School of Mathematics and Statistics

Level-4 Honours project

Factorising the Hypergeometric Differential Equation

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Abstract

We obtain factorisations of the hypergeometric differential equation via the reducibility of its monodromy group. This monodromy group is derived from the ground up by explaining the analytic continuation of pairs of local solutions at each singular point of the equation in the complex plane.

Contents

1 Introduction

The hypergeometric differential equation given by

$$z(1-z)\frac{d^2y}{dz^2} + (\gamma - (\alpha+\beta+1)z)\frac{dy}{dz} - \alpha\beta y = 0 \tag{1.1}$$

has been studied for over 200 years by some of Mathematics' biggest names such as Euler and Gauss. Though further breakthroughs have been made by great mathematicians in many different aspects, the focus of this paper is specifically the reducibility and factorisation of the equation into two first order components.

Under special conditions of the complex parameters α, β and γ we can find a fundamental pair of solutions to the equation at each of its regular singular points $z = 0, 1, \infty$ in terms of the hypergeometric or Gauss hypergeometric functions. For example, our simplest solution for $|z| < 1$ is given by;

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}}{\gamma_{(n)}n!} z^n = 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha\beta}{\gamma}\frac{(\alpha+1)(\beta+1)}{(\gamma+1)(1+1)}z^2 + \cdots$$

If these special conditions are violated, more intricate logarithmic solutions may be found, but we will not consider them here due to the number of cases and their effect on finding the monodromy group.

Factorisations of second order (and higher) equations have been studied due to the ease in which solutions to the overall equation may then be found. Infeld and Hull's influential 1951 paper 'The Factorization Method' [1] tackled many such equations including the hypergeometric equation though as eigenvalue problems, rather than a global result. Additionally, Schrödinger in 1941 studied factorisations of the equation for the real variable $0 \leq x \leq 1$ first via the transformation to $\theta : \cos\theta = 2x-1$, then by introducing the further dependent variable;

$$w = (\sin\theta)^{\frac{\alpha+\beta}{2}} \left(\tan\frac{\theta}{2}\right)^{\frac{\alpha+\beta+1-2\gamma}{2}} y$$

Ironically, after introducing w, he highlights his results "must not be regarded as the factorizations of Gauss's equation." [2] due to the particular density used in w. We, however, are explicitly interested in finding a factorisation with rational functions (quotients of polynomials) which holds globally.

When looking at the Fuchsian class of differential equations (to which the hypergeometric belongs), factorisation holds a direct link to the reducibility of the equation's monodromy group. This matrix group which is determined exclusively by the differential equation, describes how the pairs of solutions at each singular point are changed by analytic continuation in the complex plane. The framework of complex analysis necessary for this is discussed in depth in the first section since as a Mathematics and Statistics student, I had not taken the 3H Complex Analysis Course.

Using the definition discussed in [3] by Haraoka, we derive the necessary 4 conditions for reducibility; $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}$, $\gamma - \alpha \in \mathbb{Z}$ and $\gamma - \beta \in \mathbb{Z}$. Though no two of these cases may occur simultaneously, in each case we can factorise the equation into two first order components which may be solved for alternative solutions.

2 Complex Analysis Results

2.1 Complex Geometry

We have briefly mentioned above that the point $z = \infty$ is important to the hypergeometric equation. To include this value, we define the extended complex plane, $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. To examine the behaviour of a function f at ∞ we create a new function $\tilde{f}$. For arbitrary $r > 0$, consider $f : \mathbb{C} \to \mathbb{C}$ defined on $\{z \in \mathbb{C} : |z| > r\}$ but not necessarily at ∞. Now defining $\tilde{f} : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ by;

$$\tilde{f}(w) := f\left(\frac{1}{w}\right) \quad , \ \{w \in \mathbb{C} : 0 < |w| < \frac{1}{r}\}$$

Notions of $\tilde{f}$ at 0 now correspond to those for f at ∞ when $f(\infty)$ is defined [4][p.61]. To more intuitively visualise $z = \infty$, we can use the Riemann Sphere. Effectively the sphere comes from wrapping the complex plane around a unit sphere with 0 at the south pole, and ∞ at the north.

Sets such as those above are open in $\mathbb{C}$. Whilst this is analogous to openness in $\mathbb{R}$, we briefly define it formally.

Definition 2.1. A set $S \subseteq \mathbb{C}$ is open if given $a \in S$, there exists $r > 0$ such that $D(a; r) \subseteq \mathrm{S}$, where $D(a; r) := \{z \in \mathbb{C} : |z - a| < r\}$ [4, p.30].

One further formality we must define for the complex plane are curves, since soon we will consider functions travelling along them.

Definition 2.2 (Curves and Properties)**.** For α and $\beta \in \mathbb{R}$ $(\alpha < \beta)$, a curve (or path) γ in the complex plane is a continuous function $\gamma : [\alpha, \beta] \to \mathbb{C}$. The curve has initial point $\gamma(\alpha)$ and final point $\gamma(\beta)$. Additionally;

1. The curve is closed if $\gamma(\alpha) = \gamma(\beta)$
2. The curve is simple if it never crosses itself except possibly at the point of closure.

We may also use the term contour for a new curve composed of smaller curves joined together at endpoints. [4, Ch.4]

2.2 Holomorphic Functions

As we are looking for solutions to a complex differential equation, we need to define complex differentiability.

Definition 2.3. A complex valued function f defined on an open subset $G \subseteq \mathbb{C}$ is differentiable at $z \in G$ if:

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{2.1}$$

exists. $(h \neq 0)$ [4, p.56]

When this limit does exist it is denoted $f'(z)$. Being complex differentiable is a stronger condition than real differentiability since h can approach zero from any direction in the complex plane rather than being restricted to the real line. Furthermore, we would like our functions to be complex differentiable everywhere on some domain for valid solutions.

Definition 2.4. A complex function f is said to be holomorphic at a point $a \in \mathbb{C}$ if $\exists$ r > 0 such that f is defined and differentiable in D(a;r). Furthermore, if f is differentiable at every point in an open set G, f is said to be holomorphic on G. The set of functions holomorphic in G is denoted H(G) [4, p.59].

In the complex plane, a function being holomorphic is equivalent to it being analytic. This means if $f \in H(G)$, for any point $a \in G$, f may be expressed by the convergent power series;

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{2.2}$$

This property is extremely useful, especially when using analytic continuation.

2.3 Multifunctions

Whilst functions are single valued in the real domain, things are not always so simple in $\mathbb{C}$. All complex numbers $z = x + iy$ may be written in polar form $z = re^{i\theta}$ where $r = |z|$ and θ is the argument of the complex number. However, as $e^{2\pi i} = 1$, for any given z, if θ is unbounded, we can add or subtract integer multiples of 2π to θ to represent the same z-value. i.e for $k \in \mathbb{Z}$;

$$\begin{aligned} z &= re^{i\theta} \\ &= re^{i\theta} \cdot 1 \\ &= re^{i\theta} \cdot e^{2k\pi i} \\ &= re^{i(\theta + 2k\pi)} \end{aligned}$$

For example, using the natural logarithm with $z = re^{i\theta}$, we find;

$$\begin{aligned} \log(z) &= \log(re^{i\theta}) \\ &= \log(r) + i\theta \end{aligned}$$

However, when $k = 1$ we also have $z = re^{i(\theta+2\pi)}$ so that;

$$\begin{aligned} \log(z) &= \log(re^{i(\theta+2\pi)}) \\ &= \log(r) + i(\theta + 2\pi) \end{aligned}$$

Hence, if no bounds exist on θ, the logarithm is a multi-valued function, which we can define as;

$$\log(z) := \{\log(r) + i(\theta + 2k\pi) : k \in \mathbb{Z}, z \neq 0\} \tag{2.3}$$

Similarly consider $f(z) = z^{1/2}$. Then for $z = re^{i\theta}$;

$$\begin{aligned} f(z) &= (re^{i\theta})^{1/2} \\ &= r^{1/2} e^{i\theta/2} \end{aligned}$$

But again, taking $z = re^{i(\theta+2\pi)}$;

$$\begin{aligned} f(z) &= (re^{i(\theta+2\pi)})^{1/2} \\ &= r^{1/2} e^{i\theta/2} e^{i\pi} \\ &= -r^{1/2} e^{i\theta/2} \end{aligned}$$

So similarly the square root and in fact all other non-integer powers are multifunctions in a complex domain. This causes problems so we must set restrictions. We begin by restricting $0 \leq \theta < 2\pi$, (or any other 2π interval) but we still have a problem. Consider the path taken by a point $z = 1$ anticlockwise around the unit circle. As z traces this path, θ increases towards 2π, though when it returns to $z = 1$, it must jump back to $\theta = 0$. Hence, we have a discontinuity in the argument of θ. Let us return to the logarithm and begin by defining its so called multibranches;

$$F_k(r,\theta) := \log(r) + i(\theta + 2k\pi) \quad (k \in \mathbb{Z}, 0 \leq \theta < 2\pi) \tag{2.4}$$

Note that these multibranches are continuous and represent the logarithm for each possible θ. We define $\{F_k\}_{k\in\mathbb{Z}}$ as a complete set of multibranches for $\log(z)$.

Now, for $r > 0$ consider the movement of $z = re^{it}$ where t increases from 0 to 2π. For each $k \in \mathbb{Z}$,

$$\begin{aligned}
[F_k(r,\theta)]_{t=0} &= \log(r) + 2k\pi i \\
[F_k(r,\theta)]_{t=2\pi} &= \log(r) + (2k\pi + 2\pi)i \\
&= \log(r) + 2(k+1)\pi i \\
&= [F_{k+1}(r,\theta)]_{t=0}
\end{aligned}$$

So when z travels in the positive direction (anticlockwise) along a curve which surrounds $z = 0$, we transfer continuously from F_k to F_{k+1}. Ideally having identified this, we may add further restrictions to prevent switching between branches.

Definition 2.5. Consider a function $f : \mathbb{C} \to \mathbb{C}$. $\underline{a}$ is a branch point of $f(z)$ for z $\in D(a;r)$ if for all $r > 0$, $f(z) = c$ has more than one solution for some constant c. If we wish to determine whether ∞ is a branch point, we check if 0 is a branch point of $\tilde{f}(\frac{1}{z})$.

Avoiding these branch points is the key to avoiding movement between multibranches. For both the complex logarithm and square root, we see that $z = 0$ must be a branch point ($z = \infty$ is also a branch point for both). To avoid encircling $z = 0$ we define a <u>cut</u> in the plane which we forbid contours to cross.

For both of these functions, by placing a branch cut $(-\infty, 0]$ along the real axis so that our domain becomes $\tilde{\mathbb{C}} \setminus (-\infty, 0]$, we find it is no longer possible to define a contour in the domain which surrounds 0. Any such contour which would now cross the cut is defined as <u>inadmissible</u>.

