MATH-UA 329: Homework 2

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1 Problem 1

1.1 Part (a)

Proof. Define $x_1, x_2 \in I$ such that $|x_1 - x_2| \leq t_1 + t_2$ and $|f(x_1) - f(x_2)| = \omega_f(t_1 + t_2)$. We will demonstrate that $|f(x_1) - f(x_2)| \leq \omega_f(t_1) + \omega_f(t_2)$.

Let $z = x_1\left(\frac{t_2}{t_1+t_2}\right) + x_2\left(\frac{t_1}{t_1+t_2}\right)$. It is clear that z lies between x_1 and x_2, so it is an element of I. Then

$$|x_1 - z| \ = \ \left|-x_1\left(\frac{t_1}{t_1+t_2}\right) - x_2\left(\frac{t_1}{t_1+t_2}\right)\right| \ = \ \frac{t_1}{t_1+t_2}|x_1 - x_2| \ = \ t_1.$$

Similarly, we have that

$$|x_2 - z| \ = \ \left|x_1\left(\frac{t_2}{t_1+t_2}\right) + x_2\left(\frac{t_2}{t_1+t_2}\right)\right| \ = \ \frac{t_2}{t_1+t_2}|x_1 - x_2| \ = \ t_2.$$

This enables us to use the moduli of continuity, $\omega_f(t_1)$ and $\omega_f(t_2)$:

$$\begin{aligned}
\omega_f(t_1 + t_2) &= |f(x_1) - f(x)_2| \\
&\leq |f(x_1) - f(z)| \ + \ |f(x_2) - z| \\
&\leq \sup_{|y,z|\leq t_1} |f(y) - f(z)| + \sup_{|y,z|\leq t_2} |f(y) - f(z)| \\
&= \omega_f(t_1) + \omega_f(t_2).
\end{aligned}$$

□

1.2 Part (b)

Proof. The result from Part (a) ensures that $\omega_f(t_1) \leq \omega_f)(t_2) + \omega_f(t_2 - t_1)$; thus we have that $\omega_f(t_1) - \omega_f(t_2) \leq \omega_f(t_1 - t_2)$. Hence we have that

$$\begin{aligned}
\omega_{\omega_f}(t) &= \sup_{|x_1 - x_2|\leq t} |\omega_f(x_1) - \omega_f(x_2)| \\
&\leq \sup_{|x_1 - x_2|\leq t} |\omega_f(x_1 - x_2)|.
\end{aligned}$$

Since $\omega_f(t)$ is a strictly increasing function, the right-hand side is equal to $\omega_f(t)$. This concludes the proof. □

1.3 Part (c)

Proof. It is clear that from the result of Part (a) that for all $t \geq 0$ and integers $n > 0$, we have

$$\omega_f(nt) = \omega_f\left(\sum_{i=1}^{n} t\right) \leq \sum_{i=1}^{n} \omega_f(t) = n\omega_f(t).$$

If $I = (a, b)$, let $|I| = b - a$. It is trivial that $t \geq |I|$ implies $\omega_f(|I|)$; thus we need only concern ourselves with $t \leq |I|$. We have two cases:

Case 1: If $|I| \leq t$, we have that

$$\omega_f(t) \geq \left(\frac{t}{|I|}\right)\omega_f(t) \geq t\left(\frac{\omega_f|I|}{|I|}\right) \geq t\left(\frac{\omega_f(|I|\,/\,2)}{|I|}\right).$$

Case 2: If $|I| > t$, there exists an integer n between $|I|\,/\,(2t)$ and $|I|\,/\,t$; thus

$$\omega_f(t) \;\geq\; \frac{n}{|I|\,/\,t}\,\omega_f(t) \;\geq\; \frac{t}{|I|}\omega_f(nt) \geq t\left(\frac{\omega_f(|I|\,/\,2)}{|I|}\right).$$

Combining these cases, we attain the theorem if we set $c = \frac{\omega_f(|I|)\,/\,2}{|I|}$. $\square$

