Atiyah-MacDonald: Rings and Ideals

James Pagan

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Contents

1 Rings

1.1 Ring Axioms

A **ring** R is a set endowed with two binary operations, here denoted "+" and "×", such that if $a, b, c \in R$, the following ten axioms are satisfied:

- **Additive Axioms**
  1. **Closure**: $a + b \in R$.
  2. **Associativity**: $a + (b + c) = (a + b) + c$.
  3. **Identity**: There is $0 \in R$ such that $a + 0 = 0 + a = a$.
  4. **Invertability**: There is $-a \in R$ such that $a + (-a) = (-a) + a = 0$.
  5. **Commutativity**: $a + b = b + a$.
- **Multiplicative Axioms**
  6. **Closure**: $ab \in R$.
  7. **Associativity**: $a(bc) = (ab)c$.
  8. **Identity**: There is $1 \in R$ such that $a1 = 1a = a$.
- **Distributive Axioms**
  9. **Left Distributivity**: $a(b + c) = ab + ac$.
  10. **Right Distributivity**: $(a + b)c = ac + bc$.

Since $(R, +)$ is an Abelian group, the following properties hold for $a, b \in R$: the additive identity 0 is unique, the additive inverse $-a$ is unique, $-(-a) = a$, and $-(a + b) = -a - b$.

Theorem 1. *The following properties hold for any ring R and $a, b \in R$:*

1. *1 is the unique multiplicative inverse of R.*
2. *If a has a multiplicative inverse a^{-1}, it is unique.*
3. $a0 = 0a = a$.
4. $-a = (-1)a$.
5. $a(-b) = (-a)b = -ab$.
6. $(-a)(-b) = ab$.

Proof. (1) and (2) follow from the monoid/group axioms. For the rest:

3. As $0 + 0 = 0$, we have that $a0 = a(0 + 0) = a0 + a0$; subtracting by $a0$ yields $a0 = 0$. Similarly, $0a = 0$.
4. We have that
$$(-1)a + a = (-1)a + 1a = (-1 + 1)a = 0a = 0,$$
so $(-1)a = -a$.
5. See that
$$a(-b) + ab = a(-b + b) = a0 = 0,$$
so $a(-b) = -ab$. Similarly, $(-a)b = -ab$.
6. Using (5), we find that
$$(-a)(-b) = -(a)(-b) = -(-ab) = ab,$$
as desired.

This yields the desired six properties. $\square$

1.2 Subrings and Ideals

A **subring** R' of R is a subset of R that is also a ring. This relation is denoted $R' \subseteq R$.

Theorem 2. *A subset R' of R is a subring if it is nonempty, closed under addition and multiplication, contains additive inverses, and contains the multiplicative identity.*

Proof. The conditions that $(R', +)$ is nonempty, closed, and contains inverses ensures that it is a group. Note that $(R', \times)$ is closed and contains the multiplicative identity.

The final properties are implied by the fact R' is a subset of R; all the elements of R' satisfy both associative and distributive laws, plus additive commutativity. We deduce that R' is a subring. $\square$

All rings contain at least two subrings: the 0 ring and R itself.

A **ideal** $\mathfrak{a}$ of R is a subset of R that satisfies the following twokproperties:

1. **Additive**: $\mathfrak{a}$ is an additive subgroup of R.
2. **Multiplicative**: For all $a \in \mathfrak{a}$ and $x \in R$, we have $ax, xa \in \mathfrak{a}$.

All rings contain at least two ideals: one is R itself, one is a maximal ideal (Section 2.3).

Theorem 3. *If R' is both a subring and an ideal of R if and only if R' is R or 0.*

Proof. Suppose that $R' \neq 0$ is both a subring and an ideal of R. As R' is a subring, $1 \in R'$; as R' is an ideal, $a = a1 \in R'$ for all $a \in R$. Then $R' = R$. Clearly, R itself and 0 are both ideals and subrings — which yields the desired result. $\square$

1.3 Ring Homomorphisms

A **ring homomorphism** between two rings R and R' is a mapping $\phi : R \to R'$ such that for all $a, b \in R$,

$$\phi(a+b) = \phi(a) + \phi(b)$$
$$\phi(ab) = \phi(a)\phi(b)$$
$$\phi(1) = 1.$$

By the group axioms, $\phi(-a) = -\phi(a)$ and $\phi(0) = 0$ for all $a \in R$. If a has a multiplicative inverse a^{-1}, then $\phi(a^{-1}) = \phi(a)^{-1}$.

The **image** of R under ϕ is the set $\{\phi(a) \mid a \in R\}$, and is denoted $\phi(R)$.

Theorem 4. *The image of any ring homomorphism $\phi : R \to R'$ is a subring of R'.*

Proof. Realize that $\phi(R)$ is nonempty, and for all $\phi(a), \phi(b) \in \phi(R)$, we have that

1. $\phi(a) + \phi(b) = \phi(ab) \in \phi(R)$.
2. $\phi(a)\phi(b) = \phi(ab) \in \phi(R)$.
3. $-\phi(a) = \phi(-a) \in \phi(R)$.
4. $\phi(1) \in R$.

Hence, $\phi(R)$ is a subring of R'. $\square$

The **kernel** of R under ϕ is the set $\{a \in R \mid \phi(r) = 0\}$ and is denoted $\operatorname{Ker}\phi$.

Theorem 5. $\operatorname{Ker}\phi$ *is an ideal of* R.

Proof. Since ϕ is a homomorphism of the Abelian groups $(R, +)$ and $(R', +)$, the kernel of ϕ is an Abelian group with respect to addition. We need only verify the multiplicative condition; for all $a \in R$ and $k \in \operatorname{Ker}\phi$,

$$\phi(ak) = \phi(a)\phi(k) = 0\phi(a) = 0 = \phi(a)0 = \phi(a)\phi(k) = \phi(ak).$$

Then $ak \in \operatorname{Ker}\phi$. Thus, $\operatorname{Ker}\phi$ is an ideal. □

Categories of group homomorphisms — like monomorphisms, epimorphisms, isomorphisms, endomorphisms, automorphisms — have equivalent formulations for ring homomorphisms. An isomorphism between R and R' is denoted the same as groups:

$$R \cong R'.$$

We can extend the notion of a quotient group to a ring R with an ideal $\mathfrak{a}$ as follows, yielding a **quotient ideal**:

Theorem 6. *The quotient group* $R\,/\,\mathfrak{a}$ *is a ring under the product* $(a+\mathfrak{a})(b+\mathfrak{a}) = ab+\mathfrak{a}$ *for* $a, b \in R$.

Proof. The quotient group $R\,/\,\mathfrak{a}$ exists, since $\mathfrak{a}$ is an additive subgroup of R and all subgroups of Abelian groups are normal. We must demonstrate that the product is well-defined.

