MATH-UA 129: Homework One

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1 Section 1.2

Problem 12

The vector $\mathbf{v} = \frac{\mathbf{5\sqrt{13}}}{\mathbf{13}}(\mathbf{3}, -\mathbf{2})$ is perpendicular to $(2, 3)$, as

$$\mathbf{v} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left(\frac{5\sqrt{13}}{13} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right) \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5\sqrt{13}}{13} \left(\begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \frac{5\sqrt{13}}{13}(0) = 0,$$

and has norm 5, as

$$\|\mathbf{v}\| = \left\| \frac{5\sqrt{13}}{13} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\| = \frac{5\sqrt{13}}{13} \left\| \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\| = \frac{5\sqrt{13}}{13} \sqrt{13} = 5.$$

Problem 15

For two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. We claim that $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\|\mathbf{w}\|$ if and only if $\theta = \mathbf{180^\circ}$ **or at least one of the vectors is** $\mathbf{0}$.

Suppose that $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\|\mathbf{w}\|$. Trivially, if one of these vectors is $\mathbf{0}$, then $\mathbf{v} \cdot \mathbf{w} = 0 = -\|\mathbf{v}\|\|\mathbf{w}\|$. If both vectors are nonzero, then their norms are nonzero; we find that

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = -\frac{\|\mathbf{v}\|\|\mathbf{w}\|}{\|\mathbf{v}\|\|\mathbf{w}\|} = -1,$$

so, $\theta = 180^\circ$. Identical means prove that both $\theta = 180^\circ$ and least one of $\mathbf{v}$ or $\mathbf{w}$ being $\mathbf{0}$ imply that $\mathbf{v} \cdot \mathbf{w} = -\|\mathbf{v}\|\|\mathbf{w}\|$, which completes the proof.

Problem 20

Using the formula, we find that the projection of $\mathbf{u} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ onto $\mathbf{v} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ is

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{-2+1-3}{1^2+2^2+(-3)^2}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = -\frac{4}{14}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = -\frac{\mathbf{4}}{\mathbf{7}}\hat{\mathbf{i}} - \frac{\mathbf{2}}{\mathbf{7}}\hat{\mathbf{j}} + \frac{\mathbf{6}}{\mathbf{7}}\hat{\mathbf{k}}.$$

Problem 24

(a) Two such vectors are $\mathbf{v_1} = (\mathbf{1}, \mathbf{1}, -\mathbf{1})$ **and** $\mathbf{v_2} = (-\mathbf{2}, \mathbf{2}, \mathbf{0})$, as $\mathbf{v}_1 \cdot \mathbf{v}_2 = -2 + 2 + 0 = 0$ and each vector lies on the plane, since the plane contains $(0,0,0)$ and the tips of each vector:

$$(1) + (1) + 2(-1) = 0 = (-2) + (2) + 2(0)$$

(b) We have that the orthogonal projection of $\mathbf{b}$ onto P is

$$\mathrm{Proj}_{\mathbf{v_1}}(\mathbf{b}) + \mathrm{Proj}_{\mathbf{v_2}}(\mathbf{b}) = \frac{\mathbf{v_1} \cdot \mathbf{b}}{\|\mathbf{v_1}\|^2}\mathbf{v_1} + \frac{\mathbf{v_2} \cdot \mathbf{b}}{\|\mathbf{v_2}\|^2}\mathbf{v_2} = \frac{3+1-1}{1^2+1^2+(-1)^2}\mathbf{v_1} + \frac{-6+2+0}{(-2)^2+(2)^2+0}\mathbf{v_2},$$

which simplfies to $\mathbf{v_1} - \frac{1}{2}\mathbf{v_2}$. Thus, the projection is $(\mathbf{2}, \mathbf{0}, -\mathbf{1})$.

Problem 25

Two such vectors are $\mathbf{v_1} = (\mathbf{3}, \mathbf{4}, -\mathbf{7})$ **and** $\mathbf{v_2} = (-\mathbf{2}, \mathbf{1}, \mathbf{1})$.

Both vectors are orthogonal to $(1,1,1)$, as $\mathbf{v_1} \cdot (1,1,1) = 3 + 4 - 7 = 0$ and $\mathbf{v_2} \cdot (1,1,1) = -2 + 1 + 1 = 0$. Further observe that if θ is the angle between $\mathbf{v_1}$ and $\mathbf{v_2}$,

$$\cos(\theta) = \frac{\mathbf{v_1} \cdot \mathbf{v_2}}{\|\mathbf{v_1}\|\|\mathbf{v_2}\|} = \frac{-6+4-7}{\sqrt{3^2+4^2+(-7)^2} \times \sqrt{(-2)^2+1^2+1^2)}} = \frac{-9}{\sqrt{74} \times \sqrt{6}} \neq 1, -1,$$

so $\theta \neq 0^\circ, 180^\circ$. Therefore, $\mathbf{v_1}$ and $\mathbf{v_2}$ are nonparallel.