Returning to our multibranches $F_k(r,\theta) = \log(r) + i(\theta + 2k\pi)$ we now restrict θ such that $-\pi \leq \theta < \pi$. This means we can no longer cross the cut placed at $(-\infty, 0]$ and hence θ will now be continuous. This means that any branch F_k is holomorphic in $\mathbb{C}_\pi = \mathbb{C} \setminus (-\infty, 0]$. We now say that $\{F_k\}_{k\in\mathbb{Z}}$ are the <u>holomorphic branches</u> of the log and by choosing one of these branches to work with, typically the principal branch when $k = 0$, no longer encounter multivaluedness.

Some multifunctions may have more than one branch point, in which case we need to define our branch cut(s) such that contours enclosing any of the points are excluded. For example, the branch cut above also prevents contours encircling ∞.

2.4 Analytic Continuation

Analytic continuation is an important technique for functions in the complex plane. The aim is to extend the domain of a function f holomorphic in a region R by its equivalence to a function g holomorphic on a larger domain. Here, by a region, we mean a non-empty, connected, open set. To prove analytic continuation can be done, we first recall limit points.

Definition 2.6 (Limit Points)**.** The point $a \in \mathbb{C}$ is a limit point or accumulation point of a set S if $\forall\epsilon > 0, D'(a,\epsilon) \cap S \neq \emptyset$. [4][p.32]

Here we have used the punctured disc $D'(a, \epsilon) = \{z \in \mathbb{C} \mid 0 < |z - a| < \epsilon\}$. Also, note that a need not be in S itself. Limit points let us prove the foundation of analytic continuation.

Theorem 2.1 (Identity Theorem). *Suppose $f, g \in H(D)$ where D is a non-empty, open and connected subset of $\mathbb{C}$. Let $S = \{z \in D \mid f(z) = g(z)\}$. If S has a limit point in D, $f \equiv g$ on D. [4][p.180]*

Proof. Letting $h = f - g$, since f and g are holomorphic on D, h must be holomorphic and hence analytic as well. Let c be a limit point of the set $S = \{z \in D \mid h(z) = 0\}$. Suppose $h(z) \neq 0$ near c. As h is a holomorphic function, near c we have;

$$h(z) = \sum_{n=0}^{\infty} c_n (z - c)^n$$

where not all coefficients are 0. Let c_m be the first non-zero coefficient, then

$$h(z) = (z - c)^m (c_m + c_{m+1}(z - c) + ...)$$

The right hand bracket is continuous and non-zero at $z = c$ so by continuity is non-zero in some small neighbourhood around c. This means, $h(z)$ is non-zero in an open disc around c which means c cannot be a limit point. We therefore must have $h(z) = 0$ in an open neighbourhood around c. Let $U \subseteq D$ be the smallest open subset with $c \in U$ and $h = 0$ on U. As U contains all its limit points, it must also be closed. Since D is connected, the only sets which are both open and closed are D and $\emptyset$. As U is non-empty, it must be that $U = D$. Therefore, h is identically 0 on D and equivalently, $f = g$ on D. □

Hence, if we find two functions which are equal in some small open subset, they must be equal everywhere on the larger region, the following result shows this choice of function is unique.

Theorem 2.2. *If f_1 is holomorphic on a region D_1 and D_2 is a region with $D_1 \cap D_2 \neq \varnothing$, there is at most one holomorphic function f_2 on D_2 such that $f_1 \equiv f_2$ on $D_1 \cap D_2$. When the function f_2 exists it is called an analytic continuation. [5, p.124]*

Proof. Suppose f_2 and g_2 are holomorphic on D_2 and both equal f_1 on $D_1 \cap D_2$. Then $f_2 \equiv g_2$ on $D_1 \cap D_2$. This intersection must be an open set, as both D_1 and D_2 are open, so by the previous theorem, $f_2 \equiv g_2$ on D_2 and hence they are not unique. □

Essentially we are saying that for f_1 defined on a region D_1 and f_2 defined on a region D_2, if $f_1 = f_2$ on $D_1 \cap D_2$, the analytic continuation f_2 is unique in acting as f_1 on the second region D_2. We can formalise this by defining a new function;

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in D_1 \\ f_2(z) & \text{if } z \in D_2 \end{cases}$$

Clearly $f(z)$ is now holomorphic on the larger region $D_1 \cup D_2$.

One way to quickly chain analytic continuations together is by using power series. Let D be a region and consider a function $f \in H(D)$. f has a power series expansion in D, so let us choose $a \in D$ and let $D_1(a, r_1)$ be the largest open disc centred at a which is fully contained in D. We know the power series;

$$f_1(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n$$

converges to f on D_1 so whilst we are not extending the domain, f_1 is still trivially an analytic continuation of f on D_1.

Proceeding in the same fashion we choose $b \in D_1$ and define $D_2(b, r_2)$ to be the largest open disc centred at b contained entirely in D_1. We can define the series;

$$f_2(z) = \sum_{n=0}^{\infty} b_n(z-b)^n$$

which coincides with f_1 on D_2, where the coefficients b_n are determined by expanding and rearranging;

$$(z-a)^n = [(z-b) + (b-a)]^n \quad [6][p.12]$$

The function f_2 certainly converges and is analytic on D_2 but may in fact converge on a larger open disc U. If U extends beyond D_2, then f_2 is analytic outside the starting region D. If this is the case, f_2 is an analytic continuation of f_1 outside of D_1 which is in turn an analytic continuation of f. There is no limit to the number of functions we could iteratively define this way, all of which are analytic continuations of the previous function in the chain. [6][p.15]

Definition 2.7 (Analytic Continuation along a curve)**.** Consider a curve γ defined on the interval $[a, b]$ with initial point z_0 and final point w. By dividing the interval into an increasing sequence of points $a = a_0, a_1, ..., a_{n+1} = b$, we also divide the curve into smaller pieces $\gamma_0, \gamma_1, ..., \gamma_n$. For example, γ_0 connects z_0 to $\gamma(a_1)$ and γ_n connects $\gamma(a_n)$ to w. Finally, we define discs D_0 centred at z_0, D_n centred at w, and intermediary discs $D_1, ...D_{n-1}$ so that the curve γ_i is contained completely in disc D_i.

Suppose we have a function $f(z)$ holomorphic on D_0 with $f_0(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ converging to $f(z)$ on D_0. Following the previous process of creating new power series at consecutive points by rearrangement, we let f_i be the power series convergent in D_i obtained by rearranging the series f_{i-1} for $i = 1, ..., n$. We define $f_n(z)$ convergent in at least D_n as the analytic continuation of the function $f(z)$ along the curve γ. [6][p.13]

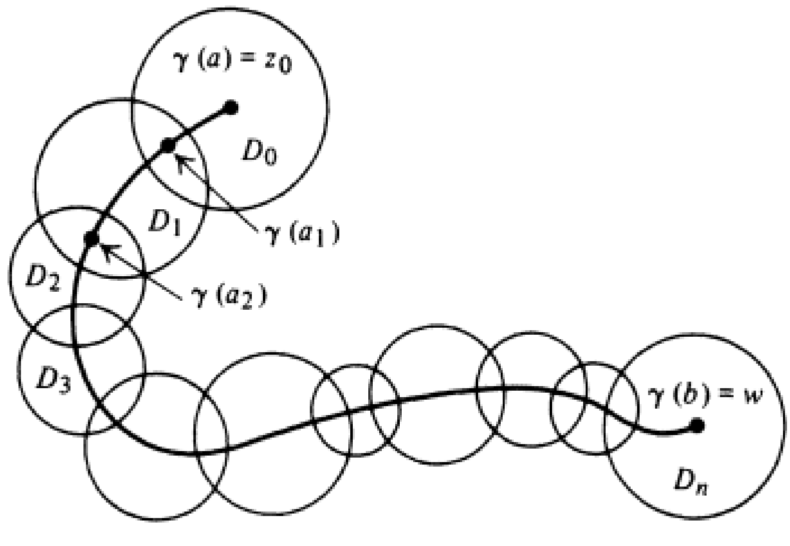

Figure 1: Curve γ covered by a series of discs [7][p.323]

For example, take $b \in U$, where U is simply connected and let γ be a curve with base point b that encircles a once in the positive direction. For $\rho \in \mathbb{C}$, if we fix any branch of the functions $\log(z-a)$ and $(z-a)^\rho$, since as defined earlier a is a branch point, the analytic continuations along γ are given in [3][p.58] by;

$$\gamma_* \log(z-a) = \log(z-a) + 2\pi i$$
$$\gamma_*(z-a)^\rho = (z-a)^\rho e^{2\pi i \rho}$$

We use the notation $\gamma_* f$ for the continuation of f along γ. One other key property highlighted here is that the result of analytic continuation along a closed curve need not be the original function.

Analytic continuation has many further useful properties. It is commutative under multiplication and differentiation whilst additionally, if two curves with identical start and end points are homotopic in a domain D, then the results of analytic continuation along both curves coincide. [3][p.24]

2.5 Singularities

Whilst dealing with functions which are holomorphic everywhere is convenient, many functions such as $\frac{1}{z(1-z)}$ which appears in the hypergeometric equation, are not fully holomorphic in $\mathbb{C}$. This leads us to deal with singularities of functions.

Definition 2.8. We say a is a regular point of f if f is holomorphic at a. The point a is a singularity of f if a is a limit point of regular points and not a regular point itself.

If a is a singularity of f, and f is holomorphic in some punctured disc $D'(a;r) = \{z \in \mathbb{C} : 0 < |z-a| < r\}$, then a is an isolated singularity. Otherwise, a is a non-isolated or essential singularity. [4][p.200]

Describing certain singularities as 'essential' implies they come in different severities. We would prefer our singularities to be reasonably well-behaved.

Definition 2.9 (Poles)**.** For an open domain D, let $f \in H(D'(a;r))$. The function f has a pole of order m at a if and only if

$$\lim_{z \to a} (z-a)^m f(z) = c$$

where c is a finite, non-zero constant. If no such m exists, a is an irregular singularity. [4, p.202]

Poles are the simplest type of isolated singularities and play an important part in our equation. We can account for the behaviour of poles in functions with the following.

Definition 2.10 (Meromorphic Functions)**.** Let G be an open subset of $\tilde{\mathbb{C}}$. A complex valued function which is holomorphic in G (except possibly at poles) is said to be meromorphic. [4, p.206]

3 Complex Differential Equations

For this chapter we will be considering the second-order linear differential equation given by:

$$\frac{d^2y}{dz^2} + p_1(z)\frac{dy}{dz} + p_2(z)y = 0 \tag{3.1}$$

where $p_1(z)$ and $p_2(z)$ are functions defined on some subset of $\mathbb{C}$. We begin with the most fundamental theorem for this equation.

