

Rudin: Basic Topology

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1 Finite, Countable, and Uncountable Sets

1.1 Functions

A **function** or **mapping** from a set A to a set B is an assignment of each element of A to an element of B . The set A is called the **domain**, B is called the **codomain**, the elements $f(x)$ are called the **values** of f , and the set of all $f(x)$ is called the **image** of f . Such a relation is notated as $f : A \rightarrow B$.

The **inverse image** $f^{-1}(E)$ of a subset $E \subset B$ is the set of all $x \in A$ such that $f(x) \in E$. $f^{-1}(y)$ for $y \in B$ denotes the set of all $x \in A$ such that $f(x) = y$. If $f^{-1}(y)$ contains of one element of A for each $y \in B$, then f is said to be an **bijective** (or one-to-one) mapping of A into B .

If there exists a bijective mapping of A onto B , we say that A and B can be put into **one-to-one correspondence** (or that A and B have the same cardinal number, or that they are equivalent), and we write $A \sim B$. Trivially, this relation has the following properties:

- **Reflexivity:** $A \sim A$.
- **Symmetry:** $A \sim B$ if and only if $B \sim A$.
- **Transitivity:** $A \sim B$ and $B \sim C$ implies that $A \sim C$.

Any relation with these three properties is called an **equivalence relation**. Intuitively, we have that $A \sim B$ if and only if A and B have the “same number of elements”.

1.2 Cardinality

Let J_n be the set whose elements are the integers $0, \dots, n-1$; let J be the set consisting of all nonnegative integers. Then for any set A , we say:

- A is **finite** if $A \sim J_n$ for some n .
- A is **infinite** if A is not finite.
- A is **countable** if $A \sim J$.
- A is **uncountable** if A is neither finite nor countable.
- A is **at most countable** if A is neither finite or countable.

Let K be the set of nonnegative integers. Then K has the same cardinal number as J :

$$\begin{aligned} K &= 0, 1, 2, 3, 4, 5, 6, 7, 8, \dots \\ J &= 0, -1, 1, -2, 2, -3, 3, -4, 4, \dots \end{aligned}$$

The function exhibited by the relation above is the following function:

$$\begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n+1}{2} & n \text{ is odd.} \end{cases}$$

A finite set cannot have the same cardinal number as one of its proper subsets. However, this is always possible for infinite sets — for instance, via a subset formed by removing one single element. This is an alternative definition of an infinite set.

1.3 Sequences

A **sequence** is a function f defined on the set $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{\geq 0}$ (which we shall denote neutrally by J). If $f(x) = x_n$ for $n \in J$, we often denote the total sequence by x_n or by $(x_0), x_1, x_2, x_3, \dots$. The values of f are called the **terms** of the sequence. If A is a set and $x_n \in A$ for all $n \in J$, then x_n is called a sequence in A .

A countable set is the range of a bijective function with domain over J ; therefore, we may regard all countable functions as the range of a sequence with distinct terms. Intuitively, a countable set can be “arranged in a sequence.”

Theorem 1. *Every infinite subset of a countable set A is countable.*

Proof. Suppose $E \subset A$ and E is infinite. Arrange the elements of A into a sequence x_n of distinct elements.

Let n_1 be the smallest integer such that $x_{n_1} \in E$, let n_2 be the smallest integer larger than n_1 such that $x_{n_2} \in E$, and so on. More formally, define n_k recursively:

$$n_k = \min\{m \in \mathbb{Z} \mid x_m \in E, m > \max\{n_1, \dots, n_n\}\}.$$

The fact E is infinite implies that x_{n_1}, x_{n_2}, \dots is an infinite sequence with distinct elements.

The function $f(m) = x_{n_m}$ for $m \in \mathbb{Z}_{>0}$ thus obtains a bijection between A and J . We conclude that A is countable. \square

In some sense, J is the “smallest infinity;” subsets of J are either finite or countable. Conversely, the Axiom of Choice implies that all uncountable sets have a countable subset.

1.4 Union and Intersection

Let A and Ω be sets, and suppose that with each element α of A , there is a corresponding subset of Ω which we denote by E_α . The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. We sometimes refer to a set of sets as a *collection* or *family* of sets.

The **union** of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

If A consists of the integers $1, \dots, n$, we use the notation $S = \bigcup_{i=1}^n E_i$ or $S = E_1 \cup \dots \cup E_n$.

