MATH-UA 129: Homework 11

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1 Section 8.1

Problem 10

Let C be the closed simple curve that bounds the disc with radius R — namely, $C(\theta) = (R\cos(\theta), R\sin(\theta))$. Then by Green's Theorem, the area of this region is

$$\frac{1}{2}\int_0^{2\pi} (R\cos(\theta))(R\cos(\theta)) - (R\sin(\theta))(-R\sin(\theta))\,\mathrm{d}\theta = \frac{2\pi R^2}{2} = \boxed{\pi R^2}.$$

Problem 12

By the Divergence Theorem,

$$\begin{aligned}
\int_{\partial D} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}\mathbf{s} &= \iint_D (\nabla\cdot\mathbf{F})\,\mathrm{d}A \\
&= \iint_D \frac{\partial}{\partial x}y - \frac{\partial}{\partial y}(-x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \iint_D 0 - 0\,\mathrm{d}y\,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Problem 13

Using Green's Theorem, the area of the region bounded by the curve is

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi} x\,\mathrm{d}y - y\,\mathrm{d}x &= \frac{1}{2}\int_0^{2\pi} a(\theta - \sin(\theta))(a\sin(\theta)) - a(1-\cos(\theta))(a - a\cos(\theta))\,\mathrm{d}\theta \\
&= \frac{a^2}{2}\int_0^{2\pi} \theta\sin(\theta) - \sin^2(\theta) - (1 - 2\cos(\theta) + \cos^2(\theta))\,\mathrm{d}\theta \\
&= \frac{a^2}{2}\int_0^{2\pi} \theta\sin(\theta) + 2\cos(\theta) - 2\,\mathrm{d}\theta \\
&= \frac{a^2}{2}\big[3\sin(\theta) - \theta\cos(\theta) - 2\theta\big]_0^{2\pi} \\
&= \frac{a^2}{2}\big[-2\pi - 4\pi\big] \\
&= -3\pi a^2.
\end{aligned}$$

The answer is the absolute value of this —namely $\boxed{3\pi a^2}$.

Problem 15

If D is the unit disc, then Green's Theorem yields that

$$\begin{aligned}\int_C (2x^3 - y^3)\,\mathrm{d}x + (x^3 + y^3)\,\mathrm{d}y &= \iint_D \left(\frac{\partial}{\partial x} x^3 + y^3\right) - \left(\frac{\partial}{\partial y}(2x^3 - y^3)\right) \mathrm{d}y\,\mathrm{d}x \\ &= \iint_D 3x^2 + 3y^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1 \int_0^{2\pi} 3r^2(r)\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 6\pi \left[\frac{r^4}{4}\right]_0^1 \\ &= \boxed{\frac{3\pi}{2}}.\end{aligned}$$

2 Section 8.2

Problem 11

Verification by flux: Realize that

$$\nabla \times \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y) \\ \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z) \\ \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \end{bmatrix} = \mathbf{0}.$$

Then if we let the upper hemisphere be Σ,

$$\iint_\Sigma \nabla \times \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \iint_\Sigma \mathbf{0} \cdot \mathrm{d}\mathbf{S} = \boxed{0}.$$

Verification by circulation: The oriented boundary of the upper hemisphere is given by $\mathbf{c}(\theta) = (\cos(\theta), \sin(\theta), 0)$ for $\theta \in [0, 2\pi)$. Then if the upper hemisphere is Σ,

$$\begin{aligned}
\int_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(\theta)) \cdot \mathbf{c}'(\theta)\,\mathrm{d}\theta \\
&= \int_0^{2\theta} (\cos(\theta), \sin(\theta), 0) \cdot (-\sin(\theta), \cos(\theta), 0)\,\mathrm{d}\theta \\
&= \int_0^{2\pi} -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta)\,\mathrm{d}\theta \\
&= \int_0^{2\pi} 0\,\mathrm{d}\theta \\
&= \boxed{0}.
\end{aligned}$$

These two integrals match, as stated by Stokes' Theorem.

Problem 24

Realize that if $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned}
\mathbf{v} \times \mathbf{r} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} v_2 z - v_3 y \\ v_3 x - v_1 z \\ v_1 y - v_2 x \end{bmatrix}.
\end{aligned}$$

To apply Stokes' Theorem, we must calculate the curl of this vector:

$$\begin{aligned}
\nabla \times (\mathbf{v} \times \mathbf{r}) &= \begin{bmatrix} \frac{\partial}{\partial y}(v_1 y - v_2 z) - \frac{\partial}{\partial z}(v_3 x - v_1 z) \\ \frac{\partial}{\partial z}(v_2 z - v_3 y) - \frac{\partial}{\partial x}(v_1 y - v_2 x) \\ \frac{\partial}{\partial x}(v_3 x - v_1 z) - \frac{\partial}{\partial z}(v_2 z - v_3 y) \end{bmatrix} \\
&= \begin{bmatrix} v_1 + v_1 \\ v_2 + v_2 \\ v_3 + v_3 \end{bmatrix} \\
&= 2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\
&= 2\mathbf{v}.
\end{aligned}$$

Then by Stokes' Theorem,

$$\int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot \mathrm{d}\mathbf{s} = \iint_S \nabla \times (\mathbf{v} \times \mathbf{r}) = \iint 2\mathbf{v} \cdot \mathrm{d}\mathbf{S} = 2 \iint_S \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}S\,.$$

Problem 25

Consider removing a "small hole of circumference ϵ" to yield the surface Σ. As ϵ goes to zero, we expect the circulation

$$\int_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$$

to go to zero. By Stokes' Theorem, we expect the equivalent quantity

$$\int_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

to approach 0 as well.