Suppose $a + \mathfrak{a} = a' + \mathfrak{a}$ and $b + \mathfrak{a} = b' + \mathfrak{a}$. Then since $a - a' \in \mathfrak{a}$ and $b - b' \in \mathfrak{a}$,

$$ab - a'b \in \mathfrak{a} \qquad \text{and} \qquad a'b - a'b' \in \mathfrak{a}.$$

Thus, $ab - a'b' \in \mathfrak{a}$ and $ab + \mathfrak{a} = a'b' + \mathfrak{a}$. Then the product is well-defined. Proving that the product is closed and associative is trivial; the multiplicative identity of $R\,/\,\mathfrak{a}$ is $1 + \mathfrak{a}$, and the distributivity with addition is trivial — so $R\,/\,\mathfrak{a}$ is a ring. □

The canonical mapping $\phi : R \to R\,/\,\mathfrak{a}$ is thus a surjective homomomorphism with kernel $\mathfrak{a}$. A similar definition exists for the quotient of two ideals — say, $\mathfrak{a}\,/\,\mathfrak{b}$ for $\mathfrak{a} \supseteq \mathfrak{b}$.

1.4 Isomorphism Theorems

All three Isomorphism Theorems and the Correspondence Theorem have their equivalencies for rings.

Theorem 7 (First Isomorphism Theorem)**.** *For all homomorphisms $\phi : R \to R'$ with kernel $\mathfrak{k}$,*

$$R\,/\,\mathfrak{k} \cong \phi(R)$$

by the mapping $\psi(a+\mathfrak{k}) = \phi(a)$.

Proof. We must first demonstrate that ψ is a homomorphism. If $a, b \in R$, then the following three identities hold:

1. $\psi(a+b+\mathfrak{k}) = \phi(a+b) = \phi(a) + \phi(b) = \psi(a+\mathfrak{k}) + \psi(b+\mathfrak{k})$.
2. $\psi(ab+\mathfrak{k}) = \phi(ab) = \phi(a)\phi(b) = \psi(a+\mathfrak{k})\psi(b+\mathfrak{k})$.
3. $\psi(1+\mathfrak{k}) = \phi(1)$.

Thus, ψ is a homomorphism. For all $\phi(a) \in \phi(R)$, realize that $\psi(a+\mathfrak{k}) = \phi(a)$; thus ψ is surjective. Finally, let $\psi(a+\mathfrak{k}) = \psi(b+\mathfrak{k})$; then $\phi(a) = \phi(b)$, so

$$\phi(a-b) = \phi(a) - \phi(b) = 0.$$

Hence, $a - b \in \mathfrak{k}$ and $a+\mathfrak{k} = b+\mathfrak{k}$. We conclude that ψ is injective, implying the desired isomorphism. $\square$

The Correspondence Theorem expands upon the result of the First Isomorphism Theorem.

Theorem 8 (Correspondence Theorem)**.** *There is a one-to-one correspondence between ideals of $\phi(R)$ and ideals of R that contain $\mathfrak{k}$.*

Proof. For an ideal $\mathfrak{a}'$ of $\phi(R)$, define $\mathfrak{a} = \{a \in R \mid \phi(a) \in \mathfrak{a}'\}$. By the Correspondence Theorem for groups, $\mathfrak{a}$ is an additive subgroup of R. For all $a \in \mathfrak{a}$ and $b \in R$, we have $\phi(a) \in \mathfrak{a}'$; thus

$$\phi(ab) = \phi(a)\phi(b) \in \mathfrak{a}'$$

since $\mathfrak{a}'$ is an ideal. Thus $ab \in \mathfrak{a}$, so $\mathfrak{a}$ is an ideal of R. Since $0 \in R'$, we have that $\mathfrak{k}$ is a subideal of $\mathfrak{a}$. It is now relatively trivial to establish a one-to-one correspondence. $\square$

Corollary 1. *There is a one-to-one correspondence between ideals of $R\,/\,\mathfrak{a}$ and ideals of R that contain $\mathfrak{a}$.*

The two remaining Isomorphism Theorems will be proven at another time.

1.5 Assorted Rings

We will consider the following three types of rings in this section:

1. A **commutative ring** is a ring R such that $ab = ba$ for all $a, b \in R$.
2. An **integral domain** is a nonzero commutative ring R such that $ab = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$.
3. A **field** is a commutative division ring.

Note that integral domains and fields must be nonzero. **Henceforth, all rings we shall define are commutative unless stated otherwise.**

Theorem 9. *All finite domains are fields.*

Proof. Let R be a finite domain. Then for nonzero $a \in R$, consider the set

$$\{a, a^2, \dots, a^{|R|+1}\}.$$

By the Pigeonhole Principle, two elements of this set must be equal: $a^i = a^j$ for $i, j \in \{1, \dots, n\}$ with $i < j$. Thus $a^j(a^{i-j} - 1) = 0$, so $a^{i-j} = 1$ and $a^{i-j-1} = a^{-1}$. Since all nonzero elements of R are invertible, we conclude that R is a field. $\square$

Theorem 10. *R is a field if and only if the only ideals of R are 0 and R itself.*

Proof. Let R be a field and let $\mathfrak{a}$ be nonzero ideal of R. Then for $a \in \mathfrak{a}$,

$$R = (a) \subseteq \mathfrak{a} \subseteq R.$$

Thus, $\mathfrak{a} = R$. Now, suppose that the only ideals of R are 0 and R itself; then for all nonzero $a \in R$,

$$(a) = R,$$

where (a) denotes the principal ideal (Section 2.1). Thus, there exists $a^{-1} \in R$ such that $aa^{-1} = 1$, so R is a field. $\square$

An element $a \in R$ is a **unit** if it is invertible. It is trivial to verify that all the units of R constitute a multiplicative Abelian group (non-units form a commutative semigroup!)

2 Types of Ideals

2.1 Principal Ideals

For an $x \in R$, the **principal ideal** of x is the ideal given by $(x) = \{ax \mid a \in R\}$. We may alternatively denote (x) by Rx.

Theorem 11. *Principal ideals are ideals.*

Proof. Let x be ay element of R. We must perform two rather routine calculations:

1. **Additivity**: For all $ax, bx \in (x)$, we have that $ax + bx = (a+b)x \in (x)$.
2. **Multiplicativity**: For all $ax \in (x)$ and $b \in R$ we have $b(ax) = (ba)x \in (x)$.

We conclude that (x) is an ideal. $\square$

The principal ideal is the smallest ideal that contains (x), in the following sense: if $x \in \mathfrak{a}$ for an ideal $\mathfrak{a}$ of R, then $rx \in \mathfrak{a}$ for all $a \in R$, so $(x) \subseteq \mathfrak{a}$.

