Continuous Logic for Discontinuous Logicians

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Introduction

Continuous logic, or continuous model theory, is a generalization of discrete first-order logic that allows the application of model theoretic machinery, such as stability and neo-stability theory, to classes of structures studied in analysis, such as Banach spaces and C^*-algebras.

As part of the business of doing model theory, model theorists develop very strong intuitions about what can and cannot be said in first-order logic. I think that it's not unreasonable to say that a big part of this is the fact that formal first-order logic was designed to resemble the informal first-order logic regularly used by mathematicians.

Continuous first-order logic is more closely related to discrete first-order logic than any other generalization of it, and I firmly believe that much of the intuition that a typical model theorist has regarding first-order logic transfers wholesale to continuous logic. Despite this, in my experience, many model theorists seem to have difficulty getting a handle on what is or isn't formalizable in continuous logic. I think that a big part of this is down to the baroqueness of the existing formalism and the largely unfamiliar notation.

With the aim of ameliorating this issue, this note presents a formalism that is good for learning about and working with continuous logic (for someone already familiar with discrete model theory) but which is not good for *defining* continuous logic. This formalism has two components:

- Instead of defining metric structures in such a way that their ultraproducts are guaranteed to exist, we opt for a simpler definition of metric structure and then take the existence of ultraproducts as an assumption.

- We shift focus from real valued formulas to formulas that correspond to closed or open subsets of type spaces and introduce notation that resembles familiar logical notation as closely as possible while still being literally readable (as opposed to using a notion of approximate satisfaction).

Another lightweight approach is the one developed by Keisler in [16], which concerns *general structures* and is a synthesis of the modern formalism of continuous logic with the earlier formalism of Chang and Keisler [10]. The easiest way to describe this approach is that it is 'continuous logic without metric,' which like our approach here has the advantage of dispensing with moduli of continuity. For countable theories, it turns out that continuous logic and continuous logic without metric are more closely related than discrete logic and discrete logic without equality in that, as shown in [16], there is always a definable pseudo-metric with regards to which all formulas are uniformly continuous. Contrast this with discrete logic without equality, in which theories don't always have definable equivalence relations respected by all formulas.

We will only include the proof of a statement if it is both relatively easy and involves methods that are likely to be unfamiliar to a typical model theorist.

1 Metric Signatures and Structures

Definition 1.1. A *metric signature* is a discrete signature with the symbol $=$ replaced with the symbol d.

Of course, the notion of a signature is only really meaningful when paired with the notion of a structure.

Definition 1.2. Given a metric signature $\mathcal{L}$, a *metric $\mathcal{L}$-structure* is a set M together with

- a complete metric $d^M : M^2 \to \mathbb{R}$,
- a continuous function $P^M : M^n \to \mathbb{R}$ for each n-ary predicate symbol $P \in \mathcal{L} \setminus \{d\}$,
- a continuous function $f^M : M^n \to M$ for each n-ary function symbol $f \in \mathcal{L}$, and
- an element $c^M \in M$ for each constant symbol $c \in \mathcal{L}$.

We will typically refer to the entire structure as M.

While the requirement that (M, d^M) be a complete metric space seems natural enough, we should mention that it is not the only possible choice of semantics for continuous logic. A lot of what we will say here would still be true if we were to remove the requirement that (M, d^M) be complete, but there are certain things that would change significantly (such as omitting types).

Some of the precursors of continuous logic were motivated by the appearance of natural notions of ultraproducts in analysis, so it is only natural that we should have a notion of the ultraproduct of metric structures.

Definition 1.3. Given a family $\{r_i\}_{i \in I}$ of real numbers and an ultrafilter $\mathcal{F}$ on the set I, we write $\lim_{i \to \mathcal{F}} r_i$, if it exists, for the unique real number s for which $\{i \in I : r_i \in U\} \in \mathcal{F}$ for every neighborhood U of s.

Given a family $\{M_i\}_{i \in I}$ of metric $\mathcal{L}$-structures and an ultrafilter $\mathcal{F}$ on the set I, the *ultraproduct of $\{M_i\}_{i \in I}$ with regards to $\mathcal{F}$*, if it exists, is the metric $\mathcal{L}$-structure $M_{\mathcal{F}}$ in which

- each predicate symbol P has the interpretation

$$P^{M_{\mathcal{F}}}(\bar{a}) \coloneqq \lim_{i \to \mathcal{F}} P^{M_i}(\bar{a}(i)),$$

- $M_{\mathcal{F}}$ is $\prod_{i \in I} M_i$ modulo the equivalence relation $d^{M_{\mathcal{F}}}(a,b) = 0$, and
- constant and function symbols have the obvious interpretations,

so that, in particular, the predicate and function interpretations are compatible with the equivalence relation $d^{M_{\mathcal{F}}}(a,b) = 0$.

An *ultrapower*, written $M^{\mathcal{F}}$, is an ultraproduct in which all factors are the same metric structure M.

Note that the 'if it exists' comments are not vacuous. This is some of the price we pay for our simpler definition of metric signature. The definition of metric signature given in [7], for instance, is constructed in such a way that ultraproducts of $\mathcal{L}$-structures always exist.

In the interest of sweeping irrelevant detail under the rug, we will largely be restricting our attention to classes of metric structures in which ultraproducts always exist.

Definition 1.4. A class $\mathbf{K}$ of metric structures is *ultraproductive* if for any family $\{M_i\}_{i \in I}$ of members of $\mathbf{K}$ and any ultrafilter $\mathcal{F}$ on I, the ultraproduct $M_{\mathcal{F}}$ exists.

Note that we are not actually requiring that the ultraproducts themselves be in $\mathbf{K}$, but also note that for any ultraproductive class $\mathbf{K}$, the class of ultraproducts of members of $\mathbf{K}$ is also ultraproductive. Ultimately, ultraproductivity is a blanket assumption we will be making for the sake of developing continuous first-order logic (which is also implicit in most developments of continuous logic). This does, however, limit the kinds of metric structures we can talk about, and it should be noted that there are some model theoretic results regarding these more broadly conceived metric structure [3, 6, 15].

Exercises

Exercise 1.5. Give an example of a compact ultraproduct of non-compact metric spaces.

Exercise 1.6. Temporarily modify the definition of metric structure so that the interpretations of predicate and function symbols need not be continuous. Show that if M is in an ultraproductive class, then the interpretation of any predicate or function symbol on M is continuous.

Exercise 1.7. Show that if $\mathbf{K}$ is an ultraproductive class of metric structures, then for every predicate symbol P (including d), there is a real number r such that $|P^M(\bar{a})| \leq r$ for every $M \in \mathbf{K}$ and $\bar{a} \in M$.

Exercise 1.8. Give an example of a metric structure M satisfying the conclusions of Exercise 1.7 but for which $M^{\mathcal{F}}$ does not exist for any non-principal $\mathcal{F}$.

2 Formulas

The close analogies between continuous logic and discrete logic really only come into play at the level of (partial) types. Formulas simply do not and cannot work as they do in discrete logic. That said, defining types directly without defining formulas is difficult and not necessarily the best approach, so we will still need a notion of formula.

In their standard conception in continuous logic, formulas take on real values and types are thought of in terms of equalities or inequalities of formulas (referred to as *conditions*). In an abstract sense, real valued formulas correspond to continuous real valued functions on type spaces. While this, of course, works as a rigorous framework for continuous logic and is conceptually pure, I find in my own research that I often want to be able to state things in a way that more closely resembles discrete logic. I think it's safe to say that I'm not alone in this, in that in the existing literature there are many instances of continuous logicians breaking down and stating things in informal or semi-formal language that resembles discrete logic,[1] and there is a precedent for this kind of notation in some of the precursors of continuous logic [1, 15].

With this in mind I have decided to present a framework in which the formulas we will primarily work with correspond to open or closed subsets of type space, rather than continuous real valued formulas on type space. Although just as how in topology open and closed sets and real valued functions are all important concepts, in continuous logic it is not prudent to fully eschew real valued formulas, and we will return to them later.

Just as with our definition of metric signature, there is some price to be paid in order to move ahead with this decision. A common strategy is to write formulas that are to be interpreted 'approximately' in the sense of only necessarily being true in sufficiently saturated models (used in [15], for instance). Rather than force the reader to constantly check their own intuition of what $M \models \exists x \varphi(x)$ means, we will solve this problem by using two new quantifiers, $\forall\mkern-7.2mu\forall$ and ə,[2] to precisely indicate which parts of formulas and sentences are only true in some modified sense.

In order to avoid overwhelming the reader with an overly technical definition of formulas in continuous logic, we will give an abridged—but still logically complete—version of it and reserve the right to extend it later, although we will only do so twice.

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[1] See, for instance, the descriptions of the theories of atomless probability algebras and L^p lattices in [7], the axioms for randomizations in [8], the proof of Lemma 3.3 in [13], or the proof of Lemma 2.12 in [5], which contains the connectives **and** and **or**.

