Axler: Bases

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1 Span and Linear Independence

1.1 Linear Combinations and Span

Let $(\mathbf{v}_\alpha)$ be vectors in an F-vector space V. Any vector of the form $\lambda_1 \mathbf{v}_{\alpha_1} + \cdots + \lambda_n \mathbf{v}_{\alpha_n}$ for $\mathbf{v}_{a_i} \in (\mathbf{v}_\alpha)$ and $\lambda_i \in F$ is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The set of all linear combinations constitutes the **span** of the vectors:

$$\operatorname{span}(\mathbf{v}_\alpha) \quad \stackrel{\text{def}}{=} \quad \{\lambda_1 \mathbf{v}_{\alpha_i} + \cdots + \lambda_n \mathbf{v}_{\alpha_n} \mid n \in \mathbb{Z}_{>0}, \mathbf{v}_{\alpha_i} \in (\mathbf{v}_\alpha), \lambda_i \in F\}.$$

It is quite clear that $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is the smallest subspace of V that contains $(\mathbf{v}_\alpha)$. The vectors **span** V if $\operatorname{span}(\mathbf{v}_\alpha) = V$; if V is spanned by some finite list of vectors, it is **finite-dimensional**. Otherwise, V is **infinite-dimensional**. These are the classical terms for V being a finitely-generated F-module.

1.2 Linear Independence and Bases

A list of vectors $(\mathbf{v}_a)$ in V is **linearly independent** if for every nonempty finite subset $\mathbf{v}_{\alpha_1}, \ldots, \mathbf{v}_{\alpha_n} \in (\mathbf{v}_\alpha)$, the only solution to the equation

$$\lambda_1 \mathbf{v}_{\alpha_1} + \cdots + \lambda_n \mathbf{v}_{\alpha_n} = \mathbf{0}$$

is when $\lambda_1 = \cdots = \lambda_n = 0$. We declare the empty list $\varnothing$ to be linearly independent. A **linearly independent subset** is a list of vectors $(\mathbf{v}_\alpha)$ which are linearly independent.

Lemma 1 (Linear Dependence Lemma)**.** *Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a dependent list of vectors in V. Then there exists $\mathbf{v}_k$ from the list such that*

$$\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n),$$

and if one removes $\mathbf{v}_k$ from the list, the span of the remaining list equals $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Proof. Let $k \geqslant 2$ be the smallest integer such that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is dependent; there exist $\lambda_1, \ldots, \lambda_n$ not all zero such that $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0}$. We must have $\lambda_k \neq 0$, since $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ are dependent; thus

$$\mathbf{v}_k = -\frac{\lambda_1}{\lambda_k}\mathbf{v}_1 - \cdots - \frac{\lambda_{k-1}}{\lambda_k}\mathbf{v}_k,$$

so $\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$. If we have any $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, we can substitute this expression for $\mathbf{v}_k$ into the equation to express $\mathbf{w}$ as a linear combination of the $\mathbf{v}_i$ excluding $\mathbf{v}_k$; hence the span remains unchanged. □

Proposition 1 (Finite-Dimensional Case)**.** *Let* V *be a finite-dimensional vector space. Suppose that* $\mathbf{u}_1, \dots, \mathbf{u}_m$ *is independent in* V *and* $\mathbf{w}_1, \dots, \mathbf{w}_n$ *spans* V*. Then* $m \leqslant n$

Proof. We present an algorithmic proof:

1. **Step 1**: The list $\mathbf{u}_1, \mathbf{w}_1, \dots, \mathbf{w}_n$ must be linearly dependent, since $\mathbf{u}_1$ lies in the span of the $\mathbf{w}_i$. Hence we may remove some $\mathbf{w}_i$ from this list via the Linear Dependence Lemma to attain a new list which spans V.
2. **Step 2**: Now consider the list $\mathbf{u}_2, \mathbf{u}_1, \mathbf{w}_1, \dots, \mathbf{w}_n$. The Linear Dependence Lemma allows us to remove some $\mathbf{w}_i$ to attain a list which spans V.

We continue this process for m steps. Along each step, the Linear Dependence Lemma allows us to pluck out a $\mathbf{w}_i$ — a $\mathbf{v}_i$ is never removed. Thus $m \leqslant n$. □

This theorem generalizes to infinite lengths: suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ spans V. Any infinite independent list $(\mathbf{u}_\beta)$ in V contains an independent sublist with length greater than n — a contradiction. Hence all independent lists in V are finite.

