MATH-UA 140: Assignment 7

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1 Problem 1

We have that

$$\begin{aligned} Q\mathbf{x} \cdot Q\mathbf{y} &= (Q\mathbf{x})^\top Q\mathbf{y} \\ &= (\mathbf{x}^\top Q^\top)Q\mathbf{y} \\ &= \mathbf{x}^\top (Q^\top Q)\mathbf{y} \\ &= \mathbf{x}^\top (I)\mathbf{y} \\ &= \mathbf{x}^\top \mathbf{y} \\ &= \mathbf{x} \cdot \mathbf{y} \end{aligned}$$

2 Problem 2

Let the three vectors yielded by Gram-Schmidt ***without normalization*** be $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$, and denote the three given vectors by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ respectively.

We have that

$$\begin{aligned}
\mathbf{g}_1 &= \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \\
\mathbf{g}_2 &= \mathbf{v}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{v}_2) \\
&= \mathbf{v}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\
&= \mathbf{v}_2 + \frac{1}{5}\mathbf{v}_1 \\
&= \begin{bmatrix} -0.8 \\ 0.4 \\ 1 \end{bmatrix}, \\
\mathbf{g}_3 &= \mathbf{v}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{v}_3) - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{v}_3) \\
&= \mathbf{v}_3 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_3}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_3}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\
&= \mathbf{v}_3 - \frac{2}{5}\mathbf{v}_1 + \mathbf{v}_2 \\
&= \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix},
\end{aligned}$$

We must now normalize these vectors, which yields the following three:

$$\boxed{\begin{bmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4\sqrt{5}}{15} \\ \frac{2\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}}.$$

3 Problem 3

Part (a): We have that

$$A^\top A = \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 2 \end{bmatrix}.$$

It is now trivial to verify via that this matrix has determininat $\frac{1}{4}$, so its inverse is

$$(A^\top A)^1 \begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix}.$$

Therefore, the projection matrix is

$$\begin{aligned} A(A^\top A)^{-1} A^\top &= \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}. \end{aligned}$$

Part (b): Performing the Graham-Schmidt process, we find that the first vector is

$$\mathbf{v}_1 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

and the second vector is

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \mathrm{proj}_{\mathbf{v}_1}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The normalization of these vectors is $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 1, 0)$, so

$$\mathbf{u}\mathbf{u}^\top + \mathbf{v}\mathbf{v}^\top = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}.$$

4 Problem 4

Part (a): For four collinear points (a, b), (c, d), (e, f), (g, h) on the line $y = \alpha x + \beta$, we have that

$$\begin{bmatrix} a & 1 \\ c & 1 \\ e & 1 \\ g & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b \\ d \\ f \\ h \end{bmatrix}.$$

We can therefore compute the least-squares regression line by projecting

$$\begin{bmatrix} 3 \\ 1 \\ -2 \\ -5 \end{bmatrix} \qquad \text{onto} \qquad \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = M.$$

We can do this via computing the projection matrix $M(M^\top M)^{-1}M^\top$:

$$M^\top M = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}.$$

The inverse of this matrix is clearly

$$(M^\top M)^{-1} = \begin{bmatrix} 1/10 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Hence,

