Artin: Symmetry

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Contents

1 Isometries

1.1 Definition

An **isometry** of $\mathbb{R}^n$ is a distance preserving map f from $\mathbb{R}^n$ to itself — a map such that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\|f(\mathbf{v}) - f(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\|.$$

Isometries will map figures to congruent figures. It is easy to see that the composition of isometries is an isometry.

1.2 Orthogonal Linear Operators

Theorem 1. *The following three conditions on a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ are equivalent:*

1. *φ is an isometry that fixes the origin: $\varphi(0) = 0$.*
2. *φ preserves dot products: $\varphi(\mathbf{v}) \cdot \varphi(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.*
3. *φ is an orthogonal linear operator.*

Proof. Suppose (1). As φ is an isometry, $\|\varphi(\mathbf{u})\| = \|\mathbf{u}\|$ for all $\mathbf{u} \in V$. Then for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we utilize an identity expressing the dot product as a norm:

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= \frac{\|\mathbf{v} - \mathbf{w}\|^2 - \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2}{2} \\
&= \frac{\|\varphi(\mathbf{v}) - \varphi(\mathbf{w})\|^2 - \|\varphi(\mathbf{v})\|^2 - \|\varphi(\mathbf{w})\|^2}{2} \\
&= \varphi(\mathbf{v}) \cdot \varphi(\mathbf{w}),
\end{aligned}$$

which implies (2). We now utilize the following claim:

Claim 1. *If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{a} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{b}$, then $\mathbf{a} = \mathbf{b}$.*

Proof. Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{b}$. Then

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a}) - 2(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{b}) = 0.$$

Hence, $\mathbf{a} - \mathbf{b} = \mathbf{0}$ and $\mathbf{a} = \mathbf{b}$.

Suppose (2). Let $\mathbf{v}, \mathbf{w}$ be arbitrary vectors in $\mathbb{R}^n$, and define $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Then

$$\begin{aligned}
\varphi(\mathbf{u}) \cdot \varphi(\mathbf{u}) &= \mathbf{u} \cdot \mathbf{u} \\
&= \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \\
&= \varphi(\mathbf{u}) \cdot \varphi(\mathbf{v}) + \varphi(\mathbf{u}) \cdot \varphi(\mathbf{w}) \\
&= \varphi(\mathbf{u}) \cdot (\varphi(\mathbf{v}) + \varphi(\mathbf{w})).
\end{aligned}$$

Similarly, we may deduce that

$$\begin{aligned}
\varphi(\mathbf{u}) \cdot (\varphi(\mathbf{v}) + \varphi(\mathbf{w})) &= \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \\
&= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) \\
&= (\mathbf{v} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{w}) + (\mathbf{w} \cdot \mathbf{w}) \\
&= \varphi(\mathbf{v}) \cdot \varphi(\mathbf{v}) + 2\varphi(\mathbf{v}) \cdot \varphi(\mathbf{w}) + \varphi(\mathbf{w}) \cdot \varphi(\mathbf{w}) \\
&= (\varphi(\mathbf{v}) + \varphi(\mathbf{w})) \cdot (\varphi(\mathbf{v}) + \varphi(\mathbf{w})).
\end{aligned}$$

Piecing these two equalities together, we conclude through our claim that $\varphi(\mathbf{v}+\mathbf{w}) = \varphi(\mathbf{u}) = \varphi(\mathbf{v}) + \varphi(\mathbf{w})$. Thus φ is a linear operator; it is trivial to prove that φ is orthogonal using the images of the canonical basis of $\mathbb{R}^n$, which yields (3).