Theorem 3.1. *Let D be a domain in $\mathbb{C}$ and assume $p_1(z)$ and $p_2(z)$ are both holomorphic on D. Take $a \in D$ and $r > 0$ such that $D(a;r) \subset D$. For any y_0 and $y_1 \in \mathbb{C}$, there exists a unique solution $y(z)$ satisfying* (3.1) *with $y(a) = y_0$, $y'(a) = y_1$ which is holomorphic on $D(a;r)$. [3, p.26]*

As with real differential equations, this establishes that we can find unique solutions given suitable initial conditions.

Theorem 3.2 (Fuchs' Theorem)**.** *A singular point $z = z_0$ of* (3.1) *is said to be regular singular if and only if for $j = 1,2$, $z = z_0$ is a pole of $p_j(z)$ of order at most j. [3, p.31]*

Lazarus Fuchs also gives his name to the Fuchsian class of differential equations. These are equations where all singularities of the equation are regular singular. We can check that $z = z_0$ is a regular singular point of (3.1) using the definition of poles. i.e. if;

$$\lim_{z \to z_0} (z - z_0)p_1(z) \text{ and } \lim_{z \to z_0} (z - z_0)^2 p_2(z) \tag{3.2}$$

are both finite [8, p.148]. Notice that as we only require the order of the pole to be at most j, we allow these limits to equal 0.

3.1 Monodromy

Having established that complex functions may change when analytically continued along a curve, we have to consider the effect of this on our solutions. Fortunately, by its commutative properties, analytically continued solutions remain solutions.

Theorem 3.3. *For $r > 0$,let $D = D'(a;r)$, and consider* (3.1) *defined on D where $p_1, p_2 \in H(D)$ and there exists a regular singular point at $z = a$. Let $\mathcal{Y}(z) = (y_1(z), y_2(z))$ be a fundamental set of solutions to the equation on a simply connected domain $U \subset D$. Lastly, consider a closed curve Γ with start and end point $b \in U$ that encircles a once in the positive direction. Then, there exists a matrix $L \in GL_2(\mathbb{C})$ such that;*

$$\Gamma_* \mathcal{Y}(z) = \mathcal{Y}(z)L \tag{3.3}$$

where $\Gamma_* \mathcal{Y}(z)$ is the analytic continuation of $\mathcal{Y}(z)$ along Γ. L is called the circuit matrix for Γ and $GL_2(\mathbb{C})$ is the group of 2x2 invertible matrices with complex coefficients. [3][p.55]

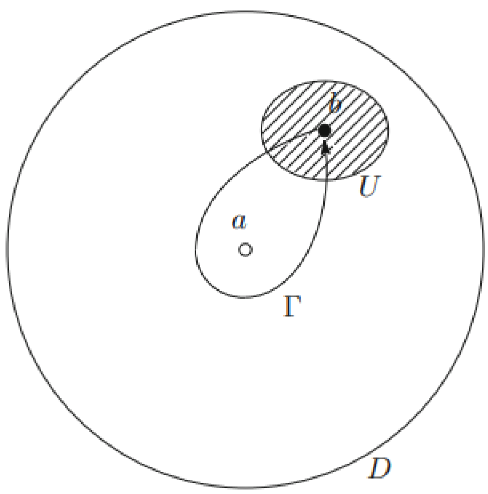

Figure 2: Closed curve Γ in D. [3][p.55]

Proof. As $\mathcal{Y}(z) = (y_1(z), y_2(z))$ is a vector, we gather that the result of analytic continuation must be a vector of the same size, so the effect is that of multiplying by a 2x2 matrix. Lastly as $y_1(z), y_2(z)$ are linearly independent, the Wronskian of both sides is non-zero and hence L must be invertible. $\square$

We therefore define the local monodromy of (3.1) at $z = a$ by the conjugacy class in $GL_2(\mathbb{C})$ of the circuit matrix L. We can further consider global analytic continuation of our solutions.

Keeping D, U and $\mathcal{Y}(z)$ as before, if we now take any loop Γ in D with base point $b \in U$, there exists a matrix M $\in GL_2(\mathbb{C})$ such that

$$\Gamma_* \mathcal{Y}(z) = \mathcal{Y}(z) M \tag{3.4}$$

As with analytic continuations of functions before, the result, in this case M, is uniquely determined by the homotopy class $[\Gamma] = \gamma$ in D which we denote by $M = M_\gamma$.

If we consider briefly the fundamental group $\pi_1(D, b)$, the group of homotopy classes of curves with starting point b in the domain D, we then have the natural mapping;

$$\begin{aligned} \rho : \pi_1(D, b) &\to GL_2(\mathbb{C}) \\ \gamma &\mapsto M_\gamma \end{aligned}$$

For analytic continuation along the product of curves, letting $\Gamma_1\Gamma_2$ be the curve with base point b which follows Γ_1 and then Γ_2 (the end point of Γ_1 and start point of Γ_2 must coincide). We have;

$$\begin{aligned} (\Gamma_1\Gamma_2)_* \mathcal{Y}(z) &= \Gamma_{2*}(\Gamma_{1*}\mathcal{Y}(z)) \\ &= \Gamma_{2*}\mathcal{Y}(z) M_{\gamma_1} \\ &= \mathcal{Y}(z) M_{\gamma_2} M_{\gamma_1} \end{aligned}$$

Hence, $\rho(\gamma_1\gamma_2) = \rho(\gamma_2)\rho(\gamma_1)$. As the order of 1 and 2 has reversed, we may call ρ an anti-homomorphism.

Definition 3.1. The function ρ is called a monodromy representation of the differential equation with respect to the fundamental set of solutions $\mathcal{Y}(z)$. The image of ρ is a subgroup of $GL_2(\mathbb{C})$ and is called the monodromy group of the equation.

Importantly, as mentioned this is with respect to a fundamental set of solutions. If we consider a second set of solutions $\tilde{\mathcal{Y}}(z)$ on U, as a linear combination of our previous solutions we must have;

$$\tilde{\mathcal{Y}}(z) = \mathcal{Y}(z) C$$

for $C \in GL_2(\mathbb{C})$. Now considering the same curve Γ from 3.4;

$$\begin{aligned} \Gamma_* \tilde{\mathcal{Y}}(z) &= \Gamma_* \mathcal{Y}(z) C \\ &= \mathcal{Y}(x) M_\gamma C \\ &= \tilde{\mathcal{Y}}(z) C^{-1} M_\gamma C \\ &= \tilde{\mathcal{Y}}(z) \tilde{M}_\gamma \end{aligned}$$

Since $\tilde{M}_\gamma = C^{-1} M_\gamma C$ we conclude that the monodromy representation with respect to $\tilde{\mathcal{Y}}(z)$ denoted $\tilde{\rho}$ satisfies;

$$\tilde{\rho}(\gamma) = C^{-1} \rho(\gamma) C \tag{3.5}$$

This means the monodromy representations are conjugate under different solution sets. It is similarly shown that the conjugacy class of ρ is not affected by the choice of base point $b \in D$ and hence, the conjugacy class of a monodromy representation must be uniquely determined by the differential equation. [3, p.64].

For equation (3.1), in the case that p_1 and p_2 are rational functions (such as in the hypergeometric equation), then there are only a finite number of singular points $a_0, a_1, ..., a_p$ and the equation is hence defined on $D = \tilde{\mathbb{C}} \setminus \{a_0, a_1, ..., a_p\}$. Once more taking a start point b in a simply connected neighbourhood of D, we can show that any two curves which encircle exactly one singular point a_j and no others are conjugate within the fundamental group. Hence, $\pi_1(D, b)$ has presentation;

$$\pi_1(D, b) = \langle \gamma_0, \gamma_1, ..., \gamma_p \mid \gamma_0\gamma_1 \cdots \gamma_p = 1 \rangle$$

where for each j $(0 \leq j \leq p), \gamma_j$ is a loop encircling a_j exactly once in the positive direction and no other singular points.

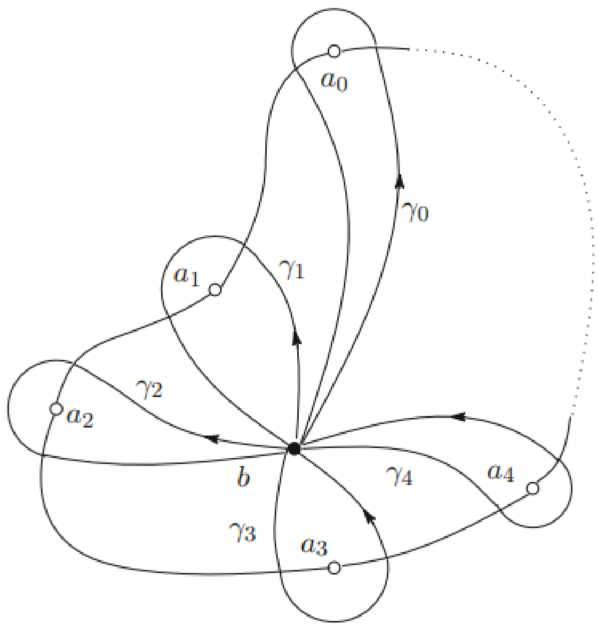

Figure 3: Example loops circling exactly one singular point. [3][p.69]

Theorem 3.4. *For the monodromy representation of* (3.1) *with singular points* $a_0, a_1, ..., a_p$ *defined on* $D = \tilde{\mathbb{C}} \setminus \{a_0, a_1, ..., a_p\}$ *with respect to* $\mathcal{Y}(z)$

$$\rho : \pi_1(D, b) \to GL_2(\mathbb{C})$$
$$\gamma_j \mapsto M_j$$

we have the following:

1. *The monodromy representation* ρ *is uniquely determined by the matrix tuple* $(M_0, M_1, \cdots, M_p)$.
2. *For the tuple of matrices* $(M_0, M_1, \cdots, M_p)$, *the following relation holds. [3, p.69]*

$$M_p \cdots M_1 \cdot M_0 = I$$

Proof. Using the presentation of the fundamental group, clearly the representation must then be determined by

$$(\rho(\gamma_0), \rho(\gamma_1), \cdots, \rho(\gamma_p)) = (M_0, M_1, \cdots, M_p)$$

similarly, by the relation in the presentation we must have

$$\rho(\gamma_0\gamma_1\cdots\gamma_p) = \rho(1)$$

Recalling that ρ is an anti-homomorphism and that $\rho(1) = I$, we must have

$$M_p \cdots M_1 \cdot M_0 = I$$

Alternatively, one can see on the Riemann Sphere that a loop around all singular points can be deformed from the opposite side to surround no singularities, hence having trivial monodromy. $\square$

4 Solving the Hypergeometric Equation

4.1 Hypergeometric Functions

Before we proceed to solve the hypergeometric equation we must discuss the hypergeometric functions known to solve it. For $\alpha, \beta, \gamma \in \mathbb{C}$, consider an infinite series of the form

$$f(z) = 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha\beta}{\gamma}\frac{(\alpha+1)(\beta+1)}{(\gamma+1)(1+1)}z^2 + \frac{\alpha\beta}{\gamma}\frac{(\alpha+1)(\beta+1)}{(\gamma+1)(1+1)}\frac{(\alpha+2)(\beta+2)}{(\gamma+2)(1+2)}z^3 + \cdots$$

which we may more concisely write as;

$$f(z) = \sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}}{\gamma_{(n)}n!}z^n$$

where the subscript (n) is the Pochammer symbol denoting the rising factorial;

$$\alpha_{(n)} = \begin{cases} 1 & n = 0 \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & n > 0 \end{cases}$$

Note: This means that $1_{(n)} = n!$ and that as n tends to infinity, the rising factorial behaves the same as a regular factorial.

