MTH108 — Linear Algebra

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Content by Week

1 Euclidean Spaces

A Euclidean Space is a mathematical space in which points and lines can be represented by a set of coordinates in the respective dimension of the space, and every point can be represented in a defined set. For example:

$$\mathbb{R}^3 = (x, y, z); x, y, z \in \mathbb{R}$$

is three-dimensional space represented using coordinates in terms of (x, y, z), where $x, y, z \in \mathbb{R}$.

Theorem 1.1. *Given two vectors $\vec{a}, \vec{b}$ and some constant k, $\vec{a}$ and $\vec{b}$ are called **parallel** if:*

$$\vec{a} = k\vec{b} \Leftrightarrow \vec{a} // \vec{b}$$

1.1 Products of Vectors with Constants

Theorem 1.2. *Given a constant k in $\mathbb{R}$ and some vector $\vec{a}$ in $\mathbb{R}^2$, the product of $k\vec{a}$ is:*

$$k\vec{a} = k(x_1, y_1) = (k \cdot x_1, k \cdot y_1), x, y \in \mathbb{R}$$

To represent a vector in Linear Algebra, we can use the following notation (*using the previously mentioned vector $\vec{a}$ as an example*):

$$\vec{a} = (x_1, y_1) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

1.1.1 Examples:

Take $\vec{a} = \left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix} \right)$ and $\vec{b} = \left(\begin{smallmatrix} 4 \\ -1 \end{smallmatrix} \right)$, compute $-2\vec{a} + 3\vec{b}$:

$$\begin{aligned} -2\vec{a} + 3\vec{b} &= 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 12 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ -1 \end{pmatrix} \end{aligned} \tag{1}$$

1.2 Products of two Vectors

Theorem 1.3. *Given two vectors in $\mathbb{R}^n, \vec{a} = \left(\begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix} \right), \vec{b} = \left(\begin{smallmatrix} y_1 \\ \vdots \\ y_n \end{smallmatrix} \right)$, the product of $\vec{a} \cdot \vec{b}$ is:*

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\ &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \end{aligned}$$

*This is known as the **Dot Product**.*

1.2.1 Examples:

Take $\vec{A} = \left(\begin{smallmatrix} 1 \\ -1 \\ 2 \\ 3 \end{smallmatrix} \right)$ and $\vec{b} = \left(\begin{smallmatrix} 2 \\ 1 \\ -1 \\ 1 \end{smallmatrix} \right)$, Find the dot product of $\vec{a} \cdot \vec{b}$:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (2) + (-1) + (-2) + (3) \\ &= 2 \end{aligned} \tag{2}$$

1.3 Other Properties of Vector Products

Given some $\vec{a}, \vec{b}$ in $\mathbb{R}^n$ and some constant k, the following properties apply:

- $(\vec{a}+\vec{b})(\vec{a}+\vec{b}) = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b}+\vec{c}) = \vec{a}\vec{b} + \vec{a}\vec{c}$
- $\vec{a}(k\vec{b}) = k\vec{a}\vec{b}$
- $\vec{a} \cdot \vec{a} = \vec{a}^2 = x_1^2 + x_2^2 + \cdots + x_n^2$

The midpoint $C(z_1, z_2, \ldots, z_n)$ of a line from $A(x_1, x_2, \ldots, x_n)$ to $B(y_1, y_2, \ldots, y_n)$ is calculated using the following forumla:

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = (1-t)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + t\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{for t such that } 0 \leq t \leq 1$$

We can simplify this to calculate every z_i:

$$z_i = \frac{x_i + y_i}{2}$$

1.3.1 Examples:

Given $A(1, -2)$ and $B(-3, 4)$, find the midpoint $C(x, y)$:

$$\begin{aligned} x &= \frac{1+(-3)}{2} \\ &= -1 \\ y &= \frac{(-2)+4}{2} \\ &= 1 \end{aligned} \tag{3}$$

Therefore $C = (-1, 1)$.

1.4 Norm and Angle

The magnitude of a vector in $\mathbb{R}^n$ is called the **Norm**, it is notated and defined as:

$$||\vec{a}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \quad (\vec{a} \text{ is some vector in } \mathbb{R}^n)$$

We can use this definition to demonstrate some inequalities and properties (*Given some $\vec{a}, \vec{b} \in \mathbb{R}^n$ and some constant k*):

- $||\vec{a}+\vec{b}|| \leq ||\vec{a}|| + ||\vec{b}||$ *(Triangle Inequality)*
- $||\vec{a}+\vec{b}|| \geq ||\vec{a}|| - ||\vec{b}||$
- $||k\vec{a}|| = |k| \cdot ||\vec{a}||$
- $||\frac{\vec{a}}{||\vec{a}||}|| = 1$
- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ *(Orthagonal)*
- $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}$

1.5 Determinants

The determinant of a matrix is a number that can be calculated using the following formula (*in* $\mathbb{R}^2$):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Given $\vec{a}, \vec{b} \in \mathbb{R}^n$, the **Gram Determinant** of $\vec{a}\vec{b}$ is defined as:

$$G(\vec{a}, \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix} = ||\vec{a}||^2 \ ||\vec{b}||^2 - (\vec{a}\vec{b})^2$$

Given some vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$, the Gram Determinant can be calculated using the following formula:

$$G(\vec{a}, \vec{b}) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}^2 + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}^2$$

Lemma 1.3.1. *The **Cauchy Inequality** states:*

$$G(\vec{a}, \vec{b}) \geq 0 \Rightarrow |\vec{a} \cdot \vec{b}| \leq ||\vec{a}|| \cdot ||\vec{b}||$$

2 Projections and Spanning Spaces

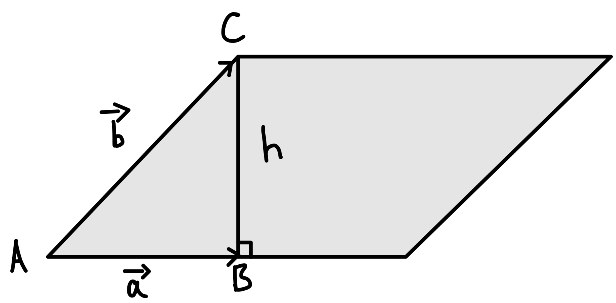
For some $\vec{a}, \vec{b} \in \mathbb{R}^n$, a projection of $\vec{a}$ on $\vec{b}$ is defined as:

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} \cdot \vec{a}$$

The norm of a projection is defined as:

$$||proj_{\vec{a}}\vec{b}|| = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} \cdot ||\vec{a}||$$

2.1 Projection and Area of a parallelogram



Given the above figure, we can determine formulas for calculating properties of the parallelogram:

1. $h = ||\vec{BC}|| = ||\vec{b} - proj_{\vec{a}}\vec{b}||$
2. $Area = ||\vec{a}|| \cdot ||\vec{b} - proj_{\vec{a}}\vec{b}||$
3. $\sin(\theta) = \dfrac{||\vec{b} - proj_{\vec{a}}\vec{b}||}{||\vec{b}||}$
4. $\cos(\theta) = \dfrac{||proj_{\vec{a}}\vec{b}||}{||\vec{b}||} = \dfrac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}$

The generalized formula for calculating the area of some $\vec{a}, \vec{b} \in \mathbb{R}^n$ is as follows:

$$A(\vec{a}, \vec{b}) = \sqrt{G(\vec{a}, \vec{b})}$$

3 Placeholder

4 Placeholder

5 Placeholder

6 Placeholder

7 Placeholder

8 Placeholder

9 Placeholder

10 Placeholder