The momentum of light in media: the Abraham-Minkowski controversy

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January 2005

Abstract

The controversy arises due to Abraham's and Minkowski's calculations disagreeing as to whether the momentum carried by an electromagnetic field is increased or decreased by the presence of a refractive medium. This paper starts by providing an overview of the life of Abraham and Minkowski. The opinions on the controversy, and solutions for the controversy proposed by physicists over the past century are considered, and the arguments they put forward to support their position are examined. Finally, some of the experiments undertaken are discussed, along with whether they have shed light on the controversy. The paper concludes by considering proposed future work on the topic.

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Introduction

For nearly 100 years physicists and mathematicians have been debating the correct form of the energy-momentum tensor required to describe the behaviour of light at the interface between two dielectric materials of different refractive indices. The two main ‘competing’ theories during this time have been those proposed by Minkowski (1908) [1] and Abraham (1909) [2]. Put simply, the dilemma is whether the momentum of a photon in a medium is equal to $n\hbar k$ (Minkowski) or $\hbar k/n$ (Abraham), where n is the refractive index of the material, ħ is Planck’s constant divided by 2π, and k is the wavevector.

While interest in the problem has waxed and waned over the years, it has never completely disappeared, and recently the topic has come back into vogue, with many papers published over the past 10 years and more due in the next year.

It is interesting to note how people's views have changed over the years. Minkowski was originally thought to be correct, although whether this was due to the convincing nature of his proposal, or people's dislike of Abraham is unclear. In 1950 M. V. Laue [3] was quite conclusive that Minkowski had developed the correct tensorial solution, and his views continued to be held through to the early 1970s without questioning. At this time experiments were undertaken which proved the existence of the Abraham force, and the opinion began to swing in favour of Abraham's tensor as being the correct solution. Recently though views have changed again, towards understanding that both are correct but in different circumstances.

It appears Minkowski derived his energy-momentum tensor independent of any other work; however Abraham's was not formulated separately, but was an attempt to reformulate Minkowski's tensor without the extra 'Minkowski force' term. His goal was to preserve the form from classical mechanics - a derivative with respect to time. In a previous paper Abraham had developed a system for the electrodynamics of objects in motion, which, while consistent with Maxwell and Hertz, also incorporated ideas by Lorentz and Minkowski. While dealing with Minkowski's ponderomotive force, Abraham derived another ponderomotive force and stated that his satisfied relativity.

At first sight it seems strange that such a small problem from the early twentieth century is still of interest. The author believes there are two contributing factors:

1. Physicists do not like unsolved problems. It spoils the 'neatness' (well, whatever neatness we have left after Heisenberg and quantum theory!) of the subject. There must be one definite mathematical formula for a particular problem. A case like this is particularly annoying, where the two equations are equivalent sometimes, yet at other times only one will solve a problem.
2. As optics becomes ever-increasingly important due to its use in telecommunications etc, any area with unanswered questions is worth investigating. We cannot tell what future technological breakthroughs it may lead to.

Many physicists and mathematicians have proposed alternative tensors that they claim do not suffer the same problem and will describe all situations. The author is not mathematically accomplished enough to comment on the accuracy of their claims. However, most have gone un-noticed and unreferenced by other papers, from which the author deduces that none have solved the problem completely; rather they reformulate it, sometimes clarifying areas, other times providing yet another layer of complexity over the problem.

This paper will start by providing an overview of the life of Abraham and Minkowski, followed by background theory. Next the opinions of physicists over the past century will be considered and the arguments they put forward to support their position will be examined. Finally, some of the experiments undertaken will be discussed, along with whether they have given additional insight to the controversy.

In this paper the problem will be both considered tensorially and using a simplified form. Both approaches have been considered widely in the literature on the subject, and the latter approach provides a more accessible route into the subject for those with less mathematical knowledge.

Biographies

Abraham (1875 –1922)

Max Abraham was born to a wealthy Jewish family and studied Physics at the University of Berlin under Planck. Abraham was appointed as a Privatdozent (an unpaid lecturer) at Göttingen in 1900, a position which lasted until 1909. The reason for his failure to obtain a permanent university position during this period was not due to any lack of ability but rather to his personality. Goldberg writes [4]:

> “...he had no patience with what he considered to be silly or illogical argumentation. Abraham had a penchant for being critical and had no hesitation in publicly chastising his colleagues, regardless of their rank or position. His sharp wit was matched by an equally sharp tongue, and as a result he remained a Privatdozent at Göttingen for nine years.”

In 1909 Abraham accepted a post at the University of Illinois in the United States. However, he disliked the atmosphere of Illinois, and returned within a few months to Göttingen. He then moved to Italy at the invitation of Levi-Civita, where he became professor of rational mechanics at the University of Milan, a position he held until 1914. While he was here Abraham and Einstein disagreed strongly about the theory of relativity.

The onset of World War 1 forced Abraham to return to Germany, where he worked on the theory of radio transmission. After the war he was unable to return to Milan and so he worked at Stuttgart until 1921, substituting for the professor of physics at the Technische Hochschule. He accepted a chair in Aachen but on the journey there he was taken ill and a brain tumour was diagnosed. He never recovered and died in agony six months later.

Almost all of Abraham's work was related to Maxwell's theory. His consistent use of vectors in his text on the subject was a significant factor in the rapid acceptance of vector notation in Germany. One of the most noteworthy features of his text was that in each new edition Abraham included not only the latest experimental work but also the latest theoretical contributions, even if these contributions were in dispute. For better or worse, he had no hesitation, after explaining both sides of a question, to use the book to argue his own point of view.

He developed a theory of the electron in 1902, but in 1904 Lorentz and Einstein produced a different theory. Abraham's study of the structure and nature of the electron led him to the idea of the electromagnetic nature of its mass, and consequently to the dependence of the velocity of electromagnetic waves in a gravitational field. It appeared that Abraham's model was correct as his ideas were at first supported by experiments, particularly work carried out by Wilhelm Kaufmann. However later work favoured the theory developed by Lorentz and Einstein.

Abraham opposed relativity all his life. At first he objected both to the postulates on which relativity was based and also to the fact that he felt that the experimental evidence did not support the theory. However by 1912 Abraham, who despite his objections, was one of those who best understood relativity theory, was prepared to accept that the theory was logically sound. In spite of this, he did not accept that the theory accurately described the physical world.

