MATH-UA 129: Homework 4

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1 Section 2.6

1.1 Problem 2

Part (a): The gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x}\left(x + 2xy - 3y^2\right) \\ \frac{\partial}{\partial y}\left(x + 2xy - 3y^2\right) \end{bmatrix} = \begin{bmatrix} 1 + 2y \\ 2x - 6y \end{bmatrix}.$$

At $(x_0, y_0) = (1, 2)$, this vector evaluates to $(5, -10)$. Therefore, the directional derivative we seek is

$$\begin{bmatrix} 5 \\ -10 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = 5\left(\tfrac{3}{5}\right) - 10\left(\tfrac{4}{5}\right) = \boxed{-5}.$$

Part (b): Assuming the logarithm is base e, the gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \ln \sqrt{x^2 + y^2} \\ \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{bmatrix}.$$

At $(x_0, y_0) = (1, 0)$, this vector evaluates to $(1, 0)$. Therefore, the directional derivative we seek is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \frac{2}{\sqrt{5}} = \boxed{\frac{2\sqrt{5}}{5}}.$$

Part (c): The gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} e^x \cos(\pi y) \\ \frac{\partial}{\partial y} e^x \cos(\pi y) \end{bmatrix} = \begin{bmatrix} e^x \cos(\pi y) \\ -\pi e^x \sin(\pi y) \end{bmatrix}.$$

At $(x_0, y_0) = (0, -1)$, this vector evaluates to $(-1, 0)$. Therefore, the directional derivative we seek is

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \boxed{\frac{\sqrt{5}}{5}}.$$

Part (d): The gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} xy^2 + x^3 y \\ \frac{\partial}{\partial y} xy^2 + x^3 y \end{bmatrix} = \begin{bmatrix} y^2 + 3x^2 y \\ 2xy + x^3 \end{bmatrix}.$$

At $(x_0, y_0) = (4, -2)$, this evaluates to $(-20, 48)$. Therefore, the directional derivative we seek is

$$\begin{bmatrix} -92 \\ -24 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} = -\frac{164}{\sqrt{10}} = \boxed{-\frac{82\sqrt{10}}{5}}.$$

1.2 Problem 5

Part (a): We have that if the angle between $\mathbf{x}_0$ and $\nabla f(\mathbf{v})$ is θ

$$\nabla_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f = \|v\| \|\nabla f\| \cos(\theta) = \|\nabla f\| \cos(\theta) \leq \|\nabla f\|.$$

The maximum possible value of $\nabla_{\mathbf{v}} f$ is thus $\|\nabla f\|$, attained when $\cos(\theta) = 1$ or $\theta = 0$.

Part (b): Via our work in Part (a), the maximum value of the directional derivative of f is the gradient of f. Observe that

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} x^3 - y^3 + z^3 \\ \frac{\partial}{\partial y} x^3 - y^3 + z^3 \\ \frac{\partial}{\partial z} x^3 - y^3 + z^3 \end{bmatrix} = \begin{bmatrix} 3x^2 \\ -3y^2 \\ 3z^2 \end{bmatrix},$$

which evaluates to $(3, -12, 27)$ when $(x, y, z) = (1, 2, 3)$. The maximum is thus

$$\|\nabla f\| = \sqrt{3^2 + (-12)^2 + 27^2} = \boxed{21\sqrt{2}}.$$

1.3 Problem 10

Part (a): The gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial}{\partial y} \frac{1}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial}{\partial z} \frac{1}{\sqrt{x^2+y^2+z^2}} \end{bmatrix} = \begin{bmatrix} -\frac{x}{\sqrt{(x^2+y^2+z^2)^3}} \\ -\frac{y}{\sqrt{(x^2+y^2+z^2)^3}} \\ -\frac{z}{\sqrt{(x^2+y^2+z^2)^3}} \end{bmatrix}.$$

Part (b): The gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} xy + yz + zx \\ \frac{\partial}{\partial y} xy + yz + zx \\ \frac{\partial}{\partial z} xy + yz + zx \end{bmatrix} = \begin{bmatrix} y + z \\ z + x \\ x + y \end{bmatrix}.$$

Part (c): The gradient of f is

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \frac{1}{x^2+y^2+z^2} \\ \frac{\partial}{\partial y} \frac{1}{x^2+y^2+z^2} \\ \frac{\partial}{\partial z} \frac{1}{x^2+y^2+z^2} \end{bmatrix} = \begin{bmatrix} -\frac{2x}{(x^2+y^2+z^2)^2} \\ -\frac{2y}{(x^2+y^2+z^2)^2} \\ -\frac{2z}{(x^2+y^2+z^2)^2} \end{bmatrix}.$$

