# MATH-UA 129: Homework 3

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## Contents



## 1 Section 2.4

### 1.1 Problem 6

**Part (a)**: For $t \in \mathbb{R}$, the parametrization is trivially

$$\boxed{\mathbf{v} = (1,2,3) + t(-2,0,7)}.$$

**Part (b)**: For $t \in \mathbb{R}$, the parametrization we seek is trivially

$$\boxed{\mathbf{v} = (t, t^2)}.$$

**Part (c)**: For $t \in \mathbb{R}$, define the fractional part of $t$ by $\{t\} = t - \lfloor t \rfloor$. The parametrization we seek is given by a piecewise function:

$$\mathbf{v} = \begin{cases} (\{t\}, 0) & \text{If } \lfloor t \rfloor \equiv 0 \bmod 4 \\ (1, \{t\}) & \text{If } \lfloor t \rfloor \equiv 1 \bmod 4 \\ (1 - \{t\}, 1) & \text{If } \lfloor t \rfloor \equiv 2 \bmod 4 \\ (0, 1 - \{t\}) & \text{If } \lfloor t \rfloor \equiv 3 \bmod 4 \end{cases}$$

It is trivial to verify that:

- $\mathbf{v}$ attains all the vectors on the side $(0, 0)$ to $(0, 1)$ when $0 \leq t \leq 1$,
- $\mathbf{v}$ attains all the vectors on the side $(0, 1)$ to $(1, 1)$ when $1 \leq t \leq 2$,
- $\mathbf{v}$ attains all the vectors on the side $(1, 1)$ to $(1, 0)$ when $2 \leq t \leq 3$,
- $\mathbf{v}$ attains all the vectors on the side $(1, 0)$ to $(0, 0)$ when $3 \leq t \leq 4$.

Therefore, the curve of $v$ traced by all $0 \leq t \leq 4$ is the box we desire; $t$-values outside this range simply retrace the box, ensured by the modulo 4 in the definition of $\mathbf{v}$.

**Part (d)**: We claim the ellipse traced by $\frac{x^2}{9} + \frac{y^2}{25} = 1$ may be parametrized by the path $\mathbf{c}(t) = (3\cos(t), 5\sin(t))$ across all $t \in \mathbb{R}$. Observe that for all reals $t$,

$$\frac{(3\cos(t))^2}{9} + \frac{(5\sin(t))^2}{25} = \cos^2(t) + \sin^2(t) = 1,$$

so all $\boxed{\mathbf{c}(t) = (3\cos(t), 5\sin(t))}$ lie on the ellipse. We wish to establih the converse — that every point on the ellipse may be parametrized by our path.

Let $(x_0, y_0)$ lie on the ellipse such that $\frac{x_0^2}{9} + \frac{y_0^2}{25} = 1$. Observe that $x_0^2 < 9$, so $-3 < x_0 < 3$. Then we may define $t_0 \in [0, 2\pi)$ such that two conditions hold: $3\cos(t_0) = x_0$ and that $5 \sin t_0$ and $y_0$ have the same sign. Hence,

$$1 = \frac{x_0^2}{9} + \frac{y_0^2}{25} = \frac{(3\cos(t_0))^2}{9} + \frac{y_0^2}{25} = \cos^2(t_0) + \frac{y_0^2}{25},$$

so

$$y_0^2 = 25 - 25\cos^2(t_0) = 25\sin^2(t_0).$$

Then as we defined $t_0$ such that $y_0$ and $5\sin(t_0))$ have the same sign, we find that $y_0 = 5\sin(t_0)$; hence, $(x_0, y_0) = (3\cos(t_0), 5\sin(t_0)) = \mathbf{c}(t_0)$. Every point on the ellipse may thus be paramtrized by our path; this completes the proof.

