RES: Introduction to Fractal Geometry

Notes

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I Hausdorff Measure and Dimension

I.1 Hausdorff Measure

Definition I.1.1. Diameter.

Let $U \subset \mathbb{R}^n$ be non-empty, the *diameter* of U is defined as:

$$|U| := \sup\{|x-y| : x, y \in U\}.$$

In particular, we have $|\varnothing| = 0$. ⌟

The diameter is the greatest distance apart of any pair of points in U.

Definition I.1.2. δ-Cover.

If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover $F \subset \mathbb{R}^n$, *i.e.*, $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \le |U_i| \le \delta$ for each i, then $\{U_i\}$ is a *δ-cover* of F. ⌟

Definition I.1.3. s-dimensional Hausdorff Measure for Diameter δ.

Suppose F is a subset of $\mathbb{R}^n$ and s is a non-negative number, for any $\delta > 0$, we define:

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}.$$

⌟

The definition of the s-dimensional Hausdorff Measure is an immediate extension from above.

Definition I.1.4. s-dimensional Hausdorff Measure.

Suppose F is a subset of $\mathbb{R}^n$ and s is a non-negative number, the s-dimensional Hausdorff measure is:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

⌟

Here, the measure is well defined, as the limit exists.

Proof. Note that for any $\delta' < \delta$, any δ'-cover of F is a δ-cover of F. Hence, by the definition of infimum:

$$\mathcal{H}^s_\delta(F) \le \mathcal{H}^s_{\delta'}(F).$$

As $\delta \to 0$, the measure is monotonic, hence the limit always exists. □

On the other hand, we also want to show that $(\mathbb{R}^n, \mathcal{M}, \mathcal{H}^s)$ is a measure space.

Proof. Here, we simply want to show that $(\mathbb{R}^n, \mathcal{H}^s)$ is an outer measure space, by the following verification:

- Consider $\varnothing \subset \mathbb{R}^n$, it can be covered simply by sets of diameter 0, hence $\mathcal{H}^s(\varnothing) = 0$.
- Suppose $E_1 \subset E_2 \subset \mathbb{R}^n$, for all $\delta > 0$, a δ-cover of E_2 is a δ-cover of E_1, then we have $\mathcal{H}^s_\delta(E_1) \le \mathcal{H}^s_\delta(E_2)$, hence when $\delta \to 0$, we must have $\mathcal{H}^s(E_1) \le \mathcal{H}^s(E_2)$.
- Consider $E = \bigcup_{k=1}^{\infty} E_k$, where E_k's are subsets of $\mathbb{R}^n$, for all $\delta > 0$, the (countable) union of the (countable) sets for each δ-cover for E_i is a δ-cover of E, that is $\mathcal{H}^s_\delta(E) \le \sum_{k=1}^{\infty} \mathcal{H}^s_\delta(E_k)$, hence when $\delta \to 0$, we must have $\mathcal{H}^s(E) \le \sum_{k=1}^{\infty} \mathcal{H}^s(E_k)$.

With the above proven, $(\mathbb{R}^n, \mathcal{H}^s)$ is an outer measure, the collection of all (Carathéodory) measurable sets restricts the $\mathcal{H}$ to be a measure space. □

Remark I.1.5. Alternative Definition by Covering Balls.

The Hausdorff measure can be alternatively be defined as:

$$\mathcal{B}_\delta^s(F) = \inf\left\{\sum_i |B_i|^s : \{B_i\} \text{ is a } \delta\text{-cover of } F \text{ by balls}\right\},$$

where ad the measure is $\mathcal{B}^s(F) = \lim_{\delta\to 0} \mathcal{B}_\delta^s(F)$, and it is equivalent to the definition with diameters. ⌟

Proposition I.1.6. Hausdorff Measure is Metric Outer Measure.

Let $E, F \subset \mathbb{R}^n$ be such that $d(E,F) := \inf\{d(x,y) : x \in E, y \in F\} > 0$, we have:

$$\mathcal{H}^s(E \sqcup F) = \mathcal{H}^s(E) + \mathcal{H}^s(F).$$

Proof. Note that when $\delta < d(E,F)$, for any δ-cover, say $\mathcal{S}$, we can do the following partition:

$$\mathcal{E} = \{U \in \mathcal{S} : U \cap E \neq \emptyset\} \text{ and } \mathcal{F} = \{U \in \mathcal{S} : U \cap F \neq \emptyset\},$$

in which we are guaranteed that $\mathcal{E}$ is a δ-cover of E and $\mathcal{F}$ is a δ-cover of F. More importantly, by $\delta < d(E,F)$, we must have $\mathcal{E}$ and $\mathcal{F}$ being disjoint. Now, the following inequality holds:

$$\sum_{\tilde{E}\in\mathcal{E}} \tilde{E} + \sum_{\tilde{F}\in\mathcal{F}} \tilde{F} \le \sum_{\tilde{S}\in\mathcal{S}} \tilde{S},$$

since there could be sets in $\mathcal{S}$ that does not intersect E nor F. Therefore, by definition of infimum, we may conclude that:

$$\mathcal{H}_\delta^s(E) + \mathcal{H}_\delta^s(F) \le \sum_{\tilde{S}\in\mathcal{S}} \tilde{S}.$$

Note that the cover $\mathcal{S}$ is arbitrary, so we conclude that:

$$\mathcal{H}_\delta^s(E) + \mathcal{H}_\delta^s(F) \le \mathcal{H}_\delta^s(E \sqcup F).$$

Since this holds for $\delta > 0$, it holds for the limit as well, that is:

$$\mathcal{H}^s(E) + \mathcal{H}^s(F) \le \mathcal{H}^s(E \sqcup F).$$

By the nature of outer measure, countable stability implies the other direction of inequality, hence:

$$\mathcal{H}^s(E) + \mathcal{H}^s(F) = \mathcal{H}^s(E \sqcup F),$$

completing the proof for metric outer measure. □

Remark I.1.7. Borel Sets are (Hausdorff) Measurable.

$\mathcal{H}^s$ satisfies all the properties of a metric Carathéodory outer measure, hence it is countably additive measure when restricted to the Borel sets. Thus we restrict to Borel sets to be the set of (Carathéodory) measurable set. ⌟

Notice that the measure space for Hausdorff measure and Lebesgue measure has the same measurable space being Borel subsets of $\mathbb{R}^n$. Especially given their similarities in definition, one could assume similarities between them.

Proposition I.1.8. Hausdorff Measure corresponding to Lebesgue Measure.

Suppose $F \subset \mathbb{R}^n$ is a Borel set, then the n-dimensional Hausdorff measure is a constant multiple of

Lebesgue measure, *i.e.*:

$$\mathcal{H}^n(F) = c_n^{-1} m_{\mathbb{R}^n}(F),$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. Notice that for Lebesgue measure, we have a (equivalent) definition using balls to cover. Note that the balls can also be sub-divided into smaller balls, thus:

$$\begin{aligned} m_{\mathbb{R}^n}(F) &= \inf\left\{\sum_i v(B_i) : \{B_i\} \text{ is a cover of } F\right\} \\ &= \inf\left\{\sum_i c_n |B_i|^n : \{B_i\} \text{ is a } \delta\text{-cover of } F\right\} \\ &= c_n \inf\left\{\sum_i |B_i|^n : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\} = c_n \mathcal{H}^n(F), \end{aligned}$$

hence the equality holds immediately. □

At the same time, Hausdorff Measures has many special properties for nice sets in special cases.

Remark I.1.9. Special Cases of Hausdorff Measures.

The following holds for specific dimensions of Hausdorff Measures:

(i) For hyper-planes (or lower-dimensional subsets) of $F \subset \mathbb{R}^n$, $\mathcal{H}^0(F)$ is the number of points in F.

(ii) Let $F \subset \mathbb{R}^n$ be a smooth curve, $\mathcal{H}^1(F)$ gives its length.

(iii) Let $F \subset \mathbb{R}^n$ be a smooth manifold, $\frac{\pi}{4} \cdot \mathcal{H}^2(F)$ gives its length.

Likewise, for $F \subset \mathbb{R}^n$ being a smooth m-dimensional submanifold of $\mathbb{R}^n$, then $c_m \mathcal{H}^m(F)$ gives its volume, where c_m is the volume of unit ball in $\mathbb{R}^m$. ⌟

Theorem I.1.10. Scaling Property for Hausdorff Measure.

Let S be a similarity transformation of scale factor $\lambda > 0$. Suppose $F \subset \mathbb{R}^n$, then:

$$\mathcal{H}^s\big(S(F)\big) = \lambda^s \mathcal{H}^s(F).$$

Proof. Suppose $\{U_i\}$ is a δ-cover of F, then $\{S(U_i)\}$ is a $\lambda\delta$-cover of $S(F)$, and:

$$\sum_i |S(U_i)|^s = \lambda^s \sum_i |U_i|^s,$$

therefore, by taking the infimum of all covers, we have:

$$\mathcal{H}^s_{\lambda\delta}\big(S(F)\big) \leq \lambda^s \mathcal{H}^s_\delta(F).$$

Now, we take the limit that $\delta \to 0$, we must have:

$$\mathcal{H}^s_{\lambda\delta}\big(S(F)\big) \leq \lambda^s \mathcal{H}^s(F).$$

For the other inequality, we replace S by S^{-1}, which implies that $\{S^{-1}(U_i)\}$ is a $\lambda^{-1}\delta$-cover of $S^{-1}(F)$, and as well as the following equality:

$$\sum_i |S^{-1}(U_i)|^s = \lambda^{-s} \sum_i |U_i|^s,$$

and therefore if $\{S(U_i)\}$ is a δ-cover of F, then $\{S(S^{-1}(U_i))\} = \{U_i\}$ is a $\lambda^{-1}\delta$-cover of $S(F)$, and by taking the infimum of all covers, we have:

$$\mathcal{H}^s_{\delta/\lambda}\big(S^{-1}(F)\big) \leq \lambda_\delta^{-s}\mathcal{H}^s(F).$$

Again, we take the limit that $\delta \to 0$, which results in:

$$\lambda^s\mathcal{H}^s\big(S^{-1}(F)\big) \leq \mathcal{H}^s(F).$$

With both inequalities shown, we may conclude the equality. □

Theorem I.1.11. Map of Hölder Condition $\Longrightarrow$ Bounded Measure after Mapping.

Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping satisfying Hölder condition of exponent α, *i.e.*:

$$|f(x) - f(y)| \leq c|x-y|^\alpha,$$

for $x, y \in F$ and constants $c > 0$ and $\alpha > 0$. Then, for each s, we have:

$$\mathcal{H}^{s/\alpha}\big(f(F)\big) \leq c^{s/\alpha}\mathcal{H}^s(F).$$

Proof. If $\{U_i\}$ is a δ-cover of F, then $\{F \cap U_i\}$ is still a cover, and we note that:

$$|f(F \cap U_i)| \leq c|F \cap U_i|^\alpha \leq c|U_i|^\alpha,$$

therefore, $\{f(F \cap U_i)\}$ is a $c\delta^\alpha$ cover of $f(F)$. Hence, we obtain the relationship that:

$$\sum_i |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha}\sum_i |U_i|^s.$$

Thence, by the definition of infimum, we obtain that:

$$\mathcal{H}^{s/\alpha}_{c\delta^\alpha}\big(f(F)\big) \leq c^{s/\alpha}\mathcal{H}^s_\delta(F).$$

Note that $\delta \to 0$, we also have that $c\delta^\alpha \to 0$, as desired. □

The Hölder condition of exponent α could lead to the following immediate results with specific cases.

Remark I.1.12. Lipschitz Mapping of Hausdorff Measure.

When $\alpha = 1$, *i.e.*:

$$|f(x) - f(y)| \leq c|x-y|,$$

for $x, y \in F$ and constant c, we have that:

$$\mathcal{H}^s\big(f(F)\big) \leq c^s\mathcal{H}^s(F).$$

Also, any *differentiable* functions with bounded derivatives is Lipschitz, hence satisfying the above inequality as an immediate result of the *mean value theorem*. ⌟

Remark I.1.13. Isometry $\Longrightarrow$ Invariant in Hausdorff Measure.

If f is an isometry, then:

$$\mathcal{H}^s\big(f(F)\big) = \mathcal{H}^s(F),$$

as an immediate result that $|f(x) - f(y)| = |x-y|$ for all $x, y \in F$. ⌟

Hausdorff measure is *translation invariant* and *rotation invariant* as translations and rotations are isometries.

I.2 Hausdorff Dimension

Prior to introducing the Hausdorff dimension, we need a few more properties on Hausdorff measure.

Proposition I.2.1. Monotonicity for Increasing s.

For any $F \subset \mathbb{R}^n$, $\mathcal{H}^s(F)$ is non-increasing with increasing s.

Proof. Let $\delta < 1$ be fixed, $\mathcal{H}^s_\delta(F)$ is non-increasing with increasing s, as for all $0 \le \gamma \le \delta < 1$, we have $\gamma^s = \exp\big(\log(\gamma)s\big)$, which is a monotonically decreasing function, since $\log(\gamma) < 0$. Hence, as $\delta \to 0$, the monotonicity still holds. $\square$

Proposition I.2.2. Dimension of Finite Measure $\implies$ Lower Dimensions have Zero Measure.

Suppose $F \subset \mathbb{R}^n$ and $\mathcal{H}^s(F) < \infty$. If $t > s$, then $\mathcal{H}^t(F) = 0$.

Proof. Let $\{U_i\}$ be a δ-cover of F, we then have:

$$\sum_i |U_i|^t \le \sum_i |U_i|^{t-s}|U_i|^s \le \delta^{t-s}\sum_i |U_i|^s.$$

Hence, by the definition of infimum, we immediately have:

$$\mathcal{H}^t_\delta(F) \le \delta^{t-s}\mathcal{H}^s_\delta(F).$$

Note that when $\delta \to 0$ and if $\mathcal{H}^s(F)$ is bounded, then $\mathcal{H}^t(F) = 0$. $\square$

Remark I.2.3. Graph of $\mathcal{H}^s(F)$ against s.

The graph of $\mathcal{H}^s(F)$ against s has some "jump" discontinuity from ∞ to 0, as there could only be at most one s with non-zero and finite Hausdorff measure. If there are more than one point whose measure is within $(0, \infty)$, the Dimension of Finite Measure (Proposition I.2.2) would incur a contradiction that the smaller s has to be zero. Therefore, the graph could only have one "jump" discontinuity. ⌟

Hence, this leads to the definition of the Hausdorff dimension.

Definition I.2.4. Hausdorff Dimension.

Let $F \subset \mathbb{R}^n$ be arbitrary, the Hausdorff dimension, denoted $\dim_{\mathrm{H}} F$, is:

$$\dim_{\mathrm{H}} F = \inf\{s \ge 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

In particular, the supremum of the empty set is 0. ⌟

With the definition of Hausdorff Dimension, we have:

$$\mathcal{H}^s(F) = \begin{cases} \infty, & \text{if } 0 \le s < \dim_{\mathrm{H}} F, \\ 0, & \text{if } s > \dim_{\mathrm{H}} F. \end{cases}$$

When $s = \dim_{\mathrm{H}} F$, then $\mathcal{H}^s(F)$ may be any non-negative number or infinity. Hence, the graph looks like:

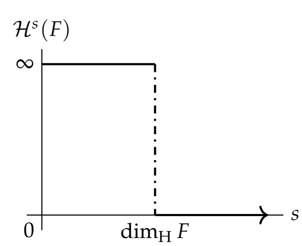

Figure I.1. Graph of $\mathcal{H}^s(F)$ against F with jump at its dimension.

Definition I.2.5. ***s*-Set.**

A Borel set $F \subset \mathbb{R}^n$ is a s-set if:

$$0 < \mathcal{H}^s(F) < \infty,$$

when $s = \dim_{\mathrm{H}} F$. ⌟

In general, s-sets have nice properties and most sets are s-sets.

Proposition I.2.6. Properties of Hausdorff Dimension.

Hausdorff dimension satisfies the following properties:

(i) *Monotonicity*. If $E \subset F$, then $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$.

(ii) *Countably stability*. Let $\{F_i\}$ be a countable sequence of sets, then $\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty}\{\dim_{\mathrm{H}} F_i\}$.

(iii) *Countably sets*. If F is countable, then $\dim_{\mathrm{H}} F = 0$.

(iv) *Open sets*. If $G \subset \mathbb{R}^n$ is a non-empty, open set, then $\dim_{\mathrm{H}} F = n$.

(v) *Smooth sets*. If F is a smooth (*i.e.*, continuously differentiable) m-dimensional sub-manifold (*i.e.*, m-dimensional surface) of $\mathbb{R}^n$, then $\dim_{\mathrm{H}} F = m$.

Proof. (i) *Monotonicity*: It is trivially an result of monotonicity of measure, *i.e.*, $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$.

(ii) *Countably stability*: By (i) and $\bigcup_{i=1}^{\infty} F_i \supset F_i$ for all i, then $\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i \leq \dim_{\mathrm{H}} F_i$ for all i, so does the supremum, therefore implying that $\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i \leq \sup_{1 \leq i < \infty}\{\dim_{\mathrm{H}} F_i\}$. For the other inequality, suppose that $s = \sup_{1 \leq i < \infty}\{F_i\} > \dim_{\mathrm{H}} F_i$ for all i, then $\mathcal{H}^s(F_i) = 0$ for all i, thus $\mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i) = 0$ by countable stability, thus $\mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) = 0$, and $\dim_{\mathrm{H}}(\bigcup_{i=1}^{\infty} F_i) \leq \sup_{1 \leq i < \infty}\{F_i\}$. Hence the equality holds.

(iii) *Countable sets*: We can write a countable set as the (countable) union of a sequence of single points. Note that each single point has dimension 0 (as $\mathcal{H}^0(\{p\}) = 0$), and by (ii), we know that the (countable) union must have dimension 0.

(iv) *Open sets*: For a non-empty, open set $G \subset \mathbb{R}^n$, there exists a open ball $B \subset G$ in which B has positive n-dimensional volume, so $\dim_{\mathrm{H}} G \geq \dim_{\mathrm{H}} B \geq n$. On the other hand, since G is contained in a (countable) union of open balls with dimension n, we must have $\dim_{\mathrm{H}} G \leq n$.

(v) *Smooth sets*: For F being a smooth sub-manifold, its Lebesgue measure is non-zero and finite in terms of $m_{\mathbb{R}^m}$. Therefore, by the relationship between Hausdorff measure and Lebesgue measure, $0 < \mathcal{H}^m(F) < \infty$, hence $\dim_{\mathrm{H}} F = m$. □

Theorem I.2.7. Hausdorff Dimension with Hölder Condition.

Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies a Hölder condition, *i.e.*:

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

for $x, y \in F$ and constants c and α. Then $\dim_{\mathrm{H}} f(F) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} F$.

Proof. If $s > \dim_{\mathrm{H}} F$, then by Hölder's condition for Hausdorff measure, we have:

$$\mathcal{H}^{s/\alpha}\big(f(F)\big) \le c^{s/\alpha}\mathcal{H}^{s}(F) = 0.$$

Thus, we must have $\mathcal{H}^{s/\alpha}\big(f(F)\big) = 0$, which implies that $\dim_{\mathrm{H}} f(F) \le s/\alpha$ for all $s > \dim_{\mathrm{H}} F$. Thus, the inequality holds when $s \to \dim_{\mathrm{H}} F$. □

Proposition I.2.8. Hausdorff Dimension after Lischitz Transformation.

Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ is a transformation.

