For $2/3^{12}$, it is in $\mathcal{C}$, since it will be on the boundary of level $\left[\frac{2}{3^{12}}, \frac{1}{3^{11}}\right]$, and the boundaries were never removed when removing middle open intervals.

Then, we want to consider the **Cantor set** represented as **ternary expansions**, *i.e.*, the number system with only $\{0, 1, 2\}$, where the n-th digit before the decimal point represents 3^{n-1} and the m-th digit after the decimal point represents 3^{-n}.

Here, we give an example of converting a ternary expressions to decimal expression.

Example. Consider the ternary expression 120.12, it is correspondingly:

$$1 \times 3^2 + 2 \times 3^1 + 0 \times 3^0 + 1 \times 3^{-1} + 2 \times 3^{-2} = 9 + 6 + 0 + 0.\overline{3} + 0.\overline{2} = 15.\overline{5},$$

as a decimal number. $\diamondsuit$

Problem 5. [5 marks] Consider the ternary expression $0.0\overline{2}$:

(a) Express it in terms of decimal expression.

(b) Is there another ternary expressions that corresponds to the same number?

Solution.
(a) To express it in decimal expression, we have:

$$2 \times 3^{-2} + 2 \times 3^{-3} + \cdots = \sum_{n=2}^{\infty} \frac{2}{3^{-n}} = 2 \times \sum_{n=2}^{\infty} \frac{1}{3^{-n}} = 2 \times \frac{1/9}{1 - 1/3} = \frac{1}{3} = 0.\overline{3}.$$

(b) There is definitely another expressions, namely 0.1 in ternary expression.

Note: The example in (b) appears in decimal numbers as well, which is $0.\overline{9} = 1$, so the representation of a number may not be unique.

Now, with the ternary expressions, we want to consider expressing the Cantor Set in terms of ternary expression.

Problem 6. [8 marks] Prove that every number in the **Cantor set** $\mathcal{C}$ can be represented as:

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}, \text{ where } a_n \in \{0, 2\},$$

that is, we can express them as $0.a_1a_2a_3\cdots$ in ternary expression, where a_n can be only 0 or 2.
Hint: The expression might not be unique, such as the example in **Problem 5**.

Solution.
First, in terms of the ternary number system, all the numbers in $[0, 1]$ can be represented as some:

$$\sum_{n=1}^{\infty} \frac{a_n}{3^n}, \text{ where } a_n \in \{0, 1, 2\}.$$

Now, as we reexamine the removal on the level, the removal on level C_i removes all the numbers whose 3^{-i} digit is 1, so up to level C_i, there will be only 0's and 2's over the first i digits. Note that for expressions like $\frac{1}{3^n}$, it can be written as $\sum_{i=n+1}^{\infty} \frac{2}{3^i}$, just like the example in **Problem 5**, so we are still good. Therefore, all the numbers left would be only composed of 0's and 2's in place. $\square$

Then, we want to fit the **Cantor set** as one of the "categories" defined above.

Problem 7. [5 marks] Show that the Cantor set is not countable.
Hint: Use **Problem 6** and the *proof* of **Theorem I.4**.

Solution.
Here, from **Problem 5**, we basically have that all the **Cantor** set components are composed with 0's and 2's. Moreover, for any numbers composed of 0's and 2's, it will be on the boundary so it will never be removed. Thus, we can apply the same *proof* with **Theorem I.2**, that we first, for the sake of contradiction, suppose that $\mathcal{C}$ is countable. Hence, we may enumerate $\mathcal{C}$ in terms of:

$$\begin{aligned} x_1 &= 0.a_{1,1}a_{1,2}\cdots, \\ x_2 &= 0.a_{2,1}a_{2,2}\cdots, \\ x_3 &= 0.a_{3,1}a_{3,2}\cdots, \\ &\vdots \end{aligned}$$

In particular, $a_{i,j}$ can be any 0 or 2, then, we can construct a decimal $x \in \mathcal{C}$ as:

$$x = 0.b_1 b_2 b_3 \cdots ,$$

such that $b_1 \neq a_{1,1}$, $b_2 \neq a_{2,2}$, $b_3 \neq a_{3,3}$, so we consequently have $x \neq x_i$ for any $i = 1, 2, \cdots$, then x is not in this enumeration, hence this is a contradiction that $\mathcal{C}$ is countable. □

Note: One will *not* be penalized if they use the conclusion of **Problem 6** without proving it.

