MATH-UA 140: Assignment 2

James Pagan, September 2023

Professor Raquépas

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1 Problem 1

Part (a): Across all $x_1, x_2 \in \mathbb{R}$, the expression

$$\begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix}$$

is the span of $(3, -2, 8)$ and $(5, 0, 9)$. However, these two vectors and $(2, -3, 8)$ are linearly independent, as

$$\begin{vmatrix} 3 & 5 & 2 \\ -2 & 0 & -3 \\ 8 & 9 & 8 \end{vmatrix} = 0 + (-120) + (-36) - (0) - (-80) - (-81) = 5 \neq 0.$$

If x_1 and x_2 existed such that $x_1(3,-2,8)+x_2(5,0,9)=(2,-3,8)$, then

$$x_1\begin{bmatrix}3\\-2\\8\end{bmatrix}+x_2\begin{bmatrix}5\\0\\9\end{bmatrix}-\begin{bmatrix}2\\-3\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

which contradicts the fact that the three vectors are linearly independent. We conclude that no such x_1 and x_2 exist.

Part (b): Adding four times the first row to the third row yields

$$\begin{bmatrix}1&-2&1\\0&2&-8\\0&-3&13\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\8\\-9\end{bmatrix}.$$

Adding two-thirds of the second row to the third row yields

$$\begin{bmatrix}1&-2&1\\0&2&-8\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\8\\3\end{bmatrix}.$$

We thus have three equations:

$$\begin{aligned}&x_3=3\\&2x_2-8x_3=8 \implies 2x_2-24=8 \implies x_2=16\\&x_1-2x_2+x_3=0 \implies x_1-32+3=0 \implies x_1=29\end{aligned}$$

The answer is thus $(x_1,x_2,x_3)=(29,16,3)$.

2 Problem 2

Part (a) For all $k\in\mathbb{R}$, the vector $(1,0,0)$ is a solution to $A\mathbf{x}=\mathbf{b}$:

$$\begin{bmatrix}1&1&1\\2&4&4\\3&7&k\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1+0+0\\2+0+0\\3+0+0\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}.$$

There is **no k-value** such that $(1,0,0)$ is not a solution to $A\mathbf{x}=\mathbf{b}$.

Part (b): Observe that $\mathbf{b}$ lies in the span of the columns of A for all $k\in\mathbb{R}$, as the first column of A is $\mathbf{b}$ itself. Therefore, there exist infinitely many solutions to $A\mathbf{x}=\mathbf{b}$ when the columns of A are linearly dependent, which occurs if $\det(A)$ is zero: namely, if

$$0=\begin{vmatrix}1&1&1\\2&4&4\\3&7&k\end{vmatrix}=4k+12+14-12-2k-28=2k-14.$$

and $\mathbf{k} = \mathbf{7}$.

Part (c): There exists exactly one solution $\mathbf{b}'$ to $A\mathbf{x} = \mathbf{b}'$ whenever the columns of A are linearly independent — which occurs if $\det(A)$ is nonzero. Our work in Part (b) establishes that this holds for all reals $\mathbf{k} \neq \mathbf{7}$

Part (d): Observe that

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

The matricies we seek are thus

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

3 Problem 3

The matrix is invertible if and only if the determinant is nonzero; namely, if

$$0 \neq \begin{vmatrix} 2 & c & c \\ c & 5 & c \\ 8 & c & c \end{vmatrix} = 10c + 8c^2 + c^3 - 40c - c^3 - 2c^2 = 6c^2 - 30c,$$

or all $\mathbf{c} \in \mathbb{R} \setminus \{\mathbf{0}, \mathbf{5}\}$.

4 Problem 4

Applying the eliminations E_{21}, E_{31}, and E_{41} yields

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}.$$

Further applying the eliminations E_{32} and E_{42} yields

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}.$$

Finally, further applying E_{43} yields an upper trianular matrix:

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

5 Problem 5

Part (a) The result is **true**. If A is an n-by-n matrix with a column of zeroes, then multiplying A with any other n-by-n matrix will clearly produce a matrix with the same column of zeroes. As the identity matrix has no column of zeroes, we deduce there is no A^{-1} such that $AA^{-1} = I$.

Part (b) The result is **true**. For every row of an n-by-n matrix A to add to zero, we must have that

$$\begin{gathered} a_{1,1} + a_{1,2} + \cdots + a_{1,n-1} = -a_{1,n}, \\ \vdots \\ a_{n,1} + a_{n,2} + \cdots + a_{n,n-1} = -a_{n,n}. \end{gathered}$$

Then adding every column vector of A *except* the final column produces the negative of the final column. Thus, the columns of A are not linearly independent, and $\det A = 0$; then A is singular.

Part (c) The result is **true**. For every column of an n-by-n matrix A to add to zero, we must have that

$$\begin{gathered}a_{1,1} + a_{2,1} + \cdots + a_{n-1,1} = -a_{n,1}, \\ \vdots \\ a_{1,n} + a_{2,n} + \cdots a_{n-1,n} = -a_{n,n}.\end{gathered}$$

Then adding every row vector of A *except* the final row produces the negative of the final row. Thus, the row of A are not linearly independent, and $\det A = 0$; then A is singular.

Part (d) The result is **false**. Consider the 2-by-2 matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It has 1's down its main diagonal, yet has determinant $1 - 1 = 0$; it is therefore not intervtible, so the answer is **false**.

Part (e) The result is **true**. If A is invertible, then the inverse of A^{-1} is A; furthermore,

$$\begin{aligned} A^2A^{-2} &= AAA^{-1}A^{-1} = A(I)A^{-1} = AA^{-1} = I, \\ A^{-2}A^2 &= A^{-1}A^{-1}AA = A^{-1}(I)A = A^{-1}A = I, \end{aligned}$$

so A^2 has inverse A^{-2}.