MATH-UA 140: Assignment 6

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1 Problem 1

It is trivial to see that the column space of A spans $\mathbb{R}^2$, and that the domain of A is $\mathbb{R}^4$, so the nullity of A is 2 by the Fundamental Theorem of Linear Maps. Thus, observe that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - 2 + 1 + 0 \\ -2 + 2 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - 2 - 1 + 1 \\ -4 + 2 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Hence, the null space of A is the column space of the matrix T defined below:

$$T = \begin{bmatrix} 1 & 2 \\ -2 & -2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

The projection of $\mathbf{b}$ into the null space of A is the projection of $\mathbf{b}$ into the column space of T. We now begin computing the projection matrix of T:

$$T^\top T = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}.$$

Now, observe that

$$\frac{1}{35}\begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix}(T^\top T) = \frac{1}{35}\begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 & 0 \\ 0 & 35 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and

$$(T^\top T)\frac{1}{35}\begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 6 & 5 \\ 5 & 10 \end{bmatrix}\begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 & 0 \\ 0 & 35 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$\frac{1}{35}\begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix} = (T^\top T)^{-1}$$

Finally,

$$\begin{aligned} T(T^\top T)^1 T^\top &= \frac{1}{35}\begin{bmatrix} 1 & 2 \\ -2 & -2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & -5 \\ -5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -2 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{35}\begin{bmatrix} 1 & 2 \\ -2 & -2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -10 & 15 & -5 \\ 7 & -2 & -11 & 6 \end{bmatrix} \\ &= \frac{1}{35}\begin{bmatrix} 14 & -14 & -7 & 7 \\ -14 & 24 & -8 & -2 \\ -7 & -8 & 26 & -11 \\ 7 & -2 & -11 & 6 \end{bmatrix}. \end{aligned}$$

We conclude that the projection we seek is

$$\frac{1}{35}\begin{bmatrix} 14 & -14 & -7 & 7 \\ -14 & 24 & -8 & -2 \\ -7 & -8 & 26 & -11 \\ 7 & -2 & -11 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \boxed{\frac{1}{35}\begin{bmatrix} 7 \\ 8 \\ -26 \\ 11 \end{bmatrix}}$$

2 Problem 2

As P and Q are projection matricies, $P^2 = P$ and $Q^2 = Q$, so $P^3 = P$; thus,

$$\begin{aligned}
PR &= P(P-Q)^2 \\
&= P(P^2 - PQ - QP + Q^2) \\
&= P^3 - P^2Q - PQP + PQ^2 \\
&= P - PQ - PQP + PQ \\
&= P - PQP \\
&= P - PQP - QP + QP \\
&= P^3 - PQP - QP^2 + Q^2P \\
&= (P^2 - PQ - QP + Q^2)P \\
&= (P-Q)^2P \\
&= RP.
\end{aligned}$$

3 Problem 3

Part (a): Observe that

$$\begin{aligned}
P_1 &= \mathbf{a}_1(\mathbf{a}_1^\top \mathbf{a}_1)^{-1}\mathbf{a}_1^\top \\
&= \frac{\mathbf{a}_1\mathbf{a}_1^\top}{\|\mathbf{a}_1\|^2} \\
&= \frac{1}{(-1)^2 + 2^2 + 2^2}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \\
&= \boxed{\frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}}
\end{aligned}$$

and

$$
\begin{aligned}
P_2 &= \mathbf{a}_2(\mathbf{a}_2^\top \mathbf{a}_2)^{-1}\mathbf{a}_2^\top \\
&= \frac{\mathbf{a}_2\mathbf{a}_2^\top}{\|\mathbf{a}_2\|^2} \\
&= \frac{1}{2^2+2^2+(-1)^2}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \\
&= \boxed{\frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}}.
\end{aligned}
$$

Multiplying these matricies, we find that

$$
P_1P_2 = \frac{1}{81}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}.
$$

This makes intuitive sense if we recall that $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal. If we project all of 3-D space onto the span of P_1, then project that line to an orthogonal line in the direction of P_2, we should project $\mathbb{R}^3$ to the origin.

Part (b): We first need to calculate:

$$
\begin{aligned}
P_3 &= \mathbf{a}_3(\mathbf{a}_3^\top \mathbf{a}_3)^{-1}\mathbf{a}_3^\top \\
&= \frac{\mathbf{a}_3\mathbf{a}_3^\top}{\|\mathbf{a}_3\|^2} \\
&= \frac{1}{2^2+(-1)^2+2^2}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \\
&= \boxed{\frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}}.
\end{aligned}
$$

We now compute the three projection vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$:

$$\mathbf{p}_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}}$$

$$\mathbf{p}_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{bmatrix}}$$

$$\mathbf{p}_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{bmatrix}}$$

Adding these projections together, we find that

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \begin{bmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{bmatrix} + \begin{bmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{bmatrix} + \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boxed{\mathbf{b}}.$$

Part (c): We have the following:

$$\begin{aligned} P_3 &= \mathbf{a}_3(\mathbf{a}_3^\top \mathbf{a}_3)^{-1}\mathbf{a}_3^\top \\ &= \frac{\mathbf{a}_3\mathbf{a}_3^\top}{\|\mathbf{a}_3\|^2} \\ &= \frac{1}{2^2 + (-1)^2 + 2^2}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \\ &= \boxed{\frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}}. \end{aligned}$$

We now verify the following calculation:

$$\begin{aligned} P_1 + P_2 + P_3 &= \frac{1}{9}\left(\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}\right) \\ &= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I. \end{aligned}$$