Assume (3). For all $\mathbf{u} \in \mathbb{R}^n$, let $u_1, \ldots, u_n \in \mathbb{R}^n$ be unique scalars such that $\mathbf{u} = u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n$. Then by the Pythagorean Theorem for inner product spaces,

$$\begin{aligned}
\|\varphi(\mathbf{u})\| &= \|\varphi(u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n)\| \\
&= \|u_1\varphi(\mathbf{e}_1) + \cdots + u_n\varphi(\mathbf{e}_n)\| \\
&= \sqrt{\|u_1\mathbf{e}_1\|^2 + \cdots + \|u_n\mathbf{e}_n\|^2} \\
&= \sqrt{u_1^2 + \cdots + u_n^2} \\
&= \|\mathbf{u}\|.
\end{aligned}$$

For all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we may substitute $\mathbf{u}$ for $\mathbf{v} - \mathbf{w}$ (and use the fact that $\varphi(\mathbf{v} - \mathbf{w}) = \varphi(\mathbf{v}) - \varphi(\mathbf{w})$) to yield that (1) — that φ is an isometry that fixes the origin.

We conclude that (1), (2), and (3) are equivalent conditions. $\square$

We conclude that isometries over $\mathbb{R}^n$ are compositions of an orthogonal linear operator and a translation. More precisely, if f is an isometry and $f(\mathbf{1}) = \mathbf{a}$, then $f = t_{\mathbf{a}}\varphi$, where $t_{\mathbf{a}}$ is a translation and φ is an orthogonal lienar operator.

Theorem 2. *The expression $f = t_{\mathbf{a}}\varphi$ for an isometry is unique.*

Proof. Let f be an isometry. Define $f(\mathbf{0}) = \mathbf{a}$ and define $\varphi = t_{-\mathbf{a}}f$. There are two observations in order:

1. φ is an isometry, since φ is a composition of the two isometries f and $t_{-\mathbf{a}}$.
2. $\varphi(\mathbf{0}) = \mathbf{0}$, since $\varphi(\mathbf{0}) = t_{-\mathbf{a}}f(\mathbf{0}) = t_{-\mathbf{a}}(\mathbf{a}) = \mathbf{0}$.

Theorem 1 thus implies that φ is an orthogonal linear operator; the unicity of $t_{\mathbf{a}}$ is apparent, and the expression $\varphi = t_{-\mathbf{a}}f$ guarantees that φ is unique. $\square$

The composition of two such expressions is defined as follows: if $f = t_{\mathbf{a}}\varphi$ and $g = t_{\mathbf{b}}\psi$ are two isometries, then

$$t_{\mathbf{a}}t_{\mathbf{b}} = t_{\mathbf{a}+\mathbf{b}} \qquad \text{and} \qquad \varphi t_{\mathbf{a}} = t_{\varphi(\mathbf{a})}\varphi;$$

the last expression is verified by $\varphi(t_{\mathbf{a}}(\mathbf{x})) = \varphi(\mathbf{x} + \mathbf{a}) = \varphi(\mathbf{x}) + \varphi(\mathbf{a}) = t_{\varphi(\mathbf{a})}\varphi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

1.3 Properties

Theorem 3. *The set M_n of all isometries of $\mathbb{R}^n$ forms a group under the operation of composition of isometries.*

Proof. We must perform four rather routine calculations:

1. **Closure**: We established earlier that if f and g are isometries, then fg is an isometry
2. **Associativity**: The associativity of compositions of isometries follows from the associativity of funcition composition.
3. **Identity**: The identity mapping $f(\mathbf{x}) = \mathbf{x}$ is trivially an isometry.
4. **Inverse**: For all isometries $f = t_{\mathbf{a}}\varphi$, note that $f^{-1} = (t_{\mathbf{a}}\varphi)^{-1} = (\varphi)^{-1}(t_{\mathbf{a}})^{-1} = \varphi^{-1}t_{-\mathbf{a}} = t_{\varphi^{-1}(-\mathbf{a})}\varphi^{-1}$; as φ^{-1} is an orthogonal linear operator and $t_{\varphi^{-1}(-\mathbf{a})}$ is a translation, f^{-1} is an isometry.