[2] The LaTeX definitions of these symbols are:

```
\newcommand{\sforall}{\forall\mkern-7.2mu\forall}
\newcommand{\wexists}{\textnormal{\larger[2]\textschwa}}
```

The ə symbol is a slightly enlarged schwa (also known as an upside down e). This requires the `tipa` package. The `\larger` command requires the `relsize` package.

Definition 2.1. For any metric signature $\mathcal{L}$, the collection of $\mathcal{L}$-*formulas* is defined inductively.

- For any n-ary predicate symbol P, any n-tuple $\bar{t}$ of $\mathcal{L}$-terms (where terms are defined exactly as they are in discrete logic), and any real number r, the following are $\mathcal{L}$-formulas.

  $$P(\bar{t}) \leq r \qquad P(\bar{t}) \geq r \qquad P(\bar{t}) = r$$
  $$P(\bar{t}) > r \qquad P(\bar{t}) < r \qquad P(\bar{t}) \neq r$$

  Formulas of this sort are called *atomic*.

- For any $\mathcal{L}$-formulas A and B, the following are $\mathcal{L}$-formulas.

  $$A \wedge B \qquad A \vee B \qquad A \to B \qquad \neg A$$

- For any $\mathcal{L}$-formula A and variable x, the following are $\mathcal{L}$-formulas.

  $$\exists x A \qquad \forall x A \qquad \ni x A \qquad \forall\!\!\!\forall x A$$

We may write $x = y$ as a shorthand for $d(x, y) = 0$ and $x \neq y$ as a shorthand for $d(x, y) > 0$. An $\mathcal{L}(\bar{x})$-*formula* is an $\mathcal{L}$-formula whose free variables are among $\bar{x}$, where free variable is defined in the obvious way. An $\mathcal{L}$-*sentence* is an $\mathcal{L}()$-formula, i.e. a formula with no free variables.

All of the interpretations of standard logical symbols are standard, and the meaning of expressions such as $M \models P(\bar{a}) < \frac{1}{2}$ should be clear. The only things we really need to define are the two new quantifiers.

Definition 2.2 (Weak Existential Quantification)**.** A formula of the form $\ni x A(x)$ is modeled by a metric structure M, written $M \models \ni x A(x)$, if and only if there exists an ultrapower $M^{\mathcal{F}}$ and an $a \in M^{\mathcal{F}}$ such that $M^{\mathcal{F}} \models A(a)$.

Definition 2.3 (Strong Universal Quantification)**.** A formula of the form $\forall\!\!\!\forall x A(x)$ is modeled by a metric structure M, written $M \models \forall\!\!\!\forall x A(x)$, if and only if for every ultrapower $M^{\mathcal{F}}$ and every $a \in M^{\mathcal{F}}$, $M^{\mathcal{F}} \models A(a)$.

Note that for any formula A, $\ni x A$ is logically equivalent to $\neg \forall\!\!\!\forall x \neg A$ and vice versa.

Notation 2.4. If $\mathbf{K}$ is some class of $\mathcal{L}$-structures and $A(\bar{x})$ and $B(\bar{x})$ are $\mathcal{L}$-formulas, then we will write $A(\bar{x}) \models_{\mathbf{K}} B(\bar{x})$ to mean that for any M in $\mathbf{K}$ and any $\bar{a} \in M$, if $M \models A(\bar{a})$, then $M \models B(\bar{a})$.

If $\Sigma(\bar{x})$ is a set of $\mathcal{L}$-formulas with free variables among $\bar{x}$, then we will also write $\Sigma(\bar{x}) \models_{\mathbf{K}} B(\bar{x})$ to mean that for any M in $\mathbf{K}$ and $\bar{a} \in M$, if $M \models A(\bar{a})$ for every $A(\bar{x}) \in \Sigma(\bar{x})$, then $M \models B(\bar{a})$.

If $\mathbf{K}$ is the class of all $\mathcal{L}$-structures, then we may omit it and write expressions such as $A \models B$.

A wrinkle in this approach is that while every formula we can write down has a clear meaning, not every formula we can write down is good, so to speak. An easy example is a formula such as $\exists x P(x) = 0$. It is not hard to construct a metric structure M such that $M \models \neg\exists x P(x) = 0$ but for which $M^{\mathcal{F}} \models \exists x P(x) = 0$ for some ultrapower $M^{\mathcal{F}}$. Since in some sense our goal is to give a logic corresponding to our notion of ultraproduct, we have to deal with this.

Definition 2.5. For any metric signature $\mathcal{L}$, the classes of *open* and *closed* *$\mathcal{L}$-formulas* are defined inductively.

- Atomic formulas involving $<$, $>$, or $\neq$ are open.
- Atomic formulas involving $\geq$, $\leq$, or $=$ are closed.

Let U and V be open formulas and F and G be closed.

- $U \wedge V$ and $U \vee V$ are open.
- $F \wedge G$ and $F \vee G$ are closed.
- $F \to U$ and $\neg F$ are open.
- $U \to F$ and $\neg U$ are closed.

Let x be a variable.

- $\exists x U$ and $\forall\!\!\!\forall x U$ are open.
- $\text{Ə} x F$ and $\forall x F$ are closed.

We will tend to write open formulas using capital Roman letters that are commonly associated with open sets, and likewise for closed formulas.

Perhaps the thing most sorely lacking here in Definition 2.5 is the $\leftrightarrow$ connective. It is impossible, in general, to express it in a way that is compatible with the following facts.

Proposition 2.6 (Preservation of Closed Sentences)**.** *If F is a closed sentence and $\{M_i\}_{i\in I}$ is a family of structures such that the ultraproduct $M_{\mathcal{F}}$ exists and $\{i \in I : M_i \models F\} \in \mathcal{F}$, then $M_{\mathcal{F}} \models F$.*

Proposition 2.7 (Co-preservation of Open Sentences)**.** *If U is an open sentence and $\{M_i\}_{i\in I}$ is a family of structures such that the ultraproduct $M_{\mathcal{F}}$ exists and $M_{\mathcal{F}} \models U$, then $\{i \in I : M_i \models U\} \in \mathcal{F}$.*

Corollary 2.8 (Preservation under Ultrapowers)**.** *If X is an open or closed formula and $M \models X$, then for any ultrapower $M^{\mathcal{F}}$ of M, $M^{\mathcal{F}} \models X$ as well.*

Now we are finally in a position to talk about theories.

Notation 2.9. Given an $\mathcal{L}$-structure M and a subset $A \subseteq M$, we write $\mathcal{L}_A$ for the signature that is $\mathcal{L}$ expanded by constants for each element of A, and we write M_A for the expansion of M to an $\mathcal{L}_A$-structure in the obvious way.

Definition 2.10. For any metric signature $\mathcal{L}$, an *$\mathcal{L}$-theory* T is a set of closed $\mathcal{L}$-sentences. An $\mathcal{L}$-structure M is a *model of* T, written $M \models T$, if $M \models F$ for every $F \in T$. The class of all models of T is written $\mathrm{Mod}(T)$. We say that a theory T is *ultraproductive* if $\mathrm{Mod}(T)$ is.

Given an $\mathcal{L}$-structure M, the *theory of* M, written $\mathrm{Th}(M)$, is the set of all closed $\mathcal{L}$-sentences F such that $M \models F$. The *elementary diagram of* M, written $\mathrm{eldiag}(M)$, is the theory of M_M. A substructure $M \subseteq N$ is an *elementary substructure* if $N_M \models \mathrm{eldiag}(M)$.

We say that M and N are *elementarily equivalent*, written $M \equiv N$, if $\mathrm{Th}(M) = \mathrm{Th}(N)$.

The statement of compactness is almost tautological at this point.

Proposition 2.11 (Compactness)**.** *Let T be an ultraproductive theory. For any set of closed sentences Σ, $T \cup \Sigma$ has a model if and only if $T \cup \Sigma_0$ has a model for every finite $\Sigma_0 \subseteq \Sigma$.*

Corollary 2.12. *Let T be an ultraproductive theory. For any set of closed sentences Σ and any open sentence U, $T, \Sigma \models U$ if and only if there is a finite $\Sigma_0 \subseteq \Sigma$ such that $T, \Sigma_0 \models U$.*

Finally, we should also pause to remark that the logic we have defined here does indeed characterize ultraproducts of metric structures (for ultraproductive classes of structures).

Definition 2.13. For any structure M, the *naïve discretization of* M, written M°, is a discrete structure whose underlying set is M in a language with a predicate symbol P_F for each closed formula F (with the obvious interpretations).

