CQF FINAL EXAMINATION

July 2020

There are two sections in this examination. Each question carries a weight of 25%.
You must answer two questions from each section.
More than four questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
Your answers must be submitted as a single pdf document. Any other format will not be marked.

You may assume throughout this examination that X_t or X is a standard Brownian motion.

Section A

A1. Consider the following model for a share price,

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	5	8	11
ω_2	5	8	5
ω_3	5	2	5
ω_4	5	2	0

The risk-free interest rate is 6%, discretely compounded.

a. Given a European put option with strike $E = 8$, construct a delta-hedged portfolio over the two periods using the above model to find the fair price of the option.

b. Find all the one period risk-neutral probabilities and confirm that the discounted expected value of the payoff is the price obtained in part **a.**

c. Hence, using the put-call parity, price a call option on an otherwise identical stock as described in **b.** with the same strike and expiry.

d. Repeat parts **a.**, **b.**, **c.** to price a binary option on the same underlying stock and same strike and expiry.

A2. a. Evaluate the Itô integral

$$\int_0^T \left(\sin t + X_t e^{X_t^2}\right) dX_t.$$

b. We know from Itô's lemma that

$$4\int_0^T X_t^3 dX_t = X_T^4 - X_0^4 - 6\int_0^T X_t^2 dt$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$4\int_0^T X_t^3 dX_t = \lim_{N\to\infty} 4\sum_{i=0}^{N-1} X_i^3 \left(X_{i+1} - X_i\right)$$

where $X_i \equiv X_{t_i}$.

c. Is the process

$$Y_t = (X_t - t)\exp\left(X_t + kt\right),\ \ t \geq 0,$$

a martingale, where k is a constant? Justify your answer.

A3. **Note: There will be no credit for simply copying from the course slides**

a. Describe (with examples) the following features of exotic options

   i. *discrete and continuous sampling*

   ii. *fixed and floating strike options.*

b. In the context of classifying exotic options, discuss

   i. *time dependence.*

   ii. *higher dimensions.*

c. Find updating rules and jump conditions for the following discretely sampled options

   i. an Asian option with payoff dependent on the geometric average of the sampled asset prices

   ii. an Asian option with payoff dependent on an exponentially-weighted arithmetic average of the sampled asset prices.

A4. **a.** Find the values of the following portfolios of European options at expiry, as a function of the share price S:

i. Long one share, long one put with exercise price E.

ii. Long one call and one put, with exercise price E.

iii. Long one call, exercise price E_1, short one call, exercise price E_2, where $E_1 < E_2$.

iv. Long one call, exercise price E_1, long one put, exercise price E_2. There are three cases to consider.

b. Complete the table for call and put options

Position	**Strategy**	**Max loss**	**Max gain**	**Breakeven**
Long Call	Bullish	–	–	–
Short Call	Bearish/neutral	–	–	–
Long Put	Bearish	–	–	–
Short Put	Bullish/neutral	–	–	–

c. Explain the difference between hedging, speculation and arbitrage. You may use an example in each case.

Section B

B1. a. Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}; \quad t \in [0, T], \quad 0 < x < L \tag{1.1}$$

where the unknown function $u = u(x,t)$; c^2 is a constant. To discretise the equation, take N and M steps for x and t respectively, so

$$\begin{aligned} x_n &= n\delta x \qquad & 0 \le n \le N \\ t_m &= m\delta t \qquad & 0 \le m \le M, \end{aligned}$$

where $\delta x = \frac{L}{N}$; $\delta t = \frac{T}{M}$. Derive the following approximations

$$\begin{aligned} \frac{\partial u}{\partial t}(n\delta x, m\delta t) &\sim \frac{u_n^{m+1} - u_n^m}{\delta t}, \\ \frac{\partial^2 u}{\partial x^2}(n\delta x, m\delta t) &\sim \frac{u_{n-1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}}{\delta x^2}. \end{aligned}$$

By writing $r = c^2 \dfrac{\delta t}{\delta x^2}$, derive the following **forward marching scheme** for (1.1)

$$A u_{n-1}^{m+1} + B u_n^{m+1} + C u_{n+1}^{m+1} = u_n^m, \tag{1.2}$$

where A, B, C should be stated.

b. Assume an initial disturbance E_n^m given by

$$E_n^m = a^m e^{in\omega}, \tag{1.3}$$

which is oscillatory of amplitude a and frequency ω; $i = \sqrt{-1}$. By substituting (1.3) in (1.2) obtain a stability condition for the forward marching scheme given by (1.2).

