MATH-UA 129: Homework 5

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Contents

1 Section 3.2

1.1 Problem 3

We have that $\frac{\partial f}{\partial x} = 2x + 2y = \frac{\partial f}{\partial y}$, so all first-order partial derivatives are 0 at (x_0, y_0), and $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y^2} = 2$ at (x_0, y_0). Then observing that $\mathbf{0} = (x_0, y_0)$ and defining that $\mathbf{h} = (h_1, h_2)$ yields

$$\begin{aligned} f(\mathbf{h}) &= f(\mathbf{0}) + \sum_{i=1}^{2} h_i \frac{\partial f}{\partial x_i}(\mathbf{0}) + \frac{1}{2} \sum_{1 \le i,j \le 2} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{0}) + R_2(\mathbf{0}, \mathbf{h}) \\ &= 0 + 0 + \frac{1}{2}\left(2h_1^2 + 4h_1 h_2 + 2h_2^2\right) + R_2(\mathbf{h}, \mathbf{0}) \\ &= (h_1 + h_2)^2 + R_2(\mathbf{0}, \mathbf{h}), \end{aligned}$$

where $\lim\limits_{\mathbf{h} \to \mathbf{0}} \frac{R_2(\mathbf{0},\mathbf{h})}{\|\mathbf{h}\|^2} = 0$. Because f is defined such that $f(\mathbf{h}) = (h_1 + h_2)^2$, we must have that $R_2(\mathbf{0}, \mathbf{h}) = 0$; thus, the second-order Taylor formula returns that $\boxed{f(\mathbf{h}) = (h_1 + h_2)^2}$.

1.2 Problem 5

Observe that all first-order and second-order partial derivatives of e^{x+y} are e^{x+y} itself; at $(x_0, y_0) = \mathbf{0}$, these evaluate to 1. We conclude that if $\mathbf{h} = (h_1, h_2)$,

$$\begin{aligned} f(\mathbf{h}) &= f(\mathbf{0}) + \sum_{i=1}^{2} h_i \frac{\partial f}{\partial x_i}(\mathbf{0}) + \frac{1}{2} \sum_{1 \le i,j \le 2} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{0}) + R_2(\mathbf{0}, \mathbf{h}) \\ &= 1 + h_i + h_2 + \frac{1}{2}(h_1^2 + 2h_1 h_2 + h_2^2) + R_2(\mathbf{0}, \mathbf{h}) \\ &= \boxed{\frac{1}{2}h_1^2 + \frac{1}{2}h_2^2 + h_1 + h_1 h_2 + h_2 + 1 + R_2(\mathbf{0}, \mathbf{h})}, \end{aligned}$$

where $\lim\limits_{\mathbf{h} \to \mathbf{0}} \frac{R_2(\mathbf{0},\mathbf{h})}{\|\mathbf{h}\|^2} = 0$.

1.3 Problem 7

We have that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y\cos(xy) - y\sin(xy),\\
\frac{\partial f}{\partial y} &= x\cos(xy) - x\sin(xy),\\
\frac{\partial^2 f}{\partial x^2} &= -y^2\sin(xy) - y^2\cos(xy),\\
\frac{\partial^2 f}{\partial x \partial y} &= \cos(xy) - \sin(xy) - xy\sin(xy) - xy\cos(xy),\\
\frac{\partial^2 f}{\partial x \partial y} &= -x^2\sin(xy) - x^2\cos(xy).
\end{aligned}$$

At the point $(x_0, y_0) = \mathbf{0}$, these compute to $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$ and $\frac{\partial^2 f}{\partial x \partial y} = 1$. We conclude that if $\mathbf{h} = (h_1, h_2)$,

$$\begin{aligned}
f(\mathbf{h}) &= f(\mathbf{0}) + \sum_{i=1}^{2} h_i \frac{\partial f}{\partial x_i}(\mathbf{0}) + \frac{1}{2} \sum_{1 \le i,j \le 2} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{0}) + R_2(\mathbf{0}, \mathbf{h})\\
&= 1 + 0 + \frac{1}{2}(0 + 2h_1h_2 + 0) + R_2(\mathbf{0}, \mathbf{h})\\
&= \boxed{h_1h_2 + 1 + R_2(\mathbf{0}, \mathbf{h})},
\end{aligned}$$

where $\lim\limits_{\mathbf{h} \to \mathbf{0}} \frac{R_2(\mathbf{0},\mathbf{h})}{\|\mathbf{h}\|^2} = 0$.

