MATH-UA 129: Homework 10

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Contents

1 Section 7.3

Problem 7

Part (a): As a cross section along the xy-plane should yield circles, as the function is unbounded along the z-axis, and as the cone is used in Part (d), the answer must be $\boxed{\textbf{(iii)}}$.

Part (b): The similarity of this function to a unit ball in spherical coordinates — with stretches of the x and y coordinates — indicates that the answer is an ellipsoid. The correct graph is thus $\boxed{\textbf{(i)}}$

Part (c): As a cross section along the xz-plane should yield a parabola, the answer is $\boxed{\textbf{(ii)}}$

Part (d): This is a common parametrization: that of a cone. The answer is $\boxed{\textbf{(iv)}}$.

Problem 8

Part (a): As this is the only problem with a constricted domain — and as we should expect the answer to resemble a circle — the answer is $\boxed{\textbf{(i)}}$.

Part (b): As the z-coordinate is bounded above by 4, the answer should be $\boxed{\textbf{(ii)}}$.

Part (c): As all components of the output vector are linear, we should expect the result to be a plane — so the answer is $\boxed{\textbf{(ii)}}$.

Part (d): By process of elimination, the answer should be $\boxed{\textbf{(iv)}}$.

Problem 9

The surface is the $\boxed{\text{unit ball}}$ in spherical coordinates. Therefore, a unit normal to the ball is $\boxed{(\cos(v)\sin(u), \sin(v)\sin(u), \cos(u))}$ itself.

Problem 15

As we seek to parametrize a function, the answer is clearly $\Phi(u, v) = (u, v, 3u^2 + 8uv)$. An easy calculation verifies that $(1, 0)$ maps to $(1, 0, 3)$. Now, as

$$\mathbf{T}_u = (1, 0, 6u + 8v) \qquad \text{and} \qquad \mathbf{T}_v = (0, 1, 8u),$$

the tangent plane should be given by

$$\begin{aligned}\mathbf{v} &= (1,0,3) + t(1,0,6) + s(0,1,8)\\ &= (1+t, s, 3+6t+8s),\end{aligned}$$

which is equivalent to the plane $\boxed{6x + 8y - z = 3}$.

Problem 22

Part (a): Let a point on the image of Φ be $(a\sin(u)\cos(v), b\sin(u)\sin(v), c\cos(u))$ for $0 \le u \le \pi$ and $0 \le v \le 2\pi$, where $b < a$. Then

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \frac{(a\sin(u)\cos(v))^2}{a^2} + \frac{(b\sin(u)\sin(v))^2}{b^2} + \frac{(c\cos(u))^2}{c^2}\\ &= \sin^2(u)\cos^2(v) + \sin^2(u)\sin^2(v) + \cos^2(u)\\ &= \sin^2(u)(\cos^2(v) + \sin^2(v)) + \cos^2(u)\\ &= \sin^2(u) + \cos^2(u)\\ &= 1,\end{aligned}$$

as desired.

Part (b): We have that

$$\begin{aligned}\mathbf{T}_u &= (a\cos(u)\cos(v), b\cos(u)\sin(v), -c\sin(u))\\ \mathbf{T}_v &= (-a\sin(u)\sin(v), b\sin(u)\cos(v), 0).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{T}_u \times \mathbf{T}_v = &\begin{vmatrix} \hat{\mathbf{i}} & a\cos(u)\cos(v) & -a\sin(u)\sin(v) \\ \hat{\mathbf{j}} & b\cos(u)\sin(v) & b\sin(u)\cos(v) \\ \hat{\mathbf{k}} & -c\sin(u) & 0 \end{vmatrix}\\ =&(bc\sin^2(u)\cos(v))\,\hat{\mathbf{i}} + (ac\sin^2(u)\sin(v))\,\hat{\mathbf{j}}\\ &+ (ab\sin(u)\cos(u)\cos^2(v) + ab\sin(u)\cos(u)\sin^2(v))\,\hat{\mathbf{k}}\\ =&(bc\sin^2(u)\cos(v))\,\hat{\mathbf{i}} + (ac\sin^2(u)\sin(v))\,\hat{\mathbf{j}} + (ab\sin(u)\cos(u))\,\hat{\mathbf{k}}.\end{aligned}$$

As this vector is never zero within the given region, the surface is regular at all points.

