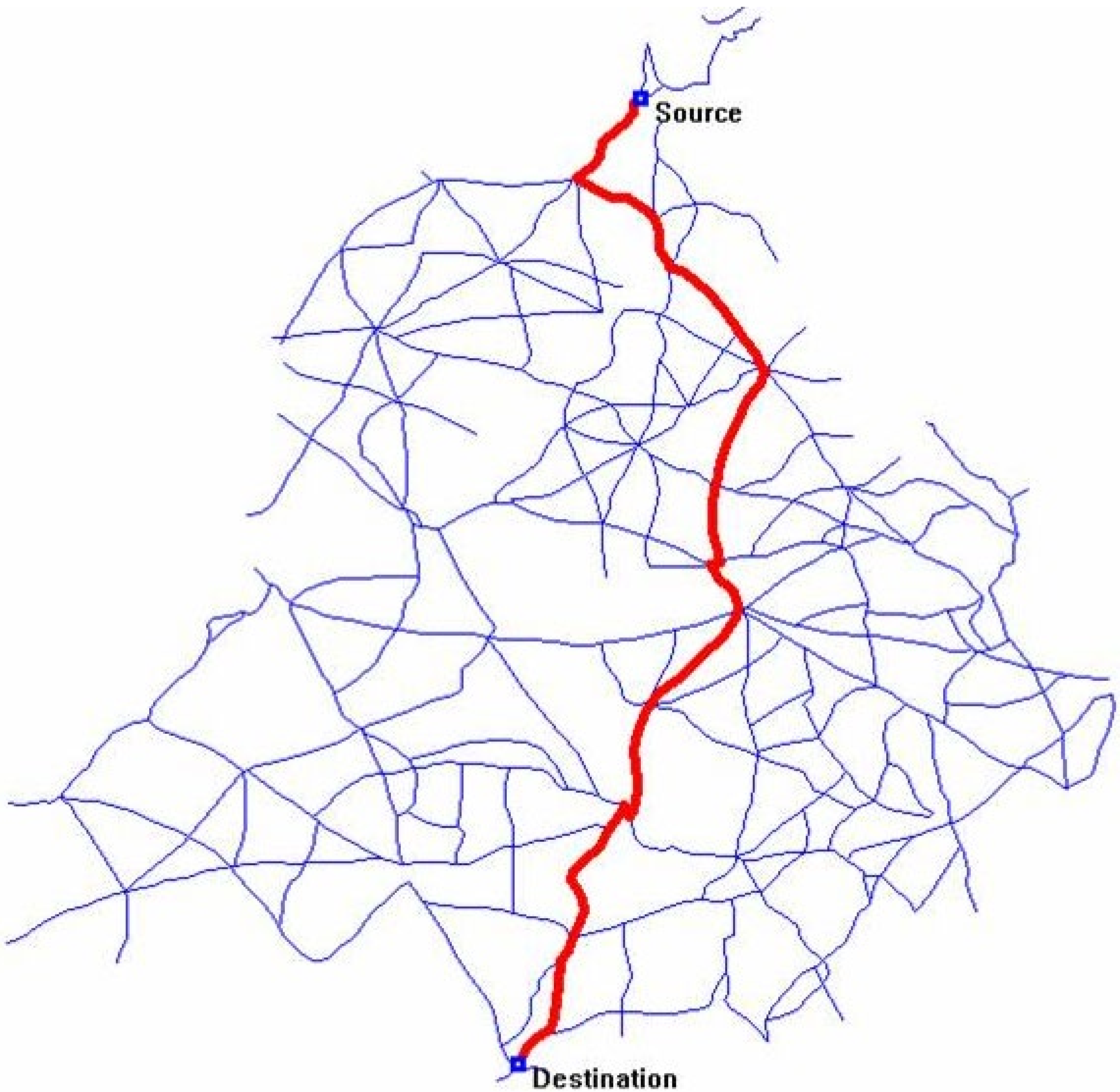
# SHORTEST PATHS
## (ON WEIGHTED GRAPHS)



# Shortest Path Problem

Basic question: find the shortest path!

- Source-to-destination: one vertex to another
- Single source: one vertex to every other
- All pairs: between all pairs of vertices

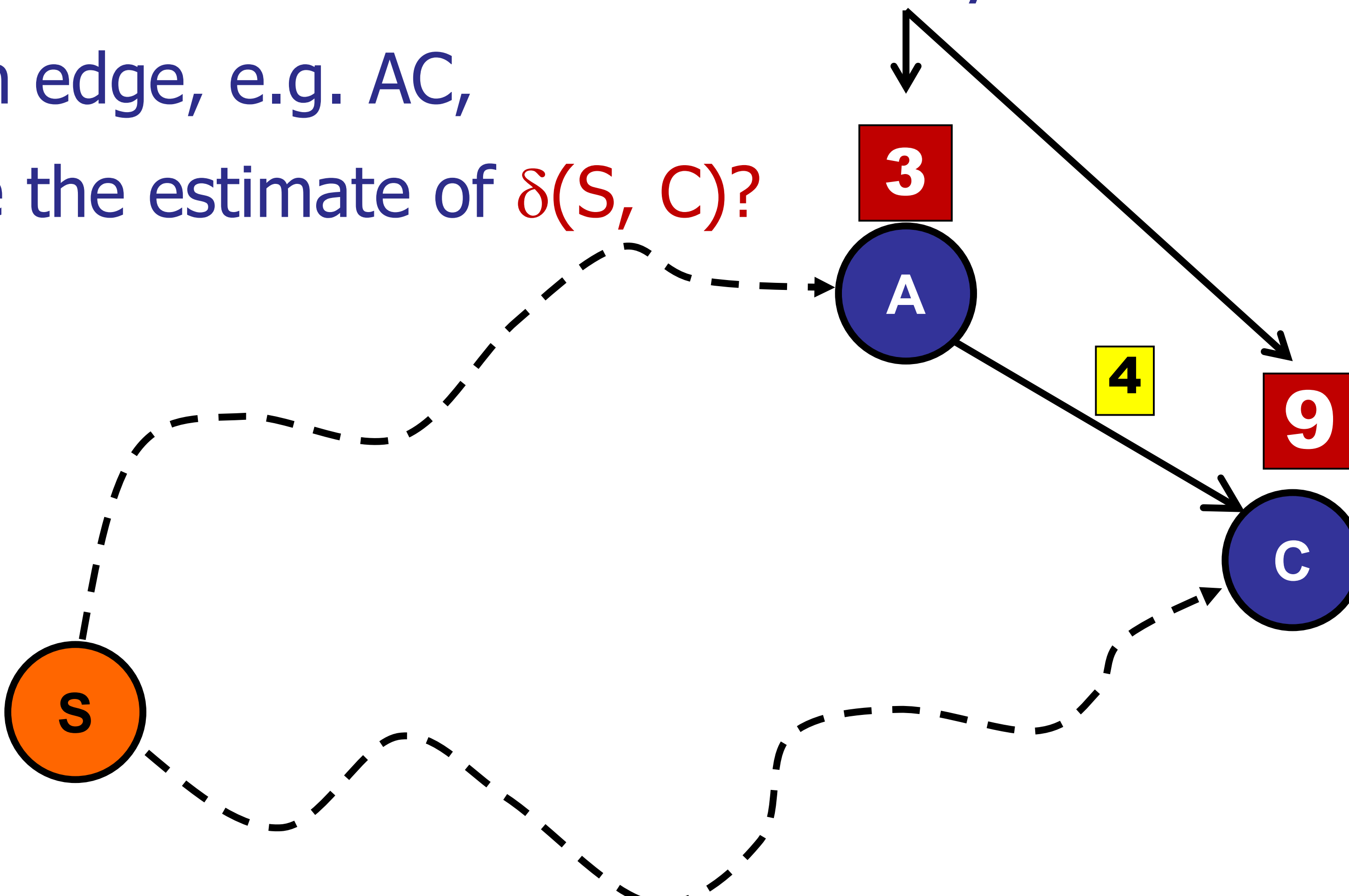
Variants:

- Edge weights: non-negative, arbitrary, Euclidean, …
- Cycles: cyclic, acyclic, no negative cycles

# Shortest Paths

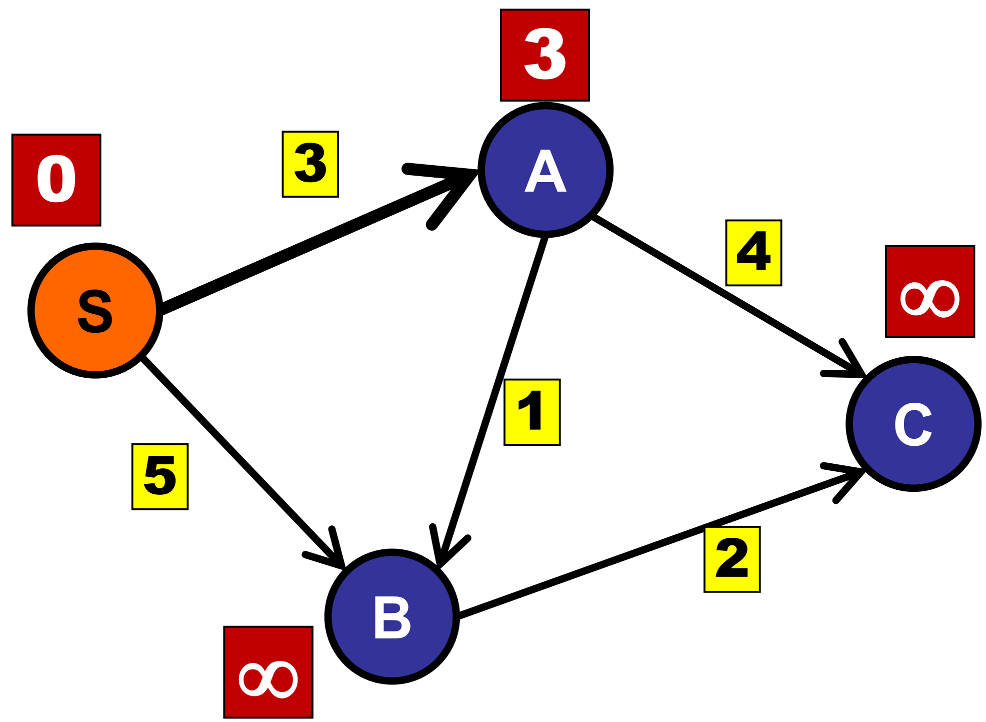
Maintain estimate for each distance:

- Lets say we have some estimate already
- For each edge, e.g. AC,
- Improve the estimate of $\delta(S, C)$?



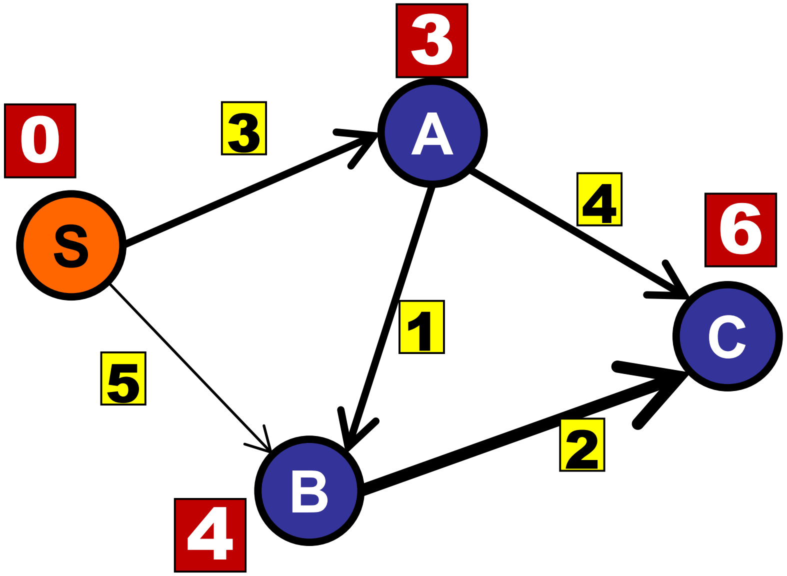
# Shortest Paths

```
relax(int u, int v){
    if (dist[v] > dist[u] + weight(u,v))
        dist[v] = dist[u] + weight(u,v);
}
```



# Bellman-Ford

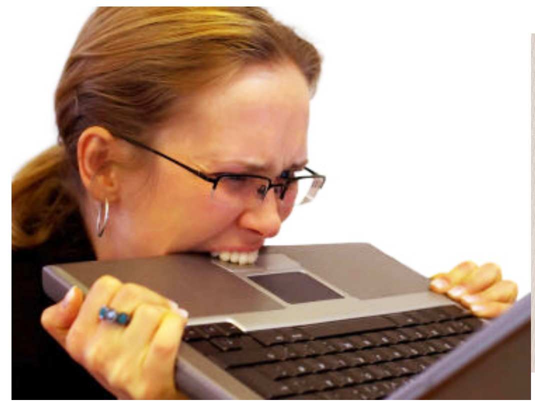
```
n = V.length;
for (i=0; i<n; i++)
    for (each edge e in the graph)
        relax(e)
```



# Bellman-Ford Summary

Basic idea:

- Repeat |V| times: relax every edge
- Stop when "converges".
- O(VE) time.




Special issues:

- If negative weight-cycle: impossible.
- Use Bellman-Ford to **detect** negative weight cycle.
- If all weights are the same, use BFS.

# Today

Key idea:

Relax the edges in the "right" order.

Only relax each edge once:

- O(E) cost (for relaxation step).


# Edsger W. Dijkstra

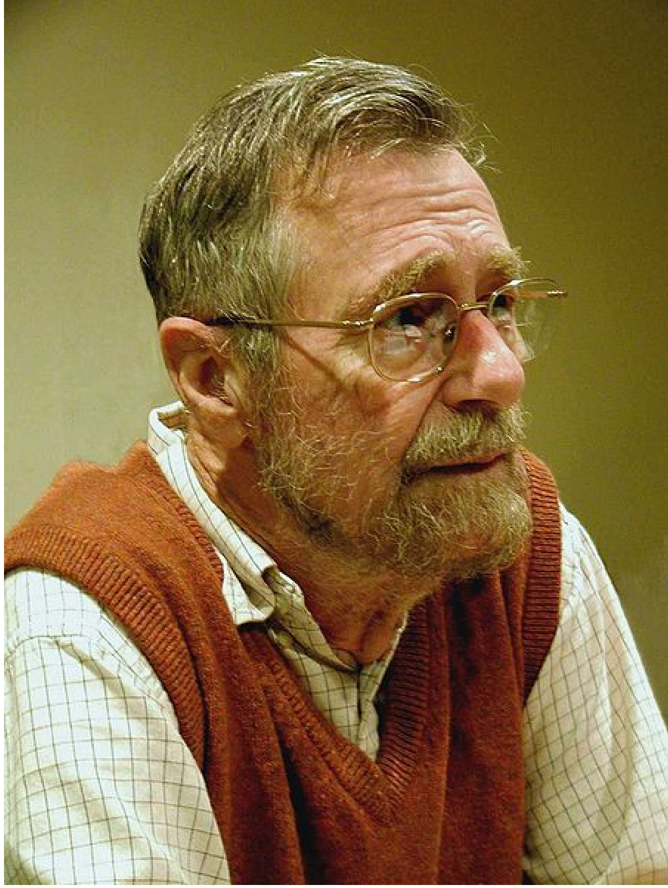
*"Computer science is no more about computers than astronomy is about telescopes."*

*"The question of whether a computer can think is no more interesting than the question of whether a submarine can swim."*

*"There should be no such thing as boring mathematics."*

*"Elegance is not a dispensable luxury but a factor that decides between success and failure."*

*"Simplicity is prerequisite for reliability."*



1930-2002

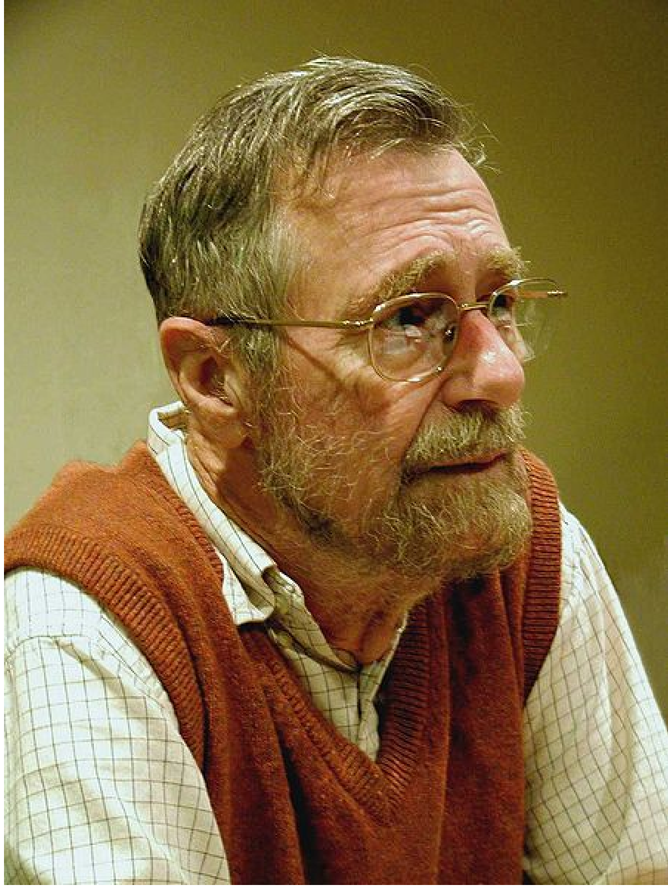
# Edsger W. Dijkstra

*"It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration."*

*"The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offense."*

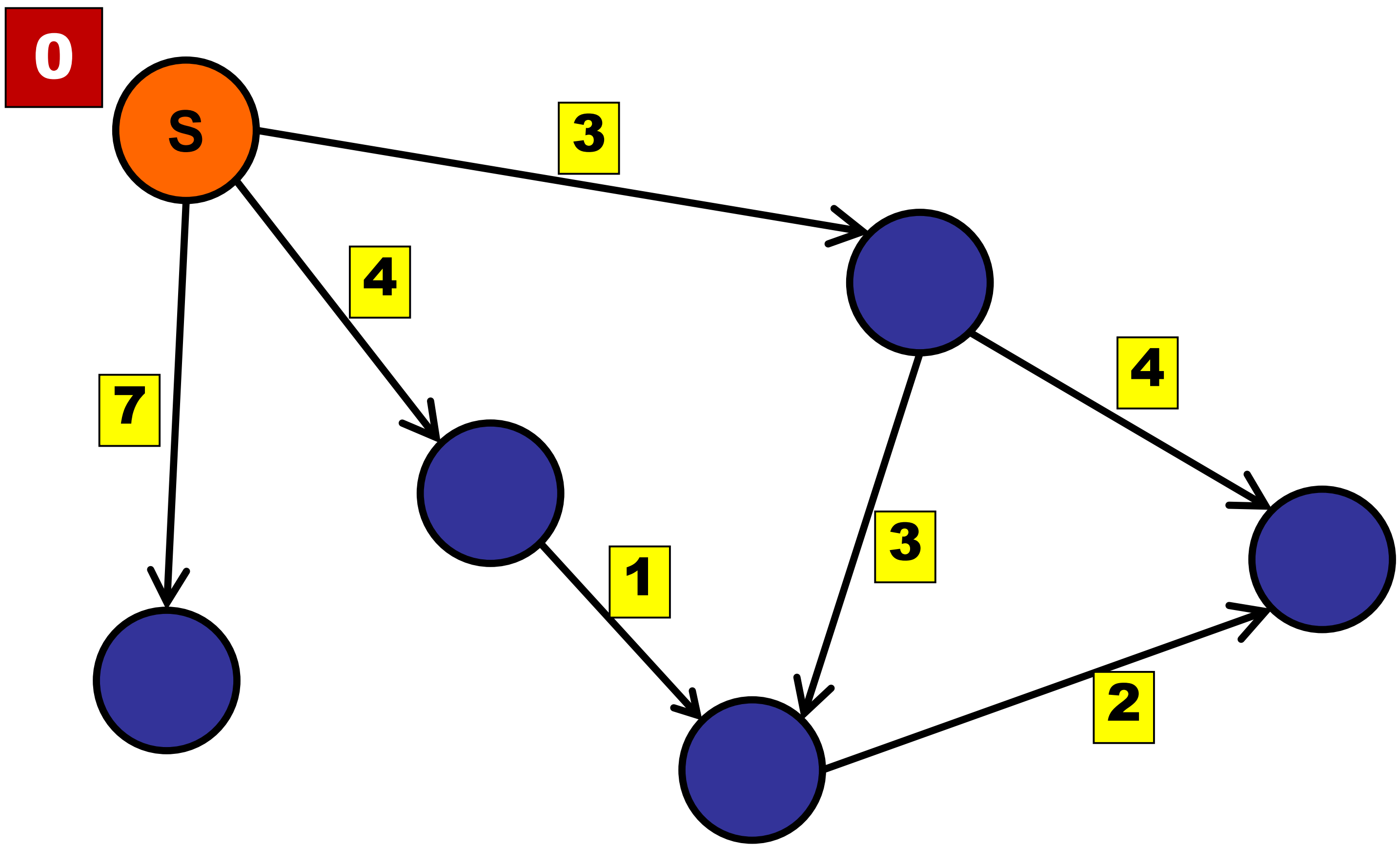
*"APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums."*