2 Problem 2

Proof. We will use algebra, as unenlightening as this may be. We have that

$$\begin{aligned}
\sum_{k=0}^{n}(nx-k)^2 P_{n,k}(x) \;&=\; n^2x^2\sum_{k=0}^{n} P_{n,k}(x) \;-\; 2nx\sum_{k=0}^{n} kP_{n,k}(x) \;+\; \sum_{k=0}^{n} k^2 P_{n,k}(x) \\
&=\; n^2x^2 \;-\; 2nx\sum_{k=0}^{n} kP_{n,k}(x) \;+\; \sum_{k=0}^{n} k^2 P_{n,k}(x)
\end{aligned}$$

Our task is to simply these summations. We have

$$\begin{aligned}
\sum_{k=0}^{n} kP_{n,k}(x) \;&=\; \sum_{k=0}^{n} k\left(\frac{n!}{k!(n-k)!}\right)x^k(1-x)^{n-k} \\
&=\; nx\sum_{k=1}^{n}\binom{n-1}{k-1}x^{k-1}(1-x)^{(n-1)-(k-1)} \\
&=\; nx\sum_{k=0}^{n-1}\binom{n-1}{k}x^{k}(1-x)^{(n-1)-k} \\
&=\; nx\big(x+(1-x)\big)^{n-1} \\
&=\; nx,
\end{aligned}$$

For the summation k^2, we find it is easier to work with $k(k-1)$ due to the factorial:

$$\begin{aligned}
\sum_{k=0}^{n} k^2 P_{n,k}(x) \;&=\; \sum_{k=0}^{n} k P_{n,k}(x) \;+\; \sum_{k=0}^{n} k(k-1) P_{n,k}(x) \\
&=\; nx \;+\; \sum_{k=0}^{n} k(k-1) P_{n,k}(x) \\
&=\; nx \;+\; \sum_{k=0}^{n} k(k-1) \left(\frac{n!}{k!(n-k)!} \right) x^k (1-x)^{n-k} \\
&= nx \;+\; n(n-1)x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{(n-2)-(k-2)} \\
&= nx + n(n-1)x^2 \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{(n-2)-k} \\
&= nx + n(n-1)x^2 \big(x + (1-x)\big)^{n-2} \\
&= nx + n(n-1)x^2.
\end{aligned}$$

We are ready to return to our original series: we have that

$$\begin{aligned}
\sum_{k=0}^{n} (nx-k)^2 P_{n,k}(x) &= n^2x^2 \;-\; 2nx(nx) \;+\; \big(nx + n(n-1)x^2\big) \\
&= n^2x^2 - 2n^2x^2 + nx + n^2x^2 - nx^2 \\
&= nx - nx^2 \\
&= nx(1-x),
\end{aligned}$$

completing the proof. $\square$

2.1 Part (b)

Proof. Let $f(x) = ax + b$ be a linear function. Then

$$\begin{aligned}
B_n(f)(x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n}\left(a\left(\tfrac{k}{n}\right)+b\right)P_{n,k}(x) \\
&= \frac{a}{n}\sum_{k=0}^{n} kP_{n,k}(x) \;+\; b\sum_{k=0}^{n} P_{n,k}(x) \\
&= \frac{a}{n}(nx) + b(1) = ax + b = f(x).
\end{aligned}$$

The story for quadratics is more complex: if we let $f(x) = ax^2 + bx + c$, we attain that

$$\begin{aligned}
B_n(f)(x) &= \sum_{k=0}^{n}\left(a\left(\tfrac{k}{n}\right)^2 + b\left(\tfrac{k}{n}\right)+c\right)\binom{n}{k}x^k(1-x)^{n-k} \\
&= \frac{a}{n^2}\sum_{k=0}^{n} k^2P_{n,k}(x) \;+\; bx + c \\
&= \frac{a}{n^2}\left(nx + n(n-1)x^2\right) + bx + c \\
&= \frac{a(n-1)}{n}x^2 + \frac{a+bn}{n}x + c.
\end{aligned}$$

We are ready to bound the difference between $B_n(f)(x)$ and $f(x)$: since $x \in [0,1]$,

$$|f(x) - B_n(f)(x)| = \frac{a}{n}x - \frac{a}{n}x^2 \le \frac{a}{n}\left(\frac{1}{2}\right) - \frac{a}{n}\left(\frac{1}{2}\right)^2 = \frac{a}{4n}.$$