Abraham had been a strong believer in the existence of the aether and that an electron was a perfectly rigid sphere with a charge distributed evenly over its surface. He was not going to give up these beliefs easily particularly since he felt that his views were based on common sense. He hoped that further astronomical data would support the aether theory and show that relativity was not in fact a good description of the real world.

Many people would still agree with Abraham that his version of the world was more in line with common sense. However, mathematics and physics during the 20th century showed that the world we inhabit is at variance with "common sense" when we examine both the large scale structure and the small scale structure. Abraham's objections were not based on misunderstanding of the theory of relativity; he was simply unwilling to accept postulates he considered contrary to his classical common sense.

Minkowski (1864–1909)

Hermann Minkowski was born in Aleksotas, Russia (now Kaunas, Lithuania), but moved to Königsberg at the age of eight. Except for three semesters at the Universities of Berlin, he attained his higher education at Königsberg, where he achieved his doctorate in 1885.

In 1883, at the age of 18 and while still a student at Königsberg, Minkowski entered the Paris Academy of Sciences' competition. Eisenstein had provided formulas for the number of representations of an integer as a sum of five squares of integers, but no proof, and the goal of the competition was to prove the topic. Minkowski produced a manuscript of 140 pages, reconstructing the entire theory of quadratic forms in *n* variables with integral coefficients from the sparse indications Eisenstein's work provided. He won the prize jointly with H. J. Smith, who had published an outline for such a proof in 1867.

After receiving his doctorate, Minkowski taught at the universities of Bonn, Göttingen, Königsberg and Zurich. In Zurich, he was one of Einstein's teachers, and described Einstein as a "lazy dog", who "never bothered about mathematics at all".

Minkowski explored the arithmetic of quadratic forms, especially that concerning n variables, and his research into that topic led him to consider certain geometric properties in a space of *n* dimensions. In 1896, he presented his geometry of numbers, a geometrical method that solved problems in number theory.

In 1902, he joined the Mathematics Department of Göttingen, where he held the third chair in mathematics, created for him at David Hilbert's request.

By 1907 Minkowski realised that the special theory of relativity, introduced by Einstein in 1905 and based on previous work of Lorentz and Poincaré, could be best understood in a non-Euclidean space, since known as "Minkowski space", in which the time and space are not separate entities but intermingled in a four dimensional space-time, and in which the Lorentz geometry of special relativity can be nicely represented. This technique certainly helped Einstein's quest for general relativity.

In 1909, at the young age of 44, Minkowski died suddenly from a ruptured appendix. Despite having an interest in mathematical physics and dabbling in this field, his main work was in the field of the geometry of number, although he is best remembered for his four-dimensional space-time.

Theory

The Abraham and Minkowski tensors

The Minkowski and Abraham tensors lead to essentially different expressions for the density (***g***) of field momentum. In vector form they are given by:

$g^M = \left(\frac{1}{4\pi c}\right)[\mathbf{DB}]$ for the Minkowski form, and

$g^A = \left(\frac{1}{4\pi c}\right)[\mathbf{EH}]$ for the Abraham form.

Where $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.

These lead to the following expressions for the photon's momentum in the medium:

$G^M = g^M l = \frac{nul}{c} = \frac{nh\nu}{c}$ according to Minkowski, and

$G^A = g^A l = \frac{h\nu}{nc}$ according to Abraham

where $l = \frac{h\nu}{u}$ and represents the length of the line of plane-polarised waves in the wavepacket

Note that in a vacuum the tensors are identical; problems arise only in connexion with electromagnetic fields in matter [5]. The Minkowski energy-momentum tensor is asymmetric, implying non-conservation of angular momentum. Abraham rendered the tensor symmetric by changing Minkowski's momentum density from $\mathbf{D}\times\mathbf{B}$ to $\mathbf{E}\times\mathbf{H}/c^2$.

Brevik's [6] comment on the two tensors was:

> "The two tensors correspond merely to different distributions of forces and torques throughout the body. According to Minkowski the torque is essentially a volume effect, described by the tensor asymmetry, while according to Abraham the torque is described completely in terms of the force density."

Radiation Pressure

The minute pressure exerted on a surface in the direction of propagation of the incident electromagnetic radiation is called radiation pressure. The fact that electromagnetic radiation exerts a pressure upon any surface exposed to it was deduced theoretically by James Clerk Maxwell in 1871, and proven experimentally by Lebedev in 1900 and by Nichols and Hull in 1901.

In quantum mechanics, radiation pressure can be interpreted as the transfer of momentum from photons as they strike a surface. Radiation pressure on dust grains in space can dominate over gravity and this explains why the tail of a comet always points away from the Sun.

The pressure is very feeble, but it can be demonstrated with a Nichols radiometer. Consider a laser beam trained upon the black face of one of the radiometer vanes. It will be absorbed (hence the surface looks black). If, before arrival, the light had some associated linear momentum, then due to conservation of momentum within the system something else now has to be moving in the direction the light was travelling because the photons have been absorbed and come to a halt. The vane therefore begins to move.

Now consider the light hitting the *shiny* side of a vane. The shininess is an indication that the light is bouncing off the surface, which means that it has completely changed direction and is now travelling the other way. In this case the momentum imparted to the vane must be twice that imparted when the photon is absorbed, so that the total momentum is conserved.

Momentum of light

Radiation pressure has shown that light must carry momentum. Three different forms of momentum have been discovered:

1. Linear momentum: the original form considered in radiation pressure.
2. Angular momentum: photons can carry an angular momentum of $\pm\hbar$ in the direction of propagation – the sign depends on which direction they have been circularly polarised.
3. Orbital angular momentum: from changing the position of the wavefront to obtain a spiral beam. It is a property of the transverse mode pattern, and each photon possesses $l\hbar$ of angular momentum, where l is the number of intertwined helices.

Poynting Vector

The Poynting vector describes the flow of energy (power) through a surface in terms of electric and magnetic properties. It is the vector product of the electric and the magnetic fields. The Poynting vector points in the direction of propagation of a travelling electromagnetic wave and has the dimensions of power per area.

