Rudin: Basic Topology

James Pagan

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Contents

1 Finite, Countable, and Uncountable Sets

1.1 Functions

A **function** or **mapping** from a set A to a set B is an assignment of each element of A to an element of Y. The set A is called the **domain**, B is called the **codomain**, the elements $f(x)$ are called the **values** of f, and the set of all $f(x)$ is called the **image** of f. SUch a relation is notated as $f : A \to B$.

The **inverse image** $f^{-1}(E)$ of a subset $E \subset B$ is the set of all $x \in A$ such that $f(x) \in E$. $f^{-1}(y)$ for $y \in B$ denotes the set of all $x \in A$ such that $f(x) = y$. If $f^{-1}(y)$ contains of one element of A for each $y \in B$, then f is said to be an **bijective** (or one-to-one) mapping of A into B.

If there exists a bijective mapping of A onto B, we say that A and B can be put into **one-to-one correspondence** (or that A and B have the same cardinal number, or that they are equivalent), and we write $A \sim B$. Trivially, this relation has the following properties:

- **Reflexivity**: $A \sim A$.
- **Symmetry**: $A \sim B$ if and only if $B \sim A$.
- **Transitivity**: $A \sim B$ and $B \sim C$ implies that $A \sim C$.

Any relation with these three properties is called an **equivalence relation**. Intuitively, we have that $A \sim B$ if and only if A and B have the "same number of elements".

1.2 Cardinality

Let J_n be the set whose elements are the integers $0, \dots, n-1$; let J be the set consisting of all nonnegative integers. Then for any set A, we say:

- A is **finite** if $A \sim J_n$ for some n.
- A is **infinite** if A is not finite.
- A is **countable** if $A \sim J$.
- A is **uncountable** if A is neither finite nor countable.
- A is **at most countable** if A is neither finite or countable.

Let K be the set of nonnegative integers. Then K has the same cardinal number as J:

$$K = 0, 1, 2, 3, 4, 5, 6, 7, 8, \dots$$
$$J = 0, -1, 1, -2, 2, -3, 3, -4, 4, \dots$$

The function exhibited by the relation above is the following function:

$$\begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n+1}{2} & n \text{ is odd.} \end{cases}$$

A finite set cannot have the same cardinal number as one of its proper subsets. However, this is always possible for infinite sets — for instance, via a subset formed by removing one single element. This is an alternative definition of an infinite set.

1.3 Sequences

A **sequence** is a function f defined on the set $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{\geqslant 0}$ (which we shall denote neutrally by J). If $f(x) = x_n$ for $n \in J$, we often denote the total sequence by x_n or by $(x_0), x_1, x_2, x_3, \ldots$. The values of f are called the **terms** of the sequence. If A is a set and $x_n \in A$ for all $n \in J$, then x_n is called a sequence in A.

A countable set is the range of a bijective function with domain over J; therefore, we may regard all countable functions as the range of a sequence with distinct terms. Intuitively, a countable set can be "arranged in a sequence."

Theorem 1. *Every infinite subset of a countable set A is countable.*

> *Proof.* Suppose $E \subset A$ and E is infinite. Arrange the elements of A into a sequence x_n of distinct elements.
>
> Let n_1 be the smallest integer such that $x_{n_1} \in E$, let n_2 be the smallest integer larger than n_1 such that $x_{n_2} \in E$, and so on. More formally, define n_k recursively:
>
> $$n_k = \min\{m \in \mathbb{Z} \mid x_m \in E, m > \max\{n_1, \ldots, n_n\}\}.$$
>
> The fact E is infinite implies that $x_{n_1}, x_{n_2}, \ldots$ is an infinite sequence with distinct elements.
>
> The function $f(m) = x_{n_m}$ for $m \in \mathbb{Z}_{>0}$ thus obtains a bijection between A and J. We conclude that A is countable. $\square$

In some sense, J is the "smallest infinity;" subsets of J are either finite or countable. Conversely, the Axiom of Choice implies that all uncountable sets have a countable subset.

1.4 Union and Intersection

Let A and Ω be sets, and suppose that with each element α of A, there is a corresponding subset of Ω which we denote by E_α. The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. We sometimes refer to a set of sets as a *collection* or *family* of sets.

The **union** of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If A consists of the integers $1, \ldots n$, we use the notation $S = \bigcup_{i=1}^{n} E_i$ or $S = E_1 \cup \cdots \cup E_n$.

The **intersection** of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in \alpha$ for all $\alpha \in A$. We will use the notation

$$S = \bigcap_{\alpha \in A} E_\alpha,$$

with similar notation above if A is the positive integers or a subset thereof. It is trivial that unions and intersections are associative and commutative.