Problem 26

Let $P = (3, 1, -2)$ be the given vector; let ℓ be the given line and define m as shown (where the equation of ℓ is also shown):

$$\ell = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad m = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Further define two vectors: $Q = (1, 0, 1)$ and $S = (0, -1, 0)$. Observe that Q is the intersection of ℓ and m, obtained at $t = 2$ and $t = 0$ respectively; further note that P is on m at $t = 1$ and S is on ℓ at $t = 1$.

The vector $\mathbf{QP}$ is thus $(2, 1, -3)$ and $\mathbf{QS}$ is $(-1, -1, -1)$. Therefore, $\mathbf{QP} \cdot \mathbf{QS} = -2 - 1 + 3 = 0$; we find that $\mathbf{QS}$ and $\mathbf{QP}$ are perpendicular. By deduction, ℓ and m are perpendicular — then as m is perpendicular to ℓ and contains $P = (3, 1, -2)$, **m is our desired line**.

2 Section 1.3

Problem 8

The volume of the parallelepiped is the (absolute value of the) cross product of the vectors that constitute its sides. We compute this quantity using the Rule of Sarrus:

$$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 3 & 2 \\ 0 & -1 & -1 \end{vmatrix} = -3 + 0 + 0 - 0 - 0 - (-2) = -1,$$

so the desired area is **1**.

Problem 10

The two unit vectors orthogonal to $-5\,\hat{\mathbf{i}} + 9\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ and $7\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}$ are

$$\frac{1}{\sqrt{23667}} \left(113\,\hat{\mathbf{i}} + 17\,\hat{\mathbf{j}} - 103\,\hat{\mathbf{k}}\right) \qquad \text{and} \qquad -\frac{1}{\sqrt{23667}} \left(113\,\hat{\mathbf{i}} + 17\,\hat{\mathbf{j}} - 103\,\hat{\mathbf{k}}\right).$$

Denote these vectors by $\mathbf{v}$ and $-\mathbf{v}$ respectively. Note that

$$v \cdot (-5\,\hat{\mathbf{i}} + 9\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}) = \frac{1}{\sqrt{23667}} \begin{bmatrix} 113 \\ 17 \\ -103 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 9 \\ -4 \end{bmatrix} = \frac{-565 + 153 + 412}{\sqrt{23667}} = \frac{0}{\sqrt{23667}} = 0$$

and

$$v \cdot (7\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}) = \frac{1}{\sqrt{23667}} \begin{bmatrix} 113 \\ 17 \\ -103 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \frac{791 + 136 - 927}{\sqrt{23667}} = \frac{0}{\sqrt{23667}} = 0.$$

Similarly, the dot product of $-\mathbf{v}$ with $-5\,\hat{\mathbf{i}} + 9\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ and $7\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}$ are both zero. Finally, we have that

$$\left\| \frac{1}{\sqrt{23667}} \left(113\,\hat{\mathbf{i}} + 17\,\hat{\mathbf{j}} - 103\,\hat{\mathbf{k}} \right) \right\| = \frac{\sqrt{113^2 + 17^2 + (-103)^2}}{\sqrt{23667}} = \frac{\sqrt{23667}}{\sqrt{23667}} = 1.$$

Similarly, $\|-\mathbf{v}\| = \|\mathbf{v}\| = 1$. Thus $\mathbf{v}$ and $-\mathbf{v}$ are the two unit vectors orthogonal to $-5\,\hat{\mathbf{i}} + 9\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ and $7\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} + 9\,\hat{\mathbf{k}}$.

