

# MATH-UA 329: Honors Analysis II

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# 1 Exposition

MATH-UA 329 expands upon Honors Analysis I and will discuss two topics:

1. The theory of differentiation and integration of multi-variable functions.
2. Measure Theory and Lebesgue integration

The instructor is Sinan Gunturk, available at [gunturk@cims.nyu.edu](mailto:gunturk@cims.nyu.edu). Professor Gunturk's office hours are at WWH 829 in Courant from 2:00-3:30 PM. The TA is Keefer Rowan. The grade distribution is as follows:

- 40%: the final exam.
- 20%: the midterm exam.
- 10-15%: quizzes.
- 15-20%: homework assignments.

The course will not follow one particular textbook; potential textbooks are enumerated on Brightspace, with particular emphasis on Rudin's *Principles of Mathematical Analysis*.

## 2 Metric Spaces

### 2.1 Metric Spaces

#### Definition

A **metric space** is a set  $X$  equipped with a binary mapping  $d : X \times X \rightarrow \mathbb{R}$  called a **metric** such that the following properties are satisfied for all  $x, y, z \in X$ :

1. **Positivity:**  $d(x, y) \geq 0$ , with equality if and only if  $x = y$ .
2. **Symmetry:**  $d(x, y) = d(y, x)$ .
3. **Triangle Inequality:**  $d(x, y) \leq d(x, z) + d(z, y)$ .

Metric spaces generalize the notion of distance to arbitrary sets.

## Examples

1. **Euclidean Distance:** In  $\mathbb{R}$ , the Euclidean distance  $d(x, y) = |x - y|$  is a metric. The complex absolute value is also a metric of  $\mathbb{C}$ .

In general, the Euclidean distance over  $\mathbb{R}^n$  is defined as follows:

$$d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$$

2. **Taxicab Metric:** in  $\mathbb{R}^n$ , the taxicab metric is defined as follows for  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$ :

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

3. **Supremum Distance:** For  $\mathbb{R}^n$ , the  $d_\infty$  metric is as follows:

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max |x_i - y_i| \mid i \in \{1, \dots, n\}.$$

It is denoted by infinity since

$$\lim_{m \rightarrow \infty} d_m(x, y) = \lim_{m \rightarrow \infty} \sqrt[m]{\sum_{i=1}^n |x_i - y_i|^m} = d_\infty(x, y).$$

4. **Discrete Metric** The discrete metric over any set  $X$  is defined as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}.$$

It is easy to verify that the discrete metric is a metric; it is primarily used in examples.

## Open Balls

For a metric space  $X$ , the **open ball** of radius  $r$  centered at  $x \in X$  is the set

$$B_r(\mathbf{x}) = \{y \in X \mid d(x, y) \leq r\}.$$

Here are examples of the unit disc  $B_1(0)$  in the above metrics in  $\mathbb{R}^2$ .

- Under the Euclidean metric, the unit disc is the standard unit circle.

- Under  $d_\infty$ , it is the unit square:

$$B_1(0) = \{\mathbf{y} \in \mathbb{R}^2 \mid \max y_i < 1 \text{ for } i \in \{1, 2\}\}.$$

- Under  $d_1$ , the unit disc is a diamond:

$$B_1(0) = \{\mathbf{y} \in \mathbb{R}^2 \mid |y| \leq 1\}.$$

- Open balls under the discrete metric are defined as follows:

$$B_r(x) = \begin{cases} \{x\} & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases}$$

We encourage the reader to graph these examples for further understanding.

\*

## Continuity

Let  $X$  and  $Y$  be metric spaces. A function  $f : X \rightarrow Y$  is **continuous** at  $x \in X$  if for all  $\epsilon > 0$ , there exists  $\delta$  such that

$$0 < d(x, y) < \delta \implies d(f(x) - f(y)) < \epsilon.$$

$f$  itself is continuous on  $X$  if it is continuous at every  $x \in X$ .