1.4 Problem 11

We have that

$$\begin{aligned}
\frac{\partial g}{\partial x} &= y\cos(xy) - 6x\ln(y)\\
\frac{\partial g}{\partial y} &= x\cos(xy) - \frac{3x^2}{y}\\
\frac{\partial^2 g}{\partial x^2} &= -y^2\sin(xy) - 6\ln(y)\\
\frac{\partial^2 g}{\partial x \partial y} &= -xy\sin(xy) + \cos(xy) - \frac{6x}{y}\\
\frac{\partial^2 g}{\partial y^2} &= -x^2\sin(xy) + \frac{3x^2}{y^2}.
\end{aligned}$$

At the point $(x_0, y_0) = (\frac{\pi}{2}, 1)$, these partial derivatives evaluate to 0, $-\frac{3\pi^2}{4}$, -1, $-\frac{7\pi}{2}$, and $\frac{\pi^2}{2}$ respectively. We conlude that if $\mathbf{x} = (\frac{\pi}{2}, 1)$ and $\mathbf{h} = (h_1, h_2)$,

$$\begin{aligned} g(\mathbf{x}+\mathbf{h}) &= g(\mathbf{x}) + \sum_{i=1}^{2} h_i \frac{\partial g}{\partial x_i}(\mathbf{x}) + \frac{1}{2} \sum_{1 \le i,j \le 2} h_i h_j \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{x}) + R_2(\mathbf{x}, \mathbf{h}) \\ &= 1 + 0 - \frac{3\pi^2}{4} h_2 + \frac{1}{2}(-h_1^2 - 7\pi h_1 h_2 + \frac{\pi^2}{2} h_2^2) + R_2(\mathbf{x}, \mathbf{h}) \\ &= \boxed{-\frac{1}{2} h_1^2 + \frac{\pi^2}{4} h_2^2 - \frac{7\pi}{2} h_1 h_2 - \frac{3\pi^2}{4} h_2 + 1 + R_2(\mathbf{x}, \mathbf{h})}, \end{aligned}$$

where $\lim_{\mathbf{h} \to \mathbf{0}} \frac{R_2(\mathbf{x}, \mathbf{h})}{\|\mathbf{h}\|^2} = 0$.

1.5 Problem 12

(NOTE: I assume I only need to approximate $f(-1, -1)$ for the functions from Exercises 3, 5, and 7.)

Problem 3: The approximation computes to $(-1-1)^2 = \boxed{4}$.

Problem 5: The approximation computes to $\frac{1}{2} + \frac{1}{2} - 1 + 1 - 1 + 1 = \boxed{1}$.

Problem 7: The approximation computes to $1 + 1 = \boxed{2}$.

1.6 Problem 13

Part (a): Let $R_n(x-h)$ be the remainder of the $(n-1)$-th taylor polynomial of $f(x+h)$ — namely,

$$R_n(x+h) = f(x+h) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} h^k.$$

Observe that there is some $c \in (x, x+h)$ such that

$$R_n(x+h) = \frac{f^n(c)}{n!}(x+h-c)^n.$$

Let M be the constant such that $\left|f^{(k)}(x)\right| < M^k$ for all $k \in \mathbb{Z}_{>0}$ over the interval $[c-1, c+1]$; thus,

$$\lim_{n \to \infty} |R_n(x-h)| = \lim_{n \to \infty} \frac{|f^n(c)|}{n!} |x+h-c|^n \le \lim_{n \to \infty} \frac{|Mh|^n}{n!}.$$