Definition 4.1. A series of the above form is defined as a hypergeometric or Gauss series and equivalently the hypergeometric function where it converges. We will represent it by;

$$\sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}}{\gamma_{(n)}n!}z^n = F(\alpha, \beta; \gamma; z) \tag{4.1}$$

Here, the first semicolon separates the numerator from the denominator and the second indicates the argument at which the series is evaluated.

We can examine when this series converges with the ratio test. Looking at coefficients, we see;

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\alpha_{(n+1)}\beta_{(n+1)}}{\gamma_{(n+1)}(n+1)!} \cdot \frac{\gamma_{(n)}n!}{\alpha_{(n)}\beta_{(n)}}\right| \\ &= \left|\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)}\right| \end{aligned}$$

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, so the series converges when its argument, in this case z, has absolute value less than 1.

The hypergeometric function is known to have many special values for different parameters and arguments. We simply need that;

$$\begin{aligned} F(\alpha, \beta; \gamma; 0) &= \sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}}{\gamma_{(n)}n!}(0)^n \\ &= 1 \end{aligned}$$

and when $z = 1$, under a slight caveat seen later, we have the Gauss-Kummer identity;

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

where $\Gamma(z)$ is the gamma function evaluated at z. [9, p.387].

4.2 The Hypergeometric Equation

For $z \in \tilde{\mathbb{C}}$, consider the hypergeometric equation given by;

$$z(1-z)\frac{d^2y}{dz^2} + (\gamma - (\alpha + \beta + 1)z)\frac{dy}{dz} - \alpha\beta y = 0 \tag{4.2}$$

which we may equally rearrange to;

$$\frac{d^2y}{dz^2} + \frac{(\gamma - (\alpha + \beta + 1)z)}{z(1-z)}\frac{dy}{dz} - \frac{\alpha\beta}{z(1-z)}y = 0 \tag{4.3}$$

The equation clearly has singularities at $z = 0$ and $z = 1$ so to check these are regular singular, we use (3.2);

$$\lim_{z \to 0} \frac{z(\gamma - (\alpha + \beta + 1)z)}{z(1-z)} = \gamma \text{ and } \lim_{z \to 0} -\frac{z^2\alpha\beta}{z(1-z)} = 0$$

which are both finite. Similarly for $z = 1$;

$$\lim_{z \to 1} \frac{(z-1)(\gamma - (\alpha + \beta + 1)z)}{z(1-z)} = \alpha + \beta + 1 - \gamma \text{ and } \lim_{z \to 1} \frac{(1-z)^2\alpha\beta}{z(1-z)} = 0$$

We therefore conclude these are both regular singular points. At both points we can find a Frobenius series solution. Starting with the singular point z = 0, we consider a solution of the form;

$$\begin{aligned} y(z) &= \sum_{n=0}^{\infty} a_n z^{n+r} \\ \frac{dy}{dz} &= \sum_{n=0}^{\infty} (n+r)a_n z^{n+r-1} \\ \frac{d^2y}{dz^2} &= \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n z^{n+r-2} \end{aligned}$$

Substituting these into our equation and collecting powers of z gives;

$$\sum_{n=0}^{\infty} \{(n+r)(n+r+\gamma-1)a_n z^{n+r-1} - [(n+r)(n+r+\alpha+\beta) + \alpha\beta]a_n z^{n+r}\} = 0$$

For this to equal zero we require the coefficients of each power of z to equal zero. Looking at the coefficient of the smallest power, z^{r-1}, we have;

$$r(r+\gamma-1)a_0 = 0$$

$a_0 \neq 0$ without loss of generality in our power series so we have $r = 0$ and $r = 1-\gamma$. For the coefficient of a general power z^{n+r} to equal zero we must have;

$$(n+r+1)(n+r+\gamma)a_{n+1} - [(n+r)(n+r+\alpha+\beta)+\alpha\beta]a_n = 0$$

This corresponds to a recurrence relation we can solve for both values of r. For $r=0$ and $r=1-\gamma$ respectively we find;

$$\begin{aligned} a_{n+1} &= \frac{(n+\alpha)(n+\beta)}{(n+\gamma)(n+1)}a_n \\ b_{n+1} &= \frac{(n+\alpha+1-\gamma)(n+\beta+1-\gamma)}{(n+2-\gamma)(n+1)}b_n \end{aligned}$$

From the earlier definition, we recognise these recurrence relations as the coefficients of a hypergeometric series. Setting $a_0 = b_0 = 1$ we find two linearly independent solutions for $|z| < 1$ given by;

$$\begin{aligned} y_{0,1}(z) &= \sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}}{\gamma_{(n)}n!}z^n \\ &= F(\alpha,\beta;\gamma;z) \qquad (4.4) \\ y_{0,2}(z) &= z^{1-\gamma}\sum_{n=0}^{\infty} \frac{(\alpha+1-\gamma)_{(n)}(\beta+1-\gamma)_{(n)}}{(2-\gamma)_{(n)}n!}z^n \\ &= z^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;z) \qquad (4.5) \end{aligned}$$

Notice these solutions are restricted to the cases when $\gamma \notin \{0,-1,-2,..\} \cup \{2,3,4,...\}$ as the denominators in the series cannot equal zero. Additionally the solutions coincide when $\gamma = 1$ so we conclude $y_{0,1}$ and $y_{0,2}$ are linearly independent solutions for $\gamma \notin \mathbb{Z}$. As mentioned in the introduction, When γ does not meet this requirement different solutions can be found (see [10] or [11]), though we will not consider these cases.

We can now similarly find a solution around the regular singular point z = 1 with a Frobenius series of the form;

$$y(z) = \sum_{n=0}^{\infty} a_n(1-z)^{n+r}$$

Following the same steps as before, we find the indicial equation;

$$a_0 r(r+\alpha+\beta-\gamma)$$

which gives us the so called characteristic exponents $r = 0$ and $r = \gamma-\alpha-\beta$ for $z = 1$. Hence, after solving the recurrence relations as before, we have linearly independent solutions for $|z-1| < 1$ given by;

$$\begin{aligned} y_{1,1}(z) &= \sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}}{(\alpha+\beta-\gamma+1)_{(n)}n!}(1-z)^n \\ &= F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z) \qquad (4.6) \\ y_{1,2}(z) &= (1-z)^{\gamma-\alpha-\beta}\sum_{n=0}^{\infty} \frac{(\gamma-\alpha)_{(n)}(\gamma-\beta)_{(n)}}{(\gamma-\alpha-\beta+1)_{(n)}n!}(1-z)^n \\ &= (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z) \qquad (4.7) \end{aligned}$$

Again, due to the denominators we impose the conditions $-(\gamma-\alpha-\beta) \notin \{-1,-2,...\}$ and $\gamma-\alpha-\beta \notin \{-1,-2,...\}$. Lastly, noting that when $\gamma-\alpha-\beta=0$, the solutions coincide to become;

$$y_{1,\{1,2\}}(z) = F(\alpha,\beta;1;1-z)$$

we conclude for these solutions to exist and be linearly independent, $\gamma-\alpha-\beta \notin \mathbb{Z}$.

4.3 Solving at ∞

As we are examining the hypergeometric equation in the extended complex plane $\tilde{\mathbb{C}}$, we take the change of variables given by $w = \frac{1}{z}$ to examine a solution at $z=\infty$. We then have;

$$\frac{dy}{dz} = -w^2\frac{dy}{dw}$$
$$\frac{d^2y}{dz^2} = w^4\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw}$$

Substituting these expressions into the hypergeometric equation gives;

$$\frac{d^2y}{dw^2} + \frac{(\alpha+\beta-1)+w(2-\gamma)}{w(w-1)}\frac{dy}{dw} - \frac{\alpha\beta}{w^2(w-1)}y = 0 \tag{4.8}$$

We now see that $z=\infty$ is a singular point of the hypergeometric equation as $w=0$ is a singularity. Once more as;

$$\lim_{w\to 0}\frac{w(\alpha+\beta-1)+w(2-\gamma)}{w(w-1)} = 1-\alpha-\beta \text{ and } \lim_{w\to 0}\frac{w^2\alpha\beta}{w^2(w-1)} = -\alpha\beta$$

which are both finite, $z=\infty$ (or $w=0$) is a regular singular point. Again, using a Frobenius series of the form;

$$y(w) = \sum_{n=0}^{\infty} a_n w^{n+r}$$

we find the indicial equation;

$$a_0(r^2 - r(\alpha+\beta)+\alpha\beta) = 0$$

which when solved gives characteristic exponents $r=\alpha$ and $r=\beta$. These lead to the respective recurrence relations;

$$c_{n+1} = \frac{(n+\alpha)(n+\alpha+1-\gamma)}{(n+1)(n+\alpha+1-\beta)}c_n$$
$$d_{n+1} = \frac{(n+\beta)(n+\beta+1-\gamma)}{(n+\beta+1-\alpha)(n+1)}d_n$$

By solving these relations and reverting from w to z^{-1} we reach local solutions about ∞ (i.e. for $\left|\frac{1}{z}\right| < 1$ or $|z|>1$);

$$\begin{aligned} y_{\infty,1}(z) &= z^{-\alpha}\sum_{n=0}^{\infty}\frac{\alpha_{(n)}(\alpha+1-\gamma)_{(n)}}{(\alpha+1-\beta)_{(n)}n!}z^{-n} \\ &= z^{-\alpha}F(\alpha,\alpha+1-\gamma;\alpha+1-\beta;z^{-1}) \end{aligned} \tag{4.9}$$

$$\begin{aligned} y_{\infty,2}(z) &= z^{-\beta}\sum_{n=0}^{\infty}\frac{\beta_{(n)}(\beta+1-\gamma)_{(n)}}{(\beta+1-\alpha)_{(n)}n!}z^{-n} \\ &= z^{-\beta}F(\beta,\beta+1-\gamma;\beta+1-\alpha;z^{-1}) \end{aligned} \tag{4.10}$$

Once more we must impose the conditions $\alpha - \beta \notin \mathbb{Z}_{<0}$ and $\beta - \alpha \notin \mathbb{Z}_{<0}$. In the same pattern as before, we also have that these solutions coincide when $\alpha = \beta$ to give;

$$y_{\infty,\{1,2\}}(z) = z^{-\alpha}F(\alpha, \alpha + 1 - \gamma; 1; z^{-1})$$

Hence, for linearly independent solutions about ∞, we must have $\alpha - \beta \notin \mathbb{Z}$.

