01183830

Electrical and Electronic Engineering

Year 4

Module: Probability and Stochastic Processes

I confirm that the answers presented in the submitted pdf file are my own work and that I have not been in contact with others during the exam period.

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#1. a) $E(Y) = 1 \cdot P(Y=1) + 0 \cdot P(Y=0)$

i) $= \frac{1}{2} \cdot 1 + 0 \cdot \frac{1}{2} = \frac{1}{2}$.

$E(Y) = \sum_{y \in Y} y\, P(Y=y)$

ii)

~~P(j heads) =~~ Let X be the number of heads in n times.

P(j heads in n times) = P(j heads and n-j tail)

$P(X=j) = \binom{n}{j} p^j (1-p)^{n-j}$, $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ here $p = 1-p = \frac{1}{2}$. as it's a fair coin.

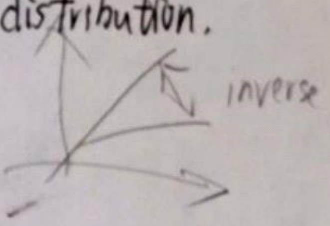
$P(X=j) = \binom{n}{j} \left(\frac{1}{2}\right)^n$

iii) P(n times until first heads)

= P(n-1 consecutive tails, then a head)

$= \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^n$ ⇒ this is a geometric distribution.

[n>0, n is a positive integer]



b) $U \sim U[0,1]$, $X = F^{-1}(U)$ $U = F(X)$

$X = F^{-1}(U) = g(U) \Rightarrow g(u) = F^{-1}(u)$, ~~assum~~

as F(x) is CDF, it is one to one.

$\therefore f_X(x) = \frac{1}{|g'(u)|} f_U(u)$ $f_U(u) = \frac{1}{1}$, $u \in [0,1]$

$g(u) = F^{-1}(u) = \cancel{\frac{1}{F(u)}}$ $F_U(u) = u$, $u \in [0,1]$

here F means CDU

$\cancel{\frac{\partial g}{\partial u} = \frac{-1 \cdot f_u(u)}{(F(u))^2}}$ $\cancel{F_X(x) = P(X \le x)}$ $\cancel{= P(F}$

$P(X \le x) = P(\overset{\text{function } F(x)}{F(X)} \le F(x))$

$= P(U \le F(x))$ ← use uniform distribution def

$= F(X)$ ← CDF of RV X ⇒

$pdf = \frac{\partial}{\partial x} CDF = \frac{\partial}{\partial x} F(x) = f(x)$

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EE4-10

#1. b) rewrite:

$U \to \mathcal{U}[0,1]$, $f_u(u) = 1$, $u \in [0.1]$

$CDF(U) = u$, $u \in [0,1]$

$P(X \leq x)$ = CDF of random variable X

$= P(F(X) \leq F(x))$: apply CDF $F(x)$ to transform both sides. as $F(x)$ is mononotic, increasing & positive ~~eq~~ inequality still holds.

$= P(U \leq F(x))$

= CDF of RV U [$F(x)$ as input]

$= F(x)$: so CDF of RV X has the correct distribution.

#2. a). $n=3$, $f(x) \sim c^4 x^3 e^{-cx}$, $x > 0$

$$f(x_1, x_2, x_3; c) = c^4 x_1^3 e^{-cx_1} c^4 x_2^3 e^{-x_2 c} c^4 x_3 e^{-cx_3}$$

$$f(\underline{x}; c) = c^{12} \cdot \left(\prod_{i=1}^{3} x_i\right)^3 e^{-c\left(\sum_{i=1}^{3} x_i\right)}$$

$$\ln(f(\underline{x}; c)) = 12 \ln c + 3 \ln\left(\prod_{i=1}^{3} x_i\right) - \left(\sum_{i=1}^{3} x_i\right) c$$

$$\frac{\partial}{\partial c}\left[\ln(f(\underline{x}; c))\right] = \frac{12}{c} - \left(\sum_{i=1}^{3} x_i\right) \Rightarrow 0$$

$$c_{ML} = \frac{12}{4.1 + 3.7 + 4.2} = \frac{12}{12} = 1$$

#2, b) $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$ $X \sim N(0,\sigma^2)$

i) Markov Inequality: $P(X \geq a) \leq \frac{E(X)}{a}$, $a>0$

Generalized: $P(g(X) \geq g(a)) \leq \frac{E(g(X))}{g(a)}$

$g(X) = |X|$, $E(|X|) = E(|X|^1)$ 1: odd $2k+1 = 1$, $k=0$

$\cancel{= 2^1 1! \sigma^{2\cdot+1}\sqrt{\frac{2}{\pi}} = \frac{\sigma^3}{\sqrt{2\pi}}}$