Proposition 2 (Infinite-Dimensional Case)**.** *Let* V *be a vector space. Suppose that* $(\mathbf{u}_\beta)$ *of length* U *is independent in* V *and* $(\mathbf{w}_\alpha)$ *of length* W *spans* V*. Then* $U \leqslant W$*.*

Proof. If one of U and W is finite, the result is implied by Proposition 1. Otherwise — suppose that $(\mathbf{u}_\beta)$ is an independent list of length U in V and $(\mathbf{w}_\alpha)$ is a spanning list of length W in V such that $U, W \geqslant \aleph_0$.

By Corollary 1, the list $(\mathbf{u}_\beta)$ may be extended to become a basis $(\mathbf{v}_\gamma)$. For each $\mathbf{w} \in (\mathbf{w}_\alpha)$, there exists a finite subset $E_\mathbf{w} \subset (\mathbf{v}_\gamma)$ such that

$$\mathbf{w} = \sum_{\mathbf{v} \in E_\mathbf{w}} \lambda_i \mathbf{v}.$$

for $\lambda_i \in F$. By the Axiom of Choice, $\bigcup_{\mathbf{w} \in (\mathbf{w}_\alpha)} E_\mathbf{w}$ has the same cardinality as $(\mathbf{w})_\alpha$.

We claim this union is equal to $(\mathbf{v}_\gamma)$. All $\mathbf{v}_\gamma$ are expressible as linear combination of some $\mathbf{w}_{\alpha_1}, \dots, \mathbf{w}_{\alpha_n}$ — which in turn are a linear combination of finitely many elements in $(\mathbf{v}_\gamma)$. As the elements in $(\mathbf{v}_\gamma)$ are independent, the only possibility is that $\mathbf{v}_\gamma \in E_{\mathbf{w}_{\alpha_i}}$ for some i. Hence $\bigcup_{\mathbf{w} \in (\mathbf{w}_\alpha)} E_\mathbf{w} = (\mathbf{v}_\gamma)$, so

$$U = \left|(\mathbf{u}_\beta)\right| \leqslant |(\mathbf{v}_\gamma)| = |(\mathbf{w}_\alpha)| = W$$

Thus the desired result holds. □

The following result is an easy corollary from Commutative Algebra. We prove it using elementary techniques as well:

Proposition 3. *Every subspace of a finite-dimensional vector space V is finite-dimensional.*

Proof. V is a finitely-generated module over a Noetherian ring, so all submodules of V are finitely generated. If desired without modules, the proof is algorithmic: let $W \subseteq V$ be a subspace. We construct a set of vectors which span V.

1. **Step 1**: If $W = 0$, we are done; otherwise, select some vector $\mathbf{w} \in W$.
2. **Step n**: If $U = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_{n-1})$, then U is finite-dimensional. Otherwise, choose a vector $\mathbf{u}_n \notin W$ such that

$$\mathbf{u}_n \notin \operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}).$$

This set constructs a linearly independent list — the length of which must be finite by Proposition 2. Thus the process must terminate, in which case $\mathbf{w}$ □

2 Bases

A **basis** of V is a list of vectors in V that are linearly independent and span V. If $(\mathbf{v}_\alpha)$ is a basis of V, each vector $\mathbf{w} \in V$ may be written as a unique combination

$$\mathbf{w} \;=\; \lambda_1 \mathbf{v}_{\alpha_1} + \cdots + \lambda_n \mathbf{v}_{\alpha_n}$$

for scalars $\lambda_1, \ldots, \lambda_n \in F$ and indices α_i. We now expand upon Axler by discussing infinite-dimensional vector spaces:

Theorem 1. *All vector spaces V have a basis.*

Proof. We furnish the tools necessary to apply Zorn's Lemma. Let $\mathcal{S}$ be the family of all linearly independent subsets of V, partially ordered by inclusion. Let $\mathcal{T}$ be a totally ordered subset of sets in $\mathcal{S}$.

Claim 1. *The union of all sets in $\mathcal{T}$ is a linearly independent subset of V.*

Proof. Let $B = \bigcup_{A \in \mathcal{T}} A$. We must demonstrate that B is linearly independent; hence, let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be some finite subset of B. Then there exist $A_1, \ldots, A_n \in \mathcal{T}$ such that $\mathbf{v}_i \in A_i$.

Since $\mathcal{T}$ is totally ordered, one of these sets is a maximal element; thus $\mathbf{v}_1, \ldots, \mathbf{v}_n \in A_j$ for some j. Because $A_j \in \mathcal{S}$, it is a linearly independent subset; hence $\mathbf{v}_1, \ldots, \mathbf{v_n}$ are linearly independent. We conclude that B is a linearly independent subset of V — hence, it lies in $\mathcal{S}$.

The set B described above thus functions as an upper bound of $\mathcal{T}$ with respect to inclusion. Zorn's lemma now implies the existence of a maximal subset $M \in \mathcal{S}$.

