MATH-UA 129: Lecture 4

Vector-Valued Functions

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Abstract

The previous document examined mappings $f : \mathbb{R}^n \to \mathbb{R}$. Now, we examine how the tools of calculus may be translated to a broader class of functions — vector-valued functions.

1 Divergence and Curl

1.1 Divergence

For the following lemmas, let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, where $\mathbf{F} = (f_1, \ldots, f_n)$; define $\mathbf{G}$ similarly. The **divergence** of f is defined as follows:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x_1} f_1 + \cdots + \frac{\partial}{\partial x_n} f_n.$$

Lemma. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$.

Proof. This is a sum rule for the divergence. We have that

$$\begin{aligned}
\nabla \cdot (\mathbf{F} + \mathbf{G}) &= \frac{\partial}{\partial x_1}(f_1 + g_1) + \cdots + \frac{\partial}{\partial x_n}(f_n + g_n) \\
&= \left(\frac{\partial}{\partial x_1} f_1 + \cdots + \frac{\partial}{\partial x_n} f_n\right) + \left(\frac{\partial}{\partial x_1} g_1 + \cdots + \frac{\partial}{\partial x_n} g_n\right) \\
&= \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}.
\end{aligned}$$

Lemma. $\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \nabla f \cdot \mathbf{F}$.

Proof. This is a product rule for the divergence. We have that

$$\begin{aligned}
\nabla \cdot (f\mathbf{F}) &= \frac{\partial}{\partial x_1}(f_1 f) + \cdots + \frac{\partial}{\partial x_n}(f_n f) \\
&= \left(f \frac{\partial}{\partial x_1} f_1 + f_1 \frac{\partial}{\partial x_1} f\right) + \cdots + \left(f \frac{\partial}{\partial x_n} f_n + f_n \frac{\partial}{\partial x_n} f\right) \\
&= f \left(\frac{\partial}{\partial x_1} f_1 + \cdots + \frac{\partial}{\partial x^n} f_n\right) + \left(f_1 \frac{\partial}{\partial x_1} f + \cdots + f_n \frac{\partial}{\partial x^n} f\right) \\
&= f(\nabla \cdot \mathbf{F}) + \nabla f \cdot \mathbf{F}.
\end{aligned}$$

Lemma. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

Proof. Of course, this presumes that $\mathbf{F}$ and $\mathbf{G}$ are three-dimensional. We have that

$$\begin{aligned}
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \nabla \cdot \left((f_2 g_3 - f_3 g_2)\, \hat{\mathbf{i}} + (f_3 g_1 - f_1 g_3)\, \hat{\mathbf{j}} + (f_1 g_2 - f_2 g_1)\, \hat{\mathbf{k}}\right) \\
&= \frac{\partial}{\partial x}(f_2 g_3 - f_3 g_2) + \frac{\partial}{\partial y}(f_3 g_1 - f_1 g_3) + \frac{\partial}{\partial z}(f_1 g_2 - f_2 g_1) \\
&= (f_2 g_3' + f_2' g_3 - f_3 g_2' - f_3' g_2) + (f_3 g_1' + f_3' g_1 - f_1 g_3' - f_1' g_3) \\
&\quad + (f_1 g_2' + f_1' g_2 - f_2 g_1' - f_2' g_1) \\
&= (f_2 g_3' - f_3 g_2') + (f_3 g_1' - f_1 g_3') + (f_1 g_2' - f_2 g_1') \\
&= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\end{aligned}$$

1.2 Curl

For the following lemmas, let $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\mathbf{F} = (f_1, f_2, f_3)$; define $\mathbf{G}$ similarly. The **curl** of f is defined as follows:

$$\nabla \times \mathbf{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{\mathbf{k}}$$

Lemma. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

Proof. This is a sum rule for the curl. We have that

$$\begin{aligned}
\nabla \times (\mathbf{F} + \mathbf{G}) &= \left(\frac{\partial}{\partial y}(f_3 + g_3) - \frac{\partial}{\partial z}(f_2 + g_2)\right)\hat{\mathbf{i}} \\
&\quad + \left(\frac{\partial}{\partial z}(f_1 + g_1) + \frac{\partial}{\partial x}(f_3 + g_3)\right)\hat{\mathbf{j}} \\
&\quad + \left(\frac{\partial}{\partial x}(f_2 + g_2) + \frac{\partial}{\partial y}(f_1 + g_1)\right) \\
&= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
&\quad + \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) \\
&= \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.
\end{aligned}$$