The right-hand side is a term of the expression $\sum_{n=0}^{\infty} \frac{|Mh|^n}{n!}$, which is convergent and defined to be $e^{|Mh|}$. By the contrapositive of the Divergence Test, we must have that $\lim_{n \to \infty} \frac{|Mh|^n}{n!} = 0$, so $\lim_{n\to\infty} |R_n(x-h)| = 0$. Then for all $\epsilon > 0$, there exists N such that

$$N < n \implies \left| f(x+h) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{x!} h^k \right| = ||R_n(x-h)| - 0| < \epsilon.$$

We conclude via the definition of a limit that

$$f(x+h) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} h^k,$$

as required.

Part (b): It is trivial that e^x and $-\frac{1}{x}$ are C^∞ for $x > 0$, so their composition $e^{-\frac{1}{x}}$ is C^∞ for $x > 0$ — and 0 is clearly C^∞ for $x \leq 0$. Clearly, all derivatives of f at 0 approaching from the left are zero; it remains to be proven that all derivatives approaching from the *right* are zero. Define

$$S = \{p\,(x)\, e^{-\frac{1}{x}} \mid p(x) \text{ is a rational function } \},$$

where $\mathcal{R}(\mathbb{R})$ is the set of all polynomials with real-valued coefficents. Observe that for all $p(x) \in \mathcal{P}(\mathbb{R})$, we may repeatedly apply l'Hospital's Rule to deduce that

$$\lim_{x\to\infty} p(x)e^x = \lim_{x\to\infty} \frac{p(x)}{e^{-x}} = \cdots = \lim_{x\to\infty} \pm\frac{1}{e^{-x}} = \lim_{x\to\infty} \pm e^x = \pm\infty.$$

The above argument may be formalized by induction. Therefore,

$$\lim_{x\to 0^+} r\,(x)\, e^{-\frac{1}{x}} = 0.$$

for all rational functions r. We claim that all derivatives of $e^{\frac{1}{x}}$ lie in S – this is true for the 0-th derivative (namely $e^{\frac{1}{x}}$ itself), and further derivatives may be proven via induction. We conclude that all n-th order derivatives of $e^{\frac{1}{x}}$ lie in S, so their limit approaching $x = 0$ from above yields 0. $e^{-\frac{1}{x}}$ is therefore C^∞.

However, constructing a taylor series of $e^{-\frac{1}{x}}$ at $x = -1$ yields $f(x) = 0$, which does not attain the positive values of $e^{-\frac{1}{x}}$. We conclude that $e^{-\frac{1}{x}}$ is not analytic.

Part (c): A function $f : \mathbb{R}^n \to \mathbb{R}$ is called an **analytic** function provided that if $\mathbf{h} = (h_1, \ldots, h_n)$,

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \frac{1}{1!}\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}) + \frac{1}{2!} \sum_{1\leq i,j\leq n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \cdots$$

ANSWER: We claim that if for all closed discs $U \subset \mathbb{R}^n$, there exists a constant M such that all n-th order partial derivatives of f at $\mathbf{x}$ are bounded above by a constant M^n for each n, then the right-hand side of this equation converges and equals $f(\mathbf{x}+\mathbf{h})$.

We define the remainder of the nth taylor polynomial as $R_k(\mathbf{x},\mathbf{h})$:

$$\begin{aligned} R_k(\mathbf{x},\mathbf{h}) = f(x+h) - f(\mathbf{x}) - \frac{1}{1!}\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}) - \cdots \\ - \frac{1}{(k-1)!}\sum_{1\le i_1,\ldots,i_{k-1}\le n} h_{i_1}\cdots h_{i_k}\frac{\partial^{k-1}\partial}{\partial x_{i_1}\ldots x_{i_{k-1}}}. \end{aligned}$$