### 1.2 Problem 11

The tangent vector to this path is

$$\mathbf{c}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}e^t, \frac{\mathrm{d}}{\mathrm{d}t}\cos(t)\right) = \boxed{(e^t, -\sin(t))}.$$

### 1.3 Problem 18

Observe that $\mathbf{c}(0) = (1, 0, 0)$. The tangent vector to the path across all $t$ is

$$\mathbf{c}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\cos^2(t), \frac{\mathrm{d}}{\mathrm{d}t}3t - t^2, \frac{\mathrm{d}}{\mathrm{d}t}t\right) = (-2\sin(t)\cos(t), 3 - 2t, 1).$$

Therefore, $\mathbf{c}'(0) = (0, 3, 1)$, and the equation of the tangent line for $t \in \mathbb{R}$ is

$$\boxed{\mathbf{v} = (1, 0, 0) + t(0, 3, 1)}.$$

### 1.4 Problem 23

**Part (a)**: Observe that

$$\mathbf{c}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\cos(t), \frac{\mathrm{d}}{\mathrm{d}t}\sin(t), \frac{\mathrm{d}}{\mathrm{d}t}t^2\right) = (-\sin(t), \cos(t), 2t).$$

Thus, $\mathbf{c}'(4\pi) = (-\sin(4\pi), \cos(4\pi), 2(4\pi)) = (0, 1, 8\pi)$. The magnitude of this vector is the speed of the particle at $t_0 = 4\pi$; namely,

$$\sqrt{0^2 + 1^2 + (8\pi)^2} = \boxed{\sqrt{64\pi^2 + 1}}.$$

**Part (b)**: We seek to solve the equation $\mathbf{c}'(t) \cdot \mathbf{c}(t) = 0$; namely,

$$(-\cos(t)\sin(t)) + (\sin(t)\cos(t)) + (2t^3) = 0,$$

or $2t^3 = 0$ and $t = 0$. Checking this solution, we find that

$$\mathbf{c}'(0) \cdot \mathbf{c}(0) = (-\sin(0), \cos(0), 2(0)) \cdot (\cos(0), \sin(0), (0)^2) = (0, 1, 0) \cdot (1, 0, 0) = 0.$$

Thus, $\mathbf{c}'(0)$ is orthogonal to $\boxed{\mathbf{c}(t) \text{ at } t = 0 \text{ only, so no}}$.

**Part (c)**: Observe that $\mathbf{c}(4\pi) = (1, 0, 16\pi^2)$. Then via our work in Part (a), the equation of the tangent line for $t \in \mathbb{R}$ is

$$\mathbf{v} = \mathbf{c}(4\pi) + t\mathbf{c}'(4\pi) = (1, 0, 16\pi^2) + t(0, 1, 8\pi).$$

**Part (d)**: This line will intersect the $xy$-plane when $z = 0$; namely, when

$$16\pi^2 + 8\pi t = 0,$$

or when $\boxed{t = -2\pi}$, or at $(1, 0, 16\pi^2) - 2\pi(0, 1, 8\pi) = \boxed{(1, -2\pi, 0)}$.

# 2 Section 2.5

## 2.1 Problem 3b

By considering $f \circ \mathbf{c}$ as a function of $t$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \mathbf{c}) = \frac{\mathrm{d}}{\mathrm{d}t} e^{(3t^2)t^3} = \frac{\mathrm{d}}{\mathrm{d}t} e^{3t^5} = 15t^4 e^{3t^5}.$$

We now seek to use the Chain Rule: observe that

$$\begin{aligned} \frac{\partial}{\partial x} e^{xy} &= y e^{xy} \\ \frac{\partial}{\partial y} e^{xy} &= x e^{xy} \\ \mathbf{c}'(t) &= (6t, 3t^2). \end{aligned}$$

Then via the Chain Rule, the derivative of $f \circ \mathbf{c}$ is

$$\begin{bmatrix} t^3 e^{3t^5} & 3t^2 e^{3t^5} \end{bmatrix} \begin{bmatrix} 6t \\ 3t^2 \end{bmatrix} = \begin{bmatrix} t^3 e^{3t^5}(6t) + 3t^2 e^{3t^5}(3t^2) \end{bmatrix} = \begin{bmatrix} 15t^4 e^{3t^5} \end{bmatrix},$$

which matches our prior computation.