Theorem 2. *If A, B, and C are sets, the following distributive laws hold:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{2}$$

Proof. For (1), suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$ — we *must* have that $x \in B$ or $x \in C$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$, so in all cases, $x \in (A \cap B) \cup (A \cap C)$.

Conversely, suppose $x \in (A \cap B) \cup (A \cap B)$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$. Therefore, $x \in A$, and $x \in B$ or $x \in C$; we conclude that $x \in A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Identity (2) has a similar proof. $\square$

Several more trivial identities include $A \subset A \cup B$ and $(A \cap B) \subset A$ for all sets A and B. If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$. The empty set is denoted $\varnothing$.

Theorem 3. *Let $E_1, E_2, \ldots$ be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Proof. For each $n \in \mathbb{Z}_{>0}$, let E_n be arrange in a sequence $x_{n1}, x_{n2}, \ldots$, and consider the infinite array.

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

If we "travel in diagonal lines", we produce the sequence

$$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \ldots.$$

This can be formalized by a rather painful argument invoving $\frac{n(n+1)}{2}$. If we accept this construction, it is easy to see that each x_{ij} for $i, j \in \mathbb{Z}_{>0} \in S$ lies in the sequence — and clearly each element of the sequence is a member of S.

Then S and the sequence have the same cardinal number, so S is countable. $\square$

A corollary is that if A is at most countable — and B_α is at most countable for each $\alpha \in A$ — then put

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable. This is because T is equivalent to a subset of S defined in the prior theorem.

Theorem 4. *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ for all $k \in \{1, \ldots, n\}$. Then B is countable.*

Proof. We use induction. Clearly B_1 is countable, as $A = B_1$; we thus proceed to the assumption that B_n is countable. Realize the following one-to-one correspondence between elements of B_{n+1} and pairs of an element of B_n and A:

$$(a_1, \ldots, a_n, a_{n+1}) \iff \{(a_1, \ldots, a_n), a_{n+1}\}$$

Then define $B_\alpha = \{b, \alpha \mid b \in B_n\}$ for each $\alpha \in A$. Clearly $B_\alpha \sim B_n$ for fixed α, so each B_α is countable. Then

$$B_{n+1} \sim \bigcup_{\alpha \in A} B_a.$$

By Theorem 3, the right-hand side is countable. This completes the induction. $\square$

A corollary of this theorem is that $\mathbb{Q}$ is countable, as by the one-to-one correspondence $\frac{a}{b} \iff (a, b)$. Thus $\mathbb{Q} \subset \mathbb{Z}_2$, so $\mathbb{Q}$ is at most countable; $\mathbb{Q}$ must be countable as it contains the countable set $\mathbb{Z}$.

Theorem 5. *The set* A *of all sequences whose elements are the digits* 0 *and* 1 *is uncountable.*

Proof. Suppose $x_1, x_2, \ldots$ is a family of all sequences whose elements are the digits 0 and 1. Consider the element x formed by swapping the first digit of x_1, the second digit of x_2, the third digit of x_3, and so on. We claim that $x \notin \{x_n\}$

Suppose for contradiction that $x \in \{x_n\}$ — namely, that there exists $r \in \mathbb{Z}_{>0}$ such that $x = x_r$. We defined the r-th digit of x to be distinct from x_r, so they cannot be equal — a contradiction.

Thus, any mapping from the positive integers to A will exclude some sequence in A. We conclude that A is not countable. □

This theorem — combined with knowledge of binary notation — implies that the set of all real numbers is uncountable. We will elaborate on this proof later in this document.

2 Metric Spaces

2.1 Definition

A **metric space** is a set X equipped with a function $d : X \times X \to \mathbb{R}$ called a **metric** that satisfies the following four axioms for all $x, y, z \in X$:

1. **Positivity**: $d(x, y) \geqslant 0$, with equality if and only if $x = y$.
2. **Symmetry**: $d(x, y) = d(y, x)$.
3. **Triangle Inequality**: $d(x, z) + d(z, y) \geqslant d(x, y)$.

The elements of X are called **points**.