3 Section 8.3

Problem 4

Part (a): For $f(x) = e^x \cos(y) + z\pi$, observe that $\mathbf{F} = \nabla f$. Hence, $\boxed{f \text{ is a gradient}}$ and is thus $\boxed{\text{not a curl}}$.

Part (b): For $f(x) = \frac{xy}{z^2+4}$, observe that $\mathbf{F} = \nabla f$. Hence, $\boxed{f \text{ is a gradient}}$ and is thus $\boxed{\text{not a curl}}$.

Part (c): Observe that

$$\nabla \times \mathbf{F} = \left(x\cos(z), 2x^2y^2z - y\cos(z), ye^x - 2x^2yz^2\right)$$

is nonzero, and

$$\nabla \cdot \mathbf{F} = xy^2z^2 + e^x - xy\sin(z)j$$

is nonzero, so $\boxed{f \text{ is neither a gradient nor a curl}}$.

Part (d): See that $\nabla \cdot F = \frac{\partial}{\partial x}\left(6z^5y^5\right) + \frac{\partial}{\partial y}\left(9x^8z^2\right) + \frac{\partial}{\partial z}\left(4x^3y^3\right) = 0 + 0 + 0 = 0$, so $\boxed{f \text{ is a curl}}$ and is thus $\boxed{\text{not a gradient}}$.

Problem 5

Suppose that f and g are two potential fields of $\mathbf{F}$ — namely, that $\nabla f = \nabla g = \mathbf{F}$. Then

$$\nabla(f - g) = \nabla f - \nabla g = \mathbf{F} - \mathbf{F} = \mathbf{0},$$

so $f - g$ is constant (this is a well-known result we discussed in class), which completes the proof.

Problem 18

Part (a): Realize that as

$$\frac{\partial}{\partial y}\big(2x + y^2 - y\sin(x)\big) = 2y - \sin(x) \neq 2yz - \sin(x) = \frac{\partial}{\partial x}(2xyz + \cos(x)),$$

the vector field $\boxed{\mathbf{F} \text{ is not a gradient}}$.

Part (b): Realize that as

$$\frac{\partial}{\partial z}6x^2z^2 = 12x^2z \neq 0 = \frac{\partial}{\partial x}4y^2z^2,$$

the vector field $\boxed{\mathbf{F} \text{ is not a gradient}}$.

Part (c): Realize that if $f(x, y) = xy^3 + x + y$, that $\mathbf{F} = \nabla f$. Thus, $\boxed{\nabla(xy^3 + x + y) = \mathbf{F}}$ — and of course, we can add a constant to f.

Part (d): Realize that as

$$\frac{\partial}{\partial y}\Big(xe^{x^2+y^2} + 2xy\Big) = 2xye^{x^2+y^2} + 2x \neq 2xye^{x^2+y^2} = \frac{\partial}{\partial x}\Big(ye^{x^2+y^2} + 4y^3z\Big),$$

the vector field $\boxed{\mathbf{F} \text{ is not a gradient}}$.

Problem 22

We have that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}(y) = z - z + 0 = 0.$$

We conclude that $\mathbf{F}$ is the curl of some vector field — an example of such a field is $\boxed{\mathbf{G}(x, y, z) = (0, xy, xyz)}$, as verified by a trivial computation.

Problem 29

We have that if $\mathbf{r} = (x, y, z)$,

$$\mathbf{F} = -\frac{GmM\mathbf{r}}{r^3} = -\frac{GmM}{\sqrt{(x^2+y^2+z^2)^3}}(x, y, z).$$

Thus,

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= -\frac{GmM(-2x^2+y^2+z^2)}{\sqrt{(x^2+y^2+z^2)^5}} - \frac{GmM(x^2-2y^2+z^2)}{\sqrt{(x^2+y^2+z^2)^5}} - \frac{GmM(x^2+y^2-2z^2)}{\sqrt{(x^2+y^2+z^2)^5}} \\
&= -\frac{GmM(0)}{\sqrt{(x^2+y^2+z^2)^5}} \\
&= 0.
\end{aligned}$$

