Special coheirs and model-theoretic trees

James E. Hanson

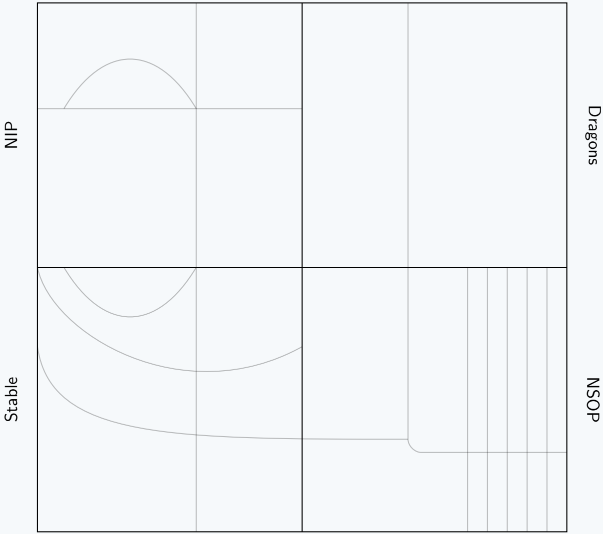
University of Maryland

April 20, 2024

5:00 PM CDT

AMS Special Session on Model Theory II

Drawing a new line on Conant's map



Examples:

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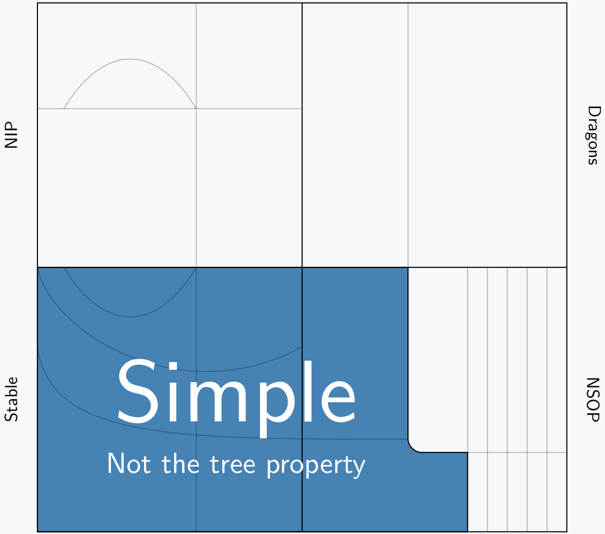

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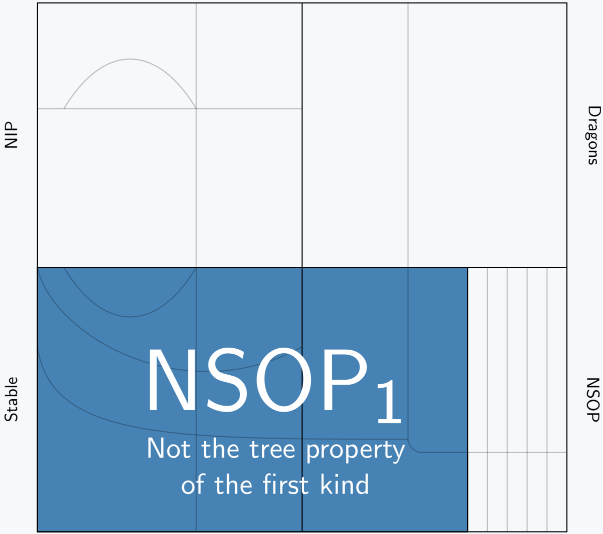