2.2 Multiple Complex Variables

The most critical metric spaces are $\mathbb{R}^n$ (particularly $\mathbb{R}$) and $\mathbb{C}$. To elaborate upon both simultaneously, these documents will expand upon $\mathbb{C}^n$, equipped with the Euclidean norm for all $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$:

$$\|\mathbf{z}\| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

Theorem 6. *The Triangle Inequality holds for all* $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$*:*

$$\|\mathbf{z}\| + \|\mathbf{w}\| \geqslant \|\mathbf{z} + \mathbf{w}\|.$$

Proof. Let $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$. Assuming the Triangle Inequality in $\mathbb{C}$, we have that

$$\begin{aligned}
\|\mathbf{z}\| + \|\mathbf{w}\| &= \sqrt{\sum_{i=1}^{n} |z_i|^2} + \sqrt{\sum_{i=1}^{n} |w_i|^2} \\
&= \sqrt{\left(\sqrt{\sum_{i=1}^{n} |z_i|^2} + \sqrt{\sum_{i=1}^{n} |w_i|^2}\right)^2} \\
&= \sqrt{\sum_{i=1}^{n} |z_i|^2 + 2\sqrt{\left(\sum_{i=1}^{n} |z_i|^2\right)\left(\sum_{i=1}^{n} |w_i|^2\right)} + \sum_{i=1}^{n} |w_i|^2} \\
&\geqslant \sqrt{\sum_{i=1}^{n} |z_i|^2 + 2\sum_{i=1}^{n} |z_i w_i| + \sum_{i=1}^{n} |w_i|^2} \\
&= \sqrt{\sum_{i=1}^{n} \left(|z_i|^2 + 2|z_i w_i| + |w_i|^2\right)} \\
&= \sqrt{\sum_{i=1}^{n} (|z_i| + |w_i|)^2} \\
&\geqslant \sqrt{\sum_{i=1}^{n} |z_i + w_i|^2} \\
&= \|\mathbf{z} + \mathbf{w}\|.
\end{aligned}$$

Thus, $\mathbb{C}^n$ is a metric space. □

$\mathbb{C}^n$ is also equipped with a dot product that maps vectors to scalars:

$$\mathbf{z} \cdot \mathbf{w} = z_1 \bar{w_1} + \cdots + z_n \bar{w_n}.$$

The properties of the complex dot product are expanded in my Linear Alebra notes: the document AbstractAlgebra/axler6.tex.

In $\mathbb{R}^k$, a **k-cell** is a multi-dimensional analogue of a box, defined if $a_i < b_i$ for all $i \in \{1, \ldots, k\}$ as

$$\{(x_1, \ldots, x_k) \mid a_i \leqslant x_i \leqslant b_i \text{ for all } i \in \{1, \ldots, k\}\}.$$

A set $E \in \mathbb{R}^n$ is **convex** if the line collecting any two points of E lines within E; namely if

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

for all $\mathbf{x}, \mathbf{y} \in E$ and $\lambda \in [0, 1]$. Trivially, open balls are convex.

2.3 Topological Notions

Let X be a metric space. Then the natural topology upon X is as follows for $x \in X$ and $E \subseteq X$:

- An **open ball** of radius $r \in \mathbb{R}_{>0}$ situated at x (denoted $B_r(x)$) is the set of all $y \in X$ such that $d(x, y) < r$.
- x is a **limit point** of E if every open ball at x contains a point inside E.
- x is an **interior point** of E if there exists an open ball N at x such that $N \subseteq X$.
- x is an **isolated point** of E if $p \in E$ and x is not a limit point of E.
- E is an **open set** if all $y \in X$ are interior points.
- E is a **closed set** if it contains all its limit points.
- The **complement** of E (denoted $E^{\complement}$) is the set of all points $x \in X$ such that $x \notin E$.
- E is **perfect** if E is closed and if every point of E is a limit point of E.
- E is **bounded** if there exists an open ball $B_r(x)$ for $x \in X$ such that $E \subseteq B_r(x)$.
- E is **dense** in X if $E = X$ or every point of X is a limit point of E.

Rudin uses the term **neighborhood** to speak of an open ball; I will use it to speak of a set that contains an open ball.

Theorem 7. *Every open ball is an open set.*

Proof. Denote N by the open ball centered at $x \in X$ with radius $r > 0$, and let $y \in N$ — that is, $d(x, y) < r$.

If $x \neq y$, denote M as the open ball centered at y with radius $r - d(x, y)$. If $z \in M$, then $d(z, y) < r - d(x, y)$, so

$$d(z, x) \leqslant d(z, y) + d(y, x) < r - d(x, y) + d(y, x) = r.$$

Hence, $z \in N$ and $M \subseteq N$. Then each $y \in N$ is an interior point; the case $x = y$ is trivial. $\square$

Theorem 8. *If x is a limit point of E, then every open ball at x contains infinitely many points of E.*

Proof. Suppose for contradiction that there exists an open ball N at x that contains only a finite number of points of E. Denote the points $x_1, \ldots, x_n$ as the points of $N \cap E$. Then we define:

$$r = \min\{d(x_1, x), \ldots, d(x_n, x)\}$$

The open ball at x with radius r contains none of these points, and is entirely within N; we deduce it should not contain a point in E. This contradicts the fact x is a limit point. $\square$