As f is C^∞, we have that

$$R_k(\mathbf{x},\mathbf{h}) = \frac{1}{k!}\sum_{1\le i_1,\ldots,i_k\le n} h_{i_1}\cdots h_{i_k}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\mathbf{c}_{i_1\cdots i_k},$$

where $\mathbf{c}_{i_1\cdots i_k}$ lies somewhere on the line joining $\mathbf{x}$ to $\mathbf{x}+h$. Define $h = \max\{h_1,\ldots,h_n\}$, and let M be the constant such that M^n bounds (above) the n-th order partial derivatives of f over the closed disc with radius 1 centered at $\mathbf{c}$; thus,

$$\begin{aligned} \lim_{k\to\infty}|R_k(\mathbf{x},\mathbf{h})| &= \lim_{k\to\infty}\left|\frac{1}{k!}\sum_{1\le i_1,\ldots,i_k\le n} h_{i_1}\cdots h_{i_k}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}\mathbf{c}_{i_1\cdots i_k}\right| \\ &\le \lim_{k\to\infty}\left|\frac{1}{k!}\sum_{1\le i_1,\ldots,i_n\le n} h^k M^k\right| \\ &= |n^n|\lim_{k\to\infty}\frac{|hM|^k}{k!}. \end{aligned}$$

The right-hand side is a term of the expression $\sum_{k=0}^\infty \frac{|Mh|^k}{k!}$, which is convergent and defined to be $e^{|Mh|}$. By the contrapositive of the Divergence Test, we must have that $\lim\limits_{k\to\infty}\frac{|Mh|^k}{k!} = 0$, so $\lim\limits_{k\to\infty} R_k(x-h) = 0$. Then for all $\epsilon > 0$, there exists N such that

$$\begin{aligned} N < k \implies & ||R_k(\mathbf{x},\mathbf{n})| - 0| \\ &= \left|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \cdots - \frac{1}{(k-1)!}\sum_{1\le i_1,\ldots,i_{k-1}\le n} h_{i_1}\cdots h_{i_k}\frac{\partial^{k-1}\partial}{\partial x_{i_1}\ldots x_{i_{k-1}}}(\mathbf{x})\right| \\ &< \epsilon. \end{aligned}$$

We conclude via the definition of a limit that

$$
\begin{aligned}
f(\mathbf{x}+\mathbf{h}) &= \lim_{k\to\infty}\left(f(\mathbf{x}) + \cdots + \frac{1}{(k-1)!}\sum_{1\le i_1,\ldots,i_{k-1}\le n} h_{i_1}\cdots h_{i_k}\frac{\partial^{k-1}\partial}{\partial x_{i_1}\ldots x_{i_{k-1}}}(\mathbf{x})\right) \\
&= f(\mathbf{x}) + \frac{1}{1!}\sum_{i=1}^{n} h_i\frac{\partial f}{\partial x_i}(\mathbf{x}) + \frac{1}{2!}\sum_{1\le i,j\le n} h_ih_j\frac{\partial^2 f}{\partial x_i\partial x_j} + \cdots,
\end{aligned}
$$

as required.

Part (d): Observe that all higher partial derivatives of e^{x+y} evaluate to 1 at $(x_0, y_0) = (0,0)$. Then via our work in Part (c), the taylor series at $\mathbf{h} = (h_1, h_2)$ is

$$
\begin{aligned}
e^{h_1+h_2} &= 1 + \frac{1}{1!}\sum_{i=1}^{2} h_i + \frac{1}{2!}\sum_{1\le i,j\le 2} h_ih_j + \frac{1}{3!}\sum_{1\le i,j,k\le 2} h_ih_jh_k + \cdots \\
&= 1 + \frac{1}{1!}\sum_{i=1}^{2} h_i + \frac{1}{2!}\left(\sum_{i=1}^{2} h_i\right)^2 + \frac{1}{3!}\left(\sum_{i=1}^{2} h_i\right)^2 + \cdots \\
&= \boxed{\sum_{i=0}^{\infty}\frac{(h_1+h_2)^i}{i!}}.
\end{aligned}
$$