Theorem 12. $(x) = R$ *for* $x \in R$ *if and only if* x *is a unit.*

Proof. Suppose that $(x) = R$. Then $1 \in (x)$, so there exists $x^{-1} \in R$ such that $xx^{-1} = x^{-1}x = 1$; x is a unit. If we suppose that x is a unit, then $x \in (x)$ implies $1 = x^{-1}x \in (x)$ implies $a = a1 \in (x)$ for all $a \in R$; thus $(x) = R$. $\square$

2.2 Prime Ideals

A **prime ideal** $\mathfrak{p}$ of R is a principal ideal such that $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. This condition generalizes to a finite amount of elements; $a_1 \cdots a_n \in \mathfrak{p}$ if and only if $a_i \in \mathfrak{p}$ for some i.

Theorem 13. *An ideal* $\mathfrak{p}$ *of* R *is prime if and only if* $R\,/\,\mathfrak{p}$ *is an integral domain.*

Proof. Suppose that $\mathfrak{p}$ is prime, and define $\phi : R \to R\,/\,\mathfrak{p}$ by $\phi(a) = a + \mathfrak{p}$. Since the kernel of ϕ is $\mathfrak{p}$, we have that

$$\phi(ab) = 0 \implies ab \in \mathfrak{p} \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{p} \implies \phi(a) = 0 \text{ or } \phi(b) = 0.$$

Conversely, suppose that $R\,/\,\mathfrak{p}$ is an integral domain. Then

$$ab \in \mathfrak{p} \implies \phi(ab) = 0 \implies \phi(a) = 0 \text{ or } \phi(b) = 0 \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}.$$

This completes the proof. $\square$

2.3 Maximal Ideals

A **maximal ideal** $\mathfrak{m}$ of R is a proper ideal such that the only ideals of R that contain $\mathfrak{m}$ are itself and R. Maximal ideals (along with prime and proper ideals) need not be mutually exclusive; they do not partition the non-units of R.

Theorem 14. *An ideal $\mathfrak{m}$ of R is maximal if and only if $R/\mathfrak{m}$ is a field.*

Proof. By the Correspondence Theorem, there is a one-to-one correspondence between ideals of R that contain $\mathfrak{m}$ and ideals of $R/\mathfrak{m}$. Then using Theorem 10,

$$\begin{aligned}\mathfrak{m} \text{ is maximal} &\iff \text{The only ideals of } R/\mathfrak{m} \text{ are } (0) \text{ and } R/\mathfrak{m} \text{ itself.}\\ &\iff R/\mathfrak{m} \text{ is a field,}\end{aligned}$$

yielding the desired result $\square$

All maximal ideals are prime. The following theorem ensures a wealth of maximal ideals:

Theorem 15 (Krull's Theorem)**.** *Every nonzero ring has a maximal ideal.*

Proof. The set of all proper ideals under $\subseteq$ forms a partially ordered set — it is nonempty, as (0) is an ideal. To construct upper bounds, define $(\mathfrak{a}_n)$ as a chain of ideals such that for indicies α and β, we have $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\alpha \supseteq \mathfrak{a}_\beta$.

Claim 1. *$\bigcup \mathfrak{a}_n$ is an ideal.*

Proof. We must perform two rather routine calculations:

1. **Additivity**: If $x, y \in \bigcup \mathfrak{a}_n$, let $x \in \mathfrak{a}_\alpha$ and $y \in \mathfrak{a}_\beta$ for indicies α and β. Without loss of generality, let $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$; then $x \in \mathfrak{a}_\beta$. Thus $x + y \in \mathfrak{a}_\beta \subseteq \bigcup \mathfrak{a}_n$.
2. **Multiplicativity**: Suppose $x \in \bigcup \mathfrak{a}_n$ and $a \in R$. Then $x \in \mathfrak{a}_\alpha$ for some index; we have $ax \in \mathfrak{a}_\alpha \subseteq \bigcup \mathfrak{a}_n$.

We deduce that $\bigcup \mathfrak{a}_n$ is an ideal.

Zorn's Lemma thus applies. The set of all proper ideals contains a maximal element with respect to inclusion — namely, a maximal ideal. $\square$

Two corollaries follow from Krull's Theorem:

Corollary 2. *All proper ideals $\mathfrak{a}$ are contained within some maximal ideal $\mathfrak{m}$.*

Proof. If $\mathfrak{a}$ is a proper ideal, then the quotient ring $R \,/\, \mathfrak{a}$ is nonzero — hence it contains a maximal ideal $\mathfrak{a}'$. By the Correspondence Theorem, there exists a corresponding ideal $\mathfrak{a}$ in R that contains $\mathfrak{a}$. The maximality of $\mathfrak{m}$ is ensured by the maximality of $\mathfrak{m}'$ (say, via a contradiction argument). □

Corollary 3. *Each non-unit $a \in R$ lies within some maximal ideal of R.*

3 Special Rings and Ideals

3.1 Local Rings

A **local ring** is a ring with exactly one maximal ideal. They may have an arbitrary number of prime ideals. The following two theorems test whether R is local with maximal ideal $\mathfrak{m}$:

Theorem 16. *R is a local ring if and only if $R - \mathfrak{m}$ consists of units.*

Proof. Suppose that $R - \mathfrak{m}$ consists of units. Then $\mathfrak{m}$ constitues all units of R; as all ideals are composed of non-units, ideals of R must lie within $\mathfrak{m}$. Then $\mathfrak{m}$ is the sole maximal ideal of the local ring R.

Suppose that $R - \mathfrak{m}$ contains a non-unit $a \in R$. Then (a) is a proper ideal, and lies within some maximal ideal $\mathfrak{n}$. As $a \in \mathfrak{n}$ and $a \notin \mathfrak{m}$, the ring R has two maximal ideals and is not local. □

Theorem 17. *R is a local ring if and only if $\mathfrak{m} + 1$ consists of units for maximal $\mathfrak{m}$.*

Proof. Suppose that R is a local ring. Then if $m \in \mathfrak{m}$, we must have $m + 1 \notin \mathfrak{m}$; otherwise, $1 \in \mathfrak{m}$ implies that $\mathfrak{m}$ is not a proper ideal. Hence, $\mathfrak{m} + 1 \subseteq R - \mathfrak{m}$, so $\mathfrak{m} + 1$ consists of units.

Suppose that $\mathfrak{m} + 1$ consists of units for maximal $\mathfrak{m}$. Let $a \notin \mathfrak{m}$; then $(a) + \mathfrak{m} = R$, so there exists $ab \in (a)$ and $m \in \mathfrak{m}$ such that $ab + m = 1$. Then $1 - m$ is a unit, so

$$R = (1 - m) = (ab) \subseteq (a) \subseteq R$$

We deduce that $(a) = R$, so a is a unit. As $R - \mathfrak{m}$ consists of non-units, Theorem 16 implies that R is a local ring with maximal ideal $\mathfrak{m}$. □

A **semilocal ring** is a ring with a finite number of maximal ideals.

