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AN A POSTERIORI ERROR ESTIMATOR FOR FINITE ELEMENT APPROXIMATIONS OF COULOMB'S FRICTION PROBLEM

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ABSTRACT

This paper deals with an adaptative strategy for F.E. computations of the frictional unilateral contact problem governed by Coulomb's law. This strategy uses an *a posteriori* error estimator which is based on constitutive relation residuals to quantify the accuracy of a finite element approximation of the problem. The numerical implementation of the error estimator and the procedure of adaptivity are carried out.

KEYWORDS

Coulomb friction, unilateral contact, elasticity, *a posteriori* estimator, error in the constitutive relation.

INTRODUCTION

Very often, the finite element method is chosen in the numerical simulation of complex problems in engineering, including frictional contact problems. In these computations, an important and interesting study consists of evaluating the quality of the finite element solution, which is performed by the (*a posteriori*) error estimators. Several approaches exist concerning estimators (see in particular [5, 1, 6]): our aim is to use an *a posteriori* error estimator based on the constitutive relation residuals to measure the quality of the finite element computations of frictional contact problems. The use of such an estimator allows to build an optimized mesh furnishing to the user a chosen quality at lowest computational costs.

THE COULOMB FRICTION PROBLEM

We consider an elastic body occupying a domain Ω in $\mathbb{R}^2$. The boundary Γ of Ω is assumed to be Lipschitz and it is divided as follows: $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$ where Γ_D, Γ_N and Γ_C are three open disjoint parts and $\mathrm{meas}(\Gamma_D) > 0$. We assume that the displacement field is given on Γ_D (i.e. $\boldsymbol{u} = \boldsymbol{U}$) and that the boundary part Γ_N is submitted to a density of surface forces denoted $\boldsymbol{F}$. The third part is Γ_C, which comprises all the points candidate to be in frictional contact with a rigid foundation (see Figure 1). The body Ω is submitted to a given density of volume forces $\boldsymbol{f}$. Let the notation $\boldsymbol{n} = (n_1, n_2)$ represent the unit outward normal vector on Γ and define the unit tangent vector $\boldsymbol{t} = (-n_2, n_1)$. We denote by $\mu > 0$ the friction coefficient on Γ_C.

The Coulomb frictional unilateral contact problem consists of finding the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ and the stress tensor field $\boldsymbol{\sigma}(\boldsymbol{u}) : \Omega \to \mathscr{S}_2$ satisfying (1)–(8):

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{div}\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} \quad \text{on } \Gamma_N, \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{U} \quad \text{on } \Gamma_D, \tag{4}$$

where $\mathscr{S}_2$ stands for the space of second order symmetric tensors on $\mathbb{R}^2$, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\nabla u} + \boldsymbol{\nabla}^T\boldsymbol{u})/2$ denotes the linearized strain tensor field, $\boldsymbol{C}$ is a fourth order symmetric and elliptic tensor of linear elasticity and $\boldsymbol{div}$ represents the divergence operator of tensor valued functions.
In order to introduce the equations on Γ_C, let us adopt the following notation: $\boldsymbol{u} = u_n\boldsymbol{n} + u_t\boldsymbol{t}$ and $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \sigma_t(\boldsymbol{u})\boldsymbol{t}$. The equations modelling contact and friction are as follows on Γ_C:

$$u_n \leq 0, \tag{5}$$

$$\sigma_n(\boldsymbol{u}) \leq 0, \tag{6}$$

$$\sigma_n(\boldsymbol{u})\, u_n = 0, \tag{7}$$

$$\begin{cases} u_t = 0 \implies |\sigma_t(\boldsymbol{u})| \leq \mu|\sigma_n(\boldsymbol{u})|, \\ u_t \neq 0 \implies \sigma_t(\boldsymbol{u}) = -\mu|\sigma_n(\boldsymbol{u})|\dfrac{u_t}{|u_t|}. \end{cases} \tag{8}$$

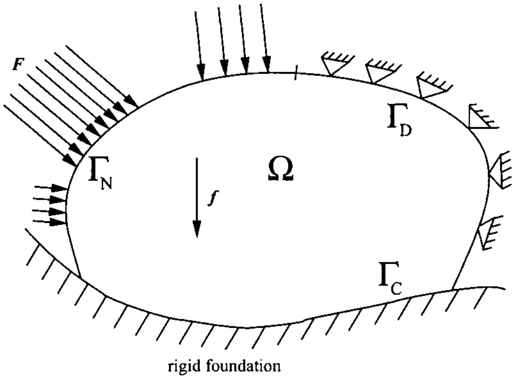

Figure 1: Setting of the problem

THE CONVENIENT SETTING

In order to introduce the error in the constitutive relation, the contact part Γ_C is considered like in [4, 3, 2] as an interface on which two unknowns $\boldsymbol{w}$ (displacement field) and $\boldsymbol{r}$ (density of surfacic forces due to the frictional contact with the rigid foundation) are to be determined. Let $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{t} = (-n_2, n_1)$ denote the unit outward normal and tangent on Γ. Next, we choose the notation $\boldsymbol{z} = z_n\boldsymbol{n} + z_t\boldsymbol{t}$ for any vector $\boldsymbol{z}$.
The unilateral contact problem with Coulomb's friction law (1)–(8) is rewritten by using these quantities (see [2]) and it consists of finding the displacement field $\boldsymbol{u}$ on Ω, the stress tensor field $\boldsymbol{\sigma}$ on Ω and $\boldsymbol{w}$, $\boldsymbol{r}$ on Γ_C satisfying the following equations (9)–(15).