*"Object-oriented programming is an exceptionally bad idea which could only have originated in California."*



# Dijkstra's Algorithm (First Try)

Relax shortest edge first



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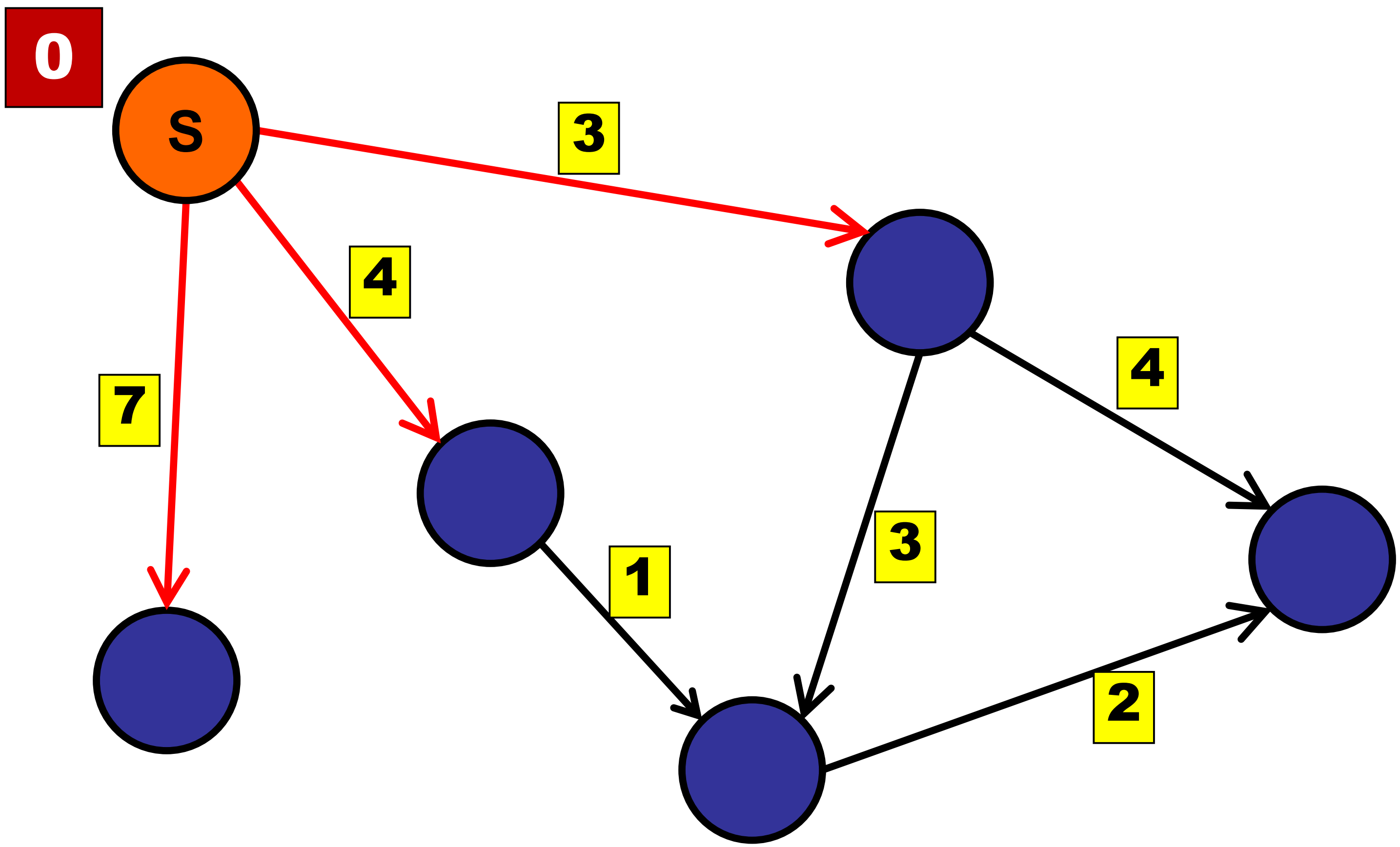

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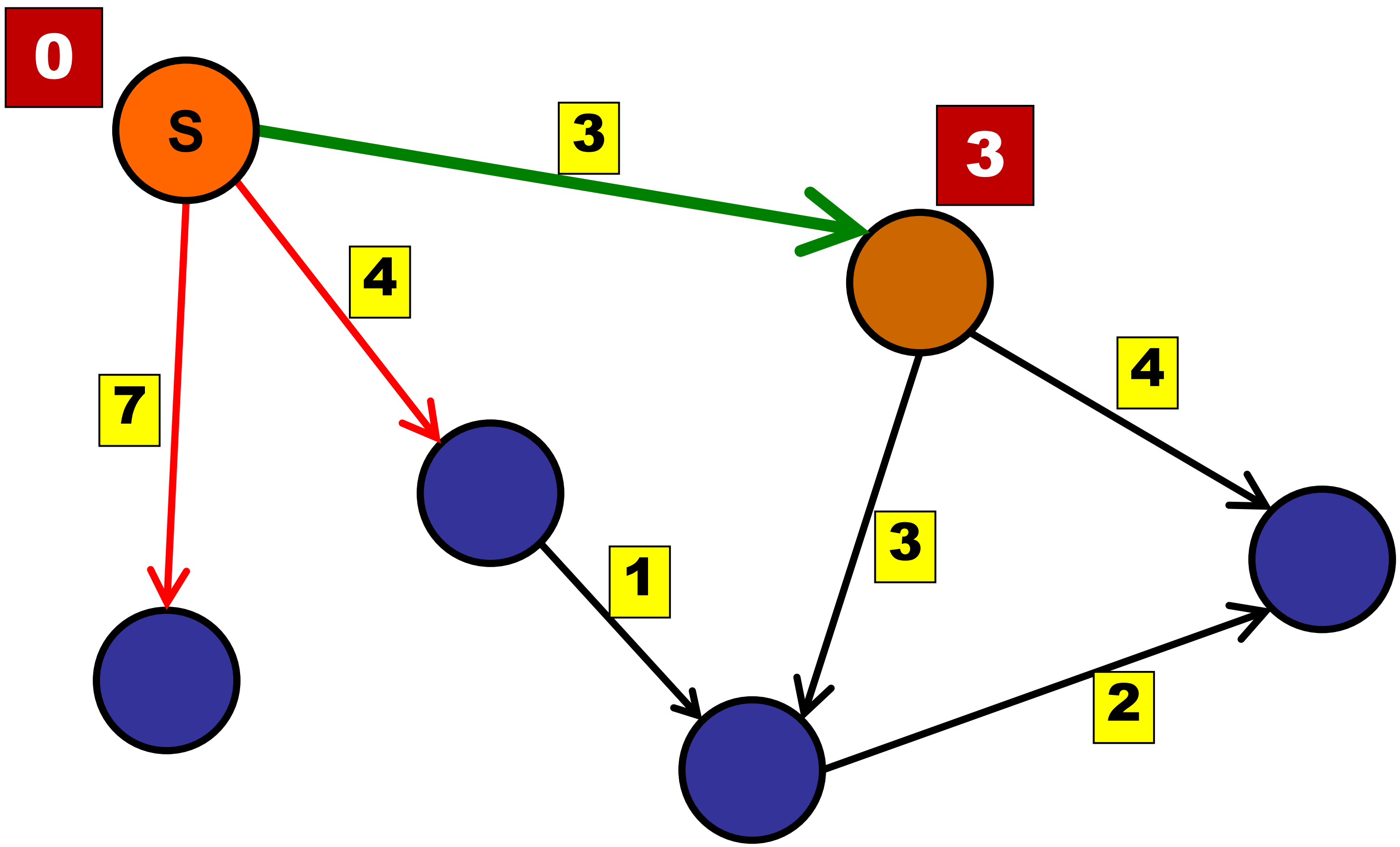
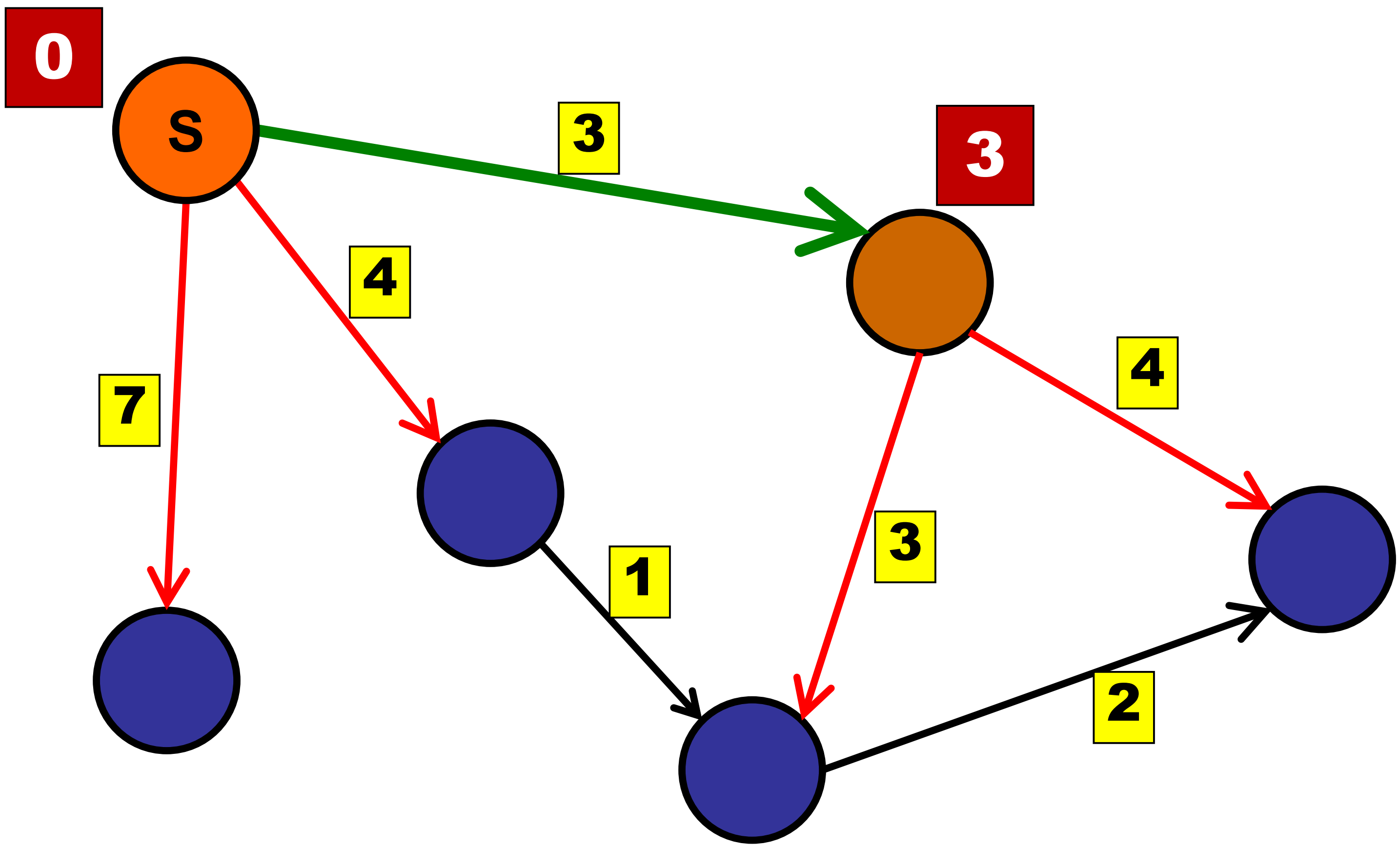


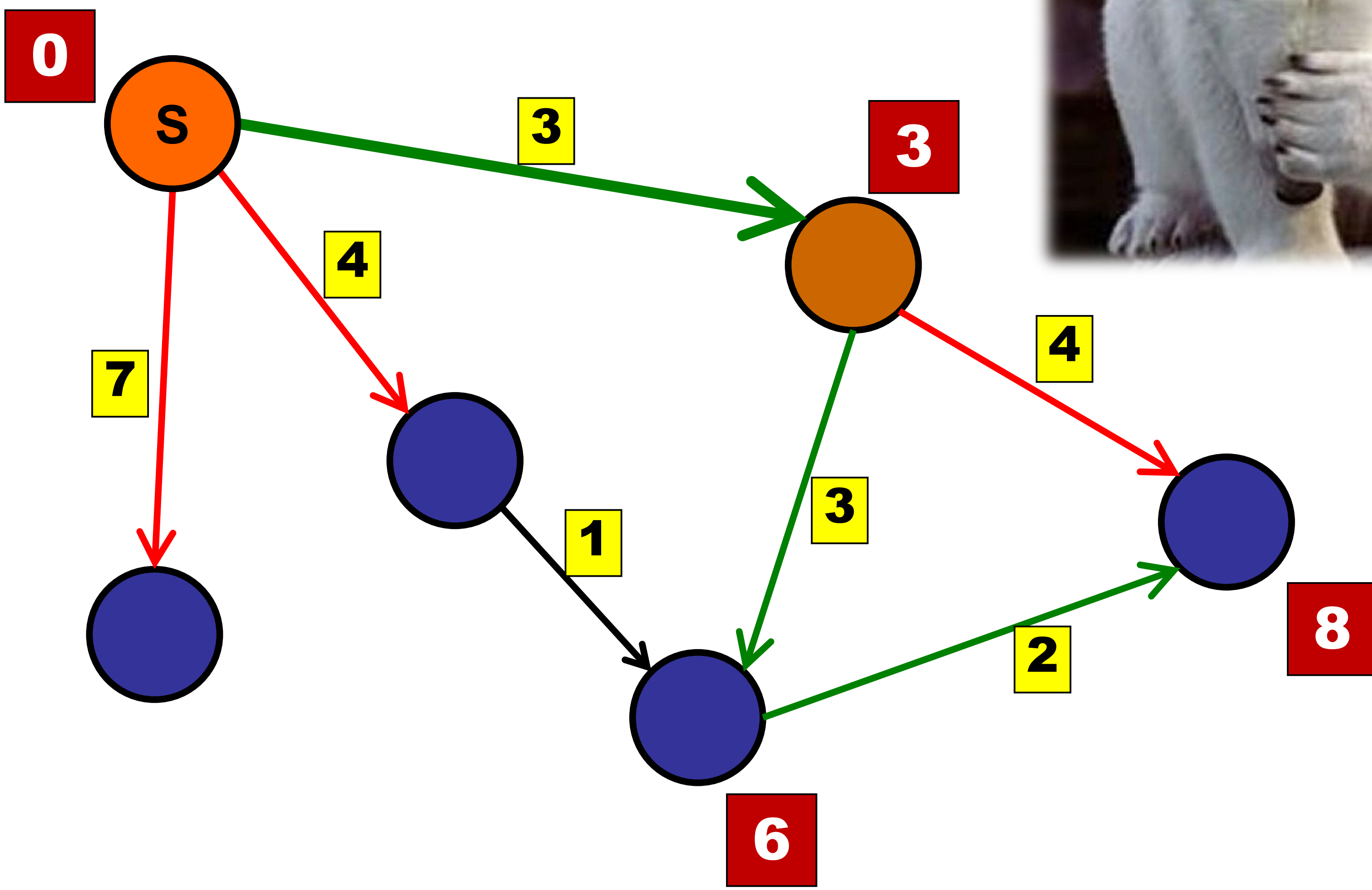
# Dijkstra's Algorithm (First Try)

Relax shortest edge first



# Dijkstra's Algorithm (Failed Try)

Oops….



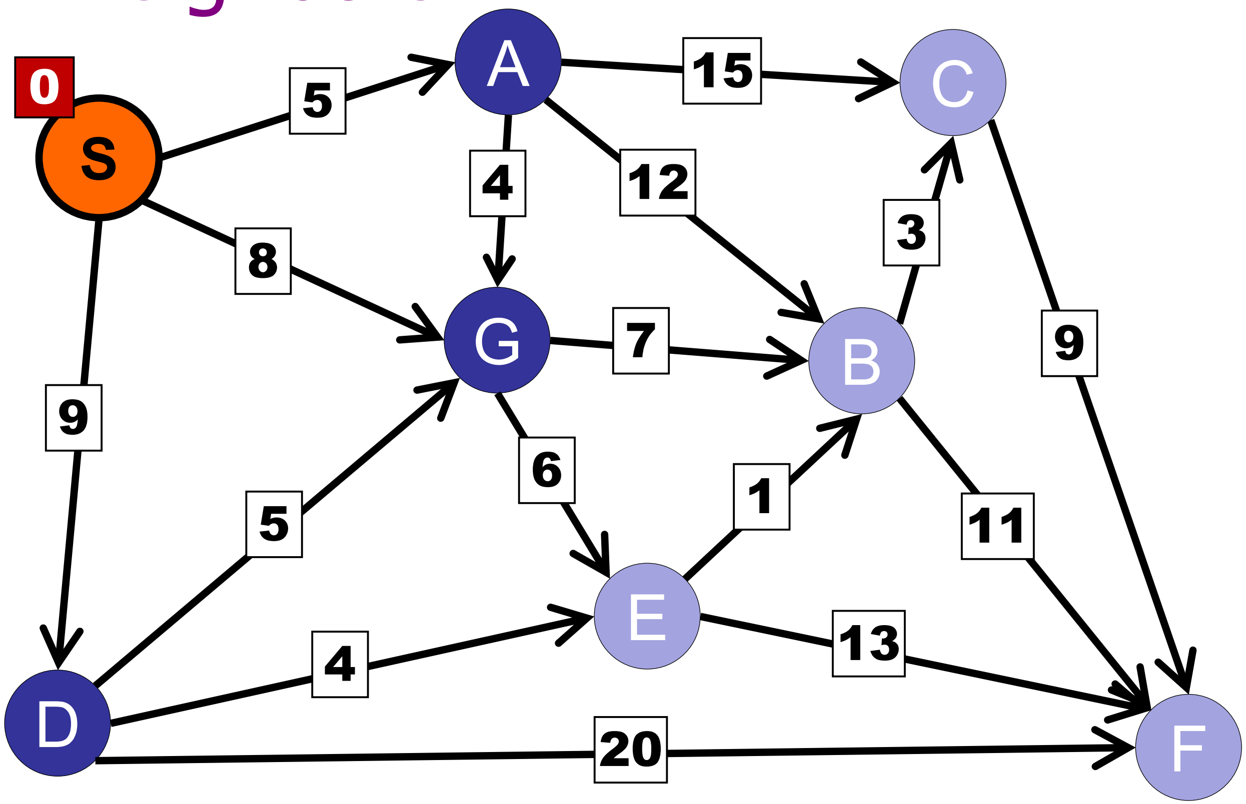
# Dijkstra’s Algorithm

Basic idea:

- Initialize:
  - Put all vertices into a priority queue
  - Set all priorities to estimated distances as infinity
  - Set the starting vertex estimated distance as 0
- Repeat until the priority is empty:
  - Extract the vertex v in the priority queue with the shortest estimated distance
  - Relax all the neighbors of v in the priority queue and update their estimated distance

