MATH-UA 329: Lecture 1

James Pagan

January 2024

Contents

1 Exposition

MATH-UA 329 expands upon the topics of Honors Analysis I and will discuss two topics:

1. The theory of differentiation and integration of multiavariable functions.
2. Measure Theory and Lebesgue integration

The instructor is Sinan Gunturk, available at gunturk@cims.nyu.edu. Professor Gunturk's office hours are at WWH 829 in Courant from 2:00-3:30 PM. The TA is Keefer Rowan. The grade distribution is as follows:

- 40%: the final exam.
- 20%: the midterm exam.
- 10-15%: quizzes.
- 15-20%: homework assignments.

The course will not follow one particular textbook; potential textbooks are enumerated on Brightspace, with particular emphasis on Rudin's *Principles of Mathematical Analysis.*

2 L1: Metric Spaces

2.1 Definition

A **metric space** is a set X equipped with a binary mapping $d : X \times X \to \mathbb{R}$ called a **metric** such that the following properties are satisfied for all $x, y, z \in X$:

1. **Positivity**: $d(x, y) \geq 0$, with equality if and only if $x = y$.
2. **Symmetry**: $d(x, y) = d(y, x)$.
3. **Triangle Inequality**: $d(x, y) \leq d(x, z) + d(z, y)$.

Metric spaces generalize the notion of distance to arbitrary sets.

2.2 Examples

1. **Euclidean Distance**: In $\mathbb{R}$, the Euclidean distance $d(x, y) = |x - y|$ is a metric. The complex absolute value is also a metric of $\mathbb{C}$.

   In general, the Euclidean distance over $\mathbb{R}^n$ is defined as follows:

   $$d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

2. **Taxicab Metric**: in $\mathbb{R}^n$, the taxicab metric is defined as follows for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$:

   $$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

3. **Supremum Distance**: For $\mathbb{R}^n$, the d_∞ metric is as follows:

   $$d_\infty(\mathbf{x}, \mathbf{y}) = \max |x_i - y_i \mid i \in \{1, \ldots, n\}|.$$

   It is denoted by infinity since

   $$\lim_{m \to \infty} d_n(x, y) = \lim_{m \to \infty} \sqrt[m]{\sum_{i=1}^{n} |x_i - y_i|^m} = d_\infty(x, y).$$

2.3 Open Balls

For a metric space X, the **open ball** of radius r centered at $x \in X$ is the set

$$B_r(\mathbf{x}) = \{y \in X \mid d(x, y) \leq 1\}.$$

Here are examples of the unit disc $B_1(0)$ in the above metrics in $\mathbb{R}^2$.

- Under the Euclidean metric, the unit disc is the standard unit circle.
- Under d_∞, it is the unit square:

  $$B_1(0) = \{\mathbf{y} \in \mathbb{R}^2 \mid \max y_i < 1 \text{ for } i \in \{1, 2\}\}.$$

- Under d_1, the unit disc is a diamond:

  $$B_1(0) = \{\mathbf{y} \in \mathbb{R}^2 \mid |y| \leq 1\}.$$

We encourage the reader to graph these examples for further understanding.

2.4 Discrete Metric

We also must discuss the **discrete metric** over any set X, defined as follows:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}.$$

It is easy to verify that the discrete metric is a metric; it is primarily used in examples. Open balls in under the discrete metric are as follows:

$$B_r(x) = \begin{cases} \{x\} & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases}$$

3 L1: Analysis in Metric Spaces

3.1 Definition

Let X and Y be metric spaces. A function $f : X \to Y$ is **continuous** at $x \in X$ if for all $\epsilon > 0$, there exists δ such that

$$0 < d(x,y) < \delta \implies d(f(x) - f(y)) < \epsilon.$$

f itself is continuous on X if it is continuous at every $x \in X$. The next section will utilize the following definition:

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous on } X\}$$

3.2 An Excursion to Linear Algebra

$C(X)$ is a vector space over $\mathbb{R}$ under addition of functions and scalar multiplication. For a vector space V, recall the definition of an inner product space; any norm $\|\cdot\|_V : V \to \mathbb{R}$ satisfies positivity, symmetry, and the Triangle Inequality.

We deduce that every norm induces a metric on an inner product spaces for $\mathbf{v}, \mathbf{w} \in V$:

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

Define a subset B of $C(X)$ as follows:

$$BC(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and bounded on } X\}.$$

Under this space, we may define a norm:

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

We encourage that the reader perform the routine calculations that verify $\|f\|_\infty$ is indeed a norm — hence a metric over BC.

Recall the **extreme value theorem**: that if X is compact, then every bounded and continuous $f : X \to \mathbb{R}$ is odd. This is because $C(X) = BC(X)$ when X is compact.

