P346 Project

Adaptive Quadrature

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Abstract

Numerical methods are often used to approximate complex integrals. Standard protocol dictates sub-dividing the domain of interest into smaller intervals of equal width so as to get a good approximation of the integral. However, depending on the function, the behavior in each sub-interval may not be the same. Thus, the error in approximation becomes different in each of these intervals. In order to tackle this issue, adaptive measures are sought after. A good adaptive method estimates number of sub divisions needed for different intervals where the curve lies, thus reducing the error.

Introduction

Numerical Integration is a useful tool to calculate definite integrals which would be tougher to do analytically or are known only at certain points. When the integral is performed along only one dimension, it is usually referred to as *numerical quadrature*. The numerical integration step can be understood by this formula;

$$I = \int_a^b f(x)dx \approx \sum_{i=1}^{N} w(x_i)f(x_i) \tag{1}$$

here, we assume f to be a piece-wise continuous, bounded and well-defined (smooth) function over $[a, b]$. $w(x_i)$ is the weight function and N is the number of integration points in the sub-division of the interval.

. In numerical integration, we choose the interval length and subsequently, the number of intervals N. Our choice depends on the error or tolerance we are willing to accept. We will be looking at the Trapezoid method, whose maximum error is known to us. We will then see analogous result in the Simpson method.

Drawbacks

Since, our main concern is the acceptable tolerance, it is important to know the form of error that comes with each quadrature method. For the most part, the error depends on higher derivatives of the function. The generalized form would be;

$$\epsilon \leq \frac{(b-a)^k}{\lambda \cdot N^{k-1}} \cdot |f^{(k-1)}|_{max} \tag{2}$$

Now, the length of each interval is given by;

$$h = \frac{(b-a)}{N} \tag{3}$$

Using this in equation 2, we have;

$$\epsilon \leq \frac{(h)^k N}{\lambda} \cdot |f^{(k-1)}|_{max} \tag{4}$$

This seems counter-intuitive since we expect the error to be minimum as N increases. This problem could come up due to many reasons. The two main reasons would be:

1. h is taken to be constant over all intervals.

2. f behaves differently over each interval.

The second reason is intrinsic to the function and not much could be done about it. We see an example where f has different behaviour over a same interval length.

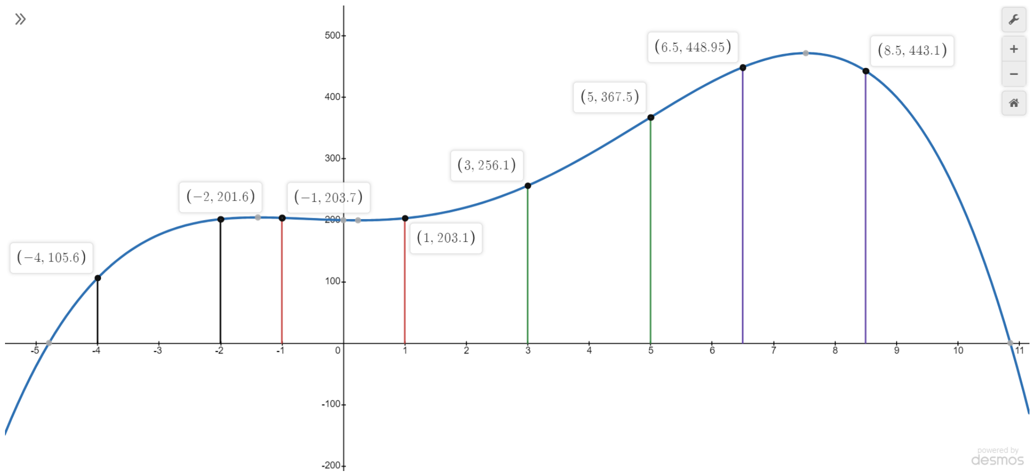

Figure 1: A cubic function with subdivided intervals of same length

We see that the red interval is almost like perfect trapezoid. Thus, integrating over few sub-intervals would give us a pretty good result. The black and green intervals display monotonous behaviours, however, the curvature is different. Applying simple trapezoid algorithms would have very different error approximations on both the intervals. In the black interval, the code that would underestimate the area would overestimate the area in the green interval. The ends of the purple interval are not that far apart i.e $f(x_i) - f(x_i + h)$ is small. However, from the graph we know that if we consider this interval length then we would get a large deviation from the analytical values.

We can overcome this by choosing smart intervals for each section of the graph. A good code would do this choosing and identifying itself.

Adaptive Advantage

The main principle underlying the adaptive measures will be that number and length of intervals will be chosen so be within the acceptable tolerance levels. We look at this through the Trapezoid method.

Consider the single interval $[a, b]$. Thus, $N = 1$ and the integration value will be;

$$I \approx \frac{h}{2}(f(a) + f(b)) - \frac{(b-a)^3}{24} \cdot f^{''}(\gamma) \tag{5}$$

where, $\gamma \in [a, b]$. For ease, we use h, thus;

$$I \approx \frac{h}{2}(f(a) + f(b)) - \frac{h^3}{24} \cdot f^{''}(\gamma) = I_1 + \epsilon_1 \tag{6}$$

where, I_k is the Trapezoid integral with $'k'$ intervals and ϵ_k is the corresponding error of the interval.

Now, if we bisect the interval at a point, say c, we don't increase N by 1 since we consider this new interval as it's own. Rather,h gets reduced by half. The integration value of one interval would thus be;

$$I_2^{[a,c]} + \epsilon_2 = \frac{h}{4}(f(a) + f(c)) - \frac{(h/2)^3}{24} \cdot f^{''}(\gamma) \tag{7}$$

the complete integral over $[a, b]$ will then be;

$$I \approx \frac{h}{4}(f(a) + 2f(c) + f(b)) - \frac{(h/2)^3}{24} \cdot (f^{''}(\gamma_1) + f^{''}(\gamma_2)) \tag{8}$$

where, $\gamma_1 \in [a, c]$ & $\gamma_2 \in [c, b]$.

We now assume that a function, for our purpose, is well-defined if it does not jump rapidly. This means that its higher derivatives don't increase or decrease rapidly. So for a large number of intervals we can say $f^{''}(\gamma_m) \approx f^{''}(\gamma_k)$, $k \neq m$. We then use a general γ.

For now, we make this approximation for 2 intervals as well. Also, we must have equations 5 and 8 to be equal since they are the same integral, with different step-sizes. Thus, we have;

$$\frac{h}{2}(f(a) + f(b)) - \frac{(b-a)^3}{24} \cdot f^{''}(\gamma) = \frac{h}{4}(f(a) + 2f(c) + f(b)) - \frac{(h/2)^3}{24}(2 \cdot f^{''}(\gamma)) \tag{9}$$

We may go one step further and take $I_1 \approx I_2$, thus, relating the two errors as;

$$\boxed{\epsilon_1 = 4 \times \epsilon_2} \tag{10}$$

we infer two things from this;

1. The error has reduced.
2. We can obtain ϵ_2 in terms of I_1 & I_2, which is easy to code.

Explicitly;

$$|\epsilon_2| = \frac{|I_1 - I_2|}{3} \tag{11}$$

by setting a tolerance, we can run an iterative loop till we get $|\epsilon_k| \leq \sigma$, where σ is our tolerance. We do this for each interval to get ideal interval lengths for each section of the graph of our given function.