Lemma 2.14. *If a discrete structure N satisfies $N \equiv M^\circ$ for some metric structure M, then there is a unique metric structure $N^\bullet$, called the* continuous reduct of N*, with a function $f : N \to N^\bullet$ such that the image of f is dense in $N^\bullet$ and for each $\bar{a} \in N$, if $N \models P_F(\bar{a})$ then $N^\bullet \models F(f(\bar{a}))$. Furthermore, $N^\bullet \equiv M$, and if $N_0 \cong N_1$, then $N_0^\bullet \cong N_1^\bullet$.*

Proof Idea. The function $d(a, b) = \inf\{r : N \models P_{d \leq r}(a, b)\}$ defines a pseudo-metric on N. The completion of the quotient by this pseudo-metric gives $N^\bullet$. The values of other predicates are defined similarly. □

Proposition 2.15 (Continuous Keisler-Shelah Theorem)**.** *If M and N are structures that model ultraproductive theories, then $M \equiv N$ if and only if there is an ultrafilter $\mathcal{F}$ such that $M^\mathcal{F} \cong N^\mathcal{F}$.*

Proof. Let $\mathcal{F}_0$ be a non-principal ultrafilter on ω. Argue that $M \equiv N$ if and only if $(M^{\mathcal{F}_0})^\circ \equiv (N^{\mathcal{F}_0})^\circ$. Use the Keisler-Shelah theorem to find an ultrafilter $\mathcal{F}_1$ such that $((M^{\mathcal{F}_0})^\circ)^{\mathcal{F}_1} \cong ((N^{\mathcal{F}_0})^\circ)^{\mathcal{F}_1}$. Argue that $(((M^{\mathcal{F}_0})^\circ)^{\mathcal{F}_1})^\bullet$ is isomorphic to $M^{\mathcal{F}_0 \otimes \mathcal{F}_1}$ (and likewise for N), and conclude that $M^{\mathcal{F}_0 \otimes \mathcal{F}_1} \cong M^{\mathcal{F}_0 \otimes \mathcal{F}_1}$. □

This machinery of passing to naïve discretizations allows for many classical model theoretic results to be transferred to continuous logic directly, such as the Löwenheim-Skolem theorem and the existence of indiscernible sequences and Ehrenfeucht-Mostowski models. Even much of the proof of Morley's theorem presented in Chang and Keisler [11] can be translated using this method.[3] These aren't the prettiest proofs of such results, but they are extremely cheap.

Note that, much like Skolemization in discrete logic, passing to a discretization of a continuous theory can be very destructive to model theoretic tameness. For an extreme example consider the theory of atomless probability algebras APA. As a continuous theory APA is ω-stable [7], but any discretization of APA which still has the probability algebra operations as functions would necessarily interpret an infinite Boolean algebra and hence would lie on the wild side of every known dividing line.

Exercises

Exercise 2.16. Fix an open formula $U(\bar{x})$. Show that there is a sequence of closed formulas $\{F_i(\bar{x})\}_{i<\omega}$ such that for any $\bar{a} \in M \models T$, with T ultraproductive, $M \models U(\bar{a})$ if and only if $M \models \bigvee_{i<\omega} F_i(\bar{a})$. (Hint: Replace $P(\bar{x}) < r$ with $P(\bar{x}) \leq r - \varepsilon$.) Use this to prove Corollary 2.8.

Exercise 2.17. Fix a non-principal ultrafilter $\mathcal{F}$ on ω. Let M be a structure such that $M^{\mathcal{F}}$ exists. For any open or closed formula $X(\bar{x})$, let X^* be the formula with each instance of $\ni$ replaced with $\exists$ and each instance of $\forall\!\!\!\forall$ replaced with $\forall$. Show that for any $\bar{a} \in M$, $M \models X(\bar{a})$ if and only if $M^{\mathcal{F}} \models X^*(\bar{a})$.

Exercise 2.18. Let T be an ultraproductive theory. Show that the following are equivalent.

- Every model of T is compact.
- For every $\varepsilon > 0$, there is an $n < \omega$ with $n > 1$ such that

$$T \models \forall x_0 \dots x_{n-1} \bigvee_{i<k<n} d(x_i, x_k) \leq \varepsilon.$$

- For every $\varepsilon > 0$, there is an $n < \omega$ with $n > 1$ such that

$$T \models \forall\!\!\!\forall x_0 \dots x_{n-1} \bigvee_{i<k<n} d(x_i, x_k) < \varepsilon.$$

Notation 2.19. In any metric structure M and for any $n < \omega$, we will write $d(\bar{x}, \bar{y})$ for the max metric on n-tuples, i.e. $d(\bar{x}, \bar{y}) \coloneqq \max_{i<n} d(x_i, y_i)$.

[3] Although the resulting proof still requires a great deal of the ideas from [2], especially regarding the correct generalization of Morley rank.

Definition Extension 2.20. From now on we will treat expressions of the form $d(\bar{x}, \bar{y})$ as if they were predicate symbols in our language. We will also now consider expressions such as $|P(\bar{t})| = r$ or $|P(\bar{t}) - Q(\bar{s})| < r$ as formulas. These formulas are either open or closed, depending on the binary relation they contain.

Exercise 2.21 (Characterization of Ultraproductive Theories; or, Return of the Moduli)**.** Show that a theory T is ultraproductive if and only if

- for every predicate symbol P (including d), there is a real number r such that $T \models \forall \bar{x} |P(\bar{x})| \leq r$,
- for every predicate symbol P and every $\varepsilon > 0$, there is a $\delta > 0$ such that $T \models \forall \bar{x}\bar{y}(d(\bar{x}, \bar{y}) < \delta \to |P(\bar{x}) - P(\bar{y})| \leq \varepsilon)$, and
- for every function symbol f and every $\varepsilon > 0$, there is a $\delta > 0$ such that $T \models \forall \bar{x}\bar{y}(d(\bar{x}, \bar{y}) < \delta \to d(f(\bar{x}), f(\bar{y})) \leq \varepsilon)$.

Conclude that we wouldn't need the concept of ultraproductivity if this data were somehow baked into the definition of metric signature.

Exercise 2.22. Fix a finite metric signature $\mathcal{L}$, computable positive real numbers r_P for each predicate symbol $P \in \mathcal{L}$ (including d), and computable functions $\varepsilon_P : \mathbb{R} \to \mathbb{R}$ and $\varepsilon_f : \mathbb{R} \to \mathbb{R}$ for each predicate symbol P and each function symbol f in $\mathcal{L}$.

Assume that each ε_s satisfies $\varepsilon_s(0) = 0$ and $\varepsilon_s(x) > 0$ for all $x > 0$, and let T be the theory axiomatized by

- $\forall \bar{x} |P(\bar{x})| \leq r_P$ for each predicate symbol $P \in \mathcal{L}$ (including d),
- $\forall \bar{x}\bar{y}(d(\bar{x}, \bar{y}) < \delta \to |P(\bar{x}) - P(\bar{y})| \leq \varepsilon_P(\delta))$ for each predicate symbol $P \in \mathcal{L} \setminus \{d\}$ and each $\delta > 0$, and
- $\forall \bar{x}\bar{y}(d(\bar{x}, \bar{y}) < \delta \to d(f(\bar{x}), f(\bar{y})) \leq \varepsilon_f(\delta))$ for each function symbol $P \in \mathcal{L}$ and each $\delta > 0$.

Show that the set of pairs of closed formulas F and open formulas U such that $T, F \models U$ is computably enumerable (where we are only considering formulas whose atomic sub-formulas contain rational real numbers).

3 Type Spaces I

Definition 3.1. For any tuple of variables $\bar{x}$, any structure M, and any tuple $\bar{a} \in M$ of the same length as $\bar{x}$, the *type of* $\bar{b}$, written $\mathrm{tp}(\bar{b})$, is the set of all closed $\mathcal{L}$-formulas $F(\bar{x})$, with free variables contained in $\bar{x}$, such that $M \models F(\bar{a})$.

The $\bar{x}$*-type space of* T (also called a *Stone space of* T), written $S_{\bar{x}}(T)$, is the set of all types of tuples $\bar{a}$ (of the appropriate length) in models M of T. We endow this set with the topology generated by sets of the form $[\![U(\bar{x})]\!] \coloneqq \{p \in S_{\bar{x}}(T) : \neg U(\bar{x}) \notin p(\bar{x})\}$. For a closed formula $F(\bar{x})$, we will also define $[\![F(\bar{x})]\!] \coloneqq \{p \in S_{\bar{x}}(T) : F(\bar{x}) \in p(\bar{x})\}$.

A *partial* $\bar{x}$*-type* is any set of closed formulas with free variables contained in $\bar{x}$.

Often in this notation and terminology we may replace $\bar{x}$ with n, where n is the length of the tuple $\bar{x}$.

Proposition 3.2. *If T is an ultraproductive theory, then for any tuple $\bar{x}$, $S_{\bar{x}}(T)$ is a compact Hausdorff space.*

This is, of course, a familiar fact, but notably the type spaces $S_{\bar{x}}(T)$ are not always totally disconnected.

Notation 3.3. If we have a particular structure M in mind with some subset $A \subseteq M$, we may write $S_{\bar{x}}(A)$ for $S_{\bar{x}}(\mathrm{Th}(M_A))$. For a tuple $\bar{b} \in M$, we will write $\mathrm{tp}(\bar{b}/A)$, read as the *type of* $\bar{b}$ *over* A, for $\mathrm{tp}(\bar{b})$ relative to the theory $\mathrm{Th}(M_A)$.