2 Section 3.3

2.1 Problem 1

The partial derivatives of f are as follows:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= 2x + y \\
\frac{\partial f}{\partial y} &= x - 2y \\
\frac{\partial^2 f}{\partial x^2} &= 2 \\
\frac{\partial^2 f}{\partial x\partial y} &= 1 \\
\frac{\partial^2 f}{\partial y^2} &= -2.
\end{aligned}
$$

The critical points (x_0, y_0) of f satisfy $2x_0 + y_0 = x_0 - 2y_0 = 0$ — the only solution of which is $(0, 0)$. At this point,

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -4 - 1 = -5 < 0,$$

so $(0, 0)$ is a saddle point of f. The only critical point of f is $\boxed{(0,0),\ \text{a saddle point of } f}$.

2.2 Problem 4

The partial derivatives of f are as follows:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x + 3y \\
\frac{\partial f}{\partial y} &= 2y + 3x \\
\frac{\partial^2 f}{\partial x^2} &= 2 \\
\frac{\partial^2 f}{\partial x \partial y} &= 3 \\
\frac{\partial^2 f}{\partial y^2} &= 2.
\end{aligned}$$

The critical points (x_0, y_0) of f satisfy $2x_0 + 3y_0 = 2y_0 + 3x_0 = 0$ — the only solution of which is $(0, 0)$. At this point,

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4 - 9 = -5 < 0$$

so $(0, 0)$ is a saddle point of f. The only critical point of f is $\boxed{(0,0),\ \text{a saddle point of } f}$.

2.3 Problem 6

The partial derivatives of f are as follows:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x - 3y + 5\\ \frac{\partial f}{\partial y} &= 12y - 3x - 2\\ \frac{\partial^2 f}{\partial x^2} &= 2\\ \frac{\partial^2 f}{\partial x \partial y} &= -3\\ \frac{\partial^2 f}{\partial y^2} &= 12.\end{aligned}$$

The only point (x_0, y_0) that satisfies $2x_0 - 3y_0 = -5$ and $-3x_0 + 12y_0 - 2 = 0$ is clearly $(-\frac{18}{5}, -\frac{11}{15})$. At this point,

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 24 - 9 = 15 > 0$$

and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$, so this point is a local minimum of f. The only critical point of f is thus $\boxed{(-\frac{18}{5}, -\frac{11}{15}), \text{ a local minimum of } f.}$

2.4 Problem 9

Observe that $\boxed{(0,0) \text{ is a local maximum}}$, as for all $(x, y) \in \mathbb{R}^2$,

$$\cos\big(x^2 + y^2\big) \leq 1 = \cos\big(0^2 + 0^2\big).$$

Similarly, $\boxed{(0, \sqrt{\pi}) \text{ and } \left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right) \text{ are local minima}}$, as for all $(x, y) \in \mathbb{R}^2$,

$$cos(x^2 + y^2) \geq -1 = \cos\left(\sqrt{\frac{\pi}{2}}^2 + \sqrt{\frac{\pi}{2}}^2\right) = \cos\Big(0^2 + \sqrt{\pi}^2\Big).$$

2.5 Problem 17

The partial derivatives of f are as follows:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 24x - 24y\\ \frac{\partial y}{\partial y} &= 24y^2 - 24x\\ \frac{\partial^2 f}{\partial x^2} &= 24\\ \frac{\partial^2 f}{\partial x \partial y} &= -24\\ \frac{\partial^2 f}{\partial y^2} &= 48y\end{aligned}$$

All critical points of f thus satisfy $24x - 24y = 0 = 24y^2 - 24x$, of which the only solutions are clearly $(0,0)$ and $(1,1)$. For $(0,0)$, see that

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0 - (-24)^2 < 0,$$

so $(0,0)$ is a saddle point. As for $(1,1)$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 24(48) - (-24)^2 > 0$$

and $\frac{\partial^2 f}{\partial x^2} > 0$, so the only local extremum of f is $\boxed{(1,1), \text{ a local minimum}}$.

