MATH-UA 349: Homework 4

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6.1 Part (a)

Proof. Let M be a simple R-module. Since $(M, +)$ is a simple Abelian group, it must be isomorphic to a finite cyclic group of prime order — say C_p. Hence M is generated by one element x.

Lemma 1. *M is isomorphic to a quotient of R.*

Proof. Define a mapping $f : R \to M$ by the rule $f(a) = ax$. This is an R-module homomorphism, since $a, b \in R$ implies

$$\begin{aligned} f(a+b) &= (a+b)x = ax + bx = f(a) + f(b) \\ f(ab) &= abx = a(bx) = af(b). \end{aligned}$$

ϕ is surjective, since $f(1+\cdots+1) = x+\cdots+x$, which generates the entirety of M. Thus if we let $\mathfrak{m} = \operatorname{Ker} f$, the First Isomorphism Theorem yields the desired $R/\mathfrak{m} \cong M$.

Because the ring $R/\mathfrak{m}$ has prime order, it contains no proper nonzero ideals. Thus the quotient is a field, so $\mathfrak{m}$ is maximal. This completes the proof. □

6.2 Part (b)

Proof. Suppose $\phi : V \to V'$ is a homomorphism of simple R-modules. Then V and V' must be finite, and the submodules $\operatorname{Ker} \phi \subseteq V$ and $\operatorname{Im} \phi \subseteq V'$ must be either 0 or the module itself.

1. If $\operatorname{Ker} \phi = V$: then ϕ is the zero homomorphism.

2. If $\operatorname{Ker} \phi = 0$: then $V \cong \operatorname{Im} \phi$ by the First Isomorphism Theorem. Thus $\operatorname{Im} \phi$ is a nonzero submodule of V', so $\operatorname{Im} \phi = V'$. We conclude that $V \cong V'$.

This yields the desired result. $\square$

7 Problem 7

7.1 Part (a)

7.2 Part (b)