Exercises

Exercise 3.4. Let T be the empty theory in the empty signature (i.e. the language of pure metric spaces). Show that $S_0(T)$ is not compact.

Exercise 3.5. Given an example of an ultraproductive theory T such that the Stone space $S_1(T)$ is not a Stone space (i.e. not the Stone space of any Boolean algebra).

Exercise 3.6. Use Exercise 2.16 to show that $S_n(T)$ is Hausdorff for any ultraproductive T.

4 Definable Sets I

The closed formulas we have described so far are really more correctly thought of as the analog of countable partial types or countably type-definable sets in discrete logic, rather than as a true analog of discrete formulas.[4] In discrete logic, formulas and definable sets have a variety of nice properties and can be characterized in a few different ways. As is typical with generalizations, formerly equivalent characterizations now fail to be so. The most direct generalization of definable would be type-definable and co-type-definable. This concept still makes sense in continuous logic. We could define *clopen formulas* to be those that are logically equivalent to both a closed formula and an open formula, but this condition is too strong in the sense that it is not true that types are always

[4]Although in truth for this analogy to be entirely correct, we would need to close the class of closed formulas under countable conjunctions, which can be done but is unnecessary for the current presentation.

axiomatized by the clopen formulas they contain. In the following, we will discuss a weaker condition which has many of the nice properties of definable sets in discrete logic.

Definition 4.1. Let $\mathcal{L}$ be a metric signature with a unary predicate symbol D. Let T be an ultraproductive $\mathcal{L}$-theory containing the axiom $\forall xy d(x,y) \leq r$, for some fixed $r > 0$. We say that D is a *distance predicate over* T if T logically entails $\forall xy |D(x) - D(y)| \leq d(x,y)$ and

$$\forall x D(x) = r \vee \forall x \ni y \left(D(y) = 0 \wedge d(x,y) = D(x)\right).$$

Proposition 4.2. *Let T be an ultraproductive theory containing the axiom $\forall xy d(x,y) \leq r$. D is a distance predicate over T if and only if for every $M \models T$, $D^M(a) = \inf\{d(a,b) : D^M(b) = 0\}$, with the understanding that* $\inf \varnothing = r$.

Proof. The $\Leftarrow$ direction is left as an exercise for the reader.

For the $\Rightarrow$ direction, if $M \models \forall x D(x) = r$, then we are done, so assume that M satisfies $\forall x \ni y \left(D(y) = 0 \wedge d(x,y) = D(x)\right)$, and fix $a \in M$.

For any $b \in M$ with $D(b) = 0$, we have that $D(a) \leq D(b) + d(a,b) = d(a,b)$, so $D(a) \leq \inf\{d(a,b) : D(b) = 0\}$.

Fix $\varepsilon > 0$ with $\varepsilon < 1$ and $a \in M$. Let $a_0 = a$. For each $i < \omega$, given a_i satisfying $D(a_i) \leq \varepsilon^i D(a)$. By assumption there must exists an a_{i+1} such that $D(a_{i+1}) < \varepsilon^{i+1} D(a)$ and $|D(a_i) - d(a_i, a_{i+1})| < \varepsilon^{i+1}$, implying that $d(a_i, a_{i+1}) < \varepsilon^i D(a) + \varepsilon^{i+1} = \varepsilon^i (D(a) + \varepsilon)$.

By construction, $\{a_i\}_{i<\omega}$ is a Cauchy sequence whose limit, a_ω, satisfies $d(a, a_\omega) < \sum_{i<\omega} \varepsilon^i (D(a) + \varepsilon) = \frac{D(a)+\varepsilon}{1-\varepsilon}$, implying that $\inf\{d(a,b) : D(b) = 0\} < \frac{D(a)+\varepsilon}{1-\varepsilon}$. Since we can do this for arbitrarily small $\varepsilon > 0$, we have that $\inf\{d(a,b) : D(b) = 0\} \leq D(a)$.

Therefore $D(a) = \inf\{d(a,b) : D(b) = 0\}$. □

We first defined the concept of a distance predicate for a predicate symbol out of simplicity. Of course, definable sets aren't always given by an atomic formula, but in order to develop this concept fully, we will need real valued formulas.

Exercises

Exercise 4.3. Show that if $D_0(x)$ and $D_1(x)$ are distance predicates over an ultraproductive theory T, which logically entails

$$\forall x E(x) \leq D_0(x) \qquad \forall x E(x) \leq D_1(x)$$
$$\forall x E(x) = D_0(x) \vee E(x) = E_1(x)$$

(i.e. for any $M \models T$, $E^M = \min\{D_0^M, D_1^M\}$), then E is a distance predicate over T. What set is E the distance predicate of?

Exercise 4.4. Give an example of a metric structure M with two distance predicates $D(x)$ and $E(x)$ such that $\max\{D(x), E(x)\}$ is not a distance predicate.

Exercise 4.5 (Uniformly Definable Family)**.** Suppose that T is an ultraproductive theory in a language with a binary predicate $D(x, y)$ such that T logically entails $\forall xy d(x, y) \leq r$, $\forall xyz |D(x, z) - D(y, z)| \leq d(x, y)$, and

$$\forall z \left[\forall x D(x, z) = r \vee \forall x \exists y \left(D(y, z) = 0 \wedge d(x, y) = D(x, z)\right)\right],$$

i.e. for any parameter $a \in M \models T$, $D^M(x, a)$ is a distance predicate. Show that if $M \models T$ is connected (in the topology induced by its metric) and for some $a \in M$, $D^M(x, a)$ is the distance predicate of a non-empty definable set, then for every $b \in M$, $D^M(x, b)$ is the distance predicate of a non-empty definable set.

5 Real Formulas

The following definition is—like much of the rest of the formalism in this article—not the best way to define this concept if one wishes to prove things as efficiently as possible. That said, while this may not be the correct way to define real valued formulas, I believe it is the correct way to think about real valued formulas.

Definition 5.1. Given an ultraproductive $\mathcal{L}$-theory T and a tuple of variables $\bar{x}$, a *real $\mathcal{L}(\bar{x})$-formula (over T)*[5] is a continuous function $\varphi : S_{\bar{x}}(T) \to \mathbb{R}$. A *real $\mathcal{L}$-formula (over T)* is a real $\mathcal{L}(\bar{x})$-formula over T for some tuple of variables $\bar{x}$. A *real $\mathcal{L}$-sentence* is a real $\mathcal{L}()$-formula, i.e. a real formula with no free variables.

For any model M of T and any tuple $\bar{a} \in M$ of the same length as $\bar{x}$, $\varphi^M(\bar{a}) \coloneqq \varphi(\mathrm{tp}(\bar{a}))$.

Really these are more accurately thought of as formulas up to logical equivalence, but maintaining this distinction is more trouble than it is worth for the current presentation.

An advantage of this lightweight definition of real formula is that Łoś's Theorem is almost trivial.

Proposition 5.2 (Łoś's Theorem for Real Formulas)**.** *For any ultraproductive theory T, family $\{M_i\}_{i \in I}$ of models of T, ultrafilter $\mathcal{F}$ on I, and sentence φ over T, we have that*

$$\varphi^{M_{\mathcal{F}}} = \lim_{i \to \mathcal{F}} \varphi^{M_i}.$$

The downside, however, is that we have to do more work to use it, proving things that would be automatic with a more nitty-gritty syntactic definition.

Proposition 5.3 (Existence of Atomic Formulas)**.** *For type $p(\bar{x}) \in S_{\bar{x}}(T)$ and any expression of the form $P(\bar{t}(\bar{x}))$, with P a predicate symbol and $\bar{t}$ a tuple of terms with free variables among $\bar{x}$, there is a unique real number r such that $p(\bar{x}) \models P(\bar{t}(\bar{x})) = r$. Furthermore, the function $p \mapsto r$ is continuous.*

[5] What we are referring to as 'real formulas' here correspond to what are called 'definable predicates' in [7], with their notion of formula being more restrictive in effect than what we have given here. Note, also, that definable predicates in [7] are attached to a particular structure, rather than a particular theory.

Given an expression of the form $P(\bar{t}(\bar{x}))$, we will write the corresponding real formula as $P(\bar{t}(p))$, or just $P(p)$ if $\bar{t}$ is just a tuple of variables.

In discrete logic it is automatic that any formula only depends on finitely many variables and non-logical symbols in the signature in question. Since real formulas in continuous logic are naturally closed under uniformly convergent limits, the best one could hope for in principle is that formulas depend on at most countably many variables and non-logical symbols. This is indeed the case.