3.2 Principal Ideal Domain

A **principal ideal domain** is an integral domain in which all ideals are principal.

Theorem 18. *Let R be a principal ideal domain. Then all nonzero prime ideals of R are maximal.*

Proof. Let $(a) \neq 0$ be prime and define (b) as the maximal ideal that contains (a). Then $a \in (b)$, so there exists $x \in R$ such that $a = bx$. We have $bx \in (a)$; then either $b \in (a)$ or $x \in (a)$.

Suppose for contradiction that $x \in (a)$. Then there exists $y \in R$ such that $x = ay$; substituting this into our earlier equation,

$$a = b(ay) \implies a(1 - by) = 0.$$

Since R is an integral domain — and since $a \neq 0$ — we must have $1 = by$. Then b is a unit, so $(b) = R$; this contradicts the fact that the maximal ideal (b) is proper.

Thus, $b \in (a)$ and $(a) = (b)$. We conclude that (a) is maximal. $\square$

These domains are unique factorization domains, and thus the techniques discussed in AbstractAlgebra/artin12.tex apply.

3.3 The Nilradical

An element $a \in R$ is a **zero divisor** if there exists nonzero $b \in R$ such that $ab = 0$. A zero divisor a is **nilpotent** if $a^n = 0$ for some positive integer n. the set of all nonzero nilpotent elements of R is called the **nilradical** of R, often denoted by $\mathfrak{N}$.

Theorem 19. *The nilradical $\mathfrak{N}$ of R is ideal of R.*

Proof. First, we must verify that $\mathfrak{N}$ is an additive subgroup of $\mathfrak{R}$. Since $0 \in \mathfrak{R}$, we need only verify two conditions:

1. **Closure**: For $a, b \in \mathfrak{N}$, let $n, m \in \mathbb{Z}$ such that $a^n = b^m = 0$. Then

$$(ab)^{nm} = a^{nm}b^{nm} = (a^n)^m(b^m)^n = 0^m0^n = 0,$$

so $ab \in \mathfrak{N}$.

2. **Inverses**: If $a^n = 0$, then $(-a)^n = 0$ as well; thus $-a \in \mathfrak{N}$.

Now, we need only verify the multiplicative condition. For $a \in \mathfrak{N}$, define $n \in \mathbb{Z}$ such that $a^n = 0$; then for all $b \in R$,

$$(ab)^n = a^n b^n = 0b^n = 0,$$

so $ab \in \mathfrak{N}$. We deduce that $\mathfrak{N}$ is an ideal. $\square$

The following proof is my favorite in this document:

Theorem 20. *The nilradical $\mathfrak{N}$ of a commutative ring R is the intersection of all the prime ideals of R.*

Proof. Suppose $a^n = 0$ and $\mathfrak{p}$ is a prime ideal of R. Then $a^n \in \mathfrak{p}$, so one of $aa \cdots a$ must be in $\mathfrak{p}$ (the prime condition inducts!).

Now, suppose that $a^n \neq 0$ for all $n \in \mathbb{Z}_{>0}$. Let S be the set of all ideals $\mathfrak{a}$ such that $a^n \notin \mathfrak{a}$ for all $n \in \mathbb{Z}_{>0}$. This set is nonempty, since $0 \in S$; then S is a partialy ordered set under inclusion.

Using identical logic as in Theorem 15, we deduce that this set must have a maximal element $\mathfrak{p}$ — however, $\mathfrak{p}$ may not be maximal in the scale of *all* ideals of R.

Claim 2. *$\mathfrak{p}$ is a prime ideal of R.*

Proof. Suppose $b, c \notin \mathfrak{p}$. Then $(b) + \mathfrak{p}$ and $(c) + \mathfrak{p}$ are ideals that contain $\mathfrak{p}$, so they do not lie within S. Then they contain a power of a; for some $m, n \in \mathbb{Z}_{>0}$, for some $x, y \in R$, and for some p_1, p_2 in $\mathfrak{p}$,

$$a^m = bx + p_1 \qquad \text{and} \qquad a^n = cy + p_2.$$

Then $a^{mn} = bcxy + bxp_2 + cyp_1 + p_1p_2$. As $\mathfrak{p}$ is an ideal, the entire expression $bxp_2 + cyp_1 + p_1p_2$ lies within $\mathfrak{p}$; thus $a^{mn} \in (bc) + \mathfrak{p}$. Then $(bc) + \mathfrak{p}$ cannot lie within S; thus $bc \notin \mathfrak{p}$.

Taking the contrapositive yields that $bc \in \mathfrak{p}$ implies $b \in \mathfrak{p}$ or $c \in \mathfrak{p}$.

Then as a is absent from the prime ideal $\mathfrak{p}$, it cannot lie within the intersection of all the prime ideals of R. $\square$

If R is an integral domain, then $\mathfrak{N}$ is the zero ideal.

3.4 The Jacobson Radical

The **Jacobson radical** $\mathfrak{J}$ is the intersection of all the maximal ideals of R. As an intersection of ideals, $\mathfrak{J}$ is an ideal (Section 4.1) — so it is a subideal of the nilradical.

Theorem 21. *j lies in the Jacobson radical $\mathfrak{J}$ if and only if $1 - ja$ is a unit across all $a \in R$.*

Proof. Suppose that there $b \in R$ such that $1 - jb$ is not a unit. Then there is a maximal ideal $\mathfrak{m}$ that contains $(1 - jb)$; such an ideal cannot contain b, or else it contains jb and thus 1. Hence $b \notin \mathfrak{J}$.

Suppose that j is not in the Jacobson radical. Then $j \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R; thus $(j) + \mathfrak{m} = R$, so there exists $b \in R$ such that $jb + m = 1$ for an arbitrary nonzero $m \in M$. Then $1 - jb \in \mathfrak{m}$, so it cannot be a unit.

Taking the contrapositive yields the desired result. □

4 Operations on Rings and Ideals

4.1 Sum, Intersection, Product

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of a ring R, we may perform the following operations upon them to yield three new ideals.

1. **Sum**: $\mathfrak{a} + \mathfrak{b} = \{a + b \mid a \in \mathfrak{a}, b \in \mathfrak{b}\}$, the smallest ideal of R that contains $\mathfrak{a}$ and $\mathfrak{b}$.
2. **Intersection**: $\mathfrak{a} \cap \mathfrak{b}$, the largest ideal of R contained within both $\mathfrak{a}$ and $\mathfrak{b}$. In fact an infinite intesection of ideals is an ideal.
3. **Product**: $\mathfrak{ab} = \{\sum a_i b_i \mid a_i \in \mathfrak{a}, b_i \in \mathfrak{b}\}$. We denote $\mathfrak{aa} \cdots \mathfrak{a}$ as $\mathfrak{a}^n$ and set $\mathfrak{a}^0 = R$.