- The displacement fields $\boldsymbol{u}$ and $\boldsymbol{w}$ verify the kinematic conditions:

$$\boldsymbol{u} = \boldsymbol{U} \text{ on } \Gamma_D \quad \text{ and } \quad \boldsymbol{w} = \boldsymbol{u} \text{ on } \Gamma_C. \tag{9}$$

- The fields $\boldsymbol{\sigma}$ and $\boldsymbol{r}$ satisfy the equilibrium equation:

$$-\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\Omega + \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, d\Omega + \int_{\Gamma_N} \boldsymbol{F}.\boldsymbol{v}\, d\Gamma + \int_{\Gamma_C} \boldsymbol{r}.\boldsymbol{v}\, d\Gamma = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{10}$$

- The fields $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are linked by the constitutive law of linear elasticity:

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{11}$$

- The displacement field $\boldsymbol{w} = w_n\boldsymbol{n} + w_t\boldsymbol{t}$ and the density of forces $\boldsymbol{r} = r_n\boldsymbol{n} + r_t\boldsymbol{t}$ satisfy the unilateral contact conditions with Coulomb's friction law along Γ_C:

$$w_n \le 0, \tag{12}$$

$$r_n \le 0, \tag{13}$$

$$r_n w_n = 0, \tag{14}$$

$$\begin{cases} w_t = 0 \implies |r_t| \le \mu |r_n|, \\ w_t \neq 0 \implies r_t = -\mu |r_n| \dfrac{w_t}{|w_t|}. \end{cases} \tag{15}$$

Define

$$K = \Big\{ \boldsymbol{z}; \boldsymbol{z} = z_n\boldsymbol{n} + z_t\boldsymbol{t} \text{ such that } z_n \le 0 \Big\},$$

and

$$C_\mu = \Big\{ \boldsymbol{s}; \boldsymbol{s} = s_n\boldsymbol{n} + s_t\boldsymbol{t} \text{ such that } s_n \le 0 \text{ and } |s_t| \le \mu |s_n| \Big\}.$$

If I_A stands for the indicator function of the set A, it is straightforward that the frictional contact conditions (12)–(15) are equivalent to

$$I_K(\boldsymbol{w}) + I_{C_\mu}(\boldsymbol{r}) + \mu|r_n||w_t| + r_t w_t + r_n w_n = 0, \qquad \text{on } \Gamma_C.$$

ADMISSIBLE SOLUTIONS

The notion of error in the constitutive law is based on a splitting the equations of the problem into two groups with different mechanical status. The first group involves "safe" equations like kinematic and static constraints whereas the second group involves less reliable equations : constitutive relation related equations. A solution $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ is admissible if the kinematic conditions (9) and the equilibrium equations (10) are fulfilled.

DEFINITION OF THE ERROR ESTIMATOR AND RELATED QUANTITIES

The error measures the non-satisfaction of the constitutive relation in Ω and on Γ_C by an admissible solution. Let $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ be admissible. The absolute error $e(\hat{s})$ is (see [2]):

$$e(\hat{s}) = \left(\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|^2_{\sigma,\Omega} + 2\int_{\Gamma_C} \left(I_K(\hat{\boldsymbol{w}}) + I_{C_\mu}(\hat{\boldsymbol{r}}) + \mu|\hat{r}_n||\hat{w}_t| + \hat{r}_t\hat{w}_t + \hat{r}_n\hat{w}_n \right) d\Gamma \right)^{\frac{1}{2}}, \tag{16}$$

where the norm $\|.\|_{\sigma,\Omega}$ on the stress tensor fields is defined by

$$\|\boldsymbol{\sigma}\|_{\sigma,\Omega} = \left(\int_\Omega (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\sigma}\, d\Omega \right)^{\frac{1}{2}}.$$

It can be easily checked that the function in the integral term of (16) is always nonnegative and that $e(\hat{s}) = 0$ if and only if $\hat{s} = ((\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{r}}))$ is solution to the reference problem (9)–(15).