The full electromagnetic energy density in a region of space where there are both electric fields and magnetic fields is given by adding up separate contributions from the electric and magnetic fields. This implies energy is stored in the field itself. A unit cube of empty space which contains electric and magnetic fields will have some finite energy. This means electric fields and magnetic fields have a real physical existence, like particles.

This energy can also flow around, and the energy current is expressed by the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H}$$

Pseudomomentum

Despite the temptation to believe otherwise, it is important to remember that it is only by differentiating with respect to the "real" metric that we obtain "real" momentum. When we differentiate with respect to the analogy metric, we obtain the density and flux of another quantity, termed pseudomomentum.

Nelson [7] notes that momentum is a conserved quantity by virtue of the homogeneity of space, that is, as a result of the invariance of the laws of physics when spatial coordinates are moved. He states that pseudomomentum can also be a conserved quantity provided the medium is homogenous, that is the laws of physics are invariant to translations of the material coordinates.

In McIntyre's paper [8] on wave momentum, he explains a lot of the controversy as due to people's mistaken beliefs about momentum. He states: "Momentum density and momentum flux are independent entities... ...fluxes of momentum can perfectly well exist in a material medium without there being any momentum". He concludes "Abraham's momentum is the electromagnetic contribution to the actual momentum, while Minkowski's is the pseudomomentum".

There has certainly been no clear understanding of momentum of light in media and at the boundary between media and vacuum. A lot of problems stem from vague definitions – what one physicist means when he states "their momentum" need not mean the same to another physicist.

Tensors

Tensors provide a formalism that helps to solve and model certain problems more easily. A matrix is a specialised form of a tensor, occupying two dimensions.

Einstein's box

This thought experiment, dreamed up by Einstein in 1905, was designed to determine the mass equivalence of a pulse of electromagnetic radiation ($m = E/c^2$). Einstein considered a closed system (a box) of mass M, which is initially at rest in an inertial frame of reference S. The walls at either end of the box are of equal mass. A photon is emitted from a photon gun on one wall, down the central axis of the box, towards the other end wall. From the phenomenon of radiation pressure (given by $E = pc$) we know the photon emitted must have momentum. Therefore to conserve the momentum of the system the box must move in the opposite direction with velocity v. When the photon hits the far side of the box, the system will come to rest again, with a slightly different position to its starting point. This shift in position can be made arbitrarily large by repeating the process.

If the assumption that the carrier remains massless is valid then the system, which was initially at rest, will have its centre of mass shifted without any external forces acting on the system. This clearly violates the law of mechanics which states that a body, which is initially at rest, cannot undergo translational motion unless there is an external force acting on the body.

The only way that this problem is able to be solved is if the photon has transported an amount of mass – so that even though the box has moved, the centre of mass is still about the original point.

Einstein's derivation has some conceptual problems with it, namely that the box is treated as a rigid body, a concept that is inconsistent with the principles of special relativity. The approach taken by French circumvents the rigid body problem by considering only the two end walls of the box, arriving at the same result utilizing the centre-of-mass theorem.

Burt & Peierls [9] and Jones have carried out Einstein box calculations applying it to the theory of optical momentum. Burt & Peierls obtained results in agreements with the nondispersive Abraham expression, while Jones obtained the dispersive Abraham result (the equations for these are (6) and (10) respectively in Loudon's paper [10]). However, neither are happy with their calculations; Burt & Peierls worry over the rigid box assumed in their calculation and the impact on the validity of the results; and Jones adds an artificial 'forward bodily impulse' in order to attain his desired result of the Minkowski form of the momentum.

Loudon [10] notes that "the Einstein box is useful for the understanding of optical momentum but the very small shifts in position would be difficult to measure. These thought experiments seem likely to remain in the mind and not to be realised on the laboratory bench".

The proponents

At first people were swayed towards Minkowski's theory. In 1950 Laue [3] demonstrated certain limiting requirements that must be satisfied by the transformation properties of the components of the momentum-energy tensor of a light wave. Laue derived these requirements from the following considerations and his criterion was based on an incorrect assumption from another frame of reference. According to his assumption, he had shown that the Abraham tensor should no longer be considered since it contradicted the criterion he had derived. Assuming, on this basis, Minkowski's tensor to be correct, Laue drew far-reaching conclusions. For example, as the space-time components of Minkowski's tensor are antisymmetrical, Planck's postulate $g = \frac{\Phi}{c^2}$ must be recognized to be in error (and in the same way Einstein's relation $E = mc^2$).

Laue was not alone in taking this approach; Moller independently developed analogous considerations rejecting Abraham's tensor. Laue's criterion was widely accepted in the literature of the time as proof that Abraham's tensor was incorrect. These findings appeared to be borne out by experiments such as Barlow's [11] and Jones & Richards' [12], and these views were held until the mid-1970s with very little challenge.

Penfield and Haus [13] provided a very extensive and detailed discussion of the various formulations of electromagnetism, and for the major formulations they found expressions for the energy momentum tensor, for the momentum density and for the force density. The expressions they arrived at are complex – the force densities each comprise about 20 terms. In view of the fact that most of the terms correspond to barely observable mechanical effects, it is hopeless to attempt a full experimental verification.

Brevik started off agreeing with Laue that Minkowski's tensor was the correct representation. In two early papers [6, 14] he provided a discussion that "if properly interpreted the tensors of Abraham and Minkowski are 'adequate and equivalent' in most considered simple physical situations". He believed the two to be equivalent, since Abraham's force density must excite the dipoles within the dielectric material and produces a mechanical momentum which travels along with the field. By considering this mechanical momentum as well as Abraham's momentum, he obtained Minkowski's tensor.

However, from the early-1970s onwards, this position was challenged. In his little-known paper, Skobel'tsyn [15] provided a strong attack on Laue's criterion, providing an alternative criterion which the Abraham tensor satisfied and demonstrated how the Jones & Richard's experiment could also support Abraham's theory. He also dealt with Moller's criterion, showing how an asymmetrical tensor (like Minkowski's) could satisfy the criterion, while if the tensor is symmetrical and the sum of the diagonal terms is zero (per Abraham's) the criterion cannot be satisfied. He concluded that "it is in general incorrect to require that the momentum-energy tensor to satisfy the Moller criterion". In his handling of others Skobel'tsyn was not exactly gentle, as witnessed by his comments about Brevik on page 395. In Skobel'tsyn opinion "The authors of the review and original articles tend to ignore arguments that appear to lead unambiguously to the conclusion that Minkowski's postulate is not acceptable."