(i) If f is Lipschitz transformation, *i.e.*:

$$|f(x) - f(y)| \le c|x - y|,$$

for $x, y \in F$ and constant c, then $\dim_{\mathrm{H}} f(F) \le \dim_{\mathrm{H}} F$.

(ii) If f is bi-Lipschitz transformation, *i.e.*:

$$c_1|x - y| \le |f(x) - f(y)| \le c_2|x - y|,$$

for $x, y \in F$ and constants $0 < c_1 \le c_2 < \infty$, then $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$.

Proof. (i) An immediate result from the Hölder condition when $\alpha = 1$.

(ii) As we apply $f^{-1} : f(F) \to F$, (i) gives that $\dim_{\mathrm{H}} F \le \dim_{\mathrm{H}} f(F)$, so the equality holds. □

Remark I.2.9. Bi-Lipschitz Mapping is Invariant under Hausdorff Dimension.

Hausdorff dimension is invariant under bi-Lipschitz transformations. Thus, if two sets have different dimensions, there cannot be a bi-Lipschitz mapping from one onto another. ⌟

Topologically, two sets are the "same" if there is a homeomorphism between them, and a bi-Lipschitz mapping are typically between fractals.

Proposition I.2.10. Hausdorff Dimension $< 1 \implies$ Total Disconnectedness.

A set $F \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}} F < 1$ is totally disconnected.

Proof. Let x and y be distinct points of F. Defined a map $f : \mathbb{R}^n \to [0, \infty)$ by $f(t) = |t - x|$. Note that f does not increase distances:

$$|f(t) - f(s)| = \big||t - x| - |s - x|\big| \le |(t - x) - (s - x)| = |t - s|.$$

Therefore, it satisfies the Lipschitz condition with $c = 1$, thus:

$$\dim_{\mathrm{H}} f(F) \le \dim_{\mathrm{H}} F < 1.$$

Therefore, $f(F)$ is a subset of $\mathbb{R}$, in which $\mathcal{H}^1\big(f(F)\big)$ is zero. Let $r \notin f(F)$ such that $0 < r < f(y)$, we construct F in two disjoint open sets:

$$F = \{t \in F : |t - x| < r\} \sqcup \{s \in F : |s - x| > r\}.$$

Therefore, x and y lie in different connected components of F. □

Example I.2.11. Hausdorff Dimension of $1/3$ Cantor Set.

Let $\mathcal{C}$ be the middle $1/3$ Cantor set, it splits into a left and right part, *i.e.*:

$$\mathcal{C}_L = \mathcal{C} \cap \left[0, \frac{1}{3}\right] \text{ and } \mathcal{C}_R = \mathcal{C} \cap \left[\frac{2}{3}, 0\right].$$

We note that both parts are geometrically similar to $\mathcal{C}$ but scaled by $1/3$. Moreover, they are disjoint and $\mathcal{C} = \mathcal{C}_L \sqcup \mathcal{C}_R$. Moreover, we have $d(\mathcal{C}_L, \mathcal{C}_R) \geq 1/3$, so they satisfies the metric outer measure. Therefore, by the scaling property, we have:

$$\mathcal{H}^s(\mathcal{C}) = \mathcal{H}^s(\mathcal{C}_L) + \mathcal{H}^s(\mathcal{C}_R) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(\mathcal{C}) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(\mathcal{C}).$$

Here, we *claim* that at $s = \dim_{\mathrm{H}} F$, $\mathcal{C}$ is a s-set, *i.e.*, we have $0 < \mathcal{H}^s(F) < \infty$. Hence, we get $1 = 2 \cdot \left(\frac{1}{3}\right)^s$ or $s = \log 2 / \log 3$. ⌟

Proof of claim. For simplicity of notation, we denote the interval as E_k being constructed of $\mathcal{C}$ on level k, *i.e.*, E_k consists of 2^k level-k intervals each of length 3^{-k}.
To verify that $\mathcal{C}$ is a s-set, we must verify that $\mathcal{H}^s(\mathcal{C})$ is finite and non-zero.

- For finite: Consider 3^{-k}-cover of $\mathcal{C}$ on E_k, we have:
  $$\mathcal{H}^s_{3^{-k}}(\mathcal{C}) \leq 2^k 3^{-ks} = 1.$$
  As $k \to \infty$, we have $3^{-ks} \to 0$ and thus $\mathcal{H}^s(F) \leq 1$.

- For non-zero: We want to show that $\mathcal{H}^s(\mathcal{C}) \geq 1/2$, in which we want to show that:
  $$\sum_i |U_i|^s \geq \frac{1}{2} = 3^{-s} \text{ for any cover } \{U_i\} \text{ of } \mathcal{C}.$$
  Without loss of generality, we may assume that $\{U_i\}$ is a collection of intervals. Note that $\mathcal{C}$ is compact and we can expand each U_i by any $\varepsilon > 0$ so that $\{\widetilde{U_i}\}$ is a open cover of $\mathcal{C}$, hence it can be reduced to a finite sub-cover covering $\mathcal{C}$. Therefore, we only need to concern about the case when $\{U_i\}$ is consisted of a finite collection of closed sub-intervals of $[0, 1]$.
  Thus, for each U_i, we let k be the integer such that:
  $$3^{-(k+1)} \leq |U_i| < 3^{-k},$$
  and as all level-k intervals are separated by 3^{-k}, so U_i can intersect at most one level-k interval. Here, we let $j \leq k$, by construction, U_i intersects at most 2^{j-k} intervals of E_j, and note that:
  $$2^{j-k} = 2^j 2^{-k} = 2^j 3^{-sk} \leq 2^j 3^s |U_i|^s.$$
  Since we can select j to be large enough so that $3^{-(j+1)} \leq |U_i|$ for all i, thus we have that $\{U_i\}$ intersecting all 2^j intervals of length 3^{-j}, which by counting implies that:
  $$2^j \leq \sum_i 2^j 3^s |U_i|^s = 2^j 3^s \sum_i |U_i|^s,$$
  which reduces to $\sum_i |U_i|^s \geq 3^{-s} = 1/2$, as desired.

Note that we have shown that $1/2 \leq \mathcal{H}^1(\mathcal{C}) \leq 1$, we verified that $\mathcal{C}$ is a s-set. □

II Box-counting Dimension

II.1 Dimension

Remark II.1.1. Goal of Dimension.

The definition of dimension is the idea of "measurement at scale δ", *i.e.*, ignoring the irregularities of size less than δ. In particular we see how the measurements behave as $\delta \to 0$. ⌟

Example II.1.2. Divider Dimension s.

Let $M_\delta(F)$ be a measurement of F with δ being a parameter, we want to find the power law in which:

$$M_\delta(F) \sim c\delta^{-s},$$

where c and s are constants. In particular, s is the divider dimension and c is the s-dimensional length of F, which gives that:

$$\log M_\delta(F) \simeq \log c - s \log \delta,$$

and we let $\delta \to 0$, we obtain that:

$$s = \lim_{\delta \to 0} \frac{\log M_\delta(F)}{-\log \delta}.$$

⌟

Recall that Hausdorff dimension are typical properties of dimension, *i.e.*, Monotonicity, (Countable) Stability, Isometry Invariance, Lipschitz Invariance, Countable Sets, Open Sets, and Smooth Manifolds.

Remark II.1.3. Properties on Dimensions.

All definitions of dimension should be monotonic, most are stable, whereas countable conditions could often fail. In terms of invariance, the usual dimensions shall follow them. The open sets and smooth manifolds ensures that the dimension is an extension of the classical definition. ⌟

II.2 Box-counting Dimension

Definition II.2.1. Box-counting Dimension.

Let $F \subset \mathbb{R}^n$ be non-empty and bounded, and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower and upper box-counting dimensions of F, respectively, are:

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$
$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

If the lower and upper box-counting dimensions of F are equal, we defined the box-counting dimension of F as:

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

⌟

Proposition II.2.2. Equivalent Definitions to Box-counting Dimensions.

All definitions of $N_\delta(F)$ below are equivalent in terms of the box-counting dimension if it exists:

(i) The smallest number of closed balls of radius δ that cover F.

(ii) The smallest number of cubes of side δ that cover F.

(iii) The number of δ-mesh cubes (cubes with coordinates of vertices being integers multiples of δ) that intersect F.

(iv) The smallest number of sets of diameter at most δ that cover F.

(v) The largest number of disjoint balls of radius δ with centers in F.

Proof. (iii) $\iff$ (iv). Let $N_\delta(F)$ denote the number by (iv) and $N'_\delta(F)$ denote the number by (iii). $N'_\delta(F)$ immediately provide a collection of $N'_\delta(F)$ sets of diameter $\delta\sqrt{n}$ that cover F, so:

$$N_{\delta\sqrt{n}}(F) \leq N'_\delta(F).$$

Hence if $\delta\sqrt{n} < 1$, then:

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \leq \frac{\log N'_\delta(F)}{-\log\sqrt{n} - \log\delta},$$

and by taking $\delta \to 0$, we obtain that:

$$\begin{aligned}\underline{\dim}_{\mathrm{B}} F &\leq \liminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta},\\ \overline{\dim}_{\mathrm{B}} F &\leq \limsup_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta}.\end{aligned}$$

Moreover, by Vitali's Lemma:

$$N'_\delta(F) \leq 3^n N_\delta(F),$$

and by $\delta \to 0$, we can get that:

$$\begin{aligned}\underline{\dim}_{\mathrm{B}} F &\geq \liminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta},\\ \overline{\dim}_{\mathrm{B}} F &\geq \limsup_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta}.\end{aligned}$$

(ii) $\iff$ (iv). By a similar argument as above, we eventually have the radius being $\delta\sqrt{n}$, which is proven to be the same in terms of the limit of logarithmic.
(i) $\iff$ (iv). Similar argument with alternative definition of Hausdorff measure, as the infimum of the balls are the same as diameters as balls occupies the most for the same diameter.
(iv) $\iff$ (v). Again, we denote $N'_\delta(F)$ by (v), thus we have a sequence of disjoint balls:

$$\{B_1, \cdots, B_{N'_\delta(F)}\}$$

of radius δ and centers in F, which results in:

$$N_{4\delta}(F) \leq N'_\delta(F).$$

Again, let $4\delta < 1$, then:

$$\frac{\log N_{4\delta}(F)}{-\log(4\delta)} \leq \frac{\log N'_\delta(F)}{-\log 4 - \log\delta},$$

and by taking $\delta \to 0$, we obtain that:

$$\underline{\dim}_{\mathrm{B}} F \leq \liminf_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta},$$
$$\overline{\dim}_{\mathrm{B}} F \leq \limsup_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta}.$$

Moreover, we consider $\{U_1, \cdots, U_k\}$ be any collection of sets of diameter at most δ which cover F. Since U_i has a smaller radius and it must cover the centers of B_i, each B_i must contain at least one of U_j. As B_i's are disjoint, there are at least as many U_j as B_j, so:

$$N'_\delta(F) \leq N_\delta(F).$$

By letting $\delta < 1$, then:

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log N_\delta(F)}{-\log \delta},$$

and by $\delta \to 0$, we can get that:

$$\underline{\dim}_{\mathrm{B}} F \geq \liminf_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta},$$
$$\overline{\dim}_{\mathrm{B}} F \geq \limsup_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta}.$$

Therefore, we have shown that all the above definitions of $N_\delta(F)$ are equivalent when the limit exists. $\square$

Geometrically, the definitions of $N_\delta(F)$ look as follows:

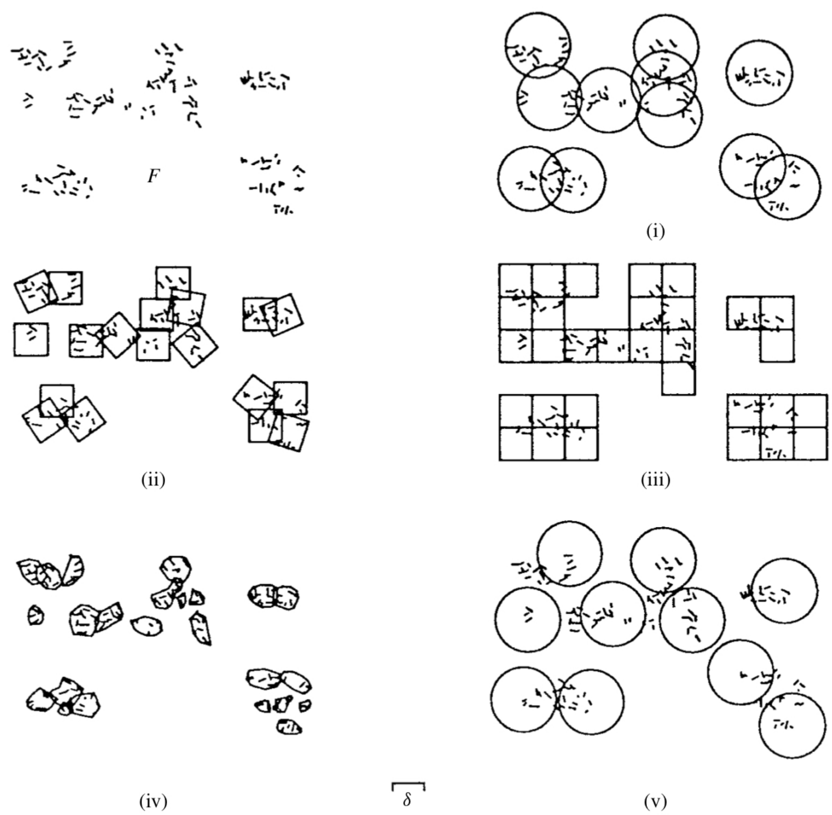

Figure II.1. Different definitions of $N_\delta(F)$.

Remark II.2.3. The Equivalent Definitions can be Extended.

The list of equivalent definitions can be extended with other definitions of $N_\delta(F)$. ⌟

Proposition II.2.4. Sequence Condition for $\limsup$ **or** $\liminf$**.**

It is sufficient to consider the limit as $\delta \to 0$ through any sequence $\{\delta_k\}$ such that $\delta_k \searrow 0$ and $\delta_{k+1} \geq c\delta_k$, for some constant $0 < c < 1$.

Proof. Here, consider $\delta_{k+1} \leq \delta < \delta_k$, then, with $N_\delta(F)$ being the least number of sets in a δ-cover of F, then:

$$\frac{\log N_\delta(F)}{-\log\delta} \leq \frac{\log N_{\delta_{k+1}}(F)}{-\log\delta_k} = \frac{\log N_{\delta_{k+1}}(F)}{-\log\delta_{k+1} + \log(\delta_{k+1}/\delta_k)} \leq \frac{\log N_{\delta_{k+1}}(F)}{-\log\delta_{k+1} + \log c}.$$

Therefore, we have:

$$\limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \leq \limsup_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k}.$$

Note that $\{\delta_k\}$ is naturally a limit (or subsequence if not considering countability), then we naturally have:

$$\limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \geq \limsup_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k}.$$

For the infimum, we have that:

$$\frac{\log N_\delta(F)}{-\log\delta} \geq \frac{\log N_{\delta_k}(F)}{-\log\delta_{k+1}} = \frac{\log N_{\delta_k}(F)}{-\log\delta_k - \log(\delta_{k+1}/\delta_k)} \geq \frac{\log N_{\delta_k}(F)}{-\log\delta_k - \log c}.$$

Therefore, we have:

$$\liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \geq \liminf_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k}.$$

Still, $\{\delta_k\}$ is naturally a limit (or subsequence if not considering countability), then we naturally have:

$$\liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \leq \liminf_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k}.$$

With all of the above relationship, we have proven the validity of the sequence. □

In particular, for some $0 < c < 1$, we can form the sequence $\{\delta_k\}$ such that $\delta_k = c^k$, which is the geometric sequence satisfying the above condition.

Definition II.2.5. δ-neighborhood.

Let $F \subset \mathbb{R}^n$ be a subset, we let:

$$F_\delta = \{x \in \mathbb{R}^n : |x - y| \leq \delta \text{ for some } y \in F\},$$

i.e., the set of points within distance δ of F. ⌟

Theorem II.2.6. Box-counting Dimension in relation to Volume.

Suppose $F \subset \mathbb{R}^n$ is bounded, then:

$$\underline{\dim}_B F = n - \limsup_{\delta\to 0} \frac{\log v_{\mathbb{R}^n}(F_\delta)}{\log\delta},$$

$$\overline{\dim}_B F = n - \liminf_{\delta\to 0} \frac{\log v_{\mathbb{R}^n}(F_\delta)}{\log\delta},$$

where F_δ is the δ-neighborhood of F.

Proof. Suppose that F can be covered by $N_\delta(F)$ balls of radius $\delta < 1$, then F_δ can be covered by those $N_\delta(F)$ with the same center and radius 2δ. Therefore, we obtain that:

$$v_{\mathbb{R}^n}(F_\delta) \leq N_\delta(F) c_n (2\delta)^n,$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$. By taking logarithmic, we have:

$$\frac{\log v_{\mathbb{R}^n}(F_\delta)}{-\log\delta} \leq \frac{\log(2^n c_n) + n\log\delta + \log N_\delta(F)}{-\log\delta},$$

and by letting $\delta \to 0$, we have:

$$\liminf_{\delta\to 0} \frac{\log v_{\mathbb{R}^n}(F_\delta)}{-\log\delta} \leq -n + \underline{\dim}_{\mathrm{B}} F,$$
$$\limsup_{\delta\to 0} \frac{\log v_{\mathbb{R}^n}(F_\delta)}{-\log\delta} \leq -n + \overline{\dim}_{\mathrm{B}} F.$$

Similarly, as we consider that there are $N_\delta(F)$ disjoint balls of radius δ with centers in F with equivalent definition (v) (Proposition II.2.2), then if we add their values, we have:

$$N_\delta(F) c_n \delta^n \leq v_{\mathbb{R}^n}(F_\delta).$$

By taking logarithmic, we obtain:

$$\frac{\log c_n + n\log\delta + \log N_\delta(F)}{-\log\delta} \leq \frac{\log v_{\mathbb{R}^n}(F_\delta)}{-\log\delta},$$

and by letting $\delta \to 0$, we have:

$$\liminf_{\delta\to 0} \frac{\log v_{\mathbb{R}^n}(F_\delta)}{-\log\delta} \geq -n + \underline{\dim}_{\mathrm{B}} F,$$
$$\limsup_{\delta\to 0} \frac{\log v_{\mathbb{R}^n}(F_\delta)}{-\log\delta} \geq -n + \overline{\dim}_{\mathrm{B}} F,$$

hence completing the proof. □

II.3 Properties of Box-counting Dimension

Proposition II.3.1. Monotonicity of Hausdorff and Box-counting Dimensions.

Suppose $F \subset \mathbb{R}^n$ is bounded, then:

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F.$$

Proof. Suppose F can be covered by $N_\delta(F)$ sets of diameter δ and $s := \dim_{\mathrm{H}} F$, then by definition, we have:

$$\mathcal{H}^s_\delta(F) \leq N_\delta(F)\delta^s.$$

Now, if $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) > 1$, then:

$$\log N_\delta(F) + s\log\delta > 0 \text{ when } \delta \text{ is sufficiently small.}$$

If it is no more than 1, the inequality holds, therefore, we have the first inequality holds:

$$s \leq \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta},$$

and the second inequality holds by the definition of infimum and supremum. □

Remark II.3.2. Hausdorff and Box-counting Dimensions in terms of Efficiency.

Hausdorff dimension assigns a different weight of $|U_i|^s$ to the covering sets whereas the box-counting dimension assigns δ^s which is same for all sets (considered more efficient). ⌟

Example II.3.3. Box-counting Dimension for Cantor Set.

