Wireless Communication Coursework 2

Xinyuan Xu

February 2020

1 System Model

In this coursework the transmission of a deterministic point to point n_r-by-n_t MIMO channel is considered. The system model is given by: $\mathbf{y} = \mathbf{Hc'}+\mathbf{n}$. $\mathbf{y}$ is a n_r-by-1 vector representing the received signal. $\mathbf{H}$ is the $n_r \times n_t$ channel matrix of i.i.d Rayleigh fading channel. $\mathbf{c'}$ is the $n_r \times 1$ transmitted symbol vector. $\mathbf{n}$ is the $n_r \times 1$ zero mean complex additive white Gaussian noise (AWGN) vector. $\rho = Es/\sigma_n^2$ represents the SNR. The input covariance matrix is given by $\mathbf{Q} = E\{\mathbf{c'c'}^H\}$ and is subject to the transmit power constraint $Tr\{\mathbf{Q}\} \leq P$. Perfect CSIT is assumed.

2 Compute analytically the capacity of simple MIMO channels

By definition, the capacity of a deterministic $n_r \times n_t$ MIMO channel with perfect channel state information at the transmitter is

$$C(\mathbf{H}) = \max_{Q \geq 0; Tr\{\mathbf{Q}\}=1} log_2 det(\mathbf{I}_{n_r} + \rho \mathbf{HQH}^H)$$

To find the optimum input covariance matrix $\mathbf{Q}$, Multiple Eignemode Transmission is used to create parellel data pipes. By singular value decomposition $\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^H$. $\Sigma = diag\{\sigma_1, ..., \sigma_n\}$, $\sigma_k^2 \triangleq \lambda_k$, $n = \min\{n_t, n_r\}$. The optimum input covariance matrix $\mathbf{Q}^*$ writes as $\mathbf{Q}^* = V diag\{s_1^*, ..., s_n^*\}\mathbf{V}^H$. After power allocation to data pipe, capacity of the channel is given by:

$$C(\mathbf{H}) = \sum_{k=1}^{n} log2(1 + \rho s_k^* \lambda_k)$$

The optimum power allocation is calculated using Water-Filling Algorithm. "Water-Level" μ at iteration i and power allocation $s_k(i)$ to each eigenmode (or virtual data pipe) is calculated by: $(\sum_{k=1}^{n} s_k^* = 1)$

$$\mu(i) = \frac{1}{n-i+1}(1 + \sum_{k=1}^{n-i+1}(\frac{1}{\rho\lambda_k}))$$

$$s_k(i) = \mu(i) - \frac{1}{\rho\lambda_k}$$

2.1 capacity of $\mathbf{H}_1$

Step 1: Singular Value Decomposition: $\mathbf{H}_1 = \mathbf{U}_1 \Sigma_1 \mathbf{V}_1^H$:
$\mathbf{U}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$, $\Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{V}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. So $\lambda_1 = 4, \lambda_2 = 0$.
Step 2: Power Allocation: Since one eigenvalue is 0, all the power is allocated to virtual channel 1. Thus, $s_1 = 1, s_2 = 0$.
Step 3: $C(\mathbf{H}_1) = log_2(1 + \rho \times 4 \times 1) + log_2(1 + 0) = log_2(1 + 4\rho)$.

2.2 capacity of $\mathbf{H}_2$

Step 1: Singular Value Decomposition: $\mathbf{H}_2 = \mathbf{U}_2 \Sigma_2 \mathbf{V}_2^H$:
$\mathbf{U}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Sigma_1 = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$, $\mathbf{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So $\lambda_1 = \lambda_2 = 2$.
Step 2: Power Allocation: Since two eigenvalues are equal, each channel would be allocated half the total power (this could be verified by using the water filling algorithm). Thus, $s_1 = s_2 = 0.5$.
Step 3: $C(\mathbf{H}_1) = 2 \times log_2(1 + \rho \times 2 \times 0.5) = 2log_2(1 + \rho)$.