We conclude that M_n is a group. We call the group of all orthogonal operators O_n $\square$

The form $f = t_{\mathbf{a}}\varphi$ depends on our choice of coordinates. If we wish to express f under some coordinate change η, the formula is familiar to Linear Algebra (defining this variant of f as f'):

$$f' = \eta^{-1}f\eta.$$

The determinantn of an orthogonal operator on $\mathbb{R}^n$ is ± 1. The operator is said to be **orientation-preserving** if its determinant is 1 and **orientation-reversing** if its determinant is -1. Rather comically, the maping

$$\sigma : M_n \to \{-1, 1\}$$

that sends an isometry to the determinant of its orthogonal operator is a group homomorphism.

1.4 The Homomorphism $M_n \to O_n$

There is an important homomorphism π defined by dropping the translation of an isometry:

Theorem 4. *The mapping $\pi : M_n \to O_n$ for an isometry $f = t_{\mathbf{a}}\varphi$ defined by $\pi(f) = \varphi$ is a surjective homomorphism. Its kernel is the set $\{t_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{R}^n\}$, which is a normal subgroup of M_n.*

Proof. Suppose that $f = t_{\mathbf{a}}\varphi$ and $g = t_{\mathbf{b}}\psi$ are two isometries. Then

$$\pi(f)\pi(g) = \varphi\psi = \pi(t_{\mathbf{a}} t_{\varphi(\mathbf{b})} \varphi\psi) = \pi(t_{\mathbf{a}}\varphi t_{\mathbf{b}}\psi) = \pi(fg),$$

so π is a homomorphism. The surjectivity of π follows from the fact that $\varphi \in O_n$ implies $\varphi \in M_n$ and $\pi(\varphi) = \varphi$. As for the kernel, $\pi(f) = I$ implies that $f = t_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{R}^n$; the kernel of any homomorphism is a normal subgroup. □

2 Isometries in $\mathbb{R}^2$

2.1 Algebraic Description

To compute in the group M_2, we choose some special isometries as generators and obtain relations between them. There are three generators of interest to us:

1. **Translation**: $t_{\mathbf{a}}$ by a vector $\mathbf{a}$: $t_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$.
2. **Rotation**: ρ_θ by an angle θ about the origin: $\rho_\theta(\mathbf{x}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
3. **Reflection**: r about the $\mathbf{e}_1$ axis: $r(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Theorem 5. *Let f be an isometry in $\mathbb{R}^2$. Then $m = t_{\mathbf{a}}\rho_\theta$ or $m = t_{\mathbf{a}}\rho(\theta)r$, for a uniquely determined vector $\mathbf{a}$ and angle θ, both possibly zero.*

Proof. It remains to be proven that all orthogonal linear operators in $\mathbb{R}^2$ are of the form ρ_θ or $\rho_\theta r$ for unique θ.

Suppose that φ is an orthogonal operator. As its columns must have norm 1, we may define:

$$\mathcal{M}(\varphi) = \begin{bmatrix} \cos(\theta) & \cos(\phi) \\ \sin(\theta) & \sin(\phi) \end{bmatrix},$$

for some $\theta, \phi \in [0, 2\pi)$. The determinant of this matrix must satisfy

$$\begin{vmatrix} \cos(\theta) & \cos(\phi) \\ \sin(\theta) & \sin(\phi) \end{vmatrix} = \cos(\theta)\sin(\phi) - \sin(\theta)\sin(\phi) = \sin(\theta - \phi) \in \{1, -1\}.$$

Thus, $\theta - \phi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. If $\theta - \phi = \frac{3\pi}{2}$, then

$$\mathcal{M}(\varphi) = \begin{bmatrix} \cos(\theta) & \cos\big(\theta - \frac{3\pi}{2}\big) \\ \sin(\theta) & \sin\big(\theta - \frac{3\pi}{2}\big) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