Let $\mathcal{C}$ be the middle $1/3$ Cantor set. By the same notation for Hausdorff dimension, we denote the interval as E_k being constructed of $\mathcal{C}$ on level k, *i.e.*, E_k consists of 2^k level-k intervals each of length 3^{-k}.
Trivially, we can cover the 2^k level-k intervals of E_k, each with length 3^{-k} gives that $N_\delta(\mathcal{C}) \le 2^k$ if $3^{-k} < \delta \le 3^{-k+1}$. Thus, we have:

$$\overline{\dim}_{\mathrm{B}}\mathcal{C} = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \le \limsup_{k\to\infty} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}.$$

Meanwhile, any interval of length δ such that $3^{-(k+1)} \le \delta < 3^{-k}$ intersects at most one of the level-k intervals of length 3^{-k}. Hence, we need at least 2^k intervals of length δ to cover $\mathcal{C}$, so $N_\delta(\mathcal{C}) \ge 2^k$. Thus, we have:

$$\underline{\dim}_{\mathrm{B}}\mathcal{C} = \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \ge \liminf_{k\to\infty} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}.$$

By the nature of lim sup and lim inf, we can conclude the equality of the lower and upper box-counting dimension. ⌟

Proposition II.3.4. Properties of Box-counting Dimension.

The box-counting dimension obeys the following properties:

(i) Let F be a smooth m-dimensional sub-manifold of $\mathbb{R}^n$, then $\dim_{\mathrm{B}} F = m$.

(ii) $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ are monotonic.

(iii) $\overline{\dim}_{\mathrm{B}}$ is finitely stable, *i.e.*:

$$\overline{\dim}_{\mathrm{B}}(E \cup F) = \max\{\overline{\dim}_{\mathrm{B}}E, \overline{\dim}_{\mathrm{B}}F\},$$

and this does not hold for $\underline{\dim}_{\mathrm{B}}$.

(iv) $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ are bi-Lipschitz invariant.

Proof. (i) The m-dimensional sub-manifold can be considered as a δ-mesh with weight $|\delta|^s$, which corresponds to the Lebesgue measure as well.

(ii) Since we can always cover a subset by the same configuration of δ-cover, this implies that $N_\delta(F)$ is monotonic, hence the limits are monotonic.

(iii) First, note that $E \subset E \cup F$ and $F \subset E \cup F$, then it is clear that $\overline{\dim}_{\mathrm{B}}E \le \overline{\dim}_{\mathrm{B}}E \cup F$ and $\overline{\dim}_{\mathrm{B}}F \le \overline{\dim}_{\mathrm{B}}E \cup F$, thus:

$$\overline{\dim}_{\mathrm{B}}E \cup F \ge \max\{\overline{\dim}_{\mathrm{B}}E, \overline{\dim}_{\mathrm{B}}F\}.$$

For the other inequality, we note that:

$$N_\delta(E \cup F) \le N_\delta(E) + N_\delta(F) \le 2\max\{N_\delta(E), N_\delta(F)\}.$$

Again, for $\delta < 1$, by taking logarithmic, we obtain that:

$$\frac{\log N_\delta(E \cup F)}{-\log\delta} \le \frac{\log 2}{-\log\delta} + \max\left\{\frac{\log N_\delta(E)}{-\log\delta}, \frac{\log N_\delta(F)}{-\log\delta}\right\},$$

which leads to the other inequality when $\delta \to 0$ that $\overline{\dim}_{\mathrm{B}}E \cup F \le \max\{\overline{\dim}_{\mathrm{B}}E, \overline{\dim}_{\mathrm{B}}F\}$.

(iv) If $|f(x) - f(y)| \leq c|x - y|$ and F can be covered by $N_\delta(F)$ sets of diameter at most δ, then the $N_\delta(F)$ images of the sets under f form a cover of $f(F)$ of set with diameters at most $c\delta$, hence $\dim_B f(F) \leq \dim_B F$. □

Proposition II.3.5. Invariant under Closure.

Suppose $F \subset \mathbb{R}^n$, then:

$$\underline{\dim}_B \overline{F} = \underline{\dim}_B F \text{ and } \overline{\dim}_B \overline{F} = \overline{\dim}_B F.$$

Proof. Note that by equivalent definition (ii) (Proposition II.2.2), we can use number of closed balls to be $N_\delta(F)$. Hence, let $\{B_1, \cdots, B_k\}$ be a finite collection of closed balls of radius δ. If $\bigcup_{i=1}^k B_i \supset F$, then $\bigcup_{i=1}^k \supset \overline{F}$ as the finite union of closed balls is a closed set. Therefore, the smallest number of closed balls of radius δ to cover F and $\overline{F}$ are the same. □

Remark II.3.6. Closures of Box-counting Dimension.

The property with the closure implies that if F is a dense subset of an open region of $\mathbb{R}^n$, then $\underline{\dim}_B F = \overline{\dim}_B F = n$.
In particular, this incurs some issues:

(i) (Dense) countable sets (e.g. $\mathbb{Q} \subset \mathbb{R}$ and $\overline{\mathbb{Q}} = \mathbb{R}$) could have non-zero box dimension.

(ii) Countable stability fails, as countable union of singleton sets (e.g. $\mathbb{Q}$) has non-zero box dimension.

These issues make the box-counting dimension not aligning with the conventional dimensions. ⌟

Example II.3.7. Countable Set ($\{0, 1, 1/2, 1/3, \cdots\}$) having Non-zero Dimension.

Let $F = \{0, 1, 1/2, 1/3, \cdots\}$. We let $0 < \delta < 1/2$ and k be integer such that:

$$\frac{1}{k(k+1)} \leq \delta < \frac{1}{k(k-1)}.$$

For any $|U| \leq \delta$, then U can cover at most one of the points in $\{1, 1/2, \cdots, 1/k\}$, as $\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} > \delta$.
Thus, we need at least k sets of diameter δ to cover F, hence:

$$N_\delta(F) \geq k,$$

and by taking the logarithmic, we have:

$$\frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log k}{\log\big(k(k+1)\big)}.$$

Hereby, as $\delta \to 0$, we have $k \to \infty$, which gives that $\underline{\dim}_B F \geq 1/2$.
On the other hand, we $(k+1)$ interval of length δ cover $[0, 1/k]$, whereas the rest $k-1$ points need at most $k-1$ points to be covered, hence:

$$N_\delta(F) \leq 2k,$$

and by taking the logarithmic, we have:

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log(2k)}{\log\big(k(k-1)\big)}.$$

Again, as $\delta \to 0$, we have $k \to \infty$, which gives that $\overline{\dim}_B F \leq 1/2$.
Therefore, by the nature of supremum and infimum, we have that $\dim_B F = 1/2$. ⌟

Example II.3.8. Construction of Set with Different Upper and Lower Dimension.

Consider the construction of $\widetilde{\mathcal{C}}$, similar to the construction of the Cantor set $\mathcal{C}$, but for each stage k:

- If $10^{2n} < k \leq 10^{2n+1}$, we remove the middle $1/3$ interval.
- If $10^{2n-1} < k \leq 10^{2n}$, we remove the middle $3/5$ interval.

Note that with $\delta_k = 3^{-k}$, and since for the k-level construction, each interval is no larger than 3^{-k}, which implies that each U_i such that $|U_i| < \delta$ can cover at most one interval, hence $N_\delta \leq 2^k$, and so by taking logarithmic:

$$\overline{\dim}_{\mathrm{B}}\widetilde{\mathcal{C}} = \limsup_{\delta\to 0} \frac{\log N_\delta(\widetilde{\mathcal{C}})}{-\log\delta} \leq \limsup_{k\to\infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3}.$$

Then, if we choose $\delta_k = 5^{-k}$, and since for the k-level construction, each separation is no less than 5^{-k}, which implies that each U_i such that $|U_i| < \delta$ can cover at most one interval, hence $N_\delta \geq 2^k$, and so by taking logarithmic:

$$\underline{\dim}_{\mathrm{B}}\widetilde{\mathcal{C}} = \liminf_{\delta\to 0} \frac{\log N_\delta(\widetilde{\mathcal{C}})}{-\log\delta} \geq \liminf_{k\to\infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 5}.$$

Now, we want to construct the following sequence (corresponding to the construction above):

$$\{a_n\} = \left\{\frac{1}{3}, \frac{1}{3^2}, \dots, \frac{1}{3^{10}}, \frac{1}{5^{11}}, \frac{1}{5^{12}}, \dots, \frac{1}{5^{100}}, \frac{1}{3^{101}}, \dots\right\}.$$

We suppose that the upper and lower dimensions are the same, then $\dim_{\mathrm{B}}\widetilde{\mathcal{C}}$ is well defined, *i.e.*:

$$\overline{\dim}_{\mathrm{B}}\widetilde{\mathcal{C}} = \underline{\dim}_{\mathrm{B}}\widetilde{\mathcal{C}}.$$

Then, all subsequences must converge for the limit to exists. However, with the example of $1/3^k$ and $1/5^k$, we know that the limit cannot exist, hence the upper and lower box dimensions cannot be the same. ⌟

II.4 Modified Box-counting Dimensions

Definition II.4.1. Modified Box-counting Dimension.

For any set $F \subset \mathbb{R}^n$, we can try to decompose F into a countable number of pieces $\{F_i\}_{i\in\mathbb{Z}^+}$ such that $F \subset \bigcup_{i=1}^\infty F_i$ we find the largest piece with as small as possible:

$$\underline{\dim}_{\mathrm{MB}} = \inf\left\{\sup_i \underline{\dim}_{\mathrm{B}} F_i : F \subset \bigcup_{i=1}^\infty F_i\right\},$$
$$\overline{\dim}_{\mathrm{MB}} = \inf\left\{\sup_i \overline{\dim}_{\mathrm{B}} F_i : F \subset \bigcup_{i=1}^\infty F_i\right\}.$$

In particular, the infimum is over all possible countable covers $\{F_i\}$ of F. ⌟

Proposition II.4.2. Monotonicity of Modified Box-counting Dimensions.

For any subset $F \subset \mathbb{R}^n$:

$$0 \leq \dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{MB}} F \leq \overline{\dim}_{\mathrm{MB}} F \leq \overline{\dim}_{\mathrm{B}} F \leq n.$$

Proof. The second inequality is by the construction of (lower) box dimension larger than the Hausdorff dimension. The third inequality is by the monotonicity of upper and lower box dimension. The forth inequality is by noting that the infimum cannot be larger than one of the construction $\{F, \emptyset, \emptyset, \cdots\}$ as a cover of F. □

Remark II.4.3. Properties of Modified Box-dimension is Good.

Modified Box-dimension follows all the desired properties of Dimensions (Remark 2.1.3). ⌟

Proposition II.4.4. Sufficient Condition for Same Box and Modified Box Dimensions.

Let $F \subset \mathbb{R}^n$ be compact. Suppose that $\overline{\dim}_{\mathrm{B}}(F \cap V) = \overline{\dim}_{\mathrm{B}} F$ for all open set V that intersects F. Then $\overline{\dim}_{\mathrm{MB}} F = \overline{\dim}_{\mathrm{B}} F$.
Likewise, suppose that $\underline{\dim}_{\mathrm{B}}(F \cap V) = \underline{\dim}_{\mathrm{B}} F$ for all open set V that intersects F. Then $\underline{\dim}_{\mathrm{MB}} F = \underline{\dim}_{\mathrm{B}} F$.

Proof. Here, we let $F \subset \bigcup_{i=1}^{\infty} F_i$ with each F_i being closed. Then, there exists an index i and an open set $V \subset \mathbb{R}^n$ such that $F \cap V \subset F_i$. Hence, for such i, $\overline{\dim}_{\mathrm{B}} F_i = \overline{\dim}_{\mathrm{B}} F$. Thus, we have:

$$\overline{\dim}_{\mathrm{MB}} F = \inf \left\{ \sup \overline{\dim}_{\mathrm{B}} F_i : F \subset \bigcup_{i=1}^{\infty} F_i \text{ where } F_i\text{'s are closed sets} \right\} \geq \overline{\dim}_{\mathrm{B}} F.$$

The other inequality is an immediate results of the Monotonicity of Modified Box-counting dimensions (Proposition II.4.2). The lower dimensions are treated the same. □

In particular, if F is a compact set with a high degree of self-similarity, and if V is any open set that intersects F, then $F \cap V$ contains a geometrically similar copy of F which must have upper box dimensions equal to F, leading to equal box and modified box dimension.

III Techniques for Calculating Dimensions

III.1 Mass Distribution Principle

Proposition III.1.1. Conclusion on Dimensions.

Suppose F can be covered by n_k sets of diameter at most δ_k in which $\delta_k \to 0$ as $k \to \infty$, then:

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \liminf_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

Moreover, if $n_k\delta_k^s$ remains bounded as $k \to \infty$, then $\mathcal{H}^s(F) < \infty$. If $\delta_k \to 0$ but $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then:

$$\overline{\dim}_{\mathrm{B}} F \leq \limsup_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

Proof. The first inequality is by Monotonicity of Hausdorff and Box-counting Dimension (Proposition II.3.1), and the latter two inequalities are by Sequence Condition for lim sup and lim inf (Proposition II.2.4).

Now, suppose that $n_k\delta_k^s$ is bounded, then $\mathcal{H}^s_{\delta_k} \leq n_k\delta_k^s$, hence $\mathcal{H}^s_{\delta_k} \to \mathcal{H}^s(F) < \infty$ as $k \to \infty$. □

Definition III.1.2. Mass Distribution.

A measure μ on a bounded subset of $F \subset \mathbb{R}^n$ such that $0 < \mu(F) < \infty$ is a mass distribution on F. ⌟

Theorem III.1.3. Mass Distribution Principle.

Let μ be a mass distribution on F and suppose that for some s, there exists $c > 0$ and $\varepsilon > 0$ such that:

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \varepsilon$. Then $\mathcal{H}^s(F) \geq \mu(F)/c$ and:

$$s \leq \dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F.$$

Proof. Here, let $\{U_i\}$ be an arbitrary cover of F, by the property of measure, we have:

$$0 < \mu(F) \leq \mu\left(\bigcup_i U_i\right) \leq \sum_i \mu(U_i) \leq c\sum_i |U_i|^s.$$

Note that the Hausdorff dimension is the infimum of sum for all covers (with δ small enough), we then have:

$$\mathcal{H}^s_\delta(F) = \inf \sum_i |U_i|^s \geq \frac{\mu(F)}{c}.$$

If we let $\delta \to 0$, we still have $\mathcal{H}^s(F) \geq \mu(F)/c$, and by $\mu(F) > 0$, we have $\dim_{\mathrm{H}} F \geq s$. □

Notice that $\mathcal{H}^s(F) \geq \mu(F)/c$ still holds if μ is a mass distribution on $\mathbb{R}^n$ and F is any subset, but the conclusion on Hausdorff dimension does not necessarily hold.

Example III.1.4. Lower Bound for Hausdorff Dimension of Cantor Set.

Let $\mathcal{C}$ be the middle 1/3 Cantor Set. We let μ be the natural mass distribution of $\mathcal{C}$, *i.e.*, have mass distributed evenly to each interval for each level of construction. Hence, each 2^k level-k intervals of length

3^{-k} of construction of $\mathcal{C}$ has mass 2^{-k}.
Here, let U be an arbitrary set such that $|U| < 1$, and let k be such that $3^{-(k+1)} \le |U| < 3^{-k}$. Thus, U can intersect at most one interval of the level-k construction, *i.e.*, E_k, hence:

$$\mu(U) \le 2^{-k} = (3^{\log 2/\log 3})^{-k} = (3^{-k})^{\log 2/\log 3} \le (3|U|)^{\log 2/\log 3}.$$

Hence, by the Mass distribution principle (Theorem 3.1.3), we know that:

$$\mathcal{H}^s(\mathcal{C}) \ge \frac{\mu(\mathcal{C})}{3^{\log 2/\log 3}} = \frac{1}{2}.$$

Therefore, we know that $\dim_{\mathrm{H}} \mathcal{C} \ge \log 2/\log 3$. ⌟

Note that finding the upper bound of the dimension is trivial, so we have $\dim_{\mathrm{H}} \mathcal{C} = \log 2/\log 3$.

Example III.1.5. Dimension of $\mathcal{C} \times [0,1]$.

Let $\mathcal{C}_1 = \mathcal{C} \times [0,1] \subset \mathbb{R}^2$ be the product of the middle $1/3$ Cantor set and the unit interval.
Here, we note that from the cantor set $\mathcal{C} \subset \mathbb{R}$, and for arbitrary k, there exists a covering of $\mathcal{C}$ by 2^k intervals of length 3^{-k}. Consider now covering a column for $\mathcal{C}_1$, we need $2^k 3^k$ squares of side length 3^{-k} hence diameter $3^{-k}\sqrt{2}$. Here, we let $s = 1 + \log 2/\log 3$, we have:

$$\mathcal{H}^s_{3^{-k}\sqrt{2}}(\mathcal{C}_1) \le 3^k 2^k (3^{-k}\sqrt{2})^s = (3 \cdot 2 \cdot 3^{-s})^k 2^{s/2} = 2^{s/2}.$$

Thus, as $k \to \infty$, *i.e.*, $3^{-k}\sqrt{2} \to 0$, we know that $\mathcal{H}^s(\mathcal{C}_1) \le 2^{s/2}$, and by definition of box-counting dimension, we have:

$$\overline{\dim}_{\mathrm{B}}\mathcal{C}_1 = \limsup_{\delta \to 0} \frac{\log N_\delta(\mathcal{C}_1)}{-\log \delta} \le \limsup_{k \to \infty} \frac{\log(2^k 3^k)}{-\log(3^{-k})} = \limsup_{k \to \infty} \frac{k(\log 2 + \log 3)}{k \log 3} = 1 + \frac{\log 2}{\log 3}.$$

Then, we define a mass distribution μ on $\mathcal{C}_1$ by the extending the definition of μ of $\mathcal{C}$ to be evenly spread to the height of the rectangle, *i.e.*, for a level-k interval of height $h \le 1$, say U, we let $\mu(U) = h2^{-k}$. Consider if $2^{-(k+1)} \le |U| < 3^{-k}$, then U lies above at most one level-k interval of $\mathcal{C}$ with side length 3^{-k}, thus:

$$\mu(U) \le |U| 2^{-k} \le |U| 3^{-k\log 2/\log 3} \le |U|(3|U|)^{\log 2/\log 3} = 2|U|^s.$$

Thus, by the Mass distribution principle (Theorem 3.1.3), we have:

$$\mathcal{H}^s(\mathcal{C}_1) > \frac{1}{2},$$

and correspondingly by:

$$s \le \dim_{\mathrm{H}} \mathcal{C}_1 \le \underline{\dim}_{\mathrm{B}}\mathcal{C}_1 \le \overline{\dim}_{\mathrm{B}}\mathcal{C}_1 \le s$$

implies that $\mathcal{H}^s(\mathcal{C}_1)$ is positive and finite, whereas:

$$\dim_{\mathrm{H}} \mathcal{C}_1 = \dim_{\mathrm{B}} \mathcal{C}_1 = s.$$

⌟

Definition III.1.6. General Construction of k-level Intervals.

Let $0 < s < 1$ and F constructed with modified construction for the cantor set to be dividing an interval I of the level k construction into the level-$k+1$ construction via separating into intervals $I_1, \cdots, I_m$ (where $m \ge 2$) contain in I with left and right endpoints coinciding and equally spaced with:

$$|I_i|^s = \frac{1}{m}|I|^s.$$

⌟

Proposition III.1.7. Dimension of General Construction of k-level Intervals.