4.4 Local Monodromy

Despite only finding these solutions locally, we may use analytic continuation to find solutions globally. Fortunately, (3.4) showed that the behaviour of analytic continuation is determined uniquely by representative curves which encircle exactly one of the singular points, so we need only consider the behaviour in these 3 cases.

Let $\gamma_0 = \epsilon e^{it}$ and $\gamma_1 = 1 + \epsilon e^{it}$ be paths around 0 and 1 respectively for some $0 < \epsilon < 1$ and $t \in [0, 2\pi]$. First let us consider what happens to solutions $y_{0,1}$ and $y_{0,2}$ as they are continued along γ_0. The hypergeometric function is composed fully of integer powers so has no multivalued behaviour, and is trivially multiplied by 1 from continuation along γ_0. However, for $y_{0,2}$, the pre-factor $z^{1-\gamma}$ is multiplied by $e^{2\pi i(1-\gamma)}$ due to complex powers being multifunctions. We may denote this by the matrix calculation;

$$\gamma_0 : \begin{bmatrix} y_{0,1} & y_{0,2} \end{bmatrix} \mapsto \begin{bmatrix} y_{0,1} & y_{0,2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i(1-\gamma)} \end{bmatrix}$$

We will call this matrix L_0, the local monodromy matrix around 0.

Similarly, let us consider what happens to our second basis of solutions $y_{1,1}$ and $y_{1,2}$ when they are continued along γ_1. Once more as our hypergeometric functions are unchanged, the only effect comes from the characteristic exponents 0 and $\gamma - \alpha - \beta$. Hence like before;

$$\gamma_1 : \begin{bmatrix} y_{1,1} & y_{1,2} \end{bmatrix} \mapsto \begin{bmatrix} y_{1,1} & y_{1,2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i(\gamma-\alpha-\beta)} \end{bmatrix}$$

Similarly, we call this matrix L_1.

Lastly, we can consider a path in the positive direction around infinity. Defining $\gamma_\infty = \lambda e^{it}$ for $\lambda > 1$ and $t \in [0, 2\pi]$, then $y_{\infty,1}$ and $y_{\infty,2}$ will be affected their characteristic exponents. However, as they are powers of $\frac{1}{z}$, we get;

$$\gamma_\infty : \begin{bmatrix} y_{\infty,1} & y_{\infty,2} \end{bmatrix} \mapsto \begin{bmatrix} y_{\infty,1} & y_{\infty,2} \end{bmatrix} \begin{bmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\beta} \end{bmatrix}$$

i.e. the negative of the characteristic exponents. Note: These 3 matrices are not strictly the same as those defined in (3.3) since for each we have used a different fundamental set of solutions. They do however, illustrate the effect of continuing local solutions around their associated singular point which we can use in the subsequent chapter.

4.5 Connection Formulas

Recall that our first two solutions $y_{0,1}$ and $y_{0,2}$ were defined for $|z| < 1$ whilst $y_{1,1}$ and $y_{1,2}$ were defined when $|z - 1| < 1$. As these domains intersect for $\{z : |z| < 1 \text{ and } |z-1| < 1\}$ and our solution pairs are linearly independent, we must have the linear relation;

$$\begin{bmatrix} y_{0,1} \\ y_{0,2} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix}$$

for constants $A, B, C, D \in \mathbb{C}$ within this domain. We can write the first relation as;

$$y_{0,1}(z) = Ay_{1,1}(z) + By_{1,2}(z)$$

or equivalently;

$$\begin{aligned} F(\alpha, \beta; \gamma; z) = AF(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z) \\ + B(1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - z) \end{aligned}$$

Assuming $\mathfrak{Re}(\gamma - \alpha - \beta) > 0$, when z tends to 1 we can use the Gauss-Kummer identity, which gives;

$$\begin{aligned} F(\alpha, \beta; \gamma; 1) &= AF(\alpha, \beta; \alpha + \beta - \gamma + 1; 0) \\ A &= F(\alpha, \beta; \gamma; 1) \\ &= \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \end{aligned}$$

Similarly letting z tend to 0 and rearranging, we find;

$$B = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}$$

Proceeding this way we find A,B,C and D which define the connection matrix;

$$\begin{bmatrix} y_{0,1} \\ y_{0,2} \end{bmatrix} = \begin{bmatrix} \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\ \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} & \frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \end{bmatrix} \begin{bmatrix} y_{1,1} \\ y_{1,2} \end{bmatrix}$$

[10, p9] We also want coefficients which define the opposite linear relation and take us back to our local solutions about $z = 0$. For constants $E, G \in \mathbb{C}$ in the intersected domain we must have;

$$\begin{aligned} y_{1,1}(z) &= Ey_{0,1}(z) + Gy_{0,2}(z) \\ F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z) &= EF(\alpha, \beta; \gamma; z) + Gz^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; z) \end{aligned}$$

As before, if $\mathfrak{Re}(1 - \gamma) > 0$, we let $z \to 0$ and find;

$$\begin{aligned} E &= F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1) \\ &= \frac{\Gamma(\alpha + \beta - \gamma + 1)\Gamma(1 - \gamma)}{\Gamma(\beta - \gamma + 1)\Gamma(\alpha - \gamma + 1)} \end{aligned}$$

We could keep following this process or invert the previous connection matrix to find;

$$\begin{bmatrix} y_{1,1}(z) \\ y_{1,2}(z) \end{bmatrix} = \begin{bmatrix} \frac{\Gamma(1-\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} & \frac{\Gamma(\gamma-1)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha)\Gamma(\beta)} \\ \frac{\Gamma(1-\gamma)\Gamma(\gamma-\alpha-\beta+1)}{\Gamma(1-\alpha)\Gamma(1-\beta)} & \frac{\Gamma(\gamma-1)\Gamma(\gamma-\alpha-\beta+1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \end{bmatrix} \begin{bmatrix} y_{0,1}(z) \\ y_{0,2}(z) \end{bmatrix}$$

[10, p.35] We will call these two connection matrices $C_{0\to1}$ and $C_{1\to0}$ respectively. This gives us the tools to examine any basis of solutions being analytically continued around each singular point. We can use the connection matrices to switch between local solutions and then use the effect of the local monodromy. For example continuing $y_{0,1}$ and $y_{0,2}$ about $z = 1$, when we move into the common domain, we have;

$$\begin{bmatrix} y_{0,1} & y_{0,2} \end{bmatrix} = \begin{bmatrix} y_{1,1} & y_{1,2} \end{bmatrix} C_{0\to1}$$

Then using the local monodromy matrix to describe continuation around $z = 1$;

$$\gamma_1 : \begin{bmatrix} y_{1,1} & y_{1,2} \end{bmatrix} C_{0\to 1} \mapsto \begin{bmatrix} y_{1,1} & y_{1,2} \end{bmatrix} L_1 C_{0\to 1}$$

Lastly, reverting to our original basis $y_{0,1}$ and $y_{0,2}$ the end point of the closed curve;

$$\gamma_1 : \begin{bmatrix} y_{0,1} & y_{0,2} \end{bmatrix} \mapsto \begin{bmatrix} y_{1,1} & y_{1,2} \end{bmatrix} C_{1\to 0} L_1 C_{0\to 1}$$

Hence, the global monodromy matrix that describes continuation of these solutions around 1 is; $C_{1\to 0} L_1 C_{0\to 1}$.

It should be noted that the gamma function used extensively here is naturally defined only for $\mathfrak{Re}(z) > 0$ however using analytic continuation we may go further. As $\Gamma(z+1) = \Gamma(z)z$, from rearrangement we have $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. Regarding this as an analytic continuation, we can define the gamma function now for $\mathfrak{Re}(z) > -1$ (excluding $z = 0$). If we follow this iteratively, the gamma function becomes defined for $\mathbb{C} \setminus \{0, -1, -2, ...\}$.

4.6 Global Monodromy

Instead consider if we used the basis of solutions $y_{0,1}$ and $y_{1,1}$. The motivation here is that they are linearly independent in an open region, and both branch around exactly one of the singular points in our domain. We shall also see that their monodromy is easier to work with. We have the connection relations as defined before, though here the key two are;

$$\begin{aligned} y_{0,1}(z) &= A y_{1,1}(z) + B y_{1,2}(z) \\ y_{1,1}(z) &= E y_{0,1}(z) + G y_{0,2}(z) \end{aligned}$$

where as before;

$$A = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \quad \text{and} \quad E = \frac{\Gamma(1-\gamma)\Gamma(\alpha + \beta - \gamma + 1)}{\Gamma(\alpha - \gamma + 1)\Gamma(\beta - \gamma + 1)} \tag{4.11}$$

As we saw looking at the local monodromy, solutions are affected locally at the corresponding singular points only by their characteristic exponents. Using these exponents with the above formulas we describe the analytic continuation of $y_{1,1}$ about $z = 0$ by;

$$\begin{aligned} \gamma_0 : y_{1,1} &= E y_{0,1} + G y_{0,2} \\ &\mapsto E y_{0,1} + e^{2\pi i(1-\gamma)} G y_{0,2} \\ &= E y_{0,1} + e^{2\pi i(1-\gamma)} (y_{1,1} - E y_{0,1}) \\ &= E y_{0,1}(1 - e^{2\pi i(1-\gamma)}) + e^{2\pi i(1-\gamma)} y_{1,1} \end{aligned}$$

where we have rearranged the connection formula for $G y_{0,2}$ to eliminate $y_{0,2}$. By the same method we can continue $y_{0,1}$ around $z = 1$;

$$\begin{aligned} \gamma_1 : y_{0,1} &= A y_{1,1} + B y_{1,2} \\ &\mapsto A y_{1,1} + e^{2\pi i(\gamma-\alpha-\beta)} B y_{1,2} \\ &= e^{2\pi i(\gamma-\alpha-\beta)} y_{0,1} + A y_{1,1}(1 - e^{2\pi i(\gamma-\alpha-\beta)}) \end{aligned}$$

Hence, for our basis we have the monodromy matrices described by;

$$\gamma_0 : \begin{bmatrix} y_{0,1} & y_{1,1} \end{bmatrix} \mapsto \begin{bmatrix} y_{0,1} & y_{1,1} \end{bmatrix} \begin{bmatrix} 1 & E(1 - e^{2\pi i(1-\gamma)}) \\ 0 & e^{2\pi i(1-\gamma)} \end{bmatrix} \tag{4.12}$$

and;

$$\gamma_1 : \begin{bmatrix} y_{0,1} & y_{1,1} \end{bmatrix} \mapsto \begin{bmatrix} y_{0,1} & y_{1,1} \end{bmatrix} \begin{bmatrix} e^{2\pi i(\gamma-\alpha-\beta)} & 0 \\ A(1-e^{2\pi i(\gamma-\alpha-\beta)}) & 1 \end{bmatrix} \tag{4.13}$$

We will call these matrices M_0 and M_1. [12, p.39-41]

Lastly, using part 2 of (3.4) we must have;

$$M_\infty M_1 M_0 = I$$

As all of these matrices belong to $GL_n(2, \mathbb{C})$, equivalently we have;

$$M_\infty = (M_1 M_0)^{-1}$$

This result can be verified visually with the Riemann sphere. By deforming a loop in the positive direction around ∞, we see it is equivalent to a loop in the negative direction around both 0 and 1. Using this fact, we then have our final matrix;

$$M_\infty = \begin{bmatrix} AE + e^{-2\pi i(\gamma-\alpha-\beta)} & E(1-e^{-2\pi i(1-\gamma)}) \\ Ae^{-2\pi i(1-\gamma)}(1-e^{-2\pi i(\gamma-\alpha-\beta)}) & e^{-2\pi i(1-\gamma)} \end{bmatrix} \tag{4.14}$$

However, as M_∞ is generated by M_0 and M_1, considering these two alone will be enough to describe the monodromy group.