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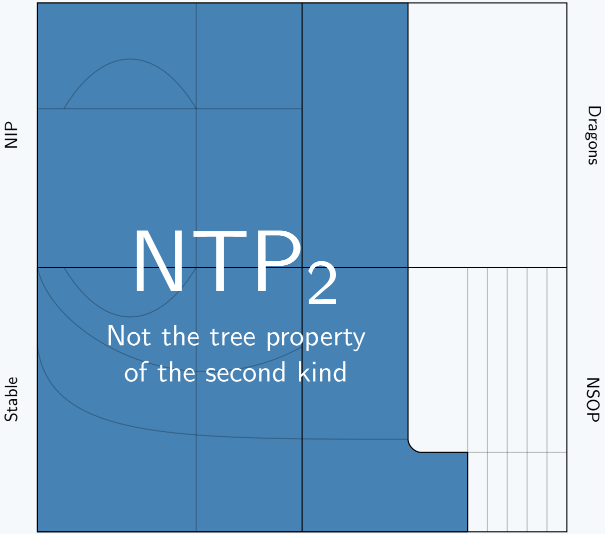

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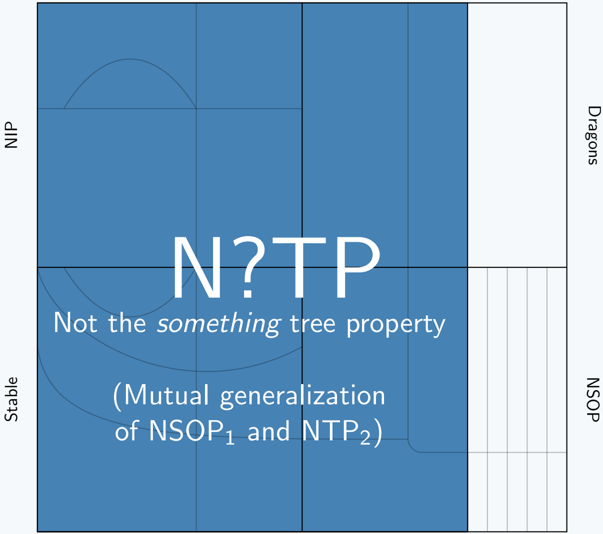

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N?TP: Generic linear order + binary function

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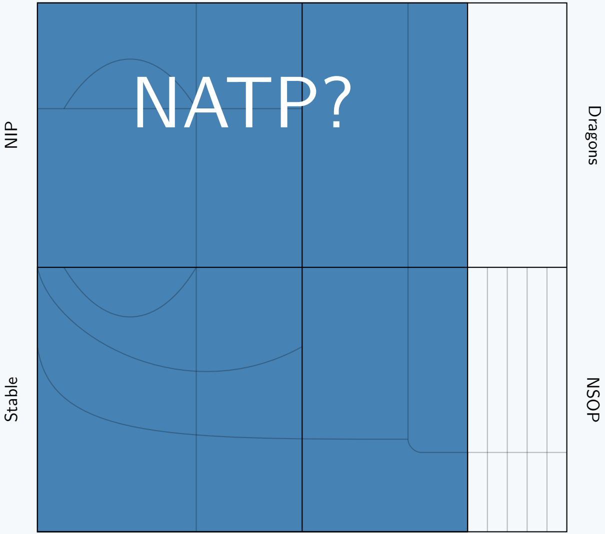

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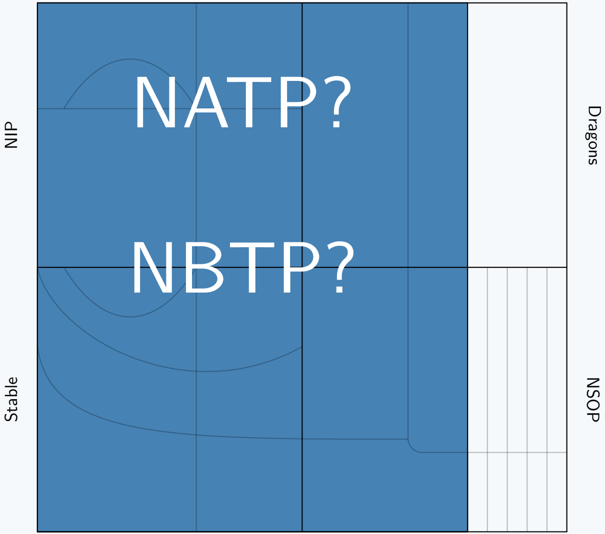

