MATH-UA 129: Homework 7

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1 Secction 4.2

Problem 4

If we define $\mathbf{c}(t) = (2\cos(t), 2\sin(t), t)$, then we seek

$$\int_0^{2\pi} \left\|\mathbf{c}'(t)\right\| \mathrm{d}t = \int_0^{2\pi} \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + 1^2}\, \mathrm{d}t = \int_0^{2\pi} \sqrt{5}\, \mathrm{d}t = t\sqrt{5}\Big|_0^{2\pi} = \boxed{2\pi\sqrt{5}}.$$

Problem 6

If we define $\mathbf{c}(t) = (t, t\sin(t), t\cos(t))$, then we seek

$$\int_0^{2\pi} \left\|\mathbf{c}'(t)\right\| \mathrm{d}t = \int_0^{2\pi} \sqrt{t^2 + (t\sin(t))^2 + (t\cos(t))^2}\, \mathrm{d}t = \int_0^{2\pi} \sqrt{2t^2}\, \mathrm{d}t = \int_0^{2\pi} 2|t|\, \mathrm{d}t = \boxed{4\pi}.$$

Problem 14

We seek to compute two integrals:

$$\int_0^t \left\|\alpha'(\tau)\right\| \mathrm{d}\tau \qquad \text{and} \qquad \int_0^t \left\|\beta'(t)\right\| \mathrm{d}\tau$$

For the first integral, observe that the arc length is

$$\int_0^t \|\alpha'(t)\| \, \mathrm{d}\tau = \int_0^t \sqrt{\sinh^2(\tau) + \cosh^2(\tau) + \tau^2} \, \mathrm{d}\tau = \boxed{\int_0^t \sqrt{2\cosh^2(\tau) - 1 + \tau^2} \, \mathrm{d}\tau}.$$

For the second integral,

$$\int_0^t \|\beta'(t)\| \, \mathrm{d}\tau = \int_0^t \sqrt{\cos^2(\tau) + \sin^2(\tau) + \tau^2} \, \mathrm{d}\tau = \boxed{\int_0^t \sqrt{1 + \tau^2} \, \mathrm{d}\tau}.$$

Problem 15

2 Section 4.3

Problem 9

Part (a): Clearly, $\mathbf{V}(x, y) = x\mathbf{i} + y\mathbf{j}$ is represented by $\boxed{\text{Graph (ii)}}$.

Part (b): Clearly, $\mathbf{V}(x, y) = y\mathbf{i} - x\mathbf{x}$ is respresented by $\boxed{\text{Graph (i)}}$.

Problem 10

Part (a): Clearly, $\mathbf{V}(x, y) = \frac{y}{\sqrt{x^2+y^2}}\mathbf{i} - \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ is represented by $\boxed{\text{Graph (i)}}$.

Part (a): Clearly, $\mathbf{V}(x, y) = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$ is represented by $\boxed{\text{Graph (ii)}}$.

Part (c): These two fields are the $\boxed{\text{normalized vector fields of Problem 9}}$. They are thus not defined at $\boxed{(x, y) = (0, 0)}$.

Problem 15

We have that

$$\mathbf{c}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}e^{2t}, \frac{\mathrm{d}}{\mathrm{d}t}\ln|t|, \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{t}\right) = \left(2e^{2t}, \tfrac{1}{t}, -\tfrac{1}{t^2}\right) = \mathbf{F}(e^{2t}, \ln|t|, \tfrac{1}{t}) = \mathbf{F}(\mathbf{c}(t)).$$

Thus, $\mathbf{c}(t)$ is a flow line of the given velocity vector field $\mathbf{F}(x, y, z)$.

Problem 20

We have that

$$\begin{aligned}
\mathbf{c}'(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}a\cos(t) - b\sin(t), \frac{\mathrm{d}}{\mathrm{d}t}a\sin(t) - b\cos(t)\right) \\
&= (-a\sin(t) - b\cos(t), a\cos(t) - b\sin(t)) \\
&= \mathbf{F}(a\cos(t) - b\sin(t), a\sin(t) + b\cos(t)) \\
&= \mathbf{F}(\mathbf{c}(t)).
\end{aligned}$$

Thus, $\mathbf{c}(t)$ is a flow line of the given velocity vector field $\mathbf{F}(x, y, z)$.