Proposition 5.4 (Countable Base)**.** *For any ultraproductive $\mathcal{L}$-theory T and any real $\mathcal{L}(X)$-formula $\varphi(X)$ (where X is some set of variables) over T, there is a countable $\mathcal{L}_0 \subseteq \mathcal{L}$ and a countable $X_0 \subseteq X$ such that there exists a real $\mathcal{L}_0(X_0)$-formula ψ over $T \upharpoonright \mathcal{L}_0$ satisfying $\varphi = \psi \circ \iota$, where $\iota : S_X(T) \to S_{X_0}(T \upharpoonright \mathcal{L}_0)$ is the natural reduct map defined by $\iota(p) = p \upharpoonright \mathcal{L}_0$.*

Proposition 5.5 (Quantification)**.** *For any ultraproductive $\mathcal{L}$-theory T and any $\mathcal{L}(\bar{x}y)$-formula $\varphi(\bar{x}, y)$ over T, there is an $\mathcal{L}(\bar{x})$-formula $\psi(\bar{x})$ over T such that for any $M \models T$ and any $\bar{a} \in M$, $\psi^M(\bar{a}) = \inf\{\varphi^M(\bar{a}, b) : b \in M\}$. There is also such a formula for $\sup\{\varphi^M(\bar{a}, b) : b \in M\}$.*

We will write these formulas as $\inf_y \varphi(\bar{x}, y)$ and $\sup_y \varphi(\bar{x}, y)$. Note that quantification of real formulas has a 'philosophical' advantage over quantification of closed or open formulas. The quantifiers $\forall\!\!\!\forall$ and $\ni$ require thinking about ultraproducts,[6] but inf and sup can be 'computed' within the structure in question.

Proposition 5.6 (Iterative Construction of Formulas)**.** *For any ultraproductive theory T, the collection of real $\mathcal{L}$-formulas over T is the smallest collection of objects containing the atomic real $\mathcal{L}$-formulas over T and closed under composition with continuous functions $f : \mathbb{R}^n \to \mathbb{R}$, quantification, and uniformly convergent limits.*[7]

Continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ in the context of real formulas are sometimes referred to as *continuous connectives.*

Exercises

Exercise 5.7 (Zerosets)**.** Fix an ultraproductive theory T. Show that for any $S_{\bar{x}}(T)$ and set $F \subseteq S_{\bar{x}}(T)$, the following are equivalent.

- There is a countable sequence $\{G_i(\bar{x})\}_{i<\omega}$ of closed $\mathcal{L}(\bar{x})$-formulas such that $F = \bigcap_{i<\omega} [\![G_i(\bar{x})]\!]$.
- There is a real $\mathcal{L}(\bar{x})$-formula $\varphi(\bar{x})$ such that $F = \varphi^{-1}(0)$ (i.e. $F = \{p : \varphi(p) = 0\}$).

[6] At least in the way in which we have defined them. It is possible to define $\forall\!\!\!\forall$ and $\ni$ without ultraproducts, but the least contrived way to do so amounts to starting with real formulas and inf and sup.

[7] We could avoid uniform limits by allowing continuous functions of the form $f : \mathbb{R}^\omega \to \mathbb{R}$.

- F is a closed G_δ set (also known as a closed Π^0_2 set).

Definition 5.8. For any theory T and any real formula φ, the *logical norm of φ over T*, written $\|\varphi\|_T$, is $\sup\{|\varphi^M(\bar{a})| : \bar{a} \in M \models T\}$.

Exercise 5.9. Fix an ultraproductive theory T. Verify that for any real $\mathcal{L}(\bar{x})$-formula φ over T, $\|\varphi\|_T$ is finite, and show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $T \models \forall \bar{x}\bar{y}(d(\bar{x},\bar{y}) < \delta \to |\varphi(\bar{x}) - \varphi(\bar{y})| \leq \varepsilon)$.

Definition 5.10. Call a real formula *restricted*[8] if it is contained in the smallest class of formulas containing atomic formulas and closed under $\sup_x$, $\inf_x$, and the connectives $x + y$, $\max\{x, y\}$, $\min\{x, y\}$, and $r \cdot x$ for each $r \in \mathbb{Q}$.

Exercise 5.11. Show that for any ultraproductive theory T and any tuple of variables $\bar{x}$, the collection of restricted real $\mathcal{L}(\bar{x})$-formulas is dense in the collection of real $\mathcal{L}(\bar{x})$-formulas over T under $\|\cdot\|_T$. Conclude that the collection of real $\mathcal{L}(\bar{x})$-formulas over T has a $\|\cdot\|_T$-dense subset of cardinality $\aleph_0 + |\mathcal{L}| + |\bar{x}|$.

Exercise 5.12 (Purely Syntactic Real Formulas)**.** Write $[x]_r^s$ for $\min\{\max\{x, r\}, s\}$. Show that if $\sum_{i<\omega} a_i < \infty$ for some sequence of positive numbers $\{a_i\}_{i<\omega}$, then for any sequence $\{\varphi_i\}_{i<\omega}$ of restricted $\mathcal{L}$-formulas, the expression $\sum_{i<\omega}[\varphi_i(\bar{x})]_{-a_i}^{a_i}$ is equivalent to a real $\mathcal{L}(\bar{x})$-formula over any ultraproductive theory T. Show that every real $\mathcal{L}(\bar{x})$-formula over such a T is equivalent to an expression of this form.

Exercise 5.13 (Prenex Form for Restricted Formulas)**.** Show that any restricted real $\mathcal{L}$ is logically equivalent to one of the form

$$\mathrm{qqq}_{x_1} \dots \mathrm{qqq}_{x_n} \min_{i<k} \max_{j<n_i} a_{i,j} + \sum_{\ell < m_{i,j}} b_{i,j,\ell} P_{i,j,\ell}(\bar{t}_{i,j,\ell}),$$

where each qqq_{x_i} is either $\inf_{x_i}$ or $\sup_{x_i}$, $a_{i,j}, b_{i,j,\ell} \in \mathbb{R}$, each $P_{i,j,\ell}$ is a predicate symbol, and each $\bar{t}_{i,j,\ell}$ is a tuple of terms.

Definition 5.14. Fix an ultraproductive theory T. A real formula φ over T is *quantifier free* if it is in the smallest class of formulas containing the atomic formulas and closed under composition with continuous connectives and uniform limits. T admits *quantifier elimination* if every real formula over it is quantifier free.

Exercise 5.15. Show that an ultraproductive theory T admits quantifier elimination if and only if every formula of the form

$$\inf_x \max_{i<k} a_i + \sum_{\ell < m_i} b_{i,\ell} P_{i,\ell}(\bar{t}_{i,\ell})$$

is equivalent to a quantifier free formula, where each $a_i, b_{i,\ell}$ is an integer, each $P_{i,\ell}$ a predicate symbol, and each $\bar{t}_{i,\ell}$ a tuple of terms.

[8]Note that, as formal expressions, restricted $\mathcal{L}$-formulas clearly have an interpretation in any $\mathcal{L}$-structure, and so don't really need to be thought of as attached to a particular theory.

Exercise 5.16. Show that there is no equivalent of the Sheffer stroke in $[0,1]$-valued propositional logic, i.e. that there does not exist a single function $f : [0,1]^n \to [0,1]$ such that functions constructed from compositions of f are dense in the space of continuous functions from $[0,1]^m \to [0,1]$ for each $m < \omega$.

6 Definable Sets II

Definition Extension 6.1. From now on we will treat expressions such as $\varphi(\bar{t}) \leq \psi(\bar{s})$, with φ and ψ real formulas, as formulas.

This final extension has a fairly significant consequence.

Proposition 6.2. *Fix an ultraproductive theory T. For any countable sequence $\{F_i(\bar{x})\}_{i<\omega}$ of closed formulas, there is a closed formula $G(\bar{x})$ logically equivalent to $\bigwedge_{i<\omega} F_i(\bar{x})$ over T. For any countable sequence $\{U_i(\bar{x})\}_{i<\omega}$, there is an open formula $V(\bar{x})$ logically equivalent to $\bigvee_{i<\omega} U_i(\bar{x})$ over T.*

So we have completely dropped the distinction between closed formulas and countable partial types. One argument that this is the correct perspective is that the original finitary definition of closed formula is highly language dependent. Appending a real formula φ as a new predicate symbol to a language can make things which were formerly only countable partial types into closed formulas in the most restrictive sense.

We can now finally give the definition of a definable set.

Definition 6.3. Fix an ultraproductive theory T. A closed formula $F(\bar{x})$ is a *definable set over T*[9] if there is a real formula $\varphi(\bar{x})$ satisfying the condition given in Definition 4.1 (called the *distance predicate of $F(\bar{x})$*) such that $F(\bar{x})$ and $\varphi(\bar{x}) = 0$ are logically equivalent over T.

A closed formula $F(\bar{x}, \bar{y})$ is a *$\bar{y}$-uniformly definable family over T* if there is a real formula $\varphi(\bar{x}, \bar{y})$ such that for any $\bar{a} \in M \models T$, $\varphi^M(\bar{x}, \bar{a})$ is the distance predicate of $F(M, \bar{a})$.