The **intersection** of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in \alpha$ for all $\alpha \in A$. We will use the notation

$$S = \bigcap_{\alpha \in A} E_\alpha,$$

with similar notation above if A is the positive integers or a subset thereof. It is trivial that unions and intersections are associative and commutative.

Theorem 2. *If A , B , and C are sets, the following distributive laws hold:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{2}$$

Proof. For (1), suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in (B \cup C)$ — we *must* have that $x \in B$ or $x \in C$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$, so in all cases, $x \in (A \cap B) \cup (A \cap C)$.

Conversely, suppose $x \in (A \cap B) \cup (A \cap C)$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$. Therefore, $x \in A$, and $x \in B$ or $x \in C$; we conclude that $x \in A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Identity (2) has a similar proof. \square

Several more trivial identities include $A \subset A \cup B$ and $(A \cap B) \subset A$ for all sets A and B . If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$. The empty set is denoted \emptyset .

Theorem 3. Let E_1, E_2, \dots be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Proof. For each $n \in \mathbb{Z}_{>0}$, let E_n be arranged in a sequence x_{n1}, x_{n2}, \dots , and consider the infinite array.

$$\begin{array}{ccccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots & & \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots & & \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots & & \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \end{array}$$

If we “travel in diagonal lines”, we produce the sequence

$$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, \dots$$

This can be formalized by a rather painful argument involving $\frac{n(n+1)}{2}$. If we accept this construction, it is easy to see that each x_{ij} for $i, j \in \mathbb{Z}_{>0} \in S$ lies in the sequence — and clearly each element of the sequence is a member of S .

Then S and the sequence have the same cardinal number, so S is countable. \square

A corollary is that if A is at most countable — and B_α is at most countable for each $\alpha \in A$ — then put

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable. This is because T is equivalent to a subset of S defined in the prior theorem.

Theorem 4. Let A be a countable set, and let B_n be the set of all n -tuples (a_1, \dots, a_n) , where $a_k \in A$ for all $k \in \{1, \dots, n\}$. Then B is countable.

Proof. We use induction. Clearly B_1 is countable, as $A = B_1$; we thus proceed to the assumption that B_n is countable. Realize the following one-to-one correspondence between elements of B_{n+1} and pairs of an element of B_n and A :

$$(a_1, \dots, a_n, a_{n+1}) \iff \{(a_1, \dots, a_n), a_{n+1}\}$$

Then define $B_\alpha = \{b, \alpha \mid b \in B_n\}$ for each $\alpha \in A$. Clearly $B_\alpha \sim B_n$ for fixed α , so each B_α is countable. Then

$$B_{n+1} \sim \bigcup_{\alpha \in A} B_\alpha.$$

By Theorem 3, the right-hand side is countable. This completes the induction. \square

A corollary of this theorem is that \mathbb{Q} is countable, as by the one-to-one correspondence $\frac{a}{b} \iff (a, b)$. Thus $\mathbb{Q} \subset \mathbb{Z}_2$, so \mathbb{Q} is at most countable; \mathbb{Q} must be countable as it contains the countable set \mathbb{Z} .

Theorem 5. *The set A of all sequences whose elements are the digits 0 and 1 is uncountable.*

Proof. Suppose x_1, x_2, \dots is a family of all sequences whose elements are the digits 0 and 1. Consider the element x formed by swapping the first digit of x_1 , the second digit of x_2 , the third digit of x_3 , and so on. We claim that $x \notin \{x_n\}$

Suppose for contradiction that $x \in \{x_n\}$ — namely, that there exists $r \in \mathbb{Z}_{>0}$ such that $x = x_r$. We defined the r -th digit of x to be distinct from x_r , so they cannot be equal — a contradiction.

Thus, any mapping from the positive integers to A will exclude some sequence in A . We conclude that A is not countable. \square

This theorem — combined with knowledge of binary notation — implies that the set of all real numbers is uncountable. We will elaborate on this proof later in this document.

2 Metric Spaces

2.1 Definition

A **metric space** is a set X equipped with a function $d : X \times X \rightarrow \mathbb{R}$ called a **metric** that satisfies the following four axioms for all $x, y, z \in X$:

1. **Positivity:** $d(x, y) \geq 0$, with equality if and only if $x = y$.
2. **Symmetry:** $d(x, y) = d(y, x)$.
3. **Triangle Inequality:** $d(x, z) + d(z, y) \geq d(x, y)$.