For the General Construction of k-level Intervals (Definition 3.1.6), $\dim_{\mathrm{H}} F = s$, and $0 < \mathcal{H}^s(F) < \infty$.

Proof. Here, with each level construction, we note that:

$$|I|^s = \sum_{i=1}^{m} |I_i|^s.$$

Now, we consider this process repetitively, when we sum over all intervals of level k, we should sum up to the 1 (as we start with $[0,1]$).

Here, note that for the level-k construction, it covers F, and since the maximum interval length tends to 0 as $k \to \infty$, we must have $\mathcal{H}^s(F) \le 1$ for sufficiently small δ, so we have $\mathcal{H}^s(F) \le 1$.

Then, we want to construct a mass distribution. For an interval I of level k, we let $\mu(I) = |I|^s$, so this divides 1 equally into each interval.

Now, consider an interval U with endpoints in F, and let I be the smallest basic interval that contains U. Suppose that I is a level-k interval, with the $k+1$-level construction containing m intervals, then the spacing between the intervals is:

$$\frac{|I| - m|I_i|}{m-1} = |I|\frac{1 - m|I_i|/|I|}{m-1} = |I|\frac{1 - m^{1-1/s}}{m-1} \ge (1 - 2^{1-1/s})|I|/m.$$

Say U intersects $j \ge 2$ intervals, we have that:

$$|U| \ge \frac{j-1}{m}(1 - 2^{1-1/s})|I| \ge \frac{j}{2m}(1 - 2^{1-1/s})|I|.$$

Thus, we have the mass as:

$$\mu(U) \le j\mu(I_i) = j|I_i|^s = \frac{j}{m}|I|^s \le 2^s(1 - 2^{1-1/s})^{-s}\left(\frac{j}{m}\right)^{1-s}|U|^s \le 2^s(1 - 2^{1-1/s})^{-s}|U|^s.$$

Since this holds for any U with endpoints in F, while we are using such that the smallest interval containing U, this holds for all U, and by the Mass distribution principle (Theorem 3.1.3), we have $\mathcal{H}^s(F) > 0$. Hence, we are able to conclude that $\dim_{\mathrm{H}} F = s$. □

Definition III.1.8. Uniform Cantor Sets.

Let $m \ge 2$ be integer and $0 < r < 1/m$. We let $\mathcal{C}$ be the set obtained by the construction where each basic interval I is replaced by m equally spaced subintervals of length $r|I|$, where the ends of I coinciding with the ends of the extrema of the subintervals. ⌟

Proposition III.1.9. Dimension of Uniform Cantor Sets.

For the Uniform Cantor Sets (Definition 3.1.8), $\dim_{\mathrm{H}} \mathcal{C} = \dim_{\mathrm{B}} \mathcal{C} = -\log m/\log r$, and specially for Hausdorff measure, $0 < \mathcal{H}^{-\log m/\log r}(\mathcal{C}) < \infty$.

Proof. The Hausdorff dimension is a special case for dimension of general construction of k-level interval (Proposition III.1.7) with m being a constant and $s = -\log m/\log r$. In particularly, $(r|I|)^s = |I|^s/m$, thus $\dim_{\mathrm{H}} \mathcal{C} = s$.

For the box counting dimensions, note that it can be covered by m^k level-k intervals of length r^k, thus we have:

$$\overline{\dim}_{\mathrm{B}}\mathcal{C} = \limsup_{\delta \to 0} \frac{\log N_\delta(\mathcal{C})}{-\log \delta} \le \limsup_{k \to \infty} \frac{\log m^k}{-\log r^k} = \frac{\log m}{-\log r}.$$

Hence by the inequality that $-\log m/\log r \le \dim_{\mathrm{H}} \mathcal{C} \le \underline{\dim}_{\mathrm{B}}\mathcal{C} \le \overline{\dim}_{\mathrm{B}}\mathcal{C} \le -\log m/\log r$, the dimension is $-\log m/\log r$. □

Remark III.1.10. Middle λ Cantor Set as a Special Case.

For the middle λ Cantor set, we remove $0 < \lambda < 1$ from the middle of the intervals, we consider this as a special case of uniform Cantor set, with $m = 2$ and $r = (1-\lambda)/2$. Hence, the Hausdorff and box dimensions are $\log 2/\log\big(2/(1-\lambda)\big)$. ⌟

Proposition III.1.11. The Covering Lemma.

Let $\mathcal{C}$ be a family of balls contained some bounded region of $\mathbb{R}^n$. Then, there exists a countable disjoint sub-collection $\{B_i\}$ such that:

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \widetilde{B}_i,$$

where $\widetilde{B}_i$ is the closed ball concentric with B_i and of four times the radius.

Proof. The set of open balls $\{B_i\}$ can be constructed in the following (recursive) way:

- For the first ball, we select (one of) the ball in $\mathcal{C}$ with the largest radius.
- Then, for the later selections, we select (one of) the ball with the largest radius which has no intersections with the previous balls.

With this constructions, we can trivially note that the balls are disjoint, and moreover, for any ball $B \in \mathcal{C}$, it is either one of the balls, or it intersects one of the balls with radius no less than that ball, and thus four times of the radius would include the ball. □

Theorem III.1.12. Bounds in Mass Locally $\Longrightarrow$ Bounds in Hausdorff Measure.

Let μ be a mass distribution on $\mathbb{R}^n$, $F \subset \mathbb{R}^n$ be a Borel set, and $0 < c < \infty$ be an arbitrary constant, then:

(i) If $\limsup_{r\to 0} \mu\big(B_r(x)\big)/r^s < c$ for all $x \in F$, then $\mathcal{H}^s(F) \geq \mu(F)/c$.

(ii) If $\limsup_{r\to 0} \mu\big(B_r(x)\big)/r^s > c$ for all $x \in F$, then $\mathcal{H}^s(F) \leq 8^s\mu(\mathbb{R}^n)/c$.

Proof. (i) For every $\delta > 0$, we let:

$$F_\delta = \{x \in F : \mu\big(B_r(x)\big) < cr^s \text{ for all } 0 < r \leq \delta\}.$$

Here, we let $\{U_i\}$ be an arbitrary δ-cover of F, and by construction, it is a δ-cover of F_δ. Thus, for each U_i containing $x \in F_\delta$, the ball $B_{|U_i|}(x) \supset U_i$. Thus, by definition of F_δ, we have:

$$\mu(U_i) \leq \mu(B) < c|U_i|^s.$$

Hence, as we consider the mass if F_δ by countable sub-additivity, we have:

$$\mu(F_\delta) \leq \sum_i \{\mu(U_i) : U_i \text{ intersects } F_\delta\} \leq c\sum_i |U_i|^s.$$

Since $\{U_i\}$ is arbitrary, then $\mu(F_\delta) \leq c\mathcal{H}^s_\delta(F) \leq c\mathcal{H}^s(F)$. Eventually, since $F_\delta \to F$ as $\delta \to 0$, so we have $\mu(F)/c \geq \mathcal{H}^s(F)$.

(ii) First, suppose that F is bounded, and let $\delta > 0$ be fixed. Here, we let $\mathcal{C}$ be the collection of balls by:

$$\mathcal{C} := \{B_r(x) : x \in F, 0 < r \leq \delta \text{ and } \mu\big(B_r(x)\big) > cr^s\}.$$

Then, by the hypothesis and by the Covering lemma (Proposition III.1.11), $\mathcal{C}$ implies the existence of disjoint balls $B_i \in \mathcal{C}$ such that $\bigcup_{B \in \mathcal{C}} B \subset \bigcup_i \widetilde{B}_i$, so we can consider $\{\widetilde{B}_i\}$ be a 8δ-cover of F, so:

$$\mathcal{H}^s_{8\delta}(F) \leq \sum_i |\widetilde{B}_i|^s \leq 4^s \sum_i |B_s|^s \leq 8^s \sum_i \mu(B_i)/c \leq 8^s \mu(\mathbb{R}^n)/c.$$

Then, let $\delta \to 0$, we can get that $\mathcal{H}^s(F) \leq 8^s \mu(\mathbb{R}^n)/c < \infty$.

Otherwise, if F is unbounded and $\mathcal{H}^s(F) > 8^s \mu(\mathbb{R}^n)/c$, then the $\mathcal{H}^s$-measure of some bounded subset of F will also exceed the value, hence is a contradiction. □

Remark III.1.13. Stronger Result for Bounds in Hausdorff Measure.

The results in Bounds in Mass Locally $\implies$ Bounds in Hausdorff Measure (Theorem III.1.12) can be stronger, *i.e.*, $\mathcal{H}^s(F) \leq 2^s \mu(\mathbb{R}^n)/c$. ⌟

Remark III.1.14. Condition for Set of Dimension s.

By Bounds in Mass Locally $\implies$ Bounds in Hausdorff Measure (Theorem III.1.12), if:

$$\lim_{r \to 0} \frac{\log \mu\big(B_r(x)\big)}{\log r} = s,$$

then $\dim_{\mathrm{H}} F = s$. ⌟

III.2 Subsets of Finite Measure

Theorem III.2.1. Existence of Compact Subset for Non-null Borel Set.

Let $F \subset \mathbb{R}^n$ be a Borel subset such that $0 < \mathcal{H}^s(F) \leq \infty$. There exists a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$.

The *proof* is omitted as it is lengthy and extraneous.

Proposition III.2.2. Set with Upper Bound for Hausdorff Measure.

Let F be a Borel set satisfying $0 < \mathcal{H}^s(F) < \infty$, then there is a constant b and a compact set $E \subset F$ with $\mathcal{H}^s(E) > 0$ such that:

$$\mathcal{H}^s\big(E \cap B_r(x)\big) \leq br^s$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

The *proof* is omitted as it is lengthy and extraneous.

Proposition III.2.3. Frostman's Lemma.

Let F be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(F) \leq \infty$, then there is a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$ and a constant b such that:

$$\mathcal{H}^s\big(E \cap B_r(x)\big) \leq br^s$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

The *proof* is omitted as it is lengthy and extraneous.

III.3 Potential Theoretic Methods

Definition III.3.1. Newtonian Potential.

For $s \geq 0$, the s-potential at a point x of $\mathbb{R}^n$ due to the mass distribution μ on $\mathbb{R}^n$ is defined as:

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s}.$$

⌟

Definition III.3.2. Newtonian Energy.

The s-energy of μ is the integral of μ, a mass distribution, as of in the Newtonian potential (Definition 3.2.1), is:

$$I_s(\mu) = \int \phi_s(x) d\mu(x) = \iint \frac{d\mu(x) d\mu(y)}{|x-y|^s}.$$

⌟

Remark III.3.3. Newtonian in $\mathbb{R}^3$.

Notice that when $n = 3$ and $s = 1$, the Newtonian potential is the gravitational potential, and when μ is the mass, the energy is the gravitational potential energy. ⌟

Theorem III.3.4. Hausdorff Dimension in relation to Mass Distribution.

Let $F \subset \mathbb{R}^n$ be arbitrary, then:

(i) If there is a mass distribution μ on F with $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_{\mathrm{H}} F \geq s$.

(ii) If F is a Borel set with $\mathcal{H}^s(F) > 0$, then there exists a mass distribution μ on F such that $I_t(\mu) < \infty$ for all $0 < t < s$.

Proof. (i) Suppose that $I_s(\mu) < \infty$ for some mass distribution μ whose support is contained in F, we define:

$$F_1 = \left\{ x \in F : \limsup_{r \to 0} \mu\big(B_r(x)\big)/r^s > 0 \right\}.$$

Now, suppose $x \in F_1$, then there exists $\varepsilon > 0$ and a sequence of numbers $\{r_i\}$ decreasing to 0 such that $\mu\big(B_{r_i}(x)\big) \geq \varepsilon r_i^s$. Since $I_s(\mu) < \infty$, then $\mu(\{x\}) = 0$.
Hence, note that μ is continuous, and by taking $0 < q_i < r_i$ to be sufficiently small and consider about annulus $A_i := B_{r_i}(x) \setminus B_{q_i}(x)$, we get:

$$\mu(A_i) \geq \frac{1}{4} \varepsilon r_i^s.$$

Since we can take subsequences, we can have $r_{i+1} < q_i$ for all i, hence A_i's are disjoint and centered on x, therefore, for $x \in F_1$, and notice that $|x-y|^{-s} \geq r_i^{-s}$, we have:

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \int_{A_i} \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \frac{1}{4} \varepsilon r_i^s r_i^{-s} = \infty.$$

However, since $I_s(\mu) = \int \phi_s(x) d\mu(x) < \infty$, then $\phi_s(x) < \infty$ for a.e. x with respect to μ. Hence, we must have $\mu(F_1) = 0$, and since $\limsup_{r \to 0} \mu\big(B_r(x)/r^s\big) = 0$ if $x \in F \setminus F_1$, then by Bounds in Hausdorff Measure (Theorem III.1.12), we have:

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(F \setminus F_1) \geq \mu(F \setminus F_1)/c \geq \big(\mu(F) - \mu(F_1)\big)/c = \mu(F)/c.$$

Hence, we must have $\mathcal{H}^s(F) = \infty$.

(ii) Now, we suppose $\mathcal{H}^s(F) > 0$, we want to use $\mathcal{H}^s$ to construct a mass distribution μ on F such that $I_t(\mu) < \infty$ for every $t < s$. Here, by Frostman's Lemma (Proposition III.2.3), there exists a compact set $E \subset F$ such that:

- $0 < \mathcal{H}^s(E) < \infty$, and
- a constant b such that $\mathcal{H}^s\big(E \cap B_r(x)\big) \le br^s$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Here, we can let μ be the measure $\mathcal{H}^s$ but limiting the measurable space to be F but restricted to $\mathcal{P}(E)$ by the definition that:

$$\mu(A) := \mathcal{H}^s(A \cap E),$$

and by construction, μ is a mass distribution to F, and for any $x \in \mathbb{R}^n$, we have:

$$m(r) = \mu\big(B_r(x)\big) = \mathcal{H}^s\big(E \cap B_r(x)\big) \le br^s.$$

Then, as we consider $0 < t < t$, then:

$$\begin{aligned}
\phi_t(x) &= \int_{|x-y|\le 1} \frac{d\mu(y)}{|x-y|^t} + \int_{|x-y|>1} \frac{d\mu(y)}{|x-y|^t} \\
&\le \int_0^1 r^{-t} dm(r) + \mu(\mathbb{R}^n) = \Big[r^{-t} m(r)\Big]_{r=0^+}^{r=1} + t\int_0^1 r^{-(t+1)} m(r) dr + \mu(\mathbb{R}^n) \\
&\le b + bt\int_0^1 r^{s-t-1} dr + \mu(\mathbb{R}^n) \\
&= b\left(1 + \frac{t}{s-t}\right) + \mathcal{H}^s(F) = c.
\end{aligned}$$

As this holds for all $x \in \mathbb{R}^n$, we have $I_t(\mu) = \int \phi_t(x) d\mu(x) \le c\mu(\mathbb{R}^n) < \infty$. □

Remark III.3.5. Hausdorff Dimension in relation to Capacity.

Hausdorff Dimension in relation to Mass Distribution (Theorem III.3.4) can be discussed in terms of capacity in potential theory, *i.e.*, the s-capacity of a set F is:

$$C_s(F) = \sup_\mu \{1/I_s(\mu) : \mu \text{ is a mass distribution on } F \text{ with } \mu(F) = 1\}.$$

Thus, we have:

$$\dim_{\mathrm{H}} F = \inf\{s \ge 0 : C_s(F) = 0\} = \sup\{s \ge 0 : C_s(F) > 0\}.$$

⌟

IV Iterated Function Systems

IV.1 Iterated Function System

Many fractals are made with ways of iterated processes, the self-similarities, defined by iterated functions, can be helpful in finding the dimensions.

Definition IV.1.1. Contraction Mapping.

Let $D \subset \mathbb{R}^n$ be closed, a mapping $S : D \to D$ is a contraction on D if there exists a number c with $0 < c < 1$ such that $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in F$. ⌟

Remark IV.1.2. Remarks for Contractions.

Let S be a contraction, the following properties hold:

(i) Any contraction is continuous, by the definition of continuity.

(ii) If the equality holds, *i.e.*, $|S(x) - S(y)| = c|x - y|$, then S transforms sets into geometrically similar sets, then S is a contracting similarity.

⌟

Definition IV.1.3. Iterated Function System.

Let $\{S_1, S_2, \cdots, S_m\}$ be a finite family of contractions ($S_i : D \to D$) with $m \geq 2$, it is an iterated function system, *or* IFS. ⌟

Definition IV.1.4. Attractor.

A compact subset $F \subset D$ is an attractor (*or* invariant set) for the IFS $\{S_i\}$, in which $S_i : D \to D$, if:

$$F = \bigcup_{i=1}^{m} S_i(F).$$

⌟

Example IV.1.5. IFS for Middle $1/3$ Cantor Set.

Let $\mathcal{C}$ be the middle $1/3$ Cantor set, then the iterated function system $\{S_1, S_2\}$ where $S_1, S_2 : D \to D$ is:

$$S_1(x) = \frac{1}{3}x \text{ and } S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Here, we can consider $S_1(\mathcal{C})$ and $S_2(\mathcal{C})$ as the left and right halves of $\mathcal{C}$, *i.e.*, $\mathcal{C} = S_1(\mathcal{C}) \cup S_2(\mathcal{C})$.
Specifically, we consider $\mathcal{C}$ as an attractor of $\{S_1, S_2\}$ as IFS, representing the self-similarities of $\mathcal{C}$. ⌟

Definition IV.1.6. Hausdorff Metric.

Let $\mathcal{S}$ denote the class of all non-empty compact subsets of $D \subset \mathbb{R}^n$, for $A, B \in \mathcal{S}$, let the Hausdorff metric distance be defined as:

$$d(A, B) = \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where the neighborhoods are defined as $A_\delta := \{x \in D : |x - a| \leq \delta \text{ for some } a \in A\}$. ⌟

Here, we want to show that δ-Neighborhood definition is a metric space.

Proof. (i) Positivity: Let $E, F \subset \mathbb{R}^n$ be non-empty compact sets, and by definition δ should be non-negative, else $A_\delta = \emptyset$. Hence, $d(E, F) \geq 0$.

(ii) Definiteness: Let $F \subset \mathbb{R}^n$ be a non-empty compact set, then we note that $F \subset F$, so we have $d(F, F) \leq 0$, and by positivity, $d(F, F) = 0$.

(iii) Symmetry: Let $E, F \subset \mathbb{R}^n$ be non-empty compact sets, we have that, by definition, that:

$$d(E,F) = \inf\{\delta : E \subset F_\delta \text{ and } F \subset E_\delta\} = \inf\{\delta : F \subset E_\delta \text{ and } E \subset F_\delta\} = d(F,E).$$

(iv) Triangular inequality: Let $E, F, G \subset \mathbb{R}^n$ be non-empty compact sets, for any δ, γ such that $E \subset F_\delta$, $F \subset E_\delta$, $F \subset G_\gamma$, and $G \subset F_\gamma$, we have:

$$E \subset F_\delta \subset G_{\delta+\gamma} \text{ and } G \subset F_\gamma \subset E_{\gamma+\delta},$$

hence:

$$d(E,G) \leq \delta + \gamma \text{ for all } d(E,F) \leq \delta \text{ and } d(F,G) \leq \gamma.$$

So the triangle inequality holds. □

Proposition IV.1.7. Hausdorff Metric Space is Complete.

The Hausdorff Metric Space $(\mathcal{S}, d)$ is complete.