2.6 Problem 27

It is $\boxed{\text{indeterminate}}$; without knowing the determinant of H, we cannot know whether $\mathbf{x}_0$ is degenerate or a saddle point of f.

2.7 Problem 28

Observe that all points on the plane $2x - y + 2z = 20$ are of the form

$$\left(x, y, 10 - x + \tfrac{1}{2}y\right)$$

for all $x, y \in \mathbb{R}$. The distance d of any point to the origin is thus a function of x and y: namely,

$$d(x,y) = \sqrt{x^2 + y^2 + \left(10 - x + \tfrac{1}{2}y\right)^2} = \sqrt{2x^2 + \tfrac{5}{4}y^2 - 20x - xy + 10y + 100}$$

for all $x, y \in \mathbb{R}$. As this function is nonnegative, it suffices to find the minimum of the square $d(x,y)$. The partial derivatives of the square of d are as follows:

$$\begin{aligned}
\frac{\partial d^2}{\partial x} &= 4x - y - 20 \\
\frac{\partial d^2}{\partial y} &= \tfrac{5}{2}y - x + 10 \\
\frac{\partial^2 d^2}{\partial x^2} &= 4 \\
\frac{\partial^2 d^2}{\partial x \partial y} &= -1 \\
\frac{\partial^2 d^2}{\partial y^2} &= \tfrac{5}{2}.
\end{aligned}$$

A critical point (x_0, y_0) of d^2 satisfies $4x_0 - y_0 - 20 = \frac{5}{2}y_0 - x_0 + 10 = 0$ — the only solution of which is clearly $\left(\frac{40}{9}, -\frac{20}{9}\right)$. At this point,

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 4\left(\frac{5}{2}\right) - (-1)^2 > 0$$

and $\frac{\partial^2 d^2}{\partial x^2} > 0$, so this point is a global minimum of d^2. As d^2 is a continuous function that has no further critical points, we conclude that this point is a *global* minimum of d^2 — and thus, a global minimum of d. We conclude that the point on the plane $2x - y + 2z + 20$ nearest the origin is

$$\left(\frac{40}{9}, -\frac{20}{9}, 10 - \frac{40}{9} - \frac{10}{9}\right) = \boxed{\left(\frac{40}{9}, -\frac{20}{9}, \frac{40}{9}\right)}.$$

2.8 Problem 30

A rectangular parallelepiped with side lengths x, y, and z has surface area $2x + 2y + 2z$ and volume xyz. We wish to prove that across all triples $a, b, c \in \mathbb{R}_{>0}$ such that $2ab + 2bc + 2ca$ equals a fixed real S, the maximum of abc is attained when $a = b = c$.

Consider all $a, b, c > 0$ such that $2a + 2b + 2c = S$; from the AM-GM Inequality, we have that

$$2\sqrt[3]{a^2b^2c^2} \leq \frac{2ab + 2bc + 2ca}{3} = \frac{S}{3},$$

with equality if and only if $a = b = c$. This rearranges to

$$abc \leq \sqrt{\left(\frac{S}{6}\right)^3}.$$

which expresses the maximum of abc as attained biconditionally when $a = b = c$. We conclude that a rectangular parallelepiped with fixed surface area has maximum volume when its side lengths are equal.

2.9 Problem 52

Note: I used Lagrange Multipliers, which we learned in class the day this assignment was due on the 17th.