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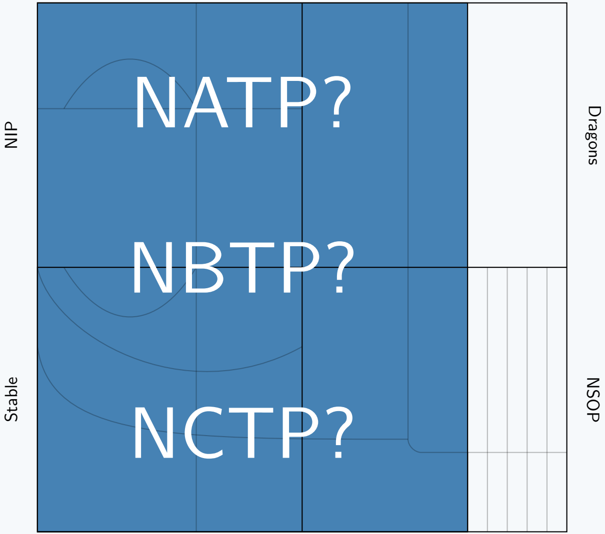

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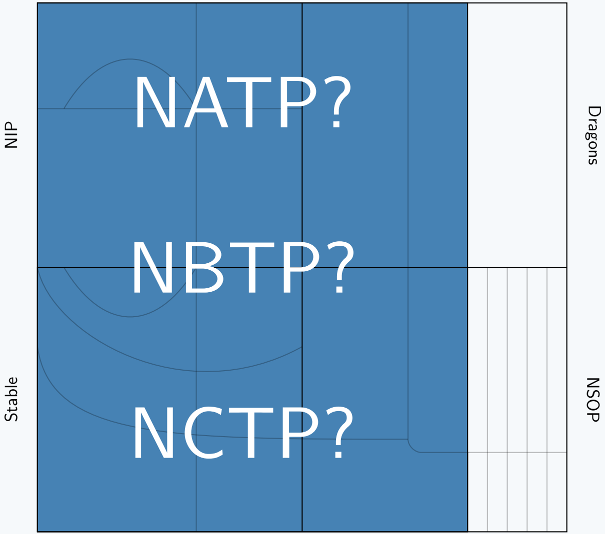

Antichain tree property (Ahn, Kim)

Bizarre tree property (Ramsey, Kruckman)

Comb tree property (Mutchnik)

Coheirs

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Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

$a_0, a_1, \ldots$ is the *Morley sequence generated by* $\mathcal{U}$.

SOP_1 in terms of coheirs

Theorem (Kaplan, Ramsey)

T has SOP_1 iff there are two coheirs $\mathcal{U}$ and $\mathcal{V}$ extending the same type and a formula $\varphi(x, y)$ that divides along $\mathcal{U}$ but not along $\mathcal{V}$.

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DLO example: Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$.

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This is non-trivial. $\mathcal{U}_{\text{pinch}}$ *does not* have this property.

TP_2 in terms of heir-coheirs

Definition

$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

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Lead them to the *bizarre tree property* or *BTP* (uses a weakening of heir-coheirdom).

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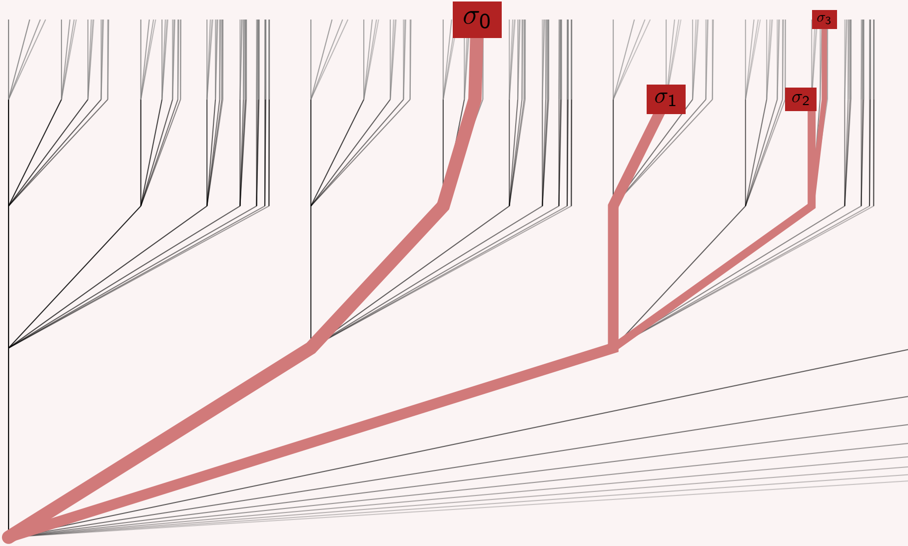
Mutchnik established the following in his proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$.