Proof. For simplicity of notation, we let the Euclidean metric space be $(\mathbb{R}^n, d)$ and the induced Hausdorff metric space be $(\mathcal{S}, d')$. Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{S}$ with respect to the Hausdorff metric d'. Since $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence, any of its subsequences converging is a sufficient condition for the sequence to converge. Hence, without loss of generality, we assume that $d'(A_n, A_{n+1}) < 2^{-n}$, *i.e.*:

$$A_n \subset (A_{n+1})_{2^{-n}} \text{ and } A_{n+1} \subset (A_n)_{2^{-n}}.$$

Now, as we consider $x \in \mathbb{R}^n$, for any positive integer N, there exists a sequence $\{x_n\}_{n=N}^{\infty}$ with elements $x_n \in A_n$ and $d(x_n, x_{n+1}) < 2^{-n}$. Since $\mathbb{R}^n$ is complete, and since $\{x_n\}$ is Cauchy, it must converge to some $x \in \mathbb{R}^n$. Note that by triangle inequality, we have that for all $n \geq N$ that $d(x_n, x) < 2^{-n+1}$.
Now, we shall construct the set A as the set of all x such that x is the limit for some sequence $\{x_n\}$ as constructed above. First, we note that A is non-empty. Moreover, for any $x \in A$, there must be some $x_n \in A_n$ such that $d(x_n, x) < 2^{-n+1}$, which implies that $A \subset (A_n)_{2^{-n+1}}$.
For the other direction, we suppose that $\varepsilon > 2^{-N} > 0$ for some positive integer N. For any $x_n \in A_n$, there exists some $x \in A$ such that $d(x_n, x) < 2^{-n+1}$ for $n \geq N+1$. Therefore, $\{A_n\}$ converges to A in $(\mathcal{S}, d')$. □

Theorem IV.1.8. Existence and Uniqueness of Attractor.

Consider the iterated function system given by the contractions $\{S_1, \cdots S_m\}$ on $D \subset \mathbb{R}^n$, so that:

$$|S_i(x) - S_i(y)| \leq c_i |x - y| \text{ for } x, y \in D.$$

with $c_i < 1$ for each i. Then there is a unique attractor F, *i.e.*, a non-empty compact set such that:

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Moreover, if we define a transformation on the class $\mathcal{S}$ of non-empty compact sets by:

$$S(E) = \bigcup_{i=1}^{m} S_i(E) \text{ for } E \in \mathcal{S},$$

and for k-th iterate of S, we have:

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

for every set $E \in \mathbb{C}$ such that $S_i(E) \subset E$ for all i.

Proof. (Existence:) Here, let $E \in \mathcal{S}$ be arbitrary such that $S_i(E) \subset E$ for all i (e.g. we can have $E = D \cap B_r(0)$). Then we have $S^k(E) \subset S^{k-1}(E)$, thus $S^k(E)$ is a sequence of non-empty compact sets. Therefore $F = \bigcap_{k=1}^{\infty} S^k(E)$ is non-empty and compact.
Since $S^k(E)$ is decreasing, then it follows that $S(F) = F$, hence F is an attractor.
(Uniqueness:) Note that sets in $\mathcal{S}$ are closed under transformed by S. Let $A, B \in \mathcal{S}$, then:

$$d\big(S(A), S(B)\big) = d\left(\bigcup_{i=1}^{m} S_i(A), \bigcup_{i=1}^{m} S_i(B)\right) \leq \max_{1\leq i\leq m} d\big(S_i(A), S_i(B)\big).$$

Also note that if the δ-neighborhood $\big(S_i(A)\big)_\delta$ contains $S_i(B)$ for all i, then $\big(\bigcup_{i=1}^{m} S_i(A)\big)_\delta$ contains $\bigcup_{i=1}^{m} S_i(B)$, and vice versa, thus:

$$d\big(S(A), S(B)\big) \leq \big(\max_{1\leq i\leq m} c_i\big) d(A, B).$$

Now, suppose that A and B are any attractors, then $S(A) = A$ and $S(B) = B$. Since $0 < \max_{1\leq i\leq m} c_i < 1$, we must have $d(A, B) = 0$, which implies that $A = B$. □

Example IV.1.9. Constructing the IFS in which Cantor Dust is attractor.

Consider the Cantor Dust, constructed in the following sense:

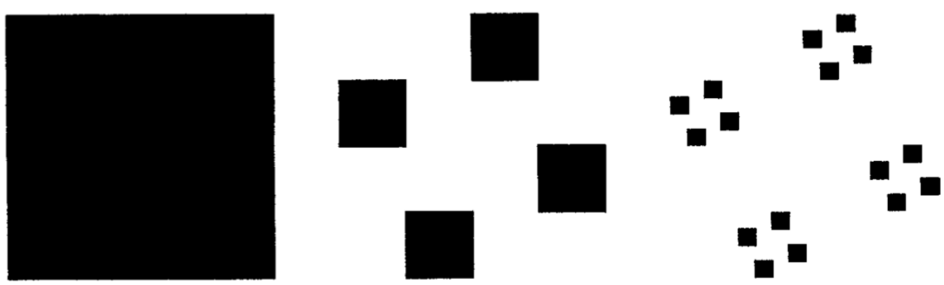

Figure IV.1. Cantor Dust in $\mathbb{R}^2$.

Notice that by observation, we can construct the four similarities as:

$$S_1(x,y) = \left(\frac{1}{4}x, \frac{1}{4}y + \frac{1}{2}\right), \qquad S_2(x,y) = \left(\frac{1}{4}x + \frac{1}{4}, \frac{1}{4}y\right),$$

$$S_3(x,y) = \left(\frac{1}{4}x + \frac{1}{2}, \frac{1}{4}y + \frac{3}{4}\right), \qquad S_4(x,y) = \left(\frac{1}{4}x + \frac{3}{4}, \frac{1}{4}y + \frac{1}{4}\right).$$

⌟

Proposition IV.1.10. Sequence of Iterations Converges to Fractal.

Let $\{S_1, \cdots, S_m\}$ be contractions on $D \subset \mathbb{R}^n$ and $\mathcal{S}$ be the class of non-empty compact sets, with the transformation:

$$S(E) := \bigcup_{i=1}^{m} S_i(E) \text{ for } E \in \mathcal{S}.$$

$S^k(E)$ converges to the attractor F for any initial set $E \in \mathcal{S}$ in the sense that $d\big(S(E), S(F)\big) \to 0$.

Proof. In the proof of Theorem IV.1.8, we have that:

$$d\big(S(A), S(B)\big) \le \big(\max_{1\le i\le m} c_i\big) d(A, B),$$

which implies that:

$$d\big(S(E), F\big) = d\big(S(E), S(F)\big) \le c d(E, F),$$

where c is the largest of all c_i's (and is less than 1). Therefore, when extended to k-th iteration, we have:

$$d\big(S^k(E), F\big) \le c^k d(E, F).$$

Hence, the converges applies as $k \to \infty$. □

Remark IV.1.11. Pre-Fractals.

Since $S^k(E)$ provide increasingly good approximations to F, hence it can be a pre-fractal of F. ⌟

IV.2 Dimensions of Self-similar Sets

Theorem IV.2.1. Lemma on Upper Bound for Intersections of Disjoint Balls.

Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbb{R}^n$ such that each V_i contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$. Then any ball B of radius r intersects at most $(1 + 2a_2)^n a_1^{-n}$ of the closures $\overline{V_i}$.

Proof. Suppose $\overline{V_i}$ intersects B, then $\overline{V_i}$ is contained in the ball concentric with B and radius $(1 + a_2)r$. Now, if there are q of the sets $\overline{V_i}$ intersecting B, then the sum of the volumes shall correspond, *i.e.*:

$$q(a_1 r)^n \le (1 + 2a_2)^n r^n,$$

which aligns with the statement. □

Definition IV.2.2. Open Set Condition.

Let $S_i : D \to D$ be contractions in IFS $\{S_i\}_{i=1}^m$, then $\{S_i\}$ satisfy the open set condition if there exists a non-empty bounded open set V such that:

$$V \supset \bigcup_{i=1}^m S_i(V).$$

⌟

Theorem IV.2.3. Open Set Condition $\implies$ Conclusion of Dimensions.

Suppose that open set condition (Definition IV.2.2) holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \le i \le m$. If F is the attractor of the IFS $\{S_1, \cdots, S_m\}$, *i.e.*:

$$F = \bigcup_{i=1}^m S_i(F),$$

then $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = s$, where s is given by:

$$\sum_{i=1}^m c_i^s = 1.$$

Moreover, for this value of s, $0 < \mathcal{H}^s(F) < \infty$.

Proof. Let s be such that:

$$\sum_{i=1}^{m} c_i^s = 1.$$

Let $\mathcal{I}_k$ be the indexed set of all k-term sequences $\{i_1, \cdots, i_k\}$ with $1 \leq i_j \leq m$. For any set A and $\{i_1, \cdots, i_k\} \in \mathcal{I}_k$, we denote $A_{i_1,\cdots,i_k} := S_{i_1} \circ \cdots \circ S_{i_k}(A)$. Here, by having the unions of contracted sets, we have:

$$F = \bigcup_{\mathcal{I}_k} F_{i_1,\cdots,i_k}.$$

Here, since we have $S_{i_1} \circ \cdots \circ S_{i_k}$ as a similarity of ratio $c_{i_1}, \cdots, c_{i_k}$, then we have:

$$\sum_{\mathcal{I}_k} |F_{i_1,\cdots,i_k}|^s = \sum_{\mathcal{I}_k} (c_{i_1} \cdots c_{i_k})^s |F|^s = \left(\sum_{i_1} c_{i_1}^s\right) \cdots \left(\sum_{i_k} c_{i_k}^s\right) |F|^s = |F|^s.$$

Hence, for any $\delta > 0$, we may choose k such that $|F_{i_1,\cdots,i_k}| \leq (\max_i c_i)^k |F| \leq \delta$, so $\mathcal{H}^s_\delta(F) \leq |F|^s$ and then $\mathcal{H}^s(F) \leq |F|^s$. Therefore, the covers of F provide a suitable upper estimate of the Hausdorff measure.
For the lower bound, we let $\mathcal{I}$ be the set of all infinite sequences $\mathcal{I} = \big\{\{i_1, i_2, \cdots\} : 1 \leq i_j \leq m\big\}$, and we denote $I_{i_1,\cdots,i_k} := \big\{\{i_1, \cdots, i_k, q_{k+1}, \cdots\} : 1 \leq q_j \leq m\big\}$.
Here, we define a mass distribution $\mu(I_{i_1,\cdots i_k}) = (c_{i_1} \cdots c_{i_k})^s$. Note that $(c_{i_1} \cdots c_{i_k})^s = \sum_{i=1}^m (c_{i_1} \cdots c_{i_k} c_i)^s$, hence the (countable) sub-additivity holds, *i.e.*:

$$\mu(I_{i_1,\cdots,i_k}) = \sum_{i=1}^{m} \mu(I_{i_1,\cdots,i_k,i}),$$

and note that $\mu(\mathcal{I}) = 1$, it is a mass distribution.
Meanwhile, we naturally induce a mass distribution $\tilde{\mu}$ on F by having:

$$\tilde{\mu}(A) = \mu\big[\{i_1, i_2, \cdots\} : x_{i_1,i_2,\cdots} \in A\big] \text{ for } A \subset F.$$

Note that $\tilde{\mu}$ is on the sets and $\tilde{\mu}(F) = 1$.
Now, we want to use the mass distribution principle (Theorem III.1.3) by checking its condition. We let V be the open set such that:

$$V \supset \bigcup_{i=1}^{m} S_i(V).$$

Since $\overline{V} \supset S(\overline{V}) = \bigcup_{i=1}^m S_i(\overline{V})$, and by sequence of iterations converges to fractal (Proposition IV.1.10), we have the decreasing sequence of iterates $S^k(\overline{V})$ converging to F. In particular, $\overline{V} \supset F$ and $\overline{V_{i_1,\cdots,i_k}} \supset F_{i_1,\cdots,i_k}$ for each finite sequence $\{i_1, \cdots, i_k\}$.
Here, let B be any ball of radius $r < 1$, we estimate $\tilde{\mu}(B)$ by considering the sets $V_{i_1,\cdots,i_k}$ with diameters comparable with that of B and closures intersecting $F \cap B$. For the infinite sequence $\{i_1, i_2, \cdots\} \in \mathcal{I}$ after the first term i_k gives that:

$$\left(\min_{1 \leq i \leq m} c_i\right)^k r \leq c_{i_1} c_{i_2} \cdots c_{i_k} r \leq r.$$

By having $\mathcal{Q}$ denoting the finite set of all (finite) sequences obtained in such way, then for every infinite sequence $\{i_1, i_2, \cdots\} \in \mathcal{I}$, there exists exactly one value of k in which $\{i, \cdots, i_k\} \in \mathcal{Q}$. This is justified, *i.e.*, it $i_1 \cdots i_k r$ will not skip the interval as the value is no more than the minimum $\min_i c_i r$. Since $V_1, \cdots, V_m$ are disjoint, so are $V_{i_1,\cdots,i_k,1}, \cdots, V_{i_1,\cdots,i_k,m}$ for each $\{i_1, \cdots, i_k\}$. By doing so recursively, we have the collection of open sets $\{V_{i_1,\cdots,i_k} : \{i_1,\cdots,i_k\} \in \mathcal{Q}\}$ is disjoint. Hence, we have similarly that:

$$F \subset \bigcup_{\mathcal{Q}} F_{i_1,\cdots,i_k} \subset \bigcup_{\mathcal{Q}} \overline{V_{i_1,\cdots,i_k}}.$$

Here, by selecting a_1 and a_2 so V contains a ball of radius a_1 and is contain in a ball of radius a_2. Then, for all $\{i_1,\cdots,i_k\} \in \mathcal{Q}$, the set $V_{i_1,\cdots,i_k}$ contains a ball of radius $c_{i_1}\cdots c_{i_k}a_1$ and therefore one of radius $(\min_i c_i)a_1 r$, and it is contain in ball of radius $c_{i_1}\cdots a_{i_k}a_2$, which is $a_2 r$. Let $\mathcal{Q}_1$ denote the sequences $\{i_1,\cdots,i_k\}$ in $\mathcal{Q}$ such that B intersects $\overline{V_{i_1,\cdots,i_k}}$, and by the lemma on upper bound for intersections of disjoint balls (Theorem IV.2.1), there are at most $q=(1+2a_2)^n a_1^n(\min_i c_i)^{-n}$ sequences in $\mathcal{Q}_1$, thus since if $x_{i_1,i_2,\cdots} \in F\cap B \subset \bigcup_{\mathcal{Q}_1}\overline{V_{i_1,\cdots,i_k}}$, then there is an integer k such that $\{i_1,\cdots,i_k\} \in \mathcal{Q}_1$, and that:

$$\tilde{\mu}(B) = \tilde{\mu}(F\cap B) = \mu\{\{i_1,i_2,\cdots\} : x_{i_1,i_2,\cdots} \in F\cap B\} \le \mu\left\{\bigcup_{\mathcal{Q}_1} I_{i_1,\cdots,i_k}\right\}.$$

Note that then:

$$\tilde{\mu}(B) \le \sum_{\mathcal{Q}_1}\mu(I_{i_1,\cdots,i_k}) = \sum_{\mathcal{Q}_1}(c_{i_1}\cdots c_{i_k})^s \le \sum_{\mathcal{Q}_1} r^s \le r^s q.$$

Since any set U is contain in a ball of radius $|U|$, we have $\tilde{\mu}(U) \le |U|^s q$, so by mass distribution principle (Theorem III.1.3), we have:

$$\mathcal{H}^s(F) \ge q^{-1} > 0,$$

hence $\dim_{\mathrm{H}} F = s$.

Eventually, consider if $\mathcal{Q}$ is any set of finite sequences such that for every $\{i_1,i_2,\cdots\} \in \mathcal{I}$, there is exactly one integer k with $\{i_1,\cdots,i_k\} \in \mathcal{Q}$, and inductively implying that:

$$\sum_{\mathcal{Q}}(c_{i_1}\cdots c_{i_k})^s = 1.$$

Therefore, if $\mathcal{Q}$ is chosen as indicated, $\mathcal{Q}$ contains at most $(\min_i c_i)^{-s}r^{-s}$ sequences, and for each sequence $\{i_1,\cdots,i_k\} \in \mathcal{Q}$, we have:

$$|\overline{V_{i_1,\cdots,i_k}}| = c_{i_1}\cdots c_{i_k}|\overline{V}| \le r|\overline{V}|,$$

so F may be covered by $(\min_i c_i)^{-s}r^{-s}$ sets of diameter $r|\overline{V}|$ for each $r<1$. By the equivalent definition (iv) (Proposition II.2.2), we have that $\overline{\dim}_{\mathrm{B}}F \le s$, and noting that by monotonicity of Hausdorff and Box-counting dimensions (Proposition II.3.1):

$$s = \dim_{\mathrm{H}} F \le \underline{\dim}_{\mathrm{B}}F \le \overline{\dim}_{\mathrm{B}}F \le s.$$

□

Example IV.2.4. Sierpiński Triangle.

The Sierpiński triangle F is constructed from a equilateral triangle by removing the inverted equilateral triangle(s) of 1/4 area, with such step iterated.

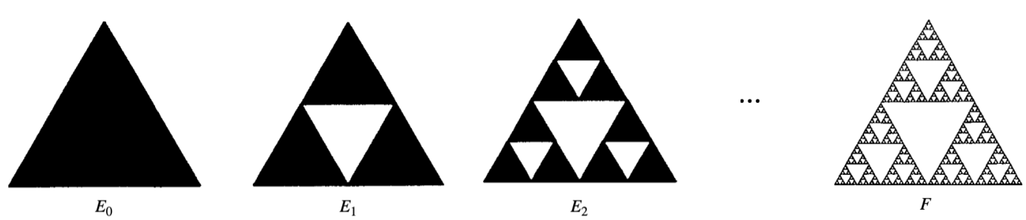

Figure IV.2. Sierpiński Triangle.

Note that we can form the IFS as:

$$S_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right), \qquad S_2(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y + \frac{1}{2}\right), \qquad S_3(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right).$$

Specifically, consider E_0° as an open set, we note that $S_1(E_0^\circ) \cup S_2(E_0^\circ) \cup S_3(E_0^\circ)$ can be covered by E_0°. Hence, F satisfies the open set conditions (Definition IV.2.2). Thus, by open set condition $\implies$ conclusion of dimensions, we have the dimensions $s = \dim_H F = \dim_B F$ that:

$$\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s = 1,$$

which implies that $s = \log_{1/2}(1/3) = \log 3 / \log 2$. ⌟

Example IV.2.5. Modified von Koch curve.

Let $0 < a \leq 1/3$ be arbitrary, we construct F from the unit interval by replacing the middle $1/3$ proportion of each interval by the other two sides of an equilateral triangle.