2 Section 7.4

Problem 10

We compute the area *for a sphere of radius* 1 by an integral — one must break this integral into two parts: a sector of a sphere and a cone. We have that

$$\begin{aligned}\int_0^{\frac{\sqrt{2}}{2}} \pi z^2 \,\mathrm{d}z + \int_{\frac{\sqrt{2}}{2}}^1 \pi(\sqrt{1-z^2})^2 \,\mathrm{d}z &= \pi\left[\frac{z^3}{3}\right]_0^{\frac{\sqrt{2}}{2}} + \pi\left[z - \frac{z^3}{3}\right]_{\frac{\sqrt{2}}{2}}^1 \\ &= \pi\left(\frac{\sqrt{2}}{12}\right) + \pi\left(\frac{2}{3} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12}\right) \\ &= \pi\left(\frac{2-\sqrt{2}}{3}\right).\end{aligned}$$

To compute the area for a sphere of radius k, we simply multiply this by R^3 to get the answer:

$$\boxed{\pi R^3\left(\frac{2-\sqrt{2}}{3}\right)}.$$

Problem 13

Inspired by spherical coordinates, one such parametrization is

$$\boxed{\Phi(\theta,\phi) = (a\cos(\theta)\sin(\phi), b\sin(\theta)\sin(\phi), c\cos(\phi))}$$

for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi)$. Now, observe that

$$\begin{aligned}\mathbf{T}_\theta &= (-a\sin(\theta)\sin(\phi), b\cos(\theta)\sin(\phi), 0) \\ \mathbf{T}_\phi &= (a\cos(\theta)\cos(\phi), b\sin(\theta)\cos(\phi), -c\sin(\phi)),\end{aligned}$$

so

$$\begin{aligned}\mathbf{T}_\theta \times \mathbf{T}_\phi &= \begin{vmatrix} \hat{\mathbf{i}} & -a\sin(\theta)\sin(\phi) & a\cos(\theta)\cos(\phi) \\ \hat{\mathbf{j}} & b\cos(\theta)\sin(\phi) & b\sin(\theta)\cos(\phi) \\ \hat{\mathbf{k}} & 0 & -c\sin(\phi) \end{vmatrix} \\ &= (-bc\cos(\theta)\sin^2(\phi))\,\hat{\mathbf{i}} + (-ca\sin(\theta)\sin^2(\phi))\,\hat{\mathbf{j}} \\ &\quad + (-ab\sin^2(\theta)\sin(\phi)\cos(\phi) - ab\cos^2(\theta)\sin(\phi)\cos(\phi))\,\hat{\mathbf{k}} \\ &= (-bc\cos(\theta)\sin^2(\phi))\,\hat{\mathbf{i}} + (-ca\sin(\theta)\sin^2(\phi))\,\hat{\mathbf{j}} + (-ab\sin(\phi)\cos(\phi))\,\hat{\mathbf{k}}.\end{aligned}$$

Thus, we have that the area of the surface integral is the integral over this curl — namely,

$$\boxed{\int_0^{\pi}\int_0^{2\pi}\sqrt{b^2c^2\cos^2(\theta)\sin^4(\phi)+c^2a^2\sin^2(\phi)\sin^4(\phi)+a^2b^2\sin^2(\phi)\cos^2(\phi)}\,\mathrm{d}\theta\,\mathrm{d}\phi}.$$