Let the two unlabeled sides of the pentagon — or equivalently, the two legs of the isosceles triangle — by z. The triangle inequality necessitates that $2z > y$. We thus seek to minimize the area A of the pentagon:

$$A(x, y, z) = xy + \frac{y}{2}\sqrt{z^2 - \left(\tfrac{y}{2}\right)^2},$$

under the restriction that the perimeter P is fixed at a real number p:

$$P(x, y, z) = 2x + y + 2z = p.$$

We seek to use Lagrange Mulitpliers. We are given that A attains a maximum; also, for all (x, y, z) such that $P(x, y, z) = p$, we have that $\nabla P(x, y, z) = (2, 1, 2) \neq \mathbf{0}$. Then for all local minima and maxima of A, there exists a real number λ such that

$$\begin{bmatrix} y \\ x + \frac{1}{2}\sqrt{z^2 - \left(\frac{y}{2}\right)^2} - \frac{y^2}{8\sqrt{z^2 - \left(\frac{y}{2}\right)^2}} \\ \frac{yz}{2\sqrt{z^2 - \left(\frac{y}{2}\right)^2}} \end{bmatrix} = \nabla f(x, y, z) = \lambda \nabla g(x, y, z) = \begin{bmatrix} 2\lambda \\ \lambda \\ 2\lambda \end{bmatrix}.$$

We thus find that

$$y = 2\lambda = \frac{yz}{2\sqrt{z^2 - \left(\frac{y}{2}\right)^2}},$$

so $\sqrt{z^2 - \left(\frac{y}{2}\right)^2} = \frac{z}{2}$. Thus,

$$\sqrt{z^2 - \lambda^2} = \tfrac{z}{2} \implies \frac{3z^2}{4} = \lambda^2 \implies z = \frac{2\lambda\sqrt{3}}{3}.$$

Finally, we solve for x:

$$\lambda = x + \frac{1}{2}\sqrt{z^2 - \left(\tfrac{y}{2}\right)^2} - \frac{y^2}{8\sqrt{z^2 - \left(\tfrac{y}{2}\right)^2}} = x + \frac{z}{4} - \frac{y^2}{4z} = x + \frac{\lambda\sqrt{3}}{6} - \frac{\lambda\sqrt{3}}{2} = x - \frac{\lambda\sqrt{3}}{3}$$

so $x = \lambda\left(\frac{3+\sqrt{3}}{3}\right)$. Therefore,

$$p = 2x + y + 2z = 2\lambda\left(\frac{3+\sqrt{3}}{3}\right) + 2\lambda + \frac{4\lambda\sqrt{3}}{3} = \lambda(4 + 2\sqrt{3}),$$

so $\lambda = p\left(\frac{2-\sqrt{3}}{2}\right)$. As we are given that A attains a maximum, it must be achieved at this λ-value, where x, y, and z are

$$\begin{aligned}
\boxed{x = p\left(\frac{3-\sqrt{3}}{6}\right)} &= p\left(\frac{2-\sqrt{3}}{2}\right)\left(\frac{3+\sqrt{3}}{3}\right) = \lambda\left(\frac{3+\sqrt{3}}{3}\right), \\
\boxed{y = p(2-\sqrt{3})} &= 2p\left(\frac{2-\sqrt{3}}{2}\right) = 2\lambda, \\
\boxed{z = p\left(\frac{2\sqrt{3}-3}{3}\right)} &= \frac{2p\sqrt{3}}{3}\left(\frac{2-\sqrt{3}}{2}\right) = \frac{2\lambda\sqrt{3}}{3}.
\end{aligned}$$

It is trivial to verify that $2z > y$. We conclude that the area we seek is

$$\begin{aligned}
xy + \frac{y}{2}\sqrt{z^2 - \left(\tfrac{y}{2}\right)^2} &= xy + \frac{yz}{4} \\
&= p^2\left(\frac{3-\sqrt{3}}{6}\right)\left(2-\sqrt{3}\right) + \frac{p^2}{4}\left(2-\sqrt{3}\right)\left(\frac{2\sqrt{3}-3}{3}\right) \\
&= p^2\left(\frac{9-5\sqrt{3}}{6} + \frac{7\sqrt{3}-12}{12}\right) \\
&= p^2\left(\frac{6-3\sqrt{3}}{12}\right) \\
&= \boxed{p^2\left(\frac{2-\sqrt{3}}{4}\right)}.
\end{aligned}$$