

# Jänich: Fundamental Concepts

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## Contents

<b>1</b>	<b>The Concept of a Topological Space</b>	<b>2</b>
1.1	Definition . . . . .	2
1.2	Basic Consequences . . . . .	2
<b>2</b>	<b>Metric Spaces</b>	<b>4</b>

# 1 The Concept of a Topological Space

We proceed assuming basic familiarity with sets, as seen in RealAnalysis/babyrudin2.tex.

## 1.1 Definition

A **topological space**  $(X, \mathcal{O})$  is a set  $X$  and a set  $\mathcal{O}$  of subsets of  $X$ , called **open sets**, such that the following three axioms hold:

1. Any arbitrary union of open sets in  $\mathcal{O}$  lies in  $\mathcal{O}$ .
2. Any finite intersection of open sets in  $\mathcal{O}$  lies in  $\mathcal{O}$ .
3.  $X$  itself and  $\emptyset$  are open sets in  $\mathcal{O}$ .

The set  $\mathcal{O}$  is a **topology** in  $X$ . Henceforth, we will speak of a topological space  $X$  instead of the pair  $(X, \mathcal{O})$ . The following concepts are critical to Point-Set Topology:

1. A set  $F \subseteq X$  is **closed** if its complement is open; that is, if  $F^c \in \mathcal{O}$ .
2. A set  $N \subset X$  is a **neighborhood** of  $x$  if  $N$  contains an open set  $U$  which contains  $x$ .
3. A point  $x$  is an **interior**, **exterior**, or **boundary point** of a set  $S$  according to whether  $S$ ,  $S^c$ , or neither is a neighborhood of  $x$ .
4. The set  $\overset{\circ}{S}$  of all the interior points of  $S$  is the **interior** of  $S$ .
5. The set  $\overline{S}$  of all the interior and boundary points of  $S$  is the **closure** of  $S$ .

Naturally, exterior points of  $S$  are interior points of  $\overline{S}$  and the complement of  $\overset{\circ}{S}$  is  $\overline{S^c}$ .

## 1.2 Basic Consequences

The following theorem allows for an alternative definition of topological spaces  $(X, \mathcal{O})$  by *closed sets*:

**Theorem 1.** *Let  $X$  be a topological space. Then*

1. *Any arbitrary intersection of closed sets is closed.*
2. *Any finite union of closed sets is closed.*
3.  *$X$  itself and  $\emptyset$  are closed.*

*Proof.* Let  $F_\alpha$  be a collection of closed sets. Since their compliments are open, we have

$$F_\alpha^c \text{ are open} \implies \bigcup_\alpha F_\alpha^c \text{ is open} \implies \left( \bigcup_\alpha F_\alpha^c \right)^c \text{ is closed} \implies \bigcap_\alpha F_\alpha \text{ is closed.}$$

If we let  $F_n$  be closed for  $n \in \{1, \dots, k\}$ , a similar argument follows:

$$F_n^c \text{ are open} \implies \bigcap_{n=1}^k F_n^c \text{ is open} \implies \left( \bigcap_{n=1}^k F_n^c \right)^c \text{ is closed} \implies \bigcup_{n=1}^k F_n \text{ is closed.}$$

The sets  $X$  and  $\emptyset$  are clearly closed, which completes the proof.  $\square$

**Theorem 2.** *A set is open if and only if all of its points are interior.*

*Proof.* If each point of  $S$  is interior, then  $x \in S$  implies the existence of an open set  $U_x$  such that  $x \in U_x \subseteq S$ . Thus we define:

$$U = \bigcap_{x \in \overset{\circ}{S}} U_x.$$

Observe that  $U$  is open. We claim that  $S = U$  by a two-part argument:

1. Suppose  $x \in S$ . Then clearly  $x \in U_x \subseteq U$ , so  $S \subseteq U$ .
2. Suppose  $x \in U$ . Then  $x \in U_y$  for some  $y$ ; since  $U_y \subseteq S$ , we have  $x \in S$ . Thus  $U \subseteq S$ .

Hence  $S$  is open. The reverse direction follows naturally: if  $S$  is open and  $x \in S$ , then  $S$  is a neighborhood of all  $x$ . Hence all  $x$  is interior.  $\square$

**Theorem 3.** *The interior of  $S$  is the union of all open sets contained in  $S$ .*

*Proof.* Let  $U$  be the union of all open sets contained in  $S$ . We claim that  $\overset{\circ}{S} = U$  by a two-part argument:

1. Suppose  $x \in \overset{\circ}{S}$ . Then there exists an open set  $U_x$  such that  $x \subseteq U_x \subseteq S$  — hence  $x \in U_x \in U$ .
2. Suppose  $x \in U$ . Then  $x$  lies in an open set contained in  $S$ , so  $x \in S$ .

Hence  $\overset{\circ}{S} = U$   $\square$

**Corollary 1.** *The interior of a set is open.*

By taking complements of these results about open sets and interiors, we find:

1. A set is closed if and only if all of its points are interior or boundary points.
2. The closure of  $S$  is the intersection of all closed sets containing  $S$ .
3. The closure of a set is closed.

## **2 Metric Spaces**