Rudin: Sequences and Series of Functions

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Contents

1 Discussion of the Main Problem

1.1 Exposition

Suppose $\{f_n\}$ is a sequence of functions defined on a set E. If the sequence of numbers $\{f_n(x)\}$ converges for each $x \in E$, we may define a function

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We say that $\{f_n\}$ converges to f **pointwise** on E. An important special case is series of functions: where if $\sum f_n(x)$ converges for eac $x \in E$, we may define

$$f = \sum_{n=1}^{\infty} f_n(x).$$

The critical question posed by this question is: which properties of functions are preserved under the limit operation? If each f_n is continuous or differentiable, is the same true of f? For continuity: recall that f is continuous at x if

$$\lim_{t\to x} f(t) = f(x);$$

then the continuity of f is equivalent to

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

The following examples will illustrate how this question fails in a general context.

1.2 Motivating Examples

Example 1: For $m, n \in \mathbb{Z}_{>0}$, define

$$a_{m,n} = \frac{m}{m+n}.$$

This example fails to satisfy the continuity condition established above, as demonstrated by the following computation:

$$\lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{m}{m+n}\right) = \lim_{n\to\infty} 1 = 1 \neq 0 = \lim_{m\to\infty} 0 = \lim_{m\to\infty}\lim_{n\to\infty}\left(\frac{m}{m+n}\right)$$

Example 2: For $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$, let

$$f_n(x) = \frac{x^2}{(1+x^2)^n}.$$

and consider the infinite series

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(x^2+1)^n}.$$

Since $f_n(0) = 0$ for all n, we have $f(0) = 0$. For nonzero n, this is a geometric series that converges to $x^2 + 1$. We deduce from pointwise convergence that

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^2 + 1 & \text{if } x \neq 0. \end{cases}$$

Then f is continuous everywhere *except* the origin.

Example 3: For $n \in \mathbb{Z}_{>0}$, consider the functions

$$f_n(x) = \lim_{m\to\infty} (\cos \pi n! x)^{2m} \qquad \text{and} \qquad f(x) = \lim_{n\to\infty} f_n(x).$$

If x is rational, let $x = \frac{p}{q}$. Then $m > q$ implies $m!x$ is an integer, so $f_n(x) = 1$; hence $f(x) = 1$. If x is irrational, then $n!x$ is never an integer. Thus $\cos \pi n! x < 1$, so (by nontrivial but irrelevant techniques) $f_n(0) = 0$ and $f(x) = 0$. We deduce that

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We have obtained a limit function which is *nowhere continuous*. It thus fails to be Riemann integral — though the Lebesgue integral returns 0, as will be determined in Chapter 11.

2 Uniform Convergence

2.1 Definition

Let $\{f_n\}$ be a sequence of functions from E to a metric space X. We say the sequence **converges uniformly** to a function f if for all $\epsilon > 0$, there is an integer N such that

$$N \leq n \implies d\big(f_n(x) - f(x)\big) < \epsilon$$

for all $x \in E$. It is natural that each uniformly convergent sequnce is pointwise convergent.

2.2 Uniform Cauchy Criterion

Theorem 1. *Suppose $\{f_n\}$ is a uniformly continuous sequence of functions from E to a metric space X. Then $\{f_n\}$ satisfies the uniform Cauchy criterion.*

Proof. Let $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in E$. For all $\epsilon > 0$, there exists integers N_1 and N_2 such that

$$N_1 \leq n \implies d\big(f_n(x), f(x)\big) < \frac{\epsilon}{2}$$
$$N_2 \leq m \implies d\big(f_m(x), f(x)\big) < \frac{\epsilon}{2}$$

for all $x \in E$. Let $N = \max\{N_1, N_2\}$; then $N \leq n, m$ implies that

$$d\big(f_n(x) - f_m(x)\big) \;\leq\; d\big(f_n(x), f(x)\big) + d\big(f_m(x), f(x)\big) \;<\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.$$

for all $x \in E$. We conclude that $\{f_n\}$ satisfies the Cauchy criterion. □

Theorem 2. *Suppose $\{f_n\}$ is a sequence of functions from E to a complete metric space Y that satisfies the uniform Cauchy criterion. Then $\{f_n\}$ converges uniformly.*

Proof. Since Y is complete, $\{f_n\}$ converges pointwise for each $x \in E$. We must demonstrate that this convergence is uniform.