The maximality of M entails that for all $\mathbf{w} \in V$, we have $\operatorname{span}(M) \cup \{\mathbf{w}\} = \operatorname{span}(M)$. Hence $\mathbf{w} \in \operatorname{span}(M)$; we conclude that $V \subseteq \operatorname{span}(M)$, which entails $V = \operatorname{span}(M)$. Hence M is a basis of V. $\square$

Hence all F-modules are free. Theorem 1 is actually *equivalent* to the Axiom of Choice.

Corollary 1. *Any linearly independent list $(\mathbf{v}_\alpha)$ can be extended to become a basis.*

Proof. The argument follows Theorem 1 precisely — except we define $\mathcal{S}$ as the set of all linearly independent subsets of V *that contains* $(\mathbf{v}_\alpha)$. The argument demonstrates the existence of a basis which contains $(\mathbf{v}_\alpha)$. $\square$

Its sister theorem is proven below:

Corollary 2. *Any spanning list $(\mathbf{v}_\alpha)$ can be reduced to become a basis.*

Proof. In the proof of Theorem 1, let $\mathcal{S}$ be the all the linearly independent subsets of $(\mathbf{v}_\alpha)$. The same argument demonstrates that an element of $\mathcal{S}$ is a basis, as desired. $\square$

If V is finite-dimensional, algorithms can prove the above results by elementary means. The fact that all finite-dimensional vector spaces have a basis follows from Corollary 2.

Proposition 4. *Let $W \subseteq V$ be a subspace. Then there exist a subspace $U \subseteq V$ such that $V \cong W \oplus U$.*

Proof. Let $(\mathbf{w}_\alpha)$ be a basis of W; extend it to become $(\mathbf{v}_\alpha)$, a basis of V. Define $(\mathbf{u}_\gamma) = (\mathbf{v}_\alpha) \setminus (\mathbf{w}_\alpha)$, and let $U = \operatorname{span}(\mathbf{u}_\gamma)$. Two things:

1. Clearly $W + U = V$, since we defined these spaces by splitting the bases.
2. $W \cap U = \mathbf{0}$, since the contrary would provide a nontrivial equation which expresses a linear combination of $(\mathbf{v}_\alpha)$ as $\mathbf{0}$.

We conclude via the results of LinearAlgebra/axler1.tex that $V \cong W \oplus U$. $\square$

3 Dimension

Before we may define dimension, we need the assistance of the following theorem:

Theorem 2 (Dimension Theorem)**.** *Let V be a vector space. All bases of V have the same cardinality.*

Proof. Let $(\mathbf{u}_\alpha)$ and $(\mathbf{w}_\beta)$ be bases of V with cardinalities U and W. We apply Proposition 2 in two ways:

1. Since $(\mathbf{u}_\alpha)$ is independent and $(\mathbf{w}_\beta)$ is spanning, $U \leqslant W$.
2. Since $(\mathbf{w}_\beta)$ is independent and $(\mathbf{u}_\alpha)$ is spanning, $W \leqslant U$.

We conclude that $U = W$. □

The cardinality of s bases of V is called the **dimension** of V. Clearly finite-dimensional vector spaces have finite dimension, and otherwise for infinite-dimensional vector spaces.

Proposition 5. *Suppose $W \subseteq V$ is a subspace. Then* $\dim W \leqslant \mathtt{dim} V$.

Proof. Let $(\mathbf{w}_\beta)$ and $(\mathbf{v}_\alpha)$ be bases of W and V respectively. Observe that $(\mathbf{w}_\beta)$ is independent in V and $(\mathbf{v}_\alpha)$ spans V; hence the result is implied by Proposition 2. □

Unfortunately, the next two results do not generalize to infinite-dimensional vector spaces.

Proposition 6. *Let V be finite-dimensional. Then any independent list or spanning list of of length* $\dim V$ *is a basis.*

Proof. Let $\dim V = n$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ be an independent list. We may extend it to become a basis, yielding a list of length n. Thus it must add no new vectors; $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis. Similarly, if $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ is spanning, it may be reduced to attain a basis — a reduction which eliminates no vectors, so $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis. □

Corollary 3. *Let V be finite dimensional and let $W \subseteq V$ be a subspace. Then $V = W$ if and only if* $\dim V = \dim W$.

For a counterexample, consider the vector space of polynomials in real coefficients and countably many variables: $\mathbb{R}[x_1, x_2, x_3, \ldots]$ The list $x_2, x_3, \ldots$ is independent and the list $x_1, x_2, x_3, \ldots, x_1 + x_2$ spans. Both have cardinality $\aleph_0$, but neither is a basis.