MATH-UA 148: Homework 1

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Contents

1 1B Problems

1.1 Problem 1B.2

If $a\mathbf{v} = 0$ and $a \neq 0$, then $\frac{1}{a}$ exists; we deduce that

$$\mathbf{v} = 1\mathbf{v} = \left(\tfrac{1}{a} \cdot a\right)\mathbf{v} = \tfrac{1}{a}(a\mathbf{v}) = \tfrac{1}{a}(\mathbf{0}) = \mathbf{0},$$

as desired.

1.2 Problem 1B.5

Denote a *quasi-space* as a set V over a field F that satisfies all the vector axioms — except *potentially* the existence of additive inverses — and where $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in V$. If V is a quasi-space, $-1 \in \mathbb{F}$, so $(-1)\mathbf{v} \in V$; we have that

$$(-1)\mathbf{v} + \mathbf{v} = (-1)\mathbf{v} + 1\mathbf{v} = (-1+1)\mathbf{v} = 0\mathbf{v} = \mathbf{0}.$$

All elements of V thus have an additive inverse, which implies that V is a vector space. Conversely, let V be a vector space. For all $\mathbf{v} \in V$, we have that $0\mathbf{v} = (0+0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$. Adding $-0\mathbf{v}$ to both sides yields $\mathbf{0} = 0\mathbf{v}$, so V is a quasi-space.

We conclude that quasi-spaces and vector spaces are equivalent structures.

1.3 Problem 1B.6

The set $R \cup \infty \cup -\infty$ is not a vector space, as addition of infinities is not associative:

$$\begin{gathered} \infty + (-\infty + (-\infty)) = \infty + (-\infty) = 0 \\ (\infty + (-\infty)) + (-\infty) = 0 + (-\infty) = -\infty. \end{gathered}$$

2 1C Problems

2.1 Problem 1C.4

Let the set of all continuous real-valued functions with integral b from 0 to 1 be X_b.

For X_b to be a vector space, we must have that $f(x)+g(x) \in X_b$ — or equivalently, that the integral of $f(x)+g(x)$ is b — for all $f(x), g(x) \in X_b$. However, the integral of $f(x)+g(x)$ from 0 to 1 is $2b$ — so $2b$ must equal b, and $b = 0$. We deduce that only X_0 can be a vector space.

We now prove that X_0 is a subspace of $\mathbb{R}^{[0,1]}$:

- **Additive Closure**: If $f, g \in X_0$, then $f + g$ is continuous, real-valued, and the integral of $f + g$ is $0 + 0 = 0$, so $f + g \in X_0$.
- **Multiplicative Closure**: If $a \in \mathbb{R}$ and $f \in X_0$, then af is continuous, real-valued, and its integral from 0 to 1 is $a0 = 0$, so $af \in X_0$.
- **Additive Identity**: The function $h = 0$ is trivially in X_0, and for all $f \in X_0$, we have that $f + 0 = 0 + f = f$.

Therefore, X_0 is a subspace of $\mathbb{R}^{[0,1]}$. We deduce that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

2.2 Problem 1C.6

(a) **Yes**: Let $S = \{(a, b, c) \in \mathbb{R}^3 \mid a^3 = b^3\}$. If $a^3 = b^3$ for $a, b \in \mathbb{R}$, we have that $a = b$. Thus, $S = \{(a, a, b) \in \mathbb{R}^3\}$. Now, we verify the conditions that S is a subspace of $\mathbb{R}^3$:

- **Additive Closure**: If (a, a, b) and $(c, c, d) \in S$, then their sum is $(a+c, a+c, b+d)$; as the first and second coordinates of this vector are the same, it belongs to S.
- **Multiplicative Closure**: If $(a, a, b) \in S$ and $x \in \mathbb{R}$, then their product is (xa, xa, xb); as the first and second coordinates of this vector are the same, it belongs to S.
- **Additive Identity**: Clearly $(0, 0, 0) \in S$ such that $\mathbf{v} + (0, 0, 0) = (0, 0, 0) + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in S$.

Hence, S is a subspace of $\mathbb{R}^3$.

(b) **No**: Let $S = \{(a, b, c) \in \mathbb{C}^3 \mid a^3 = b^3\}$ and let ω be a nontrivial cubic root of unity as follows:

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \qquad \text{and} \qquad \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

Clearly $\omega^3 = 1$ and $\omega - \omega^2 = \sqrt{3}i$. See that $(-1, -1, 0)$ and $(\omega^2, \omega, 0)$ are elements of S. However, their sum $(\omega^2 - 1, \omega - 1, 0)$ fails to meet the conditions to belong in S, as the cubes of its first and second coordinates are not equal:

$$\begin{aligned}(\omega - 1)^3 &= \omega^3 - 3\omega^2 + 3\omega - 1 = 3\omega - 3\omega^2 = 3\sqrt{3}i \\ (\omega^2 - 1)^3 &= \omega^6 - 3\omega^4 + 3\omega^2 - 1 = 3\omega^2 - 3\omega = -3\sqrt{3}i.\end{aligned}$$

As S is not closed under addition, it cannot be a subspace of $\mathbb{C}^3$.