Ideals under sums and intersections form a complete lattice. Sums may be infinite; products must be finite. All of the above are commutative and associative; products and sums of ideals satisfy the distributive law. $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, with equality if $\mathfrak{a} + \mathfrak{b} = R$ (Theorem 22).

4.2 Relatively Prime Ideals

Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are **relatively prime** if $\mathfrak{a} + \mathfrak{b} = R$. Clearly, this holds if and only if there exists $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$.

We have invokea facts about relatively prime ideals several times thus far throughout this document — notably that if $\mathfrak{m}$ is maximal and $a \notin \mathfrak{m}$, then $\mathfrak{m} + (a) = R$.

Theorem 22. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals of R. If $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are coprime whenever $i \neq j$, then $\prod \mathfrak{a}_i = \cap \mathfrak{a}_i$*

Proof. **Base case**: Consider ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R. and let $ab \in \mathfrak{ab}$. Then as $\mathfrak{a}$ is an ideal, $ab \in \mathfrak{a}$; likewise, $ab \in \mathfrak{b}$. Them $ab \in \mathfrak{a} \cap \mathfrak{b}$. Now if $x \in \mathfrak{a} \cap \mathfrak{b}$, then $x \in \mathfrak{a}$ and $x \in \mathfrak{b}$. Let $a + b = 1$; then $xa \in \mathfrak{ba}$ and $xb \in \mathfrak{ab}$, so $x = xa + xb \in \mathfrak{ab}$. We conclude that $\mathfrak{ab} = \mathfrak{a} \cap \mathfrak{b}$ (this proof is wrong, $\mathfrak{ab}$ is consists of sums).

Inductive step: Let the theorem be true for n; we wish to prove that if $\mathfrak{a}_1, \ldots, \mathfrak{a}_n, \mathfrak{b}$ are all pairwise coprime, then

$$\left(\bigcup_{i=1}^{n} \mathfrak{a}_i\right) \mathfrak{b} = \left(\bigcup_{i=1}^{n} \mathfrak{a}_i\right) \cap \mathfrak{b}$$

We have a sequence of equations from $a_1 + b_1 = 1$ to $a_n + b_n = 1$, where $a_i \in \mathfrak{a}_i$ and $b_i \in \mathfrak{b}$ ($i \in \{1, \ldots, n\}$). We argue by cosets:

$$\left(\prod_{x=1}^{n} a_i\right) + \mathfrak{b} = \left(\prod_{x=1}^{n} (1 - b_i)\right) + \mathfrak{b} = 1 + \mathfrak{b}.$$

Thus there exists $b \in \mathfrak{b}$ such that $a_1 \cdots a_n + b = 1$; thus $\mathfrak{b}$ is coprime to $\prod \mathfrak{a}_i$, which implies the given result by the base case. □

A rather trivial result is that if $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are principal ideals, then their product is the ideal of all products $a_1 \cdots a_n$ — no summations required.

4.3 Direct Product of Rings

For rings $R_1, \ldots, R_n$, their **direct product**

$$R = \prod_{i=1}^{n} R_i$$

is the set of all sequences $a = (a_1, \ldots, a_n)$ with $a_i \in R_i$ for $i \in \{1, \ldots, n\}$, endowed with componentwise addition and multiplication. It is a commutative ring; the mappings $\phi : R \to R_i$ defined by $\phi(a_1, \ldots, a_n)$ are homomorphisms.

In the following theorem, let R be a ring with ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$; define a homomorphism

$$\phi : R \to \prod_{i=1}^{n} R \,/\, \mathfrak{a}_i$$

by $\phi(a) = (a + \mathfrak{a}_1, \ldots, a + \mathfrak{a}_n)$.

Theorem 23. *The following two properties of ϕ hold:*

1. *ϕ is injective if and only if $\cap\mathfrak{a}_i = 0$.*
2. *ϕ is surjective if and only if $\mathfrak{a}_i$ and $\mathfrak{a}_\mathrm{i}$ are relatively prime whenever $i \neq j$.*

Proof. For (1), the following sequence of claims is easy to verify:

$$\begin{aligned} k \in \operatorname{Ker}\phi &\iff \phi(k) = 0 \\ &\iff k \in \mathfrak{a}_i \text{ for each } i \in \{1, \dots, n\} \\ &\iff k \in \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n. \end{aligned}$$

Thus, $\operatorname{Ker} f = 0$ if and only if $\cup\mathfrak{a}_i = 0$. Now for (2): suppose that ϕ is surjective. For $\mathfrak{a}_i$ and $\mathfrak{a}_j$, there exists $a \in R$ such that $\phi(a)$ returns $(\dots, 0, 1, 0, \dots)$, where 1 is in the i-th place. Then $a - 1 \in \mathfrak{a}_i$ and $a \in \mathfrak{a}_j$, so

$$1 = (1 - a) + a \in (\mathfrak{a}_i + \mathfrak{a}_j),$$

so $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are relatively prime. Now, suppose that $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are relatively prime for each $i \neq j$. We need only show that the element $(\dots, 0, 1, 0, \dots)$ lies in the image of ϕ; the 1 may be anywhere by similarity, so we can generate all elements of $\prod R \,/\, \mathfrak{a}_i$.

For each $i \in \{1, \dots, n\}$, we have $\mathfrak{a}_i$ and $\prod_{j \neq i} \mathfrak{a}_j$ are coprime; thus there exists a_i in the former and a in the latter such that

$$a_i + a = 1.$$

Thus, $a \in (1 + \mathfrak{a}_i)$. We conclude that $\phi(a) = (\dots, 0, 1, 0, \dots,)$, from which we construct as aforementioned and demonstrate the surjectivity of ϕ. □

4.4 Inclusion and Prime Ideals

In general, the union of ideals is rarely an ideal — yet there is much to be said about them:

Theorem 24. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals in R and let $\mathfrak{a}$ be an ideal contained in $\bigcup_{i=1}^n \mathfrak{p}_i$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some $i \in \{1, \dots, n\}$.*

Proof. We prove the contrapositive — that if $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for each i, then $\mathfrak{a} \not\subseteq \bigcup \mathfrak{p}_i$. The result is clearly true for $n = 1$, so we utilize induction: let the result be true for $n - 1$, and consider the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.