Theorem (Mutchnik)

The above condition without the switcheroo is equivalent to SOP_1.

A right-comb in $\omega^{<\omega}$



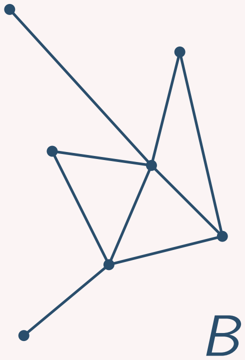
Characterization of CTP

Theorem (H.)

A theory has k-CTP if and only if there is a formula $\varphi(x, b)$ and an heir-coheir $\mathcal{U}$ and a coheir $\mathcal{V}$ extending the type of b such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

What's special about heir-coheirs?

If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.



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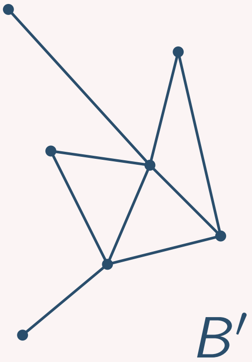

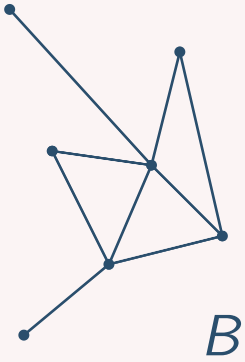

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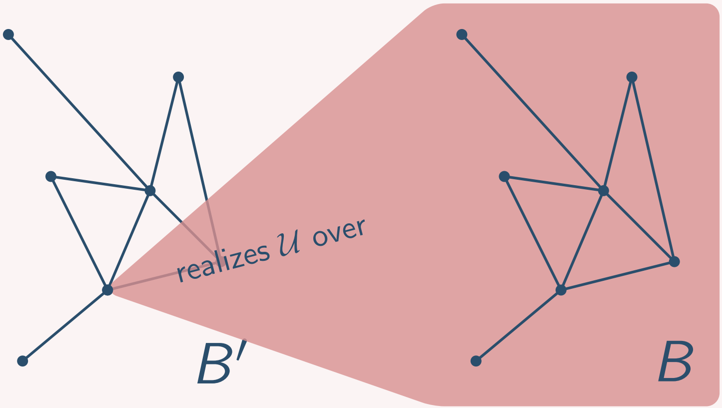

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

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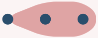
Realize $\mathcal{U}$ 

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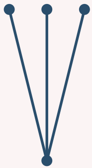

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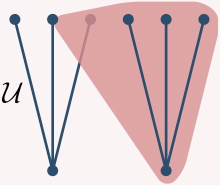

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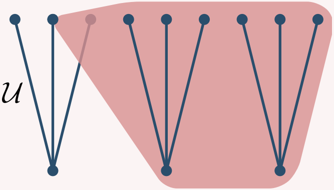
Super realize $\mathcal{U}$



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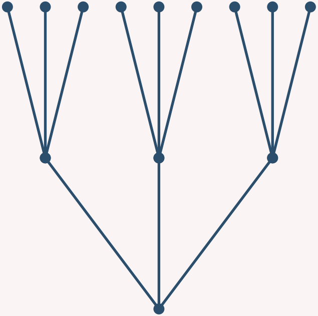

