MATH-UA 129: Homework 9

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Contents

1 Section 6.1

Problem 3

We claim the following linear transformation maps D^* to D:

$$\boxed{T = \begin{bmatrix} 1 & 0 \\ -1/3 & 2/3 \end{bmatrix}}.$$

This may be verified by the following computations:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1+0 \\ -1/3+4/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} 1+0 \\ -1/3-2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
\begin{bmatrix} 1 & 0 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2+0 \\ -2/3+2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.
\end{aligned}$$

This transformation also preserves the order of the verticies of the parallelogram, which completes the proof.

Problem 11

Clearly, $D = T(D^*)$ is $\boxed{\text{the unit ball of } \mathbb{R}^3}$. T is not one-to-one if one or more of the following occurs:

- $\rho = 0$: in which case, $(0, \phi, \theta) = (0, 0, 0)$ for all $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$.
- $\phi = 0$: in which case, $(\rho, 0, \theta) = (0, 0, \rho)$ for all $\theta \in [0, 2\pi]$.
- $\phi = \pi$: in which case, $(\rho, \pi, \theta) = (0, 0, -\rho)$ for all $\theta \in [0, 2\pi]$.
- $\theta \in \{0, 2\pi\}$: in which case, $(\rho, \phi, \theta) = (\rho, \phi, 2\pi - \theta)$ and $\theta \neq 2\pi - \theta$.

With this in mind, T is one-to-one on the following subset of D^*:

$$\boxed{\{(\rho, \phi, \theta) \mid \rho \in (0, 1], \phi \in (0, \pi), \theta \in [0, 2\pi)\}}$$

Problem 14

If T is not injective, then $\operatorname{null} T$ contains a nonzero vector $\mathbf{v}$. As $\mathbf{0}$ lies on the span of $\mathbf{v}$, this vector would be an eigenvector with eigenvalue 0. Hence, the determinant of A — the product of its eigenvalues — would be 0.

Conversely, if A has determinant zero, one of its eigenvalues must be 0; then a nonzero eigenvector $\mathbf{v}$ is mapped to zero, and $\operatorname{null} T \neq \{\mathbf{0}\}$. Thus, T is not injective.

Taking the contrapositive yields the desired result: that $\det A \neq 0$ if and only if T is injective.

2 Section 6.2

Problem 4

Observe that

$$\begin{aligned} 0 \leq& u - v \leq u + v && (1) \\ 0 \leq& u + v \leq 1. && (2) \end{aligned}$$

From $u - v \leq u + v$, we find that $0 \leq 2v$ and $0 \leq v$. We also find from (1) that $v - u \leq 0$ — which when combined with $u + v \leq 1$ from (2) yields $2v \leq 1$, so $v \leq \frac{1}{2}$.

We deduce from (1) that $v \leq u$ and from (2) that $u \leq 1 - v$. We therefore have that all solutions to (1) and (2) satisfy the following set of equations:

$$\begin{aligned} 0 \leq& v \leq \frac{1}{2} \\ v \leq& u \leq 1 - v. \end{aligned}$$

A quick verification yields that all such u and v satisfy (1) and (2). These equations thus form our bounds of integration.

Now clearly $x + y = 2u$; then

$$\begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$$

We may now use the Change of Variables Theorem to integrate the function in terms of u and v:

$$\begin{aligned}
\iint_D (x+y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\frac{1}{2}} \int_v^{1-v} 2u(2)\,\mathrm{d}u\,\mathrm{d}v \\
&= 2\int_0^{\frac{1}{2}} \Big[u^2\Big]_v^{1-v}\,\mathrm{d}u \\
&= 2\int_0^{\frac{1}{2}} (1-v)^2 - v^2\,\mathrm{d}v \\
&= 2\int_0^{\frac{1}{2}} 1 - 2v\,\mathrm{d}v \\
&= 2\Big[v - v^2\Big]_0^{\frac{1}{2}} \\
&= 2\left(\frac{1}{4}\right) \\
&= \boxed{\frac{1}{2}}.
\end{aligned}$$

