Handbook of Mathematical Formulae

Certificate in Quantitative Finance

General Mathematics

Exponential and Logarithmic Functions

$$\lim_{n \longrightarrow \infty} \left(1 + \frac{x}{n}\right)^n \equiv e^x = \exp(x)$$

$$\exp(x+y) = e^x e^y, \quad \exp(xy) = (e^x)^y = (e^y)^x$$

$$a^p = N \Rightarrow p = \log_a N$$

$$\log(xy) = \log(x) + \log(y), \quad \log\left(\frac{x}{y}\right) = \log(x) - \log(y), \quad \log(x^y) = y \log x$$

$$\log(e^x) = x, \quad \exp(\log x) = x$$

$$e^0 = 1, \quad e^1 \approx 2.71828$$

$$\lim_{x \to \infty} e^x \longrightarrow \infty, \quad \lim_{x \to \infty} e^{-x} \longrightarrow 0$$

Trigonometric Functions and Identities

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x$$

$$\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
\sin 2x &= 2 \sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \\
\sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} \\
\sin x - \sin y &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} \\
\cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} \\
\cos x - \cos y &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \\
\sin\left(x + \frac{\pi}{2}\right) &= \cos x, \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x \\
\tan x &= \frac{\sin x}{\cos x}
\end{aligned}$$

If $i \in \mathbb{C}$

$$\text{Euler's Identity} \quad e^{ix} = \cos x + i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Hyperbolic Functions and Identities

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right), \quad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= 1 \\
\sinh 2x &= 2 \sinh x \cosh x \\
\cosh 2x &= \cosh^2 x + \sinh^2 x \\
\sinh (x + y) &= \sinh x \cosh y + \cosh x \sinh y \\
\cosh (x + y) &= \cosh x \cosh y + \sinh x \sinh y \\
\operatorname{ar}\cosh x &= \log\left(x + \sqrt{x^2 - 1}\right) \\
\operatorname{ar}\sinh x &= \log\left(x + \sqrt{x^2 + 1}\right) \\
\operatorname{ar}\tanh x &= \frac{1}{2}\log\left(\frac{1 + x}{1 - x}\right)
\end{aligned}$$

Standard Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}e^{ax} = ae^{ax}$$
$$\frac{d}{dx}\log x = \frac{1}{x}$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}arc\sin(ax) = \frac{a}{\sqrt{1-a^2x^2}}$$
$$\frac{d}{dx}arc\cos(ax) = -\frac{a}{\sqrt{1-a^2x^2}}$$
$$\frac{d}{dx}arctan(ax) = \frac{a}{1+a^2x^2}$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x$$
$$\frac{d}{dx}\mathrm{cosec} x = -\mathrm{cosec} x \cot x$$
$$\frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\tanh x = \mathrm{sech}^2 x$$
$$\frac{d}{dx}(\mathrm{ar}\sinh x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\mathrm{ar}\cosh x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\mathrm{ar}\tanh x) = \frac{1}{1-x^2}$$

Rules for Differentiation

If $f(x)$ and $g(x)$ are differentiable functions of x and λ and μ are constants then

$$\frac{d}{dx}\left(\lambda f(x) + \mu g(x)\right) = \lambda \frac{df}{dx} + \mu \frac{dg}{dx},$$

$$\frac{d}{dx}\left(f(x)\, g(x)\right) = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx},$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{\left(g(x)\right)^2}$$

If $f(u)$ is a differentiable function of u and $u = u(x)$ is a differentiable function of x then the **chain rule** gives

$$\begin{aligned}\frac{d}{dx} f(u) &= \frac{du}{dx}\frac{d}{du} f(u) = u'(x)\, f'(u) \\ &= u'(x)\, f'(u(x)).\end{aligned}$$

Chain Rules for Partial Derivatives

If $x = x(u,v)$, $y = y(u,v)$ and $F(u,v) = F\ (x(u,v),\ y(u,v))$, then

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v}.$$

or in operator form

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}.$$

Binomial Series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\times 2}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{1\times 2\times\ldots\times r}x^r+\ldots \qquad (|x|<1, n\in\mathbb{R})$$

Taylor Series

If $f(x)$ is an analytic function of x near $x = x_0$ then

$$f(x_0+\delta x) = f(x_0) + f'(x_0)\,\delta x + \tfrac{1}{2!}f''(x_0)\,\delta x^2 + \tfrac{1}{3!}f'''(x_0)\,\delta x^3 + \ldots.$$

If $f(x,y)$ is an analytic function of x, y then

$$\begin{aligned} f(x_0+\delta x, y_0+\delta y) = {} & f(x_0,y_0) + f_x(x_0,y_0)\,\delta x + f_y(x_0,y_0)\,\delta y \\ & \tfrac{1}{2!}f_{xx}(x_0,y_0)\,\delta x^2 + \tfrac{1}{2!}f_{yy}(x_0,y_0)\,\delta y^2 + f_{xy}(x_0,y_0)\,\delta x\delta y \\ & + \ldots. \end{aligned}$$

Standard Taylor Series Expansions about $x_0 = 0$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots \\ arc\tan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \\ \operatorname{ar}\tanh x &= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \end{aligned}$$