# Dijkstra's Algorithm

## Extract S and relax/update neighbors

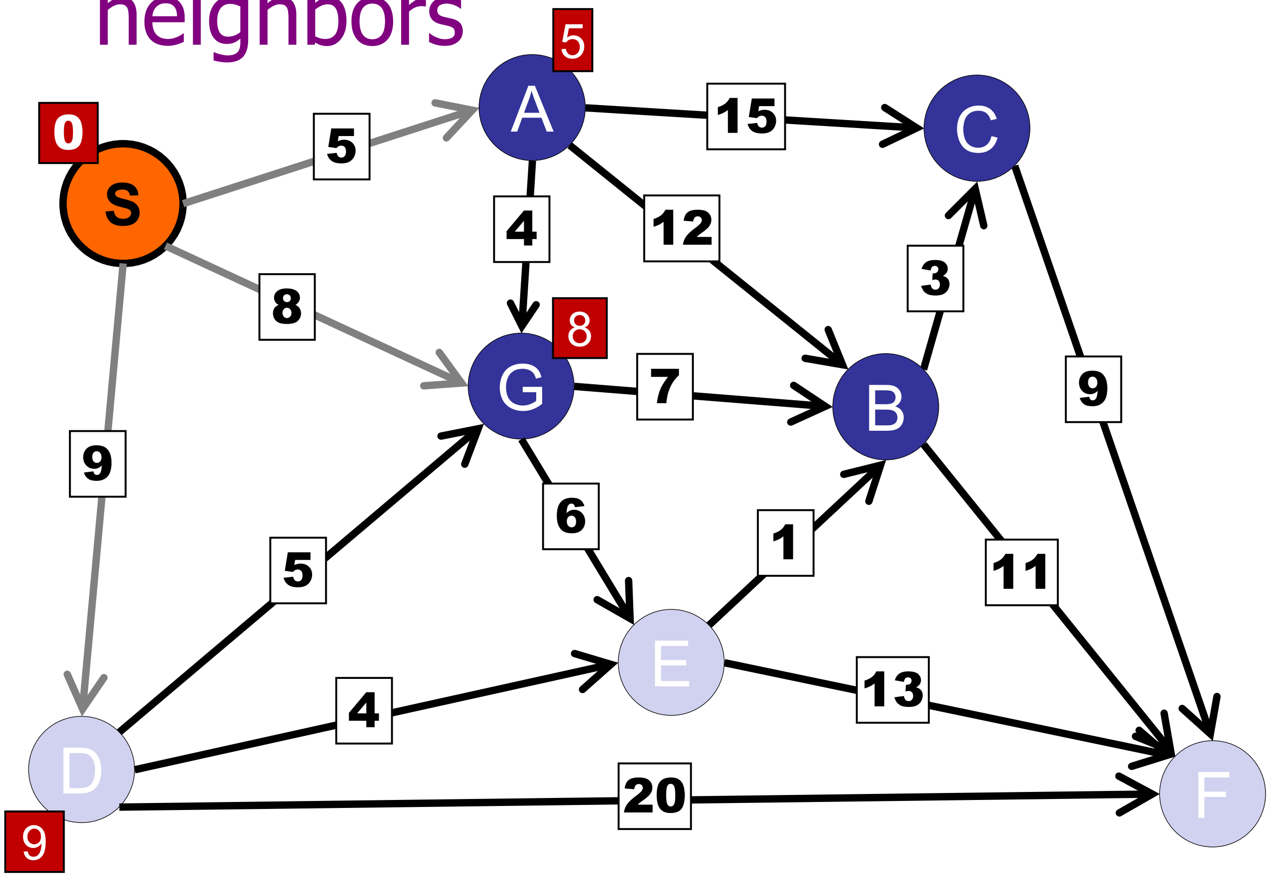



| Vertex | Dist. |
|---|---|
| S | 0 |
| | |
| | |
| | |
| | |
| | |
| | |

(Not showing vertices with distance = infinity)

# Dijkstra's Algorithm

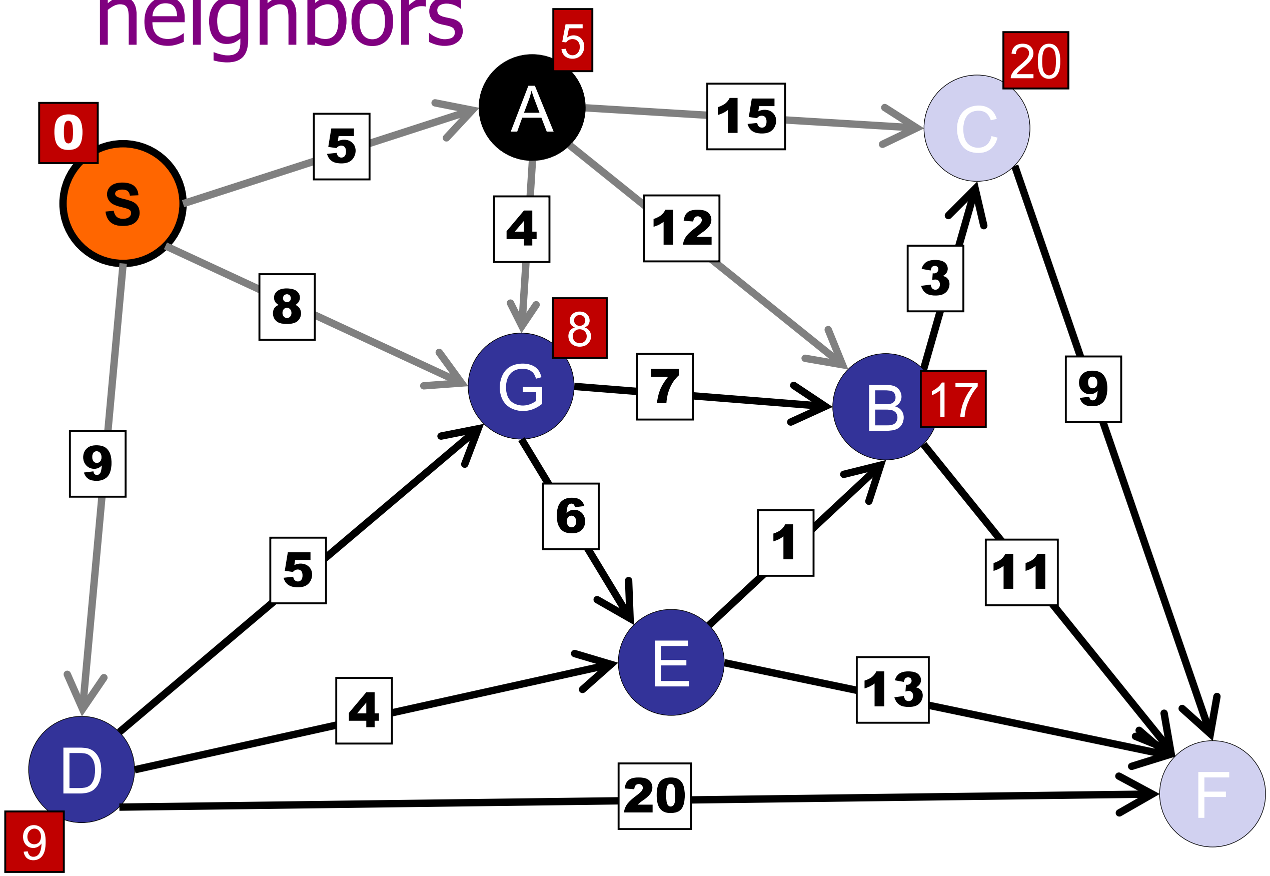
Extract A and relax/update neighbors




| Vertex | Dist. |
|---|---|
| A | 5 |
| G | 8 |
| D | 9 |
| | |
| | |
| | |
| | |

# Dijkstra's Algorithm

## Extract G and relax/update neighbors

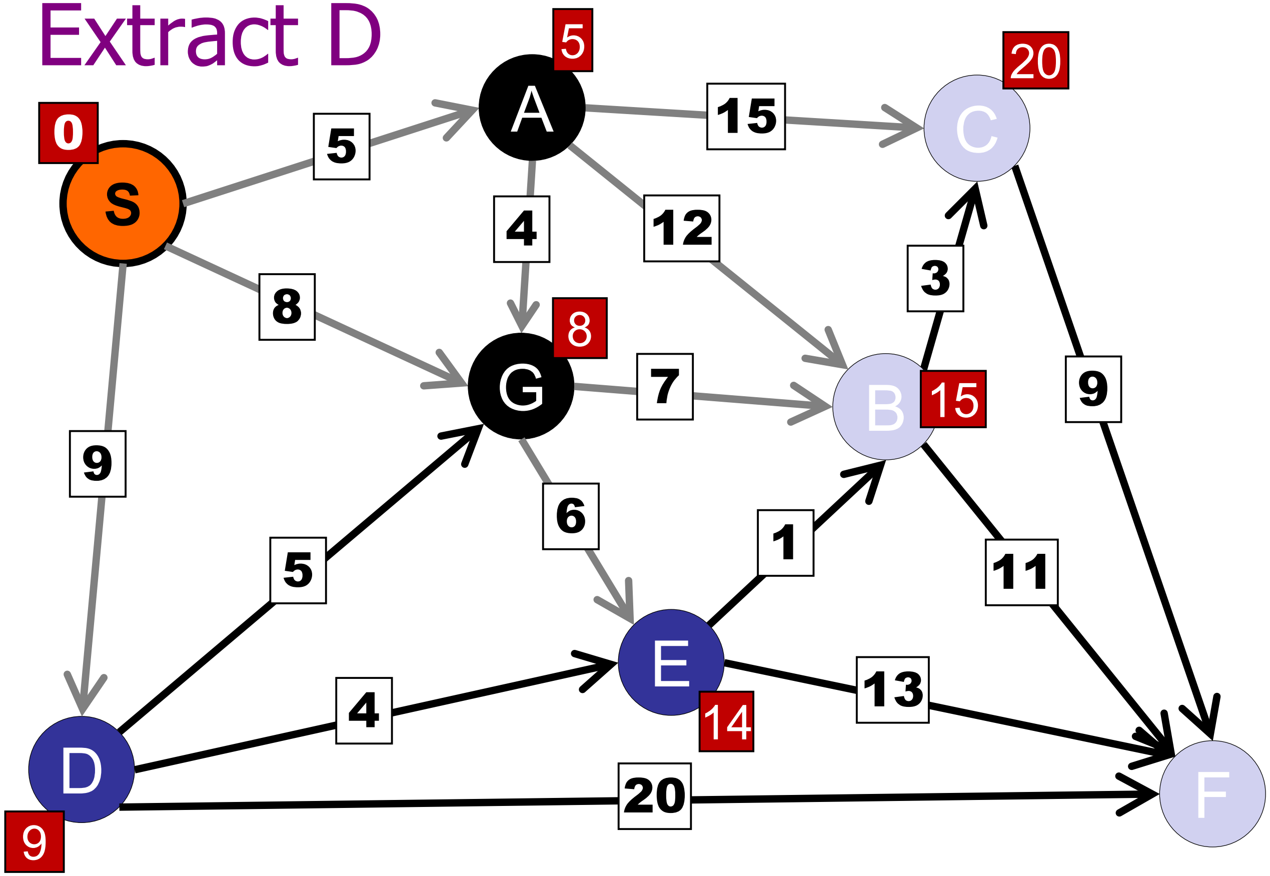



| Vertex | Dist. |
|---|---|
| G | 8 |
| D | 9 |
| B | 17 |
| C | 20 |
| | |
| | |

# Dijkstra’s Algorithm

## Dist. of B updated to 15

## Extract D

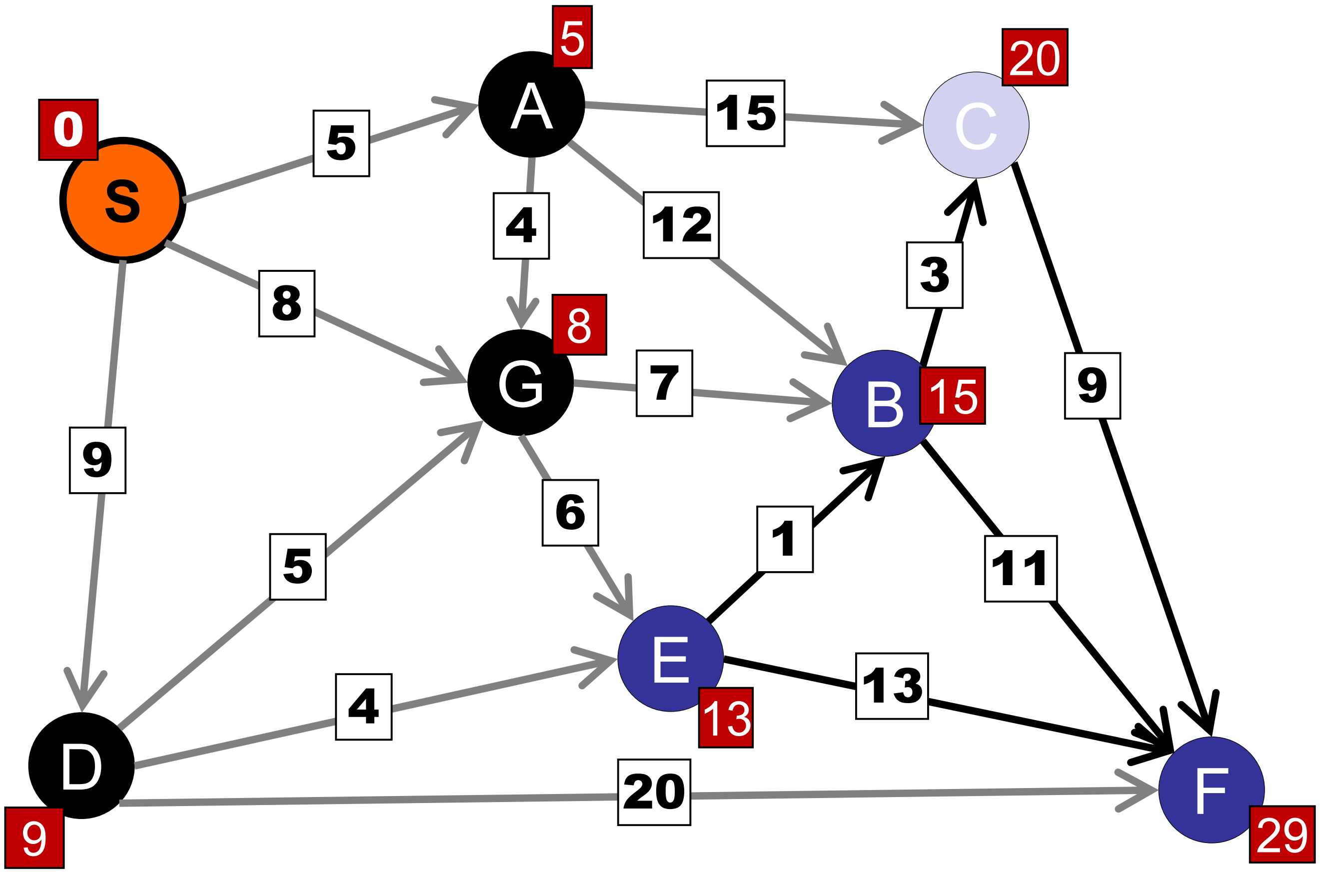



| Vertex | Dist. |
|---|---|
| D | 9 |
| E | 14 |
| B | 15 |
| C | 20 |
| | |
| | |

G is a neigbor of D. However, since G is already ”dequeued”, G won’t be added back to the PQ anymore

# Dijkstra's Algorithm

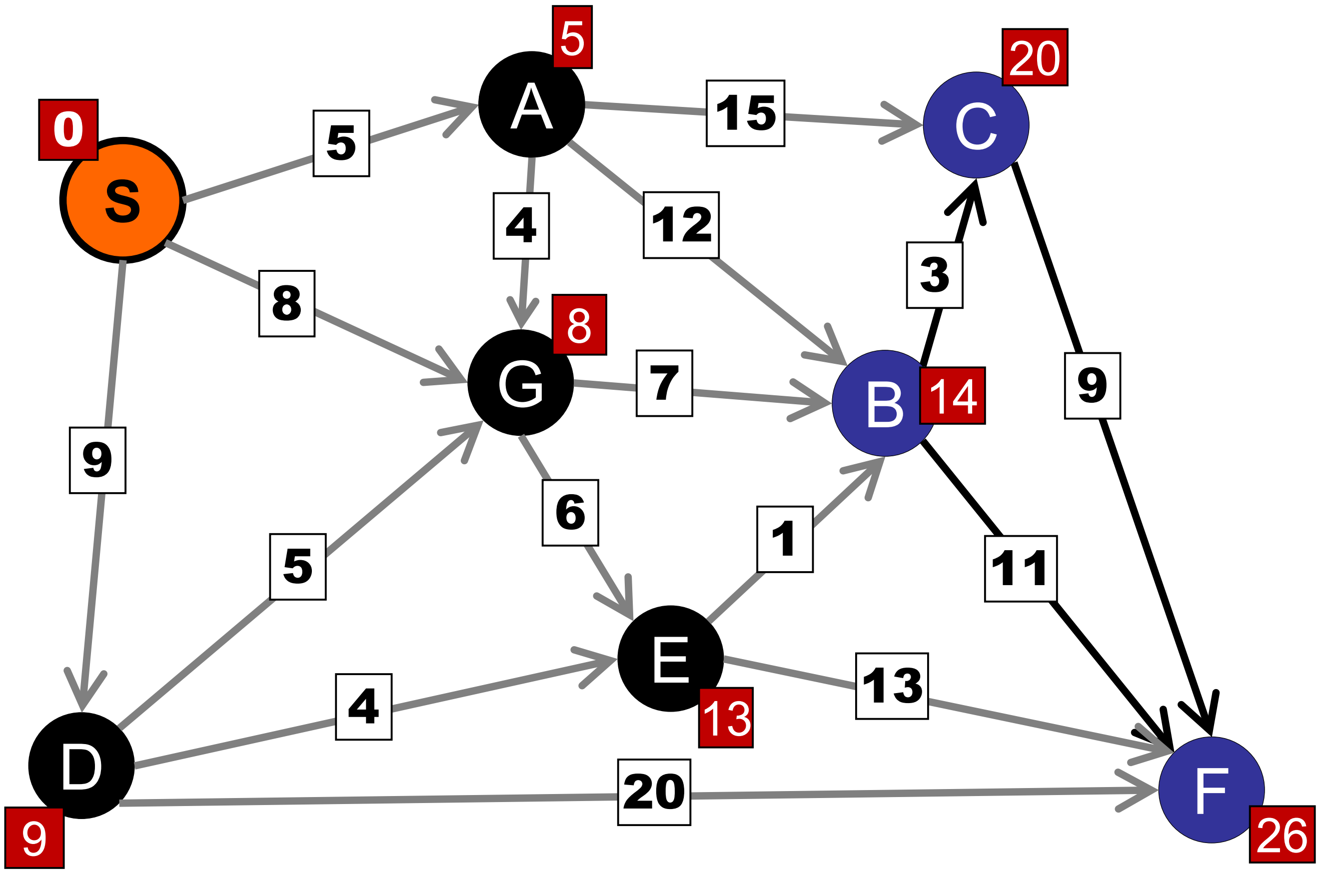
## Extract E




| Vertex | Dist. |
|---|---|
| **E** | **13** |
| B | 15 |
| C | 20 |
| **F** | **29** |
| | |