Claim 1. *Suppose that $\{a_n\}$ is a Cauchy sequence in a metric space Y with limit A: that for all $\epsilon > 0$, there exists an integer N such that*

$$N \leq n, m \implies d(a_n, a_m) < \epsilon.$$

Then $N \leq n$ implies that $d(a_n, A) < \epsilon$.

Proof. Suppose for contradiction that some $N \leq k$ that $d(a_k, A) \geq \epsilon$. Then $N \leq n$ implies

$$d(a_n, a_k) \geq d(a_k, A) - d(a_n, A) \geq d(a_k, A) \geq \epsilon,$$

which yields the desired contradiction.

By definition, for all $\epsilon > 0$, there exists an integer N such that

$$N \leq n, m \implies d(f_n(x), f_m(x)) < \epsilon$$

for all $x \in E$. Then via our claim, the same choice of N demonstrates uniform continuity: namely, $N \leq n$ implies $d(f_n(x), f(x)) < \epsilon$ for all $x \in E$. □

Corollary 1. *A sequence of functions $\{f_n\}$ from a set E to a Banach space Y converges uniformly if and only if it satisfies the uniform Cauchy criterion.*

2.3 Uniform Convergence of Series

Theorem 3. *Suppose $\{\mathbf{f}_n\}$ is a sequence of functions from E to a Banach space Y such that*

$$\|\mathbf{f}_n(x)\| \leq M_n$$

for all $x \in E$ and $n \in \mathbb{Z}_{>0}$. If $\sum M_n$ converges, then $\sum \mathbf{f}_n$ converges uniformly.

Proof. Suppose that $\sum M_n$ converges. Then it satisfies the Cauchy criterion: for each $\epsilon > 0$, there exists an integer N such that

$$N \leq n, m \implies \sum_{i=n+1}^{m} M_i < \epsilon.$$

We deduce that

$$\left\| \sum_{i=n+1}^{m} \mathbf{f}_i(x) \right\| \leq \sum_{i=n+1}^{m} M_i < \epsilon$$

for all $x \in E$ as well, so $\mathbf{f}_n$ satisfies the uniform Cauchy criterion. Its uniform convergence is hence guaranteed by Theorem 2. □

If E is $\mathbb{C}^n$, and Y is $\mathbb{C}^m$, then functions $\{\mathbf{f}_n\}$ are matricies.

2.4 The Metric Space $\mathcal{C}(\mathbf{X})$

If X is a metric space, we denote by $\mathcal{C}(X)$ the set of continuous and bounded functions from X to a Banach space Y. If X is compact, then the boundedness condition is redundant. We associate with each $\mathbf{f} \in \mathcal{C}(X)$ its **supremum norm**

$$\|\mathbf{f}\|_X = \sup_{x \in X} \|\mathbf{f}(x)\|.$$

Since f is assumed to be bounded, $\|\mathbf{f}\|_X < \infty$. A similar definition exists if Y is substituted with any metric space; only a few properties about $\mathcal{C}(X)$ hold in such a setting.

Theorem 4. *$\mathcal{C}(X)$ equipped with the supremum norm is a metric space.*

Proof. We must perform three rather routine calculations:

- **Positivity**: It is natural that $\|\mathbf{f} - \mathbf{g}\|_X \geq 0$, with equality if and only if $\mathbf{f} = \mathbf{g}$.
- **Symmetry**: For all $\mathbf{f}, \mathbf{g} \in \mathcal{C}(X)$, we have

$$\|\mathbf{f} - \mathbf{g}\|_X = \sup_{x \in X} \|\mathbf{f}(x) - \mathbf{f}(x)\| = \sup_{x \in X} \|\mathbf{g}(x) - \mathbf{f}(x)\| = \|\mathbf{g} - \mathbf{f}\|_X.$$

- **Triangle Inequality**: For all $\mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathcal{C}(X)$, we have

$$\begin{aligned}
\|\mathbf{f}-\mathbf{g}\|_X &= \|\mathbf{f}-\mathbf{h}+\mathbf{h}-\mathbf{g}\|_X \\
&= \sup_{x\in X} \|\mathbf{f}(x)-\mathbf{h}(x)+\mathbf{h}(x)-\mathbf{g}(x)\| \\
&\le \sup_{x\in X} \Big(\|\mathbf{f}(x)-\mathbf{h}(x)\| + \|\mathbf{h}(x)-\mathbf{g}(x)\| \Big) \\
&\le \sup_{x\in X} \|\mathbf{f}(x)-\mathbf{h}(x)\| + \sup_{x\in X} \|\mathbf{h}(x)-\mathbf{g}(x)\| \\
&= \|\mathbf{f}-\mathbf{h}\|_X + \|\mathbf{h}-\mathbf{g}\|_X.
\end{aligned}$$