5 Reducibility and Factorisation

5.1 Reducibility

In [3, p72] we are given the following definition.

Definition 5.1 (Reducibility)**.** A subgroup G of $GL(n, \mathbb{C})$ is said to be <u>reducible</u> if there exists a G-invariant, linear subspace $W \subset \mathbb{C}^n$ which is proper and non-trivial (i.e. $W \neq 0, W \neq \mathbb{C}^n$). Namely we require W such that;

$$gW \subset W \quad , \forall g \in G \tag{5.1}$$

Otherwise G is said to be irreducible.

Let us consider the reducibility of our monodromy group. As we are working with 2x2 matrices, W may be of dimension 0,1 or 2. If W is dimension 0 then it is trivial ($W = \{0\}$) and likewise if W has dimension 2, by which we mean the span of two linearly independent vectors in $\mathbb{C}$, these vectors must span $\mathbb{C}^2$ so W is not proper. Therefore, W must have dimension 1 so that $W = span\{\mathbf{w}\}$ for $\mathbf{w} \in \mathbb{C}^2$.

Another motivation of defining the monodromy group by $y_{0,1}$ and $y_{1,1}$ is to utilise the following theorem from Oshima. [11, p18].

Theorem 5.1. *Let*

$$A_0 = \begin{bmatrix} \lambda_{0,1} & a_0 \\ 0 & \lambda_{0,2} \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} \lambda_{1,1} & 0 \\ a_1 & \lambda_{1,2} \end{bmatrix} \text{ in } GL(2, \mathbb{C})$$

Then there exists a non-trivial proper, simultaneous invariant subspace under the linear transformations of $\mathbb{C}^2$ defined by A_0 and A_1 if and only if

$$a_0 a_1 (a_0 a_1 + (\lambda_{0,1} - \lambda_{0,2})(\lambda_{1,1} - \lambda_{1,2})) = 0$$

Proof. We may assume $a_0 a_1 \neq 0$ as this case is trivial. If a 1-dimensional invariant subspace exists, then it must be of the form $\begin{bmatrix}1\\c\end{bmatrix}$ for $c \in \mathbb{C}$. By using the definition of reducibility we must have that;

$$A_0 \begin{bmatrix}1\\c\end{bmatrix} = \begin{bmatrix}\lambda_{0,1} + a_0 c\\ \lambda_{0,2} c\end{bmatrix} \text{ and } A_1 \begin{bmatrix}1\\c\end{bmatrix} = \begin{bmatrix}\lambda_{1,1}\\ a_1 + \lambda_{1,2} c\end{bmatrix}$$

are both multiples of $\begin{bmatrix}1\\c\end{bmatrix}$. We therefore require $\lambda_{0,1} + a_0 c = \lambda_{0,2}$ and $a_1 + \lambda_{1,2} c = \lambda_{1,1} c$. This means that;

$$c = \frac{\lambda_{0,2} - \lambda_{0,1}}{a_0} = \frac{a_1}{\lambda_{1,1} - \lambda_{1,2}} \tag{5.2}$$

. Rearranging this and multiplying by $a_0 a_1$ to include the trivial case gives;

$$a_0 a_1 (a_0 a_1 + (\lambda_{0,1} - \lambda_{0,2})(\lambda_{1,1} - \lambda_{1,2})) = 0$$

□

Hence, for reducibility we require $a_0 a_1 = 0$ or $a_0 a_1 + (\lambda_{0,1} - \lambda_{0,2})(\lambda_{1,1} - \lambda_{1,2}) = 0$. Using our monodromy matrices M_0 (4.12) and M_1 (4.13) we calculate $a_0 a_1$ as;

$$AE(1 - e^{2\pi i(1-\gamma)})(1 - e^{2\pi i(\gamma-\alpha-\beta)})$$

Recalling the conditions we placed to find linearly independent solutions on pages 14 and 15, neither exponential can equal 1 so we need $AE = 0$. Recalling (4.11) and using the property $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$;

$$\begin{aligned} AE &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \frac{\Gamma(1-\gamma)\Gamma(\alpha+\beta-\gamma+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \\ &= \frac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)} \end{aligned}$$

Examining this expression, we note the denominator cannot equal 0 because of our earlier restrictions so this is well defined. We infer from the numerator that $AE = 0$ when $\gamma - \alpha \in \mathbb{Z}$ or $\gamma - \beta \in \mathbb{Z}$. For the second reducibility condition we have;

$$\begin{aligned} 0 &= a_0 a_1 + (\lambda_{0,1} - \lambda_{0,2})(\lambda_{1,1} - \lambda_{1,2}) \\ &= AE(1 - e^{2\pi i(1-\gamma)})(1 - e^{2\pi i(\gamma-\alpha-\beta)}) + (1 - e^{2\pi i(1-\gamma)}(e^{2\pi i(\gamma-\alpha-\beta)} - 1) \\ &= (AE - 1)(1 - e^{2\pi i(1-\gamma)})(1 - e^{2\pi i(\gamma-\alpha-\beta)}) \end{aligned}$$

Again, neither exponent may equal 1 so we require $AE = 1$. For this we first note that using Euler's sine formula, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$;

$$\begin{aligned} AE &= \frac{\sin\pi(\gamma-\alpha)\sin\pi(\gamma-\beta)}{\sin\pi\gamma\sin\pi(\gamma-\alpha-\beta)} \\ &= \frac{(e^{\pi i(\gamma-\alpha)} - e^{\pi i(\alpha-\gamma)})(e^{\pi i(\gamma-\beta)} - e^{\pi i(\beta-\gamma)})}{(e^{\pi i(\gamma-\alpha-\beta)} - e^{\pi i(\alpha+\beta-\gamma)})(e^{\pi i\gamma} - e^{-\pi i\gamma})} \cdot \frac{e^{\pi i(\gamma-\alpha)} e^{\pi i(\beta-\gamma)} e^{\pi i(\gamma-\alpha-\beta)} e^{-\pi i\gamma}}{e^{\pi i(\gamma-\alpha)} e^{\pi i(\beta-\gamma)} e^{\pi i(\gamma-\alpha-\beta)} e^{-\pi i\gamma}} \\ &= \frac{(e^{2\pi i(\gamma-\alpha)} - 1)(1 - e^{2\pi i(\beta-\gamma)})}{(e^{2\pi i(\gamma-\alpha-\beta)} - 1)(1 - e^{-2\pi i\gamma})} \cdot \frac{e^{\pi i(\gamma-\alpha-\beta)} e^{-\pi i\gamma}}{e^{\pi i(\gamma-\alpha)} e^{\pi i(\beta-\gamma)}} \\ &= \frac{(1 - e^{2\pi i(\gamma-\alpha)})(1 - e^{2\pi i(\beta-\gamma)})}{e^{2\pi i\beta}(1 - e^{2\pi i(\gamma-\alpha-\beta)}(1 - e^{-2\pi i\gamma})} \end{aligned}$$

When this equals 1 we have;

$$
\begin{aligned}
(1-e^{2\pi i(\gamma-\alpha)})(1-e^{2\pi i(\beta-\gamma)}) &= e^{2\pi i\beta}(1-e^{2\pi i(\gamma-\alpha-\beta)}-e^{-2\pi i\gamma}+e^{2\pi i(-\alpha-\beta)}) \\
1-e^{2\pi i(\gamma-\alpha)}-e^{2\pi i(\beta-\gamma)}+e^{2\pi i(\beta-\alpha)} &= e^{2\pi i\beta}-e^{2\pi i(\gamma-\alpha)}-e^{2\pi i(\beta-\gamma)}+e^{2\pi i(-\alpha)} \\
1 &= e^{2\pi i\beta}-e^{2\pi i(\beta-\alpha)}+e^{-2\pi i\alpha} \\
0 &= (1-e^{2\pi i\beta})(1-e^{-2\pi i\alpha})
\end{aligned}
$$

From this we see that when $\beta \in \mathbb{Z}$ or $\alpha \in \mathbb{Z}$ the monodromy group is also reducible. We therefore have 4 separate conditions on our parameters which give us a reducible monodromy group. Note: No two of these conditions can occur at once because of our parameter restrictions. For example, if $\gamma-\alpha \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}$, then γ must be an integer which violates our original restriction: $\gamma \notin \mathbb{Z}$. Any other combination of conditions similarly violates the earlier rules.