3 Section 4.4

Problem 2

If we let $\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$, then

$$\begin{aligned}
\nabla \cdot \mathbf{V}(x, y, z) &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
&= \frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}zx + \frac{\partial}{\partial z}xy \\
&= 1 + 1 + 1 \\
&= \boxed{3}.
\end{aligned}$$

Problem 4

If we let $\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$, then

$$\begin{aligned}
\nabla \cdot \mathbf{V}(x, y, z) &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
&= \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(x + y)^2 + \frac{\partial}{\partial z}(x + y + z)^2 \\
&= 2x + 2(x + y) + 2(x + y + z) \\
&= \boxed{6x + 4y + 2z}.
\end{aligned}$$

Problem 17

If we let $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$, then the scalar curl is as follows:

$$\nabla \times \mathbf{F} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} = \left(\frac{\partial}{\partial x}\cos(x) - \frac{\partial}{\partial y}\sin(x)\right)\mathbf{k} = \left(-\sin(x) - 0\right)\mathbf{k} = \boxed{-\sin(x)\mathbf{k}}.$$

Problem 18

If we let $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$, then the scalar curl is as follows:

$$\nabla \times \mathbf{F} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} = \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}y\right)\mathbf{k} = \left(-1 - 1\right)\mathbf{k} = \boxed{-2\mathbf{k}}.$$

Problem 21

Part (a): If we let $F(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$, then

$$\nabla \times f(x,y,z) = \begin{bmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y}(z+zx) - \frac{\partial}{\partial z}x^2y \\ \frac{\partial}{\partial z}x^2 - \frac{\partial}{\partial x}(z+zx) \\ \frac{\partial}{\partial x}x^2y - \frac{\partial}{\partial y}x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -z \\ 2xy \end{bmatrix}.$$

Therefore,

$$\nabla \cdot (\nabla \times f) = \nabla \cdot \begin{bmatrix} 0 \\ -z \\ 2xy \end{bmatrix} = \frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}z + \frac{\partial}{\partial z}2xy = 0.$$

Part (b): Suppose for contradiction that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Then

$$\frac{\partial f}{\partial x} = x^2$$
$$\frac{\partial f}{\partial y} = x^2y$$
$$\frac{\partial f}{\partial z} = z + zx$$

We may integrate each of these equations with respect to x, y, and z respectively to yield that

$$f = \frac{x^3}{3} + P(y,z)$$
$$f = \frac{x^2y^2}{2} + Q(z,x)$$
$$f = \frac{z^2}{2} + \frac{z^2x}{2} + S(x,y)$$

for some functions $P(y,z)$, $Q(z,x)$, and $S(x,y)$. Observe the term $\frac{z^2x}{2}$ in the third equation; it cannot be cancelled by $S(x,y)$, as S does not contain terms with z. Therefore, $\frac{z^2x}{2}$ is a term of f.

However, the first equation reveals that f cannot contain any terms with both z and x — the only term with an x is $\frac{x^3}{3}$, and all other terms exclusively contain y and z. This implies that $\frac{z^2x}{2}$ cannot be a term of f, which yields the desired contradiction.

We conclude that $\boxed{f \text{ does not exist}}$.

Problem 24

Observe that ∇f maps $\mathbb{R}^3$ to $\mathbb{R}^3$, $\nabla \cdot f$ maps $\mathbb{R}^3$ to $\mathbb{R}$, and $\nabla \times f$ is not defined.

Part (a): The expression $\nabla \times (\nabla f)$ is $\boxed{\text{a meaningful vector-valued function}}$.

Part (b): The expression $\nabla(\nabla \times f)$ is $\boxed{\text{not meaningful}}$, as $\nabla \times f$ is only defined if f maps to $\mathbb{R}^3$.

Part (c): The expression $\nabla \cdot (\nabla f)$ is $\boxed{\text{a meaningful scalar-valued function}}$.

Part (d): The expression $\nabla(\nabla \cdot f)$ is $\boxed{\text{a meaningful vector-valued function}}$.

Part (e): The expression $\nabla \times (\nabla \cdot f)$ is $\boxed{\text{not meaningul}}$, as the curl accepts 3-D vectors and $\nabla \cdot f$ maps to $\mathbb{R}$.

Part (f): The expression $\nabla \cdot (\nabla \times f)$ is $\boxed{\text{not meaningful}}$, as $\nabla \times f$ is only defined if f maps to $\mathbb{R}^3$.