Realize $\mathcal{V}$

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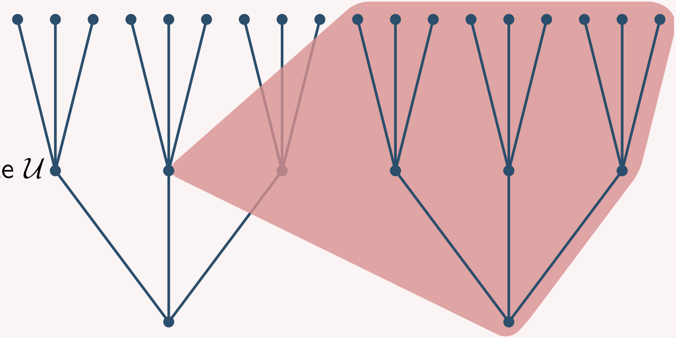

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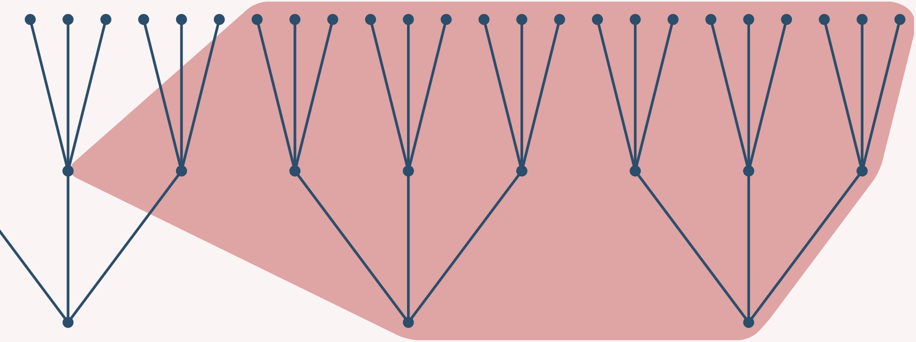

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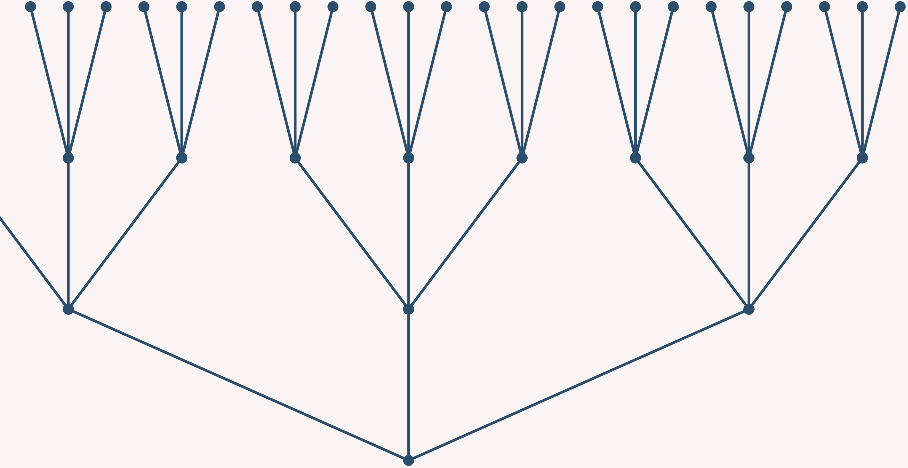

