MATH-UA 329: Homework 3A

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1 Problem 1

Let $\mathbf{x}$ be the vector in X such that $\|\mathbf{x}\|_X = 1$ and $\|\mathbf{STx}\|_Z = \|\mathbf{ST}\|_{X\to Z}$. The existence of this vector is ensured by Extreme Value Theorem, since $\|\mathbf{ST}\|_{X\to Z}$ is a supremum of the image of a compact set. Thus we have

$$\begin{aligned} \|\mathbf{ST}\|_{X\to Z} &= \|\mathbf{STx}\|_Z \\ &\leq \|\mathbf{S}\|_{Y\to Z}\|\mathbf{Tx}\|_Y \\ &\leq \|\mathbf{S}\|_{Y\to Z}\|\mathbf{T}\|_{X\to Y}\|\mathbf{x}\|_X \\ &= \|\mathbf{S}\|_{Y\to Z}\|\mathbf{T}\|_{X\to Y}. \end{aligned}$$

This completes the proof.

2 Problem 2

2.1 Part (a)

Let the matrix $\mathbf{M}$ have the form

$$\mathbf{M} \stackrel{\text{def}}{=} \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix}$$

For each $i \in \{1, \ldots, m\}$, define the constants $s_{i1}, \ldots, s_{in} \in \{1, -1\}$ such that $s_{ij}M_{ij} = |M_{ij}|$. Let

$$L \stackrel{\text{def}}{=} \max_{1\leq i\leq m} \sum_{j=1}^{n} |M_{ij}| = \max_{1\leq i\leq m} \sum_{j=1}^{n} M_{ij}s_{ij}. \tag{1}$$

We will show that for all $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x}\|_\infty = 1$, we have $\|\mathbf{Mx}\| \leq L$ and equality is attained. So, suppose that $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ has supremum norm 1. Hence $|x_i| \leq 1$ for all i; this yields

$$\mathbf{Mx} = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} M_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} M_{mj}x_j \end{bmatrix}.$$

We deduce that

$$\|\mathbf{Mx}\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} M_{ij}x_j \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |M_{ij}||x_j| \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |M_{ij}| = L.$$

We conclude that L is an upper bound of $\|\mathbf{Mx}\|$ across all $\mathbf{x} \in \mathbb{R}^n$ with supremum norm 1. To demonstrate that L is achieved, let k be an integer in $\{1, \ldots, m\}$ such that the maximum in Equation (1) is achieved. Consider the vector $\mathbf{s} = (s_{k1}, \ldots, s_{kn})$:

$$\mathbf{Ms} = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} s_{k1} \\ \vdots \\ s_{kn} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} M_{1j} s_{kj} \\ \vdots \\ \sum_{j=1}^{n} M_{mj} s_{kj} \end{bmatrix}.$$

Observe that one of the entries of $\mathbf{Ms}$ is

$$\sum_{j=1}^{n} M_{kj} s_{kj} = \sum_{j=1}^{n} |M_{kj}| = L.$$

Thus the supremum norm of $\mathbf{Ms}$ is $|L| = L$ or greater; since $\mathbf{s}$ clearly has supremum norm 1, we proved it must be precisely L. We conclude that

$$\max_{1 \le i \le m} \sum_{j=1}^{n} |M_{ij}| = L = \sup_{\|\mathbf{x}\|_\infty = 1} \|\mathbf{Mx}\|_\infty = \|\mathbf{M}\|_{\infty,\infty}.$$

We now describe all $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x}\|_\infty = 1$ and $\|\mathbf{Mx}\|_\infty = \|\mathbf{M}\|_{\infty,\infty}$. It is quite simple: they are all vectors with components satisfying the following properties. For each $j \in \{1, \ldots, n\}$, select a k such that the maximum of Equation 2 is satisfied.

1. If $M_{kj} = 0$, then the j-th coordinate of $\mathbf{x}$ can be any number from -1 to 1. We impose these bounds on the coordinate so that $\mathbf{x}$ has ∞-norm 1.
2. If $M_{kj} \neq 0$, then the j-th coordinate of $\mathbf{x}$ must be s_{kj} — namely, the element of $\{1, -1\}$ such that $M_{kj} s_{kj} = |M_{kj}|$.