Problem 25

We can define a parametrized surface Φ for f as $\Phi(x,y)=\left(x,y,\frac{2}{3}(x^{3/2}+y^{3/2})\right)$. Then $\mathbf{T}_x=(1,0,x^{1/2})$ and $\mathbf{T}_y=(0,1,y^{1/2})$, so $\mathbf{T}_x\times\mathbf{T}_y=\left(-x^{1/2},-y^{1/2},1\right)$ and $\|\mathbf{T}_x\times\mathbf{T}_y\|=\sqrt{x+y+1}$ The area of this surface is thus

$$\begin{aligned}\int_0^1\int_0^1\|\mathbf{T}_x\times\mathbf{T}_y\|\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1\int_0^1\sqrt{x+y+1}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\left[\tfrac{2}{3}\,(x+y+1)^{3/2}\right]_0^1\mathrm{d}y\\ &= \frac{2}{3}\int_0^1(y+2)^{3/2}-(y+1)^{3/2}\,\mathrm{d}y\\ &= \frac{2}{3}\left[\tfrac{2}{5}(y+2)^{5/2}-\tfrac{2}{5}(y+1)^{5/2}\right]_0^1\\ &= \frac{4}{15}\left(\sqrt{3^5}-\sqrt{2^5}-\sqrt{2^5}+1\right)\\ &= \boxed{\frac{36\sqrt{3}-32\sqrt{2}+4}{15}}.\end{aligned}$$

3 Section 7.5

Problem 6

The surface can be parametrized by the mapping $\Phi(r,\theta)=(r\cos(\theta),r\sin(\theta),4+r\cos(\theta)+r\sin(\theta))$ under the domain $\{(r,\theta)\mid r\in[0,2],\theta\in[0,2\pi)\}$. Now, see that

$$\begin{aligned}\mathbf{T}_r &= (\cos(\theta),\sin(\theta),\cos(\theta)+\sin(\theta))\\ \mathbf{T}_\theta &= (-r\sin(\theta),r\cos(\theta),-r\sin(\theta)+r\cos(\theta)).\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{T}_r \times \mathbf{T}_\theta &= \begin{bmatrix} \sin(\theta)(-r\sin(\theta) + r\cos(\theta)) - r\cos(\theta)(\cos(\theta) - \sin(\theta)) \\ (\cos(\theta) + \sin(\theta))(-r\sin(\theta)) - \cos(\theta)(-r\sin(\theta) + r\cos(\theta)) \\ \cos(\theta)(r\cos(\theta)) - \sin(\theta)(-r\sin(\theta)) \end{bmatrix} \\
&= \begin{bmatrix} -r \\ -r \\ r \end{bmatrix},
\end{aligned}$$

so $\|\mathbf{T}_r \times \mathbf{T}_\theta\| = r\sqrt{3}$. We are now ready to compute the surface area of the function:

$$\begin{aligned}
\iint_S x^2 z + y^2 z &= \int_0^{2\pi} \int_0^2 \left(r^2\cos^2(\theta) + r^2\sin(\theta)\right)(4 + r\cos(\theta) + r\sin(\theta))\left(\sqrt{3}r\right) \mathrm{d}r \, \mathrm{d}\theta \\
&= \sqrt{3} \int_0^{2\pi} \int_0^2 4r^3 + r^4(\cos(\theta) + \sin(\theta)) \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \sqrt{3} \int_0^{2\pi} \left[r^4 + \frac{r^5}{5}(\cos(\theta) + \sin(\theta)) \right]_0^2 \mathrm{d}\theta \\
&= \sqrt{3} \int_0^{2\pi} 16 + \frac{32}{5}(\sin(\theta) + \cos(\theta)) \, \mathrm{d}\theta \\
&= \sqrt{3} \left[16\theta + \frac{32}{5}(\sin(\theta) - \cos(\theta)) \right]_0^{2\pi} \\
&= \boxed{32\sqrt{3}\pi}.
\end{aligned}$$

Problem 10

Let X be the portion of B below the xy-plane and let Y be the portion of B above teh xy-plane. Then

$$\begin{aligned}
\iint_S (x + y + z) \, \mathrm{d}S &= \iint_X (x + y + z) \, \mathrm{d}S + \iint_Y (x + y + z) \, \mathrm{d}S \\
&= \iint_X (x + y + z) \, \mathrm{d}S - \iint_X (x + y + z) \, \mathrm{d}S \\
&= 0.
\end{aligned}$$