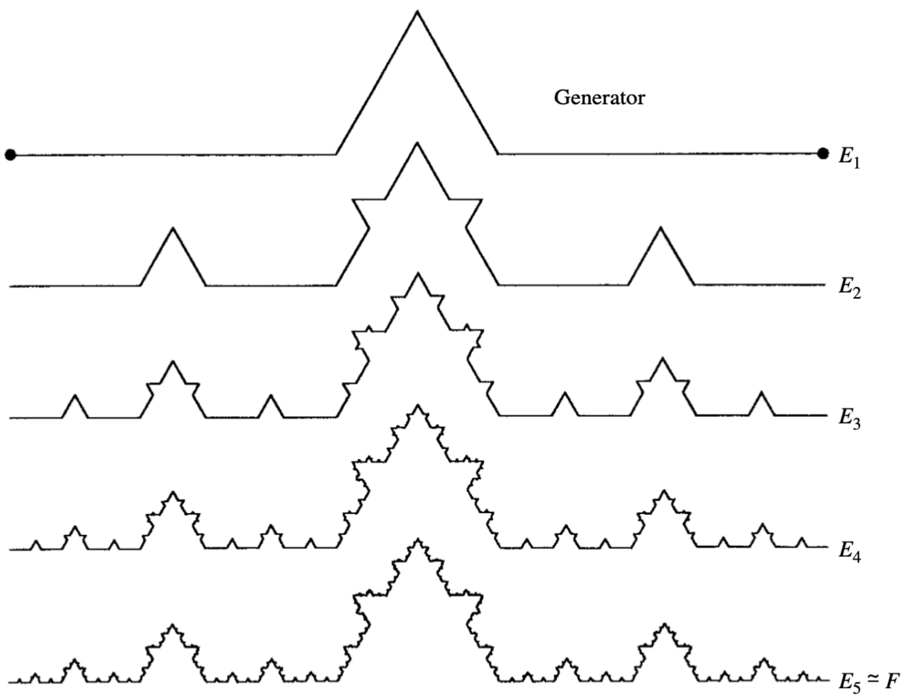

Figure IV.3. von Koch curve.

Clear, we observe that we can let the IFS be mapping F to the 4 sections in E_1. Note that we can take the open set V as the interior of the (upper) equilateral with base being the unit interval, in which the union of its transformations is trivially a subset of V. Hence, F satisfies the open set conditions (Definition IV.2.2). Then, consider that the mapping is mapping distances to ratios $\frac{1}{2}(1-a)$, a, a, and $\frac{1}{2}(1-a)$, so by open set condition $\implies$ conclusion of dimensions, we have the dimensions $s = \dim_H F = \dim_B F$ satisfies:

$$2\left(\frac{1-a}{2}\right)^s + 2a^s = 1.$$

⌟

IV.3 Dimensions on Contractions

Proposition IV.3.1. Upper Bound of Contraction $\implies$ Upper Bound of Dimension.

Let F be the attractor of an IFS consisting of contractions $\{S_1, \cdots, S_m\}$ on a closed subset D of $\mathbb{R}^n$ such that:

$$|S_i(x) - S_i(y)| \leq c_i|x-y| \text{ for } x, y \in D$$

with $0 < c_i < 1$ for each i. Then $\dim_{\mathrm{H}} F \leq s$ and $\overline{\dim}_{\mathrm{B}} F \leq s$, where $\sum_{i=1}^m c_i^s = 1$.

Proof. Again, let s satisfies that $\sum_{i=1}^m c_i^s = 1$, and let $\mathcal{I}_k$ denote the set of all k-term sequences $\{i_1, \cdots, i_k\}$ in which $1 \leq i_j \leq m$. For any set A and the sequence $\{i_1, \cdots, i_k\} \in \mathcal{I}_k$, denote $A_{i_1, \cdots, i_k} = S_{i_1}(A) \circ \cdots \circ S_{i_k}(A)$, hence, by the definition of attractor, we have that:

$$F = \bigcup_{\mathcal{I}_k} F_{i_1, \cdots, i_k}.$$

Note that the above is a cover of F, hence giving an upper estimate of the Hausdorff measure, and by the definition of contraction, we have:

$$\sum_{\mathcal{I}_k} |F_{i_1, \cdots, i_k}|^s \leq \sum_{\mathcal{I}_k} (c_{i_1} \cdots c_{i_k})^s |F|^s = \left(\sum_{i_1} c_{i_1}^s\right) \cdots \left(\sum_{i_k} c_{i_k}^s\right) |F|^s = |F|^s.$$

Hence, we can still choose k so that $|F_{i_1, \cdots, i_k}| \leq (\max_i c_i)^k |F| \leq \delta$, hence implying that $\mathcal{H}^s_\delta \leq |F|^s$ and hence $\mathcal{H}^s(F) \leq |F|^s$.

Likewise, let $\mathcal{Q}$ be any set of finite sequence such that for all $\{i_1, i_2, \cdots\} \in \mathcal{I}$, there exists exactly one integer k such that $\{i_1, \cdots, i_k\} \in \mathcal{Q}$ such that $\sum_{\mathcal{Q}} (c_{i_1} \cdots c) i_k)^s = 1$. Hence, by selecting the first i_k terms for which:

$$\left(\min_{1 \leq i \leq m} c_i\right) r \leq c_{i_1} \cdots c_{i_k} \leq r.$$

Then $\mathcal{Q}$ contains at most $(\min_i c_i)^{-s} r^{-s}$ sequences. For each sequence, we have $|\overline{V_{i_1, \cdots i_k}}| \leq c_{i_1} \cdots c_{i_k} |\overline{V}| \leq r|\overline{V}|$, so F can be covered by $(\min_i c_i)^{-s} r^{-s}$ sets of diameter $r|\overline{V}|$ for each $r < 1$. By the equivalent definition (iv) (Proposition II.2.2), we have that $\overline{\dim}_{\mathrm{B}} F \leq s$, $\overline{\dim}_{\mathrm{B}} F \leq s$. $\square$

Proposition IV.3.2. Lower Bound of Contraction $\implies$ Lower Bound of Dimension.

Consider the IFS of contractions $\{S_1, \cdots, S_m\}$ on a closed subset $D \subset \mathbb{R}^n$ such that:

$$b_i|x-y| \leq \big|S_i(x) - S_i(y)\big| \text{ for all } x, y \in D$$

in which $0 < b_i < 1$ for each i. Assume that the (non-empty and compact) attractor F satisfies that:

$$F = \bigsqcup_{i=1}^m S_i(F),$$

in which the union is disjoint, then F is totally disconnected and $\dim_{\mathrm{H}} F \geq s$ in which s satisfies that:

$$\sum_{i=1}^m b_i^s = 1.$$

Proof. Denote d by the minimum distance between any pair of disjoint compact sets $S_1(F), \cdots, S_m(F)$, *i.e.*:

$$d := \min_{i \neq j} \big(\inf\{|x-y| : x \in S_i(F), y \in S_j(F)\}\big).$$

Again, we denote $F_{i_1,\cdots,i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(F)$ and define μ by:

$$\mu(F_{i_1,\cdots,i_k}) = (b_{i_1} \cdots b_{i_k})^s.$$

Notice that:

$$\begin{aligned}\sum_{i=1}^{m} \mu(F_{i_1,\cdots,i_k,i}) &= \sum_{i=1}^{m} (b_{i_1} \cdots b_{i_k} b_i)^s \\ &= \left(\sum_{i=1}^{m} b_i^s\right)(b_{i_1} \cdots b_{i_k})^s = (b_{i_1} \cdots b_{i_k})^s \\ &= \mu(F_{i_1,\cdots,i_k}) = \mu\left(\bigsqcup_{i=1}^{k} F_{i_1}, \cdots, i_k, i\right),\end{aligned}$$

so μ is a mass distribution of F and $\mu(F) = 1$.

The for any $x \in F$, since the contractions are disjoint, there exists a unique infinite sequence $i_1, i_2, \cdots$ such that $x \in F_{i_1,\cdots,i_k}$ for each k. Thus, for $0 < r < d$, let k be the least integer such that:

$$b_{i_1} \cdots b_{i_k} d \le r < b_{i_1} \cdots b_{i_{k-1}} d.$$

Then, we consider two distinct infinite sequences $i_1, \cdots, i_k$ and $i'_1, \cdots, i'_k$ and the sets $F_{i_1,\cdots,i_k}$ with $F_{i'_1,\cdots,i'_k}$. The two sets must be disjoint and separated by a gap of at lease $b_{i_1} \cdots b_{i_{k-1}} d > r$, since let j be the smallest integer that is distinct, we have:

$$F_{i_j,\cdots,i_k} \subset F_{i_j} \text{ and } F_{i'_j,\cdots,i'_k} \subset F_{i'_j},$$

and since they are separated by d, we have $F_{i_1,\cdots,i_k}$ and $F_{i'_1,\cdots,i'_k}$ are separated by at least $b_{i_1} \cdots b_{i_{j-1}} d$, so by $F \cap B_r(x) \subset F_{i_1,\cdots,i_k}$, we must have:

$$\mu\big(F \cap B_r(x)\big) \le \mu(F_{i_1,\cdots,i_k}) = (b_{i_1} \cdots b_{i_k})^s \le d^{-s} r^s.$$

Now, if U intersects F, then $U \subset B_r(x)$ for some $x \in F$ and $r = |U|$. Therefore, $\mu(U) \le d^{-s}|U|^s$, and by the mass distribution principle (Theorem III.1.3), we have $\mathcal{H}^s(F) > 0$ and $\dim_{\mathrm{H}} F \ge s$.

Note that we have shown that for any $x \in F$, there is a unique sequence whereas it has a positive distance with any other sequences, thus we know that the attractor F is totally disconnected. $\square$

Example IV.3.3. "Non-linear" Cantor Set.

Let the domain be:

$$D := \left[\frac{1}{2}(1+\sqrt{3}, 1+\sqrt{3})\right],$$

and let the contractions $S_1, S_2 : D \to D$ be given by:

$$S_1(x) = 1 + \frac{1}{x} \qquad \text{and} \qquad S_2(x) = 2 + \frac{1}{x}.$$

Here, we note the domain can be transformed into:

$$S_1(D) = \left[\frac{1}{2}(1+\sqrt{3}, \sqrt{3})\right] \qquad \text{and} \qquad S_2(x) = \left[\frac{1}{2}(3+\sqrt{3}), 1+\sqrt{3}\right].$$

Note that S_1 and S_2 are continuous on D, we can apply the mean value theorem, so for any distinct $x, y \in D$ such that $x < y$ we have some $z \in (x, y)$ such that:

$$S_i(x) - S_i(y) = S'_i(z_i) \cdot (x - y).$$

Moreover, we can extend the results that:

$$\inf_{x \in D} |S'_i(x)| \le \frac{|S_i(x) - S_i(y)|}{|x - y|} \le \sup_{x \in D} |S'_i(x)|.$$

By taking derivative, $S_1' = S_2' = -1/x^2$, and consider D, we have the infimum and supremum as:

$$\inf_{x\in D}|S_i'(x)| = \inf_{x\in D}\left|\frac{1}{x^2}\right| = \frac{1}{(1+\sqrt{3})^2} = \frac{2-\sqrt{3}}{2},$$
$$\sup_{x\in D}|S_i'(x)| = \sup_{x\in D}\left|\frac{1}{x^2}\right| = \frac{1}{\left[\frac{1}{2}(1+\sqrt{3})\right]^2} = 2(2-\sqrt{3}).$$

Now, by noting the bounds of the differences, we have that the upper bound being the solution to:

$$2\left(\frac{1}{2}(2-\sqrt{3})\right)^s = 1,$$

and the lower bound being the solution to:

$$2\big(2(2-\sqrt{3})\big)^s = 1,$$

hence by upper bound of contraction $\implies$ upper bound of dimensions (Proposition IV.3.1) and lower bound of contraction $\implies$ lower bound of dimensions (Proposition IV.3.2) implying that $0.34 \lessapprox \dim_{\mathrm{H}} F \lessapprox 1.11$. However, note that the "non-linear" Cantor set is a subset of a 1 dimension set, so the conclusion that $\dim_{\mathrm{H}} F \lessapprox 1.11$ is not necessarily helpful.

A way to improve the estimation is to consider more functions, *i.e.*, compositing the functions:

$$S_i \circ S_j = i + \frac{1}{j+1/x} = i + \frac{x}{jx+1}, \text{ for } i,j = 1,2.$$

Note that the derivative is:

$$\frac{d}{dx}\big[S_i \circ S_j(x)\big] = \frac{1}{(jx+1)^{-2}}.$$

Again, if we consider the mean value theorem, we have:

$$\big(j(1+\sqrt{3})+1\big)^{-2}|x-y| \le |S_i\circ S_j(x) - S_i \circ S_j(y)| \le \left(\frac{1}{2}j(1+\sqrt{3})+1\right)^{-2}|x-y|.$$

Hence, we have the lower bound as the solution to:

$$2(2+\sqrt{3})^{-2s} + 2(2+2\sqrt{3})^{-2s} = 1,$$

whereas the lower upper is the solution to:

$$2\left(\frac{1}{2}(3+\sqrt{3})\right)^{-2s} + 2(2+\sqrt{3})^{-2s} = 1.$$

Hence, we can conclude that $0.44 \lessapprox \dim_{\mathrm{H}} \lessapprox 0.66$. ⌟

V Examples: Number Theory

V.1 Distribution of Digits

Definition V.1.1. Proportions and Base-m Representations.

Let $m \geq 2$ be a fixed integer, and for the base-m expansions, let $p_0, p_1, \cdots, p_{m-1}$ be proportions summing to 1, *i.e.*, $0 \leq p_i \leq 1$ and $\sum_{i=0}^{m-1} p_i = 1$.
Here, $F(p_0, \cdots, p_{m-1})$ is the set of numbers $x \in [0,1)$ with base-m expansions containing digits 0, 1, $\cdots$, $m-1$ in proportions p_0, p_1, $\cdots$, p_{m-1} respectively. ⌟

Remark V.1.2. Representation with Number of Occurrences.

Let $n_j(x|_k)$ denotes the number of times the digit j occurs in the first k places of base-m expression of x, then:

$$F(p_0, \cdots, p_{m-1}) = \left\{ x \in [0,1) : \lim_{k\to\infty} \frac{n_j(x|_k)}{k} = p_j \text{ for all } j = 0, 1, \cdots, m-1 \right\}.$$

⌟

Definition V.1.3. Normal Number.

A number $x \in \mathbb{R}$ is normal in base m if for all positive integer n, all string with n digits have density b^{-n}. If x is normal for all integer bases $m \geq 2$, it is *absolute normal*. ⌟

Proposition V.1.4. Borel-Cantelli Lemma.

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{E_n\}_{n\in\mathbb{Z}^+}$ be a sequence in $\mathcal{M}$, suppose that:

$$\sum_{n=1}^{\infty} \mu(A_n) < +\infty,$$

then:

$$\mu\left(\limsup_{n\to\infty} A_n\right) = 0.$$

Proof. Notice that:

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,$$

hence, $\limsup_{n\to\infty} A_n$ is a countable intersection of countable unions of elements in σ-algebra $\mathcal{M}$, hence $\limsup_{n\to\infty} A_n \in \mathcal{M}$, and is thus measurable. Moreover, by monotonicity, we have that:

$$\bigcup_{k=1}^{\infty} A_k \supset \bigcup_{k=2}^{\infty} A_k \supset \cdots \supset \limsup_{n\to\infty} A_n,$$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \mu\left(\bigcup_{k=2}^{\infty} A_k\right) \geq \cdots \geq \mu\left(\limsup_{n\to\infty} A_n\right).$$

Since the sum $\sum_{n=1}^{\infty} \mu(A_n)$ converges, then $\sum_{n=k}^{\infty} \mu(A_k)$ converges to 0 as $k \to 0$. Therefore, by countable sub-additivity, we have:

$$\mu\left(\bigcup_{k=n}^{\infty} A_k\right) \to 0,$$

meaning that $\mu\left(\limsup_{n\to\infty} A_n\right) = 0$. ☐

Theorem V.1.5. Borel's Strong Law of Large Numbers.

The strong law of large numbers states that the sample average converges almost surely to the expected value $\mu = \mathbb{E}(X_i)$, *i.e.*:

$$\mathbb{P}\left(\lim_{n\to\infty} \overline{X_n} = \mu\right) = 1.$$

Sketch on Proof. The case for which $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, and $\mathbb{E}(X^4)$ is finite is trivial, but the results holds generally with Borel-Cantelli Lemma (Proposition V.1.4). □

Theorem V.1.6. Borel's Normal Theorem.

Almost every number is absolutely normal, with the exceptions being a set of (Lebesgue) measure zero.

Proof. First, let X be chosen uniformly at random from the interval $[0,1)$, we note that for any Borel set $A \subset [0,1)$, we have that:

$$\mathbb{P}\{X \in A\} = m(A).$$

Here, we write X in m-nary form, *i.e.*, $\{X_j\}_{j=1}^{\infty}$. By the Borel's Strong Law of Large Numbers (Theorem V.I.5), we note that for all letters $j \in \{0, \cdots, m-1\}$ that:

$$\mathbb{P}\left\{\lim_{n\to\infty} \frac{\chi_{X_1=j} + \cdots + \chi_{X_n=j}}{n} = \frac{1}{b}\right\} = 1.$$

Therefore, for any positive integer base m, we have the normality in base m, as we let $\mathscr{N}_m$ denote collections of all numbers normal in base m, we have $\mathbb{P}\{X \in \mathscr{N}_m\} = 1$, hence proving the statement. □

Example V.1.7. Normal Number in Base 10.

$0.1234567891011\cdots$ is normal in base 10, and any string of length n would appear uniformly. ⌟

Proposition V.1.8. Dimensions of Proportionated Digits.

Let $F := F(p_1, \cdots, p_{m-1})$ in base m, then:

$$\dim_{\mathrm{H}} F = -\frac{1}{\log m}\sum_{i=1}^{m-1} p_i \log p_i.$$

Proof. Here for the base-m numbers, suppose $x = 0.i_1 i_2 \cdots$ are selected at random such that the kth digit i_k takes the value j with probability p_j independently. Here, we define the probability measure as $\big([0,1), \mathcal{M} \cap \mathcal{P}([0,1)), \mathbb{P}\big)$, in which for interval $I_{i_1,\cdots,i_k}$ being the kth basic interval, we have:

$$\mathbb{P}(I_{i_1,\cdots,i_k}) = p_{i_1} \cdots p_{i_k}.$$

Now, given j, and the events that "the kth digit of x is j" are independent for $k = 1, 2, \cdots$. By Borel's strong law of large numbers (Theorem V.1.5), we have:

$$\frac{1}{k} n_j(x|_k) = \frac{1}{k}(\text{\# of occurrences of } j \text{ in the first } k \text{ digits}) \to p_j \text{ as } k \to \infty.$$

Here, we denote $I_k(x)$ for the kth interval (of length m^{-k}) in which x belongs to. Therefore, for fixed y, we have the probability as measure:

$$\mathbb{P}\big\{x \in I_k(y)\big\} = \mathbb{P}\big(I_k(y)\big),$$

and moreover, we take the logarithms as:

$$\log \mathbb{P}\big(I_k(y)\big) = n_0\big(y|_k\big)\log p_0 + \cdots + n_{m-1}\big(y|_k\big)\log p_{m-1}.$$

Thus, if $y \in F$, then $n_j\big(y|_k\big)/k \to p_k$ as $k \to \infty$, hence:

$$\frac{1}{k}\log\frac{\mathbb{P}\big(I_k(y)\big)}{|I_k(y)|^s} = \frac{1}{k}\log\mathbb{P}\big(I_k(y)\big) - \frac{1}{k}\log m^{-ks} \to \sum_{i=0}^{m-1} p_i \log p_i + s\log m.$$

Here, we denote:

$$\vartheta := -\frac{1}{\log m}\sum_{i=0}^{m-1} p_i \log p_i,$$

we have that for all $y \in F$ that the interval density as:

$$\lim_{k\to\infty}\frac{\mathbb{P}\big(I_k(y)\big)}{|I_k(y)|^s} = \begin{cases} 0, & \text{if } s < \vartheta, \\ \infty, & \text{if } s > \vartheta. \end{cases}$$

Note that this is the case for bounds in mass locally $\Longrightarrow$ bounds in Hausdorff measure (Theorem III.1.12), and thus, we have $\mathcal{H}^s(F) = \infty$ when $s < \vartheta$ and $\mathcal{H}^s(F) = 0$ when $s > \vartheta$, as desired. □

V.2 Continued Fractions

Definition V.2.1. Continued Fractions.