The elements of X are called **points**.

2.2 Multiple Complex Variables

The most critical metric spaces are \mathbb{R}^n (particularly \mathbb{R}) and \mathbb{C} . To elaborate upon both simultaneously, these documents will expand upon \mathbb{C}^n , equipped with the Euclidean norm for all $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$:

$$\|\mathbf{z}\| = \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2}$$

Theorem 6. *The Triangle Inequality holds for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$:*

$$\|\mathbf{z}\| + \|\mathbf{w}\| \geq \|\mathbf{z} + \mathbf{w}\|.$$

Proof. Let $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$. Assuming the Triangle Inequality in \mathbb{C} , we have that

$$\begin{aligned} \|\mathbf{z}\| + \|\mathbf{w}\| &= \sqrt{\sum_{i=1}^n |z_i|^2} + \sqrt{\sum_{i=1}^n |w_i|^2} \\ &= \sqrt{\left(\sqrt{\sum_{i=1}^n |z_i|^2} + \sqrt{\sum_{i=1}^n |w_i|^2} \right)^2} \\ &= \sqrt{\sum_{i=1}^n |z_i|^2 + 2\sqrt{\left(\sum_{i=1}^n |z_i|^2\right)\left(\sum_{i=1}^n |w_i|^2\right)} + \sum_{i=1}^n |w_i|^2} \\ &\geq \sqrt{\sum_{i=1}^n |z_i|^2 + 2\sum_{i=1}^n |z_i w_i| + \sum_{i=1}^n |w_i|^2} \\ &= \sqrt{\sum_{i=1}^n (|z_i|^2 + 2|z_i w_i| + |w_i|^2)} \\ &= \sqrt{\sum_{i=1}^n (|z_i| + |w_i|)^2} \\ &\geq \sqrt{\sum_{i=1}^n |z_i + w_i|^2} \\ &= \|\mathbf{z} + \mathbf{w}\|. \end{aligned}$$

Thus, \mathbb{C}^n is a metric space. □

\mathbb{C}^n is also equipped with a dot product that maps vectors to scalars:

$$\mathbf{z} \cdot \mathbf{w} = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n.$$

The properties of the complex dot product are expanded in my Linear Algebra notes: the document `AbstractAlgebra/axler6.tex`.

In \mathbb{R}^k , a **k-cell** is a multi-dimensional analogue of a box, defined if $a_i < b_i$ for all $i \in \{1, \dots, k\}$ as

$$\{(x_1, \dots, x_k) \mid a_i \leq x_i \leq b_i \text{ for all } i \in \{1, \dots, k\}\}.$$

A set $E \in \mathbb{R}^n$ is **convex** if the line collecting any two points of E lies within E ; namely if

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in E$$

for all $\mathbf{x}, \mathbf{y} \in E$ and $\lambda \in [0, 1]$. Trivially, open balls are convex.

2.3 Topological Notions

Let X be a metric space. Then the natural topology upon X is as follows for $x \in X$ and $E \subseteq X$:

- An **open ball** of radius $r \in \mathbb{R}_{>0}$ situated at x (denoted $B_r(x)$) is the set of all $y \in X$ such that $d(x, y) < r$.
- x is a **limit point** of E if every open ball at x contains a point inside E .
- x is an **interior point** of E if there exists an open ball N at x such that $N \subseteq E$.
- x is an **isolated point** of E if $p \in E$ and x is not a limit point of E .
- E is an **open set** if all $y \in E$ are interior points.
- E is a **closed set** if it contains all its limit points.
- The **complement** of E (denoted E^c) is the set of all points $x \in X$ such that $x \notin E$.
- E is **perfect** if E is closed and if every point of E is a limit point of E .
- E is **bounded** if there exists an open ball $B_r(x)$ for $x \in X$ such that $E \subseteq B_r(x)$.
- E is **dense** in X if $E = X$ or every point of X is a limit point of E .

Rudin uses the term **neighborhood** to speak of an open ball; I will use it to speak of a set that contains an open ball.

Theorem 7. *Every open ball is an open set.*

Proof. Denote N by the open ball centered at $x \in X$ with radius $r > 0$, and let $y \in N$ — that is, $d(x, y) < r$.