We conclude that $\mathcal{C}(X)$ is a metric space with respect to the supremum norm. □

Theorem 5. *A series of functions $\{\mathbf{f}_n\}$ from X to a Banach space Y uniformly converges to $\mathbf{f}$ if and only if $\{\mathbf{f}_n\}$ converges to $\mathbf{f}$ in $\mathcal{C}(X)$.*

Proof. We have the following: for all $\epsilon > 0$,

$$\begin{aligned}
\{\mathbf{f}_n\} \to \mathbf{f} \text{ is uniform} \quad &\iff \exists N \text{ such that } N \le n \implies \|\mathbf{f}_n(x) - \mathbf{f}(x)\| < \epsilon \\
&\qquad \text{for all } x \in X \\
&\iff \exists N \text{ such that } N \le n \implies \sup_{x\in X} \|\mathbf{f}_n(x) - \mathbf{f}(x)\| \le \epsilon \\
&\iff \exists N \text{ such that } N \le n \implies \|\mathbf{f}_n - \mathbf{f}\|_X \le \epsilon \\
&\iff \{\mathbf{f}_n\} \text{ converges to } \mathbf{f} \text{ in } \mathcal{C}(X).
\end{aligned}$$

A corollary is that $\{\mathbf{f}_n\}$ uniformly converges to $\mathbf{f}$ if and only if $\lim_{n\to\infty} \|\mathbf{f}_n - \mathbf{f}\|_X = 0$. □

Theorem 6. *$\mathcal{C}(X)$ under the supremum norm is complete — hence a Banach space.*

Proof. Let $\{\mathbf{f}_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$: for all $\epsilon > 0$, there exists an integer N such that

$$N \le n, m \implies \|\mathbf{f}_n - \mathbf{f}_m\|_X < \epsilon.$$

This implies that

$$N \le n, m \implies d\big(\mathbf{f}_n(x) - \mathbf{f}_m(x)\big) < \epsilon$$

for all $x \in X$. By Theorem 2, such a uniform Cauchy sequence in Y converges uniformly; by Theorem 5, $\{\mathbf{f}_n\}$ converges in $\mathcal{C}(X)$. The continuity of $\mathbf{f}$ is ensured by Theorem 7, and $\mathbf{f}$ is bounded since there is n such that

$$\|\mathbf{f}(x) - \mathbf{f}_n(x)\| < 1$$

for all $x \in X$, and $\mathbf{f}_n$ is bounded. This completes the proof. □

3 Uniform Convergence and Continuity

3.1 The Theorem

Theorem 7. *Suppose $\{f_n\}$ is a sequence of functions from E to a metric space X that converges uniformly to f. If x is a limit point of E and*

$$\lim_{t \to x} f_n(t) = F_n$$

for each n, then $\{F_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} F_n$.

Proof. Since $\{f_n\}$ is uniformly continuous, it satisfies the uniform Cauchy sequence. For all $\epsilon > 0$, there exist δ_2, δ_3, and an integer N such that each of the following is satisfied:

$$\begin{aligned} 0 < d(x,t) < \delta_1 &\implies d(f_i(t), F_i) < \frac{\epsilon}{3} \\ 0 < d(x,t) < \delta_2 &\implies d(f_j(t), F_j) < \frac{\epsilon}{3}. \\ N \leq i, j &\implies d(f_i(t), f_j(t)) < \frac{\epsilon}{3}. \end{aligned}$$

Suppose we consider $0 < d(x,t) < \min\{\delta_1, \delta_2\}$. Then $N \leq i, j$ implies

$$\begin{aligned} d(F_i, F_j) &\leq d(F_i, f_i(t)) + d(f_i(t), f_j(t)) + d(f_j(t), F_j) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Let $\{F_n\}$ converge to F. For all $\epsilon > 0$, there exist δ_1 and δ_2 such that

$$\begin{aligned} t \in E, \quad N_1 \leq n &\implies d(f(t), f_n(t)) < \frac{\epsilon}{3} \\ 0 < d(x,y) \leq \delta &\implies d(f_n(t), F_n) < \frac{\epsilon}{3} \\ N_2 \leq n &\implies d(F_n, F) < \frac{\epsilon}{3}. \end{aligned}$$

