# Real Analysis: Self-Discovered Proofs

James Pagan

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## 1 Bolzano-Weierstrauss Theorem

**Theorem.** *All bounded sequences $a_n$ in $\mathbb{R}$ contain a convergent subsequence.*

*Proof.* Let the first term of $a_n$ be $a_0$, the lower bound of $a_n$ be $m$, and the upper bound of $a_n$ be $M$ such that $m \leq a_n \leq M$ for all $n \in \mathbb{Z}_{\geq 0}$. If $m = M$, then $a_n$ trivially converges.

Otherwise, $m \neq M$; let the set $S_n$ for $n \in \mathbb{Z}_{\geq 0}$ be the $2^n$ closed intervals of size $\frac{M-m}{2^n}$ in the interval $[m, M]$, and $I_n$ for $n \in \mathbb{Z}_{\geq 0}$ be the unique interval — of those in $S_n$ that contain infinitely many values of $a_n$ — that has the greatest upper endpoint.

> **Claim.** *$I_N$ exists for all $\mathbb{Z}_{\geq 0}$*
>
> *Proof.* Suppose for contradiction that all the intervals in $S_n$ for some $n \in \mathbb{Z}_{\geq 0}$ contain finitely many values of $a_n$. Then the union of all these intervals — $[m, M]$ itself — must contain finitely many values of $a_n$, which contradicts the definition of $a_n$. Hence, at least one interval in $S_n$ must contain infinitely many values of $a_n$ for all $n \in \mathbb{Z}_{\geq 0}$.
>
> As all the intervals in $S_n$ for each $n$ have distinct upper endpoints, there exists a unique interval among those in $S_n$ that contains infinitely many values of $a_n$ that has the greatest upper endpoint. $I_n$ thus exists for all $n \in \mathbb{Z}_{\geq 0}$.

We will now establish that $I_n$ meets the two conditions of the Nested Intervals Theorem.

**Claim 1.** *$I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}_{\geq 0}$*

*Proof.* Of all the intervals in $S_{n+1}$, it is trivial that no interval with upper endpoint greater than that of $I_n$ contains infinitely many values of $a_n$. Now, at least one of the two intervals of size $\frac{M-m}{2^{n+1}}$ inside $I_n$ must contain infinitely many values of $a_n$; therefore, one of these is $I_{n+1}$, and so $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}_{\geq 0}$

**Claim 2.** *The limit of the width of $I_n$ as $n$ approaches infinity is* $0$.

*Proof.* $I_n$ has width $\frac{M-m}{2^n}$, which trivially approaches 0 as $n$ approaches infinity.

Then by the Nested intervals Theorem, there exists a unique real number $x$ such that $x \in I_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

We now construct an explicit convergent subsequence of $a_n$. Consider $a_{b_n}$, where $b_n$ is a sequence of integers defined for $n \in \mathbb{Z}_{\geq 0}$ recursively: $b_0 = 0$ and $b_n = \min\{k \mid k \in \mathbb{Z}_{\geq 0}, a_k \in I_n, k > b_{n-1}\}$ for $n \in \mathbb{Z}_{>0}$. It is trivial to prove that $b_n$ always exists for $n \in \mathbb{Z}_{\geq 0}$, and that as $b_n$ is strictly increasing, $a_{b_n}$ is an infinite subsequence. By definition, $a_{b_n} \in I_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

**Claim.** $\lim\limits_{n\to\infty} a_{b_n} = x$.

*Proof.* For all $n \in \mathbb{Z}_{\geq 0}$, observe that $a_{b_n}$ and $x$ both lie inside $I_n$ — so $|a_{b_n} - x| < \frac{M-m}{2^n}$. Then for all $\epsilon > 0$, we have that $\log_2(\frac{M-m}{\epsilon}) < n$ implies

$$|a_{b_n} - x| < \frac{M-m}{2^n} < \frac{M-m}{2^{\log_2\left(\frac{M-m}{\epsilon}\right)}} = \frac{M-m}{\frac{M-m}{\epsilon}} = \epsilon$$

.

Hence, $\lim\limits_{n\to\infty} a_{b_n} = x$.

Therefore, if $a_n$ is a bounded sequence, it is always possible to construct $a_{b_n}$ — a convergent subsequence of $a_n$.