$= 2^0 \cdot 0! \sigma^1 \sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{\pi}}\sigma$

$P(|X| > 3\sigma) \leq \frac{E(|X|)}{3\sigma} = \frac{\sqrt{\frac{2}{\pi}}\sigma}{3\sigma} = \frac{1}{3}\sqrt{\frac{2}{\pi}} \simeq 0.26596$

ii) Chebyshev Inequality: $P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$. $a>0$

$P(|X-0| > 3\sigma) \leq \frac{\sigma^2}{(3\sigma)^2} = \frac{1}{9} \simeq 0.1111\ldots$

iii) Chernoff Bound. $P(X>a) \leq \min_{\lambda>0} e^{-\lambda a} E(e^{\lambda X})$

$P(x>a) \leq e^{-a\lambda} e^{\frac{\sigma^2\lambda^2}{2}}$ $= \min_{\lambda>0} e^{\lambda a}\phi_X(\lambda)$, where ϕ is the characteristic function

$= e^{-a\lambda + \sigma^2\lambda^2/2}$, $\Rightarrow \lambda^* = \frac{a}{\sigma^2}$.

$P(X>a) \leq e^{-\frac{a^2}{2\sigma^2}}$

$P(|X|>a) = 2P(X>a)$ as two sides of Normal distribution is symmetric with respect to $\mu=0$.

$\leq 2e^{-\frac{a^2}{2\sigma^2}}$

$P(|X|>3\sigma) \leq 2\cdot e^{-\frac{9\sigma^2}{2\sigma^2}} = 2\cdot e^{-\frac{9}{2}} \simeq 0.0222$

#3. a) time unit: hour. intensirty $\lambda=0.25$

i) probability of number of failures in different period could be seen as independent, as failure is modeled by Poisson process.

$$P(N\leq 1,[0,4)) = P(N=0,[0,4)) + P(N=1,[0,4))$$

$$\lambda T = 0.25\times 4 = 1. \quad = e^{-1}\cdot\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!}$$

$$= e^{-1} + e^{-1} = 2e^{-1} \simeq 0.736$$

$$P(N\geq 2,[4,8)) = 1-P(N=0,[0,4)) - P(N=1,[0,4))$$

$$= 1-2e^{-1} \simeq 0.264$$

$$P(N\leq 1,[8,12)) = P(N\leq 1,[0,4)) = 2e^{-1}$$

$$P(N\leq 1\,[0,4)\cap N\geq 2,[4,8)\cap N\leq 1,[8,12))$$

$$= P(N\leq 1,[0,4))\cdot P(N\geq 2,[4,8))\cdot P(N\leq 1,[8,12))$$ → due to independence nature of Poisson process

$$= 2e^{-1}\cdot(1-2e^{-1})\cdot 2e^{-1}$$

$$= 4e^{-2}(1-2e^{-1})$$

$$= 4e^{-2} - 8e^{-3} \simeq 0.143$$

ii) P(3rd failure after 4 hours) = P(at most 2 failures in 4 hours)

$$= P(N_4=0) + P(N_4=1) + P(N_4=2)$$

$$= e^{-1}\frac{1^0}{0!} + e^{-1}\frac{1^1}{1!} + e^{-1}\frac{1^2}{2!}$$

$$= e^{-1} + e^{-1} + \frac{1}{2}e^{-1} = \frac{5}{2}e^{-1} \simeq 0.9197$$

$$P(X=k \text{ in 4hours}) = e^{-4\lambda}\frac{(4\lambda)^k}{k!}, \quad k=0,1,2$$

$$= e^{-1}\frac{1}{k!} \quad (\lambda=0.25)$$

\# 3, (b) define the split of output $y(t)$ as: Fourier Transform

$$y(t) = y_s(t) + n(t) \qquad S(\omega) = F\{s(t)\} \ , \ H(\omega) = F(h(t))$$

$y_s(t) = s(t) \circledast h(t)$: the output signal part; $\circledast$: convolution operation.

