Calculus III: Lesson 1

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1 Introduction

In the context of Calculus III, a vector is an arrow in $\mathbb{R}^n$, notated as follows:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Note that $\mathbb{R}^n$ possess a vector space structure, and obey the standard vector axioms as defined over my physical notebook; I don't feel the need to rearticulate the vector axioms here.

the **canonical basis** of $\mathbb{R}^n$ consist of the unit vectors as follows:

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{e_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Decomposing a vector into a linear combination of the canonical basis of $\mathbb{R}^n$ is trivial. Vectors are said to be collinear if one is a scalar multiple of the other — or if the two are linearly dependent. The dot product (as you know it) for vectors in $\mathbb{R}^n$ is called the *canonical inner product* — likewise for the *Euclidean norm*.

You already know about the parametric representation of a line — namely, that $\mathbf{u} + t\mathbf{v}$ crap. v is called the *direction vector* or *velocity vector*, while u is just a starting vector.

For a vector $\mathbf{v}$, the *normed vector* of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Clearly it has length 1.

2 Dot and Cross Products

The (canonical) dot product is said to *encode* the geometry of $\mathbb{R}^n$ due to several ubuitous formulas:

- $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$,
- $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$,
- $\mathbf{u} \cdot \mathbf{v} = \mathrm{Proj}_{\mathbf{u}}(\mathbf{v}) = \mathrm{Proj}_{\mathbf{v}}(\mathbf{u})$.

The dot product is thus particularly essential in higher geometry (like topology or differential geometry).

The cross product of two three-dimensional vectors $\mathbf{u} \times \mathbf{v}$ is a vector that that — if you ignore the weird determinant notation — computes a vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$ with length of the determinant of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ as follows:

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & v_1 & u_1 \\ \mathbf{j} & v_2 & u_2 \\ \mathbf{k} & v_3 & u_3 \end{vmatrix},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the basis vectors of $\mathbb{R}^3$. The computation of 2×2 and 3×3 determinants is standard; higher-dimensional computation will be covered in the future. Cross products satisfy the essential identity

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| |\sin(\theta)|.$$

Clearly $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ implies $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0$ We also have the cutesy identity $\|\mathbf{u} \times \mathbf{v}\|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2$. Clearly also, through a really nice proof involving projection, we have that for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$,

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

Ah, skipping planes and distance formulas is really coming back to bite you. Have fun with that over the weekend.

3 Planes and Lines in 3-D

Distance between Lines: Suppose we would like to compute the distance between two lines $\ell_1 = \vec{v}_1 + t\vec{w}_1$ and $\ell_2 = \vec{v}_2 + t\vec{w}_2$. If we suppose *the lines are nonparallel*, here's how:

1. Construct a plane that contains ℓ_1 and is parallel to ℓ_2 — namely, the plane $t\vec{w}_1 + s\vec{w}_2 + \vec{v}_1$.

2. Compute the distance between a point ℓ_2 and the plane. (there exists a point on ℓ_2 that projects to a point on ℓ_2, as we assumed ℓ_1 and ℓ_2 are nonparallel).

From such a process emerges the formula: if $\vec{v_1}$ and $\vec{v_2}$ are vectors from the lines, and $\vec{a}_1, \vec{a}_2$ are any two nonparallel vectors on the plane, the distance is

$$\frac{|(\vec{v}_1 - \vec{v}_2)\vec{a}_1 \times \vec{a}_2|}{\|\vec{a}_1 \times \vec{a}_2\|}.$$

Distance between Planes: If we wish to compute the distance between two *parallel* planes, we can use the following steps:

1. Select a point on one of the planes.

2. Find the distance of the point to the other plane, *using the formula.*

Performing this process yields the formula

$$\frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}},$$

where A, B, C are the coefficents in the equations and D_1, D_2 are the differing constant terms in their equations.

4 Cylindrical and Spherical Coordinates

You're already intimately familiar with polar coordinates — including their complex representations. Below are its analogues in $\mathbb{R}^3$.

Cylindrical Coordinates: Representing a point (x, y, z) as r, θ, z, where r and θ are just like in polar coordinates. Converting between these coordinates is straightfowards, no memorization required.

Spherical Coordinates: Representing a point (x, y, z) as (ρ, θ, φ), where ρ denotes the distance of the point to the origin, θ the angle with the x-axis if the point is projected onto the xy-plane, and φ the angle made from the z-axis.

The following formulas articulate the conversion from spherical to cylindrical or Cartesian coordiantes:

- $x = \rho \cos(\theta) \sin(\varphi)$.
- $y = \rho \sin(\theta) \sin(\varphi)$.
- $z = \rho \cos(\varphi)$.
- $r = \rho \sin(\varphi)$.

I remember this as the y-coordinate "liking" sines, and the rest of the three formulas follow.

5 Multivariable Functions

A real-valued multivariable function can be of many types, for $A \subset \mathbb{R}^n$:

- **Scalar-Valued**: A function $f : A \to \mathbb{R}$, mapping vectors to scalars.
- **Vector-Valued**: A function $f : A \to \mathbb{R}^n$, mapping vectors to vectors.

We say f maps its *domain space* to its *target space*. The graph of a function $f : A \to \mathbb{R}^n$ is a surface defined as the set of points $(x_1, x_2, \ldots x_{n-1},$

$$f(x_1, x_2, \ldots, x_{n-1})) \in \mathbb{R}^n$$

Level Set: For $f : A \to \mathbb{R}^n$ and $c \in \mathbb{R}$, the *level set* of f for the value c is the set of points $\{x \in A \mid f(x) = c\}$. The level set is a critical way to visualize a multivariable function.