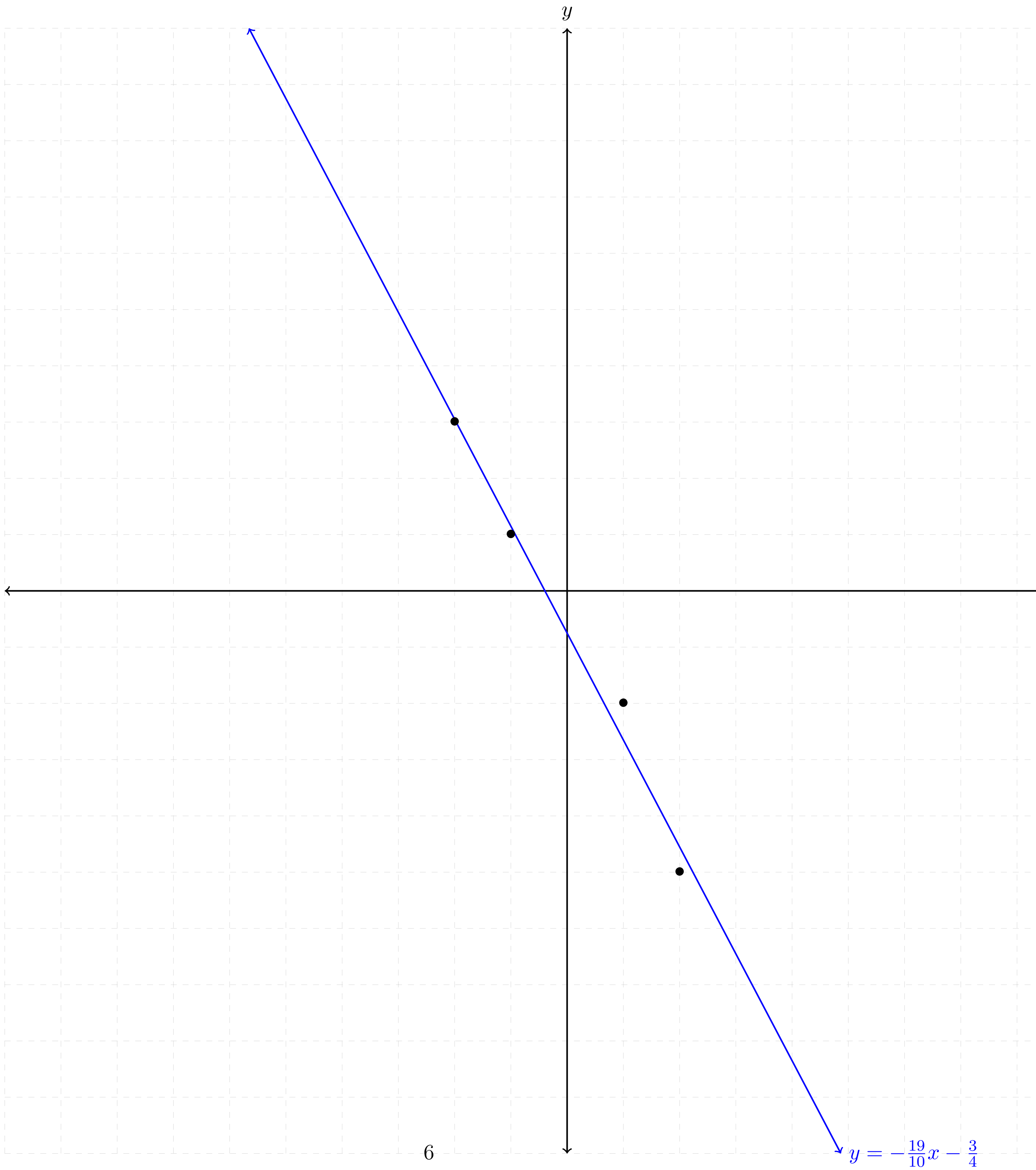
$$\begin{aligned} (M^\top M)^{-1} M^\top &= \begin{bmatrix} 1/10 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1/5 & -1/10 & 1/10 & 1/5 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{aligned}$$

Therefore, we seek

$$\begin{bmatrix} -1/5 & -1/10 & 1/10 & 1/5 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} -19/10 \\ -3/4 \end{bmatrix}$$

Therefore, our line has equation $\boxed{\frac{-19}{10}x - \frac{3}{4}}$.

Part (b): The graph is given by (pardon the oversized diagram)



y
$y = -\frac{19}{10}x - \frac{3}{4}$

5 Problem 5

We have that $P^2 = P$ and $P^\top = P$, so

$$\begin{aligned} B^\top B &= (I - 2P)^\top (I - 2P) \\ &= (I^\top - 2P^\top)(I - 2P) \\ &= (I - 2P)(I - 2P) \\ &= I - 4P + 4P^2 \\ &= I - 4P + 4P \\ &= I. \end{aligned}$$

Thus, B is an orthogonal matrix.