1.4 Problem 13

Let $w = \cos(xy) - e^z$. Then

$$\nabla w = \begin{bmatrix} \frac{\partial}{\partial x} \cos(xy) - e^z \\ \frac{\partial}{\partial y} \cos(xy) - e^z \\ \frac{\partial}{\partial z} \cos(xy) - e^z \end{bmatrix} = \begin{bmatrix} -y\sin(xy) \\ -x\sin(xy) \\ -e^z \end{bmatrix}$$

At the point 3

1.5 Problem 16

Several level curves are graphed via Desmos on a second file attatched to this submission. We have that

$$\nabla T = \begin{bmatrix} \frac{\partial}{\partial x} x\sin(y) \\ \frac{\partial}{\partial y} x\sin(y) \end{bmatrix} = \begin{bmatrix} \sin(y) \\ x\cos(y) \end{bmatrix}.$$

The direction of this vector is the direction in which traveling from (x, y) ''increases the value of T the most"; its norm is the amount by which the T-value changes instantaneously.

1.6 Problem 17

Part (a): We have that

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} xy + yz + zx \\ \frac{\partial}{\partial y} xy + yz + zx \\ \frac{\partial}{\partial z} xy + yz + zx \end{bmatrix} = \begin{bmatrix} y+z \\ z+x \\ x+y \end{bmatrix}$$

and

$$\mathbf{g}' = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} e^t \\ \frac{\mathrm{d}}{\mathrm{d}t} \cos(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} \sin(t) \end{bmatrix} = \begin{bmatrix} e^t \\ -\sin(t) \\ \cos(t) \end{bmatrix}.$$

Observing that $\mathbf{g}(1) = (e, \cos(1), \sin(1))$ and $\mathbf{g}'(1) = (e, -\sin(1), \cos(1))$, we find that ∇f at $\mathbf{g}(1)$ is $(\cos(1) + \sin(1), \sin(1) + e, e + \cos(1))$. Therefore,

$$(f \circ \mathbf{g})' = \nabla f(\mathbf{g}) \cdot \mathbf{g}' = \begin{bmatrix} \cos(1) + \sin(1) \\ \sin(1) + e \\ e + \cos(1) \end{bmatrix} \cdot \begin{bmatrix} e \\ -\sin(1) \\ \cos(1) \end{bmatrix} = \boxed{\cos(1)^2 - \sin(1)^2 + 2e\cos(1)}.$$

Part (b): We have that

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} e^{xyz} \\ \frac{\partial}{\partial y} e^{xyz} \\ \frac{\partial}{\partial z} e^{xyz} \end{bmatrix} = \begin{bmatrix} yze^{xyz} \\ zxe^{xyz} \\ xye^{xyz} \end{bmatrix}$$

and

$$\mathbf{g}' = \begin{bmatrix} \frac{\partial}{\partial t} 6t \\ \frac{\partial}{\partial t} 3t^2 \\ \frac{\partial}{\partial t} t^3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6t \\ 3t^2 \end{bmatrix}$$

Observing that $\mathbf{g}(1) = (6, 3, 1)$ and $\mathbf{g}'(1) = (6, 6, 3)$, we find that ∇f at $\mathbf{g}(1)$ evaluates to $(3e^{18}, 6e^{18}, 18e^{18})$. Therefore,

$$(f \circ \mathbf{g})' = \nabla f(\mathbf{g}) \cdot \mathbf{g}' = \begin{bmatrix} 3e^{108} \\ 6e^{108} \\ 18e^{108} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \boxed{324e^{108}}.$$

Part (c): We have that

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \left(x^2 + y^2 + z^2\right) \ln\left(\sqrt{x^2 + y^2 + z^2}\right) \\ \frac{\partial}{\partial y} \ln\left(\sqrt{x^2 + y^2 + z^2}\right) \\ \frac{\partial}{\partial z} \left(x^2 + y^2 + z^2\right) \ln\left(\sqrt{x^2 + y^2 + z^2}\right) \end{bmatrix} = \begin{bmatrix} x \ln\left(x^2 + y^2 + z^2\right) \\ y \ln\left(x^2 + y^2 + z^2\right) + y \\ z \ln\left(x^2 + y^2 + z^2\right) + z \end{bmatrix}.$$

and

$$\mathbf{g}' = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} e^t \\ \frac{\mathrm{d}}{\mathrm{d}t} e^{-t} \\ \frac{\mathrm{d}}{\mathrm{d}t} t \end{bmatrix} = \begin{bmatrix} e^t \\ -e^{-t} \\ 1 \end{bmatrix}.$$

