# Hartshorne: Varieties

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# 1 Affine Varieties

## 1.1 Familiar Definitions

Let $k$ be an algebraically closed field. We define **affine space** over $k$, denoted $\mathbb{A}_k^n$ or $\mathbb{A}^n$, as the set of all $n$-tuples with components in $k$. The elements $P \in \mathbb{A}^n$ are called **points** — and if $P = (a_1, \ldots, a_n)$, the elements $a_1, \ldots, a_n$ are called **components**.

The set $R = k[x_1, \ldots, x_n]$ denotes the commutative ring of polynomials with variables $x_1, \ldots, x_n$ with coefficents in $k$. We may interpret each $f \in R$ as a function from $\mathbb{A}^n$ to $k$, defined by $f(P) = f(a_1, \ldots, a_n)$. We may thus define the **zeroes** of $f$, given by the set $Z(f) = \{P \in \mathbb{A}_k^n \mid f(P) = 0\}$. More generally, for any subset $T$ of polynomials $R$, its **zeroes** are given by

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for each } f \in T\}.$$

Ideals in $R$ are quite elegant: since $R$ is Noetherian, each ideal $\mathfrak{a}$ has a finite set of generators $f_1, \ldots, f_n$. Thus $\mathfrak{a}$ may be expressed as the common zeroes of $f_1, \ldots, f_n$. It is easy to verify that if $\mathfrak{b}$ is the ideal generated by $T$, then $Z(T) = Z(\mathfrak{b})$.

## 1.2 The Zariski Topology

A subset $Y \subseteq \mathbb{A}^n$ is an **algebraic set** if there exists a subset $T \subseteq Y$ such that $Y = Z(T)$.

**Theorem 1.** *The following two results hold:*

1. *The union of two algebraic sets $X, Y \subseteq \mathbb{A}^n$ is algebraic.*
2. *The intersection of any family of algebraic sets $Y_\alpha$ is an algebraic set.*

*Proof.* For (1), let $X = Z(T)$ and $Y = Z(S)$. We claim that $Z(T \cup S) = Z(TS)$, where $TS$ is the set of all $ts$ with $t \in T$ and $s \in S$.

1. Suppose $P \in Z(TS)$ — that is, $ts(P) = 0$ for all $ts \in TS$. Since $R$ is an integral domain, we have $t(P)$ or $s(P) = 0$, so $P \in Z(T)$ or $P \in Z(S)$. Hence $P \in Z(T) \cup Z(S)$.
2. Suppose $P \in Z(T) \cup Z(S)$. Then $P \in Z(T)$ or $P \in S(T)$ — in which case, $t(P) = 0$ for all $t$ or $s(P) = 0$ for all $s$. In either case, $ts(P) = 0$ for all $ts \in TS$, so $P \in Z(TS)$.

Thus $X \cup Y = Z(T) \cup Z(S) = Z(TS)$. We deduce that $X \cup Y$ is an algebraic set. For (2), let $Y_\alpha = T_a$. It is easy to verify that $\bigcap_{\alpha \in A} Z(T_\alpha) = Z\big(\bigcup_{\alpha \in A} T_\alpha\big)$; hence $\bigcup_{\alpha \in A} Y_\alpha$ is an algebraic set. □

Noting that ;0 and $\mathbb{A}^n$ are algebraic sets (since $\{0\} = Z(1)$ and $\mathbb{A}^n = Z(0)$), we deduce that algebraic sets in $R$ satisfy the closed set axioms to be a topological space. The ensuing topology is called the **Zariski topology**.