## 2.2 The Metric Space $\mathcal{BC}(X)$

### On Metric Sets

The next section will utilize the following definition:

$$\mathcal{C}(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous on } X\}$$

$\mathcal{C}(X)$  is a vector space over  $\mathbb{R}$  under addition of functions and scalar multiplication. The natural question is: is  $\mathcal{C}(X)$  a metric space? Since a norm on a vector space  $V$  satisfies positivity, the symmetry Triangle Inequality, it induces a metric for  $\mathbf{v}, \mathbf{w} \in V$ :

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

$\mathcal{C}(X)$  does not possess a clear norm. We must define a subspace  $B$  of  $\mathcal{C}(X)$  as follows:

$$\mathcal{BC}(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous and bounded on } X\}.$$

The natural norm of this space is the **supremum norm**, defined as follows:

$$\|f\|_X = \sup_{x \in X} |f(x)|.$$

This norm fashions  $\mathcal{BC}(X)$  into a metric space. The supremum norm encapsulates the concept of uniform convergence quite precisely.

### For General Sets

For any set  $E$ , we may define a similar function space:

$$\mathcal{B}(E) = \{f : E \rightarrow \mathbb{R} \mid f \text{ is bounded on } E\}.$$

This set  $\mathcal{B}(E)$  is a normed vector space under the supremum norm:

$$\|f\|_E = \sup_{x \in E} |f(x)|.$$

**Theorem 1.**  $\mathcal{B}(E)$  is a complete metric space — hence a Banach space.

*Proof.* Suppose  $(f_n)$  is a Cauchy sequence under the supremum norm: that for all  $\epsilon > 0$ , there exists  $N_\epsilon$  such that

$$N_\epsilon \leq i, j \implies \|f_i - f_j\|_E < \epsilon.$$

Then for all  $x \in E$ ,

$$N_\epsilon \leq i, j \implies \|f_i(x) - f_j(x)\|_E < \epsilon.$$

Then the sequence  $f_1(x), f_2(x), \dots$  is a Cauchy sequence in  $\mathbb{R}$  under the supremum norm. Then let  $f$  be the function that maps  $x$  to the limit of  $f_1(x), f_2(x), \dots$ . Clearly,  $f \in \mathbb{R}^E$ . We must demonstrate that this convergence is uniform.

Now, let  $N_\epsilon \leq i, j$ . Then

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| \\ &< |f(x) - f_m(x)| + \epsilon. \end{aligned}$$

Observe that  $\inf_{N_\epsilon \leq m} |f(x) - f_m(x)| = 0$  by the convergence. Therefore, we may take the infimum of both sides of the above equation:

$$\begin{aligned} |f(x) - f_n(x)| &= \inf_{N_\epsilon \leq m} |f(x) - f_m(x)| \\ &< \inf_{N_\epsilon \leq m} |f(x) - f_m(x)| + \epsilon \\ &= \epsilon. \end{aligned}$$

Thus,  $N_\epsilon < i$  implies  $\|f - f_n\| = \sup_{x \in E} |f(x) - f_n(x)|_E < \epsilon$ . We conclude that  $(f_n)$  converges, so  $\mathcal{B}(E)$  is complete.  $\square$

If we would like to prove that  $\mathcal{BC}(X)$  is continuous, we only need demonstrate that the limit of a Cauchy sequence  $(f_n)$  is continuous — which is true, since  $\mathcal{BC}(X)$  is a closed subspace of the complete metric space  $\mathcal{B}(X)$ .

### Uniform Continuity

Let  $f : (X, d_x) \rightarrow (Y, d_y)$  map between metric spaces. Then  $f$  is **uniformly continuous** if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon.$$

This now leads us to define the following two spaces:

$$\begin{aligned}\mathcal{UC}(X) &= \{f : X \rightarrow \mathbb{R} \mid f \text{ is uniformly continuous on } X\}, \\ \mathcal{BUC}(X) &= \{f : X \rightarrow \mathbb{R} \mid f \text{ is bounded and uniformly continuous on } X\}.\end{aligned}$$

Both are subspaces of  $\mathcal{C}(X)$ , but only  $\mathcal{BUC}(X)$  is a normed vector space. The exact same proof as Theorem 1 demonstrates that  $\mathcal{BUC}(X)$  is a Banach space.