If $x \neq y$, denote M as the open ball centered at y with radius $r - d(x, y)$. If $z \in M$, then $d(z, y) < r - d(x, y)$, so

$$d(z, x) \leq d(z, y) + d(y, x) < r - d(x, y) + d(y, x) = r.$$

Hence, $z \in N$ and $M \subseteq N$. Then each $y \in N$ is an interior point; the case $x = y$ is trivial. \square

Theorem 8. *If x is a limit point of E , then every open ball at x contains infinitely many points of E .*

Proof. Suppose for contradiction that there exists an open ball N at x that contains only a finite number of points of E . Denote the points x_1, \dots, x_n as the points of $N \cap E$. Then we define:

$$r = \min\{d(x_1, x), \dots, d(x_n, x)\}$$

The open ball at x with radius r contains none of these points, and is entirely within N ; we deduce it should not contain a point in E . This contradicts the fact x is a limit point. \square

Corollary 1. *A finite set has no limit points.*

We will enumerate the topological properties of the following sets. If a property (excluding compactness and connectedness) is not listed, it fails to hold:

1. The set of all complex z such that $|z| < 1$ is open and bounded.
2. The set of all complex z such that $|z| \leq 1$ is closed, perfect, and bounded.
3. A nonempty finite set is closed and bounded.
4. The set of all integers is closed.
5. The set consisting of the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ is bounded.
6. The set consisting of all complex numbers is closed, open, and perfect.
7. The segment $(a, b) \subset \mathbb{R}^1$ is open and bounded.

Theorem 9. Let $\{E_\alpha\}$ be a collection of sets E_α . Then

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{\complement} = \bigcap_{\alpha} (E_{\alpha}^{\complement})$$

Proof. If $x \in (\bigcup_{\alpha} E_{\alpha})^{\complement}$, then $x \notin \bigcup_{\alpha} E_{\alpha}$ and $x \notin E_{\alpha}$ for all α . Then $x \in E_{\alpha}^{\complement}$ for all α , so $x \in \bigcap_{\alpha} E_{\alpha}^{\complement}$. Hence

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{\complement} \subseteq \bigcap_{\alpha} (E_{\alpha}^{\complement}).$$

Conversely, if $x \in \bigcap_{\alpha} (E_{\alpha}^{\complement})$, then $x \in E_{\alpha}^{\complement}$ for each α ; then $x \notin E_{\alpha}$ for each α , and $x \notin \bigcup_{\alpha} E_{\alpha}$. Thus $x \in (\bigcup_{\alpha} E_{\alpha})^{\complement}$, so

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{\complement} \supseteq \bigcap_{\alpha} (E_{\alpha}^{\complement})$$

We conclude that $(\bigcup_{\alpha} E_{\alpha})^{\complement} = \bigcap_{\alpha} (E_{\alpha}^{\complement})$. □

Theorem 10. E is an open set if and only if E^{\complement} is a closed set.

Proof. Let E be an open set and let x be a limit point of E^{\complement} . Suppose for contradiction that $x \in E$. Then x is an interior point of E , so there exists an open ball N such that $N \subseteq E$; this contradicts the fact that x is a limit point of E^{\complement} . We conclude that $x \in E^{\complement}$, so E^{\complement} is closed.

Now, suppose that E^{\complement} is a closed set, and let $x \in E$. Suppose for contradiction that there does not exist an open ball N at x such that $N \subseteq E$. Then x is a limit point; as E^{\complement} is closed, $x \in E^{\complement}$. This contradiction leads us to conclude that x is an interior point, so E is open. □

Corollary 2. F is a closed set if and only if F^{\complement} is open.

Theorem 11. *The following four results hold:*

1. *For any collection G_α of open sets, $G = \bigcup_\alpha G_\alpha$ is open.*
2. *For any collection F_α of closed sets, $F = \bigcap_\alpha F_\alpha$ is closed.*
3. *For any finite collection G_1, \dots, G_n of open sets, $G = \bigcap_{i=1}^n G_i$ is open.*
4. *For any finite collection F_1, \dots, F_n of closed sets, $F = \bigcup_{i=1}^n F_i$ is closed.*

Proof. For (1): If $x \in G$, then $x \in G_\alpha$ for some index α . As G_α is open, there exists an open ball N at x such that $N \subseteq G_\alpha$; thus $N \subseteq G$, so G is open.

For (2): We take complements. The set F_α^c are all open sets, so

$$\bigcup_\alpha F_\alpha^c = \left(\bigcap_\alpha F_\alpha \right)^c$$

is open. Then $\bigcap_\alpha F_\alpha = F$ is closed.

