Jänich: Fundamental Concepts

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Contents

1 The Concept of a Topological Space

We proceed assuming baisc familiarity with sets, as seen in RealAnalysis/babyrudin2.tex.

1.1 Definition

A **topological space** $(X, \mathcal{O})$ is a set X and a set $\mathcal{O}$ of subsets of X, called **open sets**, such that the following three axioms hold:

1. Any arbitrary union of open sets in $\mathcal{O}$ lies in $\mathcal{O}$.
2. Any finite intersection of open sets in $\mathcal{O}$ lies in $\mathcal{O}$.
3. X itself and $\varnothing$ are open sets in $\mathcal{O}$.

The set $\mathcal{O}$ is a **topology** in X. Henceforth, we will speak of a topological space X instead of the pair $(X, \mathcal{O})$. The following concepts are critical to Point-Set Topology:

1. A set $F \subseteq X$ is **closed** if its complement is open; that is, if $F^{\complement} \subset \mathcal{O}$.
2. A set $N \subset X$ is a **neighborhood** of x if N contains an open set U which contains x.
3. A point x is an **interior**, **exterior**, or **boundary point** of a set S according to whether S, $S^{\complement}$, or neither is a neighborhood of x.
4. The set $\mathring{S}$ of all the interior points of S is the **interior** of S.
5. The set $\overline{S}$ of all the interior and boundary points of S is the **closure** of S.

Naturally, exterior points of S are interior points of $\overline{S}$ and the complement of $\mathring{S}$ is $\overline{S^{\complement}}$.

1.2 Basic Consequences

The following theorem allows for an alternative definition of topological spaces $(X, \mathcal{O})$ by *closed sets*:

Theorem 1. *Let X be a topological space. Then*

1. *Any arbitrary intersection of closed sets is closed.*
2. *Any finite union of closed sets is closed.*
3. *X itself and $\varnothing$ are closed.*

Proof. Let F_α be a collection of closed sets. Since their compliments are open, we have

$$F_\alpha^{\complement} \text{ are open} \implies \bigcup_\alpha F_\alpha^{\complement} \text{ is open} \implies \left(\bigcup_\alpha F_\alpha^{\complement}\right)^{\complement} \text{ is closed} \implies \bigcap_\alpha F_\alpha \text{ is closed.}$$

If we let F_n be closed for $n \in \{1, \ldots, k\}$, a similar argument follows:

$$F_n^{\complement} \text{ are open} \implies \bigcap_{n=1}^{k} F_n^{\complement} \text{ is open} \implies \left(\bigcap_{n=1}^{k} F_n^{\complement}\right)^{\complement} \text{ is closed} \implies \bigcup_{n=1}^{k} F_n \text{ is closed.}$$

The sets X and $\varnothing$ are clearly closed, which completes the proof. □

Theorem 2. *A set is open if and only if all of its points are interior.*

Proof. If each point of S is interior, then $x \in S$ implies the existence of an open set U_x such that $x \in U_x \subseteq S$. Thus we define:

$$U = \bigcap_{x \in \mathring{S}} U_x.$$

Observe that U is open. We claim that $S = U$ by a two-part argument:

1. Suppose $x \in S$. Then clearly $x \in U_x \subseteq U$, so $S \subseteq U$.
2. Suppose $x \in U$. Then $x \in U_y$ for some y; since $U_y \subseteq S$, we have $x \in S$. Thus $U \subseteq S$.

Hence S is open. The reverse direction follows naturally: if S is open and $x \in S$, then S is a neighborhood of all x. Hence all x is interior. □

Theorem 3. *The interior of S is the union of all open sets contained in S.*

Proof. Let U be the union of all open sets contained in S. We claim that $\mathring{S} = U$ by a two-part argument:

1. Suppose $x \in \mathring{S}$. Then there exists an open set U_x such that $x \subseteq U_x \subseteq S$ — hence $x \in U_x \in U$.
2. Suppose $x \in U$. Then x lies in an open set contained in S, so $x \in S$.

Hence $\mathring{S} = U$ □

Corollary 1. *The interior of a set is open.*

By taking complements of these results about open sets and interiors, we find:

1. A set is closed if and only if all of its points are interior or boundary points.
2. The closure of S is the intersection of all closed sets containing S.
3. The closure of a set is closed.

2 Metric Spaces