# MATH-UA 140: Assignment 4

James Pagan, October 2023

Professor Raquépas

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## 1 Problem 1

**Part (a)**: We have that

$$\begin{aligned}
\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix} &\implies \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ -\cos(\theta) & -\frac{\cos^2(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&\implies \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ 0 & -\frac{\cos^2(\theta)}{\sin(\theta)} - \sin(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ 0 & -\frac{1}{\sin(\theta)} & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\end{aligned}$$

**Part (b)**: Performing the same actions on $I_3$ yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{\cos(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Part (c)**: Following from our work in Part (a),

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ 0 & -\frac{1}{\sin(\theta)} & 0 \\ 0 & 0 & -1 \end{bmatrix} \implies \begin{bmatrix} 1 & -\frac{\sin(\theta)}{\cos(\theta)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Part (d)**: Performing the same actions on our matrix in Part (a) yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -\frac{\cos(\theta)}{\sin(\theta)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} \frac{1}{\cos(\theta)} & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\implies \begin{bmatrix} \frac{1}{\cos(\theta)} - \frac{\sin^2(\theta)}{\cos(\theta)} & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**Part (e)**: Because $R_0$ is the identity matrix,

$$BA = (E^{-1}EI)A = E^{-1}EA = R_0 = I.$$

It is well-known that all left-inverses of matricies are also right-inverses, so $AB = 1$; we conclude that $\boxed{A^{-1} = B}$.

**Part (f)**: We invoked the fact that when $\theta$ is not an integer multiple of $\frac{1}{2}\pi$, then $\cos(\theta)$ and $\sin(\theta)$ are both nonzero, so we may divide by trigonometric functions — in short, when $\boxed{\text{division by } \cos(\theta) \text{ or } \sin(\theta) \text{ occured}}$.

## 2 Problem 2

**Part (a)**: We have that

$$\begin{bmatrix} 2 & 0 & 2 \\ -2 & 1 & 4 \\ 4 & -1 & -2 \\ 6 & 0 & 6 \end{bmatrix} \implies \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & -1 & -6 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Part (b)**: As the first two columns are pivots and the third is free, we deduce that $\boxed{\text{the rank is 2 and the nullity is 1}}$.

**Part (c)**: As noted above, the third column is a $\boxed{\text{free column}}$.

**Part (d)**: Because the nullity of the matrix is 1, the null space is the span of only a single vector — one such vector is trivially

$$\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}.$$

**Part (e)**: One such vector is $\boxed{(0, -\sqrt{7}, 0)}$, as may be verified by a trivial calculation.

**Part (f)**: The general solution to the equation is clearly all vectors of the form

$$\begin{bmatrix} 0 \\ -\sqrt{7} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

for $\lambda \in \mathbb{R}$.