This yields the same answer as standard integration:

$$\begin{aligned}
\iint_D (x+y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^x (x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{(x+y)^2}{2}\right]_0^x \mathrm{d}x \\
&= \int_0^1 \frac{(2x)^2}{2} - \frac{x^2}{2}\,\mathrm{d}x \\
&= \int_0^1 \frac{3x^2}{2}\,\mathrm{d}x \\
&= \left[\frac{x^3}{2}\right]_0^1 \\
&= \boxed{\frac{1}{2}}.
\end{aligned}$$

Problem 11

We evaluate the integral by polar substitution. Observe that if $(x,y)=(r,\theta)$, then $x^2+y^2 \leq 4$ is equiavalent to $r^2 \leq 4$, so $r \in [0,2]$. Because the Jacobian determinant from polar to

Cartesian coordinates is r, we may now use the Change of Variables Theorem to deduce that

$$\begin{aligned}\iint_D (x^2+y^2)^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^2 (r^2)^{\frac{3}{2}}\,(r)\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi\int_0^2 r^4\,\mathrm{d}r \\ &= 2\pi\left[\frac{r^5}{5}\right]_0^2 \\ &= \boxed{\frac{64\pi}{5}}.\end{aligned}$$

Problem 16

We evaluate the integral by polar substitution. Observe that as D is the unit disc — and that the Jacobian determinant from polar to Cartesian coordinates is r — we can use the change of variables formula to find that

$$\begin{aligned}\iint_D \left(1+x^2+y^2\right)^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi}\int_0^1 \left(1+r^2\right)^{\frac{3}{2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \pi\int_0^1 2r\left(1+r^2\right)^{\frac{3}{2}}\,\mathrm{d}r \\ &= \pi\left[\frac{2}{5}\left(1+r^2\right)^{\frac{5}{2}}\right]_0^1 \\ &= \frac{2\pi(\sqrt{2^5}-1)}{5} \\ &= \boxed{\frac{8\pi\sqrt{2}-2\pi}{5}}.\end{aligned}$$

Problem 20

Part (a): Observe that this equation converts spherical coordinates to Cartesian. Using a geometeric argument, we can see that any point on the unit sphere has a polar angle v and an azimuth angle w (letting $u=1$) such that T maps (u,v,w) to the point. Therefore, T is onto.

Part (b): Observe that $T(1,v,w)=T(1,v+2n_1\pi,w+2n_2\pi)$ for all integers n_1 and n_2 by the period of sine and cosine. Therefore, T is *nowhere* one-to-one on the unit sphere.

Problem 25

We evaluate the integral by spherical substitution. Observe that the Jacobian determinant from spherical to Cartesian coordinates is $\rho^2 \sin(\phi)$, so we can use the change of variables formula to find that

$$\begin{aligned}
\iiint_W \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x^2+y^2+z^2)^{\frac{3}{2}}} &= \int_0^\pi \int_0^{2\pi} \int_b^a \frac{\rho^2 \sin(\phi)}{\rho^3}\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \left(\int_0^\pi \sin(\phi)\,\mathrm{d}\phi\right)(2\pi)\left(\int_b^a \frac{1}{\rho}\,\mathrm{d}\rho\right) \\
&= 4\pi\Big[\ln|\rho|\Big]_b^a \\
&= 4\pi(\ln(a) - \ln(b)) \\
&= \boxed{4\pi \ln\left(\tfrac{a}{b}\right)}.
\end{aligned}$$

Problem 29

We evaluate the integral by spherical substitution. Observe that the Jacobian determinant from spherical to Cartesian coordinates is $\rho^2 \sin(\phi)$, so we can use the change of variables formula to find that (because ρ is positive)

$$\begin{aligned}
\iiint_W \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^\pi \int_0^{2\pi} \int_b^a \sqrt{\rho^2}e^{-\rho^2}(\rho^2 \sin(\phi))\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \left(\int_0^\pi \sin(\phi)\,\mathrm{d}\phi\right)(2\pi)\left(\int_b^a \rho^3 e^{-\rho^2}\,\mathrm{d}\rho\right) \\
&= 4\pi\Big[-\tfrac{1}{2}\rho^2 e^{-\rho^2} + \int_b^a \rho e^{\rho^2}\,\mathrm{d}\rho\Big]_b^a \\
&= 4\pi\Big[-\tfrac{1}{2}\rho^2 e^{-\rho^2} - \tfrac{1}{2}e^{-\rho^2}\Big]_b^a \\
&= -4\pi\Big[\frac{e^{-\rho^2}(r^2+1)}{2}\Big]_b^a \\
&= \boxed{2\pi\left(e^{-b^2}(b^2+1) - e^{-a^2}(a^2+1)\right)}.
\end{aligned}$$