Standard Integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad (n \neq -1),$$
$$\int \frac{dx}{x} = \log(x) + C,$$
$$\int e^{ax} dx = \frac{1}{a}e^{ax} + C \qquad (a \neq 0),$$
$$\int \cos ax dx = \frac{1}{a}\sin ax + C$$
$$\int \sin ax dx = -\frac{1}{a}\cos ax + C$$
$$\int \sec^2 ax dx = \frac{1}{a} tan ax + C$$
$$\int \tan x dx = \log|\sec x| + C$$
$$\int \cot x dx = \log|\sin x| + C$$
$$\int \mathrm{cosec} x dx = -\log|\mathrm{cosecx} + \cot x| + C$$
$$\int \sec x dx = \log|\sec x + \tan x| + C$$
$$\int \sinh x dx = \cosh x + C$$
$$\int \cosh x dx = \sinh x + C$$
$$\int \tanh x dx = \log\cosh x + C$$
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = arc\sin\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} arc\tan\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = ar\cosh\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = ar\sinh\left(\frac{x}{a}\right) + C$$

where C is an arbitrary constant of integration and $a \neq 0$ is a given constant.

Rules for Integration

Change of Variables/Substitution:
If we have an integral of the form

$$\int g\left(f\left(x\right)\right) f'\left(x\right) dx,$$

by writing $z = f\left(x\right)$ so that $dz/dx = f'\left(x\right)$ or $dz = f'\left(x\right) dx$, then the integral becomes

$$\int g\left(z\right) dz.$$

Integration by parts:
If we have an integral of the form

$$\int \frac{du}{dx} v dx$$

it can be expressed as

$$\int \frac{du}{dx} v dx = u\left(x\right) v\left(x\right) - \int u\left(x\right) \frac{dv}{dx} dx + C.$$

Numerical Integration

The trapezium rule:

$$\int_a^b y dx \approx \frac{1}{2} h\{(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\}, \quad \text{where } h = \frac{b-a}{n}$$

Leibniz Rule

$$\begin{aligned} \frac{d}{dx} \int_{\alpha(x)}^{\beta(x)} G(x,t).dt &= G\left(x, \beta(x)\right) \frac{d\beta}{dx} - G\left(x, \alpha(x)\right) \frac{d\alpha}{dx} \\ &+ \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}.dt \end{aligned}$$

Probability

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A \cap B) &= P(A)P(B \mid A) \\
P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}
\end{aligned}
$$

Expectation Algebra

For independent random variables X and Y

$$
\begin{gathered}
\mathbb{E}[aX \pm bY] = a\mathbb{E}[X] \pm b\mathbb{E}[Y] \\
\mathbb{E}[X \times Y] = \mathbb{E}[X] \times \mathbb{E}[Y] \\
Var[aX \pm bY] = a^2 Var[X] + b^2 Var[Y]
\end{gathered}
$$

Jensen's Inequality

$$\sqrt{\mathbb{E}[Var]} \geq \mathbb{E}[\sqrt{Var}]$$

Discrete Distributions

For a random variable X taking values x_i with probabilities $P(X = x_i)$

Mean, μ: $\mathbb{E}[X] = \sum x_i \times P(X = x_i)$

Variance, σ^2: $Var[X] = \sum (x_i - \mu)^2 \times P(X = x_i) = \sum x_i^2 \times P(X = x_i) - \mu^2$

For a function, g(X): $\mathbb{E}[g(x)] = \sum g(x_i) \times P(X = x_i)$

Binomial Distribution: $B(n, p)$

$$P(X = x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x (1-p)^{n-x}; \qquad \text{Mean} = np; \qquad \text{Variance} = np(1-p)$$

Poisson Distribution: $Po(\lambda)$

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}; \qquad \text{Mean} = \lambda; \qquad \text{Variance} = \lambda$$

Continuous Distributions

For a random variable X following probability density function $f(x)$

Mean, μ: $\mathbb{E}[X] = \displaystyle\int x f(x) dx$

Variance, σ^2: $Var[X] = \displaystyle\int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$

For a function, g(X): $\mathbb{E}[g(x)] = \int g(x_i)f(x)dx$

Cumulative distribution function: $F(x_0) = P(X \leq x_0) = \int_{-\infty}^{x_0} f(t)dt$

Normal Distribution: $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad \text{Mean} = \mu; \qquad \text{Variance} = \sigma^2$$

Standard Normal Distribution: $N(0, 1)$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}; \qquad \text{Mean} = 0; \qquad \text{Variance} = 1$$

Moment Generating Function

The Moment Generating Function of a random variable X is $M_X(t)$ is

$$M_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx}f(x)dx$$

The nth moment can be determined by

$$E(X^n) = M_X^{(n)}(0) = \frac{d^n M_X}{dt^n}(0)$$

Copula Function

For random variables $X_1, X_2, \cdots X_n$ with distributions $F_1, F_2, \cdots, F_n$

$$C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)) = F(x_1, x_2, \cdots, x_n)$$

where $C(u_1, u_2, \cdots, u_n; \rho)$ is the joint distribution function of uniform random variables $u_1, u_2, \cdots, u_n$