# Dijkstra's Algorithm

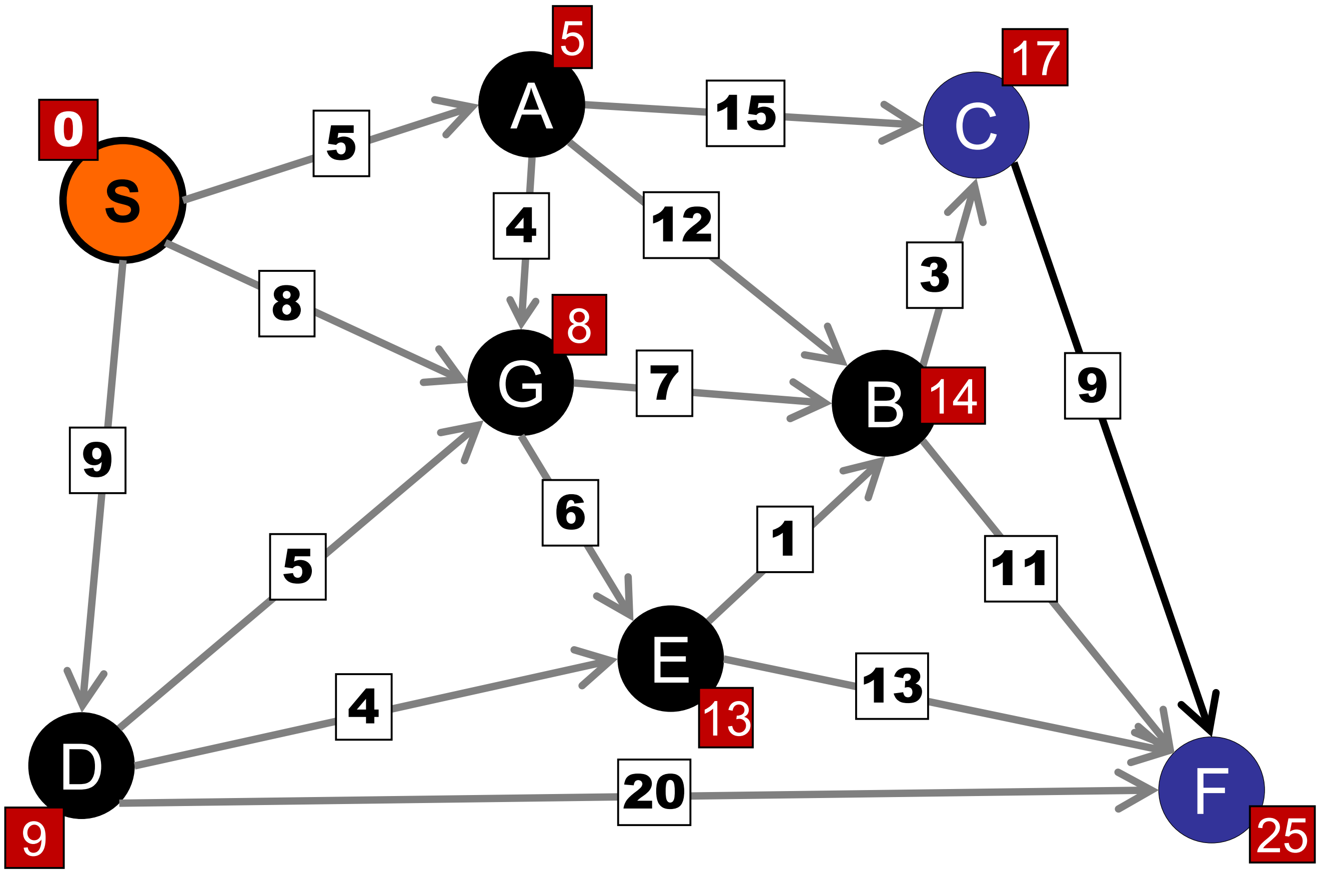
## Extract B




| Vertex | Dist. |
|---|---|
| **B** | **14** |
| C | 20 |
| **F** | **26** |
| | |

# Dijkstra's Algorithm

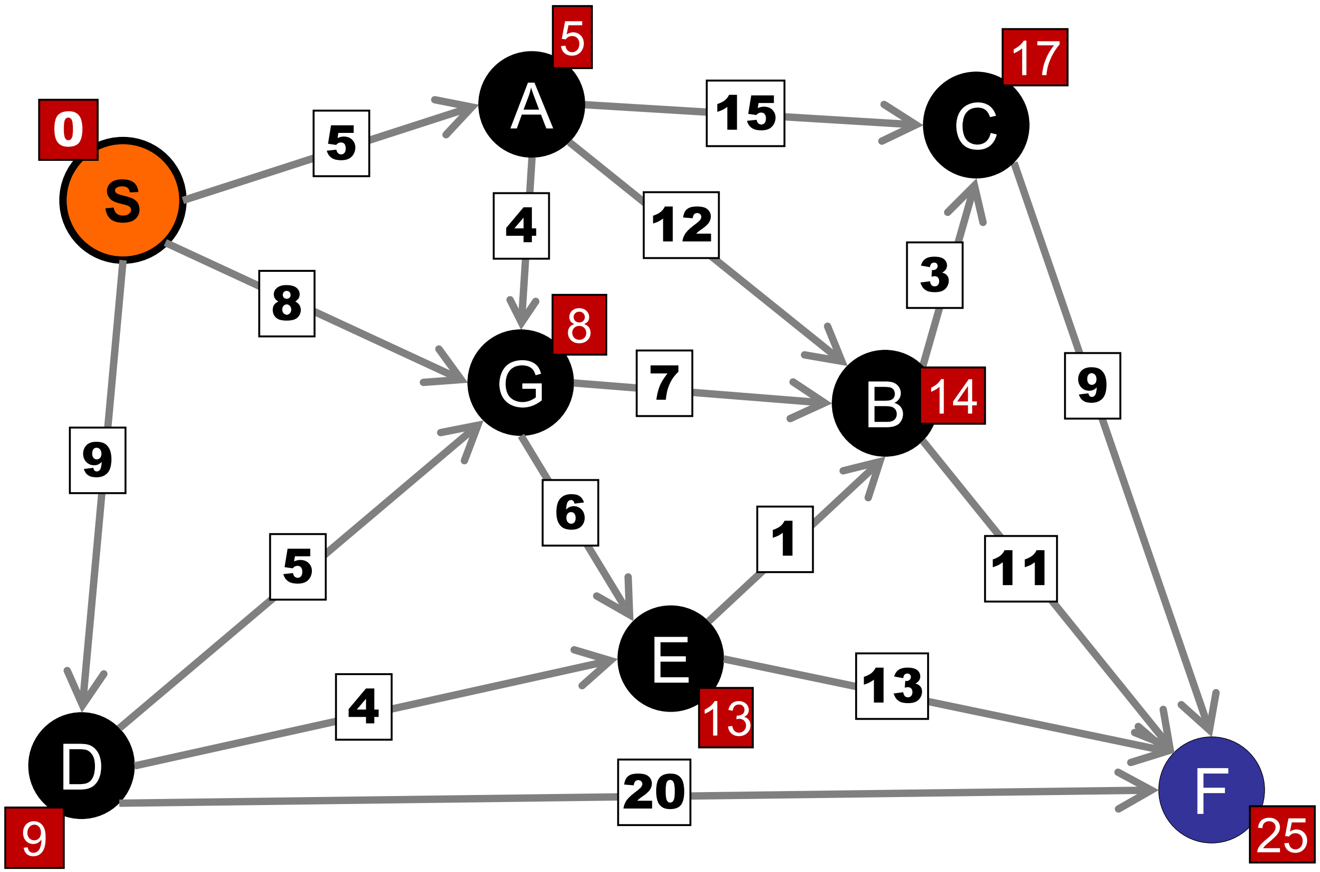
## Extract C




| Vertex | Dist. |
|---|---|
| C | 20 |
| F | 25 |
| | |

# Dijkstra's Algorithm

## Extract F

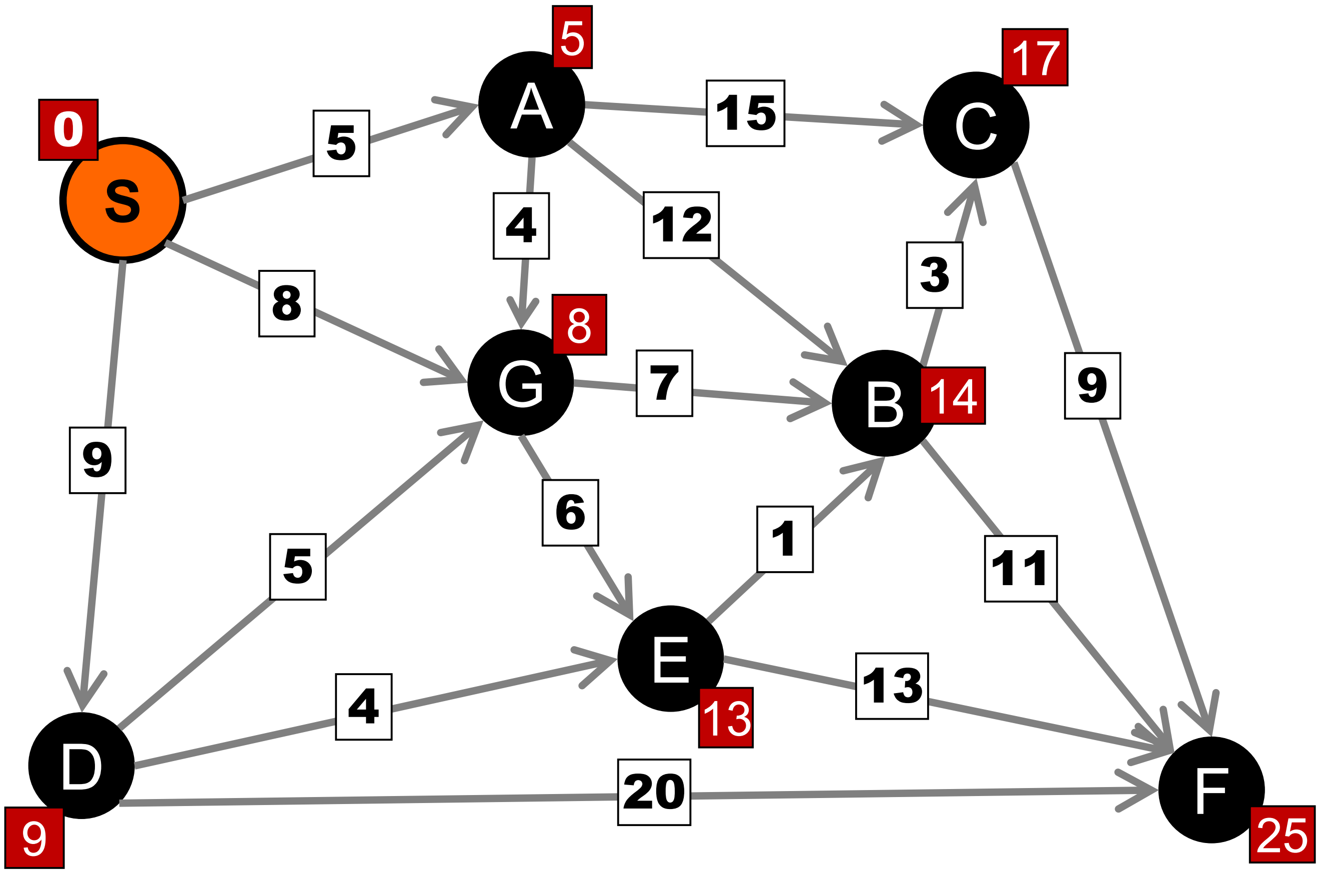



| Vertex | Dist. |
|---|---|
| F | 25 |
| | |

# Dijkstra's Algorithm

Done!

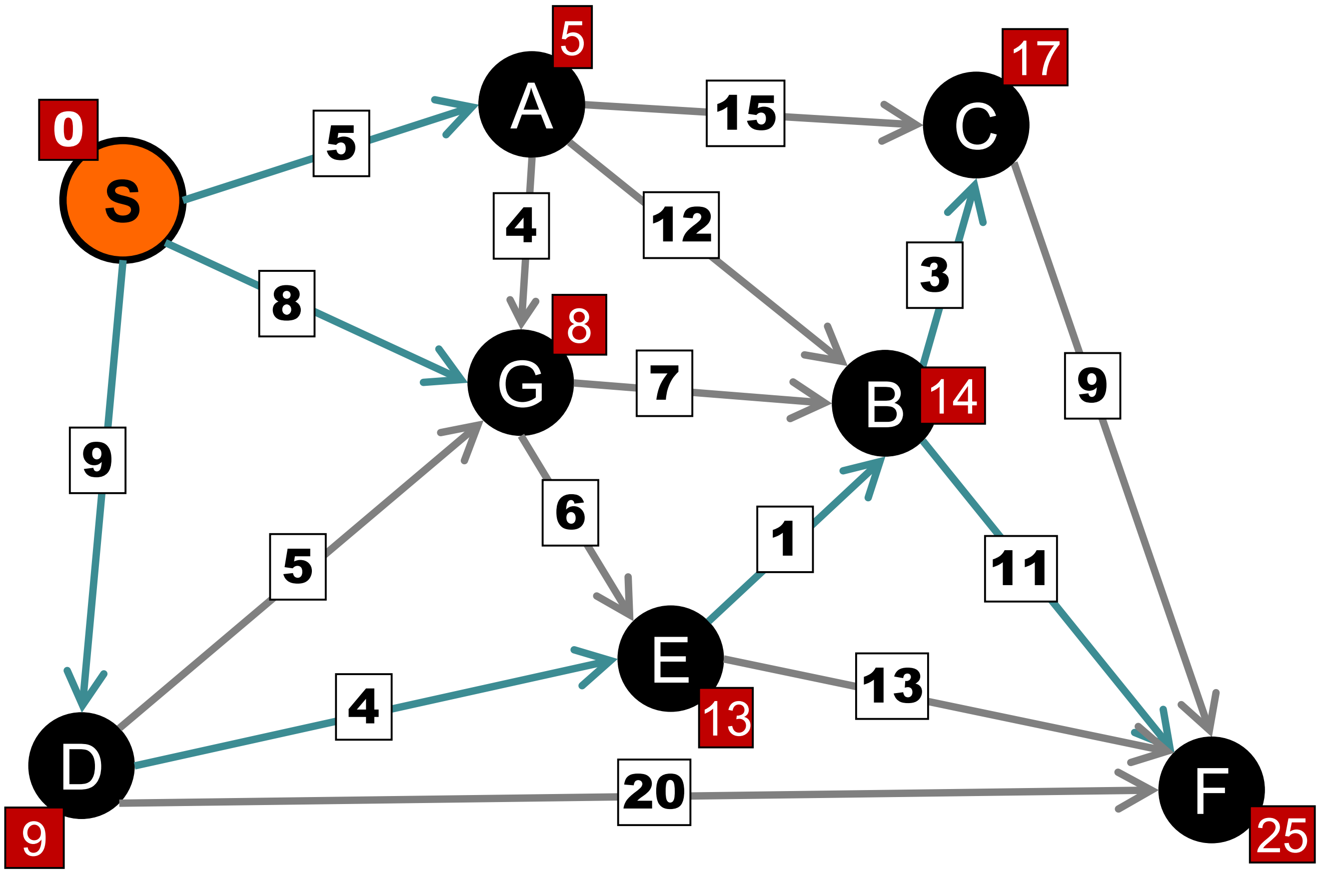
| Vertex | Dist. |
|---|---|
| | |



# Dijkstra’s Algorithm

## Done

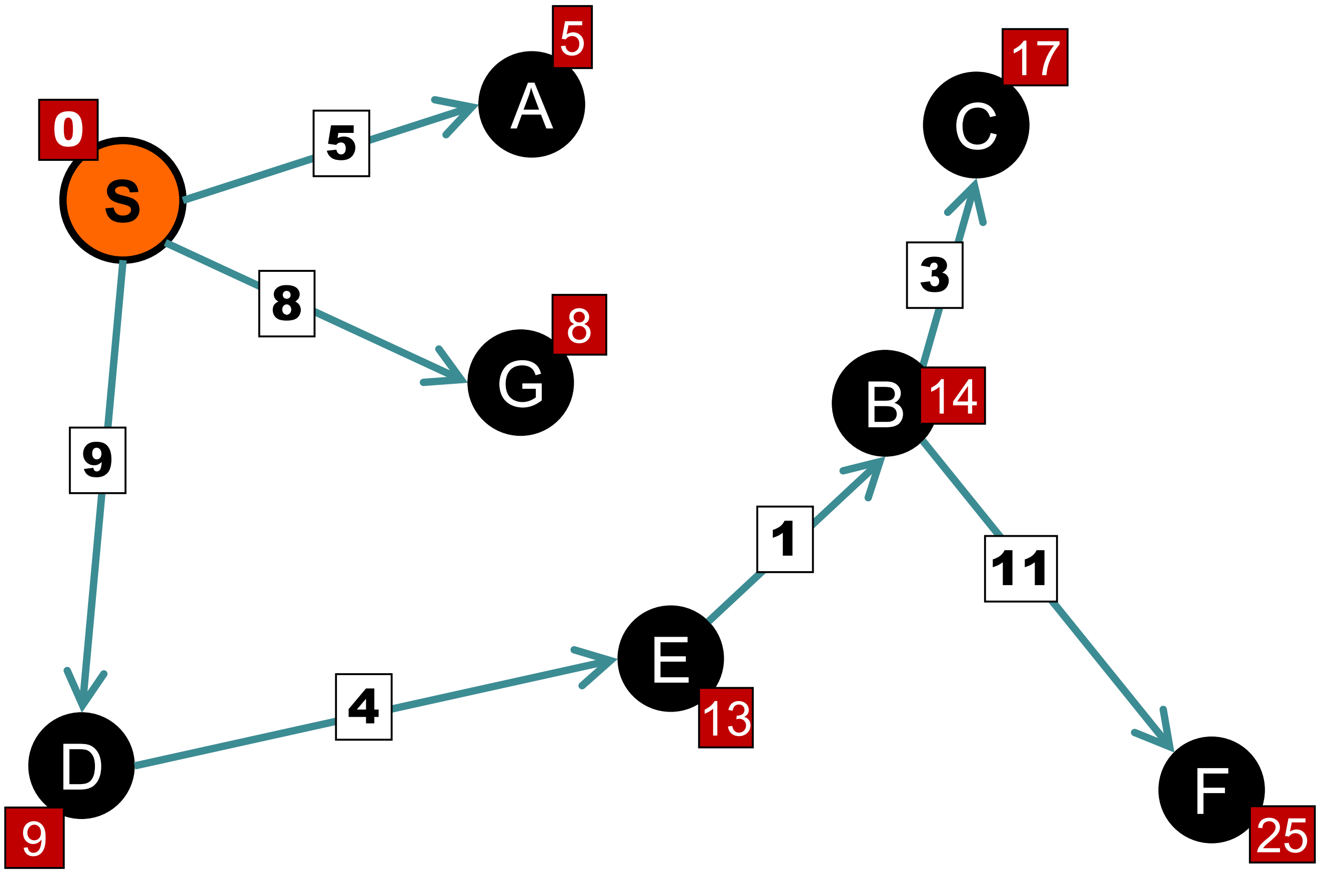
| Vertex | Dist. |
|---|---|
| | |



# Dijkstra's Algorithm

## Shortest Path Tree

| Vertex | Dist. |
|---|---|
| | |



# Abstract Data Type

## Priority Queue

| | | |
|---|---|---|
| `void` | `insert(Key k, Priority p)` | *insert k with priority p* |
| `Data` | `extractMin()` | *remove key with minimum priority* |
| `void` | `decreaseKey(Key k, Priority p)` | *reduce the priority of key k to priority p* |
| `boolean` | `contains(Key k)` | *does the priority queue contain key k?* |
| `boolean` | `isEmpty()` | *is the priority queue empty?* |

Notes:

Assume data items are unique.

