

# MTH310 Calculus & Computational Methods II

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# 1 Integration Practice

## 1.1 FToC I

Recall that we can solve a definite integral using the following definition:

$$\int_a^b f(x)dx = F(b) - F(a)$$

## 1.2 Change of Variables / U-Substitution

Suppose we need take the antiderivative of  $\int 2x \cos(x^2)dx$ , let us suppose that  $g(x) = x^2$ , then we know  $g'(x) = 2x$ . We also know that  $\int \cos(x)dx = \sin(x) + C$ . If we combine these, we can derive the answer as:

$$\sin(x^2) + C = \int 2x \cos(x^2)dx$$

**Theorem 1.1.** Let us take  $u = g(x) \rightarrow \frac{du}{dx} = g'(x)$  and  $du = g'(x)dx$ . We can then derive the following:

$$\begin{aligned} f(g(x)) &= \int (f \cdot g)'x \\ &= \int f'(g(x))g'(x)dx \\ &= \int f'(u)du \end{aligned} \tag{1}$$

### 1.2.1 Examples:

Consider the following substitution  $u = 3x \rightarrow du = 3x \rightarrow \frac{1}{3}du = dx$ , we can then solve:

$$\begin{aligned} \int \cos(3x)dx &= \frac{1}{3} \int \cos(u)du \\ &= \frac{1}{3} \sin(u) + C \\ &= \frac{1}{3} \sin(3x) + C \end{aligned} \tag{2}$$

Consider the following substitution  $u = 2x^2 + 1 \rightarrow du = 4x dx$ , we can then solve:

$$\begin{aligned} \int \frac{x}{2x^2 + 1} dx &= \frac{1}{4} \int \frac{du}{u} \\ &= \frac{1}{4} \ln(u) + C \\ &= \frac{1}{4} \ln(2x^2 + 1) + C \end{aligned} \tag{3}$$

Consider the following substitution  $u = 1 + x \rightarrow du = dx \rightarrow u - 1 = x$

$$\begin{aligned} \int x\sqrt{1+x} dx &= \int (u-1)u^{\frac{1}{2}} \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C \end{aligned} \tag{4}$$

### 1.3 Area between two curves

#### Recall:

Suppose  $f(x) \geq 0$  is the area beneath the curve  $0 \leq y \leq f(x)$ , where  $a \leq x \leq b$ , then:

$$\int_a^b f(x)dx$$

is the area of the curve between  $a, b$ .

**Theorem 1.2.** Given functions  $f(x), g(x)$  the area between the curves from  $(a, b)$  can be represented as:

$$\int_a^b (f(x) - g(x))dx$$

#### 1.3.1 Examples:

Given two functions  $f(x) = 3x^2 + 12$  and  $g(x) = 4x + 4$ , find the area between the curves from  $(-3, 3)$ .

$$\begin{aligned} \int_{-3}^3 (f(x) - g(x))dx &= \int_{-3}^3 (3x^2 + 12) - (4x + 4)dx \\ &= \int_{-3}^3 3x^2 - 4x + 8dx \\ &= x^3 - 2x^2 + 8x \Big|_{-3}^3 \\ &\vdots \\ &= 102 \end{aligned} \tag{5}$$

Given two functions  $(x^2 + 2)$  and  $(2x + 5)$ , find the **enclosed** area between these two curves.

Find  $a, b$ , where the lines intersect  $\rightarrow 2x + 5 = x^2 + 2 \dots (-1 \leq x \leq 3)$  then solve the integral.

$$\begin{aligned} \int_{-1}^3 (f(x) - g(x))dx &= \int_{-1}^3 (x^2 + 2) - (2x + 5) \\ &\vdots \\ &= \frac{32}{3} \end{aligned} \tag{6}$$

Given two functions  $\sin(x)$  and  $\cos(x)$  find the area between these two curves given that  $0 \leq x \leq \frac{\pi}{2}$ .

