MATH-UA 140: Assignment 10

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Contents

1 Problem 1

1.1 Part (a)

We have that

$$A^\top A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Therefore, all eigenvalues λ of $A^\top A$ satisfy

$$0 = \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda - 4).$$

Hence, $\lambda_1 = 0$ and $\lambda_2 = 4$. It is trivial to verify that

$$\boxed{\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}}$$

have norm 1 and posess these eigenvalues respectively.

1.2 Part (b)

Realize that

$$\begin{aligned} \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} &= \begin{bmatrix} -2\sqrt{2} & 0 \\ 2\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\ &= A^\top A. \end{aligned}$$

Furtheremore, see that

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

so

$$\boxed{U\Lambda U^\top = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}},$$

where $UU^\top = I$ and the entires along the diagonal of Λ are the eigenvalues of $A^\top A$.

1.3 Part (c)

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvalues discussed in Part (a), we have that $(A^\top A)\mathbf{v}_1 = \mathbf{0}$ and $(A^\top A)\mathbf{v}_2 = 4\mathbf{v}_2$. Thus,

$$(AA^\top)(A\mathbf{v}_1) = A(A^\top A\mathbf{v}_1) = A(\mathbf{0}) = 0(A\mathbf{v}_1)$$
$$(AA^\top)(A\mathbf{v}_2) = A(A^\top A\mathbf{v}_2) = A(4\mathbf{v}_2) = 4(A\mathbf{v}_2).$$

1.4 Part (d)

The catch is that $A\mathbf{v}_1$ is the zero vector! It cannot be an eigenvector of $AA^\top$ and cannot be normalized.

1.5 Part (e)

Recalling that $\lambda_1 = 0$ and $\lambda_2 = 4$, we have that

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \mathbf{0} = \sqrt{0} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \sqrt{\lambda_1}\mathbf{v}_2.$$
$$A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \sqrt{4} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \sqrt{\lambda_2}\mathbf{v}_1.$$

Therefore, two such $\mathbf{u}_1$ and $\mathbf{u}_2$ are $\boxed{\mathbf{v}_2 \text{ and } \mathbf{v}_1 \text{ respectively}}$.

1.6 Part (f)

Realize that

$$\begin{aligned} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} &= \begin{bmatrix} 0 & \sqrt{2} \\ 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ &= A. \end{aligned}$$

It is trivial to verify that the first and third matricies above are orthonormal. Therefore,

$$\boxed{U\Sigma V^\top = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}$$