MTH108 — Linear Algebra

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Content by Week

1 Euclidean Spaces

A Euclidean Space is a mathematical space in which points and lines can be represented by a set of coordinates in the respective dimension of the space, and every point can be represented in a defined set. For example:

$$\mathbb{R}^3 = (x, y, z); x, y, z \in \mathbb{R}$$

is three-dimensional space represented using coordinates in terms of (x, y, z), where $x, y, z \in \mathbb{R}$.

Theorem 1.1. *Given two vectors $\vec{a}, \vec{b}$ and some constant k, $\vec{a}$ and $\vec{b}$ are called* ***parallel*** *if:*

$$\vec{a} = k\vec{b} \Leftrightarrow \vec{a}//\vec{b}$$

1.1 Products of Vectors with Constants

Theorem 1.2. *Given a constant k in $\mathbb{R}$ and some vector $\vec{a}$ in $\mathbb{R}^2$, the product of $k\vec{a}$ is:*

$$k\vec{a} = k(x_1, y_1) = (k \cdot x_1, k \cdot y_1), x, y \in \mathbb{R}$$

To represent a vector in Linear Algebra, we can use the following notation (*using the previously mentioned vector $\vec{a}$ as an example*):

$$\vec{a} = (x_1, y_1) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

1.1.1 Examples:

Take $\vec{a} = \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right)$ and $\vec{b} = \left(\begin{smallmatrix} 4 \\ -1 \end{smallmatrix} \right)$, compute $-2\vec{a} + 3\vec{b}$:

$$\begin{aligned} -2\vec{a} + 3\vec{b} &= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 12 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ -1 \end{pmatrix} \end{aligned} \tag{1}$$

1.2 Products of two Vectors

Theorem 1.3. *Given two vectors in $\mathbb{R}^n, \vec{a} = \left(\begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix} \right), \vec{b} = \left(\begin{smallmatrix} y_1 \\ \vdots \\ y_n \end{smallmatrix} \right)$, the product of $\vec{a} \cdot \vec{b}$ is:*

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n \end{aligned}$$

This is known as the ***Dot Product****.*

1.2.1 Examples:

Take $\vec{A} = \left(\begin{smallmatrix} 1 \\ -1 \\ 2 \\ 3 \end{smallmatrix} \right)$ and $\vec{b} = \left(\begin{smallmatrix} 2 \\ 1 \\ -1 \\ 1 \end{smallmatrix} \right)$, Find the dot product of $\vec{a} \cdot \vec{b}$:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (2) + (-1) + (-2) + (3) \\ &= 2 \end{aligned} \tag{2}$$

1.3 Other Properties of Vector Products

Given some $\vec{a}, \vec{b}$ in $\mathbb{R}^n$ and some constant k, the following properties apply:

- $(\vec{a}+\vec{b})(\vec{a}+\vec{b}) = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b}+\vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$
- $\vec{a}(k\vec{b}) = k\vec{a}\vec{b}$
- $\vec{a} \cdot \vec{a} = \vec{a}^2 = x_1^2 + x_2^2 + \cdots + x_n^2$

The midpoint $C(z_1, z_2, \ldots, z_n)$ of a line from $A(x_1, x_2, \ldots, x_n)$ to $B(y_1, y_2, \ldots, y_n)$ is calculated using the following forumla:

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = (1-t)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + t\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{for t such that } 0 \leq t \leq 1$$

We can simplify this to calculate every z_i:

$$z_i = \frac{x_i + y_i}{2}$$

1.3.1 Examples:

Given $A(1,-2)$ and $B(-3,4)$, find the midpoint $C(x,y)$:

$$\begin{aligned} x &= \frac{1+(-3)}{2} \\ &= -1 \\ y &= \frac{(-2)+4}{2} \\ &= 1 \end{aligned} \tag{3}$$

Therefore $C = (-1, 1)$.

1.4 Norm and Angle

The magnitude of a vector in $\mathbb{R}^n$ is called the **Norm**, it is notated and defined as:

$$||\vec{a}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad (\vec{a} \text{ is some vector in } \mathbb{R}^n)$$

We can use this definition to demonstrate some inequalities and properties (*Given some* $\vec{a}, \vec{b} \in \mathbb{R}^n$ *and some constant* k):

- $||\vec{a}+\vec{b}|| \leq ||\vec{a}|| + ||\vec{b}||$ *(Triangle Inequality)*
- $||\vec{a}+\vec{b}|| \geq ||\vec{a}|| - ||\vec{b}||$
- $||k\vec{a}|| = |k| \cdot ||\vec{a}||$
- $||\frac{\vec{a}}{||\vec{a}||}|| = 1$
- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ *(Orthagonal)*
- $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}$

