Math-UA 148: Homework 2

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1 2A Problems

1.1 Problem 9

The given result is **false**. In $\mathbb{R}^2$, see that the two lists $(1,0),(0,1)$ and $(1,0),(0,-1)$ are both independent. Yet their sum of $(1,0)+(1,0),(0,1)+(0,-1)$ or $(2,0),(0,0)$ is not an independent list, as $0(2,0)+5(0,0)=\mathbf{0}$.

1.2 Problem 10

As $\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_m + \mathbf{w}$ is linearly dependent, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$, not all zero, such that

$$\lambda_1(\mathbf{v}_1 + \mathbf{w}) + \cdots + \lambda_m(\mathbf{v}_m + \mathbf{w}) = \mathbf{0}.$$

This can be rearranged to

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m = -(\lambda_1 + \cdots + \lambda_m)\mathbf{w}.$$

Now, suppose that $\lambda_1 + \cdots + \lambda_m = 0$. Then the above equation rearranges to

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m = -(\lambda_1 + \cdots + \lambda_m)\mathbf{w} = 0\mathbf{w} = \mathbf{0},$$

which implies that the list $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly dependent — a contradiction. Then $\lambda_1 + \cdots + \lambda_m$ must be nonzero. This allows us to divide both sides of the above equation by $-(\lambda_1 + \cdots + \lambda_m)$, which yields that

$$-\frac{\lambda_1\mathbf{v}_1 + \cdots + \lambda_m\mathbf{v}_m}{\lambda_1 + \cdots + \lambda_m} = \mathbf{w}.$$

Hence, $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

1.3 Problem 12

If a list of polynomials in $\mathcal{P}_4(\mathbb{F})$ is linearly independent, then the length of the list is less than or equal to 5 — the dimension of $\mathcal{P}_4(\mathbb{F})$. By contraposition, a list of 6 polynomials in $\mathcal{P}_4(\mathbb{F})$ cannot be linearly independent.

2 2B Problems

2.1 Problem 7

Consider the four polynomials $1, x+1, x^2, x^3 \in \mathcal{P}_3(\mathbb{R})$, and define

$$W = \{ax^3 + bx^2 + cx \mid a, b, c \in \mathbb{R}\}.$$

Clearly the four polynomials are a basis of $\mathcal{P}_3(\mathbb{R})$ and W is a subspace of $\mathcal{P}_3(\mathbb{R})$. Observe that $x^2, x^3 \in W$ and $1, x+1 \notin W$ — however, x^2 and x^3 do not constitute a basis of W, as no linear combination of the two generates the one-degree polynomials of W.

2.2 Problem 8

Observe that for all $\mathbf{v} \in V$, there exist $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$. Now, define $\lambda_1, \ldots, \lambda_{n+m} \in \mathbb{F}$ such that

$$\begin{aligned} \mathbf{u} &= \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_j \\ \mathbf{w} &= \lambda_{n+1} \mathbf{w}_1 + \cdots + \lambda_{n+m} \mathbf{w}_m. \end{aligned}$$

We find that

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + \lambda_{n+1} \mathbf{w}_1 + \cdots + \lambda_{n+m} \mathbf{w}_m.$$

Therefore, $\mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{w}_1, \ldots, \mathbf{w}_m$ spans V. Now, observe that if there is a nontrivial solution to

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + \lambda_{n+1} \mathbf{w}_1 + \cdots + \lambda_{n+m} \mathbf{w}_m = \mathbf{0},$$

we may rearrange this to

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n = -\lambda_{n+1} \mathbf{w}_1 - \cdots - \lambda_{n+m} \mathbf{w}_m.$$

As $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis of U — and as $\lambda_1, \ldots, \lambda_n$ are not all equal to zero — both sides of this equation are nonzero. Note that the left-hand side is in U and the right-hand side is in W; thus, their sum is a nonzero vector in U and W, so $U \cap W \neq \{\mathbf{0}\}$. This contradicts the fact $U + W$ is a direct sum. We conclude that there is no nontrivial solution to

$$\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n + \lambda_{n+1} \mathbf{w}_1 + \cdots + \lambda_{n+m} \mathbf{w}_m = \mathbf{0},$$

so $\lambda_1 \mathbf{u}_1, \ldots, \lambda_n \mathbf{u}_n, \lambda_{n+1} \mathbf{w}_1, \ldots, \lambda_{n+m} \mathbf{w}_m$ is a linearly independent list. Therefore, the list is a basis of V.