Skobel'tsyn put forward two Gedanken experiments. In the first one, he considered a "packet of light waves plus medium" and how the displacements of the centre of gravity changes as the beam propagates. He considered this for two cases, and claimed this was sufficient to exclude Minkowski's hypothesis from being correct. In the second thought experiment, he considers a cylindrical capacitor almost completely filled with a dielectric. The dielectric and outer electrode of the capacitor can rotate about a common axis freely and independently. By considering the conservation of angular momentum Skobel'tsyn again showed that Minkowski's tensor contradicts the conservation laws.

At the same time as Skobel'tsyn, J. P. Gordon [16] wrote a paper demonstrating that Abraham's form is correct for a nondispersive dielectric media, and that Minkowski's form, which deals with "crystal momentum", or "pseudomomentum", may be used to compute the radiation pressure on objects embedded in such a dielectric media. Gordon's argument followed the line that 'radiation' pressure was a combination of ponderomotive forces exerted directly by the field in the medium, and forces exerted on the medium by the dielectric in the presence of the field.

In his 1979 paper [17] in response to Walker, Lahoz and Walker's experiment [18] Brevik explained why the Abraham force could not be measured directly and why the divergence-free Minkowski tensor was perfectly satisfactory for explaining all existing torque experiments in optics. He noted that a microscopical energy-momentum tensor can be constructed for a closed system (that is, the field plus matter) and it can be averaged over suitable space-time regions, obtaining terms for pure field or matter plus a number of complicated correlation terms. This means there is no unique way to separate the total energy-momentum tensor into a field part and a matter part.

To obtain a definite energy-momentum tensor it is possible to impose extra restriction based on physical arguments, such as the Laue and Moller criterions (that is, that the propagation velocity of the energy of a light wave in a moving body shall transform like a particle velocity under Lorentz transformations) – and if they are applied, Minkowski's tensor satisfies these criterion while Abraham's does not. However, Brevik notes that such criteria are a test of a tensor's convenience rather than its correctness.

Brevik was writing at a time when the problem had come back in fashion, and in the previous few years some important work had been done on it, such as the measuring of the Abraham force directly by Walker, Lahoz and Walker [18]. Current experimental trends were moving along both microscopical and macroscopical lines. Brevik believed: "The microscopical method is advantageous from a fundamental viewpoint, but its drawback is that it easily becomes formally complicated and tends to obscure a simple physical interpretation. The advantage of the simple and less fundamental macroscopical method is its close connection with observation".

At this time Brevik's interest was lying in the consideration of the electrostriction effect (or the magnostriction effect). He stated that although the electrostriction effect does not affect many experiments it does in special cases, so one should bear its existence in mind and add the Helmholtz electrostriction term to Abraham's or Minkowski's tensor whenever necessary. He reinforces this on page 139: "Minkowski's tensor does not describe electrostriction or magnetostriction. This tensor, therefore, is unable to give a complete description of the local electromagnetic state in the medium." However, he then continued that it does not matter.

Brevik used this paper to provide a rebuff to what he described as Skobel'tsyn's "..violent attack on the present author..". By analysing the two basic Gedanken experiments Skobel'tsyn considered, he showed that the reason the first experiment does not allow Minkowski's momentum to follow naturally is due to the particular feature of this momentum in that it included the mechanical momentum that accompanies the wave. For Skobel'tysn's second experiment he demonstrated the solution to be simple: that in the Minkowski case there is no azimuthal force on the medium, only radial forces at the boundaries. Therefore no resulting angular momentum of the medium (the problem which the Minkowski tensor could not handle, leading Skobel'tsyn to rubbish it) is predicted. Brevik claimed there were no consistency problems here. This does raise the interesting feature that the physical motion of the medium is predicted differently in the two cases. Brevik compromised by saying that the Abraham tensor may be right, but "it is usually best to use the Minkowski tensor, because of its formal advantages".

Lorrain [5], stated that "the whole [Abraham-Minkowski] controversy was pointless, for the following reason. Clearly, angular momentum must be conserved, but only for the complete system composed of various subsystems (electromagnetic, kinetic, material, …), the split into subsystems being more or less arbitrary. It was thus shown that the Minkowski and Abraham tensors are both correct, as long as the various subsystems are treated correspondingly".

Lai and Brevik argued in public (Physics Letters) quite frankly about each others work. In Lai's original paper he assumed the medium to be nonmagnetic, so the two fields **B** and **H** are equal. He concluded in the follow-on paper [19] "Minkowski's expression could be mistakenly accepted should one consider the torque only on the conduction current, rather than on the total current". Brevik responded [20] that Minkowski's theory as well as Abraham's satisfied the general conservation principles of total angular momentum, but conceded that the two theories do make different predictions for some physical effects, and this is experimentally verifiable. As Brevik pointed out, Lai's example is quite similar to one considered by Skobel'tsyn some years previously. For once Brevik is brief, and in only 4 pages showed the formal complete equivalence of both momentum expressions with respect to the angular momentum conservation requirements. He summarized his results as the following points:

1. At high optical frequencies both tensors are equivalent. Thus the Minkowski tensor can effectively describe all optics experiments that exist. Plus the Minkowski tensor satisfies the experimentally detected Laue criterion.
2. At lower optical frequencies, in the range where mechanical test systems may be able to track the alternating forces predicted by time-varying electromagnetic fields, the tensors contradict each other, at least for nonmagnetic media. Brevik states that Abraham's tensor predicts an angular momentum imparted to the annular medium whereas Minkowski's tensor does not.
3. Both tensors ignore the effect of electrostriction. This had affected a couple of experiments, but mostly does not have any influence on the overall force or torque on a test body.

In 1972 de Groot and Suttorp derived the Abraham result for the momentum density from a relativistic treatment, and obtained a different result if not carried out relativistically. Nelson [7] found this puzzling, and decided that there was a fault in their argument.