Problem 32

Realize that the linear map

$$T = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

transforms the unit square to the desired square, has determinant five, and preserves orientation. If the desired square is B (and if we change the *names* of the variables in the given integral), we may use the Change of Variables Theorem to deduce that

$$\begin{aligned}\iint_B (u+v)\,\mathrm{d}u\,\mathrm{d}v &= \int_0^1\int_0^1 (x+y)(5)\,\mathrm{d}x\,\mathrm{d}y \\ &= 5\int_0^1 \left[\frac{x^2}{2}+xy\right]_0^1 \\ &= 5\int_0^1 \frac{1}{2}+y\,\mathrm{d}y \\ &= 5\left[\frac{y}{2}+\frac{y^2}{2}\right]_0^1 \\ &= 5(1) \\ &= \boxed{5}.\end{aligned}$$

3 Section 6.3

Problem 5

By the formula, the x-coordinate of the center of mass is

$$\begin{aligned}\frac{\int_0^1\int_{x^2}^x x(x+y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^1\int_{x^2}^x (x+y)\,\mathrm{d}y\,\mathrm{d}x} &= \frac{\int_0^1 \left(\int_{x^2}^x x^2\,\mathrm{d}y + \int_{x^2}^x xy\,\mathrm{d}y\right)\mathrm{d}x}{\int_0^1 \left(\int_{x^2}^x x\,\mathrm{d}y + \int_{x^2}^x y\,\mathrm{d}y\right)\mathrm{d}x} \\ &= \frac{\int_0^1 x^2(x-x^2)+\frac{1}{2}x(x^2-x^4)\,\mathrm{d}x}{\int_0^1 x(x-x^2)+\frac{1}{2}(x^2-x^4)\,\mathrm{d}x} \\ &= \frac{\int_0^1 -\frac{1}{2}x^5-x^4+\frac{3}{2}x^3\,\mathrm{d}x}{\int_0^1 -\frac{1}{2}x^4-x^3+\frac{3}{2}x^2\,\mathrm{d}x} \\ &= \frac{\left[-\frac{1}{12}x^6-\frac{1}{5}x^5+\frac{3}{8}x^4\right]_0^1}{\left[-\frac{1}{10}x^5-\frac{1}{4}x^4+\frac{1}{2}x^3\right]_0^1} \\ &= \frac{11}{18}.\end{aligned}$$

By the formula, the y-coordinate of the center of mass is

$$\begin{aligned}
\frac{\int_0^1 \int_{x^2}^x y(x+y)\,\mathrm{d}y\,\mathrm{d}x}{\int_0^1 \int_{x^2}^x (x+y)\,\mathrm{d}y\,\mathrm{d}x} &= \frac{\int_0^1 \left(\int_{x^2}^x xy\,\mathrm{d}y + \int_{x^2}^x y^2\,\mathrm{d}y\right)\mathrm{d}x}{\int_0^1 \left(\int_{x^2}^x x\,\mathrm{d}y + \int_{x^2}^x y\,\mathrm{d}y\right)\mathrm{d}x} \\
&= \frac{\int_0^1 \frac{1}{2}x(x^2-x^4) + \frac{1}{3}(x^3-x^6)\,\mathrm{d}x}{\int_0^1 x(x-x^2) + \frac{1}{2}(x^2-x^4)\,\mathrm{d}x} \\
&= \frac{\int_0^1 -\frac{1}{3}x^6 - \frac{1}{2}x^5 + \frac{5}{6}x^3\,\mathrm{d}x}{\int_0^1 -\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2\,\mathrm{d}x} \\
&= \frac{\left[-\frac{1}{21}x^7 - \frac{1}{12}x^6 + \frac{5}{24}x^4\right]_0^1}{\left[-\frac{1}{10}x^5 - \frac{1}{4}x^4 + \frac{1}{2}x^3\right]_0^1} \\
&= \frac{65}{126}.
\end{aligned}$$

Therefore, the coordinate of the center of mass is $\boxed{\left(\frac{11}{18}, \frac{65}{126}\right)}$.