## 2.2 Problem 5

We have that

$$\nabla(fg) = \begin{bmatrix} \frac{\partial}{\partial x} fg \\ \frac{\partial}{\partial y} fg \\ \frac{\partial}{\partial z} fg \end{bmatrix} = \begin{bmatrix} f\left(\frac{\partial}{\partial x} g\right) + g\left(\frac{\partial}{\partial x} f\right) \\ f\left(\frac{\partial}{\partial y} g\right) + g\left(\frac{\partial}{\partial y} f\right) \\ f\left(\frac{\partial}{\partial z} g\right) + g\left(\frac{\partial}{\partial z} f\right) \end{bmatrix} = f \begin{bmatrix} \frac{\partial}{\partial x} g \\ \frac{\partial}{\partial y} g \\ \frac{\partial}{\partial z} g \end{bmatrix} + g \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \\ \frac{\partial}{\partial z} f \end{bmatrix} = f\nabla g + g\nabla f.$$

## 2.3 Problem 6

Define $g : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$g(\rho, \theta, \phi) = \begin{bmatrix} x(\rho, \theta, \phi) \\ y(\rho, \theta, \phi) \\ z(\rho, \theta, \phi) \end{bmatrix} = \begin{bmatrix} \rho\cos(\theta)\sin(\phi) \\ \rho\sin(\theta)\sin(\phi) \\ \rho\cos(\theta) \end{bmatrix}.$$

We then have that $f(x, y, z) = f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) = f \circ g$, so via the Chain Rule,

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \end{bmatrix}.$$

Expanding this out, we find that

$$\begin{aligned}
\frac{\partial f}{\partial \rho} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho} \\
&= \left(\frac{\partial f}{\partial x}\right)\cos(\theta)\sin(\phi) + \left(\frac{\partial f}{\partial y}\right)\sin(\theta)\sin(\phi) + \left(\frac{\partial f}{\partial z}\right)\cos(\phi), \\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} \\
&= -\left(\frac{\partial f}{\partial x}\right)\rho\sin(\theta)\sin(\phi) + \left(\frac{\partial f}{\partial y}\right)\rho\cos(\theta)\sin(\phi), \\
\frac{\partial f}{\partial \phi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi} \\
&= \left(\frac{\partial f}{\partial x}\right)\rho\cos(\theta)\cos(\phi) + \left(\frac{\partial f}{\partial y}\right)\rho\sin(\theta)\cos(\phi) - \left(\frac{\partial f}{\partial z}\right)\rho\sin(\phi),
\end{aligned}$$

as desired.

### 2.4 Problem 11

**Part (a)**: The path we desire is

$$\begin{aligned}
f \circ \mathbf{c} &= f(\cos(t), \sin(t), t) \\
&= (3\sin(t) + 2, \cos^2(t) + \sin^2(t), \cos(t) + t^2) \\
&= (3\sin(t) + 2, 1, \cos(t) + t^2).
\end{aligned}$$

The velocity vector of this path is

$$\mathbf{c}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}3\sin(t) + 2, \frac{\mathrm{d}}{\mathrm{d}t}1, \frac{\mathrm{d}}{\mathrm{d}t}\cos(t) + t^2\right) = (3\cos(t), 0, -\sin(t) + 2t).$$

At $\pi$, this computes to $\boxed{(-3, 0, 2\pi)}$.