In an unpublished work written in 1971, Blount introduced the concept of pseudomomentum or crystal momentum, an idea well known in solid state physics. Blount identified by relativistic, macroscopic arguments the Abraham form as the momentum density and the Minkowski form as the pseudomomentum density. Momentum is always a conserved quantity due to the invariance of the laws of physics to displacements in the special coordinates. Pseudomomentum can also be a conserved quantity, as a result of the invariance of the laws of physics to translations of the material coordinates. Blount's concept changed work on the controversy from an issue of detecting the right or wrong answer, to one of identifying the momentum involved and experimental applicability of each basis.

Blount's ideas were accepted by Gordon and Peierls, although Peierls concluded [21] that "Abraham's formula gives correctly the part of the momentum of the medium which resides in the electromagnetic field, but not the mechanical momentum of the medium which travels with the light pulse".

Following in the footsteps of Blount, using a Lagrangian formalism of the problem, Nelson [7] showed mathematically that the electromagnetic momentum density was $\varepsilon_0 \mathbf{E} \times \mathbf{B}$ and not the Abraham form $\varepsilon_0 \mathbf{E} \times \mu_0 \mathbf{H}$.

More recently, Feigel [22] finds that the "current received results correspond to Abraham's predictions" and concludes that Abraham's expression is for the momentum of the field; while experimentally measured momentum also includes a matter contribution, and its total value coincides with Minkowski's result. He believes the origin of the controversy to be in the underestimation of the fact that field-matter interaction is impossible without the motion of the latter.

Louden has been prolific in researching this field recently. A large amount of his work has been taken up with Laguerre-Gaussian beams, and uses the Lorentz force. In [23] he writes:

> "The calculated results for the Lorentz forces on material media enable some conclusions to be drawn on the effective photon momenta that are needed to reproduce the same forces. For the interface of a transparent dielectric with free space, the surface force is the same as that obtained from conservation of momentum when the photon in the dielectric has the Abraham momentum. The same total value of the photon momentum is found for the system of a mirror suspended in a liquid dielectric, when the contributions of the forces on the liquid and the mirror are combined. However, the calculated force on the mirror alone is consistent with the Minkowski momentum, and this value agrees very well with the measurements."

Mansuripur [24] believes that Loudon's final result is close to being correct, but that he has neglected the mechanism of photon entry from the vacuum into the dielectric medium.

Recently, S. Antoci & L. Mihich [25] rediscovered a forgotten argument put forward by W. Gordon in 1923. Gordon's paper was published just before quantum mechanics hijacked the thinking of physics and relegated general relativistic ideas from centre-stage. This may go some way to explaining why the paper was neglected. Gordon's paper contains a clever way to *reductio ad vacuum* the problem and then select the stress-energy momentum tensor of the electromagnetic field in non-dispersive matter, provided the matter is homogeneous and isotropic when considered in its rest frame. The reason for these latter conditions is that in a homogeneous, isotropic medium the light rays are null geodesics with respect to the effective metric, when considered in the limit of geometrical optics.

The argument Gordon provides is macroscopic, in contrast to many of the more recent ones. The result of Gordon's argument is the selection of Abraham's tensor as the electromagnetic energy tensor for a material medium which is homogeneous and isotropic in its rest frame.

In his recent paper Mansuripur [24], using the Lorentz force approach, demonstrates that the correct form for the momentum density of the light field inside a dielectric media is one that has equal contributions from the traditional Minkowski and Abraham forms. His approach neglects the electric field contribution to the Lorentz force in a dielectric.

In a private communication to the author, Loudon notes that not only Gordon, but also such standard works as Landau & Lifshitz and Penfield & Haus show that the electric field makes an important contribution even in condensed matter. Of course, this contribution is not effective in many situations, such as plane wave light beams incident at normal incidence, when only the magnetic force contributes. Much of Mansuripur's paper deals with such situations and Loudon believes that some of the corresponding results are likely to be correct, including his analysis of the Barlow experiment. However, Mansuripur also has examples of non-plane-wave beams, where the electric force is important, for example Laguerre-Gaussian modes. In these cases Loudon does not agree with the results.

Alternative theories

Many people have attempted to solve the problem by proposing their own tensors. It is beyond the scope of this paper to provide anything more than a brief overview of the tensor solutions put forward.

People such as Livens and Tiersten and Tsai both obtained the momentum density $\varepsilon_0 \mathbf{E} \times \mathbf{B}$ by re-expressing the Lorentz force.

Peierls also came down to this, but didn't consider the result different from the Abraham form, as he worked in a nonmagnetic medium. Brevik [17] noted that no experiments had distinguished between the Peierls tensor and those of Abraham and Minkowski. He then proceeded to fill an appendix bringing the discussion up to date, using the 'new' Jones and Leslie experiment to compare the tensors. He concluded that the Peierls prediction was incorrect.

Obukhov & Hehl [26] recently re-analysed the problem, and proposed a solution in the form of a new energy-momentum tensor. Defined for a vague “arbitrary medium", they present evidence that this is the correct tensor for the electromagnetic field in material media. In considering the problem they ignore any effects of electrostriction. Their proposed tensor is symmetric, satisfies Planck’s field-theoretical generalisation, and the corresponding electromagnetic force is indeed the Lorentz force acting on the free and bound charge and current densities. They state:

> “Many authors pointed to a clearly unphysical result produced by the Minkowski energy-momentum: in the absence of free charges and currents, a homogeneous medium appears to be always subject to the zero electromagnetic force. This fact was usually taken in favour of the Abraham tensor which predicts an extra, so-called Abraham force. However, [our tensor] does not suffer from such a deficiency. Even when the free charge and current densities are vanishing, the total force is, in general, non-trivial in view of the presence of polarisation charge and current. Moreover, as compared to the rather ad hoc choice of the Abraham force, the mechanical action on the bound charge and current is in all cases described…by the well-known Lorentz force.
>
> Furthermore, the Minkowski tensor is asymmetric…usually this fact was also taken in favour of the Abraham tensor, which is symmetric. At the same time, despite its symmetry the structure of the Abraham tensor is defined in a rather ad hoc manner with opaque physical motivations.”

Obukhov & Hehl examined a couple of experiments (notably Walker & Walker’s) and concluded that these experiments can be shown to support their new tensor. For while the results have been claimed to confirm the Abraham force, the torque measured fits the understanding of electromagnetic force as the Lorentz force of the polarisation current.