# Dijkstra's Algorithm

```
relax(Edge e) {
    int v = e.from();
    int w = e.to();
    double weight = e.weight();
    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        pq.decreaseKey(w, distTo[w]);
    }
}
```

# Dijkstra's Algorithm

```
relax(Edge e) {
    int v = e.from();
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    }
}
```
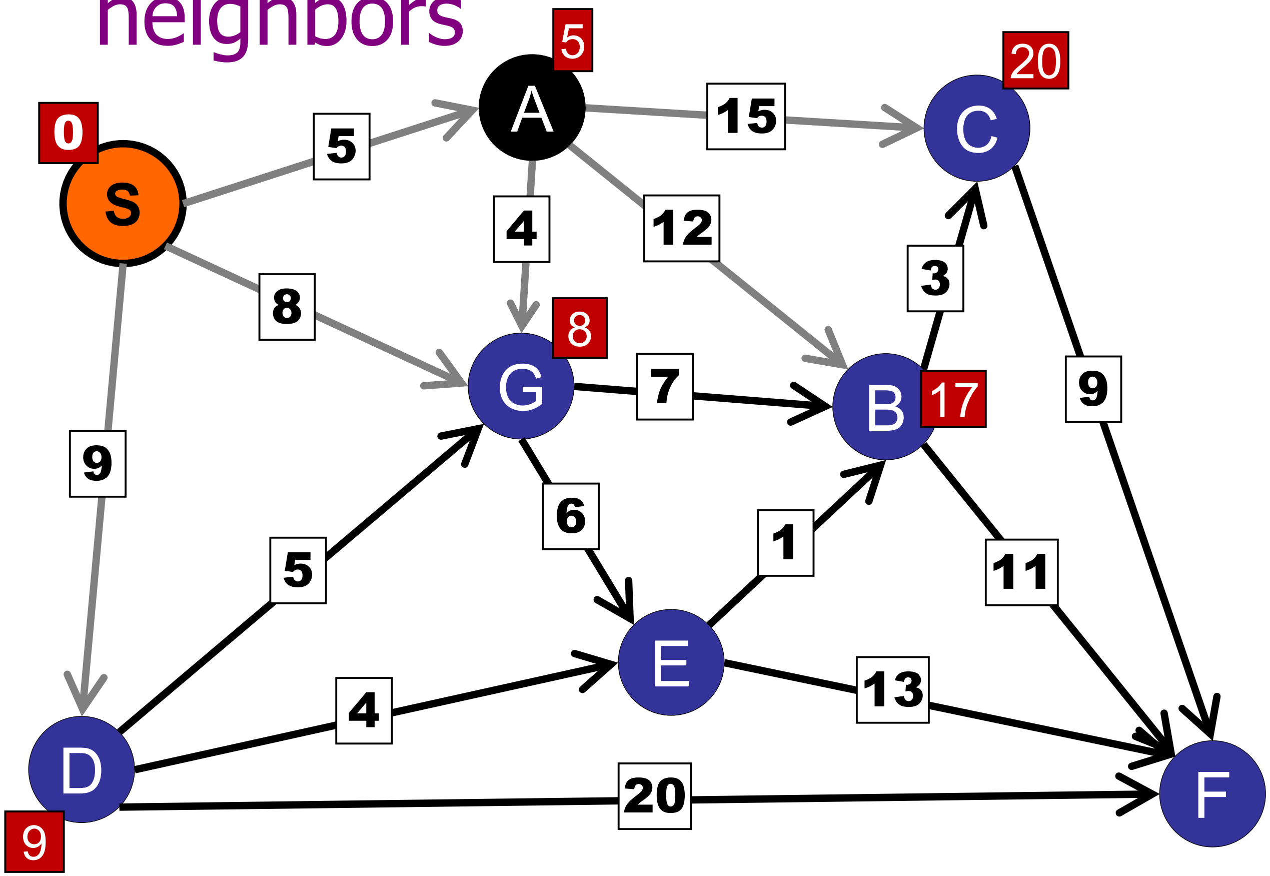

# Dijkstra's Algorithm

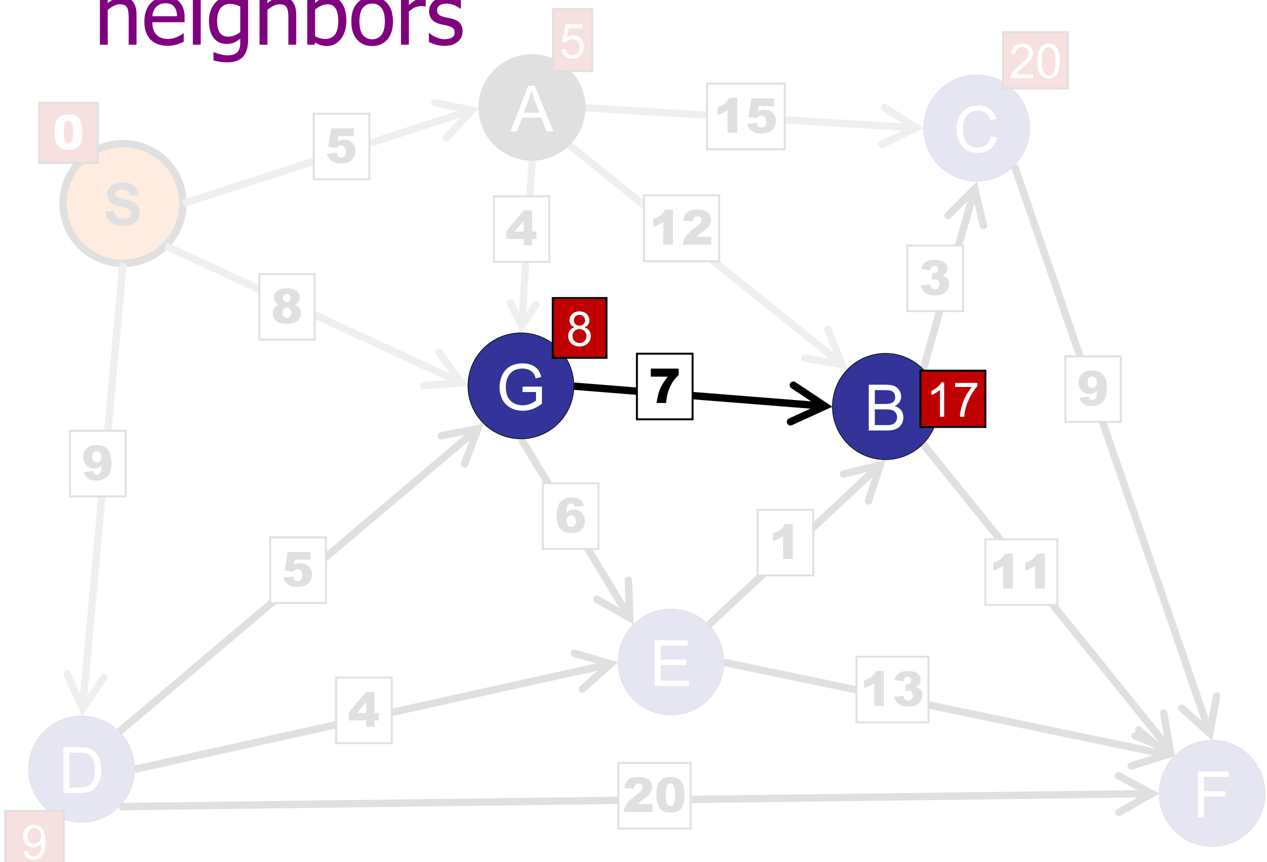
Remove G and relax/update neighbors




| Vertex | Dist. |
|---|---|
| G | 8 |
| D | 9 |
| B | 17 |
| C | 20 |
| | |
| | |

# Dijkstra's Algorithm

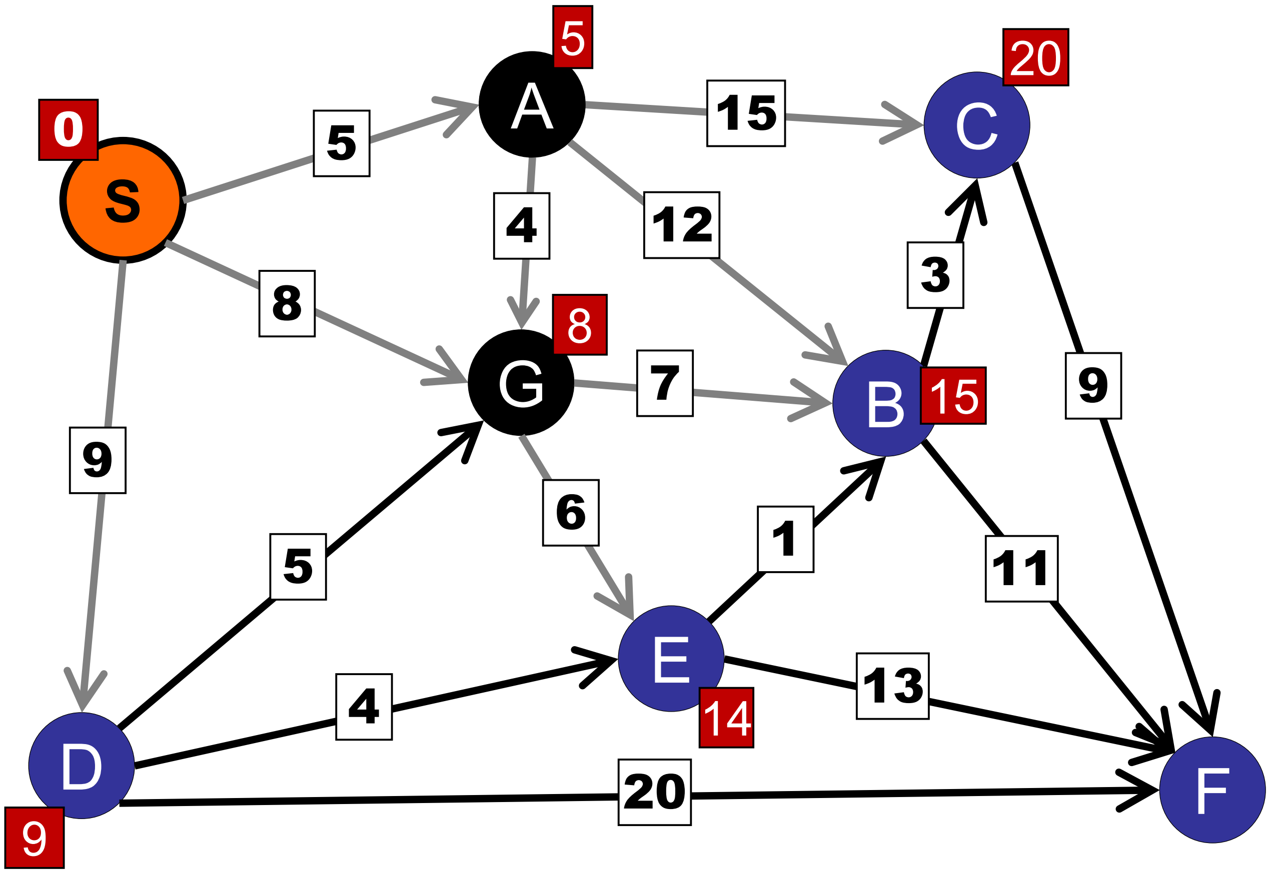
## Remove G and relax/update neighbors




| Vertex | Dist. |
|---|---|
| G | 8 |
| D | 9 |
| B | 17 |
| C | 20 |
| | |
| | |

# Dijkstra's Algorithm

Remove G and relax.




| Vertex | Dist. |
|---|---|
| D | 9 |
| E | 14 |
| B | 15 |
| C | 20 |
| | |
| | |

# Dijkstra's Algorithm

```
relax(Edge e) {
    int v = e.from();
    int w = e.to();
    double weight = e.weight();
    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        pq.decreaseKey(w, distTo[w]);
    }
}
```

# Dijkstra's Algorithm

```
relax(Edge e) {
    int v = e.from();
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    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        pq.decreaseKey(w, distTo[w]);
    }
}
```

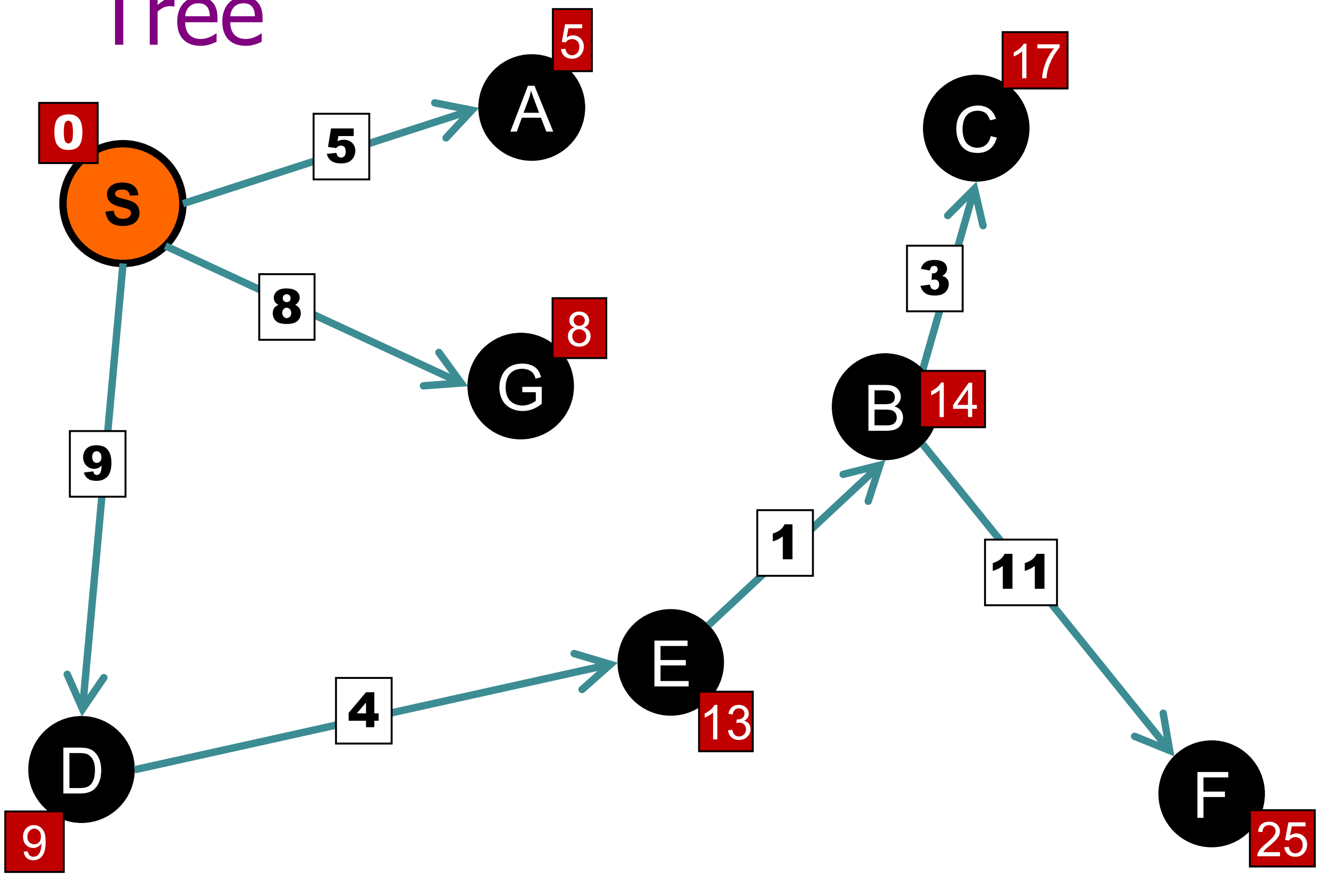
# Dijkstra’s Algorithm

Analysis:

- deleteMin: |V| times each
  - Each node is added to the priority queue **once**.
- relax / decreaseKey: |E| times
  - Each edge is relaxed once.
- Priority queue operations: O(log V)
- Total: O((V+E)log V) = O(E log V)

# Dijkstra's Algorithm

## Following the parents: Yields the Shortest Path Tree



# Dijkstra's Algorithm

Why does it work?

# Dijkstra's Algorithm

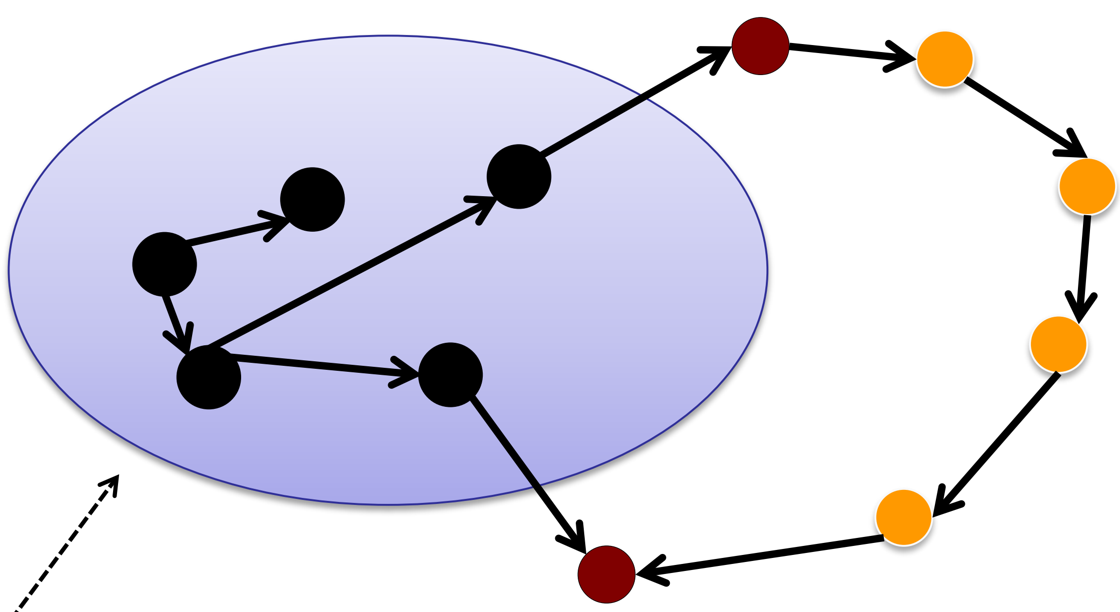
Proof by induction:

- Every "finished" (dequeued) vertex has a correct estimate.
  - Namely, shortest path is found for that vertex
- Initially: only "finished" vertex is start.

# Dijkstra's Algorithm

Every edge crossing the boundary has been relaxed.

**fringe vertices: neighbor of a finished vertex.**



**other vertices: no known estimate**

**finished vertices: distance is accurate.**

**fringe vertices: top in priority queue neighbor of a finished vertex.**

# Dijkstra's Algorithm

Proof by induction:

- Every "finished" vertex has correct estimate.
- Initially: only "finished" vertex is start.

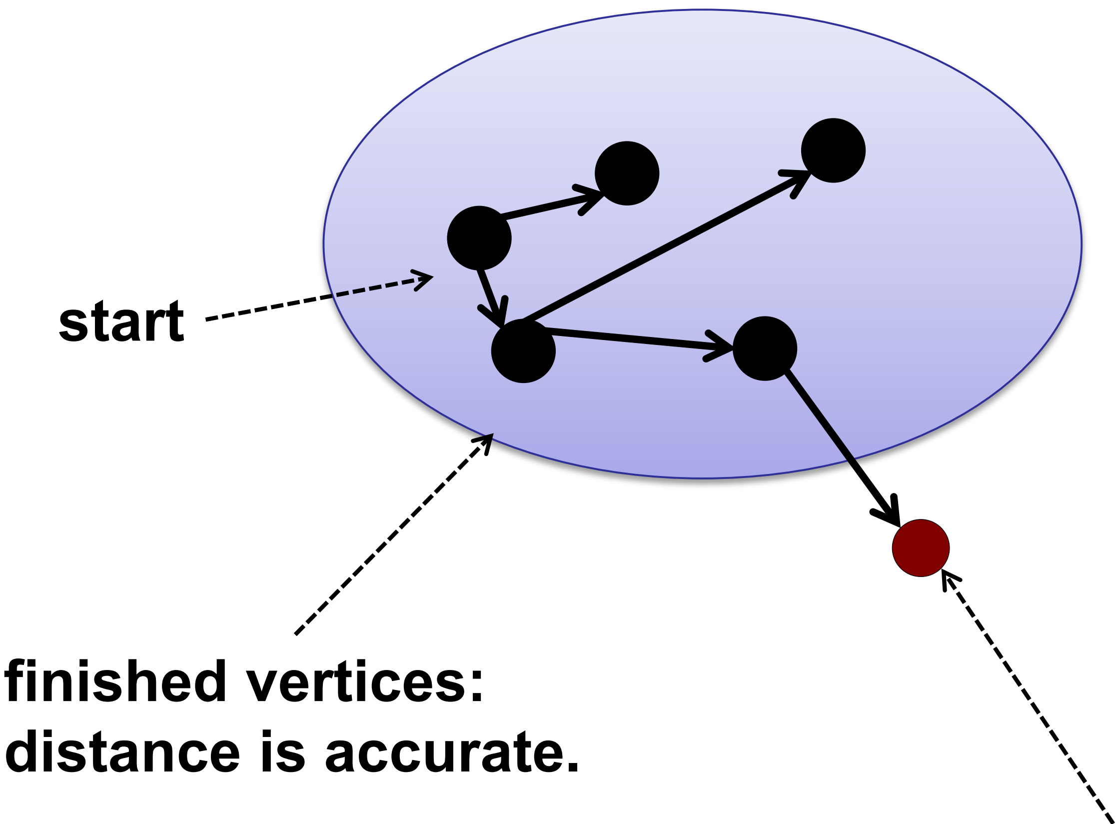
# Dijkstra's Algorithm

Proof by induction:

- Every "finished" vertex has correct estimate.
- Initially: only "finished" vertex is start.