We next define some useful quantities for the adaptative strategy. Let $\hat{s}$ be admissible. The relative error $\epsilon(\hat{s})$ is:

$$\epsilon(\hat{s}) = \frac{e(\hat{s})}{\|\hat{\boldsymbol{\sigma}} + \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|_{\sigma,\Omega}}. \tag{17}$$

If E denotes a part of Ω (typically an element), the local error contribution $\epsilon_E(\hat{s})$ is defined as

$$\epsilon_E(\hat{s}) = \frac{\left(\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|^2_{\sigma,E} + 2\int_{\Gamma_C \cap E} \left(I_K(\hat{\boldsymbol{w}}) + I_{C_\mu}(\hat{\boldsymbol{r}}) + \mu|\hat{r}_n||\hat{w}_t| + \hat{r}_t\hat{w}_t + \hat{r}_n\hat{w}_n \right) d\Gamma \right)^{\frac{1}{2}}}{\|\hat{\boldsymbol{\sigma}} + \mathcal{C}\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})\|_{\sigma,\Omega}.} \tag{18}$$

where $\|\boldsymbol{\sigma}\|_{\sigma,E} = \left(\int_E (\mathcal{C}^{-1}\boldsymbol{\sigma}) : \boldsymbol{\sigma}\, d\Omega \right)^{\frac{1}{2}}$.

Finally, we can prove that a link exists between the estimator and the others errors (see [2]).

NUMERICAL EXAMPLE

The aim of adaptive procedures is to offer the user a level of accuracy denoted ϵ_0 with a minimal computational cost. We use the h-version which is the most widespread procedure of adaptivity currently in use: the size and the topology of the elements are modified but the same kind of basis functions for the different meshes are retained. A mesh T^* is said to be optimal with respect to a measure of the error ϵ^* if :

$$\begin{cases} \epsilon^* = \epsilon_0 \\ N^*\text{minimal } (N^*\text{: number of elements of } T^*) \end{cases}$$

To solve the problem , the following procedure is applied. First, an initial analysis is performed on a relatively uniform and coarse mesh T. Next, the corresponding relative error ϵ (17) and the local contributions ϵ_E in (18) are computed. Then, the characteristics of the optimal mesh T^* are determined such as the computational costs are minimized in respect to the global error. Finally, a second finite element analysis is performed on the mesh T^* determined by the computation of a size modification coefficient r_E on each element E of the mesh T:

$$r_E = \frac{h^*_E}{h_E},$$

where h_E denotes the size of E and h_E^* represents the size that must be imposed to the elements of T^* in the region of E in order to ensure optimality. The major task is to built an admissible solution. The procedure to do that is detailled in [2].

As an application of the above concepts, we consider the structure depicted in Figure 2. The dimensions of the rectangle are 2m $\times$7m, and we choose a friction coefficient of 0.3 on Γ_C. The initial mesh is made of 490 three-node elements and 331 nodes corresponding to an accuracy ϵ of 28.90% (Fig. 3). The prescribed accuracy ϵ_0 is 10%. The optimized mesh comprising 2964 three-node elements and 1769 nodes for an accuracy ϵ of 11.78% is represented on Figure 4.

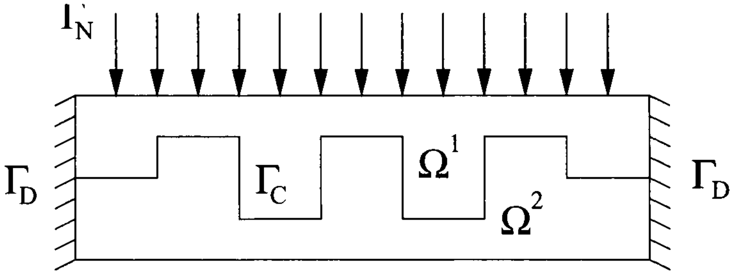

Figure 2: Mechanical problem

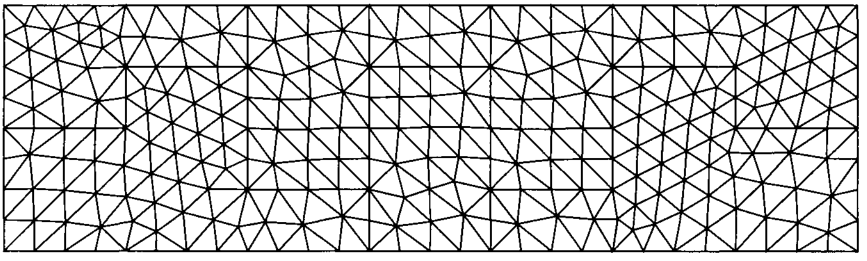

Figure 3: Initial mesh: 490 three-node elements, 331 nodes, ϵ=28.90%

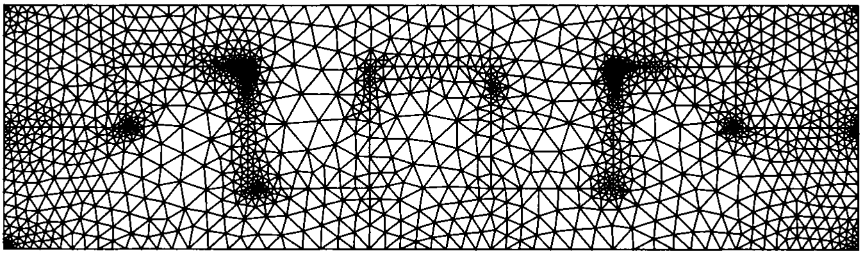

Figure 4: Optimized mesh: 2964 three-node elements, 1769 nodes, ϵ=11.78%

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