In each of the latter two cases, we have the g-invariant subspace given by the span of $\begin{bmatrix}1\\c\end{bmatrix}$ where c is found using M_0, M_1 and (5.2);

$$
c=\frac{(e^{2\pi i(1-\gamma)}-1)}{E(1-e^{2\pi i(1-\gamma)}}=-\frac{1}{E} \tag{5.3}
$$

or;

$$
c=\frac{A(1-e^{2\pi i(\gamma-\alpha-\beta)})}{e^{2\pi i(\gamma-\alpha-\beta)}-1)}=-A \tag{5.4}
$$

We are sure this equality holds as these cases were a direct result of when $AE=1$, hence, the subspace generated by the span of these vectors is the same. For the cases when $\alpha-\gamma \in \mathbb{Z}$ and $\beta-\gamma \in \mathbb{Z}$ we note these occur when $AE=0$ or, equivalently, the trivial case $a_0a_1=0$. This means the subspace is not of the form $\begin{bmatrix}1\\c\end{bmatrix}$ but instead the span of either $\begin{bmatrix}0\\c\end{bmatrix}$ or $\begin{bmatrix}c\\0\end{bmatrix}$. Utilising that $AE=0$, for both of these cases we have;

$$
\begin{aligned}
\begin{bmatrix}1 & E(1-e^{2\pi i(1-\gamma)})\\0 & e^{2\pi i(1-\gamma)}\end{bmatrix}\begin{bmatrix}E\\0\end{bmatrix} &= \begin{bmatrix}E\\0\end{bmatrix} \\
\begin{bmatrix}e^{2\pi i(\gamma-\alpha-\beta)} & 0\\A(1-e^{2\pi i(\gamma-\alpha-\beta)}) & 1\end{bmatrix}\begin{bmatrix}E\\0\end{bmatrix} &= \begin{bmatrix}Ee^{2\pi i(\gamma-\alpha-\beta)}\\0\end{bmatrix}
\end{aligned} \tag{5.5}
$$

as well as;

$$
\begin{aligned}
\begin{bmatrix}1 & E(1-e^{2\pi i(1-\gamma)})\\0 & e^{2\pi i(1-\gamma)}\end{bmatrix}\begin{bmatrix}0\\A\end{bmatrix} &= \begin{bmatrix}0\\Ae^{2\pi i(1-\gamma)}\end{bmatrix} \\
\begin{bmatrix}e^{2\pi i(\gamma-\alpha-\beta)} & 0\\A(1-e^{2\pi i(\gamma-\alpha-\beta)}) & 1\end{bmatrix}\begin{bmatrix}0\\A\end{bmatrix} &= \begin{bmatrix}0\\A\end{bmatrix}
\end{aligned} \tag{5.6}
$$

so both of these vectors produce g-invariant subspaces. As gamma functions are not defined for non-positive integers and noting that;

$$
A=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad , \quad E=\frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)\Gamma(\alpha-\gamma+1)}
$$

for $\gamma>\alpha$ or $\gamma>\beta$ respectively we use the vector $\begin{bmatrix}0\\A\end{bmatrix}$ as it is well defined. In the reverse cases that $\gamma \leq \alpha$ or $\gamma \leq \beta$, we use the vector $\begin{bmatrix}E\\0\end{bmatrix}$.

5.2 Factorisation from Reducibility

By adapting [3, p.73] for a second order equation, we have;

Theorem 5.2. *On $D = \mathbb{C} \setminus \{a_0, a_1, ..., a_p\}$, Let L be the operator defined by $L = \frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z)$ and assume that the equation $L[y] = 0$ is Fuchsian with singularities at $\{a_0, a_1, ..., a_p\}$. If the monodromy group G of the equation is reducible, there exists two differential operators*

$$K = \frac{d}{dz} + q_1(z)$$
$$M = \frac{d}{dz} + r_1(z)$$

where $q_1(z)$ and $r_1(z)$ are rational functions such that $L = MK$. Conversely if L can be decomposed into M and K, the monodromy group of $L[y] = 0$ is reducible.

Proof. Let $\mathcal{Y}(z) = (y_1(z), y_2(z))$ be a fundamental set of solutions such that G is the monodromy group with respect to it. By our assumption that G is reducible, there exists a non-trivial, G-invariant subspace $W \subset \mathbb{C}^2$ and as we have deduced, dim(W) = 1. Take $\mathbf{w_1} \in W$ and any vector $\mathbf{v_2} \in \mathbb{C}^2$ so that $span\{\mathbf{w_1}, \mathbf{v_2}\} = \mathbb{C}^2$. By defining the matrix $P = \begin{bmatrix}\mathbf{w_1} \ , \ \mathbf{v_2}\end{bmatrix}$, for any $g \in G$ we have;

$$\begin{aligned} gP &= g\begin{bmatrix}\mathbf{w_1} \ , \ \mathbf{v_2}\end{bmatrix} \\ &= \begin{bmatrix}c_g \cdot \mathbf{w_1} \ , \ \mathbf{v_3}\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{w_1} \ , \ \mathbf{v_2}\end{bmatrix}\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \end{aligned}$$

where the final matrix on the right is unknown but upper-triangular with $c_g \in \mathbb{C} \setminus \{0\}$ for each $g \in G$. Left multiplying both sides by P^{-1} therefore gives;

$$P^{-1}gP = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

As we found earlier, for two fundamental solution sets $\mathcal{Y}(z)$ and $\tilde{\mathcal{Y}}(z)$, we have $\tilde{\rho}(\gamma) = C^{-1}\rho(\gamma)C$ (3.5). Hence, the augmented solution set $\tilde{\mathcal{Y}}(z) = \mathcal{Y}(z)P$ has monodromy group;

$$P^{-1}GP = \{P^{-1}gP \mid g \in G\}$$

which must consist fully of upper-triangular matrices. Hence, for any $\gamma \in \pi_1(D, b)$;

$$\gamma_*\tilde{\mathcal{Y}}(z) = \tilde{\mathcal{Y}}(z)\begin{bmatrix} m_\gamma^{11} & * \\ 0 & * \end{bmatrix}$$

where $m_\gamma^{11} \in \mathbb{C} \setminus \{0\}$. Therefore, $\gamma_*\tilde{y}_1(z) = \tilde{y}_1(z)m_\gamma^{11}$ and similarly we must have $\gamma_*\tilde{y}_1'(z) = \tilde{y}_1'(z)m_\gamma^{11}$ as analytic continuation is invariant for derivatives. In turn we have that for $q_1(z) = -\frac{\tilde{y}_1'(z)}{\tilde{y}_1(z)}$;

$$\gamma_*q_1(z) = q_1(z)$$

Therefore, $q_1(z)$ is single valued on $D \setminus N$ where N is the set of zeroes of $\tilde{y}_1(z)$. Moreover, since $q_1(z)$ is a ratio of differentiable polynomial solutions, $q_1(z)$ is a rational function and at most regular singular at $a_0, a_1, ..., a_p$.

Now, define the operator $K = \frac{d}{dz} + q_1(z)$. Automatically, $\tilde{y}_1(z)$ is a fundamental solution to $K[y] = 0$. For our second operator, let;

$$\begin{aligned}
L_1 &= L - \frac{d}{dz}K \\
&= \frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z) - \frac{d}{dz}\left(\frac{d}{dz} + q_1(z)\right) \\
&= \frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z) - \frac{d^2}{dz^2} - q_1(z)\frac{d}{dz} - \frac{d}{dz}q_1(z) \\
&= (p_1(z) - q_1(z))\frac{d}{dz} + \left(p_2(z) - \frac{d}{dz}q_1(z)\right)
\end{aligned}$$

Ignoring the latter terms, we will keep only the derivative coefficient, $p_1(z) - q_1(z)$. Much like the previous step, we define a new operator R by;

$$R = L - \left(\frac{d}{dz} + p_1(z) - q_1(z)\right)K$$

Since $\tilde{y}_1(z)$ solves both $L[\tilde{y}_1] = 0$ and $K[\tilde{y}_1] = 0$, we must also have $R[\tilde{y}_1] = 0$. As in each step we have lowered the order of the operator, it must be that $R = 0$. Therefore by setting $M = \frac{d}{dz} + p_1(z) - q_1(z)$ we arrive at $L - MK = 0$ which completes our forward assertion.

We will conversely show that a global factorisation existing implies the monodromy group is reducible. Suppose we have the global factorisation;

$$\begin{aligned}
L &= MK \\
\frac{d^2}{dz^2} + p_1(z)\frac{d}{dz} + p_2(z) &= \left(\frac{d}{dz} + q_1(z)\right)\left(\frac{d}{dz} + r_1(z)\right)
\end{aligned}$$

Then, expanding the operators we must have that;

$$p_1(z) = q_1(z) + r_1(z)$$

as well as;

$$p_2(z) = q_1'(z) + q_1(z)r_1(z)$$

Rearranging the first condition for $r_1(z)$ and substituting into the second gives the Ricatti equation in q_1;

$$p_2(z) = q_1'(z) + p_1(z)q_1(z) - q_1(z)^2$$

Now taking the change of variable, $q_1(z) = -\frac{u'(z)}{u(z)}$, our equation becomes;

$$p_2(z) = -\frac{u''(z)}{u(z)} + \left(\frac{u'(z)}{u(z)}\right)^2 - \frac{u'(z)p_1(z)}{u(z)} - \left(\frac{u'(z)}{u(z)}\right)^2$$

The squared terms cancel out, so after multiplying by $u(z)$ and rearranging, we have;

$$u''(z) + p_1(z)u'(z) + p_2(z)u(z) = 0$$

Hence, we know $q_1(z) = -\frac{u'(z)}{u(z)}$ where $u(z)$ solves the original equation. Now suppose the monodromy group is irreducible. This means no proper, non-trivial subspace W exists such that $gW \subset W$ for all $g \in G$. As by the assumption of a global factorisation, $q_1(z)$ is single-valued and rational in D, we must have;

$$\gamma_{j_*} q_1(z) = q_1(z)$$

Furthermore;

$$\gamma_{g*}u(z) = c_g \cdot u(z)$$

for $c_g \in \mathbb{C} \setminus \{0\}$ with the same result for $u'(z)$. As $u(z)$ is one of two fundamental solutions, letting $v(z)$ be the other, we then have;

$$\gamma_{j*}\left[u(z)\ ,\ v(z)\right] = \left[u(z)\ ,\ v(z)\right] \begin{bmatrix} c_g & * \\ 0 & * \end{bmatrix}$$

so our monodromy group is exclusively upper triangular matrices. Therefore, letting $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $g\mathbf{w} \subset span\{\mathbf{w}\}$ for all $g \in G$. Lastly letting $W = span\{\mathbf{w}\}$, W is a proper, non-linear, monodromy-invariant subspace meaning the group is reducible. This contradiction completes the proof. □

5.3 Finding and Solving Factorisations

When our monodromy group is reducible, we have shown a factorisation of the hypergeometric equation exists defined by;

$$\begin{aligned} L &= MK \\ &= \left(\frac{\gamma - (\alpha+\beta+1)z}{z(1-z)} + \frac{y_1'(z)}{y_1(z)}\right)\left(\frac{d}{dz} - \frac{y_1'(z)}{y_1(z)}\right) \end{aligned}$$

for some $y_1(z)$ which solves the equation. To solve the factorised form $MK[y] = 0$, we use its equivalence to $M[K[y]]$. By construction, $K[y_1] = 0$ so we must also have $MK[y_1] = 0$. For our second solution, if we have $u(z)$ such that $M[u] = 0$, then a function y_2 such that $K[y_2] = u$ solves the factorisation. We will consider these for each reducibility case.