**Part (b)**: We have that $\mathbf{c}(\pi) = (-1, 0, \pi)$; as it is trivial that $\mathbf{c}'(t) = (-\sin(t), \cos(t), 1)$, we find that $\mathbf{c}'(\pi) = (0, -1, 1)$. Now,

$$\mathbf{D}f = \begin{bmatrix} \frac{\partial}{\partial x}3y + 2 & \frac{\partial}{\partial y}3y + 2 & \frac{\partial}{\partial z}3y + 2 \\ \frac{\partial}{\partial x}x^2 + y^2 & \frac{\partial}{\partial y}x^2 + y^2 & \frac{\partial}{\partial z}x^2 + y^2 \\ \frac{\partial}{\partial x}x + z^2 & \frac{\partial}{\partial y}x + z^2 & \frac{\partial}{\partial z}x + z^2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 2x & 2y & 0 \\ 1 & 0 & 2z \end{bmatrix}.$$

At $(x, y, z) = (-1, 0, \pi)$, this matrix is

$$\begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}.$$

**Part (c)**: We have that

$$\mathbf{D}f(-1,0,\pi)(\mathbf{c}'(\pi)) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2\pi \end{bmatrix}.$$

## 2.5 Problem 16

We have that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2\right)^{-\frac{1}{2}} = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}(2x) = -\frac{x}{\sqrt{(x^2+y^2)^3}},$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2+y^2\right)^{-\frac{1}{2}} = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}(2y) = -\frac{y}{\sqrt{(x^2+y^2)^3}}.$$

The gradient of $f$ is thus

$$\nabla f = \begin{bmatrix} -\frac{x}{\sqrt{(x^2+y^2)^3}} \\ -\frac{y}{\sqrt{(x^2+y^2)^3}} \end{bmatrix}.$$

## 2.6 Problem 17

*NOTE: In all cases, we assume that $f$ is a real-valued function.*

**Part (a)**: Define $t(x,y) = x$, so that $h(x,y) = f(t(x,y), u(x,y))$; the Chain Rule gives that

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}.$$

This comes from the matrix multiplication in the general case of the Chain Rule:

$$\begin{bmatrix} \frac{\partial f}{\partial t} & \frac{\partial f}{\partial u} \end{bmatrix} \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

**Part (b)**: Define $t(x) = x$ so that $h(x) = f(t(x), u(x), v(x))$; the Chain Rule gives that

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

This comes from the matrix multiplication in the general case of the Chain Rule:

$$\begin{bmatrix} \frac{\partial f}{\partial t} \frac{\partial f}{\partial u} \frac{\partial f}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$

**Part (c)**: The Chain Rule gives that

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}.$$

This comes from the matrix multiplication in the general case of the Chain Rule:

$$\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}.$$

## 2.7 Problem 32

We have that

$$\mathbf{D}g = \begin{bmatrix} \frac{\partial}{\partial u} e^u & \frac{\partial}{\partial v} e^u \\ \frac{\partial}{\partial u} u + \sin(v) & \frac{\partial}{\partial v} u + \sin(v) \end{bmatrix} = \begin{bmatrix} e^u & 0 \\ 1 & \cos(v) \end{bmatrix}$$

$$\mathbf{D}f = \begin{bmatrix} \frac{\partial}{\partial x} xy & \frac{\partial}{\partial y} xy & \frac{\partial}{\partial z} xy \\ \frac{\partial}{\partial x} yz & \frac{\partial}{\partial y} yz & \frac{\partial}{\partial z} yz \end{bmatrix} = \begin{bmatrix} y & x & 0 \\ 0 & z & y \end{bmatrix}.$$

Then via the Chain Rule,

$$\mathbf{D}(g \circ f) = \begin{bmatrix} e^{xy} & 0 \\ 1 & \cos(yz) \end{bmatrix} \begin{bmatrix} y & x & 0 \\ 0 & z & y \end{bmatrix} = \begin{bmatrix} ye^{xy} & xe^{x} & 0 \\ y & x + z\cos(yz) & y\cos(yz) \end{bmatrix}.$$

At $(x, y, z) = (0, 1, 0)$, this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$