Setting $C = \frac{a}{4}$ completes the proof. □

3 Problem 3

3.1 Part (a)

Proof. We must perform three routine calculations: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

1. **Positivity**: Clearly $\|\mathbf{x}\|_{\mathbf{A}} = \|\mathbf{A}\mathbf{x}\| \ge 0$. The equality condition is as follows:

$$\|\mathbf{x}\|_{\mathbf{A}} = \mathbf{0} \iff \mathbf{A}\mathbf{x} = \mathbf{0} \iff \mathbf{x} \in \operatorname{null}\mathbf{A} \iff \mathbf{x} = \mathbf{0}.$$

2. **Homogeneity**: For all $\lambda \in \mathbb{C}$, we have that

$$\|\lambda \mathbf{x}\|_{\mathbf{A}} = \|\mathbf{A}(\lambda \mathbf{x})\| = \|\lambda(\mathbf{A}\mathbf{x})\| = |\lambda|\|\mathbf{A}\mathbf{x}\| = |\lambda|\|\mathbf{x}\|_{\mathbf{A}}$$

3. **Triangle Inequality**: We have that

$$\|\mathbf{x}+\mathbf{y}\|_{\mathbf{A}} = \|\mathbf{A}\mathbf{x}+\mathbf{A}\mathbf{y}\| \le \|\mathbf{A}\mathbf{x}\| + \|\mathbf{A}\mathbf{y}\| = \|\mathbf{x}\|_{\mathbf{A}} + \|\mathbf{y}\|_{\mathbf{A}}.$$

Thus, $\|\cdot\|_{\mathbf{A}}$ defines a norm on $\mathbb{R}^d$. If $\mathbf{A}$ is not invertible, this norm fails to satisfy the positivity condition — namely, we have $\|\mathbf{x}\|_{\mathbf{A}} = \mathbf{0}$ for all nonzero vectors $\mathbf{x} \in \operatorname{null} \mathbf{A}$. $\square$

3.2 Part (b)

Proof. We must perform three rather routine calculations: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

1. **Positivity**: Clearly $\|\mathbf{x}\|'_{\mathbf{A}} = \|\mathbf{x}\| + \|\mathbf{A}\mathbf{x}\| \ge 0$. The equality condition is as follows:

$$\|\mathbf{x}\|'_{\mathbf{A}} = \mathbf{0} \iff \|\mathbf{x}\| + \|\mathbf{A}\mathbf{x}\| = \mathbf{0} \iff \|\mathbf{x}\| = \mathbf{0} \iff \mathbf{x} = \mathbf{0}.$$

2. **Homogeneity**: For all $\lambda \in \mathbb{C}$, we have that

$$\|\lambda \mathbf{x}\|'_{\mathbf{A}} = \|\lambda \mathbf{x}\| + \|\mathbf{A}(\lambda \mathbf{x})\| = |\lambda|\|\mathbf{x}\| + |\lambda|\|\mathbf{A}\mathbf{x}\| = |\lambda|\|\mathbf{x}\|'_{\mathbf{A}}.$$

3. **Triangle Inequality**: We have that

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|'_{\mathbf{A}} &= \|\mathbf{x}+\mathbf{y}\| + \|\mathbf{A}\mathbf{x}+\mathbf{A}\mathbf{y}\| \\ &\le \|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{A}\mathbf{x}\| + \|\mathbf{A}\mathbf{y}\| \\ &= \|\mathbf{x}\|'_{\mathbf{A}} + \|\mathbf{y}\|'_{\mathbf{A}}. \end{aligned}$$

We deduce that $\|\cdot\|'_{\mathbf{A}}$ is a norm on $\mathbb{R}^d$. $\square$

4 Problem 4

4.1 Part (a)

Let $B_{1/2}(x_1), \ldots, B_{1/2}(x_n)$ be a collection of open balls which cover B; without loss of generality, we may assume $x_1, \ldots, x_n \in B$. Let

$$x_i = (y_{1i}, y_{2i}, \ldots),$$

and define k_i as the unique integer such that $n_{k_i i} = \max\{y_{ji} \mid j \in \mathbb{N}\}$. Any point in B must have finitely many entries equal to or greater than $\frac{1}{2}$; thus there exists an integer m such that for each x_i, the entry y_{mi} is less than one-half. Hence

$$\max\{y_{mi} \mid i \in \{1, \dots, n\}\} + \frac{1}{2} < 1.$$

Thus the point in B with m-th coordinate equal to the above real number — and all other coordinates equal to 0 — lies in B and has ℓ_∞ norm greater than one-half. It thus lies outside the open balls $B_{1/2}(x_1), \dots, B_{1/2}(x_n)$.