$\cancel{\phi}\, n(t) = w(t) \circledast h(t)$: the output noise part

$$SNR = \frac{|y_s(t_0)|^2}{E\{|n(t)|^2\}} = \frac{|y_s(t_0)|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{nn}(\omega)\, d\omega} = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) H(\omega) e^{j\omega t_0} d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ww}(\omega) |H(\omega)|^2 d\omega}$$

since $w(t)$ is white noise ~~wh~~ with PSD $N_0 = 1$.

$$S_{ww}(\omega) = 1$$

$$SNR = \frac{\left|\int_{-\infty}^{+\infty} S(\omega) H(\omega) e^{j\omega t_0} d\omega\right|^2}{2\pi \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega}$$

using Cauchy-Schwarz's Inequality.

Parsavel's Theorem.

$$SNR \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega = \int_0^{\infty} s(t)^2 dt = E_s = \frac{E_s}{N_0}, (N_0 = 1)$$

$$\int_0^{\infty} s(t)^2 dt = \int_0^1 \sin^2(2\pi t)\, dt = \int_0^1 \frac{1}{2} - \frac{1}{2}\cos(2\pi t \cdot 2)\, dt$$

$$= \left[\frac{1}{2} t - \frac{1}{2}\sin(4\pi t)\right]_{t=0}^{t=1} \qquad \sin(4\pi) = 0,\ \sin(0) = 0$$

$$= \frac{1}{2} = E_s \Rightarrow SNR_{max} = \frac{1}{2}.$$

This is achieved only when $H(\omega) = S^*(\omega) e^{-j\omega t_0}$.

apply inverse Fourier Transform $h(t) = s(t_0 - t)$

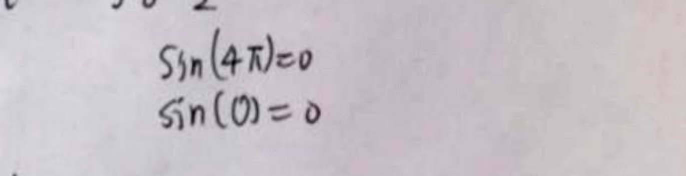
$s(t) = \sin(2\pi t)$, $0 < t < 1$ and 0 otherwise. $\qquad t_0 - t > 0$, $t < t_0$.

$h(t) = \sin[2\pi(t_0 - t)]$, $\quad 0 < t_0 - t < 1 \qquad t_0 - t < 1$, $t > t_0 - 1$

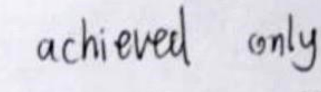
$\cancel{t_0 - 1 < t < t < t_0}$.

$t_0 - 1 < t < t_0$.

$h(t) = 0$, otherwise.



h(t)



#4. a) $i = £10$. win £1: head
lose £1: tail
stop: 0 or £50.

Given the results of first 10 flips:

6 heads → win £6.

4 tails → lose £4.

net: win £2, total asset £12.

This can be viewed as a new starting point of a new round of gamble, with new starting asset £12. $N=50$

This is assuming each flip results are independent, regardless of ~~choi~~ coin firness.

i) $p = \frac{1}{2}$. $P_i = \frac{N-i}{N} = \frac{50-12}{50} = \frac{38}{50} = 0.76$.

ii) $p = \frac{1}{3}$, $q = 1-p = \frac{2}{3}$. $\left(\frac{p}{q}\right) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$.

$$P_i = \frac{1-\left(\frac{p}{q}\right)^{N-i}}{1-\left(\frac{p}{q}\right)^{N}} = \frac{1-\frac{1}{2}^{(50-12)}}{1-\left(\frac{1}{2}\right)^{50}} \simeq 1$$