If $\theta - \phi = \frac{\pi}{2}$, then

$$\mathcal{M}(\varphi) = \begin{bmatrix} \cos(\theta) & \cos\big(\theta - \frac{\pi}{2}\big) \\ \sin(\theta) & \sin\big(\theta - \frac{\pi}{2}\big) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}.$$

The first equation is ρ_θ, while the second is $\rho_\theta r$. This completes the proof. $\square$

2.2 Geometric Description

Theorem 6. *Every isometry of the plane has one of the following forms:*

1. ***Orientation-Preserving Isometries****:*
   (a) ***Translation****: A map $t_{\mathbf{a}}$ that sends $\mathbf{x}$ to $\mathbf{x} + \mathbf{a}$.*
   (b) ***Rotation****: Rotation of the plane through a nonzero angle θ about some point.*
2. ***Orientation-Reversing Isometries****:*
   (a) ***Reflection****: A bilateral symmetry around a line ℓ.*
   (b) ***Glide Reflection****: Reflection about a line ℓ, followed by a translation by a nonzero vector parallel to ℓ.*

Proof. We must first prove (1) (b): that if $f = t_{\mathbf{a}}\rho_\theta$ and $\theta \neq 0$, then f is a rotation of the plane through a nonzero angle θ around some point.

Claim 2. *For all isometries $f = t_{\mathbf{a}}\rho_\theta$, where $\theta \neq 0$, there exists a fixed point of f: a vector $\mathbf{x}$ such that $f(\mathbf{x}) = \mathbf{x}$.*

Proof. If $t_{\mathbf{a}}(\rho_\theta(\mathbf{x})) = \mathbf{x}$, then $\rho_\theta(\mathbf{x}) = \mathbf{x} - \mathbf{a}$ and $(\rho_\theta - I)\mathbf{x} = \mathbf{a}$. This equation has a solution if $\rho_\theta - I$ is invertible, so we examine its determinant:

$$\begin{aligned}\det(\rho_\theta - I) &= \begin{vmatrix} \cos(\theta) - 1 & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - 1 \end{vmatrix} \\ &= (\cos(\theta) - 1)^2 + \sin^2(\theta) \\ &= 2 - 2\cos(\theta) \\ &= 2(1 - \cos(\theta)).\end{aligned}$$

This equals zero when $\cos(\theta) = 1$, which occurs exclusively when $\theta = 0$ in the interval $[0, 2\pi)$. This value is excluded; thus $\rho_\theta - I$ has an inverse, and $\mathbf{x} = (\rho_\theta - I)^{-1}\mathbf{a}$. A quick computation verifies that $f(\mathbf{x}) = \mathbf{x}$.

Notice that $t_{-\mathbf{x}} \circ f \circ t_{\mathbf{x}}$ is an isometry and satisfies

$$t_{-\mathbf{x}} \circ f \circ t_{\mathbf{x}}(\mathbf{0}) = t_{-\mathbf{x}} \circ f(\mathbf{x}) = t_{-\mathbf{x}}(\mathbf{x}) = \mathbf{0}.$$

Thus, Theorem 1 guarantees that $t_{-\mathbf{x}} \circ f \circ t_{\mathbf{x}} = \varphi$ for some orthogonal linear operator φ; as φ is orientation-preserving, it is a rotation. Setting $f = t_{\mathbf{x}}\varphi t_{-\mathbf{x}}$ yields that f is a rotation around some point.

Now, we prove facts if f is orientation-reversing: if $f = t_{\mathbf{a}}\rho_\theta r$.