Problem 16

A parametrization of the sphere is $\Phi(\theta, \phi) = (R\cos(\theta)\sin(\phi), R\sin(\theta)\sin(\phi), R\cos(\phi))$. Using our work in Problem 22 Part (b), we find that

$$\begin{aligned}
\|\mathbf{T}_\phi \times \mathbf{T}_\theta\| &= \left\|(R^2\sin^2(\phi)\cos(\theta))\,\hat{\mathbf{\imath}} + (R^2\sin^2(\phi)\sin(\theta))\,\hat{\mathbf{\jmath}} + (R^2\sin(\phi)\cos(\phi))\,\hat{\mathbf{k}}\right\| \\
&= R^2\sqrt{\sin^4(\phi)\cos^2(\theta) + \sin^4(\phi)\sin^2(\theta) + \sin^2(\phi)\cos^2(\phi)} \\
&= R^2\sin(\phi),
\end{aligned}$$

where $|\sin(\phi)| = \sin(\phi)$ as $\phi \in [0, \pi/2]$. Then as $x^2 + y^2 = R^2\sin^2(\phi)$, the mass density is given by the integral

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{2\pi} (R^2\sin^2(\phi))(R^2\sin(\phi)) &= 2\pi R^4 \int_0^{\pi/2} \sin^3(\phi)\,\mathrm{d}\phi \\
&= 2\pi R^4\left(\frac{2}{3}\right) \\
&= \boxed{\frac{4\pi R^3}{3}}\,/
\end{aligned}$$

Problem 17

Part (a): Realize that if we merely apply a rotation to the sphere, the Change of Variables formula may yield that

$$\iint_S x^2\,\mathrm{d}S = \iint_S y^2\,\mathrm{d}S\,.$$

Similarly, another roation yields that

$$\iint_S y^2\,\mathrm{d}S = \iint_S z^2\,\mathrm{d}S\,.$$

Part (b): For all points (x, y, z) on the unit sphere, $x^2 + y^2 + z^2 = R^2$. Thus,

$$\begin{aligned}
\iint_S x^2 \,\mathrm{d}S &= \frac{1}{3}\iint_S x^2 \,\mathrm{d}S + \frac{1}{3}\iint_S x^2 \,\mathrm{d}S + \frac{1}{3}\iint_S x^2 \,\mathrm{d}S \\
&= \frac{1}{3}\iint_S x^2 \,\mathrm{d}S + \frac{1}{3}\iint_S y^2 \,\mathrm{d}S + \frac{1}{3}\iint_S z^2 \,\mathrm{d}S \\
&= \frac{1}{3}\iint_S x^2 + y^2 + z^2 \,\mathrm{d}S \\
&= \frac{1}{3}\iint_S R^2 \,\mathrm{d}S \\
&= \frac{R^2}{3}\iint_S \mathrm{d}S \\
&= \boxed{\frac{4\pi R^4}{3}}.
\end{aligned}$$

Part (c): Yes, it does. Realize that if the sphere is S and the semisphere is X,

$$\begin{aligned}
\iint_X x^2 + y^2 \,\mathrm{d}S &= \frac{1}{2}\iint_S x^2 + y^2 \,\mathrm{d}S \\
&= \iint_S x^2 \,\mathrm{d}S \\
&= \boxed{\frac{4\pi R^4}{3}},
\end{aligned}$$

which matches our answer in Problem 16.

Problem 26

Without loss of generality, we can assume the point lies on the z-axis; this is because the desired integral only computes the distance from $\mathbf{p}$ to the unit sphere, which is symmetric across rotaions. We may also declare S to be situated at the origin.

S is parametrized by the function $\Phi(\theta, \phi) = (r\cos(\theta)\sin(\phi), r\sin(\theta)\sin(\phi), r\cos(\phi))$ for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$ Then if we represent $\mathbf{x}$ in spherical coordinates as above, we have that

$$\begin{aligned}
\|\mathbf{x} - \mathbf{p}\| &= \sqrt{r^2\cos^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\sin^2(\phi) + (r\cos(\phi) - d)^2} \\
&= \sqrt{r^2\sin^2(\phi) + r^2\cos^2(\phi) - 2rd\cos(\phi) + d^2} \\
&= \sqrt{r^2 - 2rd\cos(\phi) + d^2}.
\end{aligned}$$