We have that $\mathfrak{a} \not\subseteq \bigcup_{i=1}^{n-1} \mathfrak{p}_i$ by our inductive hypothesis, and $\mathfrak{a} \not\subseteq \mathfrak{p}_n$. Suppose for contradiction that $\mathfrak{a} \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_n$; then there exists $a_1, a_2 \in \mathfrak{a}$ such that

$$a_1 \in \bigcup_{i=1}^{n-1} \mathfrak{p}_i \text{ but } a_1 \notin \mathfrak{p}_n,$$

$$a_2 \in \mathfrak{p}_n \text{ but } a_2 \notin \bigcup_{i=1}^{n-1} \mathfrak{p}_i.$$

Their sum lies in neither; thus $a_1 + a_2 \notin \bigcup_{i=1}^{n} \mathfrak{p}_i$, which yields the desired contradiction. We conclude that $\mathfrak{a} \not\subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$; taking the contrapositive yields the required result. □

The following theorem does not concern unions, but it recasts the formulation of the above:

Theorem 25. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $\mathfrak{p}$ be a prime ideal containing $\bigcap \mathfrak{a}_i$. Then $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i.*

Proof. Suppose $\mathfrak{p} \not\supseteq \mathfrak{a}_i$ for all $i \in \{1, \ldots, n\}$. Then there exist $a_i \in \mathfrak{a}_i$ for each i that all do not belong to $\mathfrak{p}$; the product

$$a = \prod_{i=1}^{n} a_i$$

lies inside every $\mathfrak{a}_i$, so $a \in \bigcap \mathfrak{a}_i$; the primality of $\mathfrak{p}$ yields $a \notin \mathfrak{p}$, so $\mathfrak{p} \not\supseteq \bigcap \mathfrak{a}_i$. □

Corollary 4. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals. If $\bigcap \mathfrak{a}_i$ is prime, then $\bigcap \mathfrak{a}_i = \mathfrak{a}_j$ for some j.*

4.5 The Ideal Quotient

For ideals $\mathfrak{a}, \mathfrak{b}$ of R, their **ideal quotient** (which is trivially an ideal) is

$$(\mathfrak{a} : \mathfrak{b}) = \{x \mid x \in R,\, x\mathfrak{b} \subseteq \mathfrak{a}\},$$

The most important ideal quotient is the **annihalator**, defined as $(0 : \mathfrak{b})$ — the set of all $x \in R$ such that $x(\mathfrak{b}) = 0$ — and denoted as $\operatorname{Ann} \mathfrak{b}$. In this notation, the set D of all zero-divisors of R is

$$D = \bigcup_{a \neq 0} \operatorname{Ann}(a).$$

If (b) is a principal ideal, we write $(\mathfrak{a} : b)$ in place of $(\mathfrak{a} : (b))$.

Theorem 26. *For all ideals $\mathfrak{a}_i$, $\mathfrak{b}_i$ and $\mathfrak{c}$ of R for indicies $i \in I$, the following five properties hold:*

1. $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.
2. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.
3. $\big((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}\big) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = \big((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b}\big)$.
4. $\big(\bigcap_i \mathfrak{a}_i : \mathfrak{b}\big) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$.
5. $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap_i (\mathfrak{a} : \mathfrak{b}_i)$.

Proof. The proofs are as follows:

1. Let $a \in \mathfrak{a}$. Then $ab \in \mathfrak{a}$ for all $b \in \mathfrak{b}$, so $a(\mathfrak{b}) \subseteq \mathfrak{a}$; hence $a \in (\mathfrak{a} : \mathfrak{b})$. We conclude that $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.
2. Let $x \in (\mathfrak{a} : \mathfrak{b})$. By definition, $x\mathfrak{b} \subseteq \mathfrak{a}$; thus $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.
3. The two sets are equivalent, since
$$\begin{aligned} x \in \big((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}\big) &\iff x\mathfrak{c} \subseteq (\mathfrak{a} : \mathfrak{b}) \\ &\iff x\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a} \\ &\iff x \in (\mathfrak{a} : \mathfrak{b}\mathfrak{c}). \end{aligned}$$
Using this very identity yields $(\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = (\mathfrak{a} : \mathfrak{c}\mathfrak{b}) = \big((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b}\big)$.
4. The two sets are equivalent, since
$$\begin{aligned} x \in \left(\bigcap_i \mathfrak{a}_i : \mathfrak{b}\right) &\iff x\mathfrak{b} \subseteq \bigcap_i \mathfrak{a}_i \\ &\iff x\mathfrak{b} \subseteq \mathfrak{a}_i \text{ for each } i \\ &\iff x \in (\mathfrak{a}_i : \mathfrak{b}) \text{ for each } i \\ &\iff x \in \bigcap_i (\mathfrak{a}_i : \mathfrak{b}). \end{aligned}$$
5. The two sets are equivalent, since
$$\begin{aligned} x \in \left(\mathfrak{a} : \sum_i \mathfrak{b}_i\right) &\iff x\left(\sum_i \mathfrak{b}_i\right) \subseteq \mathfrak{a} \\ &\iff x\mathfrak{b}_i \subseteq \mathfrak{a} \text{ for each } i \\ &\iff x \in (\mathfrak{a} : \mathfrak{b}_i) \text{ for each } i \\ &\iff x \in \bigcap_i (\mathfrak{a} : \mathfrak{b}_i). \end{aligned}$$

This concludes the proof of all five properties. $\square$

4.6 Radicals of Ideals

The **radical** of an ideal $\mathfrak{a}$ of R

$$r(\mathfrak{a}) = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

If $\phi : R \to R/\mathfrak{a}$ is the canonical surjection, then $\phi(r(\mathfrak{a})) = \mathfrak{N}_{R/\mathfrak{a}}$, the nilradical of $R/\mathfrak{a}$; the Correspondence Theorem thus ensures that $r(\mathfrak{a})$ is an ideal.

Theorem 27. *For all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, the following six properties hold:*

1. $\mathfrak{a} \subseteq r(\mathfrak{a})$.
2. $r(r(\mathfrak{a})) = r(\mathfrak{a})$.
3. $r(\mathfrak{ab}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.
4. $r(\mathfrak{a}) = R$ *if and only if* $\mathfrak{a} = R$.
5. $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$.
6. *If $\mathfrak{p}$ is prime, then $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n \in \mathbb{Z}_{>0}$.*

Proof. Since (1) is trivial, the proofs are as follows:

2. Observe that $x \in r(r(\mathfrak{a})) \implies x^n \in r(\mathfrak{a})$ for some $n \implies x^{mn} \in \mathfrak{a}$ for some m; thus $x \in r(\mathfrak{a})$. If we suppose $x \in r(\mathfrak{a})$ and $r(r(\mathfrak{a})) \subseteq r(\mathfrak{a})$, then a usage of (1) yields $r(r(\mathfrak{a})) = \mathfrak{a}$.
3. **First Equality**: Since $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, we have $r(\mathfrak{ab}) \subseteq r(\mathfrak{a} \cap \mathfrak{b})$. If $x \in r(\mathfrak{a} \cap \mathfrak{b})$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some n; then $x^{n+1} \in \mathfrak{ab}$, so $r(\mathfrak{ab}) = r(\mathfrak{a} \cap \mathfrak{b})$.
   Second Equality: Clearly $x \in r(\mathfrak{a} \cap \mathfrak{b})$ implies $x \in r(\mathfrak{a})$ and $x \in r(\mathfrak{b})$, so $x \in r(\mathfrak{a}) \cap r(\mathfrak{b})$. If we assume the latter, then let $x^n \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$; then $x^{nm} \in \mathfrak{a} \cap \mathfrak{b}$, so $x \in r(\mathfrak{a} \cap \mathfrak{b})$. Hence, $r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.
4. Realize that
$$\begin{aligned} r(\mathfrak{a}) = R &\iff 1 \in r(\mathfrak{a}) \\ &\iff 1^n \in \mathfrak{a} \text{ for some } n \\ &\iff 1 \in \mathfrak{a} \\ &\iff \mathfrak{a} = R. \end{aligned}$$
5. We have $r(\mathfrak{a} + \mathfrak{b}) \subseteq r(r(\mathfrak{a}) + r(\mathfrak{b}))$ by (1); the other direction is simple.
6. Realize that since
$$x \in r(\mathfrak{p}) \iff x^n \in \mathfrak{p} \text{ for some } n \iff x \in \mathfrak{p},$$
   we have $r(\mathfrak{p}) = \mathfrak{p}$. The powers come from repeated application of (3).

$\square$

More generally, we can define the radical $r(E)$ for any subset $E \subseteq R$. It is not an ideal in general; it satisfies $r(\bigcup_i E) = \bigcup_i r(E)$.

Theorem 28. *The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals that contain $\mathfrak{a}$.*

Proof. Using the canonical surjection $\phi : R \to R \,/\, \mathfrak{a}$, we have for prime $\mathfrak{p}$ that

$$\mathfrak{p} \text{ contains the radical of } \mathfrak{a} \text{ in } R \iff \phi(\mathfrak{p}) \text{ contains the nilradical in } R \,/\, \mathfrak{a}.$$

The latter is guaranteed by Theorem 20. It is easy to verify that $\phi(\mathfrak{p})$ is prime. ☐

Theorem 29. *The set D of zero-divisors of R is equal to $\bigcup\limits_{a \neq 0} r(\mathrm{Ann}(a))$.*

Proof. The key is to realize that $D = r(D)$. This is because Theorem 27 ensures $D \subseteq r(D)$; now if if $x \in r(D)$, then $x^n \in D$, so $x^n y = x(x^{n-1}y) = 0$ for some $n \in \mathbb{Z}_{>0}$, and $x \in D$. Hence $D = r(D)$.

Now, we simply utilize the properties discussed in Section 4.5 and this page:

$$D = r(D) = r\left(\bigcup_{a \neq 0} \mathrm{Ann}(a)\right) = \bigcup_{a \neq 0} r(\mathrm{Ann}(a)).$$

☐

Theorem 30. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R, then $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime if and only if $r(\mathfrak{a})$ and $r(\mathfrak{b})$ are relatively prime.*

Proof. Using (4) and (5) from Theorem 27, we have that

$$\begin{aligned} \mathfrak{a} + \mathfrak{b} = R &\iff r(\mathfrak{a} + \mathfrak{b}) = R \\ &\iff r(r(\mathfrak{a}) + r(\mathfrak{b})) = R \\ &\iff r(\mathfrak{a}) + r(\mathfrak{b}) = R, \end{aligned}$$

as required. ☐

It is easy to see that $r(\mathfrak{a}) = r(\mathfrak{b})$ if and only if $\mathfrak{a} \subseteq \mathfrak{p}$ biconditionally implies $\mathfrak{b} \subseteq \mathfrak{p}$ — this is because all such $\mathfrak{p}$ satisfy $r(\mathfrak{a}) \subseteq \mathfrak{p}$.

4.7 Extension and Contraction

For a ring homomorphism $\phi : R \to S$ and an ideal $\mathfrak{a}$ of R, the image $\phi(\mathfrak{a})$ need not be an ideal of S. We define the **extension** $\mathfrak{a}^e$ as the principal ideal generated by A: namely, $\sum_{a \in R}(f(a))$. If $\mathfrak{b}$ is an ideal of S, then the Correspondence Theorem ensures that $\{a \in R \mid \phi(a) \in \mathfrak{b}\}$ is an ideal, called the **contraction** of $\mathfrak{b}$ and denoted by $\mathfrak{b}^c$.

To motivate these definitions, factorize ϕ as follows:

$$R \xrightarrow{p} \phi(R) \xrightarrow{j} S$$

The behavior of ideals under p is very simple: ideals of $\phi(R)$ correspond precisely with ideals of R that contain the kernel of ϕ. The situation with ideals under j is very complicated — in fact, it is among the central problems of Algebraic Number Theory.

Example: Consider the embedding $\mathbb{Z} \to \mathbb{Z}[i]$. For a prime ideal (p) of $\mathbb{Z}$, what is the extension of (p) in $\mathbb{Z}[i]$? Well, $\mathbb{Z}[i]$ is a principal ideal domain, and the situation is:

1. $(2)^e$ is the principal ideal $\Big((1+i)^2\Big)$, the *square* of the principal ideal $(1+i)$
2. If $p \equiv 1 \pmod 4$, then $(p)^e$ is the product of two distinct prime ideals.
3. If $p \equiv 3 \pmod 4$, then $(p)^e$ is prime in $\mathbb{Z}[i]$.

Observe the similarity between (2) and Fermat's theorem on sums of two squares.

Theorem 31. *For a homomorphism $\phi : R \to S$ and ideals $\mathfrak{a}$ and $\mathfrak{b}$ like before:*

1. $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ *and* $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$.
2. $\mathfrak{a}^e = \mathfrak{a}^{ece}$ *and* $\mathfrak{b}^c = \mathfrak{b}^{cec}$.
3. *If C is the set of contracted ideals in R and E is the set of extended ideals in S, then $C = \{\mathfrak{a} \mid \mathfrak{a}^{ec} = \mathfrak{a}\}$ and $E = \{\mathfrak{b} \mid \mathfrak{b}^{ce} = \mathfrak{b}\}$. Furthermore, $\mathfrak{a} \to \mathfrak{a}^e$ is a bijection from C to E with inverse $\mathfrak{b} \to \mathfrak{b}^c$.*

Proof. These proofs are ommitted, in the interest of remaining productive. I will comment: (1) is quite trivial, and (2) follows directly afterward. $\square$

In the interest of remaining productive, we will not prove the following fomulas:

$$\begin{aligned}
(\mathfrak{a}_1 + \mathfrak{a}_2)^e &= \mathfrak{a}_1^e + \mathfrak{a}_2^e && \text{and} && (\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c \\
(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e &\subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e && \text{and} && (\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c \\
(\mathfrak{a}_1 \mathfrak{a}_2)^e &= \mathfrak{a}_1^e \mathfrak{a}_2^e && \text{and} && (\mathfrak{b}_1 \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c \mathfrak{b}_2^c \\
(\mathfrak{a}_1 : \mathfrak{a}_2)^e &\subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e) && \text{and} && (\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c) \\
r(\mathfrak{a})^e &\subseteq r(\mathfrak{a}^e) && \text{and} && r(\mathfrak{b})^c = r(\mathfrak{b})^c.
\end{aligned}$$

The set of ideals E is thus closed under sum and product, while C is closed under ideal quotients, radicals, and intersections.