2.3 Capacity comparison between $\mathbf{H}_1$ and $\mathbf{H}_2$

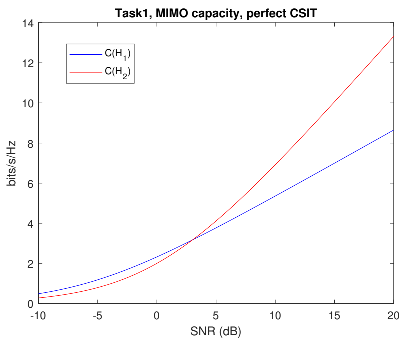

Figure 1: Capacity of $\mathbf{H}_1$ and $\mathbf{H}_2$

Solving the equation $C(\mathbf{H}_1) = C(\mathbf{H}_2)$, the solution is $\rho = 0$ or $\rho = 2$. Observing from figure 1, the two line indeed cross over at 3dB. If we call a virtual data pipe with larger λ as a "better" pipe, then $\mathbf{H}_1$ has only one but "better" pipe, compared to $\mathbf{H}_2$, which has two "equally-good" pipe. At low SNR, channel 1 could concentrate all the power to the "better" data pipe. So the capacity of channel 1 at low SNR ($\leq 3dB$) is higher. However, channel 2 enjoys the multiplexing gain of $n = 2$ (the pre-log factor). So at high SNR ($\geq 3dB$), channel 2 has higher capacity. The factor on SNR inside the log2 bracket is fixed, so it produces only an almost constant capacity gap. At low SNR, this gap is more manifest, but at high SNR, this small gap is overwhelmed by the benefit of higher gradient (multiplexing gain). For any channel H, capacity at low SNR is more likely to be achieved by allocating all the power to dominant eigenmode, like the case here in H_1;capacity at high SNR is more likely to be achieved by uniform power allocation, like the case here in H_2.

3 Evaluate the capacity of MIMO channels using Matlab

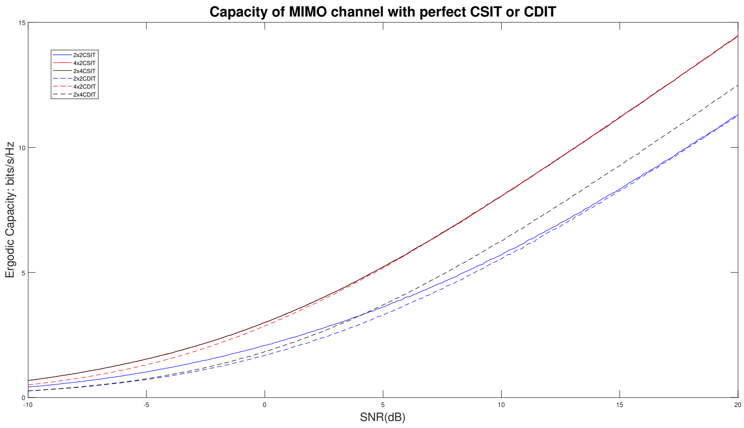

Figure 2: Capacity of i.i.d Rayleigh fading MIMO channels with full/partial knowledge at transmitter

In figure 2 The line for 2x4CSIT and 4x2CSIT channels completely overlaps. I think this is because the channel matrix of one would be line the transpose of the other, so the distribution of eigenvalues would be exactly the same. Thus the ergodic capacity would be the same. With perfect CSIT at high SNR, theoretically $C(\mathbf{H}) \cong nlog_2(\rho/n) + \sum_{k=1}^{n} log_2(\lambda_k)$, so the spatial multiplexing gain is $g_s = n = \min(n_r, n_t) = 2$ for the 3 types of channels. This is confirmed by the figure: all the solid line has the same gradient at high SNR. It can be seen that the capacity of 2x2CSIT channel is lower than the capacity of 2x4CSIT and 4x2CSIT channels. I believe this is because due to the smaller number of transmit/receive antennas, 2x2CSIT system has lower array gain, therefore the capacity line is a right shifted version of the other two.