Problem 14

The four quantities are as follows:

- $\mathbf{u} + \mathbf{v}$: Trivially, the sum is $-3\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}$.
- $\mathbf{u} \cdot \mathbf{v}$: The dot product is $3(-6) + 1(-2) + (-1)(-2) = -18$.
- $\|\mathbf{u}\|$: The norm is $\sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{11}$.
- $\|\mathbf{v}\|$: The norm is $\sqrt{(-6)^2 + (-2)^2 + (-2)^2} = 2\sqrt{11}$
- $\mathbf{u} \times \mathbf{v}$: We compute the cross product by the Rule of Sarrus:

  $$\begin{vmatrix} \hat{\mathbf{i}} & 3 & -6 \\ \hat{\mathbf{j}} & 1 & -2 \\ \hat{\mathbf{k}} & -1 & -2 \end{vmatrix} = -2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + (-6)\,\hat{\mathbf{k}} - 2\,\hat{\mathbf{i}} - (-6)\,\hat{\mathbf{j}} - (-6)\,\hat{\mathbf{k}},$$

  which is $-4\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}}$.

Problem 16

(a) The equation is $\mathbf{v} = \mathbf{t}(\mathbf{2}, \mathbf{0}, -\mathbf{1}) + \mathbf{s}(\mathbf{0}, \mathbf{4}, -\mathbf{3})$. At $t, s = 0$, this equation returns $(0, 0, 0)$; at $t = 1$, $s = 0$, this equation retuns $(2, 0, -1)$; and at $t = 0$, $s = 1$, this equation returns $(0, 4, -3)$. The plane thus contains all three points.

(b) The equation is $\mathbf{v} = (\mathbf{1}, \mathbf{2}, \mathbf{0}) + \mathbf{t}(-\mathbf{1}, -\mathbf{1}, -\mathbf{2}) + \mathbf{s}(\mathbf{3}, -\mathbf{2}, \mathbf{1})$. At $t, s = 0$, this equation returns $(1, 2, 0)$; at $t = 1$, $s = 0$, this equation returns $(0, 1, -2)$; and at $t = 0$, $s = 1$, this equation returns $(4, 0, 1)$. The plane thus contains all three points.

(c) The equation is $\mathbf{v} = \mathbf{(2, -1, 3) + t(-2, 1, 2) + s(3, 8, -4)}$. At $t, s = 0$, this equation returns $(2, -1, 3)$; at $t = 1$, $s = 0$, this equation returns $(0, 0, 5)$; and at $t = 0$, $s = 1$, this equation returns $(5, 7, -1)$. The plane thus contains all three points.

Problem 22

We claim that the line $\mathbf{v} = \mathbf{(1, 0, -1) + t(-6, 4, 10)}$ is the intersection of the planes $3(x - 1) + 2y + (z + 1) = 0$ and $(x - 1) + 4y - (z + 1) = 0$.

Observe that the tips of both $(1, 0, -1)$ and $(-5, 4, 9)$ lie on the intersection of the planes:

$$\begin{array}{lcl} 3(1-1) + 2(0) + (-1+1) = 0 & \text{and} & (1-1) + 4(0) - (-1+1) = 0. \\ 3(-5-1) + 2(4) + (9+1) = 0 & \text{and} & (-5-1) + 4(4) - (9+1) = 0. \end{array}$$

Therefore, the intersection of the planes is the line formed by these two points. As $(1, 0, 1)$ and $(-5.4, 9)$ lie on $\mathbf{v} = (1, 0, -1) + t(-6, 4, 10)$ at $t = 0$ and $t = 1$ respectively, we conclude that such a line is the intersection of the planes.

Problem 26

For two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. We claim that $\|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$ if and only if $\theta = \mathbf{30}°, \theta = \mathbf{150}°$, **or at least one of the vectors is** $\mathbf{0}$.

Suppose that $\|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$. Trivially, if one of these vectors is $\mathbf{0}$, then $\|\mathbf{v} \times \mathbf{w}\| = 0 = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$. If both vectors are nonzero, then their norms are nonzero; we find that

$$|\sin(\theta)| = \frac{\|\mathbf{v} \times \mathbf{w}\|}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{\frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{1}{2},$$

so, $\theta = 30°$ or $\theta = 150°$. Identical means prove that both $\theta = 30°$, $\theta = 150°$, and least one of $\mathbf{v}$ or $\mathbf{w}$ being $\mathbf{0}$ imply that $\|\mathbf{v} \times \mathbf{w}\| = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$, which completes the proof.

Problem 28

We claim the plane $\mathbf{3x - 2y + 4z = 20}$ satisfies the conditions of the problem.

Note that the vector $(3, -2, 4)$ is perpendicular to this plane. This is the direction vector of the line $\mathbf{v} = (1, -2, 2) + t(3, -2, 4)$, so the line is perpendicular to the plane.

Furthermore, observe that as $3(2) - 2(-1) + 4(3) = 20$, the plane passes through the point $(2, -1, 3)$. This completes the proof.