□

## 2 Boundedness Theorem

**Theorem.** *If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$, then $f(x)$ is bounded on $[a, b]$.*

*Proof.* If $a = b$, then $f(x)$ is bounded above by $f(a) + 1$ and below by $f(a) - 1$. If $a \neq b$, suppose for contradiction that $f(x)$ is not bounded on $[a, b]$ — then for all $M \in \mathbb{Z}_{\geq 0}$, there exists a $y \in [a, b]$ such that $f(y) > M$. We will prove that this assumption implies the existence of a subinterval of $[a, b]$ that is both bounded and unbounded.

Let $S_n$ for $n \in \mathbb{Z}_{\geq 0}$ be the set of the $2^n$ closed intervals of size $\frac{b-a}{2^n}$ between $a$ and $b$. Define $I_n$ as the unique closed interval that — among those in $S_n$ such that $f(x)$ is not bounded on the interval — has the largest possible upper endpoint.

**Claim.** *$I_n$ exists for all $n \in \mathbb{Z}_{\geq 0}$.*

*Proof.* Suppose for contradiction that $f(x)$ is bounded over all the closed intervals of $S_n$. Then $f(x)$ is bounded over the union of all these intervals — $[a, b]$ itself — which is a contradiction. Hence, $f(x)$ must be unbounded over least one closed interval in $S_n$.

As all the intervals in $S_n$ have distinct upper endpoints, there exists a unique interval that — among those in $S_n$ such that $f(x)$ is not bounded — has the largest possible upper endpoint. $I_n$ thus exists for all $n \in \mathbb{Z}_{\geq 0}$.

We will now establish that $I_n$ meets the two conditions of the Nested Intervals Theorem.

**Claim.** *$I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}_{\geq 0}$*

*Proof.* Of all the intervals in $S_{n+1}$, it is trivial that $f(x)$ is bounded over all interval with greater upper endpoint than $I_n$. Now, $f(x)$ must not be bounded over at least one of the two intervals of size $\frac{b-a}{2^{n+1}}$ inside $I_n$; therefore, one of these is $I_{n+1}$, and so $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}_{\geq 0}$

**Claim.** *The limit of the width of $I_n$ as $n$ approaches infinity is $0$.*

*Proof.* $I_n$ has width $\frac{1}{2^n}$, which trivially approaches 0 as $n$ approaches infinity.

Therefore, the Nested Intervals Theorem guarantees the existence of a unique real number $r$ such that $r \in I_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Because $f(x)$ is continuous at $r$, we have that there exists $\delta$ such that

$$0 < |x - r| < \delta \implies |f(x) - f(r)| < 1$$

Therefore, the interval $(r - \delta, r + \delta)$ is bounded by $f(r) + 1$.

**Claim.** *There exists $I_m$ for some $n \in \mathbb{Z}_{\geq 0}$ such that $I_m \subset (r - \delta, r + \delta)$.*

*Proof.* Consider the closed interval $I_m$ for $m = \left\lceil \log_2 \left( \frac{b-a}{\delta} \right) \right\rceil + 1$. Note that the width of $I_m$ satisfies

$$\frac{b-a}{2^{\left\lceil \log_2 \left( \frac{b-a}{\delta} \right) \right\rceil + 1}} < \frac{b-a}{2^{\log_2 \left( \frac{b-a}{\delta} \right)}} = \frac{b-a}{\frac{b-a}{\delta}} = \delta.$$

Suppose for contradiction that $I_m \not\subset (r - \delta, r + \delta)$. Then there is a real number $x$ such that $x \in I_m$ and $x \notin (r - \delta, r + \delta)$.

As $r \in I_M$, the width of $I_m$ is greater than or equal to $|x - r|$ — however, the width of $I_M$ is less than $\delta$, so $|x - r| < \delta$. Then $x$ is inside $(r - \delta, r + \delta)$, which contradicts the definition of $x$.

Therefore, no such $x$ exists; $I_m \in (r - \delta, r + \delta)$.

Hence, $I_m$ is bounded by $f(r) + 1$ as well. This contradicts the definition if $I_m$ — namely, that $f(x)$ is unbounded on $I_m$.

We deduce that $f(x)$ has an upper bound over $[a, b]$. As $-f(x)$ also has an upper bound over $[a, b]$, $f(x)$ also has a lower bound over $[a, b]$. Therefore, $f(x)$ is bounded over $[a, b]$.

$\square$