Let $x \in \mathbb{R}$ and $a_0, a_1, \cdots$ be a sequence of integers (partial quotients of x), then we can write:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

┘

Remark V.2.2. x is Rational $\Longleftrightarrow$ Sequence Terminates.

The expansion of partial quotients of x terminated if and only if x is rational. Since rational numbers can be written as fractions by definition. ┘

Example V.2.3. Examples of Continued Fractions.

The following are examples of continued fractions for $\sqrt{2}$ and $s\sqrt{3}$:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\cdots}}}.$$

$$\sqrt{3} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{\cdots}}}}}.$$

┘

Theorem V.2.4. Jarník's Theorem.

Suppose $\alpha > 2$. Let F be the set of real numbers $x \in [0, 1]$ for which the inequality $\|qx\| \le q^{1-\alpha}$ is satisfied by infinitely many positive integers q. Then $\dim_{\mathrm{H}} F = 2/\alpha$.

The *proof* is omitted as it is lengthy and complicated.

VI Examples: Graphs of Functions

VI.1 Dimensions of Graphs

Definition VI.1.1. Graph of Function.

For a functions $f : [a, b] \to \mathbb{R}$, the graph of f is:

$$\operatorname{graph} f = \{(t, f(t)) : a \leq t \leq b\}.$$

Remark VI.1.2. Remarks on Graph of Functions.

Let $f : [a, b] \to \mathbb{R}$ be a function, we may consider graph f as a fractal of the (t, x)-coordinate plane on certain circumstances.

(i) If f has a continuous derivative, then the dimension of graph f is 1.

(ii) If f is of bounded variation, *i.e.*, $\sum_{i=0}^{m-1} |f(t_i) - f(t_{i+1})| \leq c$ for some constant c for all dissections $0 = t_0 < t_1 < \cdots < t_m = 1$, then the dimension of graph f is 1.

(iii) It is possible for f to be continuous and sufficiently irregular such that the dimension of graph f is strictly larger than 1, see Weierstrass's Function (Example VI.1.6).

Definition VI.1.3. Maximum Range of Function.

Given a function f and an interval $[t_1, t_2]$, we denote R_f as the maximum range of f over the interval, *i.e.*:

$$R_f[t_1, t_2] := \sup_{t_1 \leq t, u \leq t_2} |f(t) - f(u)|.$$

Proposition VI.1.4. Number of δ-mesh of Graph.

Suppose $f : [0, 1] \to \mathbb{R}$ is continuous, $0 < \delta < 1$, and m is the least integer such that $m \geq 1/\delta$. Then, if N_δ is the number of squares of the δ-mesh that intersect graph f, then:

$$\delta^{-1} \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] \leq N_\delta \leq 2m + \delta^{-1} \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta].$$

Conceptually, the illustration of a δ-mesh is expressed as follows for $x = f(t)$:

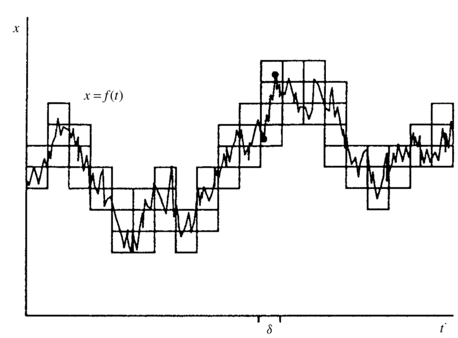

Figure VI.1. δ-mesh of the function $x = f(t)$ with the range.

Proof. Note that for the squares of side length δ, for interval $t \in [i\delta, (i+1)\delta]$, we have the squares intersecting graph f with at least $R_f[i\delta, (i+1)\delta]/\delta$ and at most $2 + R_f[i\delta, (i+1)\delta]/\delta$, given that f is continuous. Therefore, the total number of squares in the δ-mesh is:

$$\sum_{i=0}^{m-1} \delta^{-1} R_f[i\delta, (i+1)\delta] \leq N_\delta \leq \sum_{i=0}^{m-1} \left[2 + \delta^{-1} R_f[i\delta, (i+1)\delta]\right],$$

which simplifies to the desired expression. □

Proposition VI.1.5. Bounds for Box-counting Dimensions of Graph.

Let $f : [0,1] \to \mathbb{R}$ be continuous.

(i) Suppose that $|f(t) - f(u)| \leq c|t-u|^{2-s}$ for all $0 \leq t, u \leq 1$ where $c > 0$ and $1 \leq s \leq 2$. Then $\mathcal{H}^s(\text{graph } f) < \infty$ and $\dim_H \text{graph } f \leq \underline{\dim}_B \text{graph } f \leq \overline{\dim}_B \text{graph } f \leq s$.

(ii) Let numbers $c > 0$, $\delta_0 > 0$, and $1 \leq s < 2$ be fixed. Suppose that for each $t \in [0,1]$ and $0 < \delta \leq \delta_0$ there exists u such that $|t-u| \leq \delta$ and $|f(t) - f(u)| \geq c\delta^{2-s}$, then $s \leq \underline{\dim}_B \text{graph } f$.

Proof. (i) By assumption that $|f(t) - f(u)| \leq c|t-u|^{2-s}$, we have $R_f[t_1, t_2] \leq c|t_1 - t_2|^{2-s}$. Let m be the smallest integer such that $m \geq 1/\delta$, so by number of δ-mesh of graph (Proposition VI.1.4), we have:

$$N_\delta \leq 2m + \delta^{-1} m c \delta^{2-s} \leq 2\delta^{-1} + mc\delta^{-s} \leq \delta^{-s}(2\delta^{s-1} + c) \leq (2+c)\delta^{-s} = \tilde{c}\delta^{-s},$$

where $\tilde{c}$ is independent of δ. Note that from conclusion on dimensions (Proposition III.1.1), we have:

$$\dim_H \text{graph } f \leq \underline{\dim}_B \text{graph } f \leq \lim_{\delta \to 0} \frac{\log N_\delta}{-\log \delta} \leq \lim_{\delta \to 0} \frac{\log\left(\tilde{c}\delta^{-s}\right)}{-\log \delta} = s.$$

(ii) Similarly, the assumption implies that $R_f[t_1, t_2] \geq c|t_1 - t_2|^{2-s}$ for $0 \leq t_1, t_2 \leq 1$. Again, let m be the smallest integer such that $m \geq 1/\delta$, so by number of δ-mesh of graph (Proposition VI.1.4), we have:

$$N_\delta \geq \delta^{-1} m c \delta^{2-s} \geq \delta^{-1}\delta^{-1} c \delta^{2-s} = \tilde{c}\delta^{-s},$$

hence by equivalent definition (iii) (Proposition II.2.2), we have:

$$\underline{\dim}_B \text{graph } f \geq \lim_{\delta \to 0} \frac{\log(\tilde{c}\delta^{-s})}{-\log \delta} = s.$$

□

Example VI.1.6. Weierstrass Function.

Fix $\lambda > 1$ and $1 < s < 2$, let the Weierstrass function $f : [0,1] \to \mathbb{R}$ be defined by:

$$f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t).$$

Then, if λ is sufficiently large, $\dim_B \text{graph } f = s$. ⌟

The *proof* for $\underline{\dim}_B \text{graph } f \leq s$ is omitted for its lengthiness, whereas a rigorous *proof* for $\dim_H \text{graph } f \geq s$ was a very recent result.

VI.2 Iterated Function Systems with Graphs

Remark VI.2.1. Self-Affine Sets with Graphs.

For graphs, we can define $\{S_1, \cdots, S_m\}$ with matrix acting on (t, x)-coordinates. ⌟

The following is an example of self-affine curve from graph f.

Example VI.2.2. Self-Affine Curve.

Let $\{S_1, \cdots, S_m\}$ be affine transformations $S_i : [a,b] \times R_f[a,b] \to \mathbb{R}^2$ defined as:

$$S_i \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ a_i & c_i \end{pmatrix} \cdot \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} (i-1)/m \\ b_i \end{pmatrix},$$

which is equivalently as:

$$S_i(t,x) = \left(\frac{t}{m} + \frac{i+1}{m}, a_i t + c_i x + b_i \right).$$

Here, note that for any set of "vertical strips" of $0 \leq t \leq 1$, it will be transformed into vertical stripes within $\frac{i-1}{m} \leq t \leq \frac{i}{m}$. Additionally here, we suppose that $\frac{1}{m} < c_i < 1$, hence the contraction is stronger in the t-direction compared to the x-direction.

Now, we consider points $p_1 = \big(0, b_1/(1-c_1)\big)$ and $p_m = \big(1, (a_m + b_m)/(1-c_m)\big)$ as fixed points of S_1 and S_m, respectively. Moreover, we assume that the entries of the matrices are selected deliberately so that $S_i(p_m) = S_{i+1}(p_1)$ for all $1 \leq i \leq m-1$, so that the segments joint up to form a polygonal curve E_1. In particular, we avoid the trivial case in which $S_1(p_1), \cdots, S_m(p_1), p_m$ are not all collinear.

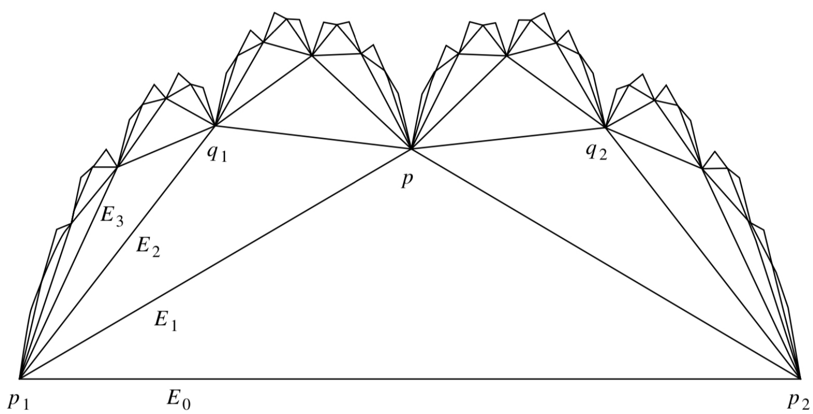

Figure VI.2. Constructions stages for self-affine curve F, which are increasingly good approximations of F, in which $m = 2$.

Specifically, note that the continuity condition allows a continuous function $f : [0,1] \to \mathbb{R}$, and $F = \operatorname{graph} f$ will be the self-affine curve.

Here, we let T_i be the linear part of S_i, *i.e.*:

$$T_i = \begin{pmatrix} 1/m & 0 \\ a_i & c_i \end{pmatrix}.$$

Now, let $I_{i_1,\cdots,i_k}$ be the interval of the t-axis consisting of those t with base-m expansion of $0.i'_1 \cdots i'_k \cdots$ where $i'_j = i_j - 1$. Here, the part of F for which $t \in I_{i_1,\cdots,i_k}$ is the affine image of $S_{i_1} \circ \cdots \circ S_{i_k}(F)$, which is the translate of $T_{i_1} \circ \cdots \circ T_{i_k}(F)$. Also, by induction, we have:

$$T_{i_1} \circ \cdots \circ T_{i_k} = \begin{pmatrix} m^{-k} & 0 \\ m^{1-k} a_{i_1} + m^{2^k} c_{i_1} a_{i_2} + \cdots + c_{i_1} c_{i_2} \cdots c_{i_{k-1}} a_{i_k} & c_{i_1} c_{i_2} \cdots c_{i_k} \end{pmatrix}.$$

Now, we observe that the vertical lines are contracted by $c_{i_1} \cdots c_{i_k}$, whereas the bottom left entry can be bounded by:

$$\begin{aligned} \left| m^{1-k} a_{i_1} + m^{2^k} c_{i_1} a_{i_2} + \cdots + c_{i_1} c_{i_2} \cdots c_{i_{k-1}} a_{i_k} \right| &\leq \big((mc)^{1-k} + (mc)^{2-k} + \cdots + 1 \big) c_{i_1} \cdots c_{i_{k-1}} a \\ &\leq r c_{i_1} \cdots c_{i_{k-1}}, \end{aligned}$$

where $a = \max |a_i|$ and $c = \min\{c_i\} > 1/m$ and $r = a/\big(1 - (mc)^{-1}\big)$. Thus, we know that the image of $T_{i_1} \circ \cdots \circ T_{i_k}(F)$ is contained in a rectangle of height $(r+h)c_{i_1} \cdots c_{i_k}$, where h is the height of F.
Moreover, note that q_1, q_2, q_3 are non-collinear points chosen from $S_1(p_1), \cdots, S_m(p_1), p_m$, then $T_{i_1} \circ \cdots \circ T_{i_k}(F)$ contains the points $T_{i_1} \circ \cdots \circ T_{i_k}(q_j)$ for $j = 1, 2, 3$. Thus, the height of the triangle with these vertices is at least $c_{i_1} \cdots c_{i_k} d$ where d is the vertical distance from q_2 to the segment $[q_1, q_3]$, thus the range of f over $I_{i_1, \cdots, i_k}$ is bounded by:

$$dc_{i_1} \cdots c_{i_k} \leq R_f[I_{i_1,\cdots,i_k}] \leq r_1 c_{i_1} \cdots c_{i_k},$$

where $r_1 = r + h$.
Therefore, with fixed k, we can sum over the m^k intervals of $I_{i_1,\cdots,i_k}$ of lengths m^{-k}, by number of δ-mesh of graph (Proposition VI.1.4), we have:

$$m^k d \sum c_{i_1} \cdots c_{i_k} \leq N_{m^{-k}}(F) \leq 2m^k + m^k d \sum c_{i_1} \cdots c_{i_k}.$$

Meanwhile, we note that the c_{i_j}'s ranges through all permutations, so we have $\sum c_{i_1} \cdots c_{i_k} = (c_1 + \cdots + c_m)^k$, thus:

$$dm^k (c_1 + \cdots + c_m)^k \leq N_{m^{-k}}(F) \leq 2m^k + r_1 m^k (c_1 + \cdots + c_m)^k.$$

Here, we take the logarithms, and by equivalent definition (iii) (Proposition II.2.2), we have:

$$\dim_B F = 1 + \frac{\log(c_1 + \cdots + c_m)}{\log m}.$$

⌟

VII Examples: Random Fractals

VII.1 Random Cantor Set

Remark VII.1.1. Construction of Random Cantor Set.

Recall that for the general construction of k-level intervals (Definition III.1.6), we can construct the cantor sets. In particular, we can construct $\mathcal{C} = \bigcap_{k=1}^{\infty} E_k$ with $[0,1] = E_0 \supset E_1 \supset \cdots$ as a decreasing sequence of closed sets, where each E_k is a union of 2^k disjoint closed level-k intervals. We construct each interval from the upper level with a left and right intervals (I_L and I_R). ⌟

Definition VII.1.2. Probability (Pre-)measure on Cantor Set.

Let $0 < a \leq b < 1/2$ be fixed, and let Ω denote the class of all decreasing sequences $[0,1] = E_0 \supset E_1 \supset E_2 \supset \cdots$ such that E_k has 2^k disjoint closed intervals with index 1 and 2. In particular, we define:

$$C_{i_1,\cdots,i_k} = \frac{|I_{i_1,\cdots,i_k}|}{|I_{i_1,\cdots,i_{k-1}}|},$$

with left or right sides coinciding respectively, and suppose that:

$$a \leq C_{i_1,\cdots,i_k} \leq b \text{ for all } i_1,\cdots,i_k.$$

The measure $\mathbb{P}$ is defined such that the ratios $C_{i_1,\cdots,i_k}$ are random variables, and note that this distribution does not have to be identical among the interval. In particular, we enforce the statistical self-similarity that $C_{i_1,\cdots,i_k,1}$ to have the same distribution as $C_1 = |I_1|$ and likewise for $C_{i_1,\cdots,i_k,2}$ with $C_2 = |I_2|$.
Note that we can naturally define the "measurable set" $\mathcal{F}$ as the collection of sets that are (Carathéodary) measurable subsets of Ω to obtain a measure. ⌟

A figure of a random Cantor set can be illustrated as follows:

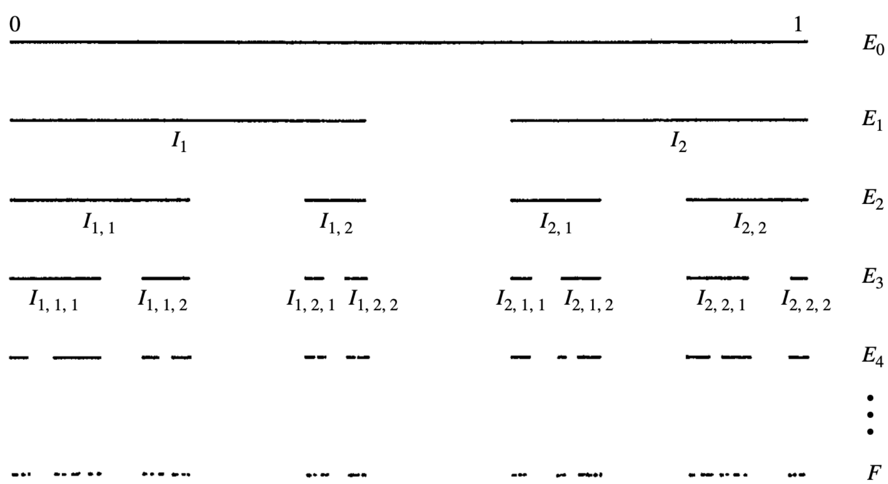

Figure VII.1. Figure of Random Cantor Set.

Theorem VII.1.3. Dimension of Random Cantor Set.

With probability 1, the random Cantor set $\mathcal{C}$ has $\dim_{\mathrm{H}} \mathcal{C} = s$, in which s satisfies that $\mathbb{E}(C_1^s + C_2^s) = 1$.

Proof. First, note that $\mathbb{E}(C_1^s + C_2^s)$ with variable s is continuous and strictly monotonically decreasing. Note that C_1 and C_2 has ratios strictly less than $1/2$, thus we know that $\mathbb{E}(C_1^1 + C_2^1) < 1$ and $\mathbb{E}(C_1^0 + C_2^0) = 2$, so the equation $\mathbb{E}(C_1^s + C_2^s) = 1$ has a unique solution.

For the simplicity of notation, we denote $I \in E_k$ for interval I as a level-k interval $I_{i_1,\cdots,i_k}$ of E_k. Moreover, we denote I_L and I_R for the left ($I_{i_1,\cdots,i_k,1}$) and right ($I_{i_1,\cdots,i_k,2}$), respectively. Additionally, consider $\mathbb{E}(X|\mathcal{F}_k)$ for the conditional expectation of a random variable X given information about $C_{i_1,\cdots,i_j}$ for all sequences such that $j \leq k$, *i.e.*, we have constructed all intervals prior to (and including) E_k.

