Fuck

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1 Day 1

An operation $*$ on a set S is something that takes in two values of S and spits out one value of S. For instance,

1. Addition, subtraction, multiplication, and division are operations on rational numbers $\mathbb{Q}$.
2. Exponentiation is an operation on positive integers: let $a * b = a^b$.
3. Logic gates are an operation: they take in two numbers (each either 0 or 1) and spit out 0 or 1.

An operation $*$ on S is called **closed** if for all $a, b \in S$, the element $a * b \in S$ too. In other words, $*$ never "leaves" S. The operation $*$ is associative if for all $a, b, c \in S$, we have $a * (b * c) = (a * b) * c$. Furthermore, $*$ is **commutative** if $a * b = b * a$ for all $a, b \in S$. It's possible to have one of these without the other!

2 Day 2

We often denote S with the operation $*$ by the pair $(S, *)$. There are two other properties of operations we care about:

1. $(S, *)$ has an **identity element** if there exists e in S such that the operation "does nothing" with e — for all a in S, we have $a * e = e * a = a$.
2. $(S, *)$ has **inverses** if for *all* a in S, there exists some b in S that "cancels" a — namely $a * b = b * a = e$, where e is the identity.

We often write the inverse of a as a^{-1}, where this is *not necessarily* 1 *divided by* a. It's like how the inverse of a matrix $\mathbf{T}$ is written $\mathbf{T}^{-1}$. Notice that to have inverses, there must be an identity. Drill these five properties — closure, associativity, identity, inverses, commutativity — into your head!

3 Day 3

Listing out the properties of $(S, *)$ can get tiring — so mathematicians gave bundles of these properties special names. $(S, *)$ has **group structure** if $*$ satisfies the following properties: for all $a, b, c \in G$,

1. **Closure**: $a * b \in G$.
2. **Associativity**: $a * (b * c) = (a * b) * c$.
3. **Identity**: There exists $e \in G$ such that $a * e = e * a = a$.
4. **Inverses**: There exists $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

The operation $*$ need not be commutative — that is, a set where $a * b$ isn't always $b * a$ can still be a group. If it is commutative, we say $(S, *)$ has **Abelian group structure** (pronounced ah-BELL-ee-an).

4 Day 4

These examples of groups have been pretty shit, in my opinion. We're gonna examine much more interesting ones from geometry, number theory, and linear algebra. **We begin with geometry — specifically symmetry**.

> **Example: The Cube**. Paint each 8 faces of a cube a different color, and imagine the set of distinct rotations of the cube. We can compose two rotations by performing one after the other. Let's denote the set of rotations by G and the operation by $*$.
>
> $(G, *)$ is closed and associate, because duh. There's an identity rotation — namely doing nothing — and each rotation can be "performed in reverse" to yield inverses. So, the symmetries $(G, *)$ of a cube constitute a group.

In fact, **the symmetries of every shape form a group**: octahedrons (this is an AMC problem!), spheres, you name it. You can always compose rotations of the shape, performing one after the other, and expect the four group properties to hold.

5 Day 5