The capacity when there is partial transmit channel knowledge is different. At low SNR, $\overline{C}_{CDIT} \geq E\{log_2[1 + \frac{\rho}{n_t}||\mathbf{H}_w||_F^2]\} \approx \frac{\rho}{n_t}E\{||\mathbf{H}_w||_F^2\}log_2(e) = n_r\rho log2(e)$. So at low SNR, 2x4 and 2x2 channel with CDIT would have approximately the same capacity. This is confirmed by the closely located blue and black dashed line smaller than 0dB. Red dashed line, representing the capacity of 4x2 channel with CDIT is much higher at low SNR as it has more receive antennas. At high SNR $\overline{C}_{CDIT} \approx E\{\sum_{k=1}^{n} log_2(\frac{\rho}{n_t}\lambda_k)\} = nlog_2(\frac{\rho}{n_t}) + E\{\sum_{k=1}^{n} log_2(\lambda_k)\}$.In figure, capacity of all the channels, with perfect CSIT or only CDIT, has the same gradient at around 20dB, because they have the same diversity gain $g_s = n = 2$.

$\overline{C}_{CSIT}$ and $\overline{C}_{CDIT}$ has a constant gap equal to $nlog_2(n_t/n)$ at high SNR. Because $n_t = 2$ for 2x2 and 4x2 channels, the dashed lines converge to solid lines at high SNR (red and blue). For 4x2 channel with CSIT, the gap theoretically equals to $2log_2(4/2) = 2$ and is confirmed by the data tips at 20dB.

4 BER vs SNR performance of Spatial Multiplexing and QPSK constellation over a 2×2 MIMO i.i.d Rayleigh fading channel

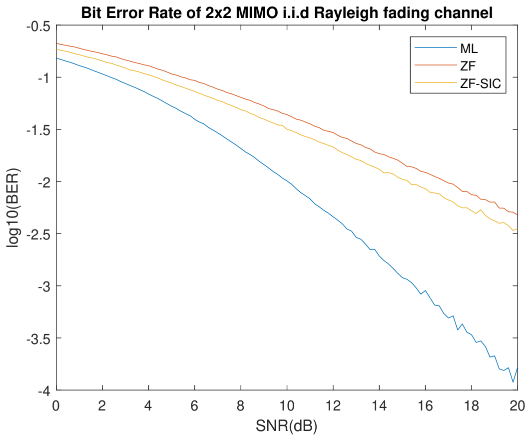

Figure 3: Bit Error Rate

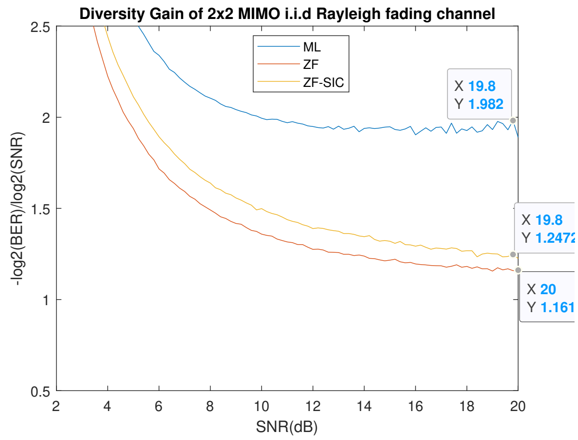

Figure 4: Diversity Gain

4.1 Maximum Likelihood Receiver

A joint ML detection is performance on the received signal. Let $\mathbf{c}_i$ be a 2×1 symbol vector inside code book $\mathbf{C}$; since there are two element in the vector and each element belongs to the QPSK constellation, there are 16 possibilities inside $\mathbf{C}$. An estimated of transmitted symbol is obtained by choosing the vector in codebook that has the smallest possibility to make and error:

$$\hat{\mathbf{c}} = \arg\min_{\mathbf{c}_i} ||\mathbf{y} - \sqrt{Es}\mathbf{H}\mathbf{c}_i||^2$$

