MATH-UA 140: Assignment 1

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1 Problem 1

(a) We have that

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 1(2) + 3(-1) + 2(1) = 1.$$

(b) Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ; as the norms of $\mathbf{u}$ and $\mathbf{v}$ are nonzero,

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{1^2 + 3^2 + 2^2} \times \sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{\sqrt{14} \times \sqrt{6}} = \frac{\sqrt{21}}{42}.$$

Therefore, $\theta = \arccos\left(\frac{\sqrt{21}}{42}\right) \approx 1.46$ radians.

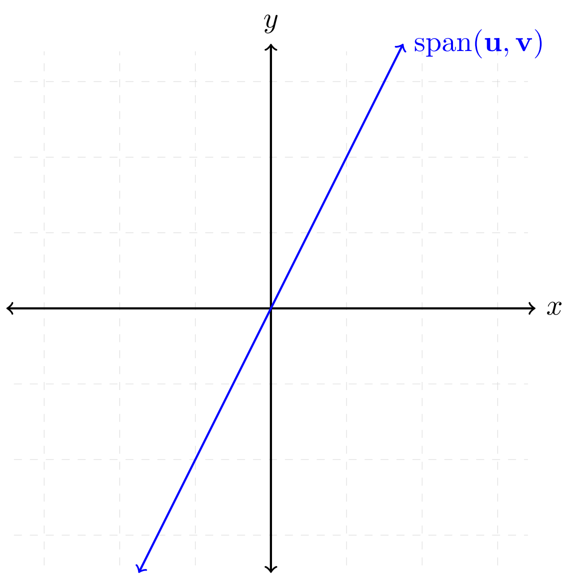
(c) No; observe that

$$\mathbf{u} + 3\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 5 \end{bmatrix} = \mathbf{w}.$$

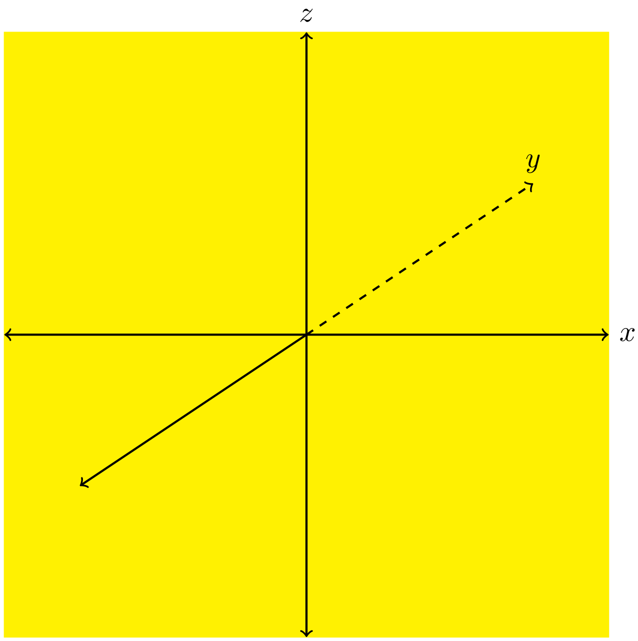
Therefore, the family $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not independent.

2 Problem 2

(a) The span of $\mathbf{u}$ and $\mathbf{v}$ is a line,, pictured on the xy-plane below:



(b) The span of $\hat{\mathbf{i}}$ and $2\,\hat{\mathbf{k}}$ is the zx-plane, pictured in yellow in $\mathbb{R}^3$ as shown:



3 Problem 3

(a) True. If $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v} = 0$. Therefore,

$$\mathbf{w} \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} - \mathbf{w} \cdot \mathbf{v} = 0 - 0 = 0,$$

so $\mathbf{w}$ is perpendicular to $\mathbf{u} - \mathbf{v}$.

(b) False. The canonical basis vector $\hat{\mathbf{i}}$ is perpendicular to $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, but $\hat{\mathbf{j}}$ is not collinear with $\hat{\mathbf{k}}$.

(c) True. If $\mathbf{v} = \mathbf{0}$, then $\mathbf{v}$ is colinear with all vectors $\mathbf{w}$. Otherwise, observe that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u}, \mathbf{v} \neq 0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent — as $\mathbb{R}^2$ has dimension 2, these two linearly independent vectors are a basis of $\mathbb{R}^2$.

Let λ_1 and λ_2 be constants such that $\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = \mathbf{w}$. Then

$$0 = \mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v}) = \lambda_1 (\mathbf{u} \cdot \mathbf{u}) + \lambda_2 (\mathbf{u} \cdot \mathbf{v}) = \lambda_1 \|\mathbf{u}\|^2.$$

However, $\|\mathbf{u}\|^2 \neq 0$ as $\mathbf{u} \neq \mathbf{0}$; then $\lambda_1 = 0$. Then

$$\mathbf{w} = \lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = 0\mathbf{u} + \lambda_2 \mathbf{v} = \lambda_2 \mathbf{v};$$

so $\mathbf{w}$ and $\mathbf{v}$ are collinear.

4 Problem 4

Using basic properties of the dot product, we have that

$$\begin{aligned}
\frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2}{4} &= \frac{(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) - (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}{4} \\
&= \frac{(\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v})}{4} \\
&= \frac{4(\mathbf{u}\cdot\mathbf{v})}{4} \\
&= \mathbf{u}\cdot\mathbf{v}.
\end{aligned}$$

5 Problem 5

(a) The form we desire is

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}.$$

(b) The form we desire is

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}.$$

6 Problem 6

(a) As the former matrix is the identity matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

(b) The matrix product we seek is

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$