**Special case:** When  $X = K$  is compact, all continuous  $f : K \rightarrow \mathbb{R}$  are bounded and uniformly continuous. Hence,

$$\mathcal{C}(K) = \mathcal{BC}(K) = \mathcal{BUC}(K)$$

For non-compact  $X$ , we can only write

$$\mathcal{C}(X) \supset \mathcal{BC}(X) \supset \mathcal{BUC}(X).$$

## 2.3 Modulus of Continuity

### Definition

Let  $f : (X, d_X) \rightarrow (Y, d_Y)$  map between metric spaces. Then the **modulus of continuity**  $\omega_f : [0, \infty) \rightarrow [0, \infty]$  is defined as

$$\omega_f(t) = \sup_{d_X(x_1, x_2) \leq t} d_Y(f(x_1), f(x_2)).$$

The modulus of continuity “measures” the uniform continuity of a function, as observed by the following facts:

**Theorem 2.**  $f$  is uniformly continuous if and only if  $\lim_{t \rightarrow 0^+} \omega_f(t) = 0$ .

*Proof.* The line of reasoning is not particularly difficult; the two expressions communicate the same idea, buried under different notation. For all  $\epsilon > 0$ ,

$$\begin{aligned}
f \text{ is uniformly continuous} &\iff \exists \delta \text{ such that } d_X(x_1, x_2) \leq \delta \text{ implies} \\
&\quad d_Y(f(x_1), f(x_2)) < \epsilon \text{ for all } x_1, x_2 \in X. \\
&\iff \exists \delta \text{ such that } d_X(x_1, x_2) \leq \delta \text{ implies} \\
&\quad \sup(d_Y(f(x_1), f(x_2))) \leq \epsilon \\
&\iff \exists \delta \text{ such that } \sup_{d_X(x_1, x_2) \leq \delta} d_Y(f(x_1), f(x_2)) \leq \epsilon. \\
&\iff \exists \delta \text{ such that } \omega_f(\delta) \leq \epsilon \\
&\iff \exists \delta \text{ such that } t < \delta \text{ implies } |\omega_f(t)| \leq \epsilon \\
&\iff \lim_{t \rightarrow 0^+} \omega_f(t) = 0.
\end{aligned}$$

We replaced  $<$  by  $\leq$  wherever necessary; their presence or absence yields an equivalent  $\epsilon - \delta$  definition of the limit.  $\square$

**Theorem 3.**  $d_Y(f(x_1), f(x_2)) \leq \omega_f(d_X(x_1, x_2))$  for all  $x_1, x_2 \in X$ .

*Proof.* Set  $t = d_X(x_1, x_2)$  when computing the modulus of continuity: we find that

$$d_Y(f(x_1), f(x_2)) \leq \sup_{d_X(y_1, y_2) < d_X(x_1, x_2)} d_Y(f(y_1), f(y_2)) = \omega_f(d_X(x_1, x_2)),$$

as required.  $\square$

To witness examples of the Modulus of Continuity, we encourage the reader to examine its implications for two types of continuity for a function  $f$ :

1. **Hölder Continuity:** If there exists  $\alpha \in (0, 1]$  such that  $\omega_f(t) \leq Ct^\alpha$ . Setting  $\alpha \geq 1$  actually implies  $f$  is constant, by Problem 2 in Homework 1.
2. **Lipschitz Continuity:** If  $\omega_f(t) \leq Ct$  for all  $t \geq 0$ , or if  $d_Y(f(x_1), f(x_2)) \leq Cd_X(x_1, x_2)$  for all  $x \in X$ .

It is clear that all Lipschitz continuous functions are Hölder continuous, by setting  $\alpha = 1$ .

## Piecewise Linear Approximation

Let  $I = [a, b]$  and  $f \in \mathcal{C}(I)$ ; clearly  $f$  is bounded on  $I$ . Let  $L$  be the affine function interpolating  $f$  at the endpoints:  $L(a) = f(a)$  and  $L(b) = f(b)$ .

**Theorem 4.** *If terms are defined like above, then*

$$\|f - L\|_I \leq \omega_f(b - a)$$

*Proof.* Recall the definition of the supremum norm:

$$\|f - L\|_I = \sup_{x \in [a, b]} |f(x) - L(x)|.$$

Let  $L(x) = y$ . Observe that since  $L$  is affine,  $y$  lies between  $L(a)$  and  $L(b)$ ; therefore, between  $f(a)$  and  $f(b)$ . The Intermediate Value Theorem implies the existence of  $c \in [a, b]$  such that  $f(c) = y$ . Then by properties discussed prior,

$$|f(x) - L(x)| = |f(x) - f(c)| \leq \omega_f|c - x| \leq \omega_f(b - a).$$

□

**Corollary 1.** *Every  $f \in \mathcal{C}(I)$  can be approximated uniformly by piecewise linear continuous functions, with arbitrarily small modulus of continuity.*

*Proof.* Relatively trivial: divide  $[b - a]$  into  $n$  segments of length  $\frac{b-a}{n}$ , and observe how  $n \rightarrow \infty$  implies  $\omega_f\left(\frac{b-a}{n}\right) \rightarrow 0$ . □

We eventually conclude that the set of piecewise linear continuous functions on  $I$  is *dense* in  $\mathcal{C}(I)$ . In fact, the set of such functions with rational values for break points is countable.