This ends our very first encounter with the **Cantor set**, and we will meet it again very soon later.

Another piece of preliminary notation is the extension of the concept of **minimum**. Consider a set of infinite number of elements, it might not necessarily have a minimum, so we define a **infimum** as follows.

Definition I.6. Infimum.

Let S be a set of real numbers, the **infimum of S**, denoted $\inf S$ is the element $y \in \mathbb{R}$ such that:

- $y \leq x$ for all $x \in S$, and
- for all $a \in \mathbb{R}$ such that $a \leq x$ for all $x \in S$, $y \geq a$.

⌟

Set infimum could account for more cases, as we may consider the following example.

Example. Let $S := \left\{\frac{1}{k} : k \in \mathbb{Z}^+\right\}$, this set has no minimum, as for any $\frac{1}{k} \in S$, there exists some $\frac{1}{k+1} \in S$ such that $\frac{1}{k+1} < \frac{1}{k}$. However, the set has infimum as 0. ♢

Problem 8. [5 marks] Prove that $\inf S = 0$ for the previous **Example**.

Solution.

Clearly, we first have that $0 \leq \frac{1}{k}$ for all $k \in \mathbb{Z}^+$. Then, suppose there exists some $\varepsilon > 0$ such that $\varepsilon \leq \frac{1}{k}$ for all $k \in \mathbb{Z}^+$, however, we can construct:

$$k = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1,$$

so that $\varepsilon > \frac{1}{k}$, so this is a contradiction, and hence $\inf S = 0$.

Now, we are finished with preliminary backgrounds, and we can move to some more interesting concepts that utilize the backgrounds.

II Measure Theory

First of all, we will first define a very common type of sets in mathematics, called *open sets*.

Definition II.1. Open Set.

Let X be a set in the Euclidean space $\mathbb{R}^n$, it is called an **open set** if for any $x \in X$, there exists some $\delta > 0$ such that for any $y \in \mathbb{R}^n$ such that $\|y - x\| < \delta$. ⌟

Again, we will provide you some examples of **open sets**.

Example. Consider the Euclidean space $\mathbb{R}^n$, the following are open sets in the Euclidean space:

- $\emptyset$ is open set, since the statement is *vacuously true*.
- $(0,1)^n$ is open set, since for any $x \in (0,1)^n$, we may let $\delta = \frac{1}{2}\min\{x_1, x_2, x_3, \cdots, x_n, (1-x_1), (1-x_2), \cdots, (1-x_n)\}$ so that for any $y \in \mathbb{R}^n$ in which $\|x - y\| < \delta$ is in $(0,1)^n$.
- $B_1(0) := \{x \in \mathbb{R}^n : \|x\| < 1\}$ is an open set, since for any $x \in B_1(0)$, we can have $\delta = \frac{1}{2}(1 - \|x\|)$ so that any $y \in \mathbb{R}^n$ in which $\|x - y\| < \delta$ has a distance to 0 less than 1. ♢

From the above examples, you should be familiar with what an **open set** is, and we will provide you with the definition of a **Borel set**.

Definition II.2. Borel Set.

The **Borel set** starts as the set of all open sets, then we do its countable union, countable intersection, and complement recursively. ⌟

For the simplicity of this part, suppose **Borel set** definition is over Euclidean space, or $\mathbb{R}^n$, denoted $\mathcal{B}(\mathbb{R}^n)$.

Here, we are going to provide you a commonly used measure for Euclidean space, namely, the **Lebesgue measure**, which measures sets that is in the **Borel set**.

Definition II.3. Lebesgue Measure.

For any $X \in \mathcal{B}(\mathbb{R}^n)$, the **Lebesgue measure**, denoted $m : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$, on X is defined to be:

$$m(X) = \inf\left\{\sum_{k=1}^{\infty} \mathrm{Vol}(C_k) : \{C_k\}_{k=1}^{\infty} \text{ is sequence of open cubes such that } \bigcup_{k=1}^{\infty} C_k \supset X\right\}.$$

Here, the volume of cubes in $\mathbb{R}^n$ is simply the product of side lengths. ⌟

Problem 9. [16 marks] Suppose $X \in \mathcal{B}(\mathbb{R}^n)$ is countable (or finite), show that $m(X) = 0$. Then, show that the converse is not necessarily true.
By converse, you should show that there exists some set X such that $m(X) = 0$, but X is uncountable.
Hint: Recall that the **Cantor set** is uncountable from **Problem 7**.

