401 HW 1

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1.1 Calculus, Taylor series

Consider the function $f(x) = \frac{\sin(x)}{x}$.

1.1.1

Compute the limit $\lim_{x \to 0} f(x)$ using l'Hopital's rule.

l'Hopital's rule states that when $\lim_{x \to c} a(x) = \lim_{x \to c} b(x) = 0$ or $\pm\infty$:

$$\lim_{x \to c} \frac{a(x)}{b(x)} = \lim_{x \to c} \frac{a'(x)}{b'(x)}$$

Therefore since the preconditions are met:

$$\lim_{x \to 0} \sin(x) = 0$$

$$\lim_{x \to 0} x = 0$$

We can apply l'Hopital's rule and get:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1}$$

Since:

$$(\sin(x))' = \cos(x)$$

$$(x)' = 1$$

Evaluating we get:

$$\lim_{x \to 0} \frac{\cos(x)}{1} = \lim_{x \to 0} \cos(x) = 1$$

Therefore:

$$\lim_{x \to 0} f(x) = 1$$

1.1.2

Use Taylor's remainder theorem to get the same result:

a) Write down $P_1(x)$, the first-order Taylor polynomial for $\sin(x)$ centered at $a = 0$.

By definition, a first order Taylor polynomial for a function $f(x)$ centered at a takes the form:

$$P_1(x) = f(a) + f'(a)(x - a)$$

So we get that for $\sin(x)$ at $a = 0$:

$$\begin{aligned} P_1(x) &= \sin(0) + \cos(0)(x - 0) \\ &= 0 + 1x \\ &= x \end{aligned}$$

Therefore:

$$P_1(x) = x$$

b) Write down a good upper bound on the absolute value of the remainder $R_1(x) = \sin(x) - P_1(x)$, using your knowledge about the derivatives of $sin(x)$. The goal here is to show that $R_1(x)/x$ is negligible.

By definition, the remainder is:

$$R_n(x) = \frac{f^{n+1}(a)}{(n+1)!}(x - a)^{n+1}$$

Since in this case $n = 1$, $a = 0$, and $f(x) = \sin(x)$ we get:

$$\begin{aligned} R_1(x) &= \frac{f''(0)}{2!}x^2 \\ &= \frac{-\sin(0)}{2}x^2 \\ &= \frac{0}{2}x^2 \\ &= 0 \end{aligned}$$

Thus, the upper bound of the remainder is 0, and we have shown that $R_1(x)/x$ is negligible since:

$$0/x = 0 \text{ when } x \to 0$$

c) Express $f(x)$ as $f(x) = \frac{P_1(x)}{x} + \frac{R_1(x)}{x}$, and compute the limits of the two terms as $x \to 0$.

By our previous calculations of $P_1(x)$ and $R_1(x)$ we get:

$$\frac{\sin(x)}{x} = \frac{x}{x} + \frac{0}{x}$$

Now applying limit rules and taking the limit of both sides we get:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \left(\frac{x}{x} + \frac{0}{x}\right) \\ &= \lim_{x\to 0} \frac{x}{x} + \lim_{x\to 0} \frac{0}{x} \\ &= 1 + 0 \\ &= 1\end{aligned}$$

Thus:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Which is the same result as derived by using l'Hopital's rule.

1.2 Asymptotic notation

Recall the definitions of the asymptotic notations. We will say that $f(x)$ has "order of growth x^α as $x \to x_0$" (where x_0 is either some fixed real number or $\pm\infty$) if $f(x) = \Theta(x^\alpha)$ as $x \to x_0$.

1.2.1

Consider the functions $f(x) = x\sin(x)$ and $g(x) = x$. Is $f(x) = \Theta(g(x))$ as $x \to \infty$? Why or why not? (Hint: As always, you should refer back carefully to the definition of $\Theta(\cdot)$.)