Observing that $\mathbf{g}(1) = (e, \frac{1}{e}, 1)$ and $\mathbf{g}'(1) = (e, -\frac{1}{e}, 1)$, we have that $\nabla f(\mathbf{g}(1)) =$

1.7 Problem 20

1.8 Problem 21

We have that

$$\nabla \left(\frac{1}{r}\right) = \nabla \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = \begin{bmatrix} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial}{\partial y} \frac{1}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial}{\partial z} \frac{1}{\sqrt{x^2+y^2+z^2}} \end{bmatrix} = \begin{bmatrix} -\frac{x}{\sqrt{(x^2+y^2+z^2)^3}} \\ -\frac{y}{\sqrt{(x^2+y^2+z^2)^3}} \\ -\frac{z}{\sqrt{(x^2+y^2+z^2)^3}} \end{bmatrix}$$

$$= -\frac{1}{\sqrt{(x^2 + y^2 + z^2)^3}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{\mathbf{r}}{r^3},$$

as desired.

1.9 Problem 24

We prove the contrapositive — that if $S_r = \{\mathbf{x} \mid x \in \mathbb{R}^3 \|x\| = r\}$ is not a level set for some $r \in \mathbb{R}_{>0}$, then there exists some $\mathbf{x}_0 \in \mathbb{R}^2$ such that $\nabla f(\mathbf{x}_0) \neq g(\mathbf{x}_0)\mathbf{x}_0$

Suppose for contradiction that S_r is not a level set and for all $\mathbf{x} \in \mathbb{R}^2$ with $\|x\| = r$,

$$\nabla f(\mathbf{x}) = g(\mathbf{x})\mathbf{x}.$$

Let $\mathbf{c}$ be a tangent vector of S_r at $\mathbf{x}_0$, and let $\mathbf{d}$ be a tangent vector of f at $\mathbf{x}_0$ — such that $\mathbf{d}$ lies in the plane formed by $\mathbf{x}_0$ and $\mathbf{c}$ (we may do this as there exist tangent vectors in every direction at f and S_r). if $\mathbf{c}$ and $\mathbf{d}$ are collinear, it implies that S_r is indeed the level set of f. We have that

$$\mathbf{d} \cdot \mathbf{x}_0 = \mathbf{d} \cdot \frac{\nabla f(\mathbf{x}_0)}{g(\mathbf{x}_0)} = \frac{1}{g(\mathbf{x}_0)}(\mathbf{d} \cdot \nabla f(\mathbf{x}_0)) = 0,$$

and

$$\mathbf{c} \cdot \mathbf{x_0} = 0.$$

This — combined with the fact that $\mathbf{c}$ and $\mathbf{d}$ are coplanar — implies that $\mathbf{c}$ and and $\mathbf{d}$ are collinear. As f and S_r are C_1, then this implies that the level set of f is precisely S_r. This contradicts our original assumption.

Taking the contrapositive yields the desired result.

2 Section 3.1

2.1 Problem 7

Part (a): We have that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}\bigg|_{\mathbf{x}_0=(1,\pi)} &= -y^2 \sin(xy)\bigg|_{\mathbf{x}_0=(1,\pi)} = \boxed{0}, \\ \frac{\partial^2 f}{\partial x \partial y}\bigg|_{\mathbf{x}_0=(1,\pi)} = \frac{\partial^2 f}{\partial y \partial x}\bigg|_{\mathbf{x}_0=(1,\pi)} &= -xy\sin(xy) + \cos(xy)\bigg|_{\mathbf{x}_0=(1,\pi)} = \boxed{-1}, \\ \frac{\partial^2 f}{\partial y^2}\bigg|_{\mathbf{x}_0=(1,\pi)} &= -x^2 \sin(xy)\bigg|_{\mathbf{x}_0=(1,\pi)} = \boxed{0}. \end{aligned}$$

