MATH-UA 140: Assignment 9

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1 Problem 1

Part (a): The trace of the matrix
$$\begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}$$
is $1 + 1 = \boxed{2}$.

Part (b): Define the matricies C and D as follows:
$$C = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix},$$
We thus deduce that
$$\begin{aligned} \operatorname{tr}(CD) &= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} \\ &= \operatorname{tr}(DC). \end{aligned}$$

Part (c): Suppose that for a 2-by-2 matrix A that there exists a diagonal 2-by-2 matrix Λ and an invertible 2-by-2 matrix J such that $A = J\Lambda J^{-1}$. Then by the result of Part (b),
$$\operatorname{tr}(A) = \operatorname{tr}\left(J\Lambda J^{-1}\right) \operatorname{tr}\left((J\Lambda)J^{-1}\right) = \operatorname{tr}\left(J^{-1}(J\Lambda)\right) = \operatorname{tr}(\Lambda).$$

Part (d): In Assignment 8, we deduced that the eigenvalues of
$$\begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}$$
are $\frac{3}{4}$ and $\frac{5}{4}$. The trace of this matrix is 2 — which is the same as the sum of its eigenvalues.

2 Problem 2

Part (a) We define:
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Then

$$A^\top = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix},$$

so

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = \operatorname{tr}\left(A^\top\right).$$

Part (b): We have that

$$\operatorname{tr}\left(\begin{bmatrix} 0 & 9 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 7 & 6 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 63 & 54 \\ 70 & 68 \end{bmatrix}\right) = 131 \neq (6)(13) = \operatorname{tr}\left(\begin{bmatrix} 0 & 9 \\ 4 & 6 \end{bmatrix}\right)\operatorname{tr}\left(\begin{bmatrix} 7 & 8 \\ 7 & 6 \end{bmatrix}\right).$$

Part (c): We have that if

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix},$$

then

$$\begin{aligned} \operatorname{tr}(A+B) &= \operatorname{tr}\left(\begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{bmatrix}\right) \\ &= \sum_{i=1}^{n}(a_{ii}+b_{ii}) \\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= \operatorname{tr}(A) + \operatorname{tr}(B). \end{aligned}$$

Part (d): We have that

$$\begin{aligned} \det\left(\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}\right) &= \det\left(\begin{bmatrix} -1 & 4 \\ 0 & 9 \end{bmatrix}\right) \\ &= -9 \\ &\neq -2 - 1 \\ &= \det\left(\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}\right). \end{aligned}$$

Part (e): We have that

$$\operatorname{tr}(\lambda A) = \operatorname{tr}\left(\begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{n1} & \cdots & \lambda a_{nn} \end{bmatrix}\right) = \sum_{i=1}^{n} \lambda a_{ii} = \lambda \sum_{i=1}^{n} a_{ii} = \lambda \operatorname{tr}(A).$$

Part (f): We have that

$$\det(2I) = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4 \neq 2 = 2\det(I).$$

3 Problem 3

Part (a): $\boxed{\text{Yes}}$, P is diagonalizable. Observe that

$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the direct sum of the eigenspaces of P has dimension 3, so P is diagonalizable.

Part (b): For all eigenvalues λ of Q,

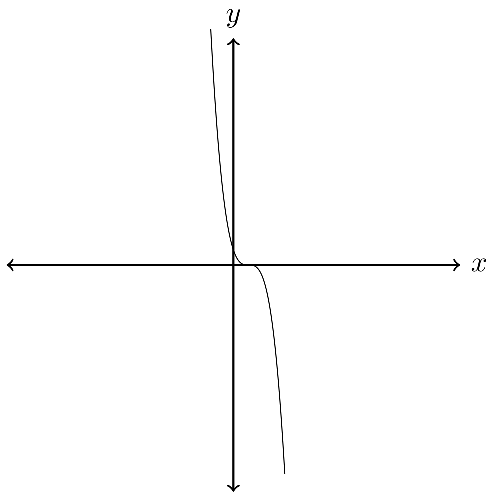
$$0 = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(3-\lambda).$$

Thus, the eigenvalues are $\boxed{\lambda = -1, 1, 3}$.

Part (c): One such eigenvector is $\hat{\mathbf{i}}$, as

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{\mathbf{i}}.$$

Part (d): The following diagram was (painfully) made on TikZ:



The eigenvalue appears to be 1. To compute this, note that all eigenvalues λ satisfy

$$0 = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3,$$

so $\lambda = 1$ is the only eigenvalue. Now, consider all vectors such that

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then $a + b = a$ and $b + c = b$; thus, $a = b = 0$. We conclude that the eigenspace of the eigenvalue 1 is the following space:

$$\boxed{\left\{ \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid \lambda \in \mathbb{R} \right\}}.$$

Part (e): We have that

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ 2+\sqrt{2} \\ 1+\sqrt{2} \end{bmatrix} = (1+\sqrt{2}) \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{2} \\ 2-\sqrt{2} \\ 1-\sqrt{2} \end{bmatrix} = (1-\sqrt{2}) \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

As per orthogonality, realize that

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = 1^2 + (\sqrt{2})(-\sqrt{2}) + 1 = 1 - 2 + 1 = 0,$$

so the two eigenvectors are orthogonal.

Part (f): Realize that S is symmetric, and recall that all eigenvectors of a symmetric matrix are orthogonal. It thus suffices to find a vector orthogonal to the two above, one of which is $\hat{\mathbf{\imath}} - \hat{\mathbf{k}}$:

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 + \sqrt{2}(0) - 1 = 0 = 1 - \sqrt{2}(0) - 1 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

To find this eigenvector's eigenvalue, we simply compute $S(\hat{\mathbf{\imath}} - \hat{\mathbf{k}})$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

so the other eigenvalue is $\boxed{1}$.