MATH-UA 329: Homework 1

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1 Problem 1

1.1 Part (a)

Proof. Let $\langle \cdot, \cdot \rangle : V \times V \to V$ be an inner product over a vector space V. Then for all $\mathbf{v}, \mathbf{w} \in V$, we have

$$\begin{aligned}
\|\mathbf{v}+\mathbf{w}\|^2 &= \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w} \rangle \\
&= \langle \mathbf{v}, \mathbf{v}+\mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v}+\mathbf{w} \rangle \\
&= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\
&= \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2.
\end{aligned}$$

We now demonstrate that the Triangle Inequality and Cauchy-Schwarz Inequality are equivalent. Suppose the Triangle Inequality holds; for all $\mathbf{v}, \mathbf{w} \in V$, we have

$$\begin{aligned}
\|\mathbf{v}\|\|\mathbf{w}\| &= \frac{(\|\mathbf{v}\| + \|\mathbf{w}\|)^2 - \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2}{2} \\
&\geq \frac{\|\mathbf{v}+\mathbf{w}\|^2 - \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2}{2} \\
&= \frac{\left(\|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2\right) - \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2}{2} \\
&= \langle \mathbf{v}, \mathbf{w} \rangle .
\end{aligned}$$

Now, suppose the Cauchy-Schwarz Inequality; for all $\mathbf{v}, \mathbf{w} \in V$, we have

$$\begin{aligned}
\|\mathbf{v}+\mathbf{w}\|^2 &= \left\|\mathbf{v}^2\right\| + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2 \\
&\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \\
&= \left(\|\mathbf{v}\| + \|\mathbf{w}\|\right)^2.
\end{aligned}$$

Taking the square root yields the Triangle Equality. $\square$

1.2 Part (b)

Proof. Suppose $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} \in (w_1, \dots, w_n)$ are vectors in $\mathbb{C}^n$ and $c \in \mathbb{C}$. Then

$$\|c\mathbf{z} - \mathbf{w}\|^2 = \sum_{i=1}^{n} (cz_1 + w_1)^2 = c^2 \left(\sum_{i=1}^{n} z_i^2 \right) + c \left(2 \sum_{i=1}^{n} z_i w_i \right) + \left(\sum_{i=1}^{n} w_i^2 \right)$$

is a quadratic which has at most one root. Its discriminant must be nonnegative:

$$0 \leq \left(2 \sum_{i=1}^{n} z_i w_i \right)^2 - 4 \left(\sum_{i=1}^{n} z_i^2 \right) \left(\sum_{i=1}^{n} w_i^2 \right) = 4(\mathbf{z} \cdot \mathbf{w})^2 - 4\|\mathbf{z}\|^2 \|\mathbf{w}\|^2.$$

Diving by 4 and rearranging yields that $(\mathbf{z} \cdot \mathbf{w})^2 \leq \|\mathbf{z}\|^2 \|\mathbf{w}\|^2$; taking the square root yields the Cauchy-Schwarz Inequality in $\mathbb{C}^n$. This proof implies Cauchy-Schwarz in $\mathbb{R}^n$ as well. $\square$

2 Problem 2

Proof. We must first unravel the notation of the expression $\frac{\omega_f(t)}{t}$. For all $\epsilon > 0$,

$$\begin{aligned}
\lim_{t \to 0^+} \frac{\omega_f(t)}{t} = 0 &\implies \exists \delta \text{ such that } 0 < t < \delta \text{ implies } \frac{\omega_f(t)}{t} < \epsilon \\
&\implies \exists \delta \text{ such that } 0 < t < \delta \text{ implies } \frac{\sup\limits_{|x_1 - x_2| \leq t} |f(x_1), f(x_2)|}{t} \leq \epsilon \\
&\implies \exists \delta \text{ such that } 0 < |x_1 - x_2| \leq t < \delta \text{ for } x_1, x_2 \in I \text{ implies} \\
&\qquad \frac{|f(x_1) - f(x_2)|}{t} \leq \epsilon.
\end{aligned}$$

Set $t = |x_1 - x_2|$. Then $\lim\limits_{t \to 0^+} \frac{\omega_f(t)}{t} = 0$ implies the existence of δ such that

$$0 < |x_1 - x_2| < \delta \implies \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right| \leq \epsilon.$$

We conclude that $\lim\limits_{x_2 \to x_1} \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(x_1) = 0$ for all $x_1 \in I$, so f is constant on I. $\square$

3 Problem 3

3.1 Part (a)

Proof. Let s be any point in S. For all $\epsilon > 0$, there exists $z \in Z$ such that

$$d(s, z) < \frac{\epsilon}{2}.$$

Since Z is dense in X: for all $\epsilon > 0$, there exists $x \in X$ corresponding to z such that

$$d(z, x) < \frac{\epsilon}{2}.$$

Then we deduce that

$$d(s, x) \leq d(s, z) + d(z, x) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so S