Corollary 1. *A finite set has no limit points.*

We will enumerate the topological properties of the following sets. If a property (excluding compactness and connectedness) is not listed, it fails to hold:

1. The set of all complex z such $|z| < 1$ is open and bounded.
2. The set of all complex z such that $|z| \leqslant 1$ is closed, perfect, and bounded.
3. A nonempty finite set is closed and bounded.
4. The set of all integers is closed.
5. The set consisting of the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is bounded.
6. The set consisting of all complex numbers is closed, open, and perfect.
7. The segment $(a, b) \subset \mathbb{R}^1$ is open and bounded.

Theorem 9. *Let* $\{E_\alpha\}$ *be a collection of sets* E_α*. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^{\complement} = \bigcap_\alpha \left(E_\alpha^{\complement}\right)$$

Proof. If $x \in (\bigcup_\alpha E_\alpha)^{\complement}$, then $x \notin \bigcup_\alpha E_\alpha$ and $x \notin E_\alpha$ for all α. Then $x \in E_\alpha^{\complement}$ for all α, so $x \in \bigcap\_E_\alpha^{\complement}$. Hence

$$\left(\bigcup_\alpha E_\alpha\right)^{\complement} \subseteq \bigcap_\alpha \left(E_\alpha^{\complement}\right).$$

Conversely, if $x \in \bigcap\limits_\alpha \left(E_\alpha^{\complement}\right)$, then $x \in E_\alpha^{\complement}$ for each α; then $x \notin E_\alpha$ for each α, and $x \notin \bigcup_\alpha E_\alpha$. Thus $x \in (\bigcup_\alpha E_\alpha)^{\complement}$, so

$$\left(\bigcup_\alpha E_\alpha\right)^{\complement} \supseteq \bigcap_\alpha \left(E_\alpha^{\complement}\right)$$

We conclude that $(\bigcup_\alpha E_\alpha)^{\complement} = \bigcap_\alpha \left(E_\alpha^{\complement}\right)$. □

Theorem 10. E *is an open set if and only if* $E^{\complement}$ *is a closed set.*

Proof. Let E be an open set and let x be a limit point of $E^{\complement}$. Suppose for contradiction that $x \in E$. Then x is an interior point of E, so there exists an open ball N such that $N \subseteq E$; this contradicts the fact that x is a limit point of $E^{\complement}$. We conclude that $x \in E^{\complement}$, so $E^{\complement}$ is closed.

Now, suppose that $E^{\complement}$ is a closed set, and let $x \in E$. Suppose for contradiction that there does not exist an open ball N at x such that $N \subseteq E$. Then x is a limit point; as $E^{\complement}$ is closed, $x \in E^{\complement}$. This contradiction leads us to conclude that x is an interior point, so E is open. □

Corollary 2. F *is a closed set if and only if* $F^{\complement}$ *is open.*

The **natural topology** of metric spaces is defined as follows: A set $U \subseteq X$ is **open** if for all $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(x)$ lies within U.

$$x \in B_\epsilon(x) \overset{\text{def}}{=} \{y \in X \mid d(x,y) < \epsilon\} \subseteq U.$$

Here, again, the novice has the opportunity to practice:

Theorem 11. *The following four results hold:*

1. *For any collection* G_α *of open sets,* $G = \bigcup_\alpha G_\alpha$ *is open.*
2. *For any collection* F_α *of closed sets,* $F = \bigcap_\alpha F_\alpha$ *is closed.*
3. *For any finite collection* $G_1, \ldots, G_n$ *of open sets,* $G = \bigcap_{i=1}^n G_i$ *is open.*
4. *For any finite collection* $F_1, \ldots, F_n$ *if closed sets,* $F = \bigcup_{i=1}^n F_i$ *is closed.*

Proof. For (1): If $x \in G$, then $x \in G_\alpha$ for some index α. As G_α is open, there exists an open ball N at x such that $N \subseteq G_\alpha$; thus $N \subseteq G$, so G is open.

For (2): We take complements. The set $F_\alpha^{\complement}$ are all open sets, so

$$\bigcup_\alpha F_\alpha^{\complement} = \left(\bigcap_\alpha F_\alpha\right)^{\complement}$$

is open. Then $\bigcap_\alpha F_\alpha = F$ is closed.

For (3): If $x \in G$, then $x \in G_i$ for all $i \in \{1, \ldots, n\}$. Then for each $i \in \{1, \ldots, n\}$, there exists an open ball N_i centered at x with radius r_i such that $N_i \subseteq G_i$. Define

$$r = \min\{r_1, \ldots, r_n\},$$

and let N be the open ball of radius N centered at x. Then $N \subseteq N_i \subseteq G_i$ for each $i \in \{1, \ldots, n\}$, so $N \subseteq G$; hence x is an interior point, and G is open.