For (3): If $x \in G$, then $x \in G_i$ for all $i \in \{1, \dots, n\}$. Then for each $i \in \{1, \dots, n\}$, there exists an open ball N_i centered at x with radius r_i such that $N_i \subseteq G_i$. Define

$$r = \min\{r_1, \dots, r_n\},$$

and let N be the open ball of radius N centered at x . Then $N \subseteq N_i \subseteq G_i$ for each $i \in \{1, \dots, n\}$, so $N \subseteq G$; hence x is an interior point, and G is open.

For (4): We take complements. The sets F_1^c, \dots, F_n^c are open, so

$$\bigcap_{i=1}^n (F_i^c) = \left(\bigcup_{i=1}^n F_i \right)^c$$

is open; thus $\bigcup_{i=1}^n F_i$ is closed. It is trivial to further deduce that results (3) and (4) fail for infinite collections of sets. \square

Let X be a metric space, E be a set in X , and E' be the set of all limit points of E in X . Then the **closure** of E is the set $\overline{E} = E \cup E'$.

Theorem 12. *Let X be a metric space and $E \subset X$. Then*

1. *\overline{E} is closed.*
2. *$E = \overline{E}$ if and only if E is closed.*
3. *$\overline{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$.*

Proof. For (1): Suppose that $x \in \overline{E}^c$, so $x \notin E$ and $x \notin E'$. Then x is not a limit point of E , so there exists an open ball N_1 at x of radius r_1 disjoint from E — that is, $N_1 \subseteq E^c$.

Suppose for contradiction that x is a limit point of E' ; then for all $\epsilon > 0$, the open ball of radius ϵ at x contains a point of E' . Denoting this point by y , we have $d(x, y) < r$; thus, consider the open ball at y of all z such that

$$d(y, z) < r - d(x, y).$$

As $y \in E'$, y is a limit point of E ; thus there exists a $z_0 \in E$ in the open ball defined above. By the Triangle Inequality,

$$d(x, z_0) \leq d(x, y) + d(y, z_0) < d(x, y) + r - d(x, y) = r.$$

Thus z_0 lies in the open ball of radius ϵ . We conclude that all open balls centered at x contain a point in E , so x is a limit point of E — a contradiction. We conclude that x is not a limit point of E' ; so, there exists an open ball N_2 at x of radius r_2 disjoint from E' — that is, $N_2 \subseteq (E')^c$.

Defining $r = \min\{r_1, r_2\}$ and N as the open ball of radius r at x , we have $N \subseteq N_1 \subseteq E^c$ and $N \subseteq N_2 \subseteq (E')^c$; thus,

$$N \subseteq E^c \cap (E')^c = (E \cup E')^c = \overline{E}^c.$$

Hence \overline{E}^c is an open set, so \overline{E} is closed. For (2): If $E = \overline{E}$, then E contains all of its limit points, so it is closed. If E is a closed set, then $E' \subseteq E$, so $\overline{E} = E \cup E' = E$.

For (3): Suppose F is a closed set in X such that $E \subseteq F$. Then $F' \subseteq F$; since limit points of E are limit points of F , $E' \subseteq F'$ — hence $E' \subseteq F$. We conclude that $\overline{E} = E \cup E' \subseteq F$. \square

This implies that \overline{E} is the smallest closed set in X that contains E .

Theorem 13. *Let E be a nonempty set of real numbers which is bounded above, and set $y = \sup E$. Then $y \in \overline{E}$; hence $y \in E$ if E is closed.*

Proof. If $y \in E$, then $y \in \overline{E}$. If $y \notin E$: by the minimality of y , there exists $x \in E$ such that $y - \epsilon < x$ for all $\epsilon > 0$. Hence, open balls at y of arbitrary radius ϵ contain a point in E , so y is a limit point. Then $y \in E' \subseteq \overline{E}$. \square

Suppose $E \subseteq Y \subseteq X$. It is important to note that Y is a metric space in its own right under the distance of X ; it is possible for E to be an open set in X , but not Y . We say that E is **open relative** to Y if for each $x \in E$ there is $r > 0$ such that $y \in Y$ and $d(x, y) < r$ implies $y \in E$.

