MATH-UA 349: Homework 6

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1 Problem 1

Proof. Let $r_1, \ldots, r_n$ be a basis of R as an F-vector space and select nonzero $r \in R$ arbitrarily; we must demonstrate that r is a unit. Define a mapping $\phi : R \to R$ by $\phi(x) = rx$. It is clear that ϕ is a linear operator on the F-vector space R.

1. ϕ is injective: $\phi(x) = 0$ implies $rx = 0$ implies $x = 0$, since R is an integral domain.
2. ϕ is surjective: this follows from the fact ϕ is a linear operator on a finite-dimensional vector space. Such operators are injective if and only if they are surjective by the Rank-Nullity Theorem.

Since ϕ is surjective, there exists $s \in R$ such that $\phi(s) = rs = 1$. We conclude that all nonzero $r \in R$ are units, so R is a field. $\square$

2 Problem 2

Proof. Simply substitute $x = \frac{-b+\delta}{2}$ into the quadratic equation:

$$\begin{aligned} x^2 + bx + c &= \left(\frac{-b+\delta}{2}\right)^2 + b\left(\frac{-b+\delta}{2}\right) + c \\ &= \frac{b^2 - 2b\delta + \delta^2}{4} + \frac{-2b^2 + 2b\delta}{4} + \frac{4c}{4} \\ &= \frac{\delta^2 - b^2 + 4c}{4} \\ &= \frac{(b^2 - 4c) - b^2 + 4c}{4} \\ &= 0 \end{aligned}$$

Similar logic demonstrates that $x = \frac{-b-\delta}{2}$ is a root of the quadratic. Now, suppose that $b^2 - 4c$ is *not* a square; then adjoin δ to F such that $\delta^2 = b^2 - 4c$. Identical logic to the above demonstrates that $\frac{-b\pm\delta}{2}$ are the two roots of f. Since $\delta \notin F$, neither of these roots are elements of F — so f has no roots in F. $\square$

3 Problem 3

Proof. Observe that $a_0 \neq 0$ by the irreducibility of f, so $\alpha \neq 0$. Hence claim the inverse has the form

$$\boxed{\alpha^{-1} = -\frac{1}{\alpha_0}\sum_{i=1}^{n} a_i\alpha^{i-1}}.$$

To verify this, we need only multiply it by α:

$$\begin{aligned}\alpha\left(-\frac{1}{a_0}\sum_{i=1}^{n}\alpha_i\alpha^{i-1}\right) &= -\frac{1}{a_0}\sum_{i=1}^{n}a_i\alpha^i \\ &= -\frac{1}{a_0}\left(-a_0+\sum_{i=0}^{n}a_i\alpha^i\right) \\ &= -\frac{1}{a_0}\left(-a_0+0\right) \\ &= 1.\end{aligned}$$

This completes the proof. □

4 Problem 4

Proof. There are two facts which propel our observations:

1. $F(\alpha)$ has prime degree, and $\alpha^2 \in F(\alpha)$.
2. $\alpha^2 \notin F$, since α has degree 5.

Thus Corollary 15.3.7 implies that α^2 has degree 5, so $F(\alpha^2) = F(\alpha)$. □

5 Problem 5

Proof. A quick examination using the Einstein criterion yields that x^4+3x+3 is irreducible over $\mathbb{Q}$; thus it is the minimal polynomial of some algebraic number α. Since 3 and 4 are relatively prime, we have that

$$12 = [\mathbb{Q}(\alpha, \sqrt[3]{2}) : \mathbb{Q}] \;=\; [\mathbb{Q}(\alpha, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})]\,[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] \;=\; [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\sqrt[3]{2})] \times 3.$$

Thus $[\mathbb{Q}(\alpha, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})] = 4$, so $\deg_{\mathbb{Q}(\sqrt[3]{2})}(\alpha) = 4$. We conclude that the minimal polynomial x^4+3x+3 cannot be reduced in $\mathbb{Q}(\sqrt[3]{2})$. □

6 Problem 6

Realize that the minimal polynomial of ζ_5 is $x^4+x^3+x^2+x+1$ and the minimal polynomial of ζ_7 is $x^6+x^5+x^4+x^3+x^2+x+1$ — hence they have degrees 4 and 6. Clearly these polynomials are irreducible in $\mathbb{Q}$.

By Corollary 15.3.8 in Artin, $[\mathbb{Q}(\zeta_5, \zeta_7) : \mathbb{Q}]$ is divisible by $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$ and $\mathbb{Q}(\zeta_7 : \mathbb{Q}) = 6$, but is less than their product — hence it is either 12 or 24. Hence.

$$[\mathbb{Q}(\zeta_5, \zeta_7) : \mathbb{Q}(\zeta_7)] \times 6 \;=\; [\mathbb{Q}(\zeta_5, \zeta_7) : \mathbb{Q}(\zeta_7)]\,[\mathbb{Q}(\zeta_7) : \mathbb{Q}] \;=\; [\mathbb{Q}(\zeta_5, \zeta_7) : \mathbb{Q}] \;\in\; \{12, 24\}.$$

Thus $[\mathbb{Q}(\zeta_5, \zeta_7) : \mathbb{Q}(\zeta_7)]$ is not 1, so $\mathbb{Q}(\zeta_5, \zeta_7) \neq \mathbb{Z}(\zeta_7)$. We conclude that $\zeta_5 \notin \mathbb{Q}(\zeta_7)$.

7 Problem 7

The polynomial $x^4 - a$ factors in $\mathbb{Q}(\sqrt[4]{2})$ as

$$(x^2 + \sqrt{a})(x + \sqrt[4]{2})(x - \sqrt[4]{2}).$$

This makes it clear that $\sqrt[4]{2}$ has degree 2 in $\mathbb{Q}[\sqrt{2}]$. Thus

$$[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] \;=\; [\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}(\sqrt{2})]\,[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \;=\; 2 \times 2 \;=\; 4,$$

which yields the required result.

8 Problem 8

Proof. Let $\deg \alpha = n$ and $\deg \alpha = m$. We start with the following observation:

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \;=\; [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)]\,[\mathbb{Q}(\alpha), \mathbb{Q}]. \tag{1}$$

Clearly $[\mathbb{Q}(\alpha), \mathbb{Q}] = n$. As per $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)]$: the set $1, \beta, \ldots, \beta^{n-1}$ spans $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}(\alpha)$, so its dimension as a $\mathbb{Q}(\alpha)$-vector space is n or smaller. Thus both terms on the right-hand side of equation are finite, so $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$ is finite. We conclude that $\mathbb{Q}(\alpha, \beta)$ is a finite extension over $\mathbb{Q}$.

Since $\alpha + \beta$ and $\alpha\beta$ are elements of the finite extension $\mathbb{Q}(\alpha, \beta)$, they must be algebraic. $\square$