For (4): We take complements. The sets $F_1^{\complement}, \ldots, F_n^{\complement}$ are open, so

$$\bigcap_{i=1}^n (F_i^{\complement}) = \left(\bigcup_{i=1}^n F_i\right)^{\complement}$$

is open; thus $\bigcup_{i=1}^n F_i$ is closed. It is trivial to further deduce that results (3) and (4) fail for infinite collections of sets. $\square$

Let X be a metric space, E be a set in X, and E' be the set of all limit points of E in X. Then the **closure** if E is the set $\overline{E} = E \cup E'$.

Theorem 12. *Let* X *be a metric space and* $E \subset X$*. Then*

1. $\overline{E}$ *is closed.*
2. $E = \overline{E}$ *if and only if* E *is closed.*
3. $\overline{E} \subseteq F$ *for every closed set* $F \subseteq X$ *such that* $E \subseteq F$*.*

Proof. For (1): Suppose that $x \in \overline{E}^{\complement}$, so $x \notin E$ and $x \notin E'$. Then x is not a limit point of E, so there exists an open ball N_1 at x of radius r_1 disjoint from E — that is, $N_1 \subseteq E^{\complement}$.

Suppose for contradiction that x is a limit point of E'; then for all $\epsilon > 0$, the open ball of radius ϵ at x contains a point of E'. Denoting this point by y, we have $d(x, y) < r$; thus, consider the open ball at y of all z such that

$$d(y, z) < r - d(x, y).$$

As $y \in E'$, y is a limit point of E; thus there exists a $z_0 \in E$ in the open ball defined above. By the Triangle Inequality,

$$d(x, z_0) \leqslant d(x, y) + d(y, z_0) < d(x, y) + r - d(x, y) = r.$$

Thus z_0 lies in the open ball of radius ϵ. We conclude that all open balls centered at x contain a point in E, so x is a limit point of E — a contradiction. We conclude that x is not a limit point of E'; so, there exists an open ball N_2 at x of radius r_2 disjoint from E' — that is, $N_2 \subseteq (E')^{\complement}$.

Defining $r = \min\{r_1, r_2\}$ and N as the open ball of radius r at x, we have $N \subseteq N_1 \subseteq E^{\complement}$ and $N \subseteq N_2 \subseteq (E')^{\complement}$; thus,

$$N \subseteq E^{\complement} \cap \left(E'\right)^{\complement} = \left(E \cup E'\right)^{\complement} = \overline{E}^{\complement}.$$

Hence $\overline{E}^{\complement}$ is an open set, so $\overline{E}$ is closed. For (2): If $E = \overline{E}$, then E contains all of its limit points, so it is closed. If E is a closed set, then $E' \subseteq E$, so $\overline{E} = E \cup E' = E$.

For (3): Suppose F is a closed set in X such that $E \subseteq F$. Then $F' \subseteq F$; since limit points of E are limit points of F, $E' \subseteq F'$ — hence $E' \subseteq F$. We conclude that $\overline{E} = E \cup E' \subseteq F$. $\square$

This implies that $\overline{E}$ is the smallest closed set in X that contains E.

Theorem 13. *Let E be a nonempty set of real numbers which is bounded above, and set $y = \sup E$. Then $y \in \overline{E}$; hence $y \in E$ if E is closed.*

Proof. If $y \in E$, then $y \in \overline{E}$. If $y \notin E$: by the minimality of y, there exists $x \in E$ such that $y - \epsilon < x$ for all $\epsilon > 0$. Hence, open balls at y of arbitrary radius ϵ contain a point in E, so y is a limit point. Then $y \in E' \subseteq \overline{E}$. $\square$

Suppose $E \subseteq Y \subseteq X$. It is important to note that Y is a metric space in its own right under the distance of X; it is possible for E to be an open set in X, but not Y. We say that E is **open relative** to Y if for each $x \in E$ there is $r > 0$ such that $y \in Y$ and $d(x, y) < r$ implies $y \in E$.