Therefore, the Change of Variables Theorem yields that (since $p \neq 0$)

$$
\begin{aligned}
\iint_S \frac{1}{\|\mathbf{x} - \mathbf{p}\|} \, \mathrm{d}S &= \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin(\phi)}{\sqrt{r^2 - 2rd\cos(\phi) + d^2}} \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= \frac{r}{2d} \int_0^{2\pi} \int_0^{\pi} \frac{2rd \sin(\phi)}{\sqrt{r^2 - 2rd\cos(\phi) + d^2}} \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= \frac{r\pi}{d} \left[2\sqrt{r^2 - 2rd\cos(\phi) + d^2} \right]_0^{\pi} \\
&= \frac{2r\pi}{d} \left(\sqrt{r^2 + 2rd - d^2} + \sqrt{r^2 - 2rd + d^2} \right) \\
&= \frac{2r\pi}{d} \big(|r + d| - |r - d| \big).
\end{aligned}
$$

If $r > d$, then $|r - d| = r - d$, so

$$
\frac{2r\pi}{d} \big(|r + d| - |r - d| \big) = \frac{2r\pi}{d} \big(r + d - (r - d) \big) = \frac{2r\pi}{d}(2d) = \boxed{4\pi r}.
$$

If $r < d$, then $|r - d| = d - r$, so

$$
\frac{2r\pi}{d} \big(|r + d| - |r - d| \big) = \frac{2r\pi}{d} \big(r + d - (d - r) \big) = \frac{2r\pi}{d}(2r) = \boxed{\frac{4\pi r^2}{d}}.
$$

4 Section 7.6

Problem 4

The cylinder may be parametrized by the mapping $\Phi(\theta, z) = (2\cos(\theta), 2\sin(\theta), z)$ for $\theta \in [0, 2\pi)$ and $z \in [0, 1]$. Then

$$
\begin{aligned}
\mathbf{T}_\theta &= (-2\sin(\theta), 2\cos(\theta), 0) \\
\mathbf{T}_z &= (0, 0, 1).
\end{aligned}
$$

Thus, $\mathbf{T}_\theta \times \mathbf{T}_z = (2\cos(x), 2\sin(x), 0)$. By substitution, $\mathbf{F}(x,y,z) = (4\cos(\theta), -4\sin(\theta), z^2)$ We are now ready to compute the required integral:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_0^1 \int_0^{2\pi} (4\cos(\theta), -4\sin(\theta), z^2) \cdot (2\cos(x), 2\sin(x), 0)\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_0^{2\pi} 8\cos^2(\theta) - 8\sin^2(\theta)\,\mathrm{d}\theta \\
&= 4\int_0^{2\pi} 2\cos(2\theta)\,\mathrm{d}\theta \\
&= 4\left[\sin(2\theta)\right]_0^{2\pi} \\
&= \boxed{0}.
\end{aligned}$$

Problem 7

Divide the surface S into two parts:

- S_1: The upper hemisphere. $\Phi_1(\phi, \theta) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$ for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi/2)$ is one parametrization. **The variables are flipped!**
- S_2: The unit disc. $\Phi_2(r, \theta) = (r\sin(\theta), r\cos(\theta), 0)$ for $r \in [0,1]$ and $\theta \in [0, 2\pi)$ is one parametrization.

It is easy to verify that these possess the same orientation. We thus have that

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathrm{d}\mathbf{S} + \iint_{S_2} \mathbf{E} \cdot \mathrm{d}\mathbf{S}\,. \tag{1}$$

First, we tackle S_1. Realize that for Φ_1,

$$\begin{aligned}
\mathbf{T}_\phi &= (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)) \\
\mathbf{T}_\theta &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0)
\end{aligned}$$

Therefore,

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = (\cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi)).$$