$0.5^{30} \simeq 9 \times 10^{-10}$

$0.5^{n} \simeq 0$ if $n > 30$

b). $0 < \lambda < 1$, λ is constant. $Y_n = \sum_{i=1}^{n} X_i \cdot \lambda^i = X_n \cdot \lambda^n + X_{n-1}\lambda^{n-1} \dots$

i). $E\{Y_{n+1} \mid Y_n, Y_{n-1} \dots Y_0\}$

$= E\{X_{n+1}\lambda^{n+1} + X_n\lambda^n + \dots \mid X_n\lambda^n + X_{n-1}\lambda^{n-1} \dots, Y_{n-1}, \dots, Y_0\}$

$= E\{X_{n+1}\lambda^{n+1} + Y_n \mid Y_n, Y_{n-1}, \dots Y_0\}$

$= E\{X_{n+1}\lambda^{n+1}\} + Y_n = \lambda^{n+1} E\{X_{n+1}\} + Y_n$.

$= \lambda^{n+1}\{P(\cancel{x(n+1)} X_{n+1} = -1) \cdot (-1) + P(X_{n+1} = 1) \cdot 1\} + Y_n$.

$= \lambda^{n+1}\{\frac{1}{2} \times 1 + \frac{1}{2} \times (-1)\} + Y_n = \lambda^{n+1} \cdot 0 + Y_n = Y_n$.

$E\{Y_{n+1} \mid Y_n, Y_{n-1}, \dots Y_0\} = Y_n$ ⇒ therefore Y_n is a martingale.

#4. b, ii) characteristic function j is complex number $\sqrt{-1}$

i is index

$$\phi_{Y_n}(w) = E(e^{jY_n w}) = \int_{-\infty}^{\infty} e^{jy_n w} f_{Y_n}(y_n)\,dy$$

$$= E\left(e^{jw\left(\sum_{i=0}^{n} X_i \cdot \lambda^i\right)}\right) = E\left\{e^{jw(X_n\lambda^n + X_{n-1}\lambda^{n-1} + \dots + X_0\lambda^0)}\right\}$$

$$= E\left\{e^{jwX_n\lambda^n} \cdot e^{jwX_{n-1}\lambda^{n-1}} \cdot \dots \cdot e^{jwX_0\lambda^0}\right\}$$

λ is a constant, so seperate out e^{λ^i} from expectation

$$= e^{\lambda^n} \cdot e^{\lambda^{n-1}} \cdot \dots e^{\lambda^0} E\left\{e^{jwX_n} \cdot e^{jwX_{n-1}} \cdots e^{jwX_0}\right\}$$

(a function of n and constant λ) X_i are i.i.d

$$= e^{\left(\sum_{i=0}^{n} \lambda^i\right)} E\{e^{jwX_n}\} E\{e^{jwX_{n-1}}\} \dots E\{e^{jwX_0}\}$$

$$= e^{\left(\sum_{i=0}^{n} \lambda^i\right)} \left(E\{e^{jwX_i}\}\right)^{n+1}$$

$\cos(-w) - j\sin(w)$
$+ \cos(w) + j\sin(w)$
$= 2\cos(w)$

$$E\{e^{jwX_i}\} = \tfrac{1}{2} e^{jw(-1)} + \tfrac{1}{2} e^{jw1} = \tfrac{1}{2}(e^{-jw} + e^{jw})$$

$$E\{e^{jwX_i}\} = \cos(w)$$

$$\phi_{Y_n}(w) = e^{\left(\sum_{i=0}^{n} \lambda^i\right)} \left(E\{e^{jwX_i}\}\right)^{n+1}$$

$$= e^{\left(\sum_{i=0}^{n} \lambda^i\right)} [\cos(w)]^{n+1}$$

#4, b, iii) $P(Y_{n+1} \in E) = P[(X_{n+1}\lambda^{n+1} + Y_n) \in E]$

$= \frac{1}{2} P[(X_{n+1} \cdot \lambda^{n+1} + Y_n) \in E \mid X_{n+1} = -1]$

$+ \frac{1}{2} P[(X_{n+1}\lambda^{n+1} + Y_n) \in E \mid X_{n+1} = 1]$

split the total probability of X_{n+1}

$= \frac{1}{2} P[(Y_n - \lambda^{n+1}) \in E] + \frac{1}{2} P[(Y_n + \lambda^{n+1}) \in E]$

$= \frac{1}{2} P[Y_n \in E.-\lambda^{n+1}] + \frac{1}{2} P[Y_n \in E.+\lambda^{n-1}]$