2.3 Problem 1C.12

Let U and W be subspaces of V. We procceed by the contrapositive — namely, we prove $U \cup W$ is *not* a subspace if and only if one of U and W does *not* contain the other.

Suppose $U \cup W$ is not a subspace. Consider which conditions to be a subspace have not been met: clearly $\mathbf{0} \in U \cup W$, and if $a \in \mathbb{F}$, then $\mathbf{v} \in U \cup W$ implies that $\mathbf{v}$ lies in at least one of the two subspaces. Thus $a\mathbf{v}$ lies in the same subspace, and $a\mathbf{v} \in U \cup W$.

We conclude that the violated condition is additive closure — namely, there exists $\mathbf{u}, \mathbf{v} \in U \cup W$ such that $\mathbf{u} + \mathbf{w} \notin U \cup W$. Note that $\mathbf{u}$ and $\mathbf{w}$ cannot both lie in U, as their sum would be in U; similarly, $\mathbf{u}$ and $\mathbf{w}$ cannot lie both in W. Then one of the two vectors lies *only* in U and the other lies *only* in W — which implies that one of U and W cannot contain the other.

Conversely, suppose one of U and W does not contain the other. Then there exists $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{u} \notin W$, and $\mathbf{w} \notin U$. Thus the sum $\mathbf{u} + \mathbf{w}$ cannot lie in either U or W, so $\mathbf{u} + \mathbf{w} \notin U \cup W$. We deduce that $U \cup W$ is not closed under vector addition, and cannot be a subspace.

Taking the contrapositive yields the desired result.

2.4 Problem 1C.20

Define the subspace $W \subseteq \mathbb{F}^4$ as follows:

$$W = \{(0, x, 0, y) \in \mathbb{F}^4 \mid x, y \in \mathbb{F}\}.$$

Observe that $(0, 0, 0, 0) \in W$, and W is closed by addition and scalar multiplication, so it is a subspace of $\mathbb{F}^4$. Clearly, U is a subspace of $\mathbb{F}^4$ as well.

Claim. $\mathbb{F}^4 = U + W$

Proof. Let (a, b, c, d) be any element of $\mathbb{F}^4$. Then $(a, a, c, c) \in U$ and $(0, b-a, 0, d-c) \in W$ such that

$$(a, b, c, d) = (a, a, c, c) + (0, b - a, 0, d - c) \in U + W.$$

Therefore, $\mathbb{F}^4 = U + W$

Trivially, $U \cap W = (0, 0, 0, 0) = \{\mathbf{0}\}$. Therefore, $U + W$ is a direct sum and $U \oplus W = \mathbb{F}^4$.

2.5 Problem 1C.23

The given result is **not true**. Consider the vector space $\mathbb{R}^2$, and define the following subspaces:

$$\begin{aligned} W &= \{(a, 0) \in \mathbb{R}^2 \mid a \in \mathbb{R}\}, \\ U_1 &= \{(0, b) \in \mathbb{R}^2 \mid b \in \mathbb{R}\}, \\ U_2 &= \{(b, b) \in \mathbb{R}^2 \mid b \in \mathbb{R}\}. \end{aligned}$$

It is trivial to verify that all three sets are subspaces of $\mathbb{R}^2$ and that $U_1 \neq U_2$. We will now prove that $\mathbb{R}^2 = W \oplus U_1 = W \oplus U_2$.

Let (x, y) be any vector in $\mathbb{R}^2$. We have that $(x, 0) \in W$ and $(0, y) \in U_1$ such that

$$(x, y) = (x, 0) + (0, y) \in W + U.$$

Thus, $W + U_1 = \mathbb{R}^2$. Comparatively, note that $(x - y, 0) \in W$ and $(y, y) \in U_2$ such that

$$(x, y) = (x - y, 0) + (y, y) \in W + U_2.$$

Thus, $W + U_2 = \mathbb{R}^2$. Clearly $W \cap U_1 = W \cap U_2 = \{(0, 0)\} = \{\mathbf{0}\}$, as if $(a, 0) = (0, b)$, then $a = 0$ and $b = 0$, and if $(a, 0) = (b, b)$, then $a = b = 0$. Therefore,

$$\mathbb{R}^2 = W \oplus U_1 = W \oplus U_2 \qquad \text{and} \qquad U_1 \neq U_2.$$

We conclude that the given result is false.