Solution.
First, we justify that a countable set has Lebesgue measure 0.
Here, since X is countable, there naturally exists an enumeration of $x \in X$, so we let it be $\{x_i\}_{i=1}^{\infty} = X$. Also, we let $\varepsilon > 0$ be fixed. For each point x_i, we can cover it via a cube of side length $\sqrt[n]{\epsilon 2^{-i}}$, so we have the sum of volume of all covering as:

$$\sum_{k=0}^{\infty} (\sqrt[n]{\varepsilon 2^{-i}})^n = \sum_{k=0}^{\infty} \varepsilon 2^{-i} = \varepsilon \sum_{k=0}^{\infty} 2^{-i} = 2\varepsilon.$$

Hence, for any $\varepsilon > 0$, we can construct a covering of X such that $m(X) \leq 2\varepsilon$, thus by the definition of infimum, $m(X) = 0$, since the sum of nonnegative numbers cannot be negative.
Then, we give a counterexample in which an uncountable set is has a measure of zero.
Note that we have shown that the **Cantor set** is uncountable, we now just need to show that **Cantor set** is Borel and it has measure 0.

- First, to show that $\mathcal{C} \in \mathcal{B}(\mathbb{R})$, we first note that for each level, we can consider $C_0 = ((-\infty, 0) \cup (1, \infty))^c$ so C_0 is in $\mathcal{B}(\mathbb{R})$, and the scaling of them can be constructed similarly, so all $C_i \in \mathcal{B}(\mathbb{R})$ for all levels of i.
  Eventually, we consider:
  $$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n = \left(\bigcup_{n=1}^{\infty} (C_n)^c \right)^c,$$
  so it naturally follows the construction of the Borel set, hence, $\mathcal{C} \in \mathcal{B}(\mathbb{R})$.

- Then, we compute the measure of $\mathcal{C}$, correspondingly, first, we note that our construction removes certain components of the set, so the covering of a lower level is guaranteed the be a covering of the later level, we can just cover the levels by the separated pieces, so the sum for each level is:
  $$m(C_k) = 2^k \cdot 3^{-k} = \left(\frac{2}{3}\right)^k.$$
  Hence, we have $m(\mathcal{C}) < \left(\frac{2}{3}\right)^k$ for all $k \in \mathbb{Z}^+$, and since $m(\mathcal{C}) \geq 0$, so $m(\mathcal{C}) = 0$.

Note: If the student come up with any other uncountable set that has Lebesgue measure 0, we will take it as long as the justification is correct.
Note: A student shall be penalized if they fails to justify that the set they propose is a Borel set.

Now, you might see a interesting point, there are sets that are uncountable but still having measure zero. As we get into the next part, we will see mathematicians giving a different definition to "measure" some of these sets.

III Hausdorff Dimension

To get around the "measure" of some special sets, a German mathematician *Felix Hausdorff* developed the **Hausdorff dimension** to deal with them, and we will first define the Hausdorff measure over dimension s.

Definition III.1. s-dimensional Hausdorff Measure for Diameter δ.

Suppose F is a subset of $\mathbb{R}^n$ and s is a non-negative number, for any $\delta > 0$, we define:

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{k=1}^{\infty} |C_k|^s : |C_k| \leq \delta \text{ for all } k \text{ and } \bigcup_{k=1}^{\infty} C_k \supset F\right\},$$

where $|C_k|$ is the diameter of the set C_k, in which you may interpret as the supremum of the all distances within the set, which is the negative of the infimum of the negative of all distances. ⌟

The definition of the s-dimensional **Hausdorff measure** is an immediate extension from above.

Definition III.2. s-dimensional Hausdorff Measure.

Suppose F is a subset of $\mathbb{R}^n$ and s is a non-negative number, the s-dimensional Hausdorff measure is:

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F).$$

⌟

Here, the measure is well defined, as the limit exists.