4 Section 8.4

Problem 5

Let the unit sphere be W. Then by the Divergence Theorem, we have that the flux on the surface is

$$\begin{aligned}
\iint_{\partial W} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iiint_W (\nabla \cdot \mathbf{F})\, \mathrm{d}V \\
&= \iiint_W \left(\frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-z) + \frac{\partial}{\partial z}(z-x) \right) \\
&= \iiint_{@} 3\, \mathrm{d}V \\
&= 3\left(\frac{4\pi}{3}\right) \\
&= \boxed{4\pi}.
\end{aligned}$$

Problem 11

Let the box be W. By Gauss' Theorem, we have that

$$\begin{aligned}\iint_{\partial W} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iiint_W (\nabla \cdot \mathbf{F})\, \mathrm{d}V \\ &= \iiint_W \left(\frac{\partial}{\partial x}(x - y^2) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x^3) \right) \mathrm{d}V \\ &= \int_0^1 \int_1^2 \int_1^4 2 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\ &= \boxed{6}.\end{aligned}$$

Problem 12

Let the unit sphere be W. Then by Gauss' Theorem, we have that

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iiint_W (\nabla \cdot \mathbf{F})\, \mathrm{d}V \\ &= \iiint_W \left(\frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial y}(3x^2 y) + \frac{\partial}{\partial z}(3z^2) \right) \mathrm{d}V \\ &= \iiint_W 3x^2 + 3y^2 + 3z^2 \, \mathrm{d}V \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 3\rho^2 (\rho^2 \sin(\phi)) \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}\rho \\ &= 3\left[\frac{\rho^5}{5}\right]_0^1 (2\pi) \Big[-\cos(\phi) \Big]_0^{\pi} \\ &= \boxed{\frac{12\pi}{5}}\end{aligned}$$

Problem 16

By Gauss' Theorem, we have that

$$\begin{aligned}
\iint_{\partial W} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A &= \iiint_W (\nabla\cdot\mathbf{F})\,\mathrm{d}V \\
&= \iiint_W \left(\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}\big(z(x^2+y^2)^2\big)\right) \\
&= \iiint_W (x^2+y^2)^2\,\mathrm{d}V \\
&= \int_0^1\int_0^1\int_0^{2\pi} (r^2)^2(r)\,\mathrm{d}\theta\,\mathrm{d}\rho\,\mathrm{d}z \\
&= 2\pi\left[\frac{r^6}{6}\right]_0^1 \\
&= \boxed{\frac{\pi}{3}}.
\end{aligned}$$

Problem 17

By the properties of divergence (and by Gauss' Theorem), we have that

$$\begin{aligned}
\iiint_W (\nabla f)\cdot\mathbf{F}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_W (\nabla f)\cdot\mathbf{F}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad + \iiint_W f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \iiint_W f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_W \nabla f\cdot\mathbf{F} + f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \iiint_W f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_W \nabla\cdot(f\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \iiint_W f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint_{\partial W} f\mathbf{F}\cdot\mathrm{d}S - \iiint_W f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint_{\partial W} f\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S - \iiint_W f(\nabla\cdot\mathbf{F})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}$$

as desired.

Problem 21

For the first formula: we have that $\nabla \cdot (f\nabla g) = f(\nabla \cdot \nabla g) + \nabla g \cdot \nabla f = f\nabla^2 g + \nabla f \cdot \nabla g$. Then by Gauss' Theorem,

$$\iint_{\partial W} f\nabla g \cdot \mathbf{n}\,\mathrm{d}S = \iiint_W \nabla \cdot (f\nabla g)\,\mathrm{d}V = \iiint_W (f\nabla^2 g + \nabla f \cdot \nabla g)\,\mathrm{d}V\,.$$

Now to prove the second formula: realize that by substituting f for g and vice versa,

$$\iint_{\partial W} g\nabla f \cdot \mathbf{n}\,\mathrm{d}S = \iiint_W (g\nabla^2 f + \nabla g \cdot \nabla f).$$

We thus deduce by substition and Gauss' Theorem that

$$\begin{aligned}
\iint_{\partial W} (f\nabla g - g\nabla f) \cdot \mathbf{n}\,\mathrm{d}S &= \iint_{\partial W} f\nabla g \cdot \mathbf{n}\,\mathrm{d}S - \iint_{\partial W} g\nabla f \cdot \mathbf{n}\,\mathrm{d}S \\
&= \iiint_W (f\nabla^2 g + \nabla f \cdot \nabla g)\,\mathrm{d}V - \iiint_W (g\nabla^2 f + \nabla f \cdot \nabla g)\,\mathrm{d}V \\
&= \iiint_W (f\nabla^2 g - g\nabla^2 f + \nabla f \cdot \nabla g - \nabla g \cdot \nabla f)\,\mathrm{d}V \\
&= \iiint_W (f\nabla^2 g - g\nabla^2 f)\,\mathrm{d}V
\end{aligned}$$

Problem 28

Let the region enclosed by S be W. Then by Gauss' Theorem (and because the curl is divergence-free),

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iiint_W \nabla \cdot (\nabla \times \mathbf{F})\,\mathrm{d}V \\
&= \iiint_W 0\,\mathrm{d}V \\
&= 0.
\end{aligned}$$