Problem 6

By the formula, the x-coordinate of the center of mass is

$$\begin{aligned}
\frac{\int_0^{\frac{1}{2}} \int_0^{x^2} x\,\mathrm{d}y\,\mathrm{d}x}{\int_0^{\frac{1}{2}} \int_0^{x^2} \mathrm{d}y\,\mathrm{d}x} &= \frac{\int_0^{\frac{1}{2}} x(x^2)\,\mathrm{d}x}{\int_0^{\frac{1}{2}} x^2\,\mathrm{d}x} \\
&= \frac{\left[\frac{1}{4}x^4\right]_0^{\frac{1}{2}}}{\left[\frac{1}{3}x^3\right]_0^{\frac{1}{2}}} \\
&= \frac{3}{8}.
\end{aligned}$$

By the formula, the x-coordinate of the center of mass is

$$\begin{aligned}
\frac{\int_0^{\frac{1}{2}} \int_0^{x^2} y \,\mathrm{d}y \,\mathrm{d}x}{\int_0^{\frac{1}{2}} \int_0^{x^2} \mathrm{d}y \,\mathrm{d}x} &= \frac{\int_0^{\frac{1}{2}} \frac{1}{2}(x^4) \,\mathrm{d}x}{\int_0^{\frac{1}{2}} x^2 \,\mathrm{d}x} \\
&= \frac{\left[\frac{1}{10}x^5\right]_0^{\frac{1}{2}}}{\left[\frac{1}{3}x^3\right]_0^{\frac{1}{2}}} \\
&= \frac{3}{30}.
\end{aligned}$$

Therefore, the coordinate of the center of mass is $\boxed{\left(\frac{3}{8}, \frac{3}{30}\right)}$.

Problem 11

We evaluate the integral by spherical substitution. Observe that the Jacobian determinant from spherical to Cartesian coordinates is $\rho^2 \sin(\phi)$, so we can use the change of variables formula to find that (because ρ is positive)

$$\begin{aligned}
\iiint_B \delta(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z &= \int_0^{\pi} \int_0^{2\pi} \int_0^5 (2\rho^2 + 1)(\rho^2 \sin(\phi)) \,\mathrm{d}\rho \,\mathrm{d}\theta \,\mathrm{d}\phi \\
&= \left(\int_0^{\pi} \sin(\phi) \,\mathrm{d}\phi\right)(2\pi)\left(\int_0^5 2\rho^4 + \rho^2\right) \\
&= 4\pi \left[\frac{2}{5}\rho^5 + \frac{1}{3}\rho^3\right]_0^5 \\
&= \boxed{\frac{15500}{3}\pi}.
\end{aligned}$$

4 Section 7.1

Problem 5

We claim the parametrization we seek is $\boxed{(3\cos(\theta), 4\sin(\theta), 3) \text{ for } \theta \in [0, 2\pi)}$. It is trivial to verify that all such points in the parametriation lie on the curve, by substitution — and a backwards construction may demonstate that all points that lie on the cylinder-plane intersection exist on the parametrization.

Problem 11

Part (a): Observe that $\mathbf{c}'(t) = (0, 0, 2t)$, so $\|\mathbf{c}'(t)\| = \sqrt{0^2 + 0^2 + (2t)^2} = 2|t|$. Thus, we seek to evaluate the following integral:

$$\begin{aligned}
\int_0^1 f(\mathbf{c}(t))\big\|\mathbf{c}'(t)\big\|\,\mathrm{d}t &= \int_0^1 e^{|t|}(2|t|)\,\mathrm{d}t \\
&= 2\int_0^1 te^t\,\mathrm{d}t \\
&= 2\Big[te^t - e^t\Big]_0^1 \\
&= \boxed{2}.
\end{aligned}$$

Part (b): Observe that $\mathbf{c}'(t) = (1, 3, 2)$, so $\|\mathbf{c}'(t)\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$. Thus, we seek to evaluate the following integral:

$$\begin{aligned}
\int_1^3 f(\mathbf{c}(t))\big\|\mathbf{c}'(t)\big\|\,\mathrm{d}t &= \int_0^1 (3t)(2t)(\sqrt{14})\,\mathrm{d}t \\
&= 2\sqrt{14}\Big[t^3\Big]_1^3 \\
&= \boxed{52\sqrt{14}}.
\end{aligned}$$