START OF LECTURE 2: More generally: for any set E, we may define

$$B(E) = \{f : E \to \mathbb{R} \mid f \text{ is bounded on } E\}.$$

This set $B(E)$ is an inner product space under the supremum norm discussed prior:

$$\|f\|_{B(E)} = \sup_{x \in E} |f(x)|.$$

There is an equivalent way to write this: that there exists a sequence of functions $f_1, f_2, \ldots$ such that $\lim_{n\to\infty} \|f_n - f\| = \limsup_{n\to\infty} |f_n - f| = 0$. Thus, this norm induced a metric that allows $B(E)$ to be a metric space. (Also, the set $\mathbb{R}^E$ denotes $\{f : E \to \mathbb{R}\}$).

Theorem 1. *$B(E)$ is a complete metric space — hence a Banach space.*

Proof. Suppose (f_n) is a Cauchy sequence under the supremum norm: that for all $\epsilon > 0$, there exists N_ϵ such that

$$N_\epsilon \leq i, j \implies \|f_i - f_j\|_E < \epsilon.$$

Then for all $x \in E$,

$$N_\epsilon \leq i, j \implies \|f_i(x) - f_j(x)\|_E < \epsilon.$$

Then the sequence $f_1(x), f_2(x), \ldots$ is a Cauchy sequence in $\mathbb{R}$ under the supremum norm. Then let f be the function that maps x to the limit of $f_1(x), f_2(x), \ldots$. Clearly, $f \in \mathbb{R}^E$. We must demonstrate that this convergence is uniform.

Now, let $N_\epsilon \leq i, j$. Then

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|. \\ &< |f(x) - f_m(x)| + \epsilon. \end{aligned}$$

Observe that $\inf_{N_\epsilon \leq m} |f(x) - f_m(x)| = 0$ by the convergence. Therefore, we may take the infimum of both sides of the above equation:

$$\begin{aligned} |f(x) - f_n(x)| &= \inf_{N_\epsilon \leq m} |f(x) - f_n(x)| \\ &< \inf_{N_\epsilon \leq m} |f(x) - f_m(x)| + \epsilon \\ &= \epsilon. \end{aligned}$$

Thus, $N_\epsilon < i$ implies $\|f - f_n\| = \sup_{x \in E} |f(x) - f_n(x)|_E < \epsilon$. We conclude that (f_n) converges, so $B(E)$ is complete. $\square$

If we would like to prove that $BC(X)$ is continuous, we only need demonstrate that the limit of a Cauchy sequence (f_n) is continuous — which is true, since $BC(X)$ is a closed subspace of the complete metric space $B(X)$.

3.3 Uniform Continuity

Let $f : (X, d_x) \to (Y, d_y)$ map between metric spaces. Then f is **uniformly continuous** if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon.$$

This now leads us to define the following two spaces:

$$\begin{aligned} UC(X) &= \{f : X \to R \mid f \text{ is uniformly continuous on } X\}, \\ BUC(X) &= \{f : X \to R \mid f \text{ is bounded and uniformly continuous on } X\}. \end{aligned}$$

Both are subspaces of $C(X)$. However, only $BUC(X)$ is a normed vector space under the supremum norm. The exact same proof as Theorem 1 demonstrates that $BUC(X)$ is a Banach space.

Special case: When $X = K$ is compact, all continuous $f : K \to \mathbb{R}$ are bounded and uniformly continuous. For compact K, in fact

$$C(K) = BC(K) = BUC(K)$$

For non compact X, we can only write

$$C(X) \supset BC(X) \supset BUC(X).$$

3.4 Modulus of Continuity

Let $f : (X, d_x) \to (Y, d_y)$ map between metric spaces. Then the **modulus of continuity** $\omega_f : [0, \infty) \to [0, \infty]$ is defined as

$$\omega_f(t) = \sup_{d_X(x_1, x_2) \leq t} d_Y(f(x_1), f(x_2)).$$

Two simple facts are in order:

1. f is uniformly continuous if and only if $\lim_{t \to 0^+} \omega_f(t) = 0$, which itself occurs if ω_f is continuous at 0.
2. $d_Y(f(x_1), f(x_2)) \leq \omega_f d_X(x_1, x_2)$, a fact observed by setting $d_X(x_1, x_2)$ to t.

As an example, consider a Lipschitz continuous function f: namely, a function f such that $d_y(f(x_1), f(x_2) \leq C d_X(x_1, x_2)$ for some constant C. It is clear that $\omega_f(t) \leq Ct$. As an example, $f(x) = \sqrt{|x|}$ is uniformly continuous but not Lipschitz continuous.

lhe2010@nyu.edu