Artin: Linear Algebra in a Ring

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0.1 Definition

An **R-module** over a commutative ring R is an abelian group M (with operation written additively) endowed with a mapping $\mu : R \times M \to M$ (written multiplicatively) such that the following axioms are satisfied for all $x, y \in M$ and $a, b \in R$:

1. $1x = x$;
2. $(ab)x = a(bx)$;
3. $a(x + y) = ax + ay$;
4. $(a + b)x = ax + bx$.

0.2 Examples of Modules

- If R is a ring, $R[x]$ is a module.
- All ideals $\mathfrak{a}$ of R are R-modules using the same additive and multiplicative operations as R — in particular R itself is an R-module.
- If R is a field, R-modules are R-vector spaces. In fact, the axioms above are identical to the vector axioms, defined over commutative rings instead of fields.
- Abelian groups G are precisely the modules over $\mathbb{Z}$.

0.3 Homomorphisms of Modules

A map $f : M \to N$ between two R-modules M and N is an **R-module homomorphism** (or is R-linear) if for all $a \in R$ and $x, y \in M$,

$$f(x + y) = f(x) + f(y)$$
$$f(ax) = af(x).$$

Thus, an R-module homomorphism f is a homomorphism of abelian groups that commutes with the action of each $a \in R$. If R is a field, an R-module homomorphism is a linear transformation. A bijective R-homomorphism is called an R-isomorphism.

The set $\mathrm{Hom}_R(M, N)$ denotes the set of all R-module homomorphisms from M to N, and is a module if we define the following operations for $a \in R$ and $f, g \in \mathrm{Hom}_R(M, N)$:

$$(f + g)(x) = f(x) + g(x)$$
$$(rf)(x) = rf(x).$$

We denote $\mathrm{Hom}_R(M, N)$ by $\mathrm{Hom}(M, N)$ if there is no ambiguity about the commutative ring R.

1 Submodules and Quotient Modules

1.1 Definition

A **submodule** M' of M is an abelian subgroup of M closed under multiplication by elements of the commutative ring R. The following proof outlines a construction of **quotient modules**:

Theorem 1. *The abelian quotient group M/M' is an R-module under the opreation $r(x + M') = rx + M'$.*

Proof. We must perform four rather routine calculations:

1. For all $x \in M$, we have that $1(x + M') = 1x + M' = x + M'$.
2. For all $r, s \in R$ and $x \in M$, we have that $r(s(x+M')) = r(sx+M') = rsx+M' = (rs)(x + M')$.
3. For all $r, s \in R$ and $x \in M$, we have that $(r + s)(x + M') = (r + s)x + M' = (rx + sx) + M' = (rx + M') + (sx + M') = r(x + M') + s(x + M')$.
4. For all $r \in R$ and $x, y \in M$, we have that $r((x + M') + (y + M')) = r((x + y) + M') = r(x + y) + M' = (rx + M') + (ry + M') = r(x + M') + r(y + M)'$.

Therefore, M/M' is an R-module. $\square$