With regard to James’ experiment, Obukhov & Hehl note that James did not measure the force itself, but a “reduced force”, which was observed to vanish in his experiment. They link his results to their new tensor, and note that “This observation is in complete agreement with the theoretical derivation based on our new energy-momentum tensor, whereas both the expressions of Minkowski and of Abraham clearly contradict this experiment.”.

They state that end corrections have never been taken into account in previous analyses of James’ experiment, and the end corrections (needed due to the deformation of the fields) are not proportional to the length of the cylinder. The point they are making is that these end corrections would not compensate the reduced force of Minkowski and Abraham, which are both proportional to the length of the cylinder used in the experiment.

Mansuripur [24] criticises Obukhov & Hehl’s paper, because when discussing the case of normal incidence from vacuum onto a semi-infinite dielectric they neglect to account for the mechanical momentum imparted to the dielectric medium. The result of this is that they find only the electromagenetic part of the momentum density. Mansuripur also considers the effect of the feeble magnetic Lorentz force over the infinite thickness of the dielectric.

Experimental Work

While theoretical arguments can be convincing to a high degree, the garland for a physicist is to find that his theory can be verified experimentally. Until that point, no matter how strong a theory he creates, it remains a flight of fancy. Of course, the fight for the theory doesn't finish there – if it does then Minkowski would have 'won' years ago, and this entire dissertation would not have needed to be written!

The simplest technique for observing the pressure of radiation is to attach a mirror to a torsion balance, and to illuminate the mirror in such a way that the radiation pressure produces a resultant couple on the suspension. This was the approach taken until the late 1970s. Originally the apparatus was filled with a gas at a suitable pressure (usually close to a vacuum) to reduce convection effects and radiometric effects.

The original experiments were designed to test the energy-momentum tensor by using a force measurement on a dielectric body in an electrostatic field.

Early Experiments

Jones & Richards [12] provide a good overview of the early experiments relating to radiation pressure and verifications of the theories in their introduction, and apart from discussing the work of G. Barlow the author will not repeat their list here.

G. Barlow

The earliest recorded attempt was by G. Barlow in 1912. He measured the torque on a dielectric cube of crown glass that resulted when an optical wave propagated through it obliquely. The plate was surrounded by hydrogen gas at low pressure. By filling the radiation pressure apparatus with hydrogen gas rather than air Barlow gained the advantage of the relatively high conductivity and low density of hydrogen.

Brevik [6] noted that any alternative for the electromagnetic energy-momentum tensor would give the same prediction for the observable torque. Thus this experiment did not clarify the choice between tensors or explain why the Abraham and Minkowski tensors are different, neither of which Barlow claimed for his experiment.

Barlow's experiment was designed to verify the theory of Poynting, a claim Barlow makes for his result. Poynting's theory is supposed to support Minkowki's form of momentum inside the dielectric, which Mansuripur [24] states cannot be correct (page 5389)

Loudon [23] notes:

> "[Barlow's was] an early experiment on radiation pressure, which was thought to support Poynting's calculations, measured the torque on a glass cube illuminated by a light beam in oblique incidence. The measured torque did indeed agree well with that calculated from the Poynting theory, which is again equivalent to the assumption of the Minkowski momentum for a photon in the cube. The same experimental arrangement can also be described by an extension of the theory given here, which is again consistent with the assumption of the Abraham effective momentum for a photon in the plate."

Jones and Richards

Jones and Richards [12] carried out an experiment in 1953. The aim of their experiment was to investigate whether the force exerted by light on an opaque body was proportional to the refractive index of the medium in which the body is immersed.

To show that the radiation pressure varies with the refractive index of the medium surrounding the mirror, Jones and Richards mounted their system in a container which could be filled with various fluids. Great care was taken in designing the equipment to prevent the expected large convection forces from interfering with the experiment.

Most experiments carried out before this time had had a relatively long response time (~1 second), and Jones & Richards intended to improve upon this.

The ways that convection forces were reduced were by:

1. Minimising the change in temperature of the mirror during the experiment. This was achieved by changing the point of action of the incident light rather than switching the light on and off
2. Making the mirror small and a good conductor of heat to keep the temperature of the mirror surfaces as uniform as possible
3. Making the container surrounding the apparatus as small as possible and a good conductor of heat, as this had been shown to reduce convection currents
4. Creating a system with a short response time. Since heating effects take time to develop, but the radiation pressure is instantaneous in action, the system would be better the shorter the response time – distinguishing between the forces is then possible.

To minimise thermal effects and the affect they would have on the results, Jones & Richards used two beams, one on either side of the mirror and striking opposite halves simultaneously. Unwanted convection forces in the liquid were eliminated by means of a chopping technique.

Jones and Richards found that the magnitude of deflexion in air was to the order of three or four times greater than that finally ascribed to radiation pressure. The extra magnitudes were due to convection currents, which arose from the heating of the mirror as a whole and produced a resultant couple because of lack of symmetry in the mirror.

They showed that the ratio of the pressure on the vane when immersed in a liquid, and the pressure when in air is equal to the refractive index of the liquid within the limits imposed by experimental error of +/- 1.2% r.m.s.

Jones and Richards' experiment was important because it showed that the result is most easily explained by attributing a momentum density $\frac{1}{c}\mathbf{D}\times\mathbf{B}$ to the optical wave travelling through a refracting fluid.

Their experiment was not widely noticed at the time, but was interpreted by Brevik in 1970 as supporting the energy-momentum tensor of Minkowski more directly than that of Abraham in the description of the electromagnetic field. In his 1979 paper [17] Brevik termed it "..one of the most important experiments in phenomenological electrodynamics".

Skobel'tsyn [15] notes that if Minkowski's expression is assumed it follows that momentum is proportional to the refractive index, as the experiment shows. However, regardless of any assumptions, it follows directly from Maxwell's theorem that light pressure on a mirror in a certain medium is proportional to the radiation density at the surface of the mirror, which in turn is proportional to the refractive index. Therefore, while Minkowski's assumption about radiation momentum can lead to the correct conclusion, the inverse conclusion, that Minkowski's expression is correct, can only be deduced if the forces due to propagation of light in a transparent medium are assumed to be equal to zero.

