MATH-UA 349: Honors Algebra II

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1 Problem 1

1.1 Part (a)

Performing reduction, it is easy to verify that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 & 17 \\ -1 & 0 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that $(x, y, z) \in \mathbb{Z}^3$ is mapped to zero by the left-hand side if and only if $x = y = 0$. Therefore, the solutions are

$$\begin{bmatrix} 2 & 1 & 17 \\ -1 & 0 & -10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \boxed{\begin{bmatrix} 17z \\ -10z \\ z \end{bmatrix}}$$

across all $z \in \mathbb{Z}$.

1.2 Part (b)

Based on the above diagonalization, a basis of the image of $\mathbf{T}$ is $\mathbf{Q}(\mathbf{e}_1)$ and $\mathbf{Q}(2\mathbf{e}_2)$, where

$$\mathbf{Q}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \implies \mathbf{Q} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the two basis vectors we seek are

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}.$$

2 Problem 2

We perform a series of operations on the matrix, after all of which the underlying Abelian group is invariant:

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 6 \end{bmatrix} \implies \begin{bmatrix} 0 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix} \implies \begin{bmatrix} -2 & -1 \\ 1 & 4 \end{bmatrix} \implies \begin{bmatrix} 0 & 7 \\ 1 & 4 \end{bmatrix} \implies \begin{bmatrix} 7 \end{bmatrix}.$$

Thus the matrix represents $\mathbb{Z} / 7\mathbb{Z} = \boxed{\mathbb{Z}_7}$.

3 Problem 3

Assembling these relations into a matrix, we obtain a presentation of V:

$$\begin{bmatrix} 7 & 5 & 2 \\ 3 & 3 & 0 \\ 13 & 11 & 2 \end{bmatrix}.$$

We now perform a similar series of operations that simplifies this presentation matrix:

$$\begin{bmatrix} 7 & 5 & 2 \\ 3 & 3 & 0 \\ 13 & 11 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 0 \\ 7 & 7 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 7 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 7 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & -12 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} -6 \\ -12 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 6 \end{bmatrix}.$$

Hence $V \cong \mathbb{Z}_6 \cong C_6$. We now utilize the following lemma:

Lemma 1. *Suppose that m and n are relatively prime, positive integers. Then $C_{mn} \cong C_m \times C_n$.*

Proof. Let a generate C_{mn}. It is clear that a^m has order n and a^n has order m, so

$$(a^m) \cong C_n \qquad \text{and} \qquad (a^n) \cong C_m.$$

We seek to use the Chinese Remainder Theorem for $\mathbb{Z}$-modules. We must verify two conditions:

1. For all $k \in \{1, \dots, mn\}$, Bézout's Identity ensures the existence of $b, c \in \mathbb{Z}$ such that $k = bm + cn$. Hence $a^k = a^{am+bn} = (a^m)^b (a^n)^c$. Thus we have the subgroup product $(a^n) \times (a^m) = C_{mn}$.
2. It is trivial that since n and m are relatively prime, $(a^n) \cap (a^m) = \{e\}$.

Thus, we conclude by the Chinese Remainder Theorem that

$$C_{mn} \quad \cong \quad C_{mn} \,/\, (a^n) \times C_{mn} \,/\, (a^m) \quad \cong \quad (a^m) \times (a^n) \quad \cong \quad C_n \times C_m.$$

This completes the proof. $\square$

We thus conclude that $\boxed{V \cong C_2 \times C_3}$.

4 Problem 4

All Abelian groups of order 400 are of the form

$$C_{n_1} \times \cdots \times C_{n_i},$$

where $n_1, \ldots, n_i > 2$ are integers such that $n_1 \cdots n_i = 400$. For our purposes, denote a cyclic group C_n as **irreducible** if there do not exist integers $a, b > 2$ such that $C_n \cong C_a \times C_b$. Several observations are in order:

1. **Order**: If $C_{n_1} \times \cdots \times C_{n_i}$ has order 400, then $n_1 \cdots n_i = 400$ implies that each $n_1, \ldots, n_k$ must divide 400. In particular, the prime factorizations of n_i must not contain primes other than 2 and 5.

2. **Irreducibility implies Prime Powers**: Suppose that C_{n_i} is cyclic, where n_i is not a prime power; then there exist relatively prime integers $a, b > 2$ such that $n_i = ab$. Hence by Lemma 1 (Problem 3), we have $C_{n_i} \cong C_a \times C_b$ — so C_{n_i} is not irreducible. Contraposition yields that all irreducible C_{n_i} must be a prime power.

3. **Prime Power implies Irreducibility**: Suppose that C_{p^n} is a cyclic group. Then C_{p^n} contains an element of order p^n, but all direct products

$$C_{p^a} \times C_{p^b}$$

   contain an element of greatest order $\max\{a, b\}$. Thus C_{p^n} is irreducible.

We deduce that G is an Abelian group of order 400 if and only if

$$G \quad \cong \quad C_{2^{n_1}} \times \cdots \times C_{2^{n_i}} \times C_{5^{m_1}} \times \cdots \times C_{5^{m_j}},$$

where n_k and m_k are positive integers such that $n_1 + \cdots + n_i = 4$ and $m_1 + \cdots + m_j = 2$. The number of distinct Abelian groups G is the number of distinct solutions to these equations — which is the number of partitions of 4 times the number of partitions of 2. Since there are 5 partitions of the former and 2 partitions of latter, the answer is $5 \times 2 = \boxed{10}$.

5 Problem 6

Let R be the ring of polynomials with complex coefficients and $2^{\aleph_0}$ variables — in particular, let x_a be a unique variable for all $a \in \mathbb{R}$. An example of an element of R is as follows:

$$i(x_\pi)^{100} + (2-i)(x_{\sqrt{2}})^9(x_0) + (x_{-1}) + i\sqrt{4}.$$

It is easy to verify that R is a ring. The principal ideal

$$\mathfrak{a} = (x_1, x_2, x_3, x_4, \ldots) \subseteq R$$

is clearly not finitely generated. For any finite set of elements $a_1, \ldots, a_k \in \mathfrak{a}$, let i be the maximum integer such that x_i appears in one of $a_1, \ldots, a_k$. Then $x_{i+1} \notin (a_1, \ldots, a_k)$, so this set of elements fails to generate $\mathfrak{a}$.