Theorem 14. *Suppose $Y \subseteq X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X .*

Proof. Suppose that E is open relative to Y ; then for each $x \in E$, there is $r_x > 0$ such that $y \in Y$ and $d(x, y) < r_x$ implies $y \in E$. Then define N_x as the open ball centered at $x \in E$ with radius r_x , and consider the set

$$G = \bigcup_{x \in E} N_x.$$

G is an open subset of X by Theorems 7 and 11. It is easy to see that $E \subseteq Y \cap G$; as per the converse, it is clear that $N_x \cap Y \subseteq E$, so performing an infinite union yields $G \cap Y \subseteq E$. We conclude that $E = G \cap Y$.

The contrary is quite easy to see: if $E = Y \cap G$, then $x \in E$ implies that $x \in Y$ and $x \in N_x$, so there exists $r_x > 0$ such that $d(x, y) < r_x$ and $x \in Y$ implies $x \in E$. By definition, E is open relative to Y . \square

An **open cover** of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \bigcup_\alpha G_\alpha$.

3 Compact Sets

3.1 Definition

A subset K of a metric space X is **compact** if every open cover of K contains a *finite* subcover — if for all open covers $\{G_\alpha\}$ of K , there exist finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

All finite sets are clearly compact — we may simply select one open set that contains each element to attain a finite subcovering. Temporarily, let K be compact relative to X if the definition above is satisfied.

3.2 In Metric Spaces

Theorem 15. *Suppose $K \subseteq Y \subseteq X$. Then K is compact to X if and only if K is compact relative to Y*

Proof. Suppose that K is compact relative to X , and let $\{G_\alpha\}$ be a collection of sets open relative to Y such that $K \subseteq \bigcup_\alpha G_\alpha$. By Theorem 14, there exist sets H_α open relative to X such that for each α ,

$$G_\alpha = H_\alpha \cap Y.$$

As $K \subseteq \bigcap_\alpha H_\alpha$, the sets H_α constitute an open covering of K in X . As K is compact, there exist indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subseteq \bigcup_{i=1}^n H_{\alpha_i}.$$

As $K \subseteq Y$, we have

$$K \subseteq \left(\bigcup_{i=1}^n H_{\alpha_i} \right) \cap Y = \bigcup_{i=1}^n (H_{\alpha_i} \cap Y) = \bigcup_{i=1}^n G_{\alpha_i}.$$

G_{α_i} constitute an open covering of K in Y , so K is compact in Y . Now, let us suppose K is compact relative to Y , and let $\{H_\alpha\}$ be an open covering of K in X . Set

$$G_\alpha = H_\alpha \cap Y.$$

for sets G_α open relative to Y . Then as $K \subseteq Y$ and $K \subseteq \bigcup_\alpha H_\alpha$, similar logic applies:

$$K \subseteq \left(\bigcup_{i=1}^n H_{\alpha_i} \right) \cap Y = \bigcup_{i=1}^n (H_{\alpha_i} \cap Y) = \bigcup_{i=1}^n G_{\alpha_i}.$$

So G_α is an open covering of K in Y ; then there exist finitely indices such that

$$K \subseteq \bigcup_{i=1}^n G_{\alpha_i} \subseteq \bigcup_{i=1}^n H_{\alpha_i},$$

so K is compact in X . This concludes the proof. \square

Theorem 16. *Any closed subset F of a compact set K is compact.*

Proof. Suppose $F \subseteq K \subseteq X$, for closed F and compact K . Suppose $\{G_\alpha\}$ is an open cover of F — then $(\bigcup_\alpha G_\alpha) \cup F^c$ is an open cover of K , and thus contains a finite subcover.

If F^c is a member of this finite subcover, we may remove it to obtain a finite subcover of F ; thus a finite subcollection of $\{G_\alpha\}$ contains F , so F is compact. \square

Theorem 17. *Any compact subset K of a metric space X is closed.*

Proof. We will prove that K^c is open — that if $x \in K^c$, there exists an open ball centered at x contained outside K . One x is selected, construct an open covering of K as follows: for any $k \in K$, let N_k be the open ball centered at k with radius $\frac{1}{2}d(x, k)$. Then

$$\bigcup_{k \in K} N_k$$

is an open covering of K ; since K is compact, there exist k_1, \dots, k_n such that $K \subseteq N_{k_1} \cup \dots \cup N_{k_n}$. Then the open ball N at x with radius $\min\{d(x, k_1), \dots, d(x, k_n)\}$ is disjoint from each N_{k_i} (say, by contradiction using the Triangle Inequality), so

$$N \cap K \subseteq N \cap (N_{k_1} \cup \dots \cup N_{k_n}) = \emptyset.$$

Thus $N \subseteq K^c$. We deduce that K^c is open, so K is closed. □

Corollary 3. *If F is closed and K is compact, then $F \cap K$ is compact.*

Proof. Since K is closed, $F \cap K$ is closed subset of K ; thus it is compact. □