Theorem 14. *Suppose $Y \subseteq X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof. Suppose that E is open relative to y; then for each $x \in R$, there is $r_x > 0$ such that $y \in Y$ and $d(x, y) < r_p$ implies $y \in E$. Then define N_x as the open ball centered at $x \in E$ with radius r_x, and consider the set

$$G = \bigcup_{x \in E} N_x.$$

G is an open subset of X by Theorems 7 and 11. It is easy to see that $E \subseteq Y \cap G$; as per the converse, it is clear that $N_x \cap Y \subseteq E$, so performing an infinite union yields $G \cap y \subseteq E$. We conclude that $E = G \cap Y$

The contrary is quite easy to see: if $E = Y \cap G$, then $x \in E$ implies that $x \in Y$ and $x \in N_x$, so there exists $r_x > 0$ such that $d(x, y) < r_x$ and $x \in Y$ implies $x \in E$. By definition, E is open relative to Y. $\square$

An **open cover** of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \bigcup_\alpha G_\alpha$.

3 Compact Sets

3.1 Definition

A subset K of a metric space X is **compact** if every open cover of K contains a *finite* subcover — if for all open covers $\{G_\alpha\}$ of K, there exist finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

The history of these sets was discovered backwards — in $\mathbb{C}^n$, the set K is compact if and only if K is closed and bounded. Thus beautiful properties of compactness were initially attributed to closure and boundedness — only later did mathematicians realize compactness is the proper formulation to generalize these properties to arbitrary metric (and topological) spaces.

All finite sets are clearly compact — we may simply select one open set that contains each element to attain a finite subcovering. Temporarily, let K be compact relative to X if the definition above is satisfied.

3.2 In Metric Spaces

Theorem 15. *Suppose* $K \subseteq Y \subseteq X$*. Then* K *is compact to* X *if and only if* K *is compact relative to* Y

Proof. Suppose that K is compact relative to X, and let $\{G_a\}$ be a collection of sets open relative to Y such that $K \subseteq \bigcup_\alpha G_\alpha$. By Theorem 14, there exist sets H_α open relative to X such that for each α,

$$G_\alpha = H_\alpha \cap Y.$$

As $K \subseteq \bigcap_\alpha H_\alpha$, the sets H_α constitute an open covering of K in X. As K is compact, there exist indicies $\alpha_1, \dots, \alpha_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} H_{\alpha_i}.$$

As $K \subseteq Y$, we have

$$K \subseteq \left(\bigcup_{i=1}^{n} H_{\alpha_i} \right) \cap Y = \bigcup_{i=1}^{n} (H_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} G_{\alpha_i}.$$

G_{α_i} constitute an open covering of K in Y, so K is compact in Y. Now, let us suppose K is compact relative to Y, and let $\{H_\alpha\}$ be an open covering of K in X. Set

$$G_\alpha = H_\alpha \cap Y.$$

for sets G_α open relative to Y. Then as $K \subseteq Y$ and $K \subseteq \bigcup_\alpha H_\alpha$, similar logic applies:

$$K \subseteq \left(\bigcup_{i=1}^{n} H_{\alpha_i} \right) \cap Y = \bigcup_{i=1}^{n} (H_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} G_{\alpha_i}.$$

So G_α is an open covering of K in Y; then there exist finitely indicies such that

$$K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i} \subseteq \bigcup_{i=1}^{n} H_{\alpha_i},$$

so K is compact in X. This concludes the proof. □

Theorem 16. *Any closed subset* F *of a compact set* K *is compact.*

Proof. Suppose $F \subseteq K \subseteq X$, for closed F and compact K. Suppose $\{G_\alpha\}$ is an open cover of F — then $(\bigcup_\alpha G_\alpha) \cup F^{\mathsf{C}}$ is an open cover of K, and thus contains a finite subcover.

If F^{C} is a member of this finite subcover, we may remove it to obtain a finite subcover of F; thus a finite subcollection of $\{G_\alpha\}$ contains F, so F is compact. □

Theorem 17. *Any compact subset* K *of a metric space* X *is closed.*

Proof. We will prove that K^{C} is open — that if $x \in K^{\mathsf{C}}$, there exists an open ball centered at x contained outside K. One x is selected, construct an open covering of K as follows: for any $k \in K$, let N_k be the open ball centered at k with radius $\frac{1}{2}d(x,k)$. Then

$$\bigcup_{k \in K} N_k$$

is an open covering of K; since K is compact, there exist $k_1, \dots, k_n$ such that $K \subseteq N_{k_1} \cup \cdots \cup N_{k_n}$. Then the open ball N at x with radius $\min\{d(x,k_1), \dots, d(x,k_n)\}$ is disjoint from each N_{k_i} (say, by contradiction using the Triangle Inequalty), so

$$N \cap K \subseteq N \cap (N_{k_1} \cup \cdots \cup N_{k_n}) = \varnothing.$$

Thus $N \subseteq K^{\mathsf{C}}$. We deduce that K^{C} is open, so K is closed. □

Corollary 3. *If* F *is closed and* K *is compact, then* $F \cap K$ *is compact.*

