Atiyah-MacDonald: Rings and Modules of Fractions

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Contents

1 Construction of Rings of Fractions

1.1 Equality of Fractions

Let R be a ring. A **multiplicatively closed subset** $S \subseteq R$ is a subset that contains 1 and is closed under multiplication — that is, if $(S, \times)$ is a submonoid of $(R, \times)$.

Lemma 1. *Define a relation* $\equiv$ *on* $R \times S$ *as follows:*

$$(a, s) \equiv (b, t) \iff (at - bs)x = 0 \text{ for some } x \in S.$$

Then $\equiv$ *is an equivalence relation.*

Proof. Let (a, s), (b, t), and (c, u) be any elements of $R \times S$. We must verify three properties:

1. **Reflexivity**: It is clear that $(a, b) \equiv (a, b)$, since $(ab - ab)1 = 0$.
2. **Symmetry**: The proof is as simple as multiplying by -1:

$$\begin{aligned}(a, s) \equiv (b, t) &\iff (at - bs)x = 0 \text{ for some } x \in S \\ &\iff (bs - at)x = 0 \text{ for some } x \in S \\ &\iff (b, t) \equiv (a, s).\end{aligned}$$

3. **Transitivity**: Suppose $(a, s) \equiv (b, t)$ and $(b, t) \equiv (c, u)$. Then there exists $x, y \in S$ such that

$$(at - bs)x = 0 \qquad \text{and} \qquad (bu - ct)y = 0.$$

   Muliplying these equations by uy and sx respectively and add them: we obtain that $(aut - cts)xy = 0$. Hence $(au - cs)xyt = 0$; since $xyt \in S$ by closure, we find $(a, s) \equiv (c, u)$.

Therefore, $\equiv$ is an equivalence relation. □

The equivalence class of (a, s) is denoted $\frac{a}{s}$ and called a **fraction**. The set $S^{-1}R$ denotes all the equivalence classes on $S \times R$. Observe that we do not impose $0 \notin S$; if S contains zero, then all fractions are equivalent and $S^{-1}R = 0$.

1.2 Operations on Fractions

We endow $S^{-1}R$ with ring structure by defining addition and multiplication of fractions:

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$$
$$\frac{a}{s} \times \frac{b}{t} = \frac{ab}{st}.$$

Lemma 2. *Addition and multiplication of fractions is well-defined.*

Proof. Suppose $\frac{a}{s} = \frac{c}{u}$ and $\frac{b}{t} = \frac{d}{v}$. First, we demonstrate that $\frac{a}{s} + \frac{b}{t} = \frac{c}{u} + \frac{d}{v}$; there exist $x, y \in S$ such that

$$(au - cs)x = 0 \qquad \text{and} \qquad (bv - dt)y = 0. \tag{1}$$

Multiply these equations by tvy and sux respectively and add them: we obtain that $(atuv + bsuv - cvst - dust)xy = 0$. Thus

$$\frac{a}{s} + \frac{b}{s} = \frac{at+bs}{st} = \frac{cv+du}{uv} = \frac{c}{u} + \frac{d}{v}.$$

Demonstrating that $\frac{a}{s} \times \frac{b}{t} = \frac{c}{u} \times \frac{d}{v}$ is a simpler story. Multiplying our equations in (1) by bvy and csx respectively and adding yields $(abuv - cdst)xy = 0$, so

$$\frac{a}{s} \times \frac{b}{s} = \frac{ab}{st} = \frac{cd}{uv} = \frac{c}{u} \times \frac{d}{v};$$

hence addition and multiplication of fractions is well-defined. □

Naturally, $S^{-1}R$ is a commutative ring. The proof of this assertion is identical to the verification that $\mathbb{Q}$ is a field — with one exception. It may not be true that each fraction has a multiplicative inverse.

Corollary 1. *If R is an integral domain and $S = R \setminus \{0\}$, then $S^{-1}R$ is the field of fractions of R.*