3.3 The Heine-Borel Theorem

Theorem 18. *Suppose $\{K_\alpha\}$ are compact in X . If the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty.*

Proof. Fix K_1 of $\{K_\alpha\}$; suppose for contradiction that for all $k \in K_1$, there exists α such that $k \notin K_\alpha$. Then $k \in K_\alpha^c$, so $\{K_\alpha^c\}$ forms an open covering of K_1 . We deduce the existence of indices $\alpha_1, \dots, \alpha_n$ such that

$$K_1 \subseteq K_{\alpha_1}^c \cup \dots \cup K_{\alpha_n}^c;$$

or equivalently,

$$K_1^c \supseteq K_{\alpha_1} \cap \dots \cap K_{\alpha_n}.$$

This yields the desired contradiction, since

$$(K_{\alpha_1} \cap \dots \cap K_{\alpha_n}) \cap K_1 = K_1^c \cap K_1 = \emptyset.$$

Then there exists $k \in K_1$ such that $k \in K_\alpha$ for all α , so $\bigcap_\alpha K_\alpha$ is nonempty. □

Corollary 4. *If $\{K_n\}$ are compact sets such that $K_n \supseteq K_{n+1}$ for each $n \in \mathbb{Z}_{>0}$, then $\bigcup_{n=1}^\infty K_n$ is nonempty.*

Theorem 19. *If E is an infinite subset of a compact set K , then E contains a limit point in K .*

Proof. If each $k \in K$ is not a limit point, then there exists an open ball N_k at k of nonzero radius that contains at most one point of E — namely, k itself.

The N_k constitute an open covering of K (and therefore E), yet no finite subcovering can contain each E ; thus it cannot contain K . Hence, K cannot be compact.

Taking the contrapositive yields the desired result. \square

Theorem 20 (Nested Intervals Theorem). *Suppose $\{I_n\}$ is a sequence of intervals in \mathbb{R} such that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}_{>0}$. Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.*

Proof. Let $I_n = [a_n, b_n]$ for sequences a_n and b_n ; define $A = \sup a_n$. Realize that for all $m, n \in \mathbb{Z}_{>0}$,

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_n.$$

Thus b_n is an upper bound of all a_n , so $A \leq b_n$. Since $a_n \leq A$, we find that $A \in I_n$ for each $n \in \mathbb{Z}_{>0}$. This concludes the proof. \square

Theorem 21. *Suppose $\{I_n\}$ is a sequence of k -cells such that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}_{>0}$. Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.*

Proof. Let I_n consist of all points $\mathbf{x}_n = (x_{n1}, \dots, x_{nk})$ such that for $i \in \mathbb{Z}_{>0}$ and $j \in \{1, \dots, k\}$,

$$a_{ij} < x_{ij} < b_{ij}.$$

For each $j \in \{1, \dots, k\}$, define I_{1j}, I_{2j}, \dots as $[a_{1j}, b_{1j}], [a_{2j}, b_{2j}], \dots$. By Theorem 20, there exists v_j in each interval. The vector $\mathbf{v} = (v_1, \dots, v_n)$ thus lies inside each k -cell, so $\bigcup_{n=1}^{\infty} I_n$ is nonempty. \square

The following proof expands upon my Bolzano-Weierstrauss reasoning found in RealAnalysis/proofs.tex; the construction will thus be simplified for brevity.

Theorem 22. *Every k -cell is compact.*

Proof. Suppose for contradiction that the k -cell I_1 is not compact. Then all open coverings $\{G_\alpha\}$ of I_1 lack a subcollection that covers I_1 .

Then define $c_j = \frac{1}{2}(a_j + b_j)$; the intervals $[a_j, c_j]$ and $[c_j, b_j]$ across all $j \in \{1, \dots, k\}$ split I_1 into 2^k subcells. At least one of these subcells is not covered by a finite subcollection of $\{G_\alpha\}$; call it I_2 .

Repeat this construction on I_2 to obtain a subcell I_3 that is not covered by $\{G_\alpha\}$; repeat this process *ad infinitum*.