Lemma. $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}$

Proof. This is a product rule for the curl. We have that

$$\begin{aligned}
\nabla \times (f\mathbf{F}) &= \left(\frac{\partial}{\partial y}(f_1 f) - \frac{\partial}{\partial z}(f_2 f)\right)\hat{\mathbf{i}} \\
&\quad + \left(\frac{\partial}{\partial z}(f_1 f) - \frac{\partial}{\partial x}(f_3 f)\right)\hat{\mathbf{j}} \\
&\quad + \left(\frac{\partial}{\partial x}(f_2 f) + \frac{\partial}{\partial y}(f_1 f)\right)\hat{\mathbf{k}} \\
&= \left(f\frac{\partial}{\partial y}f_3 + f_3\frac{\partial}{\partial y}f - f\frac{\partial}{\partial z}f_2 - f_2\frac{\partial}{\partial z}f)\right)\hat{\mathbf{i}} \\
&\quad + \left(f\frac{\partial}{\partial z}f_1 + f_1\frac{\partial}{\partial z}f - f\frac{\partial}{\partial x}f_3 - f_3\frac{\partial}{\partial x}f\right)\hat{\mathbf{j}} \\
&\quad + \left(f\frac{\partial}{\partial x}f_2 + f_2\frac{\partial}{\partial x}f + f\frac{\partial}{\partial y}f_1 - f_1\frac{\partial}{\partial y}f\right). \\
&= f\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{\mathbf{i}} + f\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{\mathbf{j}} + f\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
&\quad + \left(f_3\frac{\partial}{\partial y}f - f_2\frac{\partial}{\partial z}f\right)\hat{\mathbf{i}} + \left(f_1\frac{\partial}{\partial z}f - f_3\frac{\partial}{\partial x}f\right)\hat{\mathbf{j}} + \left(f_2\frac{\partial}{\partial x}f - f_1\frac{\partial}{\partial y}f\right) \\
&= f(\nabla \times \mathbf{F}) + \nabla f \times \mathbf{F}.
\end{aligned}$$

2 Relationships between Operators

Lemma. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Proof. We have that

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \nabla \cdot \left(\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \hat{\mathbf{k}} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \hat{\mathbf{j}} + \frac{\partial}{\partial z} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\
&= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial z \partial x} + \frac{\partial^2 f^1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial y \partial z} \\
&= 0.
\end{aligned}$$

Thus, the curl is divergence-free. The converse is true — if a vector field is divergence-free, it is a curl of some other field.

Lemma. $\nabla \times (\nabla f) = \mathbf{0}$.

Proof. This is a compact way of expressing the equality of mixed partial derivatives. We have that

$$\begin{aligned}
\nabla \times (\nabla f) &= \nabla \times \left(\frac{\partial f_1}{\partial x} \hat{\mathbf{i}} + \frac{\partial f_2}{\partial y} \hat{\mathbf{j}} + \frac{\partial f_3}{\partial z} \hat{\mathbf{k}} \right) \\
&= \left(\frac{\partial^2 f_3}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f_1}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial x \partial z} \right) \hat{\mathbf{j}} + \left(\frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial y \partial x} \right) \hat{\mathbf{k}} \\
&= 0 \, \hat{\mathbf{i}} + 0 \, \hat{\mathbf{j}} + 0 \, \hat{\mathbf{k}} \\
&= \mathbf{0}.
\end{aligned}$$

Thus, the gradiant is curl-free. The converse is true — if a vector field is curl-free, it is the gradiant of some other field.

Lemma. $\nabla \cdot (\nabla f \times \nabla g) = 0$.

Proof. By the formulas established above,

$$\nabla \cdot (\nabla f \times \nabla g) = \nabla g \cdot (\nabla \times \nabla f) - \nabla f \cdot (\nabla \times \nabla g) = \nabla f \cdot \mathbf{0} - \nabla f \cdot \mathbf{0} = \mathbf{0}.$$