MATH-UA 140: Assignment 5

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1 Problem 1

Part (a): We have that

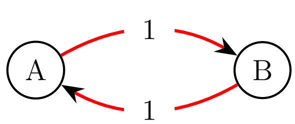
$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}.$$

Part (b): We have that

$$A^3 = A^2 A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 & 4 \\ 1 & 2 & 3 & 2 \\ 3 & 3 & 8 & 4 \\ 2 & 2 & 4 & 3 \end{bmatrix}.$$

The positivity of all entries implies that there exist paths of length 3 between any two nodes of the graph.

Part (c): Consider the graph pictured below (which I made using tikz, unfortunately):



Observe that there exists a path between any choice of nodes. However, a path connects A to B or B to A if and only if take an odd number of steps — and comparatively, a path connects A to A or B to B if and only if it takes an even number of steps.

Thus a common length cannot exist — if we suppose it does for contradiction, its parity implies it can only account for *one* of the two types of paths discussed above.

Part (d): Its corresponding adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and its powers are

$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ A^3 &= A^2 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ A^4 &= A^3 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Part (e): The transpose of the adjacency matrix of a directed graph describes the $\boxed{\text{graph created by flipping the directions of all its arrows}}$.

2 Problem 2

Part (a): The column rank of M is clearly 2, as the first and second coulums are independent, but the third and the second are not. Thus, the null space of M is the span of a single vector. One such vector is $(0, 2, -1)$, as

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2-2 \\ 6-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This vector is thus a basis of the null space of M.

Part (b): The row rank of M is clearly 2, as the first and second rows are independent, but the third and second are not. As the first and second rows constitute a list of 2 independent vectors in the row space of M, they must be a basis; these two vectors are

$$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

Part (c): We have that for all $\mu_1, \mu_2 \in \mathbb{R}$,

$$\begin{aligned}
\left(\lambda \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right) \cdot \left(\mu_1 \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right) &= \lambda\mu_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \lambda\mu_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \\
&= \lambda\mu_1(0) + \lambda\mu_2(0) \\
&= 0.
\end{aligned}$$

Thus, $\boxed{\text{the row space of } M \text{ is the orthogonal complement of the null space of } M}$ with respect to $\mathbb{R}^3$.