Problem 12

Part (a): Note that $\mathbf{c}(t) = (t, t^2, 0)$, so $\mathbf{c}'(t) = (1, 2t, 0)$; thus $\|\mathbf{c}'(t)\| = \sqrt{1^2 + (2t)^2 + 0^2} = \sqrt{1 + 4t^2}$. Therefore, we seek to evaluate the following integral:

$$\begin{aligned}
\int_0^1 f(\mathbf{c}(t))\big\|\mathbf{c}'(t)\big\|\,\mathrm{d}t &= \int_0^1 t\cos(0)\sqrt{1+4t^2}\,\mathrm{d}t \\
&= \frac{1}{8}\int_0^1 8t\sqrt{1+4t^2}\,\mathrm{d}t \\
&= \frac{1}{8}\left[\frac{2}{3}\left(1+4t^2\right)^{\frac{3}{2}}\right]_0^1 \\
&= \frac{1}{12}\left(5^{\frac{3}{2}} - 1\right) \\
&= \boxed{\frac{5\sqrt{5}-1}{12}}.
\end{aligned}$$

Part (b): Note that $\mathbf{c}(t) = \left(t, \frac{2}{3}t^{3/2}, t\right)$, so $\mathbf{c}'(t) = (1, t^{1/2}, 1)$; we deduce that $\|\mathbf{c}'(t)\| = \sqrt{1^2 + (t^{1/2})^2 + 1^2} = \sqrt{t+2}$. Therefore, we seek to evaluate the following integral:

$$\begin{aligned}
\int_1^2 f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,\mathrm{d}t &= \int_1^2 \frac{t+\frac{2}{3}t^{3/2}}{\frac{2}{3}t^{3/2}+t}(\sqrt{2+t})\,\mathrm{d}t \\
&= \int_1^2 \sqrt{2+t}\,\mathrm{d}t \\
&= \left[\tfrac{2}{3}(2+t)^{3/2}\right]_1^2 \\
&= \boxed{\frac{16}{3}-2\sqrt{3}}.
\end{aligned}$$

Problem 19

Observing that $\mathbf{c}'(t) = (2t, 1, 0)$, we seek to evaluate the following integral:

$$\int_0^1 \|\mathbf{c}'(t)\|\,\mathrm{d}t = \int_0^1 \sqrt{(2t)^2+1^2+0}\,\mathrm{d}t = \int_0^1 \sqrt{4t^2+1}\,\mathrm{d}t$$

Performing the substitution $t = \frac{1}{2}\tan(u)$, we find (after a lengthy calculation) the answer

$$\boxed{\frac{2\sqrt{5}+\ln\big(2+\sqrt{5}\big)}{4}}.$$

5 Section 7.2

Problem 4

Part (a): As $\mathbf{c}(t) = (\cos(t), \sin(t))$ and $\mathbf{c}'(t) = (-\sin(t), \cos(t))$, we have that

$$\begin{aligned}
\int_{\mathbf{c}} x\,\mathrm{d}y - y\,\mathrm{d}x &= \int_0^{2\pi} \cos(t)(\cos(t)) - \sin(t)(-\sin(t))\,\mathrm{d}t \\
&= \int_0^{2\pi} \cos^2(t)+\sin^2(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{d}t \\
&= \boxed{2\pi}.
\end{aligned}$$

Part (b): As $\mathbf{c}(t) = (\cos(\pi t), \sin(\pi t))$ and $\mathbf{c}'(t) = (-\pi \sin(\pi t), \pi \cos(\pi t))$, we have that

$$\begin{aligned}
\int_{\mathbf{c}} x\,\mathrm{d}x + y\,\mathrm{d}y &= \int_0^2 \cos(\pi t)(-\pi \sin(\pi t)) + \sin(\pi t)(\pi \cos(\pi t))\,\mathrm{d}t \\
&= \pi \int_0^2 -\cos(\pi t)\sin(\pi t) + \sin(\pi t)\cos(\pi t)\,\mathrm{d}t \\
&= \pi \int_0^2 0\,\mathrm{d}t \\
&= \boxed{0}.
\end{aligned}$$