B2. The formula for a down-and-out call option $V_{DO}(S,t)$ is given by

$$V_{DO}(S,t) = C(S,t) - \left(\frac{S}{S_d}\right)^{1-2r/\sigma^2} C\left(S_d^2/S, t\right), \qquad (2.1)$$

where $C(S,t)$ is the value of a vanilla call option with the same expiry and payoff as the barrier option. The down barrier is set at S_d. Show that (2.1) satisfies the Black-Scholes partial differential equation given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \qquad (2.2)$$

S is the underlying asset price, t is time, $r > 0$ is the constant risk-free interest rate, $\sigma > 0$ is the constant volatility.

Hint: Show that $S^{1-2r/\sigma^2} V(Y^2/S, t)$ satisfies Black-Scholes for any Y, when $V(S,t)$ satisfies (2.2).

B3. a. Consider the following SDE

$$d\sigma = a(\sigma,t)dt + b(\sigma,t)dX_t.$$

Define $a = a(\sigma,t)$ and $b = b(\sigma,t)$. The Forward Kolmogorov Equation, for the transition pdf $p(\sigma,t;\sigma',t')$ is

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial \sigma'^2}(b^2 p) - \frac{\partial}{\partial \sigma'}(ap),$$

where the primed variables refer to future states. The steady state solution is given by setting $\frac{\partial p}{\partial t'} = 0$. By considering suitable boundary conditions, show that the steady state solution is given by

$$p(\sigma') = \frac{A}{b^2}\exp\left(\frac{2a}{b^2}d\sigma'\right),$$

where A is an arbitrary constant. (During your working you may drop the primed notation).

b. Consider the process

$$d\sqrt{v} = \left(\alpha - \beta\sqrt{v}\right)dt + \delta dX_t$$

The parameters $\alpha,\ \beta,\ \delta$ are constant. Using Itô's lemma obtain a stochastic differential equation for $\exp(v)$.

c. Consider the Backward Kolmogorov equation

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + \mu S\frac{\partial p}{\partial S} = 0,$$

for the transition probability density function $p = p(t,S)$ where S satisfies $dS = \mu S dt + \sigma S dX_t$. Both μ, σ are constants. Using three substitutions, transform this Kolmogorov equation to a one-dimensional heat equation of the form

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2}$$

for the function $W = W(\tau, x)$.

B4. a. Suppose the spot interest rate r, which is a function of time t, satisfies the stochastic differential equation

$$dr = u(r,t)\,dt + w(r,t)\,dX_t.$$

The bond pricing equation for a security $Z = Z(r,t;T)$ is

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + \left(u(r,t) - \lambda(r,t)\,w(r,t)\right)\frac{\partial Z}{\partial r} - rZ = 0,$$

where T is the maturity of the bond. $\lambda(r,t)$ is the market price of interest rate risk. **You are not required to derive this equation**

The extended Hull and White model is given by

$$dr = (\theta(t) - \alpha r)\,dt + \sqrt{\beta}dX_t.$$

where $\theta(t)$ is an arbitrary function of time t; α and β are constants. Deduce that the value of a **zero-coupon bond**, $Z(r,t;T)$ which has redemption value $Z(r,T;T) = 1$, in this model is given by

$$Z(r,t;T) = e^{A(t;T) - rB(t;T)}$$

where $A(t;T)$ and $B(t;T)$ must be calculated.

b. The drift in the Hull and White model in part a. is to be adapted for the **calibration** procedure which is performed at time $t = t^*$, when the spot rate is r^*. The market prices of bonds are denoted by $Z_M(t^*;T)$. Show that this implies

$$\begin{aligned}\theta^*(t) \;\; &= \;\; -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) - \alpha\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \\ &\quad\;\; \frac{\beta}{2\alpha}\left(1 - \exp\left(-2\alpha(t - t^*)\right)\right)\end{aligned}$$