Then $\max\{N_1, N_2\} \leq N$ and $0 < d(x,y) < \delta$ implies that

$$\begin{aligned} d(f(t), F) &\leq d(f(t), f_n(t)) + (f_n(t), F_n) + d(F_n, F) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

We conclude that $\lim_{t \to x} f(t) = F$. □

Corollary 2. *Suppose that $\{f_n\}$ converges uniformly to f. Then f is continuous.*

4 The Stone-Weierstrauss Theorem

4.1 Bernstein's Proof

Theorem 8 (Weierstrauss Approximation Theorem)**.** *Let $f : [a,b] \to \mathbb{C}$ be continuous. Then there exists a sequenece of polynomials that uniformly converges to f.*

Proof. For any nonnegative integer n, define

$$B_n(f)(x) \quad \stackrel{\text{def}}{=} \quad \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}.$$

Denote the component $\binom{n}{k}x^k(1-x)^{n-k}$ by $P_{n,k}(x)$. Then the $(P_{n,k})_{k=0}^n$ satisfy the following properties for all $x \in [0,1]$:

1. $P_{n,k}(x) \geq 0$ for all $x \in [0,1]$
2. $\sum\limits_{k=0}^{n} P_{n,k}(x) = \big(x + (1-x)\big)^n = 1$, hence $(P_{n,k})_{k=0}^n$ is a partition of unity.
3. $P_{n,k}$ attains its maximum on $[0,1]$ at $\frac{k}{n}$; simply compute its derivative.
4. $(P_{n,k})_{k=0}^n$ is a basis of the vector space of polynomials of degree n or smaller.

For all $x \in [0,1]$ and $n \in \mathbb{Z}_{>0}$, we have using Property 2 that

$$\begin{aligned} f(x) - B_n(f)(x) &= \sum_{k=0}^{n} f(x)P_{n,k}(x) \; - \; \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x) \\ &= \sum_{k=0}^{n}\left(f(x) - f\left(\frac{k}{n}\right)\right) P_{n,k}(x) \end{aligned}$$

so that using the Triangle Inequality and Property 1 yields that

$$|f(x) - B_n(f)(x)| \leq \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right| P_{n,k}(x). \tag{1}$$

Select $\delta > 0$ arbitrarily. We will bound $\left|f(x) - f\left(\frac{k}{n}\right)\right|$ as follows:

1. When $\left|x - \frac{k}{n}\right| \leq \delta$, we will invoke the bound $\left|f(x) - f\left(\frac{k}{n}\right)\right| \leq \omega_f(\delta)$.
2. When $\left|x - \frac{k}{n}\right| > \delta$, we will invoke the bound $\left|f(x) - f\left(\frac{k}{n}\right)\right| \leq 2\|f\|_{[0,1]}$.

We may use the modulus of continuity, since the Heine-Cantor Theorem guarantees that continuous functions are uniformly continuous on closed intervals.

Hence equation (1) reduces to

$$\begin{aligned}
|f(x) - B_n(f)(x)| &\leq \sum_{k:\left|x-\frac{k}{n}\right|\leq\delta} \omega_f(\delta) P_{n,k}(x) \;+ \sum_{k:\left|x-\frac{k}{n}\right|>\delta} 2\|f\|_{[0,1]} P_{n,k}(x) \\
&= \omega_f(\delta) \sum_{k:\left|x-\frac{k}{n}\right|\leq\delta} P_{n,k}(x) \;+\; 2\|f\|_{[0,1]} \sum_{k:\left|x-\frac{k}{n}\right|>\delta} P_{n,k}(x) \\
&\leq \omega_f(\delta) \;+\; 2\|f\|_{[0,1]} \sum_{k:\left|x-\frac{k}{n}\right|>\delta} P_{n,k}(x)
\end{aligned}$$

Note that $\left(\frac{|x-k/n|}{\delta}\right)^2 \geq 1$ whenever $\left|k - \frac{k}{n}\right| > \delta$. Thus

$$\begin{aligned}
|f(x) - B_n(f)(x)| &\leq \omega_f(\delta) \;+\; 2\|f\|_{[0,1]} \sum_{k:\left|x-\frac{k}{n}\right|} \left(\frac{\left|x-\frac{k}{n}\right|}{\delta}\right)^2 P_{n,k}(x) \\
&\leq \omega_f(\delta) + 2\|f\|_{[0,1]} \sum_{k=0}^{n} \left(\frac{\left|x-\frac{k}{n}\right|}{\delta}\right)^2 P_{n,k}(x) \\
&\leq \omega_f(\delta) + \frac{2\|f\|_{[0,1]}}{n^2\delta^2} \sum_{k=0}^{n} (nx-k)^2 P_{n,k}(x).
\end{aligned}$$