We may write the distance predicate of a definable set $D(\bar{x})$ as $d(\bar{x}, D)$, and we may write the distance predicate of a uniformly definable family $D(\bar{x}; \bar{y})$ as $d(\bar{x}, D(\bar{y}))$.

The strongest justification that definable sets are natural is that they are characterized by relative quantification.

[9] I must apologize for an awkward type error in this terminology. Strictly speaking a definable set should be a subset of a structure, or at the very least a *set* such as a set of types. That said, this use is in line with the modern parlance of model theory, in which no strong distinction between definable sets and formulas in maintained.

Proposition 6.4 (Relative Existential Quantification)**.** *Fix an ultraproductive theory T. A closed formula $D(\bar{x})$ is a definable set over T if and only if for every open formula $U(\bar{x}, \bar{y})$, there is an open formula $V(\bar{y})$ such that for any $M \models T$ and $\bar{a} \in M$, $M \models V(\bar{a})$ if and only if there is $\bar{b} \in D(M)$ such that $M \models U(\bar{b}, \bar{a})$. The analogous statement is true for uniformly definable families.*

Notation 6.5. In the context of an ultraproductive theory T over which the closed formula $D(\bar{x}; \bar{y})$ is $\bar{y}$-uniformly definable, for any open formula $U(\bar{x}, \bar{y}, \bar{z})$, we will write $(\exists \bar{x} \in D(\bar{y}))U(\bar{x}, \bar{y}, \bar{z})$ for the open formula whose existence is guaranteed by Proposition 6.4. We will likewise write $(\forall \bar{x} \in D(\bar{y}))F(\bar{x}, \bar{y}, \bar{z})$ for $\neg(\exists \bar{x} \in D(\bar{y}))\neg F(\bar{x}, \bar{y}, \bar{z})$.

Note that for any closed formula we always get relative strong universal quantification: $(\forall\!\!\!\forall x \in F)U(x, \bar{y})$ can just be defined as $\forall\!\!\!\forall x(F(x) \to U(x, \bar{y}))$.

Now that we have spent two whole sections discussing this notion of a definable set, we have to confront an unfortunate truth: This notion is still too strong to always characterize types.

Example 6.6. *Consider a metric structure M with the metric defined by $d(x, y) = 1 \Leftrightarrow x \neq y$ and in a language with a single unary predicate U such that the set $\{U^M(a) : a \in M\}$ is a dense subset of $[0, 1]$. If T is the theory of M, then $S_1(T)$ is homeomorphic to $[0, 1]$, but the only definable subsets of $S_1(T)$ are $\varnothing$ and $S_1(T)$. More generally, for any set of parameters A, every definable subset of $S_1(A)$ is either finite or co-finite.*

Note that this example is rather nice in the sense of stability theory. It is superstable and even weakly minimal, with trivial geometry. It also has quantifier elimination and a decidable theory (once that condition is properly defined). So very strong traditional model theoretic tameness conditions can fail to give a useful collection of definable sets. It will turn out, though, that ω-stability, and more generally 'hereditary smallness,' is enough to ensure that there are many definable sets.

Exercises

Exercise 6.7. Give an example of a structure M with a definable set $D(x)$ that is not a formula in the restrictive sense in which we originally defined them in Definition 2.1. (Hint: Construct a definable set that requires countably many parameters and cannot be defined over any finite set of parameters.)

Exercise 6.8. Show that a closed formula in a relational language is clopen if and only if it admits relative existential quantification over pre-models (Definition 7.10).

Exercise 6.9 (Intersection of Definable Not Definable)**.** Give an example of two definable subsets of a structure whose intersection is not definable. (Hint: Pick D and E such that $D(M) \cap E(M) = \varnothing$, but $D(M^{\mathcal{F}}) \cap E(M^{\mathcal{F}}) \neq \varnothing$ for some ultrafilter $\mathcal{F}$, and note that if $D \cap E$ were definable, then $(\exists x \in D \cap E)d(x, x) < 1$ would be an open sentence satisfied by M.)

Exercise 6.10 (Pointwise Definable $\not\Rightarrow$ Uniformly Definable)**.** Give an example of a closed formula $F(x, y)$ and a compact structure M such that for every $a \in M$, $F(x, a)$ is definable, but such that $F(x, y)$ fails to be y-uniformly definable.

Definition 6.11 (Definable Functions)**.** Fix an ultraproductive theory T. A closed formula $F(\bar{x}, y)$ gives a *definable function over* T if it is a $\bar{x}$-uniformly definable family such that $T \models \forall \bar{x} \ni y F(\bar{x}, y)$ and $T \models \forall \bar{x}(\forall yz \in F(\bar{x}))y = z$.

If we have a closed formula giving a definable function over some theory T in question, we may write $f(\bar{x})$ to represent the function it defines, and we may freely form terms using $f(\bar{x})$.

Exercise 6.12. Show that if $f(\bar{x})$ is a definable function over an ultraproductive theory T, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $T \models \forall \bar{x}\bar{y} d(\bar{x}, \bar{y}) < \delta \to d(f(\bar{x}), f(\bar{y})) \le \varepsilon$.

Exercise 6.13 (Function Definable iff Type-Definable)**.** Fix an ultraproductive theory T and a closed formula $F(\bar{x}, y)$. Show that the following are equivalent.

- For any $M \models T$ and any $\bar{a} \in M$, there is always a unique $b \in M$ such that $M \models F(\bar{a}, b)$.
- $F(\bar{x}, y)$ gives a definable function over T.

Extend this result to arbitrary partial types $\Pi(\bar{x}, y)$.

7 Type Spaces II: Some Model Theory

Let us now turn to an actual model theoretic question: When can you omit a type in a model of a theory? It turns out that in general omitting multiple partial types is very complicated in continuous logic [12], but for omitting *complete* types the question has a pleasingly familiar answer. A type can be omitted if and only if it is not principal, although we need to know the correct definition of 'principal.'

Definition 7.1. Fix an ultraproductive theory T. For any finite tuple of variables $\bar{x}$ we define a metric on $S_{\bar{x}}(T)$, by

$$d(p, q) = \inf\{d^M(\bar{a}, \bar{b}) : \bar{a} \models p, \bar{b} \models q, \bar{a}\bar{b} \in M \models T\},$$

where we take $\inf \varnothing = \infty$. Unless otherwise specified, whenever we refer to a metric on a type space it is this metric.

Proposition 7.2. *For any ultraproductive theory T and finite tuple of variables $\bar{x}$, d is a(n extended) metric.*

Proposition 7.3 (Topological Compatibility)**.** *For any ultraproductive theory T and finite tuple of variables $\bar{x}$, the topology induced by the metric d refines the topology on $S_{\bar{x}}(T)$.*

Furthermore, for each $\varepsilon > 0$, the set $\{(p, q) \in S_{\bar{x}}(T)^2 : d(p, q) \le \varepsilon\}$ is closed. (So in particular, closed balls $B_{\le\varepsilon}(q)$ are closed.)

A topological space (X, τ) with a metric d satisfying these conditions is referred to as a *topometric space* [4]. Since we now have two topologies floating around, it will be prudent to set a careful terminological convention. Topological words, such as 'open' or 'closed,' always refer to the compact logic topology, unless qualified by the words 'metric' or 'metrically.' Metric words, such as 'Lipschitz' or 'ball,' always refer to the metric.

Notation 7.4. If (X, d) is a metric space and $A \subseteq X$ is some subset of X, then we will write $A^{<\varepsilon}$ for the set $\{x \in X : (\exists a \in A) d(x, a) < \varepsilon\}$. We will use $A^{\leq\varepsilon}$ similarly.

Note that $A^{<\varepsilon}$ is also equal to $\bigcup_{a \in A} B_{<\varepsilon}(a)$.

Proposition 7.5. *For any open set $U \subseteq S_n(T)$, $U^{<\varepsilon}$ is also open.*

Proof. If $U(\bar{x})$ is an open formula, then $\exists \bar{y} U(\bar{y}) \wedge d(\bar{x}, \bar{y}) < \varepsilon$ is also an open formula. It is not hard to show that $[\![U(\bar{x})]\!]^{<\varepsilon} = [\![\exists \bar{y} U(\bar{y}) \wedge d(\bar{x}, \bar{y}) < \varepsilon]\!]$.

Arbitrary open sets are unions of sets of the form $[\![U(\bar{x})]\!]$, so the result follows by the fact that $\left(\bigcup_{i \in I} A_i\right)^{<\varepsilon} = \bigcup_{i \in I} A_i^{<\varepsilon}$. $\square$

A trivial but important corollary of this proof is that in any model $M \models T$ (T ultraproductive) and for any $\bar{a} \in M$, if $d(\mathrm{tp}(\bar{a}), [\![U(\bar{x})]\!]) < \varepsilon$, then there is a $\bar{b} \in U(M)$ such that $d(\bar{a}, \bar{b}) < \varepsilon$.

This metric on type space is useful for reasoning about definable sets.

