Axler: Linear Maps

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1 Vector Space of Linear Maps

1.1 Definition

Let V and W be $\mathbb{F}$-vector spaces. An $\mathbb{F}$-module homomorphism is an **F-linear map** — a function $\mathbf{T} : V \to W$ such that for all $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$,

$$\begin{aligned} \mathbf{T}(\mathbf{u} + \mathbf{v}) &= \mathbf{T}\mathbf{u} + \mathbf{T}\mathbf{v} \\ \mathbf{T}(\lambda \mathbf{v}) &= \lambda \mathbf{T}\mathbf{v}. \end{aligned}$$

By the properties of module homomorphisms, $\mathbf{T}$ maps $\mathbf{0}$ to $\mathbf{0}$. Note that the indices of the following theorem depend on the Axiom of Choice:

Lemma 1 (Linear Map Lemma)**.** *Suppose $(\mathbf{v}_\alpha)$ is a basis of V and $(\mathbf{w}_\alpha)$ is a basis of W. Then there exists a unique linear map $\mathbf{T} : V \to W$ such that $\mathbf{v}_\alpha \rightsquigarrow \mathbf{w}_\alpha$.*

Proof. Define $\mathbf{T}$ as such a linear map: for all $\mathbf{u} \in V$, express $\mathbf{u} = \lambda_1 \mathbf{v}_{\alpha_1} + \cdots + \lambda_n \mathbf{v}_{\alpha_n}$, and define

$$\mathbf{T}\mathbf{u} \quad \overset{\text{def}}{=} \quad \lambda_1 \mathbf{w}_{\alpha_1} + \cdots + \lambda_n \mathbf{w}_{\alpha_n}.$$

$\mathbf{T}$ is well-defined, since representation via a basis is unique. A rather tedious argument demonstrates that it satisfies the additive and multiplicative conditions. Unicity follows from the fact that the valuation of every point is predetermined since $(\mathbf{v}_\alpha)$ and $(\mathbf{w}_\alpha)$ is a basis. In fact, $\mathbf{T}$ is an isomorphism! $\square$

1.2 Algebraic Operations on $\mathcal{L}(V, W)$

The notation $\mathcal{L}(V, W)$ denotes the set of all linear maps from V to W. Like with modules, we can define three operations on this set: if $\mathbf{T}, \mathbf{S} \in \mathcal{L}(V, W)$, then for all $\mathbf{v} \in V$ and $\lambda \in \mathbb{F}$:

1. **Addition**: $\mathbf{T} + \mathbf{S}$ is the unique linear map such that $(\mathbf{T} + \mathbf{S})\mathbf{v} = \mathbf{T}\mathbf{v} + \mathbf{S}\mathbf{v}$.
2. **Scalar Multiplication**: $\lambda\mathbf{T}$ is the unique linear map such that $(\lambda\mathbf{T})\mathbf{v} = \lambda(\mathbf{T}\mathbf{v})$.

Equipped with these operations, $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$. There is a third operation: if $\mathbf{T} \in \mathcal{L}(V, W)$ and $\mathbf{S} \in \mathcal{L}(W, U)$, then for all $\mathbf{v} \in V$ and $\lambda \in \mathbb{F}$:

1. **Multiplication**: $\mathbf{ST}$ is the unique linear map in $\mathcal{L}(V, U)$ such that $(\mathbf{ST})\mathbf{v} = \mathbf{S}(\mathbf{T}\mathbf{v})$.

Hence multiplication is simply a composition of mappings.

2 Null Spaces and Ranges

The kernel and image of an F-module homomorphism $\mathbf{T} \in \mathcal{L}(V, W)$ are given special names when F is a field: the null space and range. For all such $\mathbf{T}$, we define:

1. **Null Space**: $\operatorname{null} \mathbf{T} = \{\mathbf{v} \in V \mid \mathbf{Tv} = \mathbf{0}\}$.
2. **Range**: $\operatorname{range} \mathbf{T} = \{\mathbf{Tv} \mid \mathbf{v} \in V\}$.

As with modules, $\operatorname{null} \mathbf{T}$ is a subspace of V and $\operatorname{range} \mathbf{R}$ is a subspace of W. Furthermore, $\mathbf{T}$ is injective if and only if $\operatorname{null} \mathbf{T} = \mathbf{0}$ and surjective if and only if $\operatorname{range} \mathbf{T} = W$. A bijective linear map is called an **invertible linear map** or **isomorphism**.

Theorem 1 (Fundamental Theorem of Linear Maps)**.** *Let V be finite-dimensional and suppose $\mathbf{T} \in \mathcal{L}(V, W)$. Then $\operatorname{range} \mathbf{T}$ is a finite-dimensional subspace, and*

$$\dim V \;=\; \dim \operatorname{null} \mathbf{T} + \dim \operatorname{range} \mathbf{T}.$$

Proof. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be a basis of $\operatorname{null} \mathbf{T}$; extend it to a basis of V, namely

$$\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n.$$

The proof is finished if we prove that $\mathbf{Tv}_1, \ldots, \mathbf{Tv}_n$ is a basis of $\operatorname{range} \mathbf{T}$.

1. **Spanning**: Select $\mathbf{w} \in V$ arbitrarily; there exist constants λ_i and μ_i such that

   $$\mathbf{w} \;=\; \lambda_1 \mathbf{u}_1 + \cdots + \lambda_m \mathbf{u}_n + \mu_1 \mathbf{v}_1 + \cdots + \mu_n \mathbf{u}_n.$$

   Applying $\mathbf{T}$ to this equation, we find $\mathbf{Tw} = \mu_1 \mathbf{Tv}_1 + \cdots + \mu_n \mathbf{Tv}_n$. We conclude that $\operatorname{range} \mathbf{T} = \operatorname{span}(\mathbf{Tv}_1, \ldots, \mathbf{Tv}_n)$.
2. **Independence**: Follows from the fact that $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V.

This establishes the desired basis: hence $\operatorname{range} \mathbf{T}$ is finite-dimensional, and $\dim V = m + n = \dim \operatorname{null} \mathbf{T} + \dim \operatorname{range} \mathbf{T}$. □

The cardinals $\dim \operatorname{null} \mathbf{T}$ and $\dim \operatorname{range} \mathbf{T}$ are called the **nullity** and **rank** of $\mathbf{T}$ respectively.

Proposition 1. *Let V and W be a finite-dimensional vector spaces.*

1. *If $\dim V < \dim W$, then no linear map $\mathbf{T} \in \mathcal{L}(V, W)$ can be surjective.*
2. *If $\dim V > \dim W$, then no linear map $\mathbf{T} \in \mathcal{L}(V, W)$ can be injective.*

Proof. For (1), we have that

$$\dim \operatorname{range} \mathbf{T} \leqslant \dim \operatorname{range} \mathbf{T} + \dim \operatorname{null} \mathbf{T} = \dim V < \dim W,$$

so $\operatorname{range} \mathbf{T} \neq W$; hence $\mathbf{T}$ is not surjective. Similarly for (2), we have

$$\dim \operatorname{null} \mathbf{T} = \dim V - \dim \operatorname{range} \mathbf{T} \geqslant \dim V - \dim W > 0,$$

so $\operatorname{null} \mathbf{T} \neq 0$ and $\mathbf{T}$ is not injective. □

I do not know whether this result generalizes to infinite-dimensional vector spaces. It requires delicate cardinal arithmetic. However, it does imply this: if V and W are finite dimensional, the existence of invertible $\mathbf{T} \in \mathcal{L}(V, W)$ implies that $\dim V = \dim W$

3 Matrices