By definition of Big Theta, $f(x) = \Theta(g(x))$ iff:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c \text{ and } 0 < c < \infty$$

In our case we have $f(x) = x\sin(x)$ and $g(x) = x$ so:

$$\lim_{x\to\infty} \frac{x\sin(x)}{x}$$

$$\lim_{x\to\infty} \sin(x)$$

But we know that:

$$\lim_{x\to\infty} \sin(x) = \text{DNE}$$

and since its DNE, no c exists between 0 and ∞. Therefore $f(x)$ is not $\Theta(g(x))$

1.2.2

Suppose that we know that $f(x) = x + \Theta(x^2)$ and $g(x) = \Theta(x) > 0$ as $x \to 0$. Determine the order of growth of $f(x) + g(x)$.
(This problem is meant to get you comfortable with manipulating asymptotic notation when it appears in expressions. When I say something like "$f(x) = x + \Theta(x^2)$", this means that there is some function $h(x) = \Theta(x^2)$, and $f(x) = x + h(x)$. That is, the fact that $h(x) = \Theta(x^2)$ is the only thing you know about $h(x)$.)

First expanding $f(x) + g(x)$ we see:

$$f(x) + g(x) = x + \Theta(x^2) + g(x)$$

We know that this means there exists some function $h(x) = \Theta(x^2)$. As $x \to 0$ the $g(x)$ term which is $\Theta(x)$ dominates the unknown $h(x)$ term which is $\Theta(x^2)$, meaning the order of growth of $f(x) + g(x)$ as $x \to 0$ will be $\Theta(x)$.

1.2.3

Suppose that we know that $f(x) = e^{\Theta(x)}$ as $x \to \infty$. Does this imply that $f(x) = \Theta(e^x)$? (Hint: Think carefully about the definition of $\Theta(\cdot)$, and consider $f(x) = e^{2x}$.)

By the definition of $\Theta(\cdot)$, we see that $f(x) = e^{\Theta(x)}$ implies there exists constants c_1 and c_2 such that:

$$e^{c_1 \cdot x} \leq f(x) \leq e^{c_2 \cdot x}$$

we see that for any c_1 and c_2, $f(x)$ is still bounded by $\Theta(e^x)$ on both sides, therefore $f(x) = \Theta(e^x)$

1.3 Relative versus absolute error

1.3.1

Suppose that you are approximating a function $g(n)$ by some function $f(n)$. Suppose, further, that you know that the absolute error in approximating $g(n)$ by $f(n)$ satisfies $|f(n) - g(n)| = o(1)$ as $n \to \infty$ (that is, $\lim_{n\to\infty} |f(n) - g(n)| = 0$). Is it true that the relative error also decays to 0? If not, come up with functions $f(n)$ and $g(n)$ for which this is not true. (Hint: Come up with some $g(n)$ and $f(n)$ satisfying $g(n) = o(1)$ and $f(n)/g(n) = \Theta(1)$.)

Let $f(x) = \frac{1}{n}$ and let $g(x) = \frac{1}{3n}$ both are $o(1)$. These functions satisfy the constraint of having absolute error approach 0 when approximating $g(x)$ with $f(x)$:

$$\lim_{n\to\infty} |f(n) - g(n)| = \lim_{n\to\infty} \left| \frac{1}{n} - \frac{1}{3n} \right| = \lim_{n\to\infty} \frac{2}{3n} = 0$$

However, when computing the relative error we find:

$$\frac{\frac{2}{3n}}{\frac{1}{3n}} = 2$$

and since:

$$\lim_{n\to\infty} 2 = 2$$

we see that the relative error doesn't decay to 0, but rather stays constant.

1.4 Matlab warmup/Gentle linear algebra review

1.4.1 Exercise 2

See the attached diary file for MATLAB results

$A = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a)

$$v^T w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \end{pmatrix}$$

b)

$$vw^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 & 1 \cdot 1 \\ 2 \cdot 1 & 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

c)

$$Av = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

d)

$$A^T v = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 1 \end{pmatrix}$$

e)

$$AB = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 13 & -6 \\ 2 & 4 \end{pmatrix}$$

f)

$$BA = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 10 & -3 \\ -2 & 7 \end{pmatrix}$$

g)

$$A^2 = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 88 & -36 \\ 48 & -8 \end{pmatrix}$$

h) The vector y for which $By = w$

$$y = B^{-1}w = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i) The vector x for which $Ax = v$

$$x = A^{-1}v = \begin{pmatrix} 10 & -3 \\ 4 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{16} & \frac{3}{32} \\ -\frac{1}{8} & \frac{5}{16} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$$

1.4.2 Exercise 3