Part (c): We may represent $\mathbf{c}$ by two different paths: $(1-t, t, 0)$ for $t \in [0,1]$ and $(0, 2-t, t-1)$ for $t \in [1,2]$. It is trivial to verify that these two paths constitute the desired curve $\mathbf{c}$; therefore, we have that

$$\begin{aligned}
\int_{\mathbf{c}} yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z &= \int_0^1 (t)(0)(-1) + (0)(1-t)(1) + (1-t)(t)(0) \\
&\quad + \int_1^2 (2-t)(t-1)(0) + (t-1)(0)(-1) + (0)(2-t)(1)\,\mathrm{d}t \\
&= \int_0^1 0\,\mathrm{d}t + \int_1^2 0\,\mathrm{d}t \\
&= \boxed{0}.
\end{aligned}$$

Part (d) It is trivial to verify that the path $\mathbf{c} = (t, 0, t^2)$ from $t \in [-1,1]$ traces the given curve; we thus have from $\mathbf{c}'(t) = (1, 0, 2t)$ that

$$\begin{aligned}
\int_{\mathbf{c}} x^2\,\mathrm{d}x - xy\,\mathrm{d}y + \mathrm{d}z &= \int_{-1}^1 t^2(1) - t(0)(0) + 2t\,\mathrm{d}t \\
&= \int_{-1}^1 t^2 + 2t\,\mathrm{d}t \\
&= \left[\frac{t^3}{3} + t^2\right]_{-1}^1 \\
&= \boxed{\frac{2}{3}}.
\end{aligned}$$

Problem 8

As $\mathbf{c}(t) = (t, t^2, t^3)$ and $\mathbf{c}'(t) = (1, 2t, 3t^2)$, we have that $\mathbf{F}(\mathbf{c}(t)) = (t^2, 2t, t^2)$; hence,

$$\begin{aligned}
\int_{\mathbf{c}} \mathbf{F}\, \mathrm{d}\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, \mathrm{d}t \\
&= \int_0^1 (t^2, 2t, t^2) \cdot (1, 2t, 3t^2)\, \mathrm{d}t \\
&= \int_0^1 t^2 + 4t^2 + 3t^4\, \mathrm{d}t \\
&= \int_0^1 3t^4 + 5t^2\, \mathrm{d}t \\
&= \left[\frac{3t^5}{5} + \frac{5t^3}{3}\right]_0^1 \\
&= \boxed{\frac{34}{15}}.
\end{aligned}$$

Problem 17

Such a curve is $\mathbf{c}(t) = (1, t+1, 3t+1)$ for $t \in [0, 1]$. As $\mathbf{c}'(t) = (0, 1, 3)$, we have that

$$\begin{aligned}
\int_C 2xyz\, \mathrm{d}x + x^2 z\, \mathrm{d}y + x^2 y\, \mathrm{d}z &= \int_0^1 2(t+1)(3t+1)(0) + (3t+1) + (t+1)(3)\, \mathrm{d}t \\
&= \int_0^1 6t + 4\, \mathrm{d}t \\
&= \Big[3t^2 + 4t\Big]_0^1 \\
&= \boxed{7}.
\end{aligned}$$

Problem 19

Observe that $\mathbf{F}(x, y, z)$ is the gradient of the function $f(x, y, z) = \frac{1}{(x^2+y^2+z^2)^{1/2}}$. Therefore, the work done is represented by the following integral, where $\mathbf{c}(t)$ for $t \in [0, 1]$ is a path between (x_1, x_2, x_3) and (y_1, y_2, y_3):

$$\begin{aligned}
\int_{\mathbf{c}} \mathbf{F}\, \mathrm{d}\mathbf{s} &= \int_{\mathbf{c}} \nabla f\, \mathrm{d}\mathbf{s} \\
&= f(\mathbf{c}(1)) - f(\mathbf{c}(0)) \\
&= f(y_1, y_2, y_3) - f(x_1, x_2, x_3) \\
&= \frac{1}{(y_1^2 + y_2^2 + y_3^2)^{1/2}} - \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} \\
&= \frac{1}{R_2} - \frac{1}{R_1}
\end{aligned}$$

Therefore, the work done depends only on the two radii R_1 and R_2.