## 2.4 Separable Metric Spaces

### Definition and Examples

Suppose  $(X, d)$  is a metric space and  $Z \subseteq X$  is a subset. We say  $Z$  is **dense** in  $X$  if any of the equivalent definitions are defined:

- For all  $x \in X$  and  $\epsilon > 0$ , there exists  $z \in Z$  such that  $|x - z| < \epsilon$ .
- For all  $x \in X$  and  $\epsilon > 0$ , then  $B_\epsilon(x) \cap Z \neq \emptyset$ .
- $\bar{Z} = X$ , the closure of  $Z$ .
- For all  $x \in X$ , there exists  $(z_n) \in Z$  such that  $\lim_{n \rightarrow \infty} z_n = x$ .

Density is transitive: suppose  $S \subseteq Z \subseteq X$ , where  $S$  is dense in  $Z$  and  $Z$  is dense in  $X$ ; then  $S$  is dense in  $X$ . The metric space  $(X, d)$  is **separable** if  $X$  has a countable dense subset.



Some examples of dense subsets include:

1.  $\mathbb{R}$  with the Euclidean metric, the countable dense subset being  $\mathbb{Q}$ . We could also consider the dyadic rationals:  $\{\frac{n}{2^m}\}$ .
2.  $\mathbb{C}^n$  with the Euclidean metric, using the same methods as above.
3.  $\mathbb{R}^n$  with the Taxicab metric, using the product metric discussed below.
4.  $\mathcal{C}(I)$ , discussed prior. The set of all piecewise linear continuous functions with rational values at break points — it is countable yet dense.

For two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , the **product metric** is a metric over  $X \times Y$  defined as follows:

$$(d_1 \times d_2)((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2).$$

We could also consider  $\mathbb{R}^n$  to be dense under the product metric, considering  $\mathbb{R}^n$  as a direct product of  $\mathbb{R}^n$ . We would yield the taxicab metric, which is equivalent.

## 2.5 Polynomial Approximation

**Theorem 5** (Weierstrauss Approximation Theorem). *The set of all polynomial functions is dense on  $\mathcal{C}(I)$ : if  $f \in \mathcal{C}(I)$  and for all  $\epsilon > 0$ , there exists a polynomial  $P$  of finite degree such that  $\|f - P\|_I < \epsilon$ .*

*Proof.* The proof was discovered by Bernstein in the 1910s, found in the file RealAnalysis/babyrudin7.tex. □

Thus, polynomials are a countable dense subset of  $I$ .

## 2.6 Normed Vector Space

A **normed vector space** is a complex vector space  $X$  equipped with a mapping  $\|\cdot\| : X \rightarrow \mathbb{R}$  that satisfies the following properties:

1. **Positivity:**  $\|\mathbf{x}\| \geq 0$ , with equality if and only if  $\mathbf{x} = \mathbf{0}$ .
2. **Homogeneity:**  $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$  for all  $\lambda \in \mathbb{C}$ .
3. **Triangle Inequality:**  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

It is clear that such a norm induces a metric on  $X$ . This metric is **translation invariant** — namely, for all  $z \in X$ , we have  $d(x, y) = d(x+z, y+z)$ . In fact, we have  $B_r(x)+z = B_r(x+z)$ .

An **inner product space** is a complex vector space  $X$  equipped with a mapping  $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$  that satisfies the following properties for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$  and  $\lambda \in \mathbb{C}$ :

1. **Conjugate Symmetry:**  $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
2. **Positive-Definiteness:**  $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ , with equality if and only if  $\mathbf{x} = \mathbf{0}$ .
3. **Additivity in First Argument:**  $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ .
4. **Homogeneity in First Argument:**  $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$ .

More theorems about these spaces may be found in axler6.tex. It is clear that by setting  $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle$ , all inner product spaces are normed vector spaces. Hence,

inner product spaces  $\subseteq$  normed vector spaces  $\subseteq$  metric spaces  $\subseteq$  topological spaces.

A complete normed vector space is a **Banach space** while a complete inner product space is a **Hilbert space**. These spaces need not be finite-dimensional.