Proof. Since K is closed, $F \cap K$ is closed subset of K; thus it is compact. □

3.3 The Heine-Borel Theorem

Theorem 18. *Suppose* $\{K_\alpha\}$ *are compact in* X*. If the intersection of every finite subcollection of* $\{K_\alpha\}$ *is nonempty, then* $\bigcap_\alpha K_\alpha$ *is nonempty.*

Proof. Fix K_1 of $\{K_\alpha\}$; suppose for contradiction that for all $k \in K_1$, there exists α such that $k \notin K_\alpha$. Then $k \in K_a^{\mathsf{C}}$, so $\{K_\alpha^{\mathsf{C}}\}$ forms an open covering of K_1. We deduce the existence of indicies $\alpha_1, \dots, \alpha_n$ such that

$$K_1 \subseteq K_{\alpha_1}^{\mathsf{C}} \cup \cdots \cup K_{\alpha_n}^{\mathsf{C}};$$

or equivalently,

$$K_1^{\mathsf{C}} \supseteq K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}.$$

This yields the desired contradiction, since

$$(K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}) \cap K_1 = K_1^{\mathsf{C}} \cap K_1 = \varnothing.$$

Then there exists $k \in K_1$ such that $k \in K_\alpha$ for all α, so $\bigcap_\alpha K_\alpha$ is nonempty. □

Corollary 4. *If* $\{K_n\}$ *are compact sets such that* $K_n \supseteq K_{n+1}$ *for each* $n \in \mathbb{Z}_{>0}$*, then* $\bigcup_{n=1}^{\infty} K_i$ *is nonempty.*

Theorem 19. *If E is an infinite subset of a compact set K, then E contains a limit point in K.*

Proof. If each $k \in K$ is not a limit point, then there exists an open ball N_k at k of nonzero radius that contains at most one point of E — namely, k itself.

The N_k constitute an open covering of K (and therefore E), yet no finite subcovering can contain each E; thus it cannot contain K. Hence, K cannot be compact.

Taking the contrapositive yields the desired result. □

Theorem 20 (Nested Intervals Theorem)**.** *Suppose $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}$ for all $m \in \mathbb{Z}_{>0}$. Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.*

Proof. Let $I_n = [a_n, b_n]$ for sequences a_n and b_n; define $A = \sup a_n$. Realize that for all $m, n \in \mathbb{Z}_{>0}$,

$$a_n \leqslant a_{m+n} \leqslant b_{m+n} \leqslant b_n.$$

Thus b_n is an upper bound of all a_n, so $A \leqslant b_n$. Since $a_n \leqslant A$, we find that $A \in I_n$ for each $n \in \mathbb{Z}_{>0}$. This concludes the proof. □

Theorem 21. *Suppose $\{I_n\}$ is a sequence of k-cells such that $I_n \supseteq I_{n+1}$ for all $m \in \mathbb{Z}_{>0}$. Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.*

Proof. Let I_n consist of all points $\mathbf{x}_n = (x_{n1}, \ldots, x_{nk})$ such that for $i \in \mathbb{Z}_{>0}$ and $j \in \{1, \ldots, k\}$,

$$a_{ij} < x_{ij} < b_{ij}.$$

For each $j \in \{1, \ldots, k\}$, define $I_{1j}, I_{2j}, \ldots$ as $[a_{1j}, b_{1j}], [a_{2j}, b_{2j}], \ldots$. By Theorem 20, there exists v_j in each interval. The vector $\mathbf{v} = (v_1, \ldots, v_n)$ thus lies inside each k-cell, so $\bigcup_{n=1}^{\infty} I_n$ is nonempty. □

The following proof expands upon my Bolzano-Weierstrauss reasoning found in RealAnalysis/proofs.tex; the construction will thus be simplified for brevity.

Theorem 22. *Every k-cell is compact.*

Proof. Suppose for contradiction that the k-cell I_1 is not compact. Then all open coverings $\{G_\alpha\}$ of I_1 lack a subcollection that covers I.

Then define $c_j = \frac{1}{2}(a_j + b_j)$; the intervals $[a_j, c_j]$ and $[c_j, b_j]$ across all $j \in \{1, \ldots, k\}$ split I into 2^k subcells. At least one of these subcells is not covered by a finite subcollection of $\{G_\alpha\}$; call it I_2.

Repeat this construction on I_2 to optain a subcell I_3 that is not covered by $\{G_\alpha\}$; repeat this process *ad infinitum*.

