MTH310 Calculus & Computational Methods II

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Content by Week

1 Integration Practice

1.1 FToC I

Recall that we can solve a definite integral using the following definition:

$$\int_a^b f(x)dx = F(b) - F(a)$$

1.2 Change of Variables / U-Substitution

Suppose we need take the antiderivative of $\int 2x\cos(x^2)dx$, let us suppose that $g(x) = x^2$, then we know $g'(x) = 2x$. We also know that $\int \cos(x)dx = \sin(x) + C$. If we combine these, we can derive the answer as:

$$\sin(x^2) + C = \int 2x\cos(x^2)dx$$

Theorem 1.1. *Let us take $u = g'(x) \to \frac{du}{dx} = g'(x)$ and $du = g'(x)dx$. We can then derive the following:*

$$\begin{aligned} f(g(x)) &= \int (f \cdot g)'x \\ &= \int f'(g(x))g'(x)dx \\ &= \int f'(u)du \end{aligned} \tag{1}$$

1.2.1 Examples:

Consider the following substitution $u = 3x \to du = 3x \to \frac{1}{3}du = dx$, we can then solve:

$$\begin{aligned} \int \cos(3x)dx &= \frac{1}{3}\int \cos(u)du \\ &= \frac{1}{3}\sin(u) + C \\ &= \frac{1}{3}\sin(3x) + C \end{aligned} \tag{2}$$

Consider the following substitution $u = 2x^2 + 1 \to du = 4x\ dx$, we can then solve:

$$\begin{aligned} \int \frac{x}{2x^2+1}dx &= \frac{1}{4}\int \frac{du}{u} \\ &= \frac{1}{4}ln(u) + C \\ &= \frac{1}{4}ln(2x^2+1) + C \end{aligned} \tag{3}$$

Consider the following substitution $u = 1 + x \to du = dx \to u - 1 = x$

$$\begin{aligned} \int x\sqrt{1+x}\ dx &= \int (u-1)u^{\frac{1}{2}} \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C \end{aligned} \tag{4}$$

1.3 Area between two curves

Recall:

Suppose $f(x) \geq 0$ is the area beneath the curve $0 \leq y \leq f(x)$, where $a \leq x \leq b$, then:

$$\int_a^b f(x)dx$$

is the area of the curve between a, b.

Theorem 1.2. *Given functions $f(x), g(x)$ the area between the curves from (a, b) can be represented as:*

$$\int_a^b (f(x) - g(x))dx$$

1.3.1 Examples:

Given two functions $f(x) = 3x^2 + 12$ and $g(x) = 4x + 4$, find the area between the curves from $(-3, 3)$.

$$\begin{aligned}
\int_{-3}^{3} (f(x) - g(x))dx &= \int_{-3}^{3} (3x^2 + 12) - (4x + 4)dx \\
&= \int_{-3}^{3} 3x^2 - 4x + 8dx \\
&= x^3 - 2x^2 + 8x\|_{-3}^{3} \\
&\vdots \\
&= 102
\end{aligned} \tag{5}$$

Given two functions $(x^2 + 2)$ and $(2x + 5)$, find the **enclosed** area between these two curves.

Find a, b, where the lines intesect $\rightarrow 2x + 5 = x^2 + 2 \ldots (-1 \leq x \leq 3)$ then solve the integral.

$$\begin{aligned}
\int_{-1}^{3} (f(x) - g(x))dx &= \int_{-1}^{3} (x^2 + 2) - (2x + 5) \\
&\vdots \\
&= \frac{32}{3}
\end{aligned} \tag{6}$$

Given two functions $\sin(x)$ and $\cos(x)$ find the area between these two curves given that $0 \leq x \leq \frac{\pi}{2}$.

Find the intersection between the functions on the range given, then build the integral. This is easily done by observing the functions ***geometrically.***

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} (f(x) - g(x))dx &= \int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x) - \cos(x)dx \\
&\vdots \\
&= 2(\sqrt{2} - 1)
\end{aligned} \tag{7}$$

2 Integration in Theory/Application

2.1 Volume of Revolution

Visualize a cyclindrical shape from (a, b) built by rotating some curve defined by $f(x)$ about some x-axis

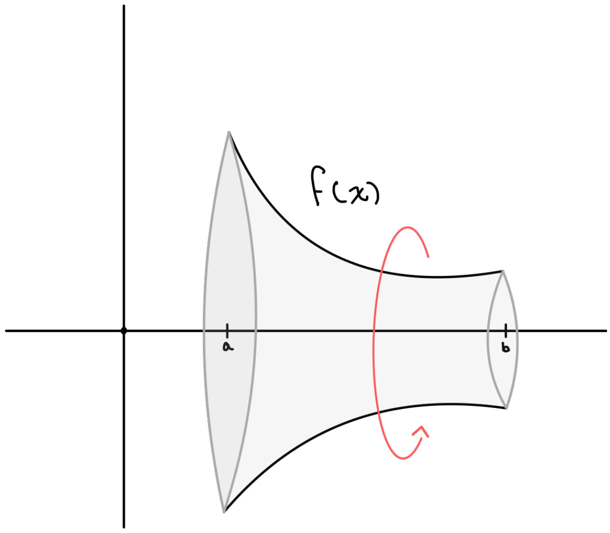

Figure 1: *Cylinder built by rotating $f(x)$ about the x-axis*

We can define the volume of this cylinder as the cross section × thickness, similar to calculating the integral in a 2-dimensional plane. The integral for the volume of this cyclinder is then written as:

$$V = \pi r^2 h$$

$$V = \int_a^b \pi f(x)^2 dx$$

Assuming another function $g(x)$ also rotated about the same axis, creating a hollow inside of the cylinder, we would then need to subtract the hollow section from the full volume:

$$V = \pi(\underbrace{r_o}_{outer} - \underbrace{r_i}_{inner})^2 h$$

$$V = \int_a^b \pi(\underbrace{f(x)}_{outer} - \underbrace{g(x)}_{inner})^2 dx$$

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