- Inductive step:
  - Remove vertex from priority queue.
  - Relax its edges.
  - Add it to finished.
  - **Claim: it has a correct estimate.**

# Dijkstra’s Algorithm

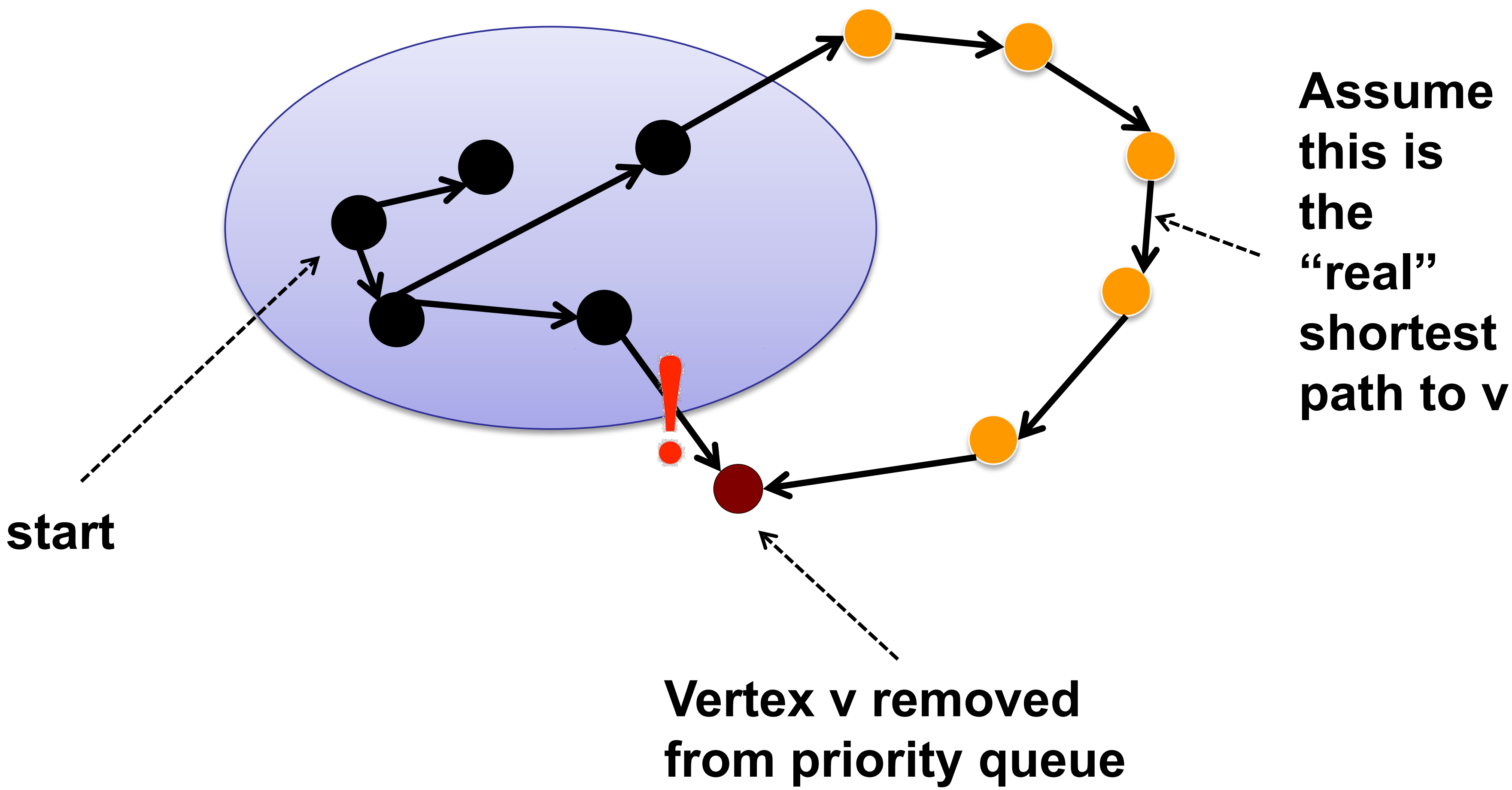



Let this distance of v be $d_v$

**Vertex v going to be removed from priority queue next. Thus, with minimum distance amount the unfinished**

# Dijkstra's Algorithm

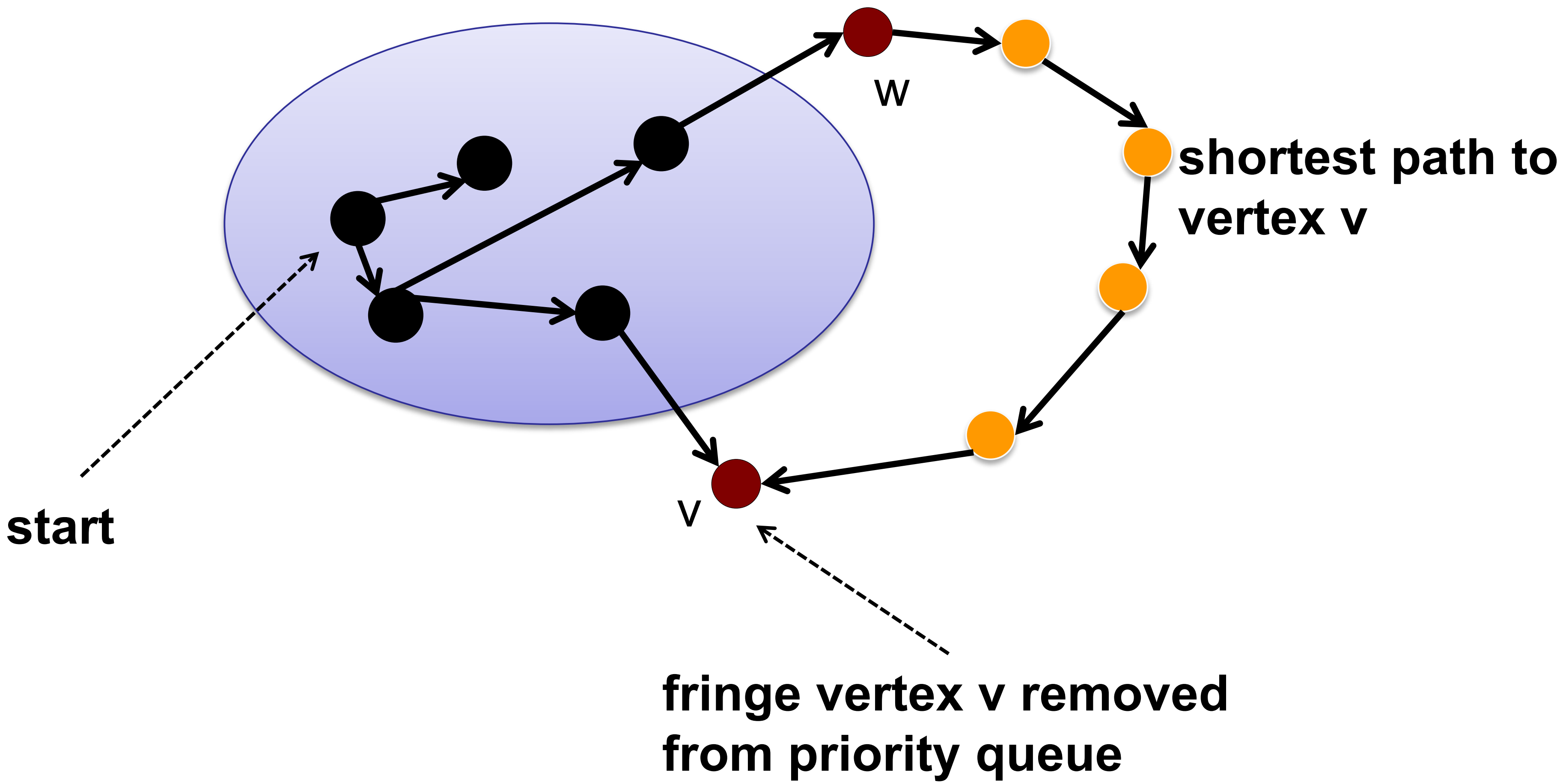
Assume NOT. The current estimate is not the shortest path. And the new path has dist. < $d_v$



# Dijkstra's Algorithm

There must be a vertex w in the current PQ on this "real" path.

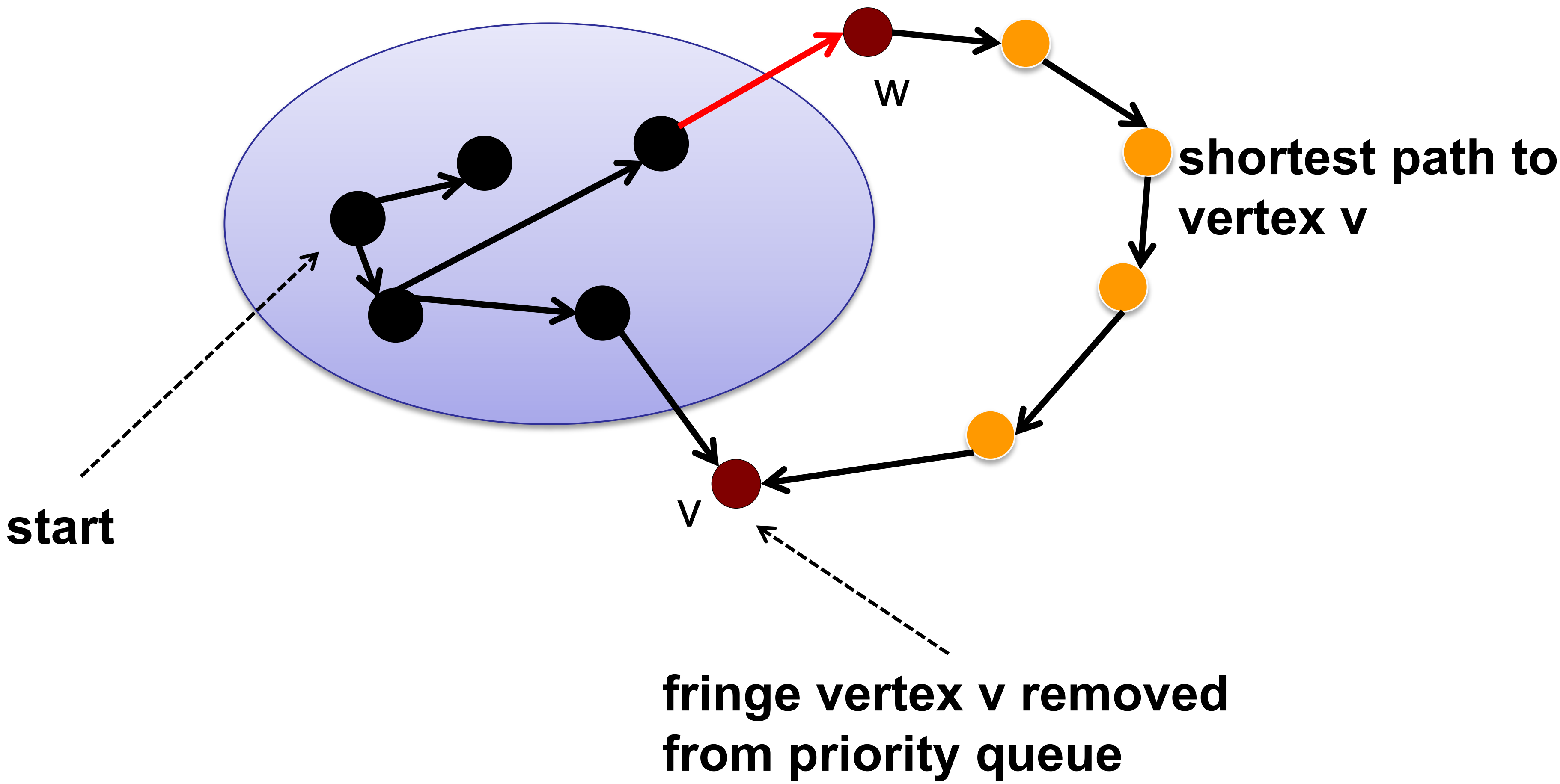
And let this "real" path has distance $r_v < d_v$



# Dijkstra's Algorithm

If P is shortest path to v, then prefix of P is shortest path to w.

Then distTo[w] is accurate. And **distTo[w] < $r_v$ < $d_v$ = distTo[v]**

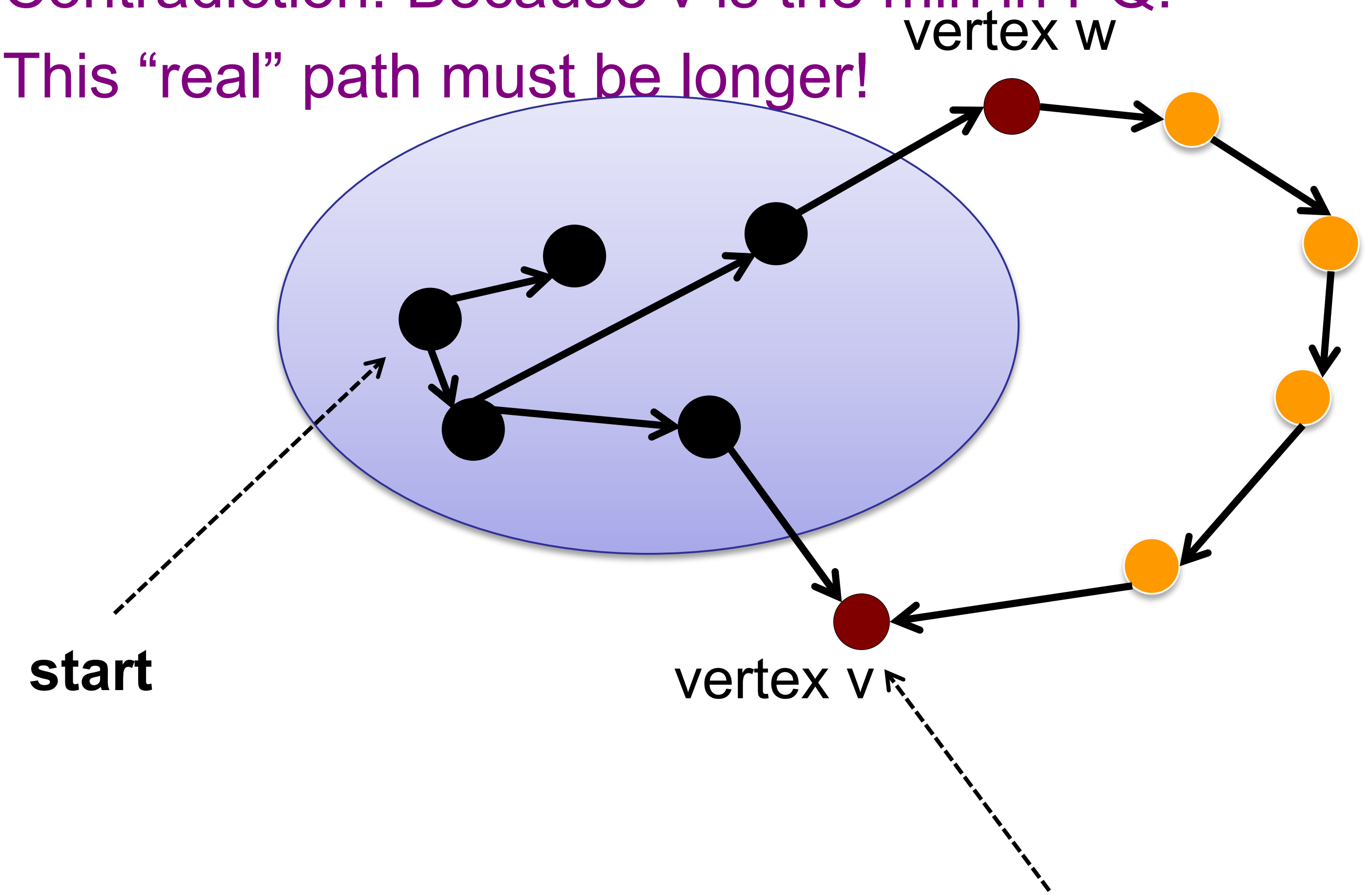


# Dijkstra's Algorithm

But `distTo[w] >= distTo[v]` according to PQ!

Contradiction! Because v is the min in PQ!

This "real" path must be longer!




**Vertex v going to be removed from priority queue next. Thus, with minimum distance amount the unfinished**

# Dijkstra's Algorithm

Proof by induction:

- Every "finished" vertex has correct estimate.
- Initially: only "finished" vertex is start.

- Inductive step:
  - Remove vertex from priority queue.
  - Relax its edges.
  - Add it to finished.
  - **Claim: it has a correct estimate.**

# Dijkstra's Algorithm

```
relax(Edge e) {
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    }
}
```

# Dijkstra's Algorithm

Analysis:

- insert / deleteMin: |V| times each
  - Each node is added to the priority queue **once**.
- decreaseKey: |E| times
  - Each edge is relaxed once.
- Priority queue operations: O(log V)
- Total: O((V+E)log V) = O(E log V)

# Dijkstra's Algorithm

Source-to-Destination:

- What if you stop the first time you dequeue the destination?

- Recall:
  - a vertex is "finished" when it is dequeued
  - if the destination is finished, then stop

# Dijkstra Summary

Basic idea:

- Maintain distance estimates.
- Repeat:
  - Find unfinished vertex with smallest estimate.
  - Relax all outgoing edges.
  - Mark vertex finished.

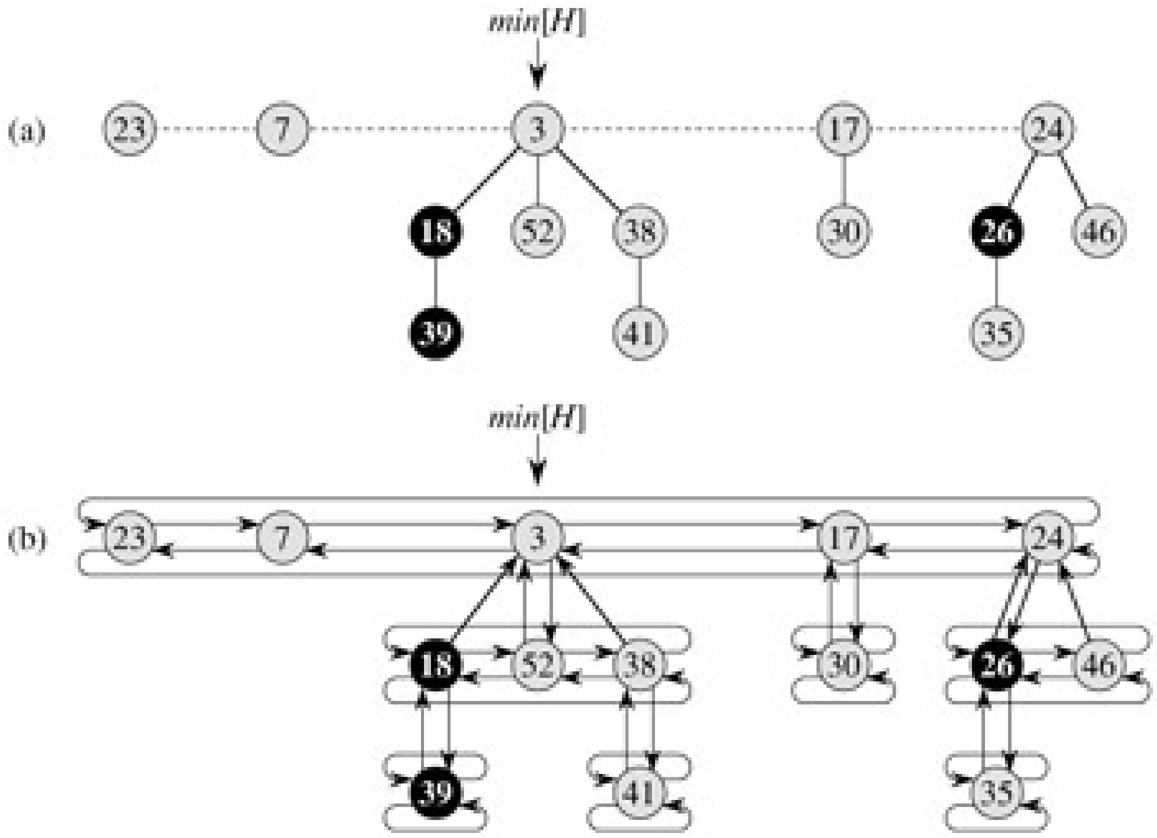
- O(E log V) time

# Dijkstra's Performance

| PQ Implementation | insert | deleteMin | decreaseKey | Total |
|---|---|---|---|---|
| Array | 1 | V | 1 | **$O(V^2)$** |
| AVL Tree | log V | log V | log V | **O(E log V)** |
| d-way Heap | $d\log_d V$ | $d\log_d V$ | $\log_d V$ | **$O(E\log_{E/V} V)$** |
| Fibonacci Heap | 1 | log V | 1 | **O(E + V log V)** |