Proof. Note that for any $\delta' < \delta$, any δ'-cover of F is a δ-cover of F. Hence, by the definition of infimum:

$$\mathcal{H}^s_\delta(F) \leq \mathcal{H}^s_{\delta'}(F).$$

As $\delta \to 0$, the measure is monotonic, hence the limit always exists. □

It is notable that the s-dimensional Hausdorff measure is defined similar to the definition of Lebesgue measure. With this in mind, answer the following problem.

Problem 10. [7 marks] Let $F \subset \mathbb{R}^n$ be a subset of the n-dimensional Euclidean space, identify the relationship between $m(F)$ and $\mathcal{H}^n(F)$.
Note: You should write out the explicit relationship for $n = 1, 2, 3$. For higher dimensions, you can explain how the relationship would be. You do not need to justify your solution.

Solution.
Here, readers should notice that the definition for Hausdorff measure assumes the measure of a unit ball just as the s-th power of the diameter. Hence, we have for some $F \in \mathbb{R}^n$ that:

$$m(F) = c_n\mathcal{H}^n(F),$$

where c_n is the volume of the n-dimensional unit ball, that is:

$$c_1 = 1, \qquad c_2 = \frac{\pi}{2}, \qquad c_3 = \frac{\pi}{6}.$$

The general formula is $c_n = \frac{\pi^{n/2}}{2^n \Gamma\left(\frac{n}{2}+1\right)}$, where $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$, but we are not anticipating the candidates to derive this formula out.

Now, we want you to verify some property of the Hausdorff measure.

Problem 11. [6 marks] Suppose $F \in \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{H}^s(F) < \infty$. If $t > s$, show that $\mathcal{H}^t(F) = 0$.

Solution.
Let $\{U_i\}$ be a δ-cover of F, we then have:

$$\sum_{i=1}^{\infty} |U_i|^t \leq \sum_{i=1}^{\infty} |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s.$$

Hence, by the definition of infimum, we immediately have:

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$

Note that when $\delta \to 0$ and if $\mathcal{H}^s(F)$ is bounded, then $\mathcal{H}^t(F) = 0$. $\square$

With the property in **Problem 11**, for any set $F \in \mathcal{B}(\mathbb{R}^n)$, we know that there exists some s such that there exists some jump discontinuity of the graph, that is:

$$\mathcal{H}^t(F) = \begin{cases} \infty, & \text{if } 0 \leq t < s, \\ 0, & \text{if } t > s. \end{cases}$$

Graphically, the t-dimensional Hausdorff measure should omit a graph that looks as follows:

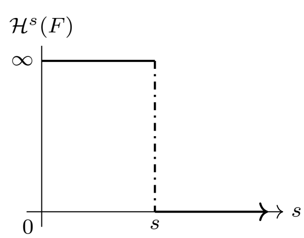

Figure 1. Graph of $\mathcal{H}^t(F)$ against F with a jump at s.

Clearly, this s is special for the set F, and this it is, in fact, the definition of **Hausdorff dimension**.

Definition III.3. Hausdorff Dimension.

Let $F \subset \mathbb{R}^n$ be arbitrary, the Hausdorff dimension, denoted $\dim_{\mathrm{H}} F$, is:

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\}.$$

To make the next argument, we give another property of the **Hausdorff measure** with the proof of it.

Proposition III.4. Hausdorff Measure is Metric Outer Measure.

Let $E, F \subset \mathbb{R}^n$ be such that $d(E, F) := \inf\{d(x, y) : x \in E, y \in F\} > 0$, we have:

$$\mathcal{H}^s(E \cup F) = \mathcal{H}^s(E) + \mathcal{H}^s(F).$$

Proof. Note that when $\delta < d(E, F)$, for any δ-cover, say $\mathcal{S}$, we can do the following partition:

$$\mathcal{E} = \{U \in \mathcal{S} : U \cap E \neq \emptyset\} \text{ and } \mathcal{F} = \{U \in \mathcal{S} : U \cap F \neq \emptyset\},$$

in which we are guaranteed that $\mathcal{E}$ is a δ-cover of E and $\mathcal{F}$ is a δ-cover of F. More importantly, by $\delta < d(E, F)$, we must have $\mathcal{E}$ and $\mathcal{F}$ being disjoint. Now, the following inequality holds:

$$\sum_{\tilde{E} \in \mathcal{E}} \tilde{E} + \sum_{\tilde{F} \in \mathcal{F}} \tilde{F} \leq \sum_{\tilde{S} \in \mathcal{S}} \tilde{S},$$

since there could be sets in $\mathcal{S}$ that does not intersect E nor F. Therefore, by definition of infimum, we may conclude that:

$$\mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) \leq \sum_{\tilde{S} \in \mathcal{S}} \tilde{S}.$$