5 The Zariski Topology

5.1 Definition

Let R be a ring and let X denote the set of prime ideals of R. For each subset $E \subseteq R$, let $V(E)$ denote the set of prime ideals which contain E. This construction should remind one of the radical $R(E)$.

Theorem 32. *Let $(E_\alpha) \subseteq R$, let $E_1, E_2 \subseteq R$. Define $\mathfrak{a}_\alpha$, $\mathfrak{a}_1$, and $\mathfrak{a}_2$ as the ideals generated by these sets. Then the following holds:*

1. $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
2. $\bigcap_\alpha V(E_\alpha) = V\big(\bigcup_\alpha E_\alpha\big)$.
3. $V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1\mathfrak{a}_2) = V(\mathfrak{a}_1 \cap \mathfrak{a}_2)$.

Proof. For (1), it is clear that

$$\mathfrak{p} \in V(E) \iff E \subseteq \mathfrak{p} \iff \mathfrak{a} \subseteq \mathfrak{p} \iff \mathfrak{p} \in V(\mathfrak{a}).$$

For (2), we similarly utilize such convenient chains of equivalencies:

$$\begin{aligned}
\mathfrak{p} \in \bigcap_\alpha V(E_\alpha) &\iff E_\alpha \subseteq \mathfrak{p} \text{ for each } \alpha. \\
&\iff \bigcup_\alpha E_\alpha \subseteq \mathfrak{p} \\
&\iff \mathfrak{p} \in V\left(\bigcup_\alpha E_\alpha\right).
\end{aligned}$$

We could also write this as $\bigcup_\alpha V(\mathfrak{a}_a) = V\big(\sum_\alpha \mathfrak{a}_a\big)$.

The story for (3) is again quite similar: we have that

$$\begin{aligned}\mathfrak{p} \in V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) &\iff \mathfrak{a}_1 \subseteq \mathfrak{p} \text{ or } \mathfrak{a}_2 \subseteq \mathfrak{p} \\ &\iff \mathfrak{a}_1 \cap \mathfrak{a}_2 \subseteq \mathfrak{p} \iff \mathfrak{p} \in V(\mathfrak{a}_1 \cap \mathfrak{a}_2) \\ &\iff \mathfrak{a}_1\mathfrak{a}_2 A \subseteq \mathfrak{p} \iff \mathfrak{p} \in V(\mathfrak{a}_1\mathfrak{a}_2).\end{aligned}$$

This last step follows from the fact $r(\mathfrak{a}_1 \cap \mathfrak{a}_2) = r(\mathfrak{a}_1\mathfrak{a}_2)$. This completes the proof. ☐

Further observe that $V(0) = X$ and $V(1) = \varnothing$. Thus the sets $V(\mathfrak{a})$ across all $\mathfrak{a} \in X$ satisfy the closed set axioms of a toplogical space. The resulting topology is called the **Zariski topology**, and the set X is called the **prime spectrum** of R, denoted $\operatorname{Spec} R$.

5.2 Open Sets in the Zariski Topology

Let $f \in R$ and $X = \operatorname{Spec} R$. We define the open set X_f as the complement of $V(f)$ in X.

Theorem 33. *The sets X_f form a basis of the Zariski topology.*

Proof. Let $V(\mathfrak{a})^\complement$ be an arbitrary open set in X. If f_α are the elements of $\mathfrak{a}$, then

$$\bigcup_\alpha X_{f_\alpha} = \bigcup_\alpha V(f_\alpha)^\complement = \left(\bigcap_\alpha V(f_\alpha)\right)^\complement = V\left(\sum_\alpha (f_a)\right)^\complement = V(\mathfrak{a})^\complement.$$

This completes the proof. ☐

Thus the sets X_f are the **basic open sets** of $\operatorname{Spec} R$. There are many more properties of open sets in the Zariski topology, including the following: since $(f) \cap (g) = (fg)$,

$$X_f \cap X_g \;=\; V(f)^\complement \cap V(g)^\complement \;=\; \big(V(f) \cup V(g)\big)^\complement \;=\; V(fg)^\complement \;=\; X_{fg}.$$

Theorem 34. *The following properties of X_f hold:*

1. *$X_f = \varnothing$ if and only if $f \in \mathfrak{N}$.*
2. *$X_f = X$ if and only if x is a unit.*
3. *$X_f = X_g$ if and only if $r\big((f)\big) = r\big((g)\big)$.*

Proof. (1) follows from the properties of the Nilradical:

$$X_f = \varnothing \iff V(f) = X \iff f \in \mathfrak{N}.$$

For (2), the answer follows from Krull's Theorem:

$$X_f = X \iff V(f) = \varnothing \iff (f) = R \iff f \text{ is a unit.}$$

Part (3) is relatively trivial from the definition of the radical:

$$X_f = X_g \iff V(f) = V(g) \iff r\big((f)\big) = r\big((g)\big).$$

This completes the proof. □

Corollary 5. *$V(f) = V(g)$ if and only if $r\big((f)\big) = r\big((g)\big)$.*

In the Zariski topology, a set $S \subseteq X$ is **quasi-compact** if each open covering of S contains a finite sub-covering. The term "compact" is reserved for sets with additional structure.

Theorem 35. *The following three facts about quasi-compactness hold:*

1. *X is quasi-compact.*
2. *Each X_f is quasi-compact.*
3. *An open subset $S \subseteq X$ is quasi-compact if and only if S is a finite union of X_f.*

Proof. We start with (2). Suppose that X_{f_α} is an open cover of X_f. Then

$$V\left(\sum_\alpha f_\alpha\right)^{\complement} = \left(\bigcap_\alpha V(f_\alpha)\right)^{\complement} = \bigcup_\alpha X_{f_\alpha} = X_f.$$

Then $\sum\limits_\alpha f_\alpha$ contains a unit, so there exist indicies $\alpha_1, \ldots, \alpha_n$ and contants $r_1, \ldots, r_n \in R$ such that

$$1 = r_1 f_{\alpha_1} + \cdots + r_n f_{\alpha_n},$$

so $(f_{\alpha_1}, \ldots, f_{\alpha_n}) = R$. Therefore,

$$V\left(\sum_{i=1}^n f_{\alpha_i}\right)^{\complement} = \bigcup_{i=1}^n X_{f_{\alpha_i}} = X,$$

so X is quasi-compact. □