Lemma 1. *$\sum_{k=0}^{n} (nx-k)^2 P_{n,k}(x) = nx(1-x)$.*

Proof. Güntürk gives a slick probibalistic proof; we will use algebra, as unenlightening as this may be. We have that

$$\begin{aligned}
\sum_{k=0}^{n}(nx-k)^2 P_{n,k}(x) \;&=\; n^2x^2 \sum_{k=0}^{n} P_{n,k}(x) \;-\; 2nx \sum_{k=0}^{n} kP_{n,k}(x) \;+\; \sum_{k=0}^{n} k^2 P_{n,k}(x) \\
&=\; n^2x^2 \;-\; 2nx \sum_{k=0}^{n} kP_{n,k}(x) \;+\; \sum_{k=0}^{n} k^2 P_{n,k}(x)
\end{aligned}$$

Our task is to simply these summations. We have

$$\begin{aligned}
\sum_{k=0}^{n} kP_{n,k}(x) &= \sum_{k=0}^{n} k\left(\frac{n!}{k!(n-k)!}\right)x^k(1-x)^{n-k} \\
&= nx\sum_{k=1}^{n}\binom{n-1}{k-1}x^{k-1}(1-x)^{(n-1)-(k-1)} \\
&= nx\sum_{k=0}^{n-1}\binom{n-1}{k}x^{k}(1-x)^{(n-1)-k} \\
&= nx\big(x+(1-x)\big)^{n-1} \\
&= nx,
\end{aligned}$$

For the summation k^2, we find it is easier to work with $k(k-1)$ due to the factorial:

$$\begin{aligned}
\sum_{k=0}^{n} k^2P_{n,k}(x) &= \sum_{k=0}^{n} kP_{n,k}(x) + \sum_{k=0}^{n} k(k-1)P_{n,k}(x) \\
&= nx + \sum_{k=0}^{n} k(k-1)P_{n,k}(x) \\
&= nx + \sum_{k=0}^{n} k(k-1)\left(\frac{n!}{k!(n-k)!}\right)x^k(1-x)^{n-k} \\
&= nx + n(n-1)x^2\sum_{k=2}^{n}\binom{n-2}{k-2}x^{k-2}(1-x)^{(n-2)-(k-2)} \\
&= nx + n(n-1)x^2\sum_{k=0}^{n-2}\binom{n-2}{k}x^{k}(1-x)^{(n-2)-k} \\
&= nx + n(n-1)x^2\big(x+(1-x)\big)^{n-2} \\
&= nx + n(n-1)x^2.
\end{aligned}$$

We are ready to return to our original series: we have that

$$\begin{aligned}
\sum_{k=0}^{n}(nx-k)^2P_{n,k}(x) &= n^2x^2 - 2nx(nx) + \big(nx+n(n-1)x^2\big) \\
&= n^2x^2 - 2n^2x^2 + nx + n^2x^2 - nx^2 \\
&= nx - nx^2 \\
&= nx(1-x),
\end{aligned}$$

completing the proof of our lemma.

Returning to our original series, we have that

$$
\begin{aligned}
|f(x) - B_n(f)(x)| \;&\le\; \omega_f(\delta) \;+\; \frac{2\|f\|_{[0,1]}}{n^2\delta^2}\big(nx(1-x)\big) \\
&\le\; \omega_f(\delta) + \frac{2\|f\|_{[0,1]}}{n\delta^2}(x^2 - x) \\
&\le \omega_f(\delta) + \frac{\|f\|_{[0,1]}}{2n\delta^2}
\end{aligned}
$$

This holds for all $\delta > 0$. To proceed, we select δ_n such that $\delta_n \to 0$ and $\frac{1}{n\delta_n^2} \to 0$ as well. The choice we will use is $\delta_n = n^{-1/3}$. Then for all $x \in [0,1]$, we have

$$
|f(x) - B_n(f)(x)| \;\le\; \omega_f(n^{-1/3}) \;+\; \frac{\|f\|_{[0,1]}}{2} n^{-1/3}.
$$

Both terms on the right-hand side converge to 0 as $n \to \infty$. We may therefore select N such that the right-hand side is less than ϵ. This completes the proof. □

We invoked properties of the modulus of continuity in our proof. If desired, we encourage the reader to explore RealAnalysis/gunturk.tex for more.