We are almost ready to integrate the flux of S_1:

$$\begin{aligned}
\mathbf{E}(\Phi_1)\cdot(\mathbf{T}_\phi\times\mathbf{T}_\theta) &= \begin{bmatrix} 2\cos(\theta)\sin(\phi) \\ 2\sin(\theta)\sin(\phi) \\ 2\cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta)\sin^2(\phi) \\ \sin(\theta)\sin^2(\phi) \\ \sin(\phi)\cos(\phi) \end{bmatrix} \\
&= 2\cos^2(\theta)\sin^3(\phi) + 2\sin^2(\theta)\sin^3(\phi) + 2\sin(\phi)\cos^2(\phi) \\
&= 2\sin^3(\phi)(\sin^2(\theta)+\cos^2(\theta)) + 2\sin(\phi)\cos^2(\phi) \\
&= 2\sin^3(\phi) + 2\sin(\phi)\cos^2(\phi) \\
&= 2\sin(\phi)(\sin^2(\phi)+\cos^2(\phi)) \\
&= 2\sin(\phi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint_{S_1} \mathbf{E}\cdot \mathrm{d}\mathbf{S} &= \iint_{S_1} \mathbf{E}(\Phi_1)\cdot(\mathbf{T}_\phi\times\mathbf{T}_\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \int_0^{2\pi}\int_0^{\pi/2} 2\sin(\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= 2\pi\Big[-2\cos(\phi)\Big]_0^{\pi/2} \\
&= 4\pi.
\end{aligned}$$

Now, we calculate the flux over S_2. Realize that for Φ_2,

$$\begin{aligned}
\mathbf{T}_r &= (\sin(\theta),\cos(\theta),0) \\
\mathbf{T}_\theta &= (r\cos(\phi),-r\sin(\phi),0).
\end{aligned}$$

Therefore,

$$\mathbf{T}_r\times\mathbf{T}_\theta = (0,0,-r),$$

so

$$\mathbf{E}(\Phi_2)\cdot(\mathbf{T}_r\times\mathbf{T}_\theta) = (2r\cos(\theta),2r\sin(\theta),0)\times(0,0,-r) = 0.$$

Hence,

$$\begin{aligned}
\iint_{S_2} \mathbf{E}\cdot \mathrm{d}\mathbf{S} &= \iint_{S_2} \mathbf{E}(\Phi_2)\cdot(\mathbf{T}_r\times\mathbf{T}_\theta)\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \iint_{S_2} 0\,\mathrm{d}\theta\,\mathrm{d}r \\
&= 0.
\end{aligned}$$

Finally, we combine this with Equation (1) to yield that

$$\iint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = 4\pi + 0 = \boxed{4\pi}.$$

Problem 9

The surface may be parametrized by $\Phi(\theta,\phi) = \Big(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \frac{\sqrt{3}}{3}\cos(\phi)\Big)$. for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi/2]$. Thus,

$$\begin{aligned}
\mathbf{T}_\theta &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\
\mathbf{T}_\phi &= \Big(\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\tfrac{\sqrt{3}}{3}\sin(\phi)\Big).
\end{aligned}$$

Therefore, we have that

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = \Big(-\tfrac{\sqrt{3}}{3}\cos(\theta)\sin^2(\phi), -\tfrac{\sqrt{3}}{3}\sin(\theta)\sin^2(\phi), -\sin(\phi)\cos(\phi)\Big),$$

We also have that $\nabla \times \mathbf{F} = (2x^3yz, -3x^2y^2z, -2)$, so

$$\nabla \times \mathbf{F}(\Phi) = \begin{bmatrix} -\frac{2\sqrt{3}}{3}\sin(\theta)\cos^3(\theta)\sin^8(\phi) \\ \sqrt{3}\sin^2(\theta)\cos^2(\theta)\sin^9(\phi)\cos(\phi) \\ -2 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
(\nabla \times \mathbf{F}(\Phi)) \cdot (\mathbf{T}_\theta \times \mathbf{T}_\phi) = &\big(\tfrac{2}{3}\sin(\theta)\cos^4(\theta)\sin^{10}(\phi)\big) + \big(-\sin^3(\theta)\cos^2(\theta)\sin^{11}(\phi)\cos(\phi)\big) \\
&- (2\sin(\phi)\cos(\phi)).
\end{aligned}$$