We obtain a sequence of k-cells $\{I_n\}$ such that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}_{>0}$, each uncovered by any finite subcollection of $\{G_\alpha\}$. Theorem 21 thus applies: there exists

$$x \in \bigcup_{n=1}^{\infty} I_n.$$

Since $\{G_\alpha\}$ covers I_1, there exists some open set G_α that contains x; inside this open set is N_r, an open ball at x of radius r.

Claim 1. *For some* $m \in \mathbb{Z}_{>0}$, *we have* $I_n \subseteq G_\alpha$.

Proof. Realize that I_1 is contained within the open ball at its centroid of the following radius:

$$\delta = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(b_i - a_i)^2}.$$

The relevant proof is long but straightforward, utilizing the Pigeonhole Principle. Thus, I_n is contained within the open ball at its centroid of radius $\delta\,/\,2^{n-1}$. Set $m = \lfloor \log_2\left(\frac{\delta}{r}\right) + 1 \rfloor + 1$. Then

$$\frac{\delta}{2^{m-1}} < \frac{\delta}{2^{\log_2(\delta\,/\,r)}} = \frac{\delta}{\delta\,/\,r} = r.$$

Then if N_m is the open ball with radius m, we conclude that

$$I_m \subseteq N_m \subseteq N_r \subseteq G_\alpha,$$

as required.

This attains the desired contradiction: that I_m cannot be covered by a finite subcollection of $\{G_\alpha\}$, yet it is covered by G_α. We conclude that I_1 must be compact. $\square$

Define a **k-pseudocell** in $\mathbb{C}^n$ as the set of all $\mathbf{z} \in \mathbb{C}^n$ such that $\operatorname{Re}\mathbf{z}$ and $\operatorname{Im}\mathbf{z}$ lie in potentially distinct k-cells. We will use this non-standard definition exclusively for Theorem 23.

Corollary 5. *Suppose* $\{I_n\}$ *is a sequence of* k*-pseudocells such that* $I_n \supseteq I_{n+1}$ *for all* $m \in \mathbb{Z}_{>0}$. *Then* $\bigcap_{n=1}^{\infty} I_n$ *is nonempty.*

Corollary 6. *All* k*-pseudocells are compact.*

Theorem 23 (Heine-Borel)**.** *For $E \subseteq \mathbb{C}^k$, the following conditions are equivalent:*

1. *E is closed and bounded.*
2. *E is compact.*
3. *Every infinite subset of E has a limit point in E.*

Furthermore, (2) *and* (3) *are equivalent in an arbitrary metric space.*

Proof. Suppose (1). As E is bounded, it is a subset of some k-pseudocell. Since E is closed, it is compact by Theorem 16 — thus establishing (2). If we assume (2), we yield (3) by Theorem 19.

Assume that E is not bounded. Then there exists a sequence of vectors $\{\mathbf{z}_n\}$ for $n \in \mathbb{Z}_{>0}$ such that

$$|\mathbf{z}_n| > n.$$

A straightforward argument verifies that no limit point for this sequence exists in E, so (3) is not met.

Assume that E is not closed. Then there exists a vector $\mathbf{z} \notin E$ which is a limit point of E but lies outside of E. Then for each $n \in \mathbb{Z}_{>0}$, there exists a sequence of vectors $\mathbf{z}_n \in E$ such that

$$\|\mathbf{z}_n - \mathbf{z}\| < \frac{1}{n}.$$

The subset $\{\mathbf{z}_n \mid n \in \mathbb{Z}_{>0}\}$ is infinite; we claim its only limit point is $\mathbf{z}$. This is because if $\mathbf{w} \in \mathbb{C}^n$ and $\mathbf{w} \neq \mathbf{z}$,

$$\begin{aligned} \|\mathbf{z}_n - \mathbf{w}\| &\geqslant \|\mathbf{z} - \mathbf{w}\| - \|\mathbf{z}_n - \mathbf{z}\| \\ &\geqslant \|\mathbf{z} - \mathbf{w}\| - \frac{1}{n} \\ &\geqslant \frac{\|\mathbf{z} - \mathbf{w}\|}{2} \end{aligned}$$

for all but finitely many n. Thus the open ball at $\mathbf{w}$ of radius $\frac{1}{2}\|\mathbf{z} - \mathbf{w}\|$ does not contain infinitely many $\mathbf{z}_n$, so it cannot be a limit point — so (3) is not met.

Then if we suppose (3), E must be closed and bounded — implying (1). This concludes the proof. □

Theorem 24 (Weierstrauss)**.** *Every bounded infinite subset of $\mathbb{C}^n$ contains a limit point.*

Proof. All bounded infinite subsets E of $\mathbb{C}^n$ are contained within an n-cell I. Since I is compact by Theorem 22, E has a limit point in E by Theorem 19 (and 23!). □