## 2.7 Equivalent Metrics

Two metrics  $d$  and  $\rho$  on  $X$  are **equivalent** if there exists  $0 < c \leq C < \infty$  such that for all  $x, y \in X$ ,

$$c\rho(x, y) \leq d(x, y) \leq C\rho(x, y).$$

Density is invariant of equivalent metrics; in fact their topologies are the same. A set  $S \subseteq X$  is open under  $d$  if and only if  $S$  is open under  $\rho$ . In particular, metrics in Banach spaces are equivalent if

$$c\|\mathbf{x}\| \leq \|\mathbf{x}\|' \leq C\|\mathbf{x}\|$$

for all  $\mathbf{x} \in X$ . As an example, the Power Mean Inequality yields in  $\mathbb{C}^n$  that

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2 \leq n \|\mathbf{x}\|_\infty.$$

These relations do *not* extend to infinite dimensional vector spaces, like  $\ell_p(\mathbb{N})$  for  $1 \leq p \leq \infty$ . A counterexample is given by  $(1, \dots, 1, 0, 0, \dots)$ . As a reminder, this norm is defined as

$$\|\mathbf{x}\|_p \stackrel{\text{def}}{=} \begin{cases} \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup_{n \in \mathbb{N}} |x_n| & \text{if } p = \infty. \end{cases}$$

Hence  $p$ -norms are not equivalent on spaces of infinite sequences.

Though worth noting, we do have the following:

$$c_{00} \subset \ell_1(\mathbb{N}) \subset \ell^2(\mathbb{N}) \subset c_0 \subset c \subset \ell^\infty(\mathbb{N})$$

All inclusions are clearly proper.

**Theorem 6.** *Let  $X$  be a finite dimensional vector space over  $\mathbb{C}$  (or  $\mathbb{R}^n$ ). Then any two norms on  $X$  are equivalent.*

*Proof.* Let  $\dim X = n$ . We first prove the theorem for  $\mathbb{C}^n$ ; let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be the canonical basis of  $\mathbb{C}^n$ , and suppose  $\|\cdot\|_1 : \mathbb{C}^n \rightarrow [0, \infty)$  is a norm. We prove that  $\|\mathbf{z}\|_1$  is equivalent to the canonical norm  $\|\mathbf{z}\|$ .

Consider the boundary of the unit ball (in the canonical norm) in  $\mathbb{C}^n$ . Since  $\|\cdot\|_1$  is continuous, the Extreme Value Theorem guarantees that there exists  $\mathbf{u}, \mathbf{s}$  with norms 1 such that

$$\|\mathbf{u}\|' = \inf_{\|\mathbf{z}\|=1} \{\|\mathbf{z}\|'\} \quad \text{and} \quad \|\mathbf{s}\|' = \sup_{\|\mathbf{z}\|=1} \{\|\mathbf{z}\|'\}$$

Then for all  $\mathbf{z} \in \mathbb{C}^n$ , the constants  $\|\mathbf{u}\|'$  and  $\|\mathbf{s}\|'$  allow for norm equivalence:

$$\|\mathbf{u}\|' \|\mathbf{z}\| \leq \left\| \frac{\mathbf{z}}{\|\mathbf{z}\|} \right\|' \|\mathbf{z}\| = \|\mathbf{z}\|' = \left\| \frac{\mathbf{z}}{\|\mathbf{z}\|} \right\|' \|\mathbf{z}\| \leq \|\mathbf{s}\|' \|\mathbf{z}\|.$$

We conclude that all norms on  $\mathbb{C}^n$  are equivalent to the canonical norm.

□

Since open sets are the same for equivalent metrics, we obtain that there is only one norm-based topology on  $\mathbb{R}^n$  — the Euclidean topology. This proof fails on  $\ell^p(\mathbb{N})$ , since the unit sphere is not compact. Realize that for all  $\mathbf{e}_i$  for  $i \in \mathbb{Z}_{>0}$ ,

$$\|\mathbf{e}_i - \mathbf{e}_j\| \geq 1.$$

Thus the set of all  $\mathbf{e}_1, \dots$  contains no convergent subsequence, so it is not compact. Thus the Heine-Borel Theorem fails for  $\ell^p(\mathbb{N})$ .

## 2.8 Linear Maps on Normed Vector Spaces

A linear map  $T : (X, d_X) \rightarrow (Y, d_Y)$  between normed vector spaces is continuous if and only if it is continuous at 0.

Also,  $c_{00}$  is not continuous at 0.