When we integrate this quantity, all terms with a θ will cancel, as this quantity is being integrated from 0 to 2π. Then only the last term will remain, and

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iint_S (\nabla \times \mathbf{F}(\Phi)) \cdot (\mathbf{T}_\theta \times \mathbf{T}_\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} -2\sin(\phi)\cos(\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= -2\pi\int_0^{\pi/2} \sin(2\phi)\,\mathrm{d}\phi \\
&= -2\pi\Big[-\frac{\cos(\phi)}{2}\Big]_0^{\pi/2} \\
&= \boxed{-2\pi}.
\end{aligned}$$

Problem 11

The following setup is identical to Problem 7. Divide the surface S into two parts:

- S_1: The upper hemisphere. $\Phi_1(\phi, \theta) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$ for $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi/2)$ is one parametrization. **The variables are flipped!**
- S_2: The unit disc. $\Phi_2(r, \theta) = (r\sin(\theta), r\cos(\theta), 0)$ for $r \in [0, 1]$ and $\theta \in [0, 2\pi)$ is one parametrization.

It is easy to verify that these possess the same orientation. Thus,

$$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathrm{d}\mathbf{S} + \iint_{S_2} \mathbf{E} \cdot \mathrm{d}\mathbf{S}\,. \tag{2}$$

Using our work from Problem 7,

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = (\cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi)).$$

Furthermore, realize that

$$\mathbf{F}(\Phi_1) = \begin{bmatrix} \cos(\theta)\sin(\phi) + 3\sin^5(\theta)\sin^5(\phi) \\ \sin(\theta)\sin(\phi) + 10\cos(\theta)\sin(\phi)\cos(\phi) \\ \cos(\phi) - \sin(\theta)\cos(\theta)\sin^2(\phi) \end{bmatrix},$$

so we have that

$$\begin{aligned} \mathbf{F}(\Phi_1) \cdot (\mathbf{T}_\phi \times \mathbf{T}(\theta)) = &\left(\cos^2(\theta)\sin^3(\phi) + 3\sin^5(\theta)\cos(\theta)\sin^7(\phi))\right) \\ &+ \left(\sin^2(\theta)\sin^3(\phi) + 10\sin(\theta)\cos(\theta)\sin^3(\phi)\cos(\phi)\right) \\ &+ \left(3\sin(\phi)\cos^2(\phi) - \sin(\theta)\cos(\theta)\sin^3(\phi)\cos(\phi)\right). \end{aligned}$$

When we integrate this quantity, all terms with a θ will cancel, as this quantity is being integrated from 0 to 2π. Then only the term $\sin(\phi)\cos^2(\phi)$ will remain, and

$$\begin{aligned} \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iint_{S_1} (\nabla \times \mathbf{F}(\Phi_1)) \cdot (\mathbf{T}_\phi \times \mathbf{T}_\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} 3\sin(\phi)\cos^2(\phi)\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= 2\pi\left[-\cos^3(\phi)\right]_0^{\pi/2} \\ &= 2\pi. \end{aligned}$$

Now, we tackle S_2. For Φ_2, our work in Problem 7 yields that

$$\mathbf{T}_r \times \mathbf{T}_\theta = (0, 0, -r),$$

Furthermore, realize that

$$\mathbf{F}(\Phi_2) = \begin{bmatrix} r\cos(\theta) + 3r^5\sin^5(\theta) \\ r\cos(\theta) \\ -r^2\sin(\theta)\cos(\theta) \end{bmatrix}.$$

Thus,

$$\mathbf{F}(\Phi^2) \cdot (\mathbf{T}_\phi \times \mathbf{T}_\theta) = r^3\sin(\theta)\cos(\theta).$$

We are now ready to compute the integral for S_2.

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{S_2} \mathbf{E}(\Phi_2) \cdot (\mathbf{T}_r \times \mathbf{T}_\theta)\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_0^1 \int_0^{2\pi} r^3\sin(\theta)\cos(\theta)\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 0. \end{aligned}$$

Finally, we combine this with Equation (1) to yield that

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 2\pi + 0 = \boxed{2\pi}.$$