3 Section 1.4

Problem 1

We claim the Cartesian point $(\sqrt{2}, -\sqrt{6}, -2\sqrt{2})$ has spherical coordinates $(\mathbf{4}, \mathbf{300}°, \mathbf{135}°)$. To verify this, we use the conversion formulas on our claimed spherical coordinates into Cartesian coordiantes:

- $4\sin(135°)\cos(300°) = 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \sqrt{2}.$
- $4\sin(135°)\sin(300°) = 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{6}.$
- $4\cos(135°) = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}.$

Thus the spherical coordinate $(4, 300°, 135°)$ and Cartesian coordinate $(\sqrt{2}, -\sqrt{6}, -2\sqrt{2})$ describe the same point.

Problem 2

We claim the Cartesian point $(\sqrt{6}, -\sqrt{2}, -2\sqrt{2})$ has spherical coordinates $(\mathbf{4}, \mathbf{330}°, \mathbf{135}°)$. To verify this, we use the conversion formulas on our claimed spherical coordinates into Cartesian coordiantes:

- $4\sin(135°)\cos(330°) = 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \sqrt{6}.$
- $4\sin(135°)\sin(330°) = 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) = -\sqrt{2}.$
- $4\cos(135°) = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}.$

Thus the spherical coordinate $(4, 330°, 135°)$ and Cartesian coordinate $(\sqrt{6}, -\sqrt{2}, -2\sqrt{2})$ describe the same point.

Problem 11

In Cartesian coordinates, the equation we seek is $x^2 + y^2 + z^2 = R^2$; we must convert this equation to cylindrical coordinates. Letting a point (x, y, z) on S have cylindrical coordinates $r, \theta, z)$, we find that

$$\begin{aligned} r^2 + z^2 &= r^2(\cos^2(\theta) + \sin^2(\theta)) + z^2 = r^2(\cos^2(\theta)) + r^2(\sin^2(\theta)) + z^2 \\ &= (r\cos(\theta))^2 + (r\sin(\theta))^2 + z^2 = x^2 + y^2 + z^2 = R^2. \end{aligned}$$

The equation we seek is thus $r^2 + z^2 = R^2$. It is trivial to verify that all points (r, θ, z) such that $r^2 + z^2 = R^2$ lie on S, which completes the proof

4 Section 1.5

Problem 7

If $\|\mathbf{v}\| = \|\mathbf{w}\|$ for two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2$, so

$$(\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}-\mathbf{w}) = (\mathbf{v}\cdot\mathbf{v}) - (\mathbf{v}\cdot\mathbf{w}) + (\mathbf{w}\cdot\mathbf{v}) - (\mathbf{w}\cdot\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 0.$$

Hence, $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are orthogonal.

Problem 11

We claim that only B is an intervible matrix. To verify, we find the determinant of all three matricies using the Rule of Sarrus:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{vmatrix} = 3 + 0 + 0 - 0 - 0 - 3 = 0, \\
\det(B) &= \begin{vmatrix} 0 & 0 & 3 \\ -1 & 1 & 19 \\ 2 & 3 & \pi \end{vmatrix} = 0 + 0 + (-9) - 6 - 0 - 0 = -15, \\
\det(C) &= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0.
\end{aligned}$$

We conclude that because B has nonzero determinant, it is an invertible matrix — and because A and C have a determinant of 0, they are not invertible.

Problem 12

The matrix A maps the vector $(2, 2, -2)$ to $\mathbf{0}$, as verified by the following computation:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+4-6 \\ 0+2-2 \\ 0+6-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 14

We have that for all real numbers $x_1, x_2, \dots x_n$ and $y_1, y_2, \dots y_n$,

$$\begin{aligned}
\left(\sum_{i=1}^{n} x_i y_i\right)^2 &= \sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{i<j} x_i y_i x_j y_j, \\
&= \left(\sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{1 \le i \ne j \le n} x_i^2 y_j^2\right) - \left(\sum_{1 \le i \ne j \le n} x_i^2 y_j^2 - \sum_{i<j} x_i y_i x_j y_j\right), \\
&= \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \sum_{i<j}(x_i y_j - x_j y_i)^2.
\end{aligned}$$

The Trivial Inequality returns that $\sum_{i<j}(x_i y_j - x_j y_i)^2 \ge 0$. Therefore,

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \sum_{i<j}(x_i y_j - x_j y_i)^2 \ge 0.$$

Rearranging this yields

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \ge \left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

which is the Cauchy-Schwarz Inequality.