Rudin: Integration of Differential Forms

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1 Integration

1.1 Definition

Suppose I_n is a **n-cell** in $\mathbb{R}^n$ — a multi-dimension analogue of a box, namely

$$I_n = \{(x_1, \dots, x_n) \mid a_i \leq x_i \leq b_i \text{ for } i \in \{1, \dots, n\}\}.$$

We define I^j as the j-cell in $\mathbb{R}^j$ defined by the first j inequalities of I^n. For a C^1 function f, put $f = f_n$, and define f_{n-1} as

$$f_{n-1}(x_1, \dots, x_{n-1}) = \int_{a_n}^{b_n} f_n(x_1, \dots, x_{n-1}, x_n) \, \mathrm{d}x_n \, .$$

We repeat this process and optain functions f_j, continuous on f_{j-1}, until we arrive at a *number* f_0, which is called the **integral** of f over I^n.

2 Differential Forms

2.1 Prerequisites

In preparation for the construction of differential forms, we develop the notion of a compact set. If $f : D \subset \mathbb{R}^k \to \mathbb{R}^n$ is C^1, then D is **compact set** if there exists an open set W containing D and a C^1 mapping $g : W \to \mathbb{R}^n$ such that $f(\mathbf{x}) = g(\mathbf{x})$ for all $x \in D$.

A **k-surface** in an open set $U \subset \mathbb{R}^n$ is a mapping from a compact set $D \subset \mathbb{R}^k$ into U.

2.2 Definition

Let $U \subset \mathbb{R}^n$ be an open set, let $D \subset \mathbb{R}^k$ be a compact set, and let $\Phi : D \to U$ be a surface. A **differential form of order k**, briefly called a **k-form**, is a mapping from surfaces Φ to real numbers, notated as

$$\omega = \sum f_{i_1,\ldots,i_k}(\mathbf{x})\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$$

that assignes to each Φ a number $\omega(\Phi) = \int_\Phi \omega$ according to the rule

$$\int_\Phi \omega = \int_D \sum f_{i_1,\ldots,i_k}(\mathbf{x}) \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)}\, \mathrm{d}\mathbf{u}\,,$$

where if the components of Φ are $\Phi_1, \ldots, \Phi_n$, the Jacobian is determined by the mapping $(u_1, \ldots, u_k) \to (\Phi_{i_1}(\mathbf{u}), \ldots, \Phi_{i_n}(\mathbf{u}))$ and the functions $f_{i_1,\ldots,i_n}$ are real and continuous.

A k-form ω is C^m for some $m \in \mathbb{Z}_{>0}$ if every function $f_{i_1,\ldots,i_n}$ is C^m. A 0-form in U is defined to be a continuous funciton in U.

2.3 Basic Properties

Let ω, ω_1, and ω_2 be k-forms. We write $\omega_1 = \omega_2$ if and only if $\omega_1(\Phi) = \omega_2(\Phi)$ for all k-surfaces Φ in U. We define $c\omega$ for $c \in \mathbb{R}^n$ by

$$\int_\Phi c\omega = c \int_\Phi \omega,$$

and we write $\omega = \omega_1 + \omega_2$ if and only if

$$\int_\Phi \omega = \int_\Phi \omega_1 + \int_\Phi \omega_2.$$

Now, consider the k-form $\omega = f(\mathbf{x})\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$. If we define $\overline{\omega}$ as the k-form obtained by swapping some pair of subscripts of ω, we swap the sign of the Jacobian — thus finding that $\overline{\omega} = -\omega$. A specal case of this is the *anticommutative relation*

$$\mathrm{d}x_i \wedge \mathrm{d}x_j = -\,\mathrm{d}x_j \wedge \mathrm{d}x_i \,.$$

With this in mind, we define $\mathrm{d}x_i \wedge \mathrm{d}x_i = 0$. More generally, if $i_r = i_k$ for some indicies over the k-form $\omega = f(\mathbf{x})\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$, we swap these indicies and obtain that $\omega = \overline{\omega} = -\omega$, so $\omega = 0$.

Consequentially, the only k-form in the open set $U \subset \mathbb{R}^n$ if $k > n$ is 0.

2.4 Basic k-forms

If $I = (i_1, \ldots, i_k)$ is an increasing sequence of k integers from $\{1, \ldots, n\}$, we call I an **increasing k-index** and use the notation

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \,.$$

The form $\mathrm{d}x_I$ is called a **basic k-form**.

There are clearly $\binom{n}{k}$ basic k-forms. Clearly we can "swap the indicies" of a form to express it as a sum of basic k-forms: namely, for all distinct $j_1, \ldots, j_k \in \{1, \ldots, n\}$, we may call its increasing permutation J and observe that

$$\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k} = \epsilon(j_{1,\ldots,j_k})\,\mathrm{d}x_J \,,$$

where $\epsilon(j_1, \ldots, j_k)$ is 1 or -1. Thus, from this point on, we will use the **standard presentation** of ω by writing

$$\omega = \sum_I f_I(\mathbf{x})\,\mathrm{d}x_I \,.$$

This is a *unique* way to represent a k-form, as we will prove in the following result:

Theorem 1. *Suppose a differential k-form ω in an open set $U \subset \mathbb{R}^n$ has the standard presentation*

$$\omega = \sum_I f_i(\mathbf{x})\,\mathrm{d}x_I \,.$$

If $\omega = 0$ in U, then $f_I(\mathbf{x}) = 0$ for every increasing k-index I and $\mathbf{x} \in U$.

Proof.