MATH-UA 148: Homework 6

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Contents

1 Section 6A

1.1 Problem 4

Part (a): As V is a real vector space, $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. Then

$$\begin{aligned}
\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u}-\mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u}-\mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.
\end{aligned}$$

Part (b): Suppose that $\|\mathbf{v}\| = \|\mathbf{u}\|$. Then by Part (a),

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0.$$

We conclude that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal.

Part (c): If we translate a rhombus such that one of its verticies is zero, we may represent all four of its verticies in order as $\mathbf{0}$, $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Then its diagonals are $\mathbf{v}-\mathbf{u}$ (or $\mathbf{u}-\mathbf{v}$) and $\mathbf{u}+\mathbf{v}$; by the result of Part (b), these diagonals are orthogonal — and thus, perpendicular under the Euclidean norm.

1.2 Problem 11

Let $\mathbf{v} = \left(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right)$ and $\mathbf{w} = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right)$. Then by Cauchy-Schwarz under the dot product,

$$\begin{aligned}
(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \\
&\geq |\mathbf{v}\cdot\mathbf{w}|^2 \\
&= \left|a\left(\frac{1}{a}\right)+b\left(\frac{1}{b}\right)+c\left(\frac{1}{c}\right)+d\left(\frac{1}{d}\right)\right|^2 \\
&= |1+1+1+1|^2 \\
&= 16,
\end{aligned}$$

as desired.

1.3 Problem 20

Lemma 1. *For all* $\mathbf{a}, \mathbf{b} \in V$,

$$\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2 = \langle 2\mathbf{a}, \mathbf{b}\rangle + \langle 2\mathbf{b}, \mathbf{a}\rangle$$

Proof. We have that

$$\begin{aligned}
\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2 &= \langle \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}\rangle - \langle \mathbf{a}-\mathbf{b}, \mathbf{a}-\mathbf{b}\rangle \\
&= \langle \mathbf{a}+\mathbf{b}, \mathbf{a}\rangle + \langle \mathbf{a}+\mathbf{b}, \mathbf{b}\rangle - \langle \mathbf{a}-\mathbf{b}, \mathbf{a}\rangle + \langle \mathbf{a}-\mathbf{b}, \mathbf{b}\rangle \\
&= \langle (\mathbf{a}+\mathbf{b}) - (\mathbf{a}-\mathbf{b}), \mathbf{a}\rangle + \langle (\mathbf{a}+\mathbf{b}) + \langle \mathbf{a}-\mathbf{b}\rangle, \mathbf{b}\rangle \\
&= \langle 2\mathbf{b}, \mathbf{a}\rangle + \langle 2\mathbf{a}, \mathbf{b}\rangle,
\end{aligned}$$

as required.

Using our lemma, we deduce that

$$\begin{aligned}
&\frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 + \|\mathbf{u}+i\mathbf{v}\|^2 i - \|\mathbf{u}-i\mathbf{v}\|^2 i}{4} \\
&= \frac{\langle 2\mathbf{u}, \mathbf{v}\rangle + \langle 2\mathbf{v}, \mathbf{u}\rangle + i\langle 2\mathbf{u}, i\mathbf{v}\rangle + i\langle 2i\mathbf{v}, \mathbf{u}\rangle}{4} \\
&= \frac{2\langle \mathbf{u}, \mathbf{v}\rangle + 2\langle \mathbf{v}, \mathbf{u}\rangle + (-2i^2)\langle \mathbf{u}, \mathbf{v}\rangle + (2i^2)\langle \mathbf{v}, \mathbf{u}\rangle}{4} \\
&= \frac{2\langle \mathbf{u}, \mathbf{v}\rangle + 2\langle \mathbf{v}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle - 2\langle \mathbf{v}, \mathbf{u}\rangle}{4} \\
&= \frac{4\langle \mathbf{u}, \mathbf{v}\rangle}{4} \\
&= \langle \mathbf{u}, \mathbf{v}\rangle,
\end{aligned}$$

as desired.

1.4 Problem 24

We must demonstrate that $\langle \cdot, \cdot \rangle_1$ satisfies the four criteria to be an inner product:

1. **Conjugate Symmetry**: For all $\mathbf{u}, \mathbf{v} \in V$, $\langle \mathbf{u}, \mathbf{v}\rangle_1 = \langle S\mathbf{u}, S\mathbf{v}\rangle = \overline{\langle S\mathbf{v}, S\mathbf{u}\rangle} = \overline{\langle \mathbf{v}, \mathbf{u}\rangle_1}$.
2. **Positive-Definiteness**: We have for all $\mathbf{v} \in V$ that $\langle \mathbf{v}, \mathbf{v}\rangle_1 = \langle S\mathbf{v}, S\mathbf{v}\rangle \geq 0$. Equality occurs if and only if $S\mathbf{v} = \mathbf{0}$, which occurs exclusively when $\mathbf{v} = \mathbf{0}$ by the injectivity of S.
3. **Additivity in First Argument**: We have for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ that $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle_1 = \langle S(\mathbf{u}+\mathbf{v})\rangle, S(\mathbf{w}) = \langle S\mathbf{u} + S\mathbf{v}, S\mathbf{w}\rangle = \langle S\mathbf{u}, S\mathbf{w}\rangle + \langle S\mathbf{v}, S\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle_1 + \langle \mathbf{v}, \mathbf{w}\rangle_1$.

4. **Homogenity in First Argument**: We have for all $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$ that $\langle \lambda \mathbf{u}, \mathbf{v} \rangle_1 = \langle S(\lambda \mathbf{u}), S\mathbf{v} \rangle = \langle \lambda (S\mathbf{u}), S\mathbf{v} \rangle = \lambda \langle S\mathbf{u}, S\mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle_1$.

Therefore, $\langle \cdot, \cdot \rangle_1$ is an inner product over V.

2 Section 6B

2.1 Problem 2

2.2 Problem 3

2.3 Problem 12

2.4 Problem 16