Note that the cover $\mathcal{S}$ is arbitrary, so we conclude that:

$$\mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) \leq \mathcal{H}^s_\delta(E \cup F).$$

Since this holds for $\delta > 0$, it holds for the limit as well, that is:

$$\mathcal{H}^s(E) + \mathcal{H}^s(F) \leq \mathcal{H}^s(E \cup F).$$

By the nature of outer measure, countable stability implies the other direction of inequality, hence:

$$\mathcal{H}^s(E) + \mathcal{H}^s(F) = \mathcal{H}^s(E \cup F),$$

completing the proof for metric outer measure. □

Now, back to our story with the **Cantor set**. Recall that it has Lebesgue measure 0, so it implies that the 1-dimensional Hausdorff measure is 0, then we know that the Hausdorff dimension of the **Cantor set** must be no more than 1.

Here is a "proof" on the Hausdorff dimension being $\log_3(2)$.

"Proof." The **Cantor set** $\mathcal{C}$ splits into a left and right part, *i.e.*:

$$\mathcal{C}_L = \mathcal{C} \cap \left[0, \frac{1}{3}\right] \text{ and } \mathcal{C}_R = \mathcal{C} \cap \left[\frac{2}{3}, 1\right].$$

We note that both parts are geometrically similar to $\mathcal{C}$ but scaled by $1/3$. Moreover, they are disjoint, $d(\mathcal{C}_L, \mathcal{C}_R) > 0$, and $\mathcal{C} = \mathcal{C}_L \sqcup \mathcal{C}_R$. Moreover, we have $d(\mathcal{C}_L, \mathcal{C}_R) \geq 1/3 > 0$, so they satisfies the metric outer measure. Therefore, by **Proposition III.4**, we have:

$$\mathcal{H}^s(\mathcal{C}) = \mathcal{H}^s(\mathcal{C}_L) + \mathcal{H}^s(\mathcal{C}_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(\mathcal{C}) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(\mathcal{C}).$$

Hence, s must satisfy that $1 = 2 \cdot \left(\frac{1}{3}\right)^s$, so $s = \log_3 2$. *End of "Proof."*

Problem 12. [8 marks] Is the above *"proof"* valid? If it is not valid, find the incorrect assumption or logic with the "proof."

Note: You do **not** have to fix the error in case the "proof" is not valid.

Another Note: The *proof* of **Proposition III.4** is legit, so you do not need to check that proof.

Solution.

The "proof" is **not** valid. It takes the assumption that $0 < \mathcal{H}^s(\mathcal{C}) < \infty$ at some s, which is not proven. Note that the "jump" discontinuity could also be 0 or ∞.

Now, we think about the **Cantor set** through a different perspective – the iterated function system (IFS).

Definition III.5. Iterated Function System.

An **iterated function system** (IFS) is a finite collection of functions $S_1, S_2, \cdots, S_m$, where $m \geq 2$ and $S_i : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$|S(x) - S(y)| = C|x - y| \text{ for all } x, y \in \mathbb{R}^n \text{ and } 0 < C < 1.$$

Problem 13. [5 marks] Find a IFS of $S_1, S_2, \cdots, S_m$ such that:

$$\mathcal{C} = \bigcup_{i=1}^{m} S_m(\mathcal{C}).$$

Hint: Consider **Definition I.5** for **Cantor set**.

Solution.

We define the iterated function system $\{S_1, S_2\}$ where $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ as:

$$S_1(x) = \frac{1}{3}x \qquad \text{and} \qquad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Here, we can consider $S_1(\mathcal{C})$ and $S_2(\mathcal{C})$ as the left and right halves of $\mathcal{C}$, *i.e.*, $\mathcal{C} = S_1(\mathcal{C}) \cup S_2(\mathcal{C})$.