$E.-\lambda^{n+1}$ or $E.+\lambda^{n+1}$

$\Rightarrow$ addition / subtraction on all elements in E

$T_1(x) = \lambda x + 1, \quad T_1(x) - 1 = \lambda x, \quad x = \frac{T_1(x) - 1}{\lambda}$

$T_1^{-1}(z) = \frac{z-1}{\lambda}, \quad T_2^{-1} = \frac{z+1}{\lambda}$

$T_1^{-1}(E) = \frac{E-1}{\lambda}, \quad \frac{E-1}{\lambda} \cdot (-\lambda^{n+1}) = E \cdot -\lambda^n - \lambda^n$

$T_1(Y_n - \lambda^{n+1}) = \lambda \cdot Y_n (-) \lambda^{n+2} + 1$

$T_2(Y_n - \lambda^{n+1}) = \lambda Y_n \lambda^{n+2} - 1$

$T_1^{-1}(E) = X \quad T_1(x) = E$

$E = \lambda x + 1$

$x = \frac{E-1}{\lambda}$

$P(Y_{n+1} \in E) = \frac{1}{2} P[Y_n \in [E - \lambda^{n+1}]] + \frac{1}{2} P[Y_n \in [E + \lambda^{n+1}]]$

$= \frac{1}{2} P(Y_n \in T_1^{-1}(E)) + \frac{1}{2} P(Y_n \in T_2^{-1}(E))$

#4. b) iv) $\phi_{Y_n}(\omega) = e^{\left(\sum_{i=0}^{n} \lambda^i\right)} [\cos(\omega)]^{n+1}$.

$$E(Y_n) = \frac{1}{j} \frac{\partial \phi_{Y_n}(\omega)}{\partial \omega}\Big|_{\omega=0}$$

$$= \frac{1}{j} e^{\left(\sum_{i=0}^{n} \lambda^i\right)} \cdot (n+1)[\cos(\omega)]^n \cdot (-)\sin(\omega)\Big|_{\omega=0}.$$

$= 0$ as $\sin(0) = 0$.

$$Y_n = Y_{n-1} + X_n \lambda^n = \begin{cases} Y_{n-1} - \lambda^n & p = \frac{1}{2} \\ Y_{n-1} + \lambda^n & p = \frac{1}{2} \end{cases}$$

total transition probability $= 1$
Y_n won't stay at state Y_{n-1}.

$$Y_n = \begin{cases} Y_{n-1} - (\frac{1}{2})^n & (p = \frac{1}{2}) \\ Y_{n-1} + (\frac{1}{2})^n & (p = \frac{1}{2}) \end{cases} \qquad Y_{n+1} = \begin{cases} Y_n - (\frac{1}{2})^{n+1} & (\frac{1}{2}) \\ Y_n + (\frac{1}{2})^{n+1} & (\frac{1}{2}) \end{cases}$$

distribution of Y_n.

limiting distribution. ~~$Y_n \to E(Y_{n+1}) = Y_n$~~ $E(Y_{n+1}) = Y_n$.

$\frac{1}{2} Y_n - (\frac{1}{2})^{n+2} + \frac{1}{2} Y_n + (\frac{1}{2})^{n+2} = Y_n$. this always holds.

for limiting distribution., distribution of $Y_n = Y_{n+1}$'s distribution

$Y_\pi - (\frac{1}{2})^n = Y_\pi - (\frac{1}{2})^{n+1}$ $\qquad \lim_{n\to\infty} (\frac{1}{2})^{n+1} \Rightarrow 0$

as ~~n goes $\to \infty$~~

as $n \to \infty$
limiting distribution of Y_n is that
it has probability 1 to ~~state~~ stay at state Y_n from state Y_n.

limiting distribution : $P(\text{no transtion out from } Y_n) = 1$
$P(\text{stay at } Y_n \text{ state}) = 1$

this is reasonable. ~~As~~ $\lambda^n \to 0$ as $\lim_{n\to\infty} \lambda^n = 0$ $(0 < \lambda < 1)$
~~Y_{n+1}~~ Y_n as little change compared to Y_{n-1}
in limit this is no change at all.
This should be true for all $0 < \lambda < 1$, including $\lambda = \frac{1}{2}$