We obtain a sequence of k -cells $\{I_n\}$ such that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}_{>0}$, each uncovered by any finite subcollection of $\{G_\alpha\}$. Theorem 21 thus applies: there exists

$$x \in \bigcup_{n=1}^{\infty} I_n.$$

Since $\{G_\alpha\}$ covers I_1 , there exists some open set G_α that contains x ; inside this open set is N_r , an open ball at x of radius r .

Claim 1. *For some $m \in \mathbb{Z}_{>0}$, we have $I_m \subseteq G_\alpha$.*

Proof. Realize that I_1 is contained within the open ball at its centroid of the following radius:

$$\delta = \frac{1}{2} \sqrt{\sum_{i=1}^n (b_i - a_i)^2}.$$

The relevant proof is long but straightforward, utilizing the Pigeonhole Principle. Thus, I_n is contained within the open ball at its centroid of radius $\delta / 2^{n-1}$.

Set $m = \lfloor \log_2 (\frac{\delta}{r}) + 1 \rfloor + 1$. Then

$$\frac{\delta}{2^{m-1}} < \frac{\delta}{2^{\log_2(\delta/r)}} = \frac{\delta}{\delta/r} = r.$$

Then if N_m is the open ball with radius m , we conclude that

$$I_m \subseteq N_m \subseteq N_r \subseteq G_\alpha,$$

as required.

This attains the desired contradiction: that I_m cannot be covered by a finite subcollection of $\{G_\alpha\}$, yet it is covered by G_α . We conclude that I_1 must be compact. \square

Define a **k-pseudocell** in \mathbb{C}^n as the set of all $\mathbf{z} \in \mathbb{C}^n$ such that $\text{Re } \mathbf{z}$ and $\text{Im } \mathbf{z}$ lie in potentially distinct k -cells. We will use this non-standard definition exclusively for Theorem 23.

Corollary 5. *Suppose $\{I_n\}$ is a sequence of k -pseudocells such that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}_{>0}$. Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.*

Corollary 6. *All k -pseudocells are compact.*

Theorem 23 (Heine-Borel). *For $E \subseteq \mathbb{C}^k$, the following conditions are equivalent:*

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E .

Furthermore, (2) and (3) are equivalent in an arbitrary metric space.

Proof. Suppose (1). As E is bounded, it is a subset of some k -pseudocell. Since E is closed, it is compact by Theorem 16 — thus establishing (2). If we assume (2), we yield (3) by Theorem 19.

Assume that E is not bounded. Then there exists a sequence of vectors $\{\mathbf{z}_n\}$ for $n \in \mathbb{Z}_{>0}$ such that

$$|\mathbf{z}_n| > n.$$

A straightforward argument verifies that no limit point for this sequence exists in E , so (3) is not met.

Assume that E is not closed. Then there exists a vector $\mathbf{z} \notin E$ which is a limit point of E but lies outside of E . Then for each $n \in \mathbb{Z}_{>0}$, there exists a sequence of vectors $\mathbf{z}_n \in E$ such that

$$\|\mathbf{z}_n - \mathbf{z}\| < \frac{1}{n}.$$

The subset $\{\mathbf{z}_n \mid n \in \mathbb{Z}_{>0}\}$ is infinite; we claim its only limit point is \mathbf{z} . This is because if $\mathbf{w} \in \mathbb{C}^n$ and $\mathbf{w} \neq \mathbf{z}$,

$$\begin{aligned} \|\mathbf{z}_n - \mathbf{w}\| &\geq \|\mathbf{z} - \mathbf{w}\| - \|\mathbf{z}_n - \mathbf{z}\| \\ &\geq \|\mathbf{z} - \mathbf{w}\| - \frac{1}{n} \\ &\geq \frac{\|\mathbf{z} - \mathbf{w}\|}{2} \end{aligned}$$

for all but finitely many n . Thus the open ball at \mathbf{w} of radius $\frac{1}{2}\|\mathbf{z} - \mathbf{w}\|$ does not contain infinitely many \mathbf{z}_n , so it cannot be a limit point — so (3) is not met.

Then if we suppose (3), E must be closed and bounded — implying (1). This concludes the proof. \square

Theorem 24 (Weierstrauss). *Every bounded infinite subset of \mathbb{C}^n contains a limit point.*

Proof. All bounded infinite subsets E of \mathbb{C}^n are contained within an n -cell I . Since I is compact by Theorem 22, E has a limit point in E by Theorem 19 (and 23!). \square