1.5 Determinants

The determinant of a matrix is a number that can be calculated using the following formula (*in* $\mathbb{R}^2$):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Given $\vec{a}, \vec{b} \in \mathbb{R}^n$, the **Gram Determinant** of $\vec{a}\vec{b}$ is defined as:

$$G(\vec{a}, \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix} = ||\vec{a}||^2 \ ||\vec{b}||^2 - (\vec{a}\vec{b})^2$$

Given some vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, the Gram Determinant can be calculated using the following formula:

$$G(\vec{a}, \vec{b}) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}^2 + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}^2$$

Lemma 1.3.1. *The* ***Cauchy Inequality*** *states:*

$$G(\vec{a}, \vec{b}) \geq 0 \Rightarrow |\vec{a} \cdot \vec{b}| \leq ||\vec{a}|| \cdot ||\vec{b}||$$

2 Projections, Linear Combinations and Span

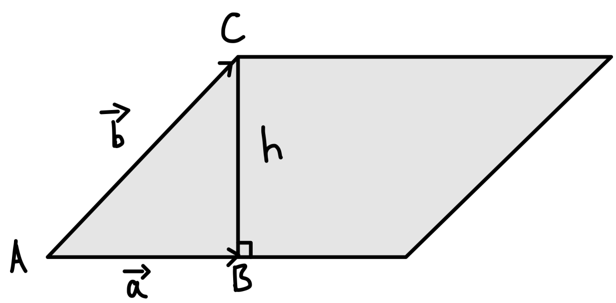
For some $\vec{a}, \vec{b} \in \mathbb{R}^n$, a projection of $\vec{a}$ on $\vec{b}$ is defined as:

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} \cdot \vec{a}$$

The norm of a projection is defined as:

$$||proj_{\vec{a}}\vec{b}|| = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} \cdot ||\vec{a}||$$

2.1 Projection and Area of a parallelogram



Given the above figure, we can determine formulas for calculating properties of the parallelogram:

1. $h = ||\vec{BC}|| = ||\vec{b} - proj_{\vec{a}}\vec{b}||$
2. $Area = ||\vec{a}|| \cdot ||\vec{b} - proj_{\vec{a}}\vec{b}||$
3. $\sin(\theta) = \dfrac{||\vec{b} - proj_{\vec{a}}\vec{b}||}{||\vec{b}||}$
4. $\cos(\theta) = \dfrac{||proj_{\vec{a}}\vec{b}||}{||\vec{b}||} = \dfrac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}$

The generalized formula for calculating the area of some $\vec{a}, \vec{b} \in \mathbb{R}^n$ is as follows:

$$A(\vec{a}, \vec{b}) = \sqrt{G(\vec{a}, \vec{b})}$$

2.2 Linear Combinations and Span

Take some vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, the linear combination$(A)$ of these vectors is simply:

$$m\vec{a} + n\vec{b} = A \qquad n, m \in \mathbb{R}$$

Span can be thought of as the set of values that you can reach by changing the constant in a linear combination of vectors, taking $\vec{a} = (1,2)$ and $\vec{b} = (2,3)$ the span would then be:

$$span(a, b) = \mathbb{R}^2$$

This is because you could theoretically reach any point in $\mathbb{R}^2$ by changing values of $m, n \in \mathbb{R}$, the exception to this is when both vectors are zero-vectors or when both vectors "align", in that case the set of values possible would only be the values extending the single line *(e.g* $\vec{a} = (1,0), \vec{b} = (-1,0)$*)* .

3 Matrices

A **matrix** A, of $n \times m$ is defined as a rectangular array of mn numbers arranged in m, n rows and columns:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}$$

If we take only either columns or rows we can define:

$$a_1 = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \end{pmatrix} \text{ This is called a } \textbf{Row Matrix}$$

$$a_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \cdots \\ a_{m,1} \end{pmatrix} \text{ This is called a } \textbf{Column Matrix}$$

3.1 Properties of Matrices

The **size** of a matrix is simply its number of rows $\times$ its number of columns:

$$size \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 2 \times 3 = 6$$

The **transpose** of a matrix is defined as the matrix itself but with the rows and columns swapped, it is denoted with ^{T}, take the previously defined matrix A, the transpose of $A = A^T$, would be denoted as:

$$A^T = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & \cdots & a_{m,2} \\ a_{1,3} & a_{2,3} & a_{3,3} & \cdots & a_{m,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & a_{3,n} & \cdots & a_{m,n} \end{pmatrix}$$

Two matrices A, B are said to be **equal**, if their size are the same and the respective entries are equal:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = B$$

Matrix **addition, subtraction and scalar multiplication** can all be achieved in the same method as their vector counterparts.

