Algebraic Geometry: Seminar 1

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1 Introduction

The undergraduate Algebraic Geometry seminar will follow two texts:

1. Hartshorne: Algebraic Geometry, and
2. THE RISING SEA: Foundations of Algebraic Geometry.

Both texts can be found on Google. The prior seminar examined the Theory of Varieties — this knowledge is critical to the Algebraic Geometry as a whole, but will not be assumed.

2 Motivation for Schemes

2.1 Spectrum of a Commutative Ring

Inutuitively, there are "three components" to a scheme.

1. Points. It is a function, after al of a Commutative Ringl.
2. A topology, specifically the Zariski topology.
3. The **structure sheaf** of the scheme.

The **spectrum** of a commutative ring denotes the set of all prime ideals of the ring:

$$\operatorname{Spec} R = \{\mathfrak{p} \subseteq R \mid \mathfrak{p} \text{ is a prime ideal of } R\}.$$

The prime ideals constitute the "points" of a scheme. Then what are elements $r \in R$? They are **functions**, that map points $\mathfrak{p}$ to r modulo $\mathfrak{p}$ (notably $\mathfrak{r} \in R\,/\,\mathfrak{p}$). We say that the function r **vanishes** at $\mathfrak{p}$ if $r \in \mathfrak{p}$.

2.2 Examples

Consider $\operatorname{Spec} \mathbb{Z}$. The prime ideals are (p) for prime p. Under the function 8, the prime ideals become $8 \pmod p$.

Consider $\operatorname{Spec}\left(k[\epsilon]\,/\,\epsilon^2\right)$, where k is a field. Consider a point $\mathfrak{p}$ in the spectrum; since $0 \in \mathfrak{p}$, we have $\epsilon^2 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we have $\epsilon \in \mathfrak{p}$ for all points $\mathfrak{p}$ (duh, because ϵ is nilpotent). ϵ vanishes everywhere on Spec, yet it clear doesn't for the ring itself.

Theorem 1. *Let R be a ring. Then* $\operatorname{Spec} R\,/\,\mathfrak{N} \cong \operatorname{Spec} R$.

Proof. prove it yourself dumbass, i think this is in atiyah-macdonald $\square$

Let k be an algebraically closed field, and consider $\operatorname{Spec} k[x_1, \ldots, x_n]$. It is a principal ideal domain; the prime/maximal ideals of this ring are exactly the principal ideals generated by linear terms.

If $\mathfrak{a}$ is an ideal of the ring R, then $\operatorname{Spec}(R / \mathfrak{a}) \subseteq \operatorname{Spec} R$. Any localization $\operatorname{Spec}(S^{-1}R)$ is also an order-preserving subset of $\operatorname{Spec} R$.

Question: Why not maximal ideals? Well, this falls apart when we consider topology: the intersection of maximal ideals is not a maximal ideal.