# Fibonacci Heap

- Not in this course

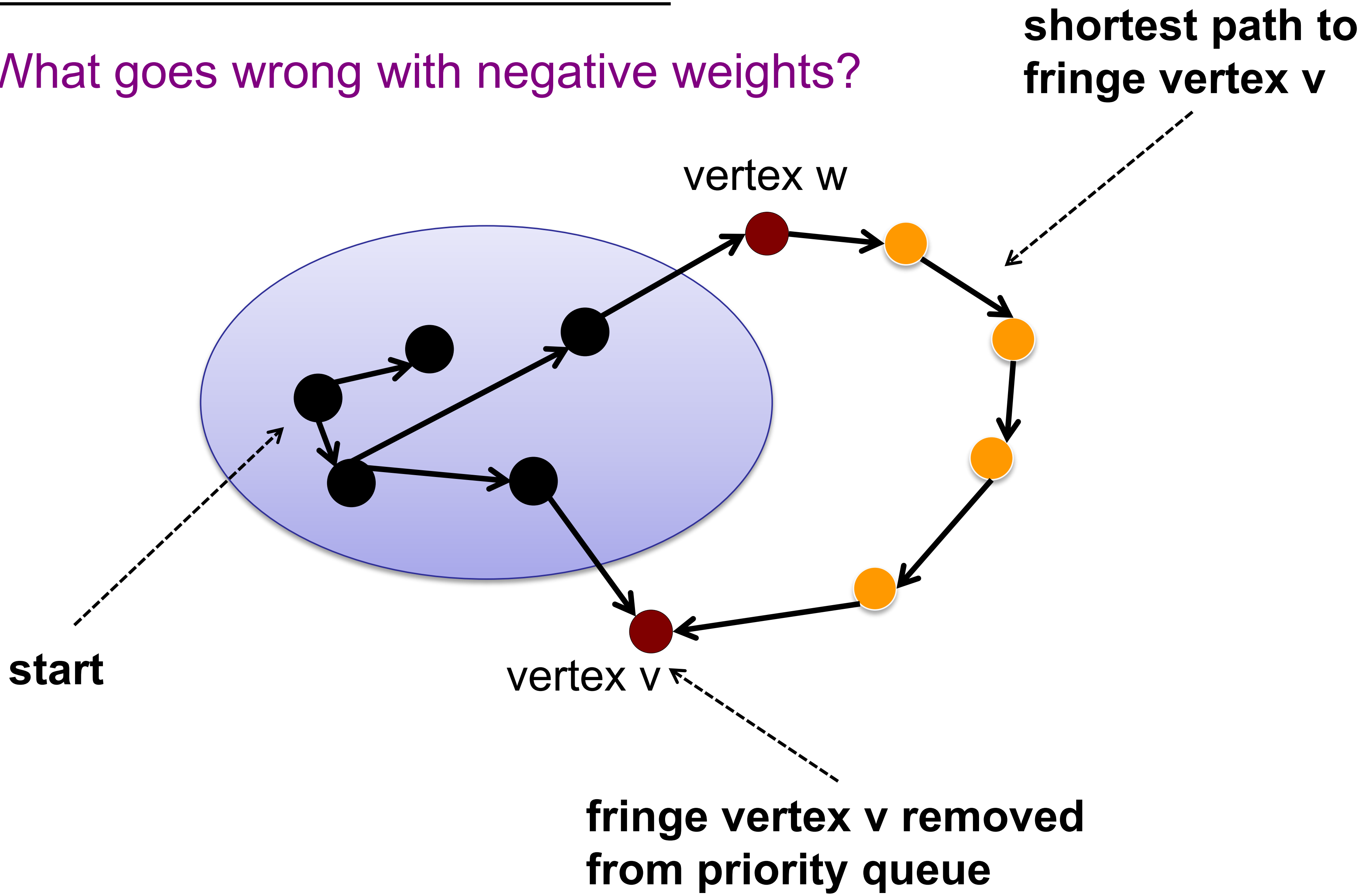


# Dijkstra Summary

Edges with negative weights?

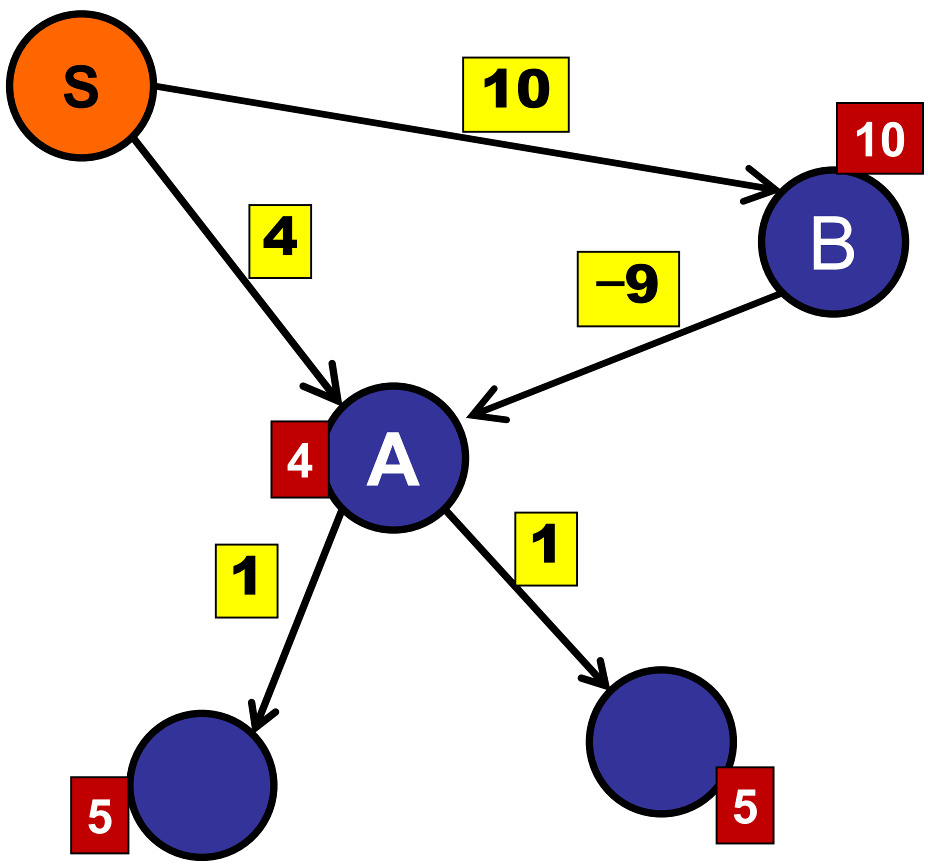
# Dijkstra's Algorithm

What goes wrong with negative weights?



# Dijkstra's Algorithm

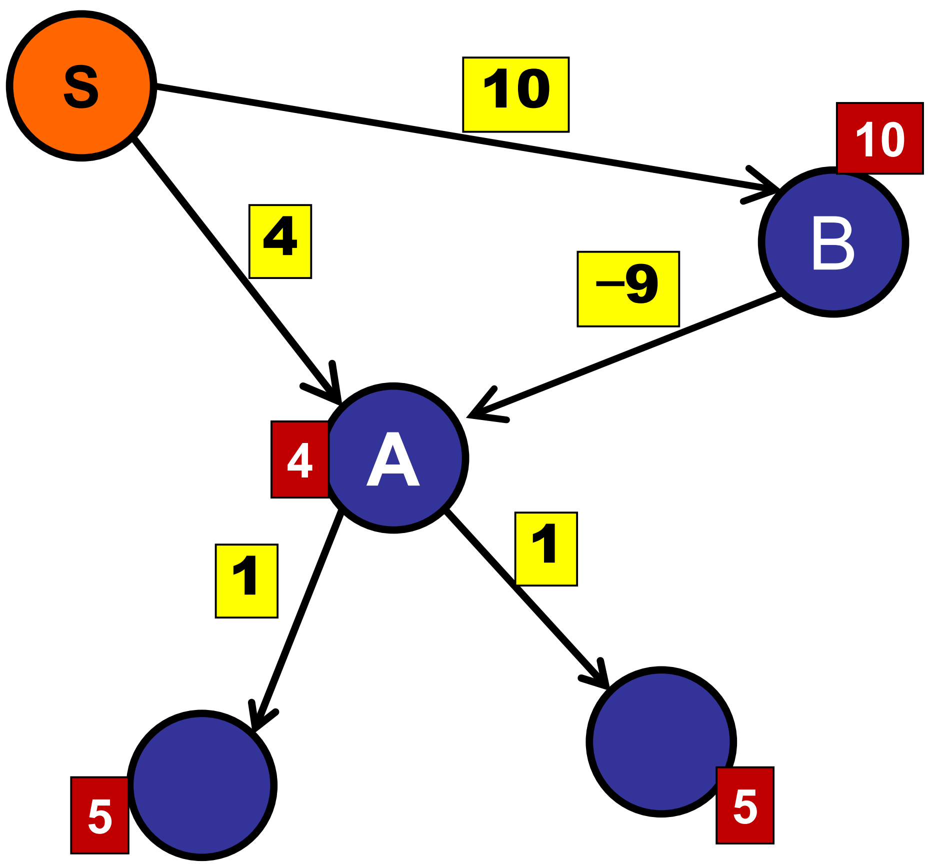
## Edges with negative weights?




Step 1: Remove A.
Relax A.
Mark A done.

# Dijkstra's Algorithm

Edges with negative weights?




Step 1: Remove A.
Relax A.
Mark A done.

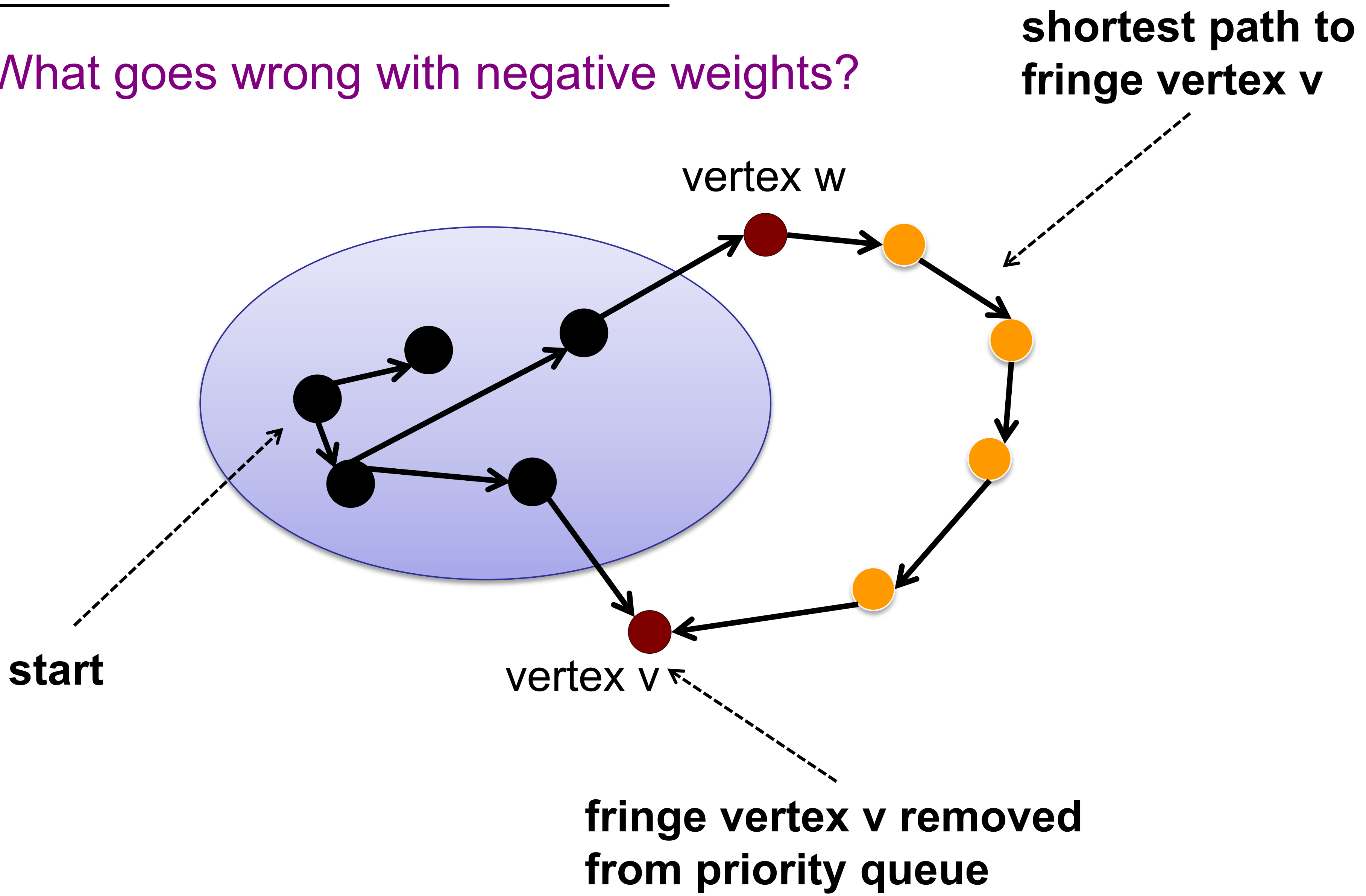
…

Step 4: Remove B.
Relax B.
Mark B done.

Oops: We need to update A.

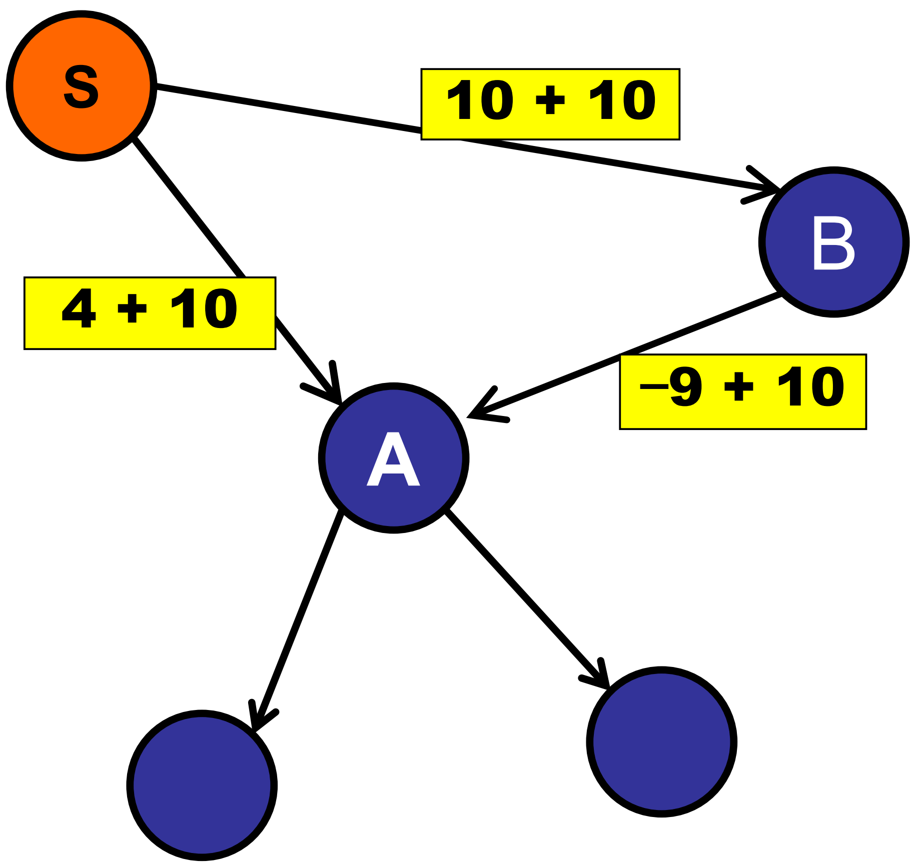
# Dijkstra's Algorithm

What goes wrong with negative weights?



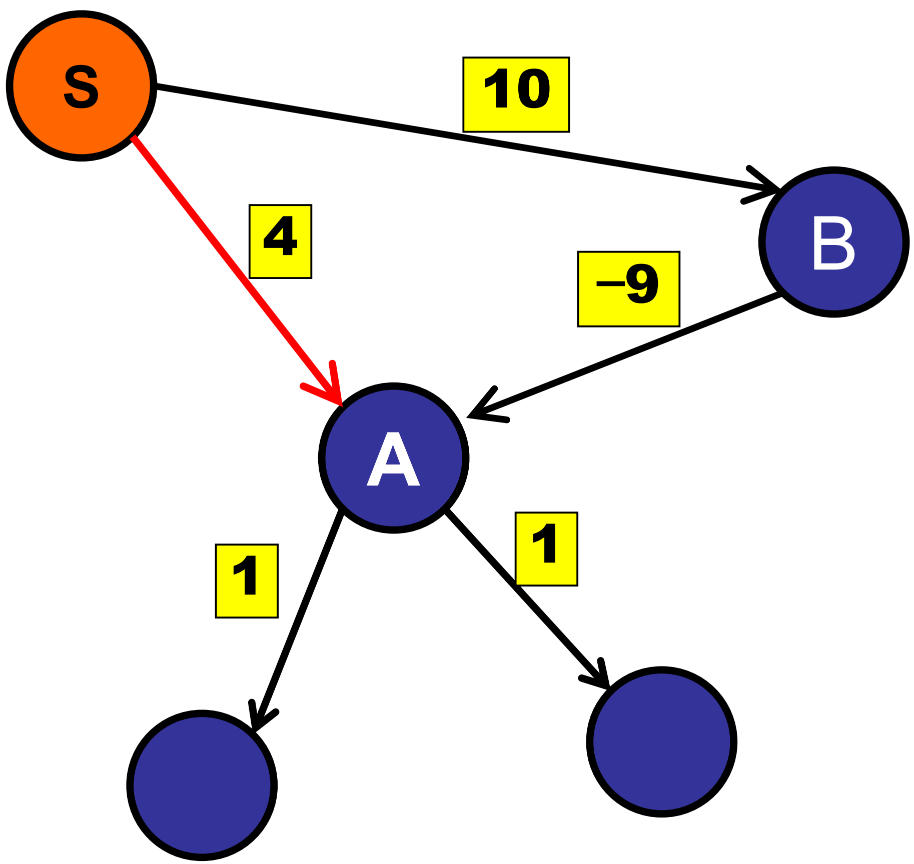
# Dijkstra's Algorithm

Can we reweight? e.g.: weight += 10



# Dijkstra's Algorithm

Can we reweight?

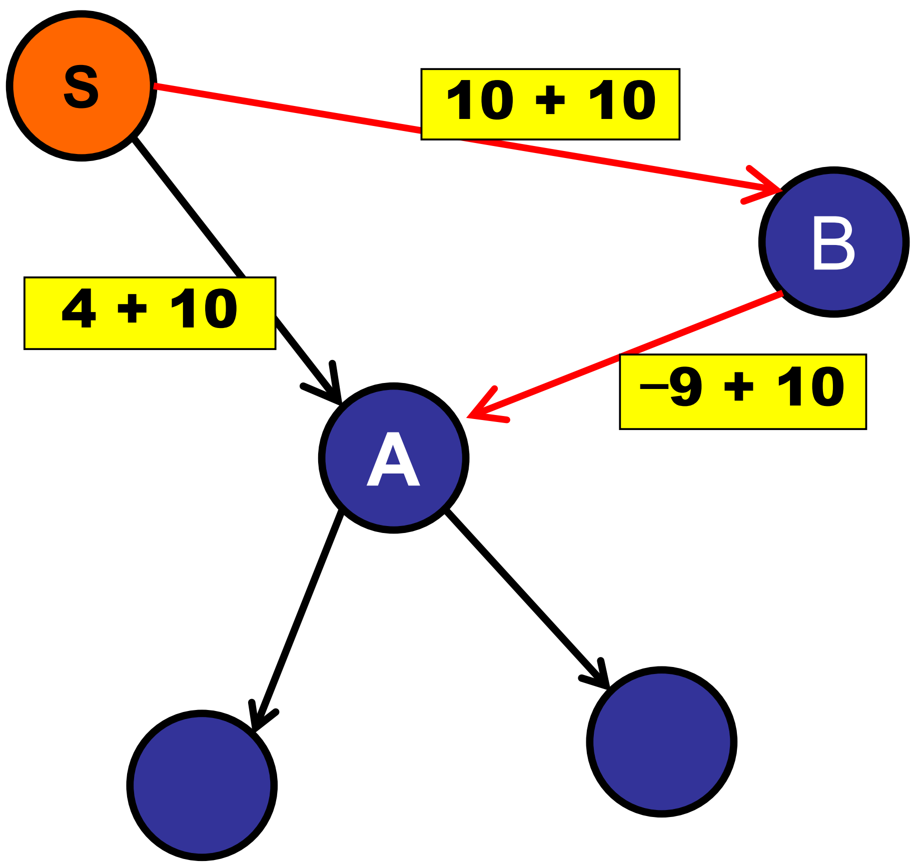



Path S-B-A: 1

Path S-A: 4

# Dijkstra's Algorithm

Can we reweight?