Claim 3. *Isomtries of the form $f = \rho_\theta r$ consist of a reflection across some line through the origin.*

Proof. First, we prove that f is constant along some line through the origin. Let $\mathbf{c}(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi)$. Now, realize that

$$f(\mathbf{c}(0)) = f(\hat{\mathbf{i}}) = -f(i\,\hat{\mathbf{i}}) = -f(\mathbf{c}(\pi)).$$

If $f(\mathbf{c}(0)) = 0$, then f is constant along the x-axis; otherwise, $f(\mathbf{c}(\mathbf{0}))$ and $f(\mathbf{c}(\pi))$ have different signs, so the Intermediate Value Theorem guarantees that $f(\mathbf{c}(t))$ attains a zero at $s \in (0, \pi)$. Then f is constant along $\operatorname{span}(\mathbf{c}(s))$.

Whatever the case, denote this line by ℓ, and change coordiantes such that ℓ is the $\mathbf{e}_1$-axis. Then f is an isometry which fixes the origin, so it is an orthogonal operator. As $\mathbf{e}_1$ is kept on its span, it is trivial that $\mathbf{e}_2$ must be mapped to its reflection across the $\mathbf{e}_1$-axis. This demonstrates that f is the desired reflection.

For an isometry $f = t_{\mathbf{a}\rho_\theta} r$, let ℓ be the line through the origin that $\rho_\theta r$ reflects across; change coordinates such that ℓ is the $\mathbf{e}_1$-axis. Our isometry is now of the form $m = t_{\mathbf{b}} r$, where $\mathbf{b}$ is the vector $\mathbf{a}$ with coordinates changed. For (x_1, x_2), we have that

$$m\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = t_{\mathbf{b}}\left(\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + b_1 \\ -x_2 + b_2 \end{bmatrix}$$

All points of the form $(*, \frac{1}{2}b_2)$ keep their y-coordinate; thus the line $y = \frac{1}{2}b_2$ remains on its span. Via the same logic in our claim, we conclude that the plane is reflected across this line — with a translation by the vector $b_1\mathbf{e}_1$. If $b_1 = 0$, then m is a reflection; otherwise, m has glide symmetry.

This completes the proof of Theorem 6. □

Corollary 1. *The glide line of the isometry $t_{\mathbf{a}}\rho_\theta r$ is parallel to the line of reflection of $\rho_\theta r$.*

By similar logic invoked in Claim 2, the group of isometries that fix a vector $\mathbf{x}$ in the plane is the group $t_{\mathbf{x}} O_2 t_{-\mathbf{x}}$.

3 Finite Groups of O_2 and M_2

The **dihedral group** D_n has order $2n$ and is generated by two elements $x, y \in D_n$ that satisfy the relations

$$x^n = e, \quad y^2 = e, \quad yx = x^{-1}y.$$

The elements of D_n are of the form $x^i y$, where $i \in \{0, \ldots, n-1\}$. When $n = 3$, the dihedral group is isomorphic to the symmetric group: that is,

$$D_3 \cong S_3.$$

This does not hold for $n > 3$, since $|S_n| = n!$ and $|D_n| = 2n$. The dihedral group encapsulates the symmetries of an n-gon.

Theorem 7. *If ρ_θ is rotation by θ and r is reflection across a line through the origin, then $r\rho_\theta = \rho_{-\theta} r$.*

Proof. Change coordinates such that the line is the $\mathbf{e}_1$ axis. Then

$$r\rho_\theta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \rho_{-\theta} r,$$

as desired. □

Theorem 8. *Let G be a finite group of O_2. Then there exists $n \in \mathbb{Z}_{>0}$ such that G is isomorphic to one of the following:*

- *The cyclic group C_n generated by a rotation ρ_θ,*
- *The dihedral group D_n generated by a rotation ρ_θ and a reflection r across the $\mathbf{e}_1$-axis.*

Proof. Let $S = \{\theta \mid \theta \in (0, 2\pi), \rho_\theta \in G\}$. This set must be finite, since $\{\rho_\theta \mid \theta \in S\}$ has the same cardinality and constitutes a subgroup of the finite group G.