3 2C Problems

3.1 Problem 6

Consider the four polynomials: 1, $(x-2)(x-5)$, $(x-2)(x-5)(x)$, and $(x-2)(x-5)(x^2)$. All four polynomials have different degrees, so they are linearly independent in U.

Consider an arbitrary $p \in U$; p has degree of four or less. Define λ such that $p(2) = p(5) = \lambda$; then $p(2) - \lambda = p(5) - \lambda = 0$. The Factor Theorem thus guarantees that $p(x) - \lambda = (x-2)(x-5)(\alpha x^2 + \beta x + \gamma)$ for some $\alpha, \beta, \gamma \in \mathbb{F}$. Then

$$\begin{aligned} p(x) &= \lambda + (x-2)(x-5)(\alpha x^2 + \beta x + \gamma) \\ &= \lambda + \alpha(x-2)(x-5)(x^2) + \beta(x-2)(x-5)(x) + \gamma(x-2)(x-5) \end{aligned}$$

We conclude that these four polynomials span U, and are thus a basis of U.

(b) Extend the basis of U with the polynomial x. Because all five polynomials have different degrees, they are linearly independent — and because $\mathcal{P}_4(\mathbb{F})$ has dimension five, our five polynomials must be a basis of $\mathcal{P}_4(\mathbb{F})$.

(c) Consider the subspace $W = \{\lambda x \mid \lambda \in \mathbb{F}\}$. The polynomial x trivially spans W. Now, consider if $p \in W \cap U$; then $p = \lambda x$ for some $\lambda \in \mathbb{F}$ and $\lambda 2 = \lambda 5$. We deduce that $\lambda = 0$. Then $U \cap W = \{\mathbf{0}\}$, and $U + V$ is a direct sum.

Observe that list x spans W and 1, $(x-2)(x-5)$, $(x-2)(x-5)(x)$, $(x-2)(x-5)(x^2)$ spam U; by the result of Section 2B Problem 8, their union is a basis of $U \oplus W$. This is the same basis of $\mathcal{P}_4(\mathbb{F})$ — then $U \oplus W = \mathcal{P}_4(\mathbb{F})$, as desired.

3.2 Problem 12

Suppose for contradiction that $U \cap W = \{\mathbf{0}\}$. Clearly, $U + W$ is thus a direct sum, and $U \oplus W$ is a subspace of V.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_5$ and $\mathbf{w}_1, \ldots, \mathbf{w}_5$ be a basis of W. Via the result of Section 2B Problem 8, $\mathbf{u}_1, \ldots, \mathbf{u}_5, \mathbf{w}_1, \ldots, \mathbf{w}_5$ is a basis of $U \oplus W$. Then $U \oplus W$ has dimension 10.

This contradicts the fact that no subspace of V has a larger dimension than V. We conclude that $U \cap W \neq \{\mathbf{0}\}$.

3.3 Problem 16

We proceed via induction.

Base case: Let U_1 and U_2 be subspaces of V such that $U_1 + U_2$ is a direct sum. We define the dimensions of U_1 and U_2 as n and m respectively and the bases of U_1 and U_2 as $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

Via the result of Section 2B Problem 8, $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_m$ is a basis of $U \oplus W$. We conclude that $\dim U_1 \oplus U_2 = n + m = \dim U_1 + \dim U_2$.

Induction step: Asssume that for all sets of m subspaces $U_1, \ldots, U_m$ of V such that $U_1 + \cdots + U_k$ is a direct sum, we have that $\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m$.

Let U_{m+1} be a subspace of W such that $U_1 + \cdots + U_{m+1}$ is a direct sum. Then by our base case,

$$\dim U_1 \oplus \cdots \oplus U_{m+1} = \dim(U_1 \oplus \cdots \oplus U_m) + \dim U_{m+1} = \dim U_1 \oplus \cdots \oplus \dim U_{m+1}.$$

This completes the induction.