3.2 Product of a Vector and Matrix

The product of matrix A with a row/column vector X is a vector where the values are each entry in the row/column of A multiplied by the nth element in X for that row/column, then summed. For example, take $X = (x_1, x_2, x_3, \dots, x_n)$

$$AX = \begin{pmatrix} a_{1,1}x_1 & + & a_{1,2}x_1 & + & a_{1,3}x_1 & + & \dots & + & a_{1,n}x_1 \\ a_{2,1}x_2 & + & a_{2,2}x_2 & + & a_{2,3}x_2 & + & \dots & + & a_{1,n}x_2 \\ a_{3,1}x_3 & + & a_{3,2}x_3 & + & a_{3,3}x_3 & + & \dots & + & a_{1,n}x_3 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m,1}x_n & + & a_{m,2}x_n & + & a_{m,3}x_n & + & \dots & + & a_{m,n}x_n \end{pmatrix}$$

If $X = \left(\begin{smallmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{smallmatrix}\right)$ instead, then AX would be:

$$AX = \begin{pmatrix} a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,3}x_3 & + & \dots & + & a_{1,n}x_n \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & a_{2,3}x_3 & + & \dots & + & a_{1,n}x_n \\ a_{3,1}x_1 & + & a_{3,2}x_2 & + & a_{3,3}x_3 & + & \dots & + & a_{1,n}x_n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & a_{m,3}x_3 & + & \dots & + & a_{m,n}x_n \end{pmatrix}$$

Note that a matrix and vector can only be multiplied if the number of columns of the matrix is equal to the number of elements in the vector. This can also be called the **image** of X under A.

3.3 Product of two Matrices

Given matrcies A, B, their product AB, can be calculated by taking the **dot product** between the first row of A with the first column of B, then the first row of A with the second column of B, until the nth column of B, then you repeat the process until you reach the nth row of A. Take the following example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$\begin{aligned} AB =& ((1,2)\cdot(5,7) + (1,2)\cdot(6,8)) \\ & ((3,4)\cdot(5,7) + (3,4)\cdot(6,8)) \\ =& \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \end{aligned}$$

Two matrices A, B with sizes $size(A) = n \times m$ and $size(B) = i \times j$, with $n, m, i, j \in \mathbb{R}$, AB can only be evaluated if $m = i$. The size of the resulting matrix AB will be $n \times j$. Keep in mind that commutativity does apply in regard to matrix multiplication, hence $AB \neq BA$ may be true. Also keep in mind the following formula:

$$(AB)^T = B^T A^T$$

3.4 Square Matrices

A square matrix is a matrix of $n \times n$ dimensions, for example:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{pmatrix}$$

The **trace** of a square matrix is defined as the sum of all the diagonal elements:

$$tr(A) = a_{1,1} + a_{2,2} + \dots + an, n$$

If all the elemnts above/below the diagonal line in a matrix is 0, then the matrix is called a **lower/upper** triangle matrix respectively. If all values on either side of the diagonal are 0, then it is called a **upper and lower** triangle matrix.

Let A be a $n \times n$ matrix, the following properties then apply:

- If $A^T = A$ then A is called **symmetric**.
- $A^n = \Pi_0^n A$
- $A^0 = I$ where I is an identity matrix with the same dimensions as A.

We define a diagonal matrix in the form $diag(a_{1,1}, a_{2,2}, \ldots, a_{n,n})$ as follows:

$$A = \begin{pmatrix} a_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} & 0 & \dots & 0 \\ 0 & 0 & a_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{pmatrix}$$

Given some identity matrix I with the same dimensions as the previous diagonal matrix, then the following property holds:

$$IA = AI$$

For a diagonal matrix, the following property also applies for some $k \in \mathbb{R}$:

$$A = \begin{pmatrix} a_{1,1} & 0 & 0 & \dots & 0 \\ 0 & a_{2,2} & 0 & \dots & 0 \\ 0 & 0 & a_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{pmatrix}^k = A = \begin{pmatrix} a_{1,1}^k & 0 & 0 & \dots & 0 \\ 0 & a_{2,2}^k & 0 & \dots & 0 \\ 0 & 0 & a_{3,3}^k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n}^k \end{pmatrix}$$

Keep in mind this **only** applies to diagonal matrices.

4 Row Echleon Form & Reduced Row Echleon Form

5 Placeholder

6 Placeholder

7 Placeholder

8 Placeholder

9 Placeholder

10 Placeholder