Univariate Trading systems:

$$\text{Cash Position} = \mu \oslash \text{Volatility}$$

Here $\oslash$ denotes the element-wise division of two vectors of the same length.

- Hedge funds embrace the construction of powerful μ. Quant Researchers, Traders
- Less effort is put into constructing a good measure for Volatility (often based on high-frequency data). Volatility shouldn't be too nervous but nevertheless respond fast.
- Combination of many such individual systems often considered to be a pure IT problem. Risk Managers or Committee.

$$
\begin{aligned}
x(t) = \arg\max_{x\in\mathbb{R}^n} & \; x^{\mathrm{T}}\mu, \\
\text{s.t. } & \; x^{\mathrm{T}}Qx \le \sigma_{\max}^2,
\end{aligned}
\tag{2.2}
$$

where $\cdot^{\mathrm{T}}$ denotes the transpose of a vector, μ is the vector of μ_i and Q is the matrix of σ_{ij}.

The set $\mathscr{F}$ of decision vectors that satisfy the constraints of an optimization problem is called the *feasible domain.* For the model problem this is the interior of an ellipsoid induced by the inequality constraint $x^{\mathrm{T}}Qx \le \sigma_{\max}^2$.

If $\mathscr{F}$ is a convex set (that is, for any pair of points $x, y \in \mathscr{F}$ the line segment $\{\xi x + (1-\xi)y : \xi \in [0,1]\}$ between x and y lies in $\mathscr{F}$), and if f is a convex function (that is, for any $x, y \in \mathscr{F}$ and $\xi \in [0,1]$, $f(\xi x + (1-\xi)y) \le \xi f(x) + (1-\xi)f(y)$), then the problem is called a *convex optimization problem*[1].

If Q is symmetric positive definite and $\mu \neq 0$, the analytic solution of this problem is given as

$$
x_* = \sigma_{\max} \frac{Q^{-1}\mu}{\sqrt{\mu^{\mathrm{T}}Q^{-1}\mu}}. \tag{2.3}
$$

We multiply the **cash position** c with 1:

$$c = \mu \oslash \text{Volatility} = (\mu \odot \text{Volatility}) \oslash (\text{Volatility} \odot \text{Volatility}).$$

Note that $\text{Volatility} \odot \text{Volatility}$ is the **variance**. We introduce the diagonal covariance matrix $\mathbf{Cov}$:

$$c = \mathbf{Cov}^{-1}(\,\mu \odot \text{Volatility}\,)$$

We introduce the variable x as:

$$x := c \odot \text{Volatility} = \mathbf{Cov}^{-1}\mu$$

Risk vs. Cash

- Asset allocation can be done both in **risk** or in **cash** space. Both approaches are equivalent.
- The risk (variance) of the portfolio is $c^T \, \mathbf{Cov} \, c = (c \odot \mathrm{Volatility})^T \mathbf{Cor} \, (\mathrm{Volatility} \odot c) = x^T \mathbf{Cor} \, x$
- The condition number for $\mathbf{Cor}$ is lower than the condition number for $\mathbf{Cov}$.
- The entries of x will spread less than the entries in c.
- The entries of x bear more insight than the entries of c.

Risk wins

Hence we solve for the risk position the Markowitz problem

$$\arg\max_{x \in \mathbb{R}^n} \; x^T \mu$$

such that

$$x^T \mathbf{Cor}\, x \leq \sigma_{\max}^2 .$$

The solution x^* is the optimal risk position. The analytic solution for x^* is given by

$$x^* = \sigma_{\max} \frac{\mathbf{Cor}^{-1} \mu}{\sqrt{\mu^T \, \mathbf{Cor} \, \mu}}$$

Note that the term in the denominator $\sqrt{\mu^T \,\mathbf{Cor}\, \mu}$ induces a norm (often called the A-norm).

If we assume there are no cross-correlations the correlation matrix boils down to the identity and we get:

$$x^* = \sigma_{\max}\, \mu \,/\, ||\mu||_2$$

The cash position $c^* = x^* \oslash \text{Volatility}$ is here a scaled version of the original univariate problem.

Note that it would be insane to use the sample correlation matrix. It is important to modify the spectrum of the observed correlation matrices.

Study in particular Sin 1 and Sin 2 from https://arxiv.org/pdf/1310.3396.pdf

Conclusions:

- We interpret $\frac{\mu}{\text{volatility}}$ as the (unscaled) solution of a convex program.
- The univariate trading system is now a special case of much richer class of multivariate trading systems.
- We work in the space of risk rather than cash positions.