If S is nonempty, let $\phi = \min S$. Define n as the order of ϕ as an element of G; then n is the minimum integer such that $n\phi = 2m\pi$ for some $m \in \mathbb{Z}_{>0}$. By the closure of G, the elements

$$\phi, 2\phi, \dots, (n-1)\phi$$

are all elements of S.

> **Lemma 1.** *If $\varphi \in S$, then $\varphi = k\phi$ for some $k \in \{1, \dots, n-1\}$.*
>
> *Proof.* Suppose to the contrary that $\varphi \notin \{\phi, 2\phi, \dots, (n-1)\phi\}$. Then for some $j \in \{0, \dots, n-1\}$,
>
> $$j\phi < \varphi < (j+1)\phi.$$
>
> Adding $(n-j)\phi = -j\phi$ to this inequality yields
>
> $$0 < \varphi - j\phi < \phi.$$
>
> We conclude by the closure of G that $\varphi - j\phi \in S$, contradicting the minimality of ϕ. Thus $\varphi = k\phi$ for some $k \in \mathbb{Z}_{>0}$.

Thus, $S = \{\phi, \dots, (n-1)\phi\}$. We conclude that if G contains no reflection, then

$$G = \{\rho_0, \rho_\phi, \dots, \rho_{(n-1)\phi}\} \cong C_n$$

if G contains a reflection, then a trivial application of Theorem 7 yields that

$$G = \{\rho_0, \dots, \rho_{(n-1)\phi}, r, \dots, \rho_{(n-1)\phi} r\} \cong D_n.$$

If S is empty, then G must contain ρ_0, and may contain terms of the form $\rho_\theta r$ or r for $\theta \in (0, 2\pi)$. We are left with five cases:

1. If $G = \{\rho_0\}$, then $G \cong C_1$.
2. If $G = \{\rho_0, r\}$, then $G \cong C_2 \cong D_1$.
3. If $G = \{\rho_0, \rho_\theta r\}$ for some $\theta \in (0, 2\pi)$, then

$$(\rho_\theta r)^2 = \rho_\theta(r\rho_\theta)r = \rho_\theta(\rho_{-\theta}r)r = \rho_0;$$

   thus, $G \cong C_2 \cong D_1$
4. If G contains r and a term of the form $\rho_\theta r$ for $\theta \in (0, 2\pi)$, then the closure of G yields that

$$\rho_\theta = (\rho_\theta r)(r) \in G.$$

   This contradicts the emptiness of S, implying no such G exists.
5. If G contains two terms of the form $\rho_\theta r$ and $\rho_\phi r$ for distinct angles $\theta, \phi \in (0, 2\pi)$ such that $\theta > \phi$, then the closure of G yields that

$$\rho_{\theta-\phi} = \rho_\theta \rho_{-\phi} rr = \rho_\theta r \rho_\phi r \in G.$$

   This contradicts the emptiness of S, implying no such G exists.

We have discussed all possibile finite groups G of O_2; in each ease, G was isomorphic to C_n or D_n for some $n \in \mathbb{Z}_{>0}$. This completes the proof. □

We could generalize the above result to *any* line — not just the $\mathbf{e}_1$-axis — if we changed coordinates to the line that performs the reflection. Intuitively, we may be rest assured that dihedral groups would coninue possessing $2n$ elements: n for rotations with preserved orientation and n for rotations with reversed orientation.

A subgroup Γ of the additive subgroup $\mathbb{R}^+$ is called **discrete** if there exists $\epsilon > 0$ such that for all nonzero $c \in \Gamma$, we have $|c| \geq \epsilon$.

Theorem 9. *A discrete subgroup of* Γ *of* $\mathbb{R}^+$ *satisfies either* $\Gamma = \{0\}$ *or* $\Gamma = \mathbb{Z}r$ *for some* $r \in \mathbb{R}$.