Path S-B-A: 21

Path S-A: 14

# Dijkstra Summary

Basic idea:

- Maintain distance estimates.
- Repeat:
  - Find unfinished vertex with smallest estimate.
  - Relax all outgoing edges.
  - Mark vertex finished.

- O(E log V) time (with AVL tree Priority Queue).

- No negative weight edges!

# Dijkstra Comparison

Same algorithm:

- Maintain a set of explored vertices.
- Add vertices to the explored set by following edges that go from a vertex in the explored set to a vertex outside the explored set.

  - BFS: Take edge from vertex that was discovered **least** recently.
  - DFS: Take edge from vertex that was discovered **most** recently.
  - Dijkstra's: Take edge from vertex that is **closest** to source.

# Dijkstra Comparison

Same algorithm:

- Maintain a set of explored vertices.
- Add vertices to the explored set by following edges that go from a vertex in the explored set to a vertex outside the explored set.

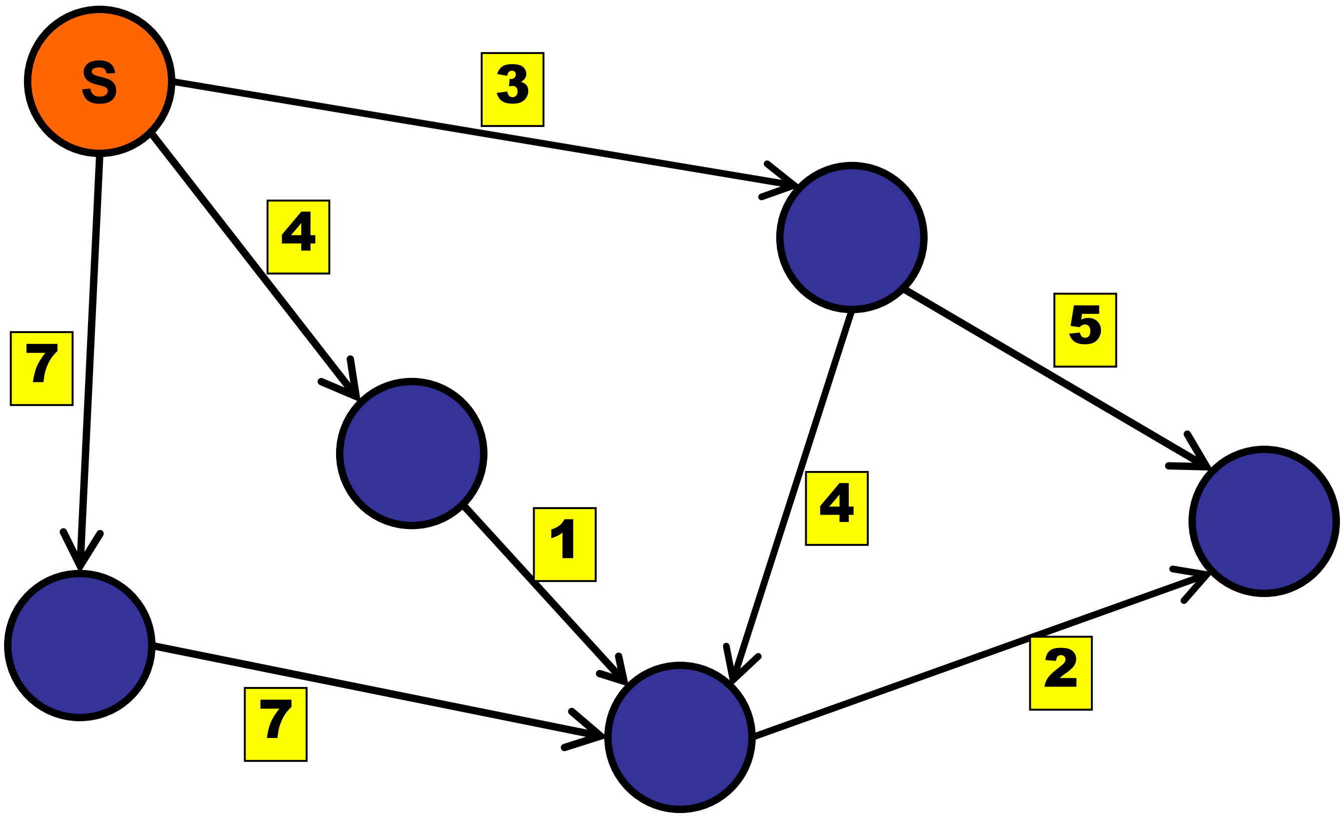
  – BFS: Use queue.

  – DFS: Use stack.

  – Dijkstra's: Use priority queue.

# Longest Paths

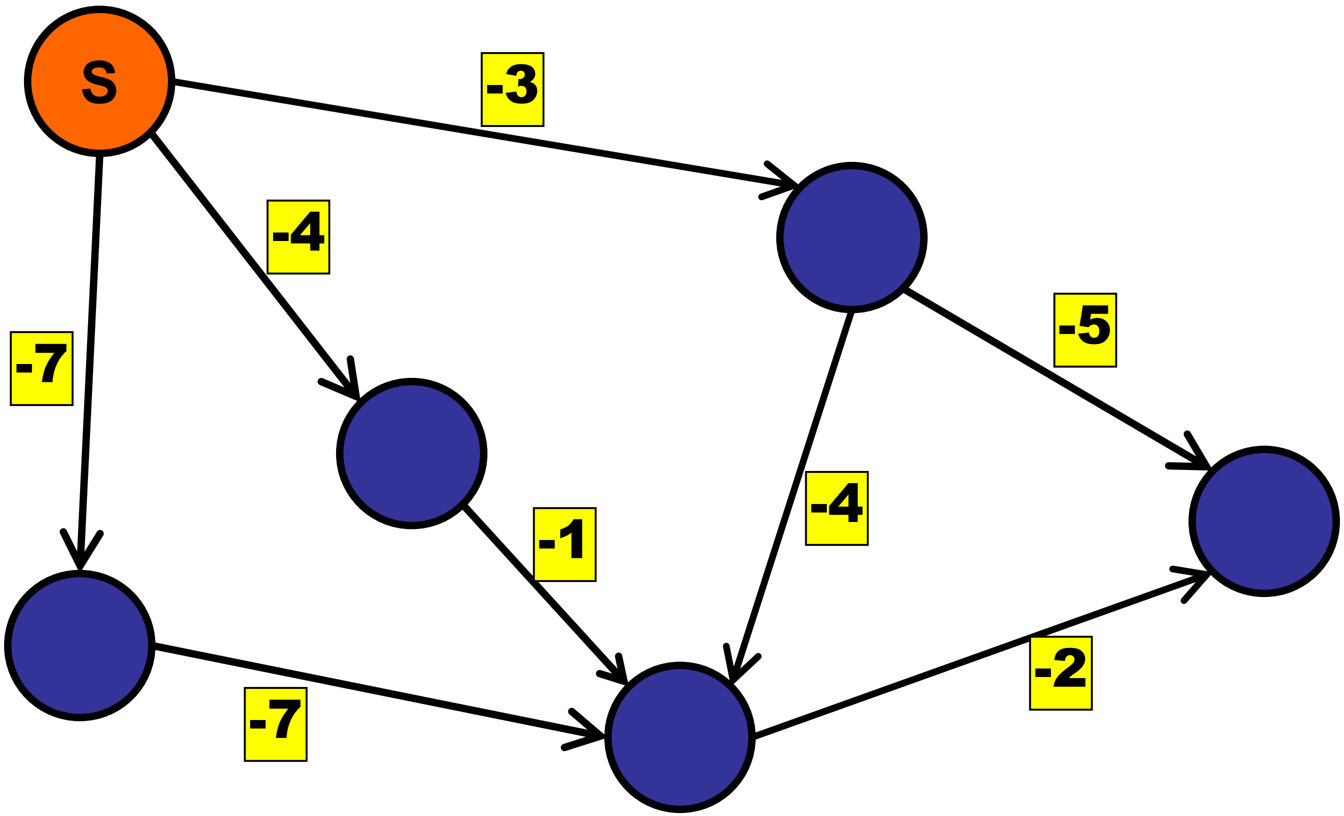
Any ideas?



# Longest Paths

Negate the edges?

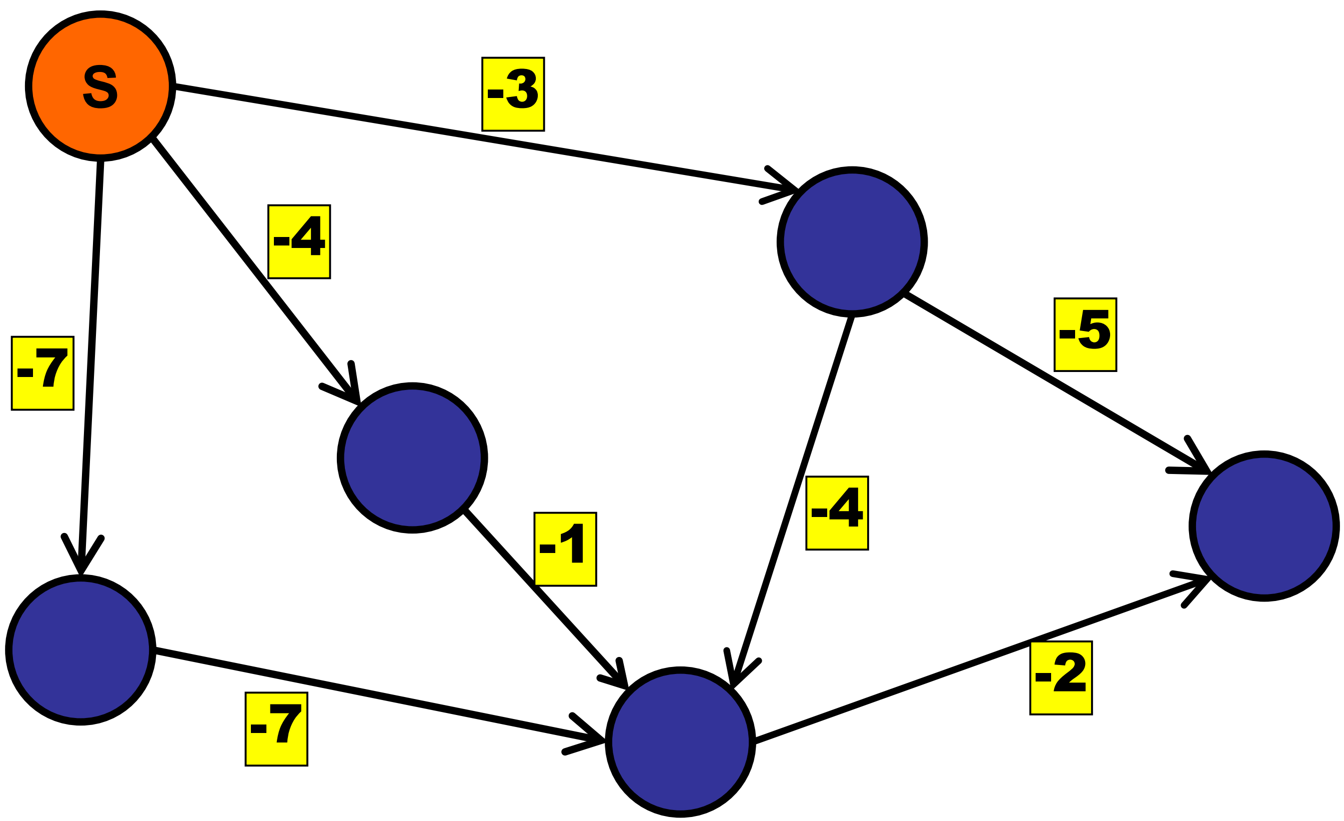
Only if your graph does not have a cycle!



# Longest Paths

Acyclic Graph:

shortest path in negated=longest path in regular



# Longest Path

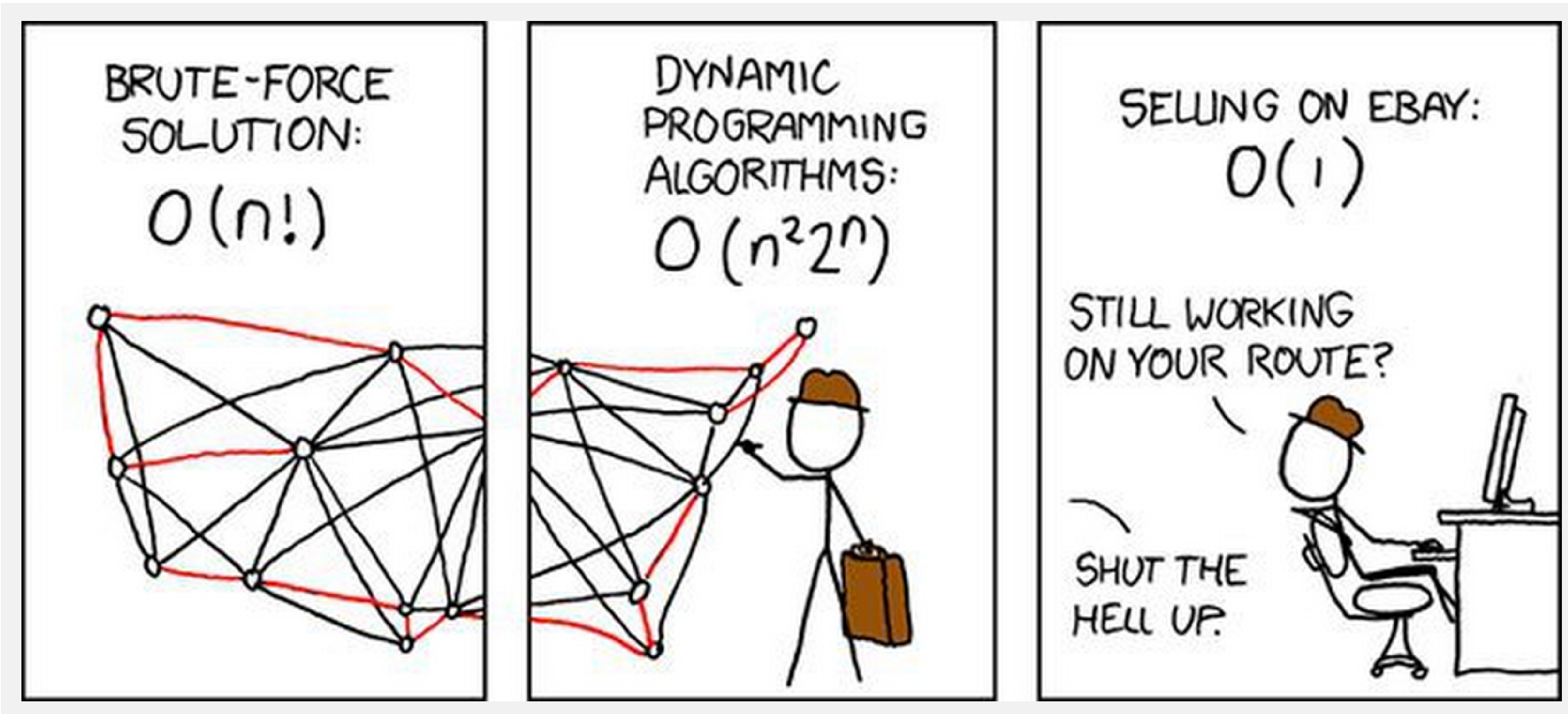
Directed Acyclic Graph:

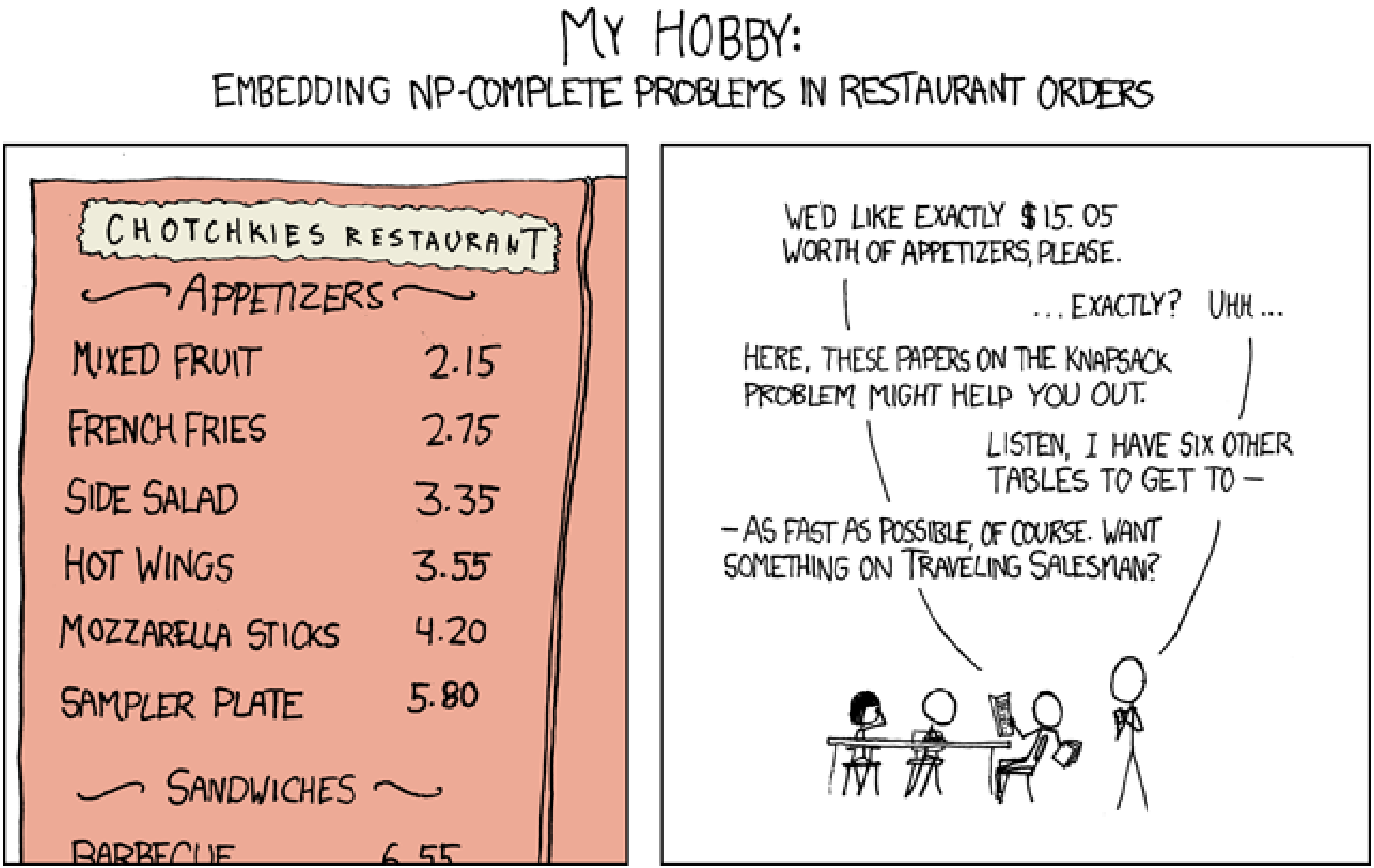
– Solvable efficiently using topological sort

General (cyclic) Graphs:

– NP-Hard
– Reduction from Hamiltonian Path:
  - If you could find the longest simple path, then you could decide if there is a path that visits every vertex.
  - Any polynomial time algorithm for longest path thus implies a polynomial time algorithm for HAMPATH.

# Also called the Travelling Salemans Problem




MY HOBBY:
EMBEDDING NP-COMPLETE PROBLEMS IN RESTAURANT ORDERS
CHOTCHKIES RESTAURANT
APPETIZERS
MIXED FRUIT 2.15
FRENCH FRIES 2.75
SIDE SALAD 3.35
HOT WINGS 3.55
MOZZARELLA STICKS 4.20
SAMPLER PLATE 5.80
SANDWICHES
WE'D LIKE EXACTLY $15.05 WORTH OF APPETIZERS, PLEASE.
...EXACTLY? UHH...
HERE, THESE PAPERS ON THE KNAPSACK PROBLEM MIGHT HELP YOU OUT.
LISTEN, I HAVE SIX OTHER TABLES TO GET TO —
— AS FAST AS POSSIBLE, OF COURSE. WANT SOMETHING ON TRAVELING SALESMAN?