Jones and Leslie

In 1977 Jones collaborated with Leslie to repeat the earlier experiment he had performed with Richards. In the intervening years there had been many developments such as the invention of the laser (allowing the production of nearly monochromatic radiation – a great improvement over the broad spectral band generated by the tungsten lamp used in the original experiment) and of multilayer reflecting mirrors of high reflectivity and low absorption.

Other researchers had also looked into the problems during the intervening years. Burt and Peierls had considered the problem and concluded that if the momentum of a photon was *p* in free space, it would decrease to *p/n* when entering a medium of refractive index *n*. This differed from the experimental result Jones and Richards had obtained, from which it appeared that the momentum associated with the photon in a denser material should increase to *np*.

Jones and Leslie decided to repeat the experiment as it appeared to be the only direct measurement of the change of momentum of a beam of light on passing into an optically dense medium, and the improvements in accuracy possible were deemed great enough to make repeating the experiment worthwhile. It was hoped to be able to decide more positively whether *n* should be the conventional phase refractive index or the 'group refractive index' – that is, the ratio of group velocities in free space and the denser medium. Peierls had also developed a theory which predicted that the momentum increase associated with electromagnetic radiation incident on a mirror suspended in a dense medium should depend on the plane of polarisation of the radiation when the incidence is oblique; hence the experiment was also looking for this.

Because of the lower absorbance of the high reflectivity multilayer mirrors, Jones and Leslie were able to simplify their equipment and only strike the mirror from one side, as they did not have to worry so much about thermal effects. Due to the developments that had occurred the precision of measurements they obtained improved more than tenfold.

Their measurements were sufficiently accurate to establish that the n in the momentum transfer was the ordinary, or phase, refractive index, not the group refractive index.

In his paper [23] Loudon shows how the Minkowski momentum, experimentally measured by Jones and Leslie, can be restored to the Abraham value by an additional unobserved transfer to the surrounding liquid.

Ashkin and Dziedzic

Ashkin and Dziedzic used an argon-ion laser source to investigate the pressure on solid dielectric spheres immersed in liquid, and the pressure on a liquid-air interface owing to the passage of a beam of radiation.

Brevik commented that like Barlow's experiment, this experiment cannot distinguish between the various tensors. In particular it gives no information about the direction of the local surface force. In his opinion the great merit of Ashkin and Dziedzic's experiment was its locality: it showed that a narrow light beam incident at the normal on a free liquid surface acts upon the surface by an outward pull. Thus the experiment bore out the Abraham-Minkowski prediction for the surface force density.

Feigel [22] notes that Ashkin and Dziedzic observed the liquid interface to bend outwards in the liquid both when the light enters and leaves the liquid. This is contrary to the conservation law he derives from a Lagrangian formalism, and which predicts inward bending. Loudon [23] arrived at the same conclusion as Feigel but by quantum analysis of the Lorentz force. Through his considerations in [10] Loudon concludes that:

> "The Ashkin and Dziedzic experiment observed an outward bulge on an illuminated water surface, apparently consistent with the sign of the Poynting surface force and the value of the Minkowski momentum. In agreement with Gordon, it is shown here that the effect is governed by a radial force and it provides no information on the longitudinal force associated with the linear momentum of light."

Walker, Lahoz and Walker

Walker, Lahoz and Walker (WLW) carried out an experiment to measure the Abraham density force in a barium titanate ceramic. $BaTiO_3$ was chosen for its high dielectric constant. They suspended a disk of the material as a torsional pendulum hung on a thin tungsten fibre and the disk was located between the poles of a powerful electromagnet.

The aim was that instead of using electromagnetic waves it should be possible to use strong, time-varying, orthogonal electric and magnetic fields. In this situation the magnitude of the Abraham term may become relatively large, and if the fields are quasi-stationary, the term itself should be measurable. WLW carried out the experiment using a varying electric field and a constant magnetic field. In the experiment oscillations were really observed and found to be in agreement with the Abraham prediction to within 10%.

The experiment was a global one, as opposed to local, and therefore ignoring electrostriction and magnetostriction in the formalism did not affect the final result.

Brevik [17] noted there were a couple of limitations to the experiment:

1. The frequencies involved were very low
2. The test body was non-magnetic
3. The **H** field was constant, whereas the **E** field was varying.

He concluded that "In spite of these limitations the experiment is very important, as it provides a direct verification of the existence of the Abraham term. Minkowski's force is contradicted by the observed oscillations of the disk. Thus in quasi-stationary fields Minkowski's tensor is inadequate".

This indeed was the merit of their experiment. Finally the Abraham term (or force) had been verified, and the existence of an experiment that Minkowski's tensor was unable to describe.

As the WLW experiment was a global experiment it gives no information about the local force density distribution within the test body. Therefore the experiment cannot distinguish between the Abraham and Einstein-Laub tensors.

The problem with experiments

The problem with experiments performed to date is that they are usually trying to measure only one term of the force equation, given by:

$$\frac{\partial P_i}{\partial t} + \frac{\partial}{\partial x_j} T^{ij} = F^i$$

where the first term is the change of the ith momentum with time, the second term is the flux of the ith component of momentum in the jth direction, and the final term is the force density exerted on the medium; or are trying to measure a boundary of a fixed block of dielectric material – so it is very difficult to consider all forces at the boundary. Instead of resolving the problem, the experiments have often clouded the picture further. It is easy to confuse the forces actually observed with those expected; likewise it is very easy when dealing with tiny forces to miss detecting them. Often the experiments haven't been well thought out in the first place, with mistakes made during implementation.

Future Experimental work

Steven Barnett et al [27] are proposing an experiment radically different to those previously carried out on the subject, and it is hoped this experiment will solve the controversy once and for all. Instead of having photons moving macroscopic material at a boundary and attempting to detect the forces exerted, the experiment will use a photon drag detector working in the far IR region. The detector works by the incident photons imparting momentum to the electrons in the tube, which in turn induces a voltage. This design of detector simplifies the problem, as the photons are not absorbed, but continue straight through, which therefore simplifies many of the forces involved. It is proposed to use light with orbital angular momentum, as it is believed this will reduce errors further.

Conclusion

Where can the author conclude? The topic is very much in vogue, with many illuminating papers having been produced over the last couple of years, with more to come. It is an exciting time for this field; with the glimmer of a final solution appearing over the horizon.