5.3.1 Case 1: $\gamma - \alpha \in \mathbb{Z}$ or $\gamma - \beta \in \mathbb{Z}$ and $\gamma > \alpha$ or $\gamma > \beta$

In this case, we showed in (5.6) that our G-invariant subspace is given by $W = span\{\begin{bmatrix} 0 \\ A \end{bmatrix}\}$. By adding the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ such that these two vectors now span $\mathbb{C}^2$, we find the matrix $P = \begin{bmatrix} 0 & 1 \\ A & 0 \end{bmatrix}$. This augments our fundamental set of solutions to $\begin{bmatrix} Ay_{1,1}(z) & y_{0,1}(z) \end{bmatrix}$ and therefore, following the theorem, we find $q_1(z) = \frac{Ay_{1,1}'(z)}{Ay_{1,1}(z)}$. Hence, we have the first factorised operator;

$$K = \frac{d}{dz} - \frac{y_{1,1}'(z)}{y_{1,1}(z)}$$

As suggested, we clearly have $K[y_{1,1}] = 0$ so this is the first solution to our factorised problem. From the theorem we also find the second operator as;

$$\begin{aligned} M &= \frac{d}{dz} + p_1(z) - q_1(z) \\ &= \frac{d}{dz} + \frac{\gamma - (\alpha+\beta+1)z}{z(1-z)} + \frac{y_{1,1}'(z)}{y_{1,1}(z)} \end{aligned}$$

We therefore conclude in this case, the hypergeometric equation may be factorised as;

$$\left(\frac{d}{dz} + \frac{(\gamma - (\alpha+\beta+1)z)}{z(1-z)} + \frac{y_{1,1}'(z)}{y_{1,1}(z)}\right)\left(\frac{d}{dz} - \frac{y_{1,1}'(z)}{y_{1,1}(z)}\right) y = 0$$

As expected all of the functions are rational and single-valued due to the upper triangular monodromy group. For our second solution, we first solve $M[u]=0$ or;

$$\frac{du}{dz}+\left(\frac{\gamma-(\alpha+\beta+1)z}{z(1-z)}+\frac{y'_{1,1}(z)}{y_{1,1}(z)}\right)u(z)=0$$

Using the integrating factor $\mu(z)$;

$$\begin{aligned}\mu(z)&=\exp\left(\int\frac{\gamma-(\alpha+\beta+1)z}{z(1-z)}+\frac{y'_{1,1}(z)}{y_{1,1}(z)}dz\right)\\&=\exp\left(\int\frac{\gamma}{z}+\frac{\gamma-\alpha-\beta-1}{1-z}+\frac{d}{dz}\ln(y_{1,1}(z))dz\right)\\&=\exp\left(\gamma\ln(z)+(\alpha+\beta+1-\gamma)\ln(1-z)+\ln(y_{1,1}(z))\right)\\&=z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}y_{1,1}(z)\end{aligned}$$

We then have;

$$\frac{d}{dz}\left(z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}y_{1,1}(z)u(z)\right)=0$$

So finally for some constant $C\in\mathbb{C}$;

$$u(z)=\frac{C}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}y_{1,1}(z)}$$

Thus our second solution comes from solving;

$$\frac{dy}{dz}-\frac{y'_{1,1}(z)}{y_{1,1}(z)}y=\frac{C}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}y_{1,1}(z)}$$

Again using an integrating factor;

$$\begin{aligned}\frac{d}{dz}\left(\frac{y}{y_{1,1}(z)}\right)&=\frac{C}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}(y_{1,1}(z))^2}\\y=y_{1,1(z)}&\int^{z}\frac{C}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}(y_{1,1}(z))^2}dz\end{aligned}$$

Hence, our fundamental solution set with these conditions is;

$$\begin{bmatrix}y_{1,1}(z) & y_{1,1}(z)\int^{z}\frac{1}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}(y_{1,1}(z))^2}dz\end{bmatrix}$$

where $y_{1,1}$ is as defined in (4.6). When using the first integrating factor, note we are solving;

$$u'+(p_1(z)+y_{1,1}(z))u(z)=0 \tag{5.7}$$

which is very close to Abel's theorem for finding the Wronskian. Checking [9][p.395], we see that;

$$\frac{C}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}} \tag{5.8}$$

is in fact the Wronskian of different fundamental solution pairs depending on the value of C. We can consider the local monodromy of these solutions. For example looping around $z=1$, we must have;

$$\gamma_1:\begin{bmatrix}y_{1,1}(z) & y_{1,1}(z)\int^{z}\frac{1}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}(y_{1,1}(z))^2}dz\end{bmatrix}\mapsto\begin{bmatrix}1&0\\0&*\end{bmatrix}\begin{bmatrix}y_{1,1}(z) & y_{1,1}(z)\int^{z}\frac{1}{z^{\gamma}(1-z)^{\alpha+\beta+1-\gamma}(y_{1,1}(z))^2}dz\end{bmatrix}$$

Under the condition $\gamma-\alpha-\beta\notin\mathbb{Z}$, the powers of z and $(1-z)$ do not have integer difference so no logarithmic term will appear in the integration and hence our monodromy matrix is 0 on the off-diagonals.

5.3.2 Case 2: $\gamma-\alpha \in \mathbb{Z}$ or $\gamma-\beta \in \mathbb{Z}$ and $\gamma \leq \alpha$ or $\gamma \leq \beta$

Similarly in this case we showed in (5.5) that an invariant subspace is given by $W = span\{\begin{bmatrix} E \\ 0 \end{bmatrix}\}$. Hence, by the same process as before we have the analogous factorisation;

$$\left(\frac{d}{dz} + \frac{(\gamma-(\alpha+\beta+1)z)}{z(1-z)} + \frac{y_{0,1}'(z)}{y_{0,1}(z)}\right)\left(\frac{d}{dz} - \frac{y_{0,1}'(z)}{y_{0,1}(z)}\right) y = 0$$

with fundamental solutions;

$$\begin{bmatrix} y_{0,1}(z) & y_{0,1}(z)\int^z \frac{1}{z^\gamma(1-z)^{\alpha+\beta+1-\gamma}(y_{0,1}(z))^2}dz \end{bmatrix}$$

5.3.3 Case 3: $\alpha \in \mathbb{Z}$ or $\beta \in \mathbb{Z}$

Finally, under these conditions, in (5.4) we found the subspace given by $W = span\{\begin{bmatrix} 1 \\ -A \end{bmatrix}\}$ so after adding the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, get the augmented solution set; $\begin{bmatrix} y_{0,1}(z) - Ay_{1,1}(z) & y_{0,1}(z) \end{bmatrix}$. This gives the factorisation:

$$\left(\frac{d}{dz} + \frac{\gamma-(\alpha+\beta+1)z}{z(1-z)} + \frac{y_{0,1}'(z) - Ay_{1,1}'(z)}{y_{0,1}(z) - Ay_{1,1}(z)}\right)\left(\frac{d}{dz} - \frac{y_{0,1}'(z) - Ay_{1,1}'(z)}{y_{0,1}(z) - Ay_{1,1}(z)}\right) y = 0$$

with general solution;

$$y = y_{0,1}(z) - Ay_{1,1}(z) + (y_{0,1}(z) - Ay_{1,1}(z))\int^z \frac{1}{z^\gamma(1-z)^{\alpha+\beta+1-\gamma}(y_{0,1}(z) - Ay_{1,1}(z))^2}dz$$

This may seem unlikely as a single valued function but we show in A the validity of this results.

6 Conclusion

We have shown that in each scenario where the monodromy group of the hypergeometric equation is reducible; $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}$, $\gamma-\alpha \in \mathbb{Z}$ or $\gamma-\beta \in \mathbb{Z}$, the equation may be factorised into first order components and solved. The factorisations given are not a complete list but rather a representative for each of these cases. For example, factorisations could be found in the case that $y_{0,2}$ and $y_{1,2}$ were the chosen fundamental solutions. Furthermore, we have not verified if these reducibility conditions hold, and hence factorisations exist, in the cases when γ or $\gamma-\alpha-\beta \in \mathbb{Z}$. This problem could be investigated using the alternative solutions given in [10].

By factorising the equation in terms of its own solutions, this method does not help in finding solutions to Fuchsian equations. Perhaps, however, interesting results could be yielded from the factorised monodromy group. In this case, it would be ideal to solve the integrals for closed form solutions so that the group could be explicitly calculated and investigated. Alternatively, it may be possible for the monodromy group of the factorised solutions to be calculated without solving the integral, since integral functions may be analytically continued as shown in [3][p.257-262] and [13]. We leave these as open problems.

A Appendix

The upper triangular monodromy group for Case 3: $\alpha \in \mathbb{Z}$ or $\beta \in \mathbb{Z}$.

$$P = \begin{bmatrix} 1 & 1 \\ -A & 0 \end{bmatrix} \to P^{-1} = \begin{bmatrix} 0 & -\frac{-1}{A} \\ 1 & \frac{1}{A} \end{bmatrix}$$

$$\begin{aligned}
\tilde{M}_0 = P^{-1} M_0 P &= \begin{bmatrix} 0 & -\frac{-1}{A} \\ 1 & \frac{1}{A} \end{bmatrix} \begin{bmatrix} 1 & E(1 - e^{2\pi i(1-\gamma)}) \\ 0 & e^{2\pi i(1-\gamma)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -A & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{e^{2\pi i(1-\gamma)}}{A} \\ 1 & E(1 - e^{2\pi i(1-\gamma)}) + \frac{e^{2\pi i(1-\gamma)}}{A} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -A & 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{2\pi i(1-\gamma)} & 0 \\ 1 - AE(1 - e^{2\pi i(1-\gamma)}) - e^{2\pi i(1-\gamma)} & 1 \end{bmatrix}
\end{aligned}$$

Recalling that in case 3, AE = 1 we have

$$\tilde{M}_0 = \begin{bmatrix} e^{2\pi i(1-\gamma)} & 0 \\ 0 & 1 \end{bmatrix}$$

Likewise,

$$\begin{aligned}
\tilde{M}_1 = P^{-1} M_1 P &= \begin{bmatrix} 0 & -\frac{-1}{A} \\ 1 & \frac{1}{A} \end{bmatrix} \begin{bmatrix} e^{2\pi i(\gamma-\alpha-\beta)} & 0 \\ A(1 - e^{2\pi i(\gamma-\alpha-\beta)}) & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -A & 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{2\pi i(\gamma-\alpha-\beta)} - 1 & -\frac{1}{A} \\ 1 & \frac{1}{A} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -A & 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{2\pi i(\gamma-\alpha-\beta)} & e^{2\pi i(\gamma-\alpha-\beta)} - 1 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Lastly,

$$\begin{aligned}
\tilde{M}_\infty = (\tilde{M}_0 \tilde{M}_1)^{-1} &= \left(\begin{bmatrix} e^{2\pi i(1-\gamma)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2\pi i(\gamma-\alpha-\beta)} & e^{2\pi i(\gamma-\alpha-\beta)} - 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \left(\begin{bmatrix} e^{2\pi i(1-\alpha-\beta)} & e^{2\pi i(1-\alpha-\beta)} - 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} e^{2\pi i(\alpha+\beta-1)} & e^{2\pi i(\alpha+\beta-1)} - 1 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

So the new monodromy group is upper triangular.

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