Hartshorne: Varieties

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1 Affine Varieties

1.1 Familiar Definitions

Let k be an algebraically closed field. We define **affine space** over k, denoted $\mathbb{A}_k^n$ or $\mathbb{A}^n$, as the set of all n-tuples with components in k. The elements $P \in \mathbb{A}^n$ are called **points** — and if $P = (a_1, \dots, a_n)$, the elements $a_1, \dots, a_n$ are called **components**.

The set $R = k[x_1, \dots, x_n]$ denotes the commutative ring of polynomials with variables $x_1, \dots, x_n$ with coefficents in k. We may interpret each $f \in R$ as a function from $\mathbb{A}^n$ to k, defined by $f(P) = f(a_1, \dots, a_n)$. We may thus define the **zeroes** of f, given by the set $Z(f) = \{P \in \mathbb{A}_k^n \mid f(P) = 0\}$. More generally, for any subset T of polynomials R, its **zeroes** are given by

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for each } f \in T\}.$$

Ideals in R are quite elegant: since R is Noetherian, each ideal $\mathfrak{a}$ has a finite set of generators $f_1, \dots, f_n$. Thus $\mathfrak{a}$ may be expressed as the common zeroes of $f_1, \dots, f_n$. It is easy to verify that if $\mathfrak{b}$ is the ideal generated by T, then $Z(T) = Z(\mathfrak{b})$.

1.2 The Zariski Topology

A subset $Y \subseteq \mathbb{A}^n$ is an **algebraic set** if there exists a subset $T \subseteq Y$ such that $Y = Z(T)$.

Theorem 1. *The following two results hold:*

1. *The union of two algebraic sets $X, Y \subseteq \mathbb{A}^n$ is algebraic.*
2. *The intersection of any family of algebraic sets Y_α is an algebraic set.*

Proof. For (1), let $X = Z(T)$ and $Y = Z(S)$. We claim that $Z(T \cup S) = Z(TS)$, where TS is the set of all ts with $t \in T$ and $s \in S$.

1. Suppose $P \in Z(TS)$ — that is, $ts(P) = 0$ for all $ts \in TS$. Since R is an integral domain, we have $t(P)$ or $s(P) = 0$, so $P \in Z(T)$ or $P \in Z(S)$. Hence $P \in Z(T) \cup Z(S)$.
2. Suppose $P \in Z(T) \cup Z(S)$. Then $P \in Z(T)$ or $P \in S(T)$ — in which case, $t(P) = 0$ for all t or $s(P) = 0$ for all s. In either case, $ts(P) = 0$ for all $ts \in TS$, so $P \in Z(TS)$.

Thus $X \cup Y = Z(T) \cup Z(S) = Z(TS)$. We deduce that $X \cup Y$ is an algebraic set. For (2), let $Y_\alpha = T_a$. It is easy to verify that $\bigcap_{\alpha \in A} Z(T_\alpha) = Z\big(\bigcup_{\alpha \in A} T_\alpha\big)$; hence $\bigcup_{\alpha \in A} Y_\alpha$ is an algebraic set. $\square$

In particular, we have the following for $T, S \subseteq \mathbb{A}^n$:

1. $Z(T) \cap Z(S) = Z(T \cup S)$.
2. $Z(T) \cup Z(S) = Z(TS)$.

Noting that $\varnothing$ and $\mathbb{A}^n$ are algebraic sets (since $\varnothing = Z(1)$ and $\mathbb{A}^n = Z(0)$), we deduce that algebraic sets in R satisfy the axioms for closed sets in a topological space. The ensuing topology is called the **Zariski topology**.

As an example, the Zariski topology on $\mathbb{A}^1$ is its finite subsets, plus $\mathbb{A}^1$ itself. Since $k[x]$ is a principal ideal domain, each ideal may be generated by one polynomial; it is clear that for each finite subset of $\mathbb{A}^1$, one can construct a polynomial with roots at the subset.

1.3 Definition of Affine Varieties

Let X be a topological space. A nonempty subset $Y \subseteq X$ is **irreducible** if it cannot be expressed as a union of two closed proper subsets of X. By convention, $\varnothing$ is not irreducible. For instance, all nonempty open sets of a topological space are irreducible.

An **affine variety** is an irreducible closed subset of $\mathbb{A}^n$. A **quasi-affine variety** is an open subset of an affine variety. Before we study affine varieties, we need to explore ideals: For a set $Y \subseteq \mathbb{A}^n$, the **ideal** of Y is

$$I(Y) \quad \overset{\text{def}}{=} \quad \{f \in R \mid f(y) = 0 \text{ for all } y \in Y\}.$$

Theorem 2. *Let $T_1, T_2 \subseteq R$ and $Y_1, Y_2 \subseteq \mathbb{A}^n$. Then the following hold:*

1. *If $T_1 \subseteq T_2$, then $Z(T_1) \supseteq Z(T_2)$.*
2. *If $Y_1 \subseteq Y_2$, then $I(Y_1) \supseteq I(Y_2)$.*
3. *$I(Y_1) \cap I(Y_2) = I(Y_1 \cup Y_2)$.*
4. *$I(Z(T)) = r(T)$, the radical of T.*
5. *$Z(I(Y)) = \overline{Y}$, the closure of Y.*

Proof. (1), (2), and (3) follow from the definitions. (4) follows from the Nullstellenstaz. □

The Nullstellenstaz will be asserted without proof.

Theorem 3 (Hilbert's Nullstellenstaz)**.** *Let k be an algebraically closed field, and let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then if f vanishes at all points of $Z(\mathfrak{a})$, we have $f^n \in \mathfrak{a}$.*