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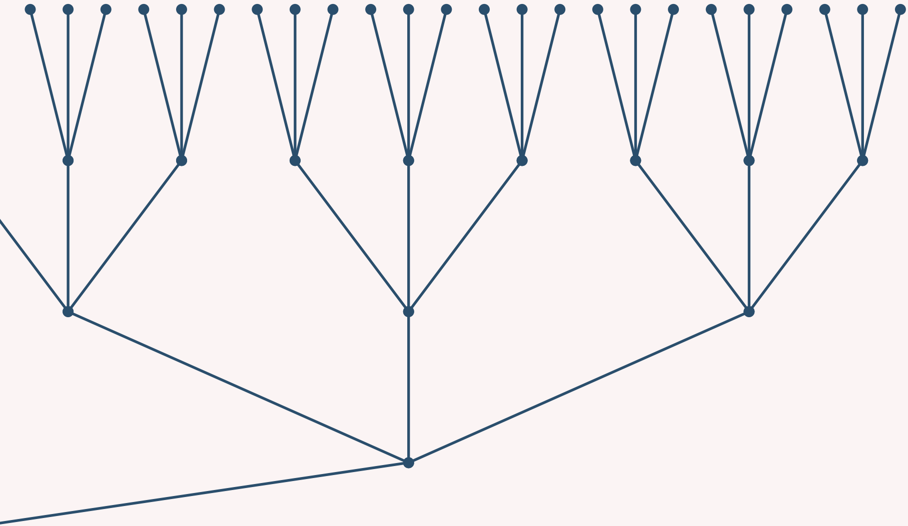

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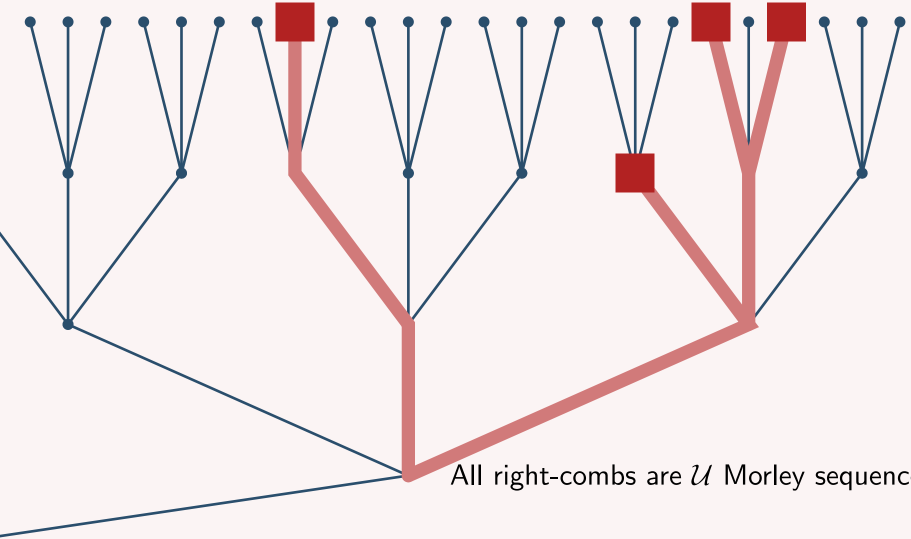

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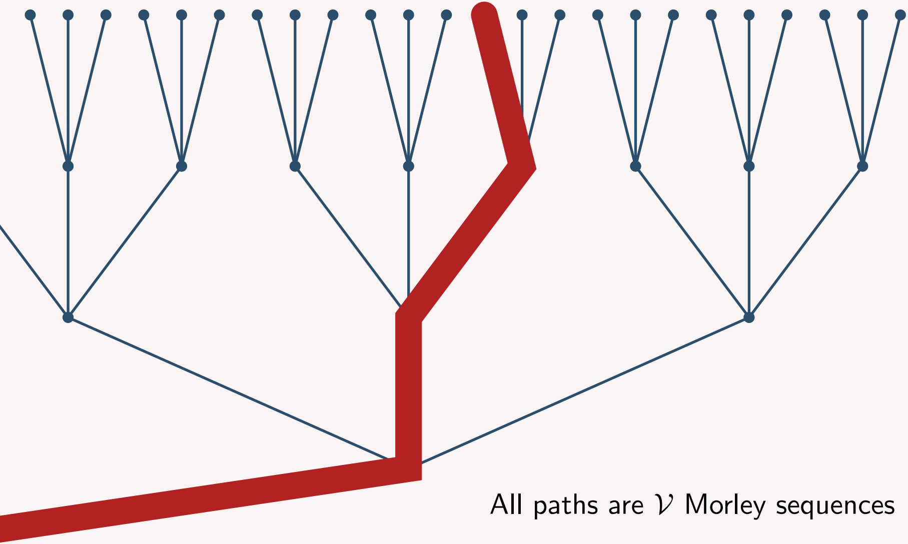