The Abraham-Minkowski controversy regarding the momentum carried by an electromagnetic wave in the presence of a refractive medium has been the subject of ongoing debate for nearly 100 years. Minkowski's theoretical work suggested the form of the momentum to be $n\hbar k$. Conversely Abraham's work, developed through the application of classical form to Minkowski's tensor, suggests that the momentum will be of the form $\hbar k/n$.

Numerous theoretical analyses have concluded in favour of both parties with a more recent tendency towards the Abraham form of the electromagnetic tensor as being the full description of the situation. Alternative forms of the electromagnetic tensor have been proposed [25, 26] to solve the controversy, with limited success. Current thinking [24, 27] would appear to be that the various forms of the tensor can be rationalised when the momentum is correctly and completely calculated, by taking into account reflections and the division of momentum between the field and the motion of charges and bulk material.

Many experiments have been performed over the years in an attempt to decide which prediction of the momentum is correct. Notoriously difficult to carry out without forgetting to account for one or more forces, these have also been subject to alternative interpretations. While early analysis tended to support the Minkowski view, many have been shown to measure the Abraham form, if certain limitations are taken into account.

It remains to be seen whether the question can ever be settled, or whether, like Simple Simon [28], people will continue to get more and more confused about the topic, clouding the issue with potential solutions that further muddy the water.

Finally, we must ask ourselves whether the theories that Abraham and Minkowski came up with nearly 100 years ago should be applied to our problem. The original papers were trying for electro-magnetic tensors in a medium. As an example, Abraham's paper has no boundary conditions, so everything is fine inside the medium until you consider the interfaces at the edge. Thus we are taking their equations and applying them to situations they were not designed to fit.

Acknowledgements
The author would like to thank Dr Moretti for his aid in translating Abraham's original paper, Prof. Steve Barnett for his discussions on the topic, and Prof. Rodney Loudon for his communication. Also thanks to my supervisor, Dr Tim Freegarde.

References:

1. Minkowski, H., Nachr. Ges. Wiss. Gottingen, 1908. **53**.
2. Abraham, M., *Sull'elettrodinamica di Minkowski.* Rend. Circ. Matem. Palermo, 1910. **XXX**: p. 33-46.
3. Laue, M.V., Zs. Phys., 1950. **128**: p. 387.
4. *Dictionary of Scientific Biography*. 1970-1990, New York.
5. Lorrain, P., *The Abraham force: comments on two recent experiments.* Can. J. Phys, 1980. **58**: p. 683-686.
6. Brevik, I., *Electromagnetic energy-momentum tensor within a material media: Discussion of various tensor forms.* Det Kongelige Danske Videnskabernes Selskab Matematisk-fysiske Meddelelser, 1970. **37**(13).
7. Nelson, D.F., *Momentum, pseudomomentum, and wave momentum: Toward resolving the Minkowski-Abraham controversy.* Physical Review A, 1991. **44**(6): p. 3985-3996.
8. McIntyre, M.E., *On the 'wave momentum' myth.* J. Fluid Mech., 1981. **106**: p. 331-347.
9. M. G. Burt and Sir Rudolf Peierls, F.R.S., *The momentum of a light wave in a refracting medium.* Proc. R. Soc. Lond, 1973. **333**: p. 149-156.
10. Loudon, R., *Radiation pressure and momentum in dielectrics.* Fortschr. Phys., 2004. **52**(11-12): p. 1134-1140.
11. Barlow, G., *On the Torque produced by a Beam of Light in Oblique Refraction through a Glass Plate.* Proc. R. Soc. Lond., 1912. **A87**: p. 1.
12. R. V. Jones, J.C.S.R., *The Pressure of Radiation in a refracting medium.* 1953. **221**.
13. P. Penfield Jr., H.A.H., *Electrodynamics of Moving Media*. 1967, Cambridge, Mass.: MIT Press. 241-243.
14. Brevik, I., *Electromagnetic energy-momentum tensor within material media: Minkowski's tensor.* Det Kongelige Danske Videnskabernes Selskab Matematisk-fysiske Meddelelser, 1970. **37**(11).
15. Skobel'tsyn, D.V., *The momentum-energy tensor of the electromagnetic field.* Sov. Phys.-Usp., 1973. **16**(3).
16. Gordon, J.P., *Radiation Forces and Momenta in Dielectric Media.* Physical Review A, 1973. **8**(1): p. 14-21.
17. Brevik, I., *Experiments in phenomenological electrodynamics and the electromagnetic energy-momentum tensor.* Physics Reports, 1979. **53**(3): p. 133-201.

18. G. B. Walker, D.G.L., G. Walker, *Measurement of the Abraham Force in a Barium Titanate Specimen.* Can. J. Phys, 1975. **53**: p. 2577-2586.
19. Lai, H.M., *Abraham-Minkowski controversy in static fields: An example.* Am. J. Phys, 1981. **49**(4).
20. Brevik, I., *Comment on "Electromagnetic momentum in static fields and the Abraham-Minkowski controversy".* Physics Letters, 1982. **88A**(7).
21. Peierls, R., *The momentum of light in a refracting medium II. Generalization.* Proc. R. Soc. Lond., 1977. **355**: p. 141-151.
22. Feigel, A., *Quantum Vacuum Contribution to the Momentum of Dielectric Media.* Physical Review Letters, 2004. **92**(2).
23. Loudon, R., *Theory of the radiation pressure on dielectric surfaces.* J. Mod. Opt., 2002. **49**: p. 821.
24. Mansuripur, M., *Radiation pressure and the linear momentum of the electromagnetic field.* Optics Express, 2004. **12**(22).
25. S. Antoci, L.M., *A forgotten argument by Gordon uniquely selects Abraham's tensor as the energy-momentum tensor for the electromagnetic field in homogeneous, isotropic matter.* Nuovo Cimento B, 1997. **112B**(7): p. 991-1001.
26. Yuri N. Obukhov, F.W.H., *Electromagnetic energy-momentum forces in matter.* Physics Letters A, 2003. **311**: p. 277-284.
27. Rodney Loudon, S.M.B., C. Baxter, *Theory of radiation pressure and momentum transfer in dielectrics: the photon drag effect*. 2004.
28. Synge, J.L., *On the present status of the electromagnetic energy-tensor.* Hermathena, 1974. **117**: p. 80-84.