Rigor in Mathematics

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0.1. Example

Define a function $f : (0,1) \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ where } m, n \in \mathbb{Z}^+ \text{ and } \gcd(m,n) = 1. \end{cases}$$

(a) Determine if f is continuous at irrational points, and if f is continuous at rational points.

(b) (If one has learned Calculus.) Determine if f is differentiable at irrational points.

0.2. Example

Note that *countable* is defined for a set when you can number each element in the set to a natural number (that is $\mathbb{N}$). Clearly, by this definition, we know that integers (that is $\mathbb{Z}$) are countable, since we can establish the following function $f : \mathbb{Z} \to \mathbb{N}$ so every number can be mapped:

$$f(x) = \begin{cases} 2x, & \text{if } x \geq 0, \\ -2x - 1, & \text{if } x < 0. \end{cases}$$

Therefore, a set could be countable even if it is infinite. Then consider the following two sets of infinite elements.

(a) Determine if $\mathbb{Q}$ is countable.

(b) Determine if $\mathbb{R}$ is countable.

1.1. Problem:

Fix $a > 1$, and let x and y be real numbers, show that:

$$a^x \cdot a^y = a^{x+y}.$$

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Natural Number ($\mathbb{N}$) Case:

This "problem" is trivially correct for all x and y being natural numbers. You can easily found it holding by applying the definition of exponentials as *multiplying things together*, that is:

$$a^x \cdot a^y = \underbrace{a \cdot a \cdot \cdots \cdot a}_{x \text{ times}} \cdot \underbrace{a \cdot a \cdot \cdots \cdot a}_{y \text{ times}} = \underbrace{a \cdot a \cdot \cdots \cdot a}_{x+y \text{ times}} = a^{x+y}.$$

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Integer ($\mathbb{Z}$) Case:

Diligent readers shall already realize how we can extend the proof for all integer, namely consider that a^{-x} being the *reciprocal* of the number and the proof shall get through.

1.2. Theorem:

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Exercise to the Readers:

The proof of this theorem would requires proof for both *existence* and *uniqueness* of y. This proof is left as an exercise to capable readers (cf. Exercise 3).

1.3. Lemma:

The case for **Problem 1.1** holds for all x and y being rational numbers. That is:
Fix $a > 1$, and let x and y be rational numbers, show that:

$$a^x \cdot a^y = a^{x+y}.$$

1.3. Lemma:

The case for **Problem 1.1** holds for all x and y being rational numbers. That is:
Fix $a > 1$, and let x and y be rational numbers, show that:

$$a^x \cdot a^y = a^{x+y}.$$

proof:

To prove this lemma, we first want to establish that for all rational x, we can write $a^x = (a^p)^{1/q}$ where p and q are integers.
Therefore, we let $r = m/n = p/q$, so we have:

$$rnq = mq = np.$$

Then, let $y_1 = (a^m)^{1/n}$ and $y_2 = (a^p)^{1/q}$, we have:

$$y_1^{rnq} = y_1^n p = ((a^m)^{1/n})^{np} = a^{mp},$$

$$y_2^{rnq} = y_2^m q = ((a^p)^{1/q})^{mq} = a^{mp}.$$

1.3. Lemma:

proof (CONT.):

Therefore, by knowing that $y_1^{rnq} = a^{mp} = y_2^{rnq}$, and based on **Theorem 1.2**, we know that:

$$y_1 = y_2 \Longrightarrow (a^m)^{1/n} = (a^p)^{1/q}.$$

Therefore, we know that for all rational x, we can write $a^x = (a^p)^{1/q}$ where p and q are integers.
Then, let $x = \frac{\alpha}{\beta}$ and $y = \frac{\gamma}{\delta}$, where $\alpha, \gamma \in \mathbb{Z}$ and $\beta, \delta \in \mathbb{Z}^+$. There, we can rewrite the expression as:

$$a^{x+y} = a^{\frac{\alpha}{\beta}+\frac{\gamma}{\delta}} = a^{\frac{\alpha\delta+\beta\gamma}{\beta\delta}} = (a^{\alpha\delta+\beta\gamma})^{1/(\beta\delta)}.$$

By definition of exponents, we have:

$$a^{x+y} = (a^{\alpha\delta}a^{\beta\gamma})^{1/(\beta\delta)}.$$

1.3. Lemma:

proof (CONT.):

Here, we want to break $a^{\alpha\delta}a^{\beta\gamma}$ apart, so we want to show that for $a, b \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ that $(ab)^{1/n} = a^{1/n}b^{1/n}$. Here, we notice that:

$$(a^{1/n}b^{1/n})^n = a^{n/n}b^{n/n} = ab,$$

then, by **Theorem 1.2** again, there exists a single positive $(a^{1/n}b^{1/n})^{n/n} = a^{1/n}b^{1/n}$ equivalent to $(ab)^{1/n}$.
Therefore, we continue with the lemma (with $a^{\alpha\delta}, a^{\beta\gamma} \in \mathbb{R}^+$ and $\beta\delta \in \mathbb{Z}^+$) that:

$$a^{r+s} = (a^{\alpha\delta}b^{\beta\gamma})^{1/(\beta\delta)} = a^{(\alpha\delta)/(\beta\delta)}a^{(\beta\gamma)/(\beta\delta)} = a^{\alpha/\beta}a^{\gamma/\delta} = a^x \cdot a^y,$$

as desired. Hence, we have completed the proof for **Lemma 1.3**, which extends the validity to all rational numbers. $\square$

1.3. Lemma

Thus, the case for **Problem 1.1** holds for all x and y being rational numbers. That is:
Fix $a > 1$, and let x and y be rational numbers, we have:

$$a^x \cdot a^y = a^{x+y}.$$

Exercise to the Readers:

Given the time constraints, the rest of the proof to **Problem 1.1** are left to the readers (cf. Exercise 5).

Exercise

1. If you have not done so, complete **Example 0.1** and **Example 0.2**.
2. Let S be an ordered set. Suppose that $E \subseteq S$ and E is bounded above. Also suppose that there exists an $\alpha \in S$ such that:
   - α is an upper bound for E, and
   - If $\gamma < \alpha$,then γ is not an upper bound for E.

   Here, α is the **supremum** of E, written as $\alpha := \sup E$.
   Given this definition, let $E := \left\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\right\} \subsetneq \mathbb{R}$, find $\sup E$.
3. Prove **Theorem 1.2** using the definition of *supremum*. Note that you need to consider both *existence* and *uniqueness*.
4. There always exists some rational numbers between any two irrationals, to show this, follow the below instructions.
   (a) Show that if $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{Z}^+$ such that $nx > y$. This is known as the Archimedean property.
   (b) If $x, y \in \mathbb{R}$ and $x < y$, then there is a $p \in \mathbb{Q}$ such that $x < p < y$.
5. Prove **Problem 1.1** for all $x, y \in \mathbb{R}$.
   Hint: You might want to establish $a^x = \sup A(x)$ where $A(x)$ is the set of all numbers a^t such that $t \leq x$. Then, you want to show that $a^x a^y \leq a^{x+y}$ and $a^x a^y \geq a^{x+y}$ to verify that $a^x a^y = a^{x+y}$.