All paths are $\mathcal{V}$ Morley sequences

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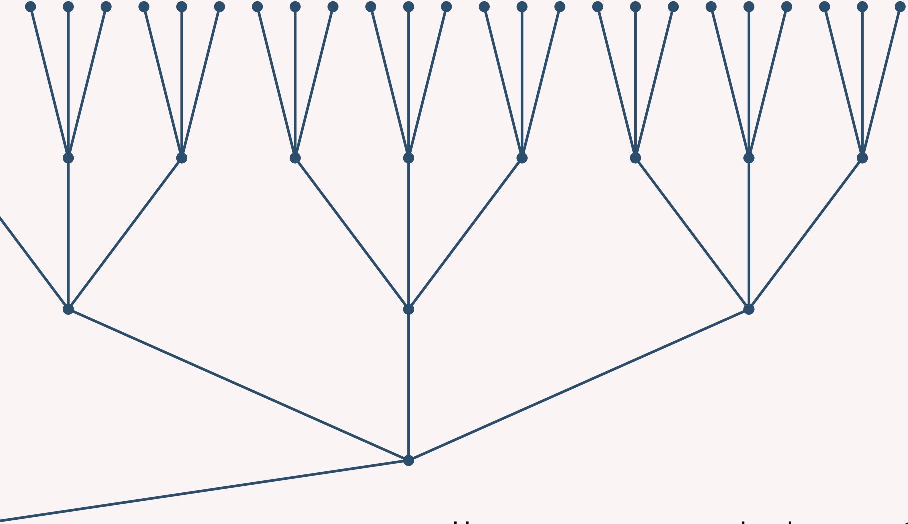

Use compactness to make the tree $\omega^{<\omega}$

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There are many heir-coheirs over $(\mathbb{Q}, <)$ (any non-realized cut). Is this generalizable?

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Argue that if $\mathcal{U}$ extends the type we built and a realizes $\mathcal{U}$ over Mb, then every formula in the type of b over Ma is already realized in M by construction. $\square$

The miniaturized argument as a blueprint for CTP

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The comb tree property (even on $2^{<\omega}$ rather than $\omega^{<\omega}$) gives you precisely what you need to generically build an heir-coheir $\mathcal{U}$ that is 'shadowed' by a coheir $\mathcal{V}$ such that the given formula divides along $\mathcal{V}$ but not along $\mathcal{U}$.

The fundamental theorem of forcing

Definition

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- For each i, σ_{i+1} extends $\sigma_i \frown 1$.
- For each $X \in \mathcal{F}$, there is an i such that $\{b_\tau \in X : \tau \succeq \sigma_i\}$ is dense above σ_i and is in $\mathcal{F}$.

Forcing with comb trees I

Suppose we have a CTP tree $(b_\sigma)_{\sigma\in 2^{<\omega}}$ (for the formula $\varphi(x,y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

- For each i, σ_{i+1} extends $\sigma_i \frown 1$.
- For each $X \in \mathcal{F}$, there is an i such that $\{b_\tau \in X : \tau \succeq \sigma_i\}$ is dense above σ_i and is in $\mathcal{F}$.
- If $\psi(x,c)$ is an M-formula (with c in the monster) such that $\{b_\sigma : \psi(b_\sigma, c)\}$ has somewhere dense intersection with every element of $\mathcal{F}$, then there is a $d \in M$ such that $\{b_\sigma : \psi(b_\sigma, d)\} \in \mathcal{F}$.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter,

Forcing with comb trees II

The second bullet point now ensures that

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generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

Forcing with comb trees II

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Forcing with comb trees II

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The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Forcing with comb trees II

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The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.
Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$.

Forcing with comb trees II

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The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$. Furthermore, the third bullet point will ensure that $\mathcal{U}$ and $\mathcal{V}$ extend the same type over M, so we have the required failure of Kim's lemma for coheirs and heir-coheirs.

Thank you

Forcing with comb trees III