Proof. If Γ, contains a nonzero element, then let $r = \sup\{\epsilon \mid c \in \Gamma \implies |c| \geq \epsilon\}$. The following lemma is unnecessary, but my analysis-loving heart enjoys the detour:

Lemma 2. *r is an element of Γ.*

Proof. Suppose for contradiction that $r \notin \Gamma$. Then as $\frac{3}{2}r$ is not an lower bound, there exists $a \in \Gamma$ such that $r < |a| < \frac{3}{2}r$. As $|a|$ is not a lower bound, there similarly exists another element $|b|$ such that $r < |b| < |a|$. As Γ is a group, it contains $|a|$ and $|b|$. Thus,

$$0 < |a| - |b| < \frac{3}{2}r - |b| < \frac{3}{2}r - r = \frac{1}{2} < r.$$

$|a| - |b|$ is an element of Γ by its closure; this contradicts the minimality of r. We conclude that r must be an element of Γ.

Thus, r is the smallest positive element of Γ. If we suppose for contradiction that there exists $s \in \Gamma$ such that $s \neq rn$ for all $n \in \mathbb{Z}$, then there exists $min\mathbb{Z}$ such that

$$rm < s < r(m+1).$$

We deduce that

$$0 < s - rm < r;$$

this contradicts the minimality of r, implying that all elements of Γ are of the form rn for $n \in \mathbb{Z}$. Then $G = \mathbb{Z}r$. □

The usage of this lemma dramatically simplifies Lemma 1 in Theorem 8. We now take a minute to extend our theorem about O_2 to any finite subgroup of M_2:

Theorem 10. *Let G be a finite group of isometries in the plane. Then there exists a vector $\mathbf{x}$ such that $g(\mathbf{v}) = \mathbf{v}$ for all $g \in G$.*

Proof. Let $\mathbf{x}$ be any point in the plane: the set $S = \{g(\mathbf{x}) \mid g \in G\}$ is called the **orbit** of $\mathbf{x}$ for the action of G. Any element of G will permute the orbit S; this is because each element of G is injective and G is closed under composition.

Lemma 3. *If $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ is a finite set of points in $\mathbb{R}^n$ with centroid $\mathbf{p}$, then the centroid of $f(S) = \{f(\mathbf{s}_1), \ldots, f(\mathbf{s}_n)\}$ is $f(\mathbf{p})$.*

Proof. Noting that $f = t_{\mathbf{a}}\varphi$ for a translation $t_{\mathbf{a}}$ and an orthogonal linear operator φ, we need only prove that these maps individually map centroids to centroids.

For translations, we have that

$$\begin{aligned} t_{\mathbf{a}}(\mathbf{p}) &= \frac{\mathbf{s}_1 + \cdots + \mathbf{s}_n}{n} + \mathbf{a} \\ &= \frac{(\mathbf{s}_1 + \mathbf{a}) + \cdots + (\mathbf{s}_n + \mathbf{a})}{n} \\ &= \frac{t_{\mathbf{a}}(\mathbf{s}_1) + \cdots + t_{\mathbf{a}}(\mathbf{s}_n)}{n}. \end{aligned}$$

For orthogonal linear operators, we have that

$$\varphi(\mathbf{p}) = \varphi\left(\frac{\mathbf{s}_1 + \cdots + \mathbf{s}_n}{n}\right) = \frac{\varphi(\mathbf{s}_1) + \cdots + \varphi(\mathbf{s}_n)}{n}.$$

Both $t_{\mathbf{a}}$ and φ map centroids to centroids; their composition yields the desired result for all isometries.

Let $\mathbf{v}$ be the centroid of S. All elements of G send S to S, so they send $\mathbf{v}$ to $\mathbf{v}$; we conclude that $g(\mathbf{v}) = \mathbf{v}$ for all $g \in G$. $\square$

Corollary 2. *Let G be a finite subgroup of M_2. Then G is a finite subgroup of O_2 under a translation; if coordiantes are chosen suitably, G becomes one of the groups C_n or D_n for $n \in \mathbb{Z}_{>0}$.*