MATH-UA 148: Homework 3

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1 Section 3A

1.1 Problem 4

We prove the contrapositive — suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly dependent. Then there exist scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n vecv_n = \mathbf{0}.$$

Therefore,

$$\lambda_1(T\mathbf{v}_1) + \cdots + \lambda_n(T\mathbf{v}_n) = T(\lambda_1 \mathbf{v}_1) + \cdots + T(\lambda_n \mathbf{v}_n) = T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) = T(\mathbf{0}) = \mathbf{0},$$

so $T\mathbf{v}_1, \ldots, T\mathbf{v}_n$ is linearly dependent. Taking the contrapositive yields the desired result.

1.2 Problem 10

We define two vectors: $\mathbf{v}$, such that $\mathbf{v} \in V$ and $\mathbf{v} \notin U$ and $\mathbf{u}$, such that $\mathbf{u} \in U$ and $S\mathbf{u} \neq \mathbf{0}$. Clearly $T\mathbf{v} = \mathbf{0}$ — and as $\mathbf{v} + \mathbf{u} \notin U$, we have that $T(\mathbf{v} + \mathbf{u}) = \mathbf{0}$, so

$$T(\mathbf{v} + \mathbf{u}) = 0 \neq T\mathbf{u} = T\mathbf{u} + T\mathbf{v}.$$

We conclude that T is not a linear map on V.

1.3 Problem 14

Let the basis of V be $\mathbf{v}_1$ and $\mathbf{v}_2$. We thus define two linear maps:

- T, such that $T\mathbf{v}_1 = \mathbf{v}_2$ and $T\mathbf{v}_2 = \mathbf{v}_1$.
- S, such that $T\mathbf{v}_1 = \mathbf{v}_1$ and $T\mathbf{v}_2 = -\mathbf{v}_2$.

Clearly $\mathbf{v}_1, \mathbf{v}_2$ are nonzero, and T and S are linear maps; thus,

$$TS\mathbf{v}_2 = T(-\mathbf{v}_2) = -T(\mathbf{v}_2) = -\mathbf{v}_1 \neq \mathbf{v}_1 = S\mathbf{v}_1 = ST\mathbf{v}_2.$$

Thus, $TS \neq ST$.

2 Section 3B

2.1 Problem 18

Surjectivity implies Dimension: Suppose that there exists a surjective linear map $T : V \to W$. Then $\operatorname{null} T = \{\mathbf{0}\}$, so $\dim \operatorname{null} T = 0$; therefore,

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim \operatorname{range} T.$$

As $\operatorname{range} T$ is a subspace of W, we have that $\dim V = \dim \operatorname{rangeT} \leq \dim W$.

Dimension implies Surjectivity: Suppose that $\dim V \leq \dim W$; let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be bases of V and W, where $n \leq m$.

Define $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ as the unique linear map such that $T\mathbf{v}_j = \mathbf{w}_j$ for each $j \in \{1, \ldots, n\}$. Suppose $\mathbf{v}$ and $\mathbf{u}$ are vectors in V such that $T\mathbf{v} = T\mathbf{u}$; let

$$\begin{aligned}
\mathbf{v} &= \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \\
\mathbf{w} &= \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n.
\end{aligned}$$

We thus dedice that

$$\begin{aligned}
(\lambda_1 - \nu_1)\mathbf{w}_1 + \cdots + (\lambda_n - \nu_n)\mathbf{w}_n &= (\lambda_1 \mathbf{w}_1 + \cdots + \lambda_n \mathbf{w}_n) - (\mu_1 \mathbf{w}_1 + \mu_n \mathbf{w}_n) \\
&= (T\lambda_1 \mathbf{v}_1 + \cdots + T\lambda_n \mathbf{v}_n) - (T\mu_1 \mathbf{v}_1 + \cdots + T\mu_n \mathbf{v}_n) \\
&= T(\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n) - T(\mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{v}_n) \\
&= T\mathbf{v} - T\mathbf{u} \\
&= T(\mathbf{v} - \mathbf{u}) \\
&= T\mathbf{0} \\
&= \mathbf{0}
\end{aligned}$$

As $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent, $\lambda_j = \mu_j$ for all $j \in \{1, \ldots, n\}$. This implies that $\mathbf{v} = \mathbf{u}$, so T is surjective.

2.2 Problem 22

By the Fundamental Theorem of Linear Maps,

$$\begin{aligned}
\dim \operatorname{null} ST &= \dim U - \dim \operatorname{range} ST \\
&= \dim \operatorname{null} T + \dim \operatorname{range} T - \dim \operatorname{range} ST \\
&\leq \dim \operatorname{null} T + \dim V - \dim \operatorname{range} ST \\
&= \dim \operatorname{null} T + \dim \operatorname{null} S + \dim \operatorname{range} S - \dim \operatorname{range} ST \\
&\leq \dim \operatorname{null} S + \dim \operatorname{null} T
\end{aligned}$$

2.3 Problem 28

3 Section 3C

3.1 Problem 6

3.2 Problem 14