MATH-UA 140: Assignment 3

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1 Problem 1

Part (a): Observe that as $B\mathbf{x} = \mathbf{0}$

$$\mathbf{x} = EB\mathbf{x} = E(B\mathbf{x}) = E\mathbf{0} = \mathbf{0};$$

thus, $\boxed{\mathbf{x} \in \operatorname{null} U \setminus \{\mathbf{0}\}}$.

Part (b): Observe that if $U\mathbf{y} = 0$, then $EB\mathbf{y} = \mathbf{0}$. Thus, $E(B\mathbf{y}) = \mathbf{0}$, so $B\mathbf{y}$ is in the null spacae of E. As E is invertible, its null space consists exclusibely of the zero vector — therefore, $B\mathbf{y} = \mathbf{0}$, and $\boxed{\mathbf{y} \in \operatorname{null} U \setminus \{\mathbf{0}\}}$.

Part (c): The above result demonstrates that $U\mathbf{x} = \mathbf{0}$ if and only if $B\mathbf{x} = 0$. We may view this matrix-vector multiplication as a linear combination of the columns of U or B — doing so yields that a linear combination of the columns of B gives $\mathbf{0}$ if and only if $\boxed{\text{the same combination of the columns of } U \text{ gives } \mathbf{0}}$.

2 Problem 2

Part (a): $\boxed{\text{yes}}$; for example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

after performing the multiplication has a different column space.

Part (b): $\boxed{\text{no}}$, performing elimination cannot change the null space of the matrix.

Part (c): $\boxed{\text{no}}$, performing elimination cannot change the row space of the matrix.

Part (d): $\boxed{\text{no}}$, performing elimination will not change the column rank.

Part (e): $\boxed{\text{no}}$, performing elimination will not change the row rank.

Part (f): $\boxed{\text{no}}$, performing elimination will not change the nullity.

3 Problem 3

Part (a): The column vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are clearly independent, and the first and second may be combined to generate the third via

$$-2\pi \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \\ 0 \\ 0 \end{bmatrix},$$

so the $\boxed{\text{first, second, and fourth}}$ are the fewest possible columns that span the column space of A.

Part (b): The dimension is $\boxed{3}$, as the first, second, and fourth rows are independent (by the placement of their zeroes), and the third row may be achieved as a trivial linear combination of the others (namely, with scalars equal to zero).

Part (c): Such a matrix is trivially the permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The Fundamental Theorem of Linear Maps guarantees that because the column space has dimension three, the null space has dimension one. Further observe that

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & \pi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2\pi \\ -\pi \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so the $\boxed{\text{span of } (2\pi, -\pi, 1, 0)}$ is the null space of A by necessity of dimension.

4 Problem 4

We have that

$$(A^{-1}B^\top)^\top A^\top = (B^\top)^\top (A^1)^\top A^\top = B(AA^{-1})^\top = BI^\top = BI = \boxed{B}.$$

5 Problem 5

Part (a): Clearly, the element of $\mathbb{R}^n$ is $\lambda\mathbf{x}+; u\mathbf{y}$.

Part (b): The dot product we seek is

$$\mathbf{x} \cdot \mathbf{y} = \boxed{\sum_{i=1}^{n} x_i y_i}$$

Part (c): The matrix-vector product is

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y^1 \\ \vdots \\ y_n \end{bmatrix} = \boxed{\sum_{i=1}^{n} x_i y_i}.$$

Part (d): The matrix product is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_3 \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_3 \end{bmatrix} = \begin{bmatrix} y_1 x_1 & \cdots & y_1 x_3 \\ \vdots & & \vdots \\ y_3 x_1 & \cdots & y_3 x_3 \end{bmatrix}$$

Part (e): The column rank is $\boxed{1}$, as every column of the matrix is a scalar multiple of the other via the x-values.