Here, we introduce another condition in which we need to have to find the **Hausdorff dimension** of a set that is invariant under IFS.

Definition III.6. Open Set Condition.

A collection of IFS, $S_1, S_2, \cdots, S_m$, satisfied the open set condition if there exists a bounded open set $V \in \mathcal{B}(\mathbb{R}^n)$ such that:

$$V \supset \bigcup_{i=1}^{m} S_m(V).$$

When a IFS satisfied the **open set condition**, we may conclude with its dimension.

Theorem III.7. Open Set Condition $\implies$ Conclusion of Dimensions.

Suppose that open set condition (**Definition III.6**) holds for the IFS $\{S_1, \cdots, S_m\}$, the set is invariant of the IFS, *i.e.*:

$$F = \bigcup_{i=1}^{m} S_i(F),$$

and all $S_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfy that:

$$|S_i(x) - S_i(y)| = C_i|x - y| \text{ for all } x, y \in \mathbb{R}^n \text{ and } 0 < C_i < 1,$$

then $\dim_{\mathrm{H}} F = s$, where s is given by:

$$\sum_{i=1}^{m} C_i^s = 1.$$

Problem 14. [5 marks] Use **Theorem III.7** to prove that $\dim_{\mathrm{H}}(\mathcal{C}) = \log_3(2)$.

Solution.
Here, we need to first verify that the IFS in **Problem 13** satisfies the open set condition, we consider $V = B_1(0) = (-1, 1)$, we have:

$$S_1(V) = B_{\frac{1}{3}}(0) = \left(-\frac{1}{3}, \frac{1}{3}\right) \qquad \text{and} \qquad S_2(V) = B_{\frac{1}{3}}\left(\frac{2}{3}\right) = \left(\frac{1}{3}, 1\right).$$

It is clear that these sets are still in $(-1, 1)$, so the set satisfies the open condition.
Then, we can clearly note that $C_1 = \frac{1}{3}$ and $C_2 = \frac{1}{3}$, so s satisfies that:

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1,$$

hence, we have $s = \log_3 2$, as desired. □

Of course, we can easily use the **Cartesian product** to construct $\mathcal{C}^n \subset \mathbb{R}^n$. Suppose that we have $\mathcal{C}^n \in \mathcal{B}(\mathbb{R}^n)$ (*you do not have to prove this*), we defined:

$$\mathcal{C}^n = \{(x_1, x_2, \cdots, x_n) : x_1, x_2, \cdots, x_n \in \mathcal{C}\}.$$

Problem 15. [6 marks] Find the Hausdorff dimension for any $\mathcal{C}^n$. Justify your answer.

Solution.
We may write the IFS for $\mathcal{C}^n$ as:

$$S_1(x) = \frac{1}{3}x, \qquad S_2 = \frac{1}{3} + \left(\frac{2}{3}, 0, \cdots, 0\right), \qquad S_{n^2} = \frac{1}{3}x + \left(\frac{2}{3}, \frac{2}{3}, \cdots, \frac{2}{3}\right),$$

where we can, more generally write it as:

$$S_i(x) = \frac{1}{3}x + x_i,$$

where x_i is any enumeration of $\left\{0, \frac{2}{3}\right\}^n$.
We can still let $V = B_1(0)$ such that $S_i(V) \subset V$, as there will be no points being mapped to any point having a larger radius by the triangle inequality.
Thus, their dimensions are respectively:

$$n^2\left(\frac{1}{3}\right)^s = \sum_{i=1}^{n^2}\left(\frac{1}{3}\right)^s = 1,$$

so we have:

$$\dim_{\mathrm{H}}(\mathcal{C}^n) = 2\log_3 2.$$

This marks the end of this chapter. While we dive deep from elementary mathematics to the world of *fractals*. Up to right now, you would be able to tackle on some unsolved problems in mathematics right now, such as the following example.

Remark. It is proven that the **$\log_3(2)$-dimensional Hausdorff measure** of $\mathcal{C}$ is 1, but the Hausdorff measure of the products of **Cantor set** $\mathcal{C}^n$ on their Hausdorff dimension is still unknown, and it is left for mathematicians, *like you*, to challenge and solve these open question! ♢

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