MATH-UA 329: Homework 3a

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1 Problem 1

Let $\mathbf{v} = (x, y)$ be any vector in $\mathbb{R}^2$: it will constitute our direction vector. Hence

$$\begin{aligned}
\lim_{\tau \to 0} \frac{f(\mathbf{0} + \tau\mathbf{v}) - f(\mathbf{0})}{\tau} &= \lim_{\tau \to 0} \frac{(\tau x)^3(\tau y)}{\tau\big((\tau x)^6 + (\tau y)^2\big)} \\
&= \lim_{\tau \to 0} \frac{\tau x^3 y}{\tau^4 x^6 + y^2} \\
&= \frac{0}{0 + y^2} \\
&= 0.
\end{aligned}$$

Thus the Gateaux derivative of f at $\mathbf{0}$ is 0. To witness the discontinuity of f at $\mathbf{0}$, consider the path $\mathbf{c}(t) = (t, t^3)$ for $t \in \mathbb{R}$. For all nonzero t, we have

$$f(\mathbf{c}(t)) = f(t, t^3) = \frac{(t)^3(t^3)}{(t)^6 + (t^3)^2} = \frac{t^6}{2t^6} = \frac{1}{2}.$$

Nonetheless, the image of the path $\mathbf{d}(t) = (t, 0)$ under f equals 0 everywhere. Thus for all $\epsilon > 0$, there exists $\mathbf{x}, \mathbf{y} \in B_\epsilon(\mathbf{0})$ such that $f(\mathbf{x}) = \frac{1}{2}$ and $f(\mathbf{y}) = 0$. We conclude that $\lim\limits_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x})$ cannot exist.

2 Problem 2

Let $u \in C[0, 1]$ be any function; it will constitute our direction vector. Hence

$$\begin{aligned}
\lim_{\tau \to 0} \frac{F(\phi + \tau u) - F(\phi)}{\tau} &= \lim_{\tau \to 0} \frac{\int_0^1 (\phi(x) + \tau u(x))^2 \, \mathrm{d}x - \int_0^1 \phi(x)^2 \, \mathrm{d}x}{\tau} \\
&= \lim_{\tau \to 0} \frac{\int_0^1 2\tau\phi(x)u(x) \, \mathrm{d}x + \int_0^1 \tau^2 u(x) \, \mathrm{d}x}{\tau} \\
&= \lim_{\tau \to 0} \int_0^1 2\phi(x)u(x) \, \mathrm{d}x - \int_0^1 \tau u(x) \, \mathrm{d}x \\
&= \int_0^1 2\phi(x)u(x).
\end{aligned}$$

Thus F is Gateaux differentiable. As per the mapping $F'_G(\phi)$ defined by

$$u(x) \mapsto \int_0^1 2\phi(x)u(x)$$

for each $u(x) \in C[0,1]$, we must compute its Gateaux derivative: for all $v(x) \in C[0,1]$, we have

$$\begin{aligned}
\lim_{\tau\to 0} \frac{F_G'(\phi)(u+\tau v) - F_G'(\phi)(u)}{\tau} &= \lim_{\tau\to 0} \frac{\int_0^1 2\phi(x)(u(x)+\tau v(x))\,\mathrm{d}x - \int_0^1 2\phi(x)u(x)\,\mathrm{d}x}{\tau} \\
&= \lim_{\tau\to 0} \frac{\int_0^1 2\tau\phi(x)v(x)\,\mathrm{d}x}{\tau} \\
&= \int_0^1 2\phi(x)v(x).
\end{aligned}$$

This is the Gateaux derivative of the mapping $F_G'(\phi)$. Now, observe that $u, v \in C[0,1]$ implies

$$\begin{aligned}
F_G'(\phi)(u+v) &= \int_0^1 2\phi(x)(u(x)+v(x))\,\mathrm{d}x \\
&= \int_0^1 2\phi(x)u(x)\,\mathrm{d}x + \int_0^1 2\phi(x)v(x)\,\mathrm{d}x \\
&= F_G'(\phi)(u) + F_G'(\phi)(v).
\end{aligned}$$

For all $u \in \mathcal{C}[0,1]$ and constants $c \in \mathbb{R}$, it is trivial that $F_G'(\phi)(cu) = cF_G'(\phi)(u)$. We deduce that $F_G'(\phi)$ is a linear map. To demonstrate that it is continuous, we need only demonstrate it is bounded: consider the unit ball of all $u \in \mathcal{C}[0,1]$ such that $\sup_{x\in\mathbb{R}} |u(x)| \leq 1$. Then

$$\begin{aligned}
\left|F'(G)(\phi)(u)\right| &= \left|\int_0^1 \phi(x)u(x)\,\mathrm{d}x\right| \\
&\leq \sqrt{\int_0^1 \phi(x)^2\,\mathrm{d}x}\sqrt{\int_0^1 u^2(x)\,\mathrm{d}x} \\
&\leq \sqrt{\int_0^1 \phi(x)^2\,\mathrm{d}x}.
\end{aligned}$$

Since $\phi^2(x) \in C[0,1]$, it is bounded; thus the final term of this inequality. We conclude that $F_G'(\phi)$ is bounded on the image of the unit ball, so it is bounded everywhere — hence it is continuous.