Proposition 7.6. *Fix an ultraproductive theory T, a finite tuple of variables $\bar{x}$, and a closed set $F \subseteq S_{\bar{x}}(T)$. The following are equivalent.*

- *There is a definable set $D(\bar{x})$ such that $F = [\![D(\bar{x})]\!]$.*
- *The function $p \mapsto d(p, F)$ is continuous.*
- *For every $\varepsilon > 0$, the set $F^{<\varepsilon}$ is open.*
- *For every $\varepsilon > 0$, F is contained in the interior of $F^{<\varepsilon}$.*
- *For every $\varepsilon > 0$, $F \subseteq (\operatorname{int} F^{<\varepsilon})^{<\varepsilon}$.*
- *No net of points in $S_{\bar{x}}(T)$ 'sneaks up on F,' i.e. for every net of types $\{p_i\}_{i \in I}$ with $\lim_{i \in I} p_i \in F$, $\lim_{i \in I} d(p_i, F) = 0$.*[10]

If $\mathcal{L}$ is countable, then 'net' can be replaced with 'sequence' in the last bullet point.

It is not hard to see that if $D(\bar{x})$ is a definable set, then for any $\bar{a}$, $d(\bar{a}, D) = d(\mathrm{tp}(\bar{a}), [\![D]\!])$.

In light of Proposition 7.6, we may sometimes refer to closed subsets of a type space as *definable* if they correspond to a definable set in this way.

The omitting types theorem is stated in terms of principal types.

[10] This is obviously equivalent to some of the other bullet points, but is often a useful perspective for showing that a set is *not* definable.

Definition 7.7. A type $p \in S_{\bar{x}}(T)$ is *principal* if the set $\{p\}$ is definable.

It is useful to restate parts of Proposition 7.6 in the special case of principal types.

Proposition 7.8. *Fix an ultraproductive theory T. For any $p \in S_{\bar{x}}(T)$, the following are equivalent.*

- *p is principal.*
- *The function $q \mapsto d(p, q)$ is continuous.*
- *For every $\varepsilon > 0$, $B_{<\varepsilon}(p)$ is open.*
- *For every $\varepsilon > 0$, $p \in \operatorname{int} B_{<\varepsilon}(p)$.*
- *For every $\varepsilon > 0$, $\operatorname{int} B_{<\varepsilon}(p)$ is non-empty.*

Many things with principal types work as they do in discrete logic, but there is one notable exception. While it is true that if $\operatorname{tp}(b)$ is principal and $\operatorname{tp}(a/b)$ is principal, then $\operatorname{tp}(ab)$ is principal, the converse fails.

Example 7.9. *Let M be a two-sorted structure with sorts A and B where each sort has as its underlying set the unit circle, S^1. Let S^1 have the discrete metric on the sort B^M, and let S^1 have the Euclidean metric on the sort A^M. Finally, let $f : A \to B$ be a function symbol such that $f^M(x) = x$.*

Let a be any element of A^M, and let $b = f(a)$. $\operatorname{tp}(ab)$ is principal, but $\operatorname{tp}(a/b)$ is not principal.

This is related to the fact that if we think of $S_x(b)$ as a subspace of $S_{xy}(T)$, then d as computed in $S_{xy}(T)$ does not agree with d as computed in $S_x(b)$.

The following definition is bad in the sense that typically we build a pre-structure and then complete it to a structure, but it gets the point across.

Definition 7.10. An *$\mathcal{L}$-pre-structure* is a dense subset of an $\mathcal{L}$-structure M closed under the interpretations of $\mathcal{L}$'s function symbols in M.[11]

A *pre-model* is a pre-structure whose metric closure is a model of some theory in question.

Proposition 7.11 (Omitting Types)**.** *For any ultraproductive theory T in a countable language and any type $p \in S_{\bar{x}}(T)$, T has a model omitting p if and only if p is not principal.*

Proof Idea. Since p is not principal, by the last bullet point of Proposition 7.6 there must be some $\varepsilon > 0$ such that $B_{\leq\varepsilon}(p)$ has empty interior. By the same argument as in discrete logic, we can build a countable pre-model M_0 which omits $B_{\leq\varepsilon}(p)$. Any type q realized in the completion of M_0 must therefore have $d(p, q) \geq \varepsilon$, so p is omitted. □

[11] The definition of pre-structure also typically allows for the metric to merely be a pseudo-metric, but we don't need this extra complexity here.

With omitting partial types, the obvious direction still works: If $\Pi(\bar{x})$ is a partial type such that $\mathrm{int}[\![\Pi(\bar{x})]\!]$ is non-empty, then $\Pi(\bar{x})$ cannot be omitted in a model of T. The difficulty is that there are partial types $\Pi(\bar{x})$ for which $[\![\Pi(\bar{x})]\!]$ has empty interior but which cannot be omitted. There is also difficulty with omitting multiple types. In discrete logic, one can simultaneously omit any countable sequence of partial types which can each individually be omitted (in a complete theory). This and similar facts regarding finite collections of partial types can fail in continuous logic [12].

With some work, the omitting types theorem leads to the continuous analog of the Ryll-Nardzewski theorem. This relies on the fact that any strict refinement of a compact Hausdorff topology fails to be compact.

Proposition 7.12 (Continuous Ryll-Nardzewski Theorem)**.** *For any ultraproductive complete theory T in a countable language, the following are equivalent.*

- *T is ω-categorical (i.e. T has a unique model of countable density character up to isomorphism).*
- *For every $n < \omega$, the metric topology agrees with the logic topology on $S_n(T)$.*
- *For every $n < \omega$, the metric topology on $S_n(T)$ is compact.*

A notable change that occurs in this context is that there are ω-categorical theories T such that for some finite tuple of parameters $\bar{a}$, $T_{\bar{a}}$ fails to be ω-categorical (see Exercise 7.27). Finite sets of parameters are not as tame in continuous logic as they are in discrete logic. As you may be able to intuit from Exercises 7.27 and 7.29, in general even a single parameter in continuous logic can encode as much information as an ω-tuple of parameters in discrete logic. Work by Ben Yaacov and Usvyatsov in [9] gives a technical condition, called d-finiteness, which recovers some of the nice properties of finite tuples of parameters.

Another classic model theoretic result is the characterization of small theories as those with ω-saturated countable models. Since ω-saturation is defined in terms of finite tuples, one should suspect that the definition will need to be modified.

At this point we arrive at another common theme which we have already seen in a few cases. In continuous logic, the correct way to count things (such as the size of models or the size of type spaces) is density character, rather than cardinality, with compact playing the role of finite.

Definition 7.13. A type space $S_n(T)$ is *small* if it has countable metric density character. A theory T is *small* if $S_n(T)$ is small for every $n < \omega$.

Definition 7.14. A structure M is *approximately ω-saturated* if for any finite tuple $\bar{a} \in M$, any $p(x, \bar{a}) \in S_x(\bar{a})$, and any $\varepsilon > 0$, there is $\bar{b}c \in M$ with $\bar{a} \equiv \bar{b}$, $d(\bar{a}, \bar{b}) < \varepsilon$, and $M \models p(c, \bar{b})$.

Proposition 7.15. *If T is a complete ultraproductive theory and M and N are two approximately ω-saturated separable models of T, then for any finite $\bar{m} \in M$ and $\bar{n} \in N$ with $\bar{m} \equiv \bar{n}$ and any $\varepsilon > 0$, there is an isomorphism $f : M \cong N$ such that $d(f(\bar{m}), \bar{n}) < \varepsilon$.*

Corollary 7.16. *A countable complete ultraproductive theory T is ω-categorical if and only if every model of it is approximately ω-saturated.*

We also get a similar picture with prime models.

Definition 7.17. A model M of an ultraproductive theory T is *prime* if for every model N of T, there is an elementary embedding of M into N.

A model M of an ultraproductive theory T is *atomic* if for every finite $\bar{a} \in M$, $\mathrm{tp}(\bar{a})$ is principal.

Proposition 7.18. *Fix a countable complete ultraproductive theory T. The following are equivalent.*

- *T has a prime model.*
- *T has an atomic model.*
- *For every $n < \omega$, principal types are dense in $S_n(T)$.*

Furthermore, a separable model of T is prime if and only if it is atomic, and if M and N are two atomic models of T, then for any finite $\bar{m} \in M$ and $\bar{n} \in N$ and any $\varepsilon > 0$, there is an isomorphism $f : M \cong N$ such that $d(f(\bar{m}), \bar{n}) < \varepsilon$.