Problem 25

Observe that ∇f does not exist, $\nabla \cdot f$ maps $\mathbb{R}^3$ to $\mathbb{R}$, and $\nabla \times f$ maps $\mathbb{R}$ to $\mathbb{R}^3$.

Part (a): The expression $\nabla \times (\nabla f)$ is $\boxed{\text{not meaningful}}$, as the gradient only accepts scalar-valued functions.

Part (b): The expression $\nabla(\nabla \times f)$ is $\boxed{\text{not meaningful}}$.

Part (c): The expression $\nabla \cdot (\nabla f)$ is $\boxed{\text{not meaningful}}$, as the gradient only accepts scalar-valued functions.

Part (d): The expression $\nabla(\nabla \cdot f)$ is $\boxed{\text{a meaningful vector-valued function}}$.

Part (e): The expression $\nabla \times (\nabla \cdot f)$ is $\boxed{\text{not meaningful}}$, as the curl accepts 3-D vectors and $\nabla \cdot f$ maps to $\mathbb{R}$.

Part (f): The expression $\nabla \cdot (\nabla \times f)$ is $\boxed{\text{a meaningful scalar-valued function}}$.

Problem 31

We have that

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} \frac{1}{x^2+y^2+z^2} \\ \frac{\partial}{\partial y} \frac{1}{x^2+y^2+z^2} \\ \frac{\partial}{\partial z} \frac{1}{x^2+y^2+z^2} \end{bmatrix} = \begin{bmatrix} -\frac{2x}{(x^2+y^2+z^2)^2} \\ -\frac{2y}{(x^2+y^2+z^2)^2} \\ -\frac{2z}{(x^2+y^2+z^2)} \end{bmatrix} = -\frac{2}{(x^2+y^2+z^2)^2} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \nabla \times (\nabla f) &= \nabla \times \left(-\frac{2}{(x^2+y^2+z^2)^2} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \\ &= -\frac{2}{(x^2+y^2+z^2)^2} \left(\nabla \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) - \left(\nabla \frac{2}{x^2+y^2+z^2} \right) \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= -\frac{2}{(x^2+y^2+z^2)^2} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) + \frac{4}{(x^2+y^2+z^2)^2} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right). \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{(x^2+y^2+z^2)^2} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Problem 34

Suppose for contradiction that there exists $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= x^2 + y^2 \\ \frac{\partial f}{\partial y} &= -2xy. \end{aligned}$$

However,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} - 2xy = -2y \neq 2y = \frac{\partial}{\partial y}\left(x^2 + y^2\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

This violates the equality of mixed partial derivatives, implying the given result.

Problem 36

Part (a): We have that

$$\nabla(\mathbf{F}+\mathbf{G}) = \nabla\mathbf{F} + \nabla\mathbf{G} = 0,$$

so the divergence of $\mathbf{F}+\mathbf{G}$ $\boxed{\text{necessarily zero}}$.

Part (b): The divergence of $\mathbf{F}\times\mathbf{G}$ is $\boxed{\text{not necessarily zero}}$.

Problem 37

Part (a): We have that

$$\nabla f = \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$$

Part (d): We have that

$$\mathbf{F}\cdot(\nabla f) = F \cdot \begin{bmatrix} 2xy \\ x^2 \end{bmatrix} = \frac{\partial}{\partial x} 2xy + \frac{\partial}{\partial y} x^2 = 2y.$$

Problem 39

$\boxed{\text{No}}$, it is not true that the curl of a vector field is perpendicular to the vector field. This is because the curl describes the activity of the vectors *around* a vector — this can be rigorously demonstrated using a counterexample.

Problem 40

Part (a): We have that

$$\nabla \times f = \begin{bmatrix} \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z}\left(x^3+y^3\right) \\ \frac{\partial}{\partial z}\left(3x^2y\right) - \frac{\partial}{\partial x} 0 \\ \frac{\partial}{\partial x}\left(x^3+y^3\right) - \frac{\partial}{\partial y}\left(3x^2y\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Part (b): The function $f(x,y) = x^3y + \frac{1}{4}y^4$ is one such function, as

$$\frac{\partial f}{\partial x} = 3x^2y$$

and

$$\frac{\partial f}{\partial y} = x^3 + y^3.$$