Gaussian Copula:

$$C(u_1, u_2, \cdots, u_n; \Sigma) = \frac{1}{\sqrt{\Sigma}}exp\left(-\frac{1}{2}U'(\Sigma^{-1} - I)U\right)$$

Frank Copula:

$$C(u_1, u_2, \cdots, u_n; \alpha) = -\frac{1}{\alpha}\ln\left[1 + \frac{\prod_{i=1}^{n}(e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right]$$

Special Integrals involving e^{-x^2}

The error function, $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds,$$

$$\operatorname{erf}(-x) = -\operatorname{erf}(x)\,; \quad \operatorname{erf}(\infty) = 1.$$

The complimentary error function $\operatorname{erf} c\,(x)$ is defined by

$$\begin{aligned} \operatorname{erf} c\,(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-s^2} ds \\ &= 1 - \operatorname{erf}(x) \end{aligned}$$

The Cumulative Density Function for the Normal Distribution, $N(x)$ is defined by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-s^2/2} ds.$$

$$\int_{-\infty}^\infty e^{-s^2} ds = \sqrt{\pi}$$

Quantitative Finance

Transition Probability Density Function

Consider the random walk

$$dy = A(y,t)dt + B(y,t)dX$$

The transition probability density function $p(y,t;y',t')$ satisfies two equations

Fokker-Planck or forward Kolmogorov equation:

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B(y',t')^2 p\right) - \frac{\partial}{\partial y'}\left(A(y',t')p\right)$$

Backward Kolmogorov equation:

$$\frac{\partial p}{\partial t} + \frac{1}{2}B(y,t)^2\frac{\partial^2 p}{\partial y^2} + A(y,t)\frac{\partial p}{\partial y} = 0$$

Itô's Lemma

Consider a function f(t,W) where t is time and W a Weiner process, then Itô's Lemma states

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W}dW$$

In integral form, this is

$$\int_0^t \frac{\partial f}{\partial W}dW = f(t,W_t) - f(0,W_0) - \int_0^t \left(\frac{\partial f}{\partial \tau} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)d\tau$$

Consider the function $V(S,t)$ where $dS = \mu S dt + \sigma S dW$, then Itô's Lemma states

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\sigma S\frac{\partial V}{\partial S}\right)dW$$

Black-Scholes Formula

Consider an asset which pays a continuous dividend yield D and evolves according to Geometric Brownian Motion

$$\frac{dS}{S} = (\mu - D)\, dt + \sigma dW.$$

The Black-Scholes pricing equation for this is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)\, S \frac{\partial V}{\partial S} - rV = 0.$$

For a European call option with strike E and expiry T, written on the above asset is

$$V(S,t) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$$

where r is the interest-rate and

$$\begin{aligned} d_1 &= \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}; \\ d_2 &= \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Greeks

$$\begin{aligned} \text{Theta, } \Theta &= \frac{\partial V}{\partial t} \\ \text{Delta, } \Delta &= \frac{\partial V}{\partial S} \\ \text{Gamma, } \Gamma &= \frac{\partial^2 V}{\partial S^2} \\ \text{Vega} &= \frac{\partial V}{\partial \sigma} \\ \text{Rho, } \rho &= \frac{\partial V}{\partial r} \end{aligned}$$

Fundamental Asset Pricing Formula

At time t, the value of a derivative maturing at time T is equal to the expected value of the discounted terminal cashflow of the contract under the risk-neutral measure $\mathbb{Q}$

$$V(t, S_t) = B_t \mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} G(S_T) | \mathcal{F}_t \right]$$

Martingale Conditions

Adapted to filtration $\mathcal{F}_s$, a martingale satisfies

$$\mathbb{E}\left[|M(t)|\right] < \infty$$

$$\mathbb{E}\left[M(t)|\mathcal{F}_s\right] = M(s) \qquad \forall t > s.$$

Exponential Martingale for GBM:

$$M(t) = exp\left(S_t - \left(\mu + \frac{1}{2}\sigma^2\right)t\right)$$

Stochastic Interest Rate Models

For the risk neutral spot rate

Vasicek model:

$$dr = (\eta - \gamma r)dt + \beta^{\frac{1}{2}} dX$$

Cox, Ingersoll & Ross:

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r} dX$$

Ho & Lee:

$$dr = \eta(t)dt + \beta^{\frac{1}{2}} dX$$

Hull & White:

$$dr = (\eta(t) - \gamma(t)r)dt + \beta^{\frac{1}{2}}(t)dX$$

$$dr = (\eta(t) - \gamma(t)r)dt + \sqrt{\alpha(t)r} dX$$

The Heath, Jarrow and Morton Model for forward rates:

$$dF(t;T) = \frac{\partial}{\partial T}\left(\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right)dt - \frac{\partial}{\partial T}\sigma(t,T)dX$$

where

$$Z(t;T) = e^{-\int_t^T F(t;s)ds}$$

and

$$dZ(t;T) = \mu(t,T)Z(t;T)dt + \sigma(t,T)Z(t;T)dX$$