Part (b): We have that

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}\bigg|_{\mathbf{x}_0=(2,-1)} &= 2\bigg|_{\mathbf{x}_0=(2,-1)} = \boxed{2}, \\
\frac{\partial^2 f}{\partial x \partial y}\bigg|_{\mathbf{x}_0=(2,-1)} = \frac{\partial^2 f}{\partial y \partial x}\bigg|_{\mathbf{x}_0=(2,-1)} &= 8y^7\bigg|_{\mathbf{x}_0=(2,-1)} = \boxed{-8}, \\
\frac{\partial^2 f}{\partial y^2}\bigg|_{\mathbf{x}_0=(2,-1)} &= 56xy^6 + 12y^2\bigg|_{\mathbf{x}_0=(2,-1)} = \boxed{124}.
\end{aligned}$$

Part (c): All the terms in the second partial derivatives of e^{xyz} will be e^{xyz} times some product of x, y, and/or z (at least one of these variables will be present). This guarantees that at $(x, y, z) = (0, 0, 0)$, $\boxed{\text{all partial derivatives of } f \text{ are } 0}$.

2.2 Problem 9

Suppose for contradiction that such a C^2 function f exists. Then the mixed second partial derivatives of f should be equal; however,

$$\frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y} 2x - 5y = -5 \neq 4 = \frac{\partial}{\partial x} 4x + y = \frac{\partial}{\partial x} f_y.$$

Thus, $\boxed{\text{no such } f \text{ exists}}$.

2.3 Problem 15

We have that

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y} y^2 + 2xy = \boxed{2x + 2y}, \\
\frac{\partial^2 f}{\partial y \partial z} &= \frac{\partial}{\partial z} x^2 + 2xy + z^2 = \boxed{2z}, \\
\frac{\partial^2 f}{\partial z \partial x} &= \frac{\partial}{\partial x} 2yz = \boxed{0}, \\
\frac{\partial^2 f}{\partial x \partial y} z &= \frac{\partial}{\partial z} 2x + 2y = \boxed{0}.
\end{aligned}$$

2.4 Problem 21

Part (a): The partial derivatives are as follows:

$$f_x = \boxed{\arctan\left(\frac{x}{y}\right) + \frac{xy}{x^2+y^2}},$$

$$f_y = \boxed{-\frac{x^2}{x^2+y^2}},$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\arctan\left(\frac{x}{y}\right) + \frac{xy}{x^2+y^2}\right) = \boxed{\frac{2y^3}{(x^2+y^2)^2}},$$

$$f_{xy} = -\frac{\partial}{\partial x}\left(\frac{x^2}{x^2+y^2}\right) = \boxed{-\frac{2xy^2}{(x^2+y^2)^2}},$$

$$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{x^2}{x^2+y^2}\right) = \boxed{\frac{2x^2y}{(x^2+y^2)^2}}.$$

Part (b): We have that

$$f_x = \boxed{\frac{x}{\sqrt{x^2+y^2}}},$$

$$f_y = \boxed{\frac{y}{\sqrt{x^2+y^2}}},$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \boxed{\frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}},$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \boxed{-\frac{xy}{\sqrt{(x^2+y^2)^3}}},$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \boxed{\frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}},$$

Part (c): We have that

$$\begin{aligned}
f_x &= \boxed{-2xe^{-x^2-y^2}},\\
f_y &= \boxed{-2ye^{-x^2-y^2}},\\
f_{xx} = \frac{\partial}{\partial x}\left(-2xe^{-x^2-y^2}\right) &= \boxed{4x^2e^{-x^2-y^2} - 2e^{-x^2-y^2}},\\
f_{xy} = \frac{\partial}{\partial y}\left(-2ye^{-x^2-y^2}\right) &= \boxed{4xye^{-x^2-y^2}},\\
f_{yy} = \frac{\partial}{\partial y}\left(-2ye^{-x^2-y^2}\right) &= \boxed{4y^2e^{-x^2-y^2} - 2e^{-x^2-y^2}}.
\end{aligned}$$

2.5 Problem 23

2.6 Problem 25

We have that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0,$$

so u is a harmonic function.

2.7 Problem 26

Part (a): We have that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 - 2 = 0,$$

so $\boxed{f \text{ is harmonic}}$.

Part (b): We have that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 + 2 = 4,$$

so $\boxed{m \text{ is not harmonic}}$

Part (c): We have that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 + 0 = 0,$$

so $\boxed{f \text{ is harmonic}}$.

Part (d): We have that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6y + 6y = 0,$$

so $\boxed{f \text{ is not harmonic}}$.

Part (e): We have that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\sin(x)\cosh(y) + \sin(x)\cosh(y) = 0,$$

so $\boxed{f \text{ is harmonic}}$.

Part (f): We have that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x \sin(y) - e^x \sin(y) = 0,$$

so $\boxed{f \text{ is harmonic}}$.