Now, let $I_{i_1,\cdots,i_k}$ be an interval of E_k, since the distributions are identical, then for $s > 0$:

$$\mathbb{E}\big((|I_{1,\cdots,i_k,1}|^s + |I_{1,\cdots,i_k,2}|^s)\big|\mathcal{F}_k\big) = \mathbb{E}(C^s_{i_1,\cdots,i_k,1} + C^s_{i_1,\cdots,i_k,2})|I_{i_1,\cdots,i_k}|^s = \mathbb{E}(C_1^s + C_2^s)|I_{i_1,\cdots,i_k}|.$$

Now, if we sum over all intervals in E_k, we have:

$$\mathbb{E}\left(\sum_{I\in E_{k+1}} |I|^s \middle| \mathcal{F}_k\right) = \sum_{I\in E_k} |I|^s \mathbb{E}(C_1^s + C_2^s) = \sum_{I\in E_k} |I|^s.$$

Here, we have $X_k = \sum_{i\in E_k} |I|^s$ as an L^2-bounded martingale with respect to $\mathcal{F}_k$, *i.e.*, there exists a number c such that $\mathbb{E}(X)k^2 \leq c$ for all k. In particular, X_k (with probability 1) converges to a random variable X as $k \to \infty$ such that $\mathbb{E}(X) = \mathbb{E}(X_0) = \mathbb{E}(1^s) = 1$, that is $0 \leq X < \infty$. However, suppose that $X = 0$ with probability $q < 1$, we have $X = 0$ if and only if both $\sum_{I\in E_k\cap I_1} |I|^s$ and $\sum_{I\in E_k\cap I_2} |I|^s$ converges to 0 as $k \to \infty$, *i.e.*, indicating that the probability happening is $q = q^2$, so we must have $q = 0$, so we have $0 < X < \infty$ with probability 1.

Particularly, this implies that with probability 1, there are (random) numbers M_1 and M_2 such that:

$$0 < M_1 \leq X_k = \sum_{I\in E_k} |I|^s \leq M_2 < \infty \text{ for all } k.$$

Since we have $|I| \leq 2^{-k}$ for all $I \in E_k$, so $\mathcal{H}^s_\delta(\mathcal{C}) \leq \sum_{i\in E_k} |I|^s \leq M_2$ of $k \geq -\log\delta/\log 2$, which gives that $\mathcal{H}^s(\mathcal{C}) \leq M_2$, *i.e.*, $\dim_{\mathrm{H}} F \leq s$ with probability 1, completing the upper bound inequality.

For the lower bound, we would use the potential theoretic method, in particular the Hausdorff dimension in relation to mass distribution (Theorem III.3.4). Specifically, our goal is to construct a mass distribution μ on the random set $\mathcal{C}$ such that $I_s(\mu) < \infty$. Here, we let μ be defined for an interval $I \in E_k$ that:

$$\mu(I) = \lim_{j\to\infty} \left\{\sum |J|^s : J \in E_j \text{ and } J \subset I\right\}.$$

By martingale, the limit exists and $0 < \mu(I) < \infty$ with probability 1. Moreover, if $I \in E_k$, we have:

$$\mathbb{E}\big(\mu(I)\big|\mathcal{F}_k\big) = |I|^s.$$

Note that by definition, we have $\mu(I) = \mu(I_L) + \mu(I_R)$ for $I \in E_k$, so we consider μ additive on the level-k sets for all k, so μ can be extended to a mass distribution, whose support is contained in $\bigcap_{k=0}^\infty E_k = \mathcal{C}$.

Then, let $0 < t < s$ be fixed, and we estimate the expectation of the t-energy of μ. Let $x, y \in \mathcal{C}$ be arbitrary, there exists a greatest integer k such that x and y belong to a common level-k interval, in which we denote this interval by $x \wedge y$. Note that I is a level-k interval, its level-$(k+1)$ intervals must be separated by a gap of at least $(1-2b)|x \wedge y|$, thus the Newtonian Energy (Definition III.3.2) is now:

$$\iint_{x\wedge y} \frac{d\mu(x)d\mu(y)}{|x-y|^t} = 2\int_{x\in(x\wedge y)_L}\int_{y\in(x\wedge y)_R} \frac{d\mu(x)d\mu(y)}{|x-y|^t} \leq \frac{2\mu\big((x\wedge y)_L\big)\mu\big((x\wedge y)_R\big)}{(1-2b)^t|x\wedge y|^t}.$$

Now, if $x \wedge y \in E_k$, we have:

$$\begin{aligned}\mathbb{E}\left(\iint_{x\wedge y} \frac{d\mu(x)d\mu(y)}{|x-y|^t}\middle|\mathcal{F}_k\right) &\leq \frac{2\mathbb{E}\left[\mu\big((x\wedge y)_L\big)\middle|\mathcal{F}_k\right]\mathbb{E}\left[\mu\big((x\wedge y)_R\big)\middle|\mathcal{F}_k\right]}{(1-2b)^t|x\wedge y|^t} \\ &\leq \frac{2|(x\wedge y)_L|^s|(x\wedge y)_R|^s}{(1-2b)^t|x\wedge y|^t} \leq \frac{2|x\wedge y|^{2s-t}}{(1-2b)^t}.\end{aligned}$$

Hence, as we consider the variation, we have the unconditional expectation as:

$$\mathbb{E}\left(\iint_{x\wedge y}\frac{d\mu(x)d\mu(y)}{|x-y|^t}\right)\leq\frac{2\mathbb{E}(|x\wedge y|^{2s-t})}{(1-2b)^t}.$$

As we sum over all $I\in E_k$, we have:

$$\mathbb{E}\left(\sum_{I\in E_k}\iint_{x\wedge y}\frac{d\mu(x)d\mu(y)}{|x-y|^t}\right)\leq\frac{2\mathbb{E}\left(\sum_{i\in E_k}|x\wedge y|^{2s-t}\right)}{(1-2b)^t}.$$

Here, note that:

$$\mathbb{E}\left(\sum_{I\in E_{k+1}}|I|^s\right)=\mathbb{E}\left(\sum_{I\in E_k}|I|^s\right)\mathbb{E}(C_1^s+C_2^s),$$

and by doing so repetitively, we have:

$$\mathbb{E}\left(\sum_{I\in E_k}\iint_{x\wedge y}\frac{d\mu(x)d\mu(y)}{|x-y|^t}\right)\leq\frac{2\mathbb{E}(C_1^{2s-t}+C_2^{2s-t})^k}{(1-2b)^t},$$

and that $\mathbb{E}(C_1^{2s-t}+C_2^{2s-t})<1$.
Therefore, as we consider the expectation for I_s, we have:

$$\begin{aligned}\mathbb{E}(I_s)=\mathbb{E}\left(\int_{\mathcal{C}}\int_{\mathcal{C}}\frac{d\mu(x)d\mu(y)}{|x-y|^t}\right)&=\mathbb{E}\left(\sum_{k=0}^{\infty}\sum_{(x\wedge y)\in E_k}\iint_{x\wedge y}\frac{d\mu(x)d\mu(y)}{|x-y|^t}\right)\\&\leq\sum_{k=0}^{\infty}\frac{2\mathbb{E}(C_1^{2s-t}+C_2^{2s-t})^k}{(1-2b)^t}=\frac{2\sum_{k=0}^{\infty}\mathbb{E}(C_1^{2s-t}+C_2^{2s-t})^k}{(1-2b)^t}<\infty.\end{aligned}$$

Hence, the t-energy of μ is finite for all $0<t<s$ with probability 1. As we have that $0<\mu(\mathcal{C})=\mu([0,1])<\infty$ with probability 1, we have $\dim_{\mathrm{H}}\mathcal{C}\geq t$ for all $0<t<s$, which implies that $\dim_{\mathrm{H}}\mathcal{C}\geq s$, completing proof for the lower bound. □

Remark VII.1.4. Generalization of Random Cantor Sets.

The construction of random cantor set (Remark VII.1.1) can be extended for an open set $V\subset\mathbb{R}^n$. For integer $m\geq 2$ and $0<b<1$, we let Ω be the class of all decreasing sequences $\overline{V}=E_0\supset E_1\supset E_2\supset\cdots$ be closed sets satisfying that E_k is a union of the m^k closed sets $\overline{V}_{i_1,\cdots,i_k}$ where $i_j=1,\cdots,m$ for $1\leq j\leq m$ and $V_{i_1,\cdots,i_k}$ is either similar to V or is the empty set.
Additional, we supposer that for all $i_1,\cdots,i_k$ that $V_{i_1,\cdots,i_k}\supset V_{i_1,\cdots,i_k,i}$ for all $1\leq i\leq m$ and these sets are disjoint, *i.e.*, the open set condition (Definition IV.2.2). Moreover, if $V_{i_1,\cdots,i_k}$ is non-empty, we can defined $C_{i_1,\cdots,i_k}=|V_{i_1,\cdots,i_k}|/|V_{i_1,\cdots,i_{k-1}}|$ for the similarity ratio, otherwise, we take the ratio as 0 when the set is empty, and thus defining the random set as $F=\bigcap_{k=0}^{\infty}E_k$.
Again, we let $\mathbb{P}$ be the probability (pre-)measure on the family of subsets of Ω such that $C_{i_1,\cdots,i_k}$ are random variables. ⌟

Theorem VII.1.5. Dimensions of Random Set.

Let $F\subset\mathbb{R}^n$ be the random set, then:

(i) It has probability q of being empty, where $t=q$ is the smallest non-negative solution to the polynomial equation:

$$f(t)\equiv\sum_{j=0}^{m}\mathbb{P}(N=j)t^j=t,$$

where N is the (random) number of $C_1,\cdots,C_m$ are positive.

(ii) It has probability $1-q$ such that F has Hausdorff and box dimensions s satisfying that:

$$\mathbb{E}\left(\sum_{i=1}^{m} C_i^s\right) = 1.$$

Proof. (i) Here, we note that if there is a positive probability that $N = 0$, then there is a positive probability that $E_1 = \varnothing$ and therefore that $F = \varnothing$, as the set "extincts". By the nature of self-similarity, the probability q_0 of this happening is $f(q_0)$, so $q_0 = f(q_0)$, and if q is the least non-negative solution of $f(q) = q$, then by the fact that f is increasing, we can show $\mathbb{P}(E_k = \varnothing) = f\big(\mathbb{P}(E_{k-1} = \varnothing)\big) \leq f(q) = q$, inductively. Hence, $q_0 \leq q$, which leads to $q_0 = q$.

(ii) The argument considers the cases given that F is non-empty, then F conditionally has probability 1 that its dimension is s. The argument is an analogy of Dimension of Random Cantor Set (Theorem VII.1.3). □

Remark VII.1.6. Conditions for the Dimensions of Random Sets.

Let $F \subset \mathbb{R}^n$ be a random set, then:

(i) F has probability 0 of being empty if and only if $N \geq 1$ with probability 1.

(ii) F has probability 1 of being empty if and only if either $\mathbb{E}(N) < 1$ or both $\mathbb{E}(N) = 1$ and $\mathbb{P}(N = 1) < 1$.

⌟

Example VII.1.7. Random von Koch Curve.

Recall for modified von Koch curve (Example IV.2.5), we have constructed F with $0 < a \leq 1/3$ and replace the middle a interval of each segment into the two other sides of the equilateral triangle. The conclusion we had was that the dimension $s = \dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F$ where s satisfies:

$$2\left(\frac{1-a}{2}\right)^s + 2a^s = 1.$$

Here, we let A be a random variable with uniform distribution on the interval $(0, 1/3)$, and we form the limiting curve F in the same manner. Likewise in modified von Koch curve (Example IV.2.5), we satisfies the open set condition, so by dimensions of random set (Theorem VII.1.5), we have probability 1 that the dimension s satisfies that:

$$\begin{aligned}
1 &= \mathbb{E}\left[2\left(\frac{1}{2}(1-A)\right)^s + 2A^s\right] \\
&= \frac{1}{1/3}\int_0^{1/3}\left[2\left(\frac{1}{2}(1-A)\right)^s + 2A^s\right] dA \\
&= 6\int_0^{1/3}\left(\frac{(1-A)^s}{2^s} + A^s\right) dA \\
&= 6\left[-\frac{(1-A)^{s+1}}{2^s(s+1)} + \frac{A^{s+1}}{s+1}\right]_{A=0}^{A=1/3} \\
&= \frac{-12\cdot 3^{-(s+1)} + 6\cdot 2^{-(s+1)} + 6\cdot 3^{-(s+1)}}{s+1} = \frac{3\cdot 2^{-s} - 2\cdot 3^{-s}}{s+1},
\end{aligned}$$

which is approximately $s \approx 1.1448$. ⌟

VII.2 Fractal Percolation

Definition VII.2.1. Construction of Random Fractal from Unit Square.

Let $E_0 = [0,1]^2$ be the unite square, and let $0 < p < 1$ be fixed, we divide E_0 into 9 squares, each of side length $1/3$ of E_0, and we select a subset of these squares to form E_1 such that each square has independent probability p of being selected. Such steps are done iteratively to form $F_p = \bigcap_{k=0}^{\infty} E_k$. ⌟

An example construction is illustrated as follows:

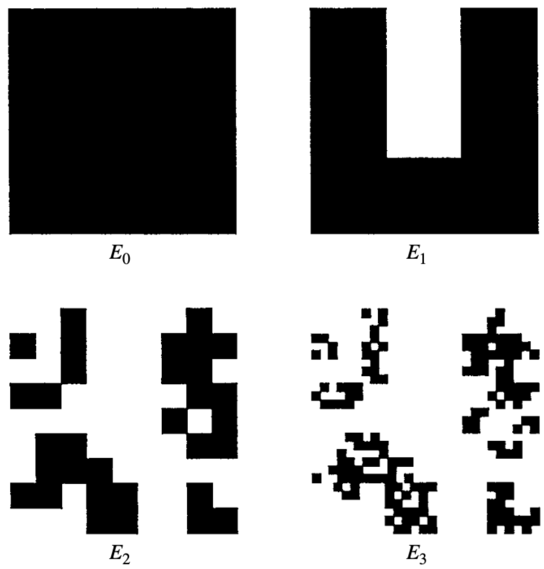

Figure VII.2. Construction of Random Fractal from Unit Square with $0 < p < 1$.

Proposition VII.2.2. Dimensions of Random Fractal from Unit Square.

Let $0 < p < 1$ be fixed, and let $t = q$ be the smallest positive solution of the polynomial equation:

$$t = (pt + 1 - p)^9.$$

(i) F_p is empty with probability q.

(ii) If $p \leq 1/9$, then $q = 1$.

(iii) If $1/9 < p < 1$, then $0 < q < 1$ and, with probability $1 - q$, $\dim_H F_p = \dim_B F_p = \log(9p)/\log 3$.

Proof. (i) Here, let N be the (random) number of squares in E_1, then we have the probability for binomial distribution as:

$$\mathbb{P}(N = j) = \binom{9}{j} p^j (1-p)^{9-j}.$$

Hence, by dimensions of random set (i) (Theorem VII.1.5), we have the probability that $F_p = \varnothing$ as the smallest positive solution of:

$$f(t) \equiv \sum_{j=0}^{9} \binom{9}{j} p^j (1-p)^{9-j} t^j = (pt + 1 - p)^9 = t.$$

(ii) Note that if $p \leq 1/9$, we have $\mathbb{E}(N) \leq 1$. In particular, if $\mathbb{E}(N) = 1$, then $p = 1/9$ and $\mathbb{P}(N = 1) = \binom{9}{1} \cdot (1/9) \cdot (8/9)^8 < 1$, so by conditions for the dimensions of Random Sets (ii) (Remark VII.1.6), we have $q = 1$.

(iii) Here, by dimensions of random set (ii) (Theorem VII.1.5), if the set is non-empty, that is almost every case for $1 - q$, the dimension s being the solution to:

$$\mathbb{E}\left(\sum_{i=1}^{N} C_i^s\right) = \mathbb{E}\left(\sum_{i=1}^{N} 3^{-s}\right) = 3^{-s}\mathbb{E}(N) = 1.$$

Note that the expectation of number of chosen ones is $9p$, hence:

$$9p = 3^s,$$

as desired. □

Remark VII.2.3. Qualitative Changes to the Random Set F_p.

Consider F_p changes as p increases from 0 to 1, case-by-case:

(i) By dimensions of random fractal from unit square (ii) (Proposition VII.2.2), if $0 < p \leq 1/9$, then $F_p = \emptyset$.

(ii) If $1/9 < p < 1/3$, we have probability 1 that either $F_p = \emptyset$ or $\dim_{\mathrm{H}} F_p = \log(9p)/\log 3 < 1$, which indicates that F_p is totally disconnected by Hausdorff dimension $< 1 \Longrightarrow$ total disconnectedness (Proposition I.2.10).

(iii) When p is close to 1, it is plausible that a high proportional of the squares are retained at each stage of the construction such that F_p will connect the left and right sides of E_0, which is described as *percolation* between the sides. In fact an example is given as jointness of random fractal F_p for large p (Proposition VII.2.4).

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Proposition VII.2.4. Jointness of Random Fractal F_p for Large p.

Suppose that $0.999 < p < 1$, then there is a positive probability that the random fractal F_p joins the left and right sides of E_0.
In fact, the probability is larger than 0.9999*.*

Proof. First, we establish the following claim: If I_1 and I_2 are abutting squares in E_k and both I_1 and I_2 contain either 8 or 9 sub-squares of E_{k+1}, then I_1 and I_2 has a connected abutting sub-squares. This is valid since two abutting squares are joined by 3 lines of the sub-squares, so the upper bound of losing 2 squares ensures the connection.
Here, we define a square of E_k *full* if it contains either 8 or 9 sub-squares of E_{k+1}. Additionally, we recursively define a square to be *m-full* if it contains either 8 or 9 squares of E_{k+1} that are $(m-1)$*-full*.
Hence, for E_0 to be m-full for $m \geq 1$, we have one of the following cases:

(i) E_1 contains 9 squares all of which are $(m-1)$-full, or

(ii) E_1 contains 9 squares of which 8 are $(m-1)$-full, or

(iii) E_1 contains 8 squares all of which are $(m-1)$-full.

Thus, let p_m be the probability in which E_0 is m-full, hence, by summing the probabilities of the alternatives and the self-similarity of the process, for $m \geq 2$ we obtain:

$$p_m = p^9 p_{m-1}^9 + 9p^9 p_{m-1}^8(1-p_{m-1}) + 9p^8(1-p)p_{m-1}^8 = 9p^8 p_{m-1}^8 - 8p^9 p_{m-1}^9.$$

Moreover, consider that $p_1 = p^9 + 9p_{m-1}^8(1-p) = 9p^8 - 8p^9$, so we have a dynamical system $p_m = f(p_{m-1})$ for $m \geq 1$, where $p_0 = 1$ and:

$$f(t) \equiv 9p^8t^8 - 8p^9t^9.$$

Now, suppose that $p = 0.999$, we have the function:

$$f(t) \approx 8.928\ 251t^8 - 7.928\ 287t^9,$$

such that $t_0 \approx 0.999\ 961$ is a fixed point of f which is stable in the sense of $0 < f(t) - t_0 \leq (t-t_0)/2$ if $t_0 < t \leq 1$. Note that p_m is decreasing and converges to t_0 as $m \to \infty$, there is a positive probability $t_0 > 0$ that E_0 is m-full for all m. □

Theorem VII.2.5. Critical Value for Connectedness.

There is a critical number p_c with $0.333 < p_c < 0.999$ such that:

(i) If $0 < p < p_c$, then F_p is totally disconnected with probability 1.

(ii) If $p_c < p < 1$, then there is a positive probability that F_p connects the left and right sides of E_0.

Sketch of proof. Here, we first suppose that p is such that there is a positive probability of F_p not being totally disconnected, which implies that there is positive probability of path connecting two arbitrary points on the two sides from the left to the right. Otherwise, when F_p has probability 1 of being totally disconnected, then $F_{p'}$ must also have probability 1 of being totally disconnected for $p' < p$, hence p_c is the supremum of p such that F_p is totally disconnected. □