Artin: Fields

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1 Fields

A **field** is a commutative division ring. If $F \subseteq K$ are a pair of fields, we say K is a **field extension** of F. This relation is denoted $K\,/\,F$; this is *not* a quotient! Examples of fields are as follows:

1. Subfields of $\mathbb{C}$ are called **number fields**. Any subfield of $\mathbb{C}$ contains the field $\mathbb{Q}$ of rational numbers. The most important number systems are **algebraic number fields**, whose elements are algebraic numbers.
2. A **finite field** is a field that contains finitely many elements. Finite fields are gorgeous and colorful objects that obey beautiful, tight-knit properties.
3. Extensions of the field $C(t)$ of rational functions are called **function fields**.

2 Algebraic and Transcendental Elements

Let $K\,/\,F$ be a field extension and let α be an element of K. The element α is **algebraic over F** if is the root of a monic polynomial with coefficients in F — say, $f(\alpha) = 0$ for

$$f(x) \;=\; x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \text{for some} \quad a_{n-1}, \ldots, a_0 \in F,$$

An element is **transcendental over F** if it is not algebraic. Both of these properties depend on the field F. Every element $\alpha \in F$ is algebraic over F due to the monomial $x - \alpha$. We can elegantly describe this as a substitution homomorphism

$$\phi : F[x] \to X \quad \text{defined by} \quad x \rightsquigarrow \alpha.$$

An element ϕ is transcendental if ϕ is injective and algebraic otherwise.

Proposition 1. *Let $\alpha \in K\,/\,F$ be an element of a field extension. The following conditions on a monic polynomial $f \in F[x]$ are equivalent:*

1. *f is the unique monic polynomial of lowest degree in $F[x]$ with α as a root.*
2. *f is an irreducible element of $F[x]$ with α as a root.*
3. *$f(\alpha) = 0$ and (f) is a maximal ideal.*
4. *If $g(\alpha) = 0$, then $f \mid g$.*

Proof. Since $F[x]$ is a Euclidean domain, the kernel of $\phi : F[x] \to K$ is a principal ideal generated by some polynomial f of smallest degree. f must be irreducible, or else a polynomial of smaller degree has a root at ϕ; the other properties are easy to deduce. $\square$

This polynomial is called the **minimal polynomial** of α. Like before, the minimal polynomial depends on both F and α. The degree of the minimal polynomial of α is called the **degree** of α. There are two distinct conversations at this point:

1. The field $F(\alpha_1, \ldots, \alpha_n)$ denotes the subfield of K generated by $\alpha_1, \ldots, \alpha_n$.

   $F(\alpha_1, \ldots, \alpha_n)$ is the smallest subfield of K that contains F and $\alpha_1, \ldots, \alpha_n$.

2. The ring $F[\alpha_1, \ldots, \alpha_n]$ denotes the subring of K generated by $\alpha_1, \ldots, \alpha_n$. The ring $F[\alpha]$ is isomorphic to the image of the substitution homomorphism $\phi : F[x] \to K$ as defined above.

The field $F(\alpha)$ is isomorphic to the field of fractions of $F[\alpha]$. If α is transcendental, then $F[\alpha] \cong F[x]$ and $F(\alpha) \cong F(\alpha)$; otherwise,

Proposition 2. *Let $\alpha \in K \,/\, F$ be an element of a field extension which is algebraic over F. Let f be the minimal polynomial of α.*

1. *The canonical map $\phi : F[x] \,/\, (f) \to F[\alpha]$ an isomorphism.*
2. *$F[\alpha]$ is a field, hence $F[\alpha] = F(\alpha)$.*
3. *More generally, $F[\alpha_1, \ldots, \alpha_n] = F(\alpha_1, \ldots, \alpha_n)$ if $\alpha_1, \ldots, \alpha_n \in K \,/\, F$ are algebraic.*

Proof. Let $\phi : F[x] \to K$ be the aforementioned substitution homomorphism. Then $F[x] \,/\, \operatorname{Ker} \phi \cong K$. By Proposition 1, the kernel of ϕ is a maximal ideal generated by the minimal polynomial f, which yields (1) and (2). As per (3), an induction argument proceeds something like

$$F[\alpha_1, \ldots, \alpha_n] = F[\alpha_1, \ldots, \alpha_{n-1}][\alpha_n] = F(\alpha_1, \ldots, \alpha_n)[a_k] = F(\alpha_1, \ldots, \alpha_n).$$

The omitted details are relatively easy to verify. □

The following proposition is a special case of one I omitted from Chapter 11.

Proposition 3. *Let $\alpha \in K \,/\, F$ be an algebraic element of a field extension. If $\deg \alpha = n$, then $\alpha_1, \ldots, \alpha_n$ is a basis for $F(\alpha)$ as a vector space over F.*

A fundamental question is: given two elements α and β — or given their minimal polynomials — when can one determine whether α and β generate equal fields? Proposition three provides a necessary non-sufficient condition: that $\deg \alpha = \deg \beta$. The following proposition answers a special case.