Find the intersection between the functions on the range given, then build the integral. This is easily done by observing the functions **geometrically**.

$$\begin{aligned} \int_0^{\frac{\pi}{4}} (f(x) - g(x))dx &= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x)dx \\ &\vdots \\ &= 2(\sqrt{2} - 1) \end{aligned} \tag{7}$$

## 2 Integration in Theory/Application

### 2.1 Volume of Revolution

Visualize a cylindrical shape from  $(a, b)$  built by rotating some curve defined by  $f(x)$  about some  $x$ -axis

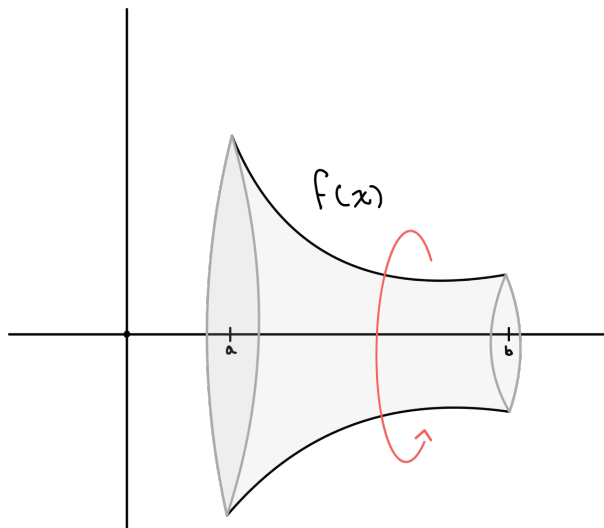


Figure 1: Cylinder built by rotating  $f(x)$  about the  $x$ -axis

We can define the volume of this cylinder as the cross section  $\times$  thickness, similar to calculating the integral in a 2-dimensional plane. The integral for the volume of this cylinder is then written as:

$$V = \pi r^2 h$$

$$V = \int_a^b \pi f(x)^2 dx$$

Assuming another function  $g(x)$  also rotated about the same axis, creating a hollow inside of the cylinder, we would then need to subtract the hollow section from the full volume:

$$V = \pi (r_{outer} - r_{inner})^2 h$$

$$V = \int_a^b \pi (f(x) - g(x))^2 dx$$

### 2.2 The Average Value of a Function

Recall that the average value of a set of numbers  $(x_1, x_2, \dots, x_n)$  can be written as:

$$Average = \frac{x_1 + x_2 + \dots + x_n}{n}$$

We can then extrapolate to a more generalized formula:

$$Average = \frac{1}{n} \sum_{i=1}^n (x_i)$$

Rewrite in terms of arbitrary even sections between  $(a, b)$  on some curve  $f(x)$ , rather than some set of numbers:

$$Average = \frac{1}{b-a} \sum_{x=a}^b f(x) \quad (\text{This can be rewritten as an integral from } (a, b))$$

$$= \frac{1}{b-a} \int_a^b f(x) dx, \quad a \leq x \leq b \quad (\text{Definition of Average Value of } f(x))$$

## 3 Integration Techniques

Techniques of integration can be thought of as methods for evaluating integrals in situations where the process or methodology required are not obvious, many integrals can only be solved by specific techniques and these techniques can generally be categorized into two types:

- Universal Techniques
- Particular Techniques

### 3.1 Universal Techniques

#### 3.1.1 Integration by Parts

Recall that  $(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$  (*Product Rule*), if we then integrate both sides of this equation:

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx$$

We can then rearrange this equation into the following:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Then we can generalize into:

$$\int u dv = uv - \int v du$$

This is known as integration by parts, the idea is that you rewrite the integrand in terms of  $u$   $dv$ , generally you will want to follow these guidelines for choosing these values:

1. Choose  $u$  to be such that  $du$  is a **simpler** value than  $u$ .
2. Choose  $dv$  such that you could easily integrate and result in  $v$ .

You will know that a integral likely requires integration by parts if the integrand is obviously a product of two functions, take the following example:

Evaluate.  $\int x \cos x dx =$

$$\begin{aligned} \int u dv &= uv - \int v du & u &= x, & dv &= \cos x dx \\ \int x \cos x dx &= x \sin x - \int \sin x dx & du &= dx, & v &= \int \cos x dx = \sin x \\ &= x \sin x + \cos x + C \end{aligned}$$

### 3.2 Particular Techniques

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