MTH210 — Discrete Mathematics II

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Content by Week

1 Sequences and Series

A **sequence** is an ordered set of numbers.

$2, 4, 6, 8, 10\dots$ is an example of a sequence of positive numbers.

$a_1, a_2, a_3, \dots, a_n$ denotes an infinite sequence.

- A sequence is defined **analytically** if each term a_i is defined by some function $f(i) = a_i$
- A sequence is defined **recursively** if the first k terms are given **explicitly** and the rest are given through a recursive function $a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-k})$ for $n > k$.
- Even if two sequences are equal for small indexes, does not indicate that they don't diverge at some further point.

A **series** is the sum of all the terms in a sequence.

If m and n are integers and $m \leq n$, the symbol $\sum\limits_{k=m}^{n} a_k$ is the summation from k, defined as:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

- We call k the **index** of the summation.
- m is the **lower limit** of the summation.
- n is the **upper limit** of the summation.

1.1 Sums and Products

If m and n are integers and $m \leq n$, the symbol $\prod\limits_{k=m}^{n} a_k$ is read as product from k equals m to n of a sub k, it can be written as:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \times \cdots \times a_n$$

Theorem 1.1. *The following properties hold for any integer $n \geq m$, given $a_m, \dots$ and $b_m, \dots$ sequences of real numbers.*

- $\sum\limits_{k=m}^{n} a_k + \sum\limits_{k=m}^{n} b_k = \sum\limits_{k=m}^{n} (a_k + b_k)$
- $c \cdot \sum\limits_{k=m}^{n} a_k = \sum\limits_{k=m}^{n} (c \cdot a_k)$, *given some constant c*
- $(\prod\limits_{k=m}^{n} a_k) \cdot (\prod\limits_{k=m}^{n} b_k) = \prod\limits_{k=m}^{n} (a_k \cdot b_k)$

Theorem 1.2. *The binomial theorem, also called n choose r is computed by using the following formula for $0 \leq r \leq n$:*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

2 Mathmatical Induction

Mathematical Induction is a two-step process, Take a statement in the form "For every integer $n \geq a$, a property P(n) holds true". We then apply the following:

1. The first step is called the **Basis Step**, this is where you show that the condition P of your starting point a, is true $\rightarrow$ Show that $P(a)$ is true.
2. The second step is called the **Inductive Step**, you show that for every integer $k \geq a$, if $P(k)$ is true, then $P(k+1)$ must also be true.
   (a) To perform this step, we must suppose that $P(k)$ holds, where $k \geq a$ (**Inductive Hypothesis**), therefore $P(k+1)$ must be true.

To write an inductive proof formally, we must clearly state all steps and assumptions, take the following example:

Prove. For every integer $n \geq 1, 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ by **Induction**:

Proof. Basis Step (*Base Case*):

$$\text{Let } n = 1, P(1) \text{ holds because } \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$$

Inductive Step:

Let $n = k$ with $k \geq 1$. Suppose that $P(k)$ is true (*inductive hypothesis*). Thus we can show:

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2+k}{2} + \frac{2k+2}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Thus $1+2+\cdots+(k+1) = \dfrac{(k+1)((k+1)+1)}{2}$ (notice how this is in the form $1+2+\cdots+n = \dfrac{n(n+1)}{2}$). Therefore we can conclude that $P(k+1)$ is true. $\square$

2.1 Recursion

Given a recursively defined sequence a_i, find an analytical formula using the following steps:

1. Find an explicit formula for a_i by making educated guesses.
2. Prove that the formula holds through induction.

Take the following example:

Prove. Given $b_0 = 1, b_n = \dfrac{b_n - 1}{1 - b_n - 1}$ for $n > 1$

Proof. Make an assumption for the formula of b_n, take the following guess:

$$b_n = \frac{1}{n+1} \text{ for } n \geq 0 \ (\textit{Conjecture})$$

Basis Case:

$$\text{Let } n = 0. P(0) \text{ is known to be true as } b_0 = 1 = \frac{1}{0+1}$$

Inductive Step:

Let $n = k$ with $k \geq 0$. Assume that $P(k)$ is true (*inductive hypothesis*). Thus we can show:

$$\begin{aligned} b_{k+1} &= \frac{b_k}{1+b_k} \\ &= \frac{\frac{1}{k+1}}{1+\frac{1}{k+1}} \\ &= \frac{\frac{1}{k+1}}{\frac{k+2}{k+1}} \\ &= \frac{1}{k+2} \\ &= \frac{1}{(k+1)+1} \end{aligned}$$

Hence $P(k+1)$ must be true. □

2.2 Strong Induction

Strong induction follows the same steps as normal/weak induction, with the difference that the basis step may contain proofs for several values and $P(n)$ is assumed not just for a single n but for all values n through k, only then is the truth of $P(k+1)$ proved. The steps for strong induction are as follows:

Let $P(n)$ be a property that is defined for integers n, let a, b be fixed integers such that $a \leq b$. Suppose the following statements:

1. $P(a), P(a+1), \ldots, P(b)$ are all true. (**Basis Step**)
2. For every integer $k \geq b$, if $P(i)$ is true for each integer i from a through k, then $P(k+1)$ is true. (**Inductive Step**)

then the statement "For every integer $n \geq a, P(n)$" is true. The supposition that $P(i)$ is true is the inductive hypothesis in this case.

Prove. Given a sequence a_i,

$$a_0 = 12, a_1 = 29 \quad a_n = 5a_{n-1} - 6a_{n-2}$$

Show that for all $n \geq 0$,

$$a_n = 5 \cdot 3^n + 7 \cdot 2^n$$

Proof. Proceed by induction on n.

Basis Step: We take that $P(0)$ and $P(1)$ are true.

Inductive Step: Let $n = k+1$. Assume $P(i)$ is true for $k \geq i \geq 0$ (*inductive Hypothesis*). Thus:

$$\begin{aligned} a_k &= 5 \cdot 3^k + 7 \cdot 2^k \\ a_{k-1} &= 5 \cdot 3^{k-1} + 7 \cdot 2^{k-1} \\ a_{k+1} &= 5(5 \cdot 3^k + 7 \cdot 2^k) - 6(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) \\ &= 25 \cdot 3^k + 35 \cdot 2^k - 30 \cdot 3^{k-1} - 42 \cdot 2^{k-1} \\ &= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3^k - 21 \cdot 2^k \\ &= 15 \cdot 3^k + 14 \cdot 2^k \\ &= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1} \end{aligned}$$

Hence $P(k+1)$ is true. □

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