This completes the proof.

2.2 Part (b)

Let $\mathbf{M}$ have the same form as above. Again, define the constant

$$L \stackrel{\text{def}}{=} \max_{1 \le j \le n} \sum_{i=1}^{m} |M_{ij}|. \tag{2}$$

Define $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that $1 = \|\mathbf{x}\|_1 = |x_1| + \cdots + |x_n|$. This yields

$$\mathbf{Mx} = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} M_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} M_{mj}x_j \end{bmatrix}.$$

We deduce through computation that

$$\begin{aligned} \|\mathbf{Mx}\|_1 &= \sum_{i=1}^{m}\sum_{j=1}^{n} |M_{ij}x_j| \\ &= \sum_{j=1}^{n} \left(|x_j| \sum_{i=1}^{m} |M_{ij}| \right) \\ &\leq \sum_{j=1}^{n} \left(|x_j| \max_{1\leq j\leq n} \sum_{i=1}^{m} |M_{ij}| \right) \\ &\leq \left(\max_{1\leq j\leq n} \sum_{i=1}^{m} |M_{ij}| \right) \left(\sum_{j=1}^{n} |x_j| \right) \\ &= \left(\max_{1\leq j\leq n} \sum_{i=1}^{m} |M_{ij}| \right) \\ &= L. \end{aligned}$$

Showing that L is achieved is again quite easy. If k is an integer in the set $\{1, \dots, n\}$ such that the maximum in Equation (2) is achieved, the vector $\mathbf{e}_k$ does the job: since $\mathbf{Me_k}$ is the vector $(M_{1k}, \dots, M_{mk})$, we have

$$\|\mathbf{Me}_k\|_1 = \sum_{i=1}^{m} |M_{mk}| = \max_{1\leq j\leq n} \sum_{i=1}^{m} |M_{ij}| = L.$$

We conclude our identification of the supremum:

$$\max_{1\leq j\leq n} \sum_{i=1}^{m} |M_{ij}| = L = \sup_{\|\mathbf{x}\|_1=1} \|\mathbf{Mx}\|_1 = \|\mathbf{M}\|_{1,1}.$$

As per the vectors $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x}\|_1 = 1$ and $\|\mathbf{Mx}\|_1 = \|\mathbf{M}\|_{1,1}$: if there are distinct values $k_1, \dots, k_\ell$ such that the maximum in Equation (2) is satisfied, we can construct all such vectors $\mathbf{x}$ as follows: The entries in $k_1, \dots, k_\ell$ may be any positive real numbers that sum to 1. All other entries must be zero.

The verification of this fact is relatively easy to deduce from the above equations.

2.3 Part (c)

The key is to examine the sums of the absolute values of all entries in $\mathbf{M}$:

$$m\|\mathbf{M}\|_{\infty,\infty} = m\left(\max_{1\le i\le m}\sum_{j=1}^{n}|M_{ij}|\right) \ge \sum_{i=1}^{m}\sum_{j=1}^{n}|M_{ij}| \ge \max_{1\le j\le n}\sum_{i=1}^{m}|M_{ij}| = \|\mathbf{M}\|_{1,1}.$$

Equality is achieved here if $\mathbf{M}$ contains one nonzero column in which all entries are equal. Similarly,

$$n\|\mathbf{M}\|_{1,1} = n\left(\max_{1\le j\le n}\sum_{i=1}^{m}|M_{ij}|\right) \ge \sum_{i=1}^{m}\sum_{j=1}^{n}|M_{ij}| \ge \max_{1\le i\le m}\sum_{j=1}^{n}|M_{ij}| = \|\mathbf{M}\|_{\infty,\infty}.$$

Equality here is similarly achieved if $\mathbf{M}$ contains one nonzero row in which all entries are equal. We conclude the existence of the desired constants:

$$\boxed{\tfrac{1}{m}\|\mathbf{M}\|_{1,1} \le \|\mathbf{M}\|_{\infty,\infty} \le n\|\mathbf{M}\|_{1,1}}.$$