In theory, the error probability is given by the equation below. The diversity gain should be n_r. This is because there is no coding across transmit antennas, so no transmitter diversity is obtained and only receive diversity is enjoyed. This is confirmed by Figure 4, the experimental diversity gain converges towards 2 at high SNR (around 20dB).

$$P(\mathbf{C} \to \mathbf{E}) \leq (\tfrac{\rho}{4n_t})^{-n_r}(\textstyle\sum_{q=1}^{n_t} |c_q - e_q|^2)^{-n_r}$$

4.2 Zero-Forcing (ZF) Linear Receiver

ZF filter tries to decouple the original channel into parallel virtual channels for each symbol, by suppressing the interference from other transmitted symbols. For each symbol period, the ZF filter could be obtained by: $\mathbf{G}_{ZF} = \sqrt{\frac{n_t}{Es}}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$, which is a scaled pseudo inverse of the channel matrix at that symbol period, assuming channel knowledge is perfectly estimated at receiver.

Transmitted symbols are separated by applying the filter: $\mathbf{z} = \mathbf{G}\mathbf{y}$. Then maximum likelihood detection is applied on each element of $\mathbf{z}$:
$\hat{\mathbf{c}_j} = \arg\min_{\mathbf{c}_i} ||\mathbf{z}_j - \mathbf{c}_i||^2, j = 1, 2$, where $\mathbf{c}_i$ belongs to the 4 possibilities in QPSK constellation.

However, suppression of inter-symbol-interference comes at the cost of noise enhancement. Channel inversion could degrade output SNR, thus bit error rate would be higher compared to ML receiver. This is confirmed by the blue and the higher red lines on figure 3.

Theoretically, the average Pairwise Error Probability (PEP) on the q-th subchannel is thus upper bounded by the equation below. So the diversity is limited to $n_r - n_t + 1 = 2 - 2 + 1 = 1$. The simulated result in Figure 4 confirms the theory.

$$P(c_q \to e_q) \leq (\tfrac{\rho}{4n_t})^{-(n_r-n_t+1)}|c_q - e_q|^{-2(n_r-n_t+1)}$$

4.3 ZF with Successive Interference Cancellation (SIC)

In this question, we assume the cancellation is perfect. The zero forcing filter is the same as previous quesion: $\mathbf{G}_{ZF} = \sqrt{\frac{n_t}{Es}}(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H = \begin{bmatrix} \leftarrow g_1 \rightarrow \\ \leftarrow g_2 \rightarrow \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix}\mathbf{h}_1, \mathbf{h}_2\end{bmatrix}$. Firstly, the row vector $\mathbf{g}_1$ is applied to the received signal, so that interference from symbol 2 should be suppressed: $\mathbf{g}_1\mathbf{h}_2 \approx 0$.
$\mathbf{z}_1 = \mathbf{g}_1\mathbf{y} = \sqrt{Es}\mathbf{g}_1\mathbf{h}_1 c_1 + n'$, ML detection is applied onto z_1 to obtain $\hat{c}_1$, the estimate of the 1st symbol. Then, assume the estimate achieves perfect cancellation, the interference of c_1 is removed from received signal: $\mathbf{y}_2 = \mathbf{y} - \sqrt{Es}\mathbf{h}_1\hat{c}_1 \approx \sqrt{Es}\mathbf{h}_1 c_2 + n''$. ML detection is applied on c_2.

At iteration i, the diversity gain experienced by that layer is $g_d(i) = n_r - n_t + i$. The system performance is limited by the layer with worst performance or highest probability of error, and this error propagates down the successive cancellation process. For unordered SIC, diversity gain is approximately the same: $g_d \approx n_r + n_t + 1$, this could be see from Figure 4. However, SIC definitely offers decoding performance gain, so ZF-SIC has slightly lower BER compared to ZF only, which is illustrated in Figure 3.