We conclude that no finite collection of open balls of radius $\frac{1}{2}$ can cover B.

4.2 Part (b)

Observe that the sequence (u_1^m) is bounded by 0 and 1. The Bolzano-Weierstrauss Theorem ensures it contains a convergent subsequence: denote it by (u_1^m). We may now utilize a contradiction argument:

Suppose for contradiction that no such subsequence of (u^m) exists. Then there exists a minimum integer $n > 1$ such that the sequence (u_n^m) does not contain a convergent subsequence.

By minimality, (u^m) contains a subsequence that converges pointwise for each components $1, \dots, n-1$: denote this subsequence by (u^{m_k}). Since the sequence $(u_n^{m_k})$ is bounded by 0, and 1, the Bolzano-Weierstrauss Theorem ensures that some infinite subsequence of (u^{m_k}) converges pointwise for component n — a contradiction.

We conclude that there exists a sequence (u^m) which converges pointwise for each component.

4.3 Part (c)

Set One: This set is $\boxed{\text{compact}}$. Given a sequence $(x^n) \in x$, it is easy to establish that it it contains a convergent subsequence; mirroring the argument in Part (b), we find through iterative Bolzano-Weierstrauss application that convergence of later entries is ensured by the convergence of each x^n itself (in the sense of its components) to zero.

Set Two: This set is $\boxed{\text{not compact}}$, since it contains the set

$$(e_n^m) \; = \; (\delta_{m,n}, n \in \mathbb{N}),$$

which clearly contains no convergent subsequence.

5 Problem 5

5.1 Part (a)

Proof. Utilizing the properties of the inner product, we have that

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2+\|\mathbf{v}-\mathbf{w}\|^2 &= \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}\rangle + \langle \mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}\rangle \\ &= \langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{w}\rangle + \langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle \\ &\quad + \langle \mathbf{v},\mathbf{v}\rangle - \langle \mathbf{v},\mathbf{w}\rangle - \langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle \\ &= 2\langle \mathbf{v},\mathbf{v}\rangle + 2\langle \mathbf{w},\mathbf{w}\rangle \\ &= 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2,\end{aligned}$$

as required. Geometrically, this corresponds to a famous theorem — that for any parallelogram, the sums of the squares of its sides equals the sums of the squares of its diagonals. □

5.2 Part (b)

We first tackle $\mathbb{R}^2$, examining the 2-vectors $\mathbf{x} = (2, 0)$ and $\mathbf{y} = (0, 1)$ under the p-norm $\|\cdot\|_p$: if we denote the 2-norm by $\|\cdot\|$, we have that

$$2\|\mathbf{x}\|_p + 2\|\mathbf{y}\|_p = 6 = \|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 > \|\mathbf{x}-\mathbf{y}\|_p^2 + \|\mathbf{x}-\mathbf{y}\|_p^2. \qquad (1)$$

The observation that the p-norm is strictly decreasing on the interval $(1, \infty)$ follows from the Power Mean Inequality (and noting that the components of neither $\mathbf{x}+\mathbf{y}$ nor $\mathbf{x}-\mathbf{y}$ are equal).

If we suppose for contradiction that an inner product on $\mathbb{R}^2$ induced a norm equal to the p-norm, the vectors $\mathbf{x}$ and $\mathbf{y}$ would violate the Parallelogram Equality. Hence this is not possible.

The cases $\mathbb{R}^n$ for $n > 2$ and $\ell^0(\mathbb{N})$ are corollaries of this result — if an inner product on them yielded a p-norm, the fact $\mathbb{R}^2$ is a subspace would yield an inner-product-induced p-norm on $\mathbb{R}^2$. This yields a desired contradiction.