Lemma 7.19. *If $S_n(T)$ (T ultraproductive) is small, then principal types are dense in T.*

Proof. For any rational $\varepsilon > 0$, let X_ε be a countable metrically dense subset of $S_n(T)$. We have that $S_n(T) = \bigcup_{p \in X_\varepsilon} B_{\leq \varepsilon}(p)$. By the Baire category theorem, this implies that $Y_\varepsilon = \bigcup_{p \in X_\varepsilon} \operatorname{int} B_{\leq \varepsilon}(p)$ is a dense open set (since closed balls are closed). By the Baire category theorem again, $Z = \bigcap_{\varepsilon \in \mathbb{Q}_{>0}} Y_\varepsilon$ is comeager and therefore dense. For any $p \in Z$, we have that for any rational $\varepsilon > 0$, there is an open set U of diameter at most 2ε such that $p \subseteq U^{\leq 3\varepsilon} \subseteq U^{<4\varepsilon}$. $U^{<4\varepsilon}$ has diameter at most 10ε, so we have that $p \in \operatorname{int} B_{<11\varepsilon}(p)$. Since we can do this for any $\varepsilon > 0$, we have that p is a principal type. $\square$

Proposition 7.20. *If a countable complete ultraproductive theory T has an approximately ω-saturated model, then it has a prime model.*

One thing that doesn't survive the generalization to continuous logic is Vaught's never-two theorem. There is a countable complete ultraproductive theory T with precisely two separable models up to isomorphism. T can even be ω-stable.

Finally, as promised, smallness ensures a prevalence of definable sets.

Lemma 7.21. *If X is a separable metric space and $f : X \to \mathbb{R}$ is any function (not necessarily continuous), then for all but at most countably many $r \in \mathbb{R}$, the metric closure of $\{f < r\} \coloneqq \{x \in X : f(x) < r\}$ contains $\{f \leq r\} \coloneqq \{x \in X : f(x) \leq r\}$.*

Proof. Assume not, then there is an uncountable set R of r for which the condition fails. For each $r \in R$, let $x_r \in X$ be such that $f(x_r) = r$ and $d(x_r, \{f < r\}) > 0$. Since this set is uncountable, there is an $\varepsilon > 0$ and an uncountable $R_0 \subseteq R$ such that for any $r \in R_0$, $d(x_r, \{f < r\}) > \varepsilon$, but this implies that for any $r, s \in R_0$ with $r < s$, $d(x_r, x_s) > \varepsilon$. So X has an uncountable $(> \varepsilon)$-separated subset and is not separable. □

Proposition 7.22. *If $S_n(T)$ is a small type space, then for any real formula φ, for any but countably many $r \in \mathbb{R}$, $\varphi \leq r$ is a definable set.*

Proof. By the lemma we have that for all but countably many $r \in \mathbb{R}$, the metric closure of $[\![\varphi < r]\!]$ is $[\![\varphi \leq r]\!]$ (since φ is continuous and therefore metrically continuous). For any such r we have that for any $\varepsilon > 0$, $[\![\varphi \leq r]\!]^{<\varepsilon} = [\![\varphi < r]\!]^{<\varepsilon}$, which is open, so $[\![\varphi \leq r]\!]$ is definable. □

Corollary 7.23. *If T is hereditarily small (i.e. for any finite tuple $\bar{a}$ of parameters, $T_{\bar{a}}$ is a small theory), then every type space $S_n(A)$ has a basis of definable neighborhoods.*[12]

Proof Idea. This follows from a compactness argument showing that for every type $p \in S_n(A)$ and every neighborhood $U \ni p$, there is an open formula V involving only finitely many parameters such that $p \in [\![V]\!] \subseteq U$. □

Some sufficient conditions for T to be hereditarily small are T being ω-stable, T being hereditarily ω-categorical, and T having an ω-saturated separable model (see Definition 7.28). In [14], I call type spaces and theories with the property of having a basis of definable neighborhood *dictionaric*. One thing to note is that, since intersections of definable sets need not be definable, this is stronger than definable sets separating *complete* types, but it is equivalent to definable sets separating partial types. This condition allows for continuous versions of many constructions that are trivial in discrete logic. I characterize it and prove some useful things about theories with it in [14]. For example: If $S_n(T)$ is a dictionaric type space and D and E are definable subsets of it, then for any $\varepsilon > 0$, there is a definable set E' with $E \subseteq E' \subseteq E^{<\varepsilon}$ such that $D \cap E'$ is definable.

Exercises

Exercise 7.24. Fix an ultraproductive theory T in a countable language. Show that every type $p \in S_{\bar{x}}(T)$ is axiomatized by a single closed formula.

Exercise 7.25. Fix an ultraproductive theory T and a finite tuple of variables $\bar{x}$. Show that d is a complete metric on $S_{\bar{x}}(T)$.

Exercise 7.26. Formulate and prove an omitting types theorem for relational pre-structures.

[12] Where a *neighborhood* of a point x is a set N such that $x \in \operatorname{int} N$.

Exercise 7.27. Let M be the metric structure whose underlying metric space is ω^ω (with the metric $d(a,b) = 2^{-n}$, where n is the smallest index at which a and b disagree) and with a function f defined by $f(a)(i) = a(i+1)$. The theory $T = \mathrm{Th}(M)$ is ω-categorical. Find a constant $a \in M$ such that T_a is not ω-categorical.

Definition 7.28. For any cardinal κ, a structure M is *κ-saturated* if for any set $A \subseteq M$ with $|A| < \kappa$, M realizes every type in $S_1(A)$.

Exercise 7.29. Fix a discrete structure M in a countable relational language and form a metric structure $M^\dagger$ whose underlying metric space is M^ω (with the same metric as in Exercise 7.27) in a signature that has the same non-logical symbols as the signature of M together with a single new unary function symbol f. For any predicate symbol P, let $P^{M^\dagger}(\bar{a})$ be $\{0,1\}$-valued and equal to 1 if and only if $M \models P(\bar{a}(0))$. Finally, let $f^{M^\dagger}(a)(n) = a(n+1)$.

Show that $M^\dagger$ is approximately ω-saturated if and only if M is ω-saturated.

Show that $M^\dagger$ is ω-saturated if and only if M is ω_1-saturated.

Show that $M^\dagger$ is κ-saturated if and only if M is κ-saturated for any $\kappa > \omega$.

Exercise 7.30. Let $\{D_i\}_{i\in I}$ be an arbitrary collection of definable sets and let $\{U_j\}_{j\in J}$ be an arbitrary collection of open sets in some type space $S_{\bar{x}}(T)$. Suppose that $\bigcup_{i\in I} D_i \cup \bigcup_{j\in J} U_j$ is closed. Show that it is definable.

Exercise 7.31. Let $\{D_i\}_{i<\omega}$ be a sequence of definable subsets of $S_{\bar{x}}(T)$. Show that if $\{D_i\}_{i<\omega}$ is a Cauchy sequence with regards to the Hausdorff metric on $S_{\bar{x}}(T)$, then its limit is a definable set.

Definition 7.32. For any ultraproductive theory T and any pair of finite tuples of variables $\bar{x}$ and $\bar{y}$, define a metric on $S_{\bar{x}\bar{y}}(T)$ by

$$d_{/\bar{y}}(p,q) \coloneqq \inf\{d^M(\bar{a},\bar{b}) : \bar{a}\bar{c} \models p, \bar{b}\bar{c} \models q, \bar{a}\bar{b}\bar{c} \in M \models T\},$$

where we take $\inf \varnothing = \infty$.

Exercise 7.33. Fix an ultraproductive theory T and a pair of finite tuples of variables $\bar{x}$ and $\bar{y}$. Show that a closed formula $D(\bar{x},\bar{y})$ gives a $\bar{y}$-uniformly definable family if and only if for every $\varepsilon > 0$, $[\![D(\bar{x},\bar{y})]\!] \subseteq \mathrm{int}[\![D(\bar{x},\bar{y})]\!]^{<\varepsilon}$, where $A \mapsto A^{<\varepsilon}$ is computed with regards to $d_{/\bar{y}}$.

Exercise 7.34. Show that for any type space $S_n(T)$ (T ultraproductive), if principal types are dense in $S_n(T)$, then for any definable set $D \subseteq S_n(T)$, principal types are dense in D as well.

Exercise 7.35. Let M be a structure with underlying set $[0,1]^3$ with the metric $d((a,b,c),(e,f,g)) = 1$ if $c \neq g$ and $d((a,b,c),(e,f,c)) = \max\{|a-e|,|b-f|\}$ and with unary predicates P_0, P_1, and Q such that $P_0^M((a,b,c)) = a$, $P_1^M((a,b,c)) = b$, and $Q^M((a,b,c)) = c$. Let $T = \mathrm{Th}(M)$. Show that $S_1(T)$ can be written as $[0,1]^3$ with $\mathrm{tp}(x) = (P_0(x), P_1(x), Q(x))$. Show that the metric on $S_1(T)$ when written this way is actually the same as the metric on M. Show that for any distinct $p, q \in S_1(T)$, there are definable sets D and E with $p \in D$ and $q \in E$, but also show that $S_1(T)$ does not have a basis of definable neighborhoods. (Hint: Show that for any non-empty definable set D and any $r \in [0,1]$, there must be a type $p \in D$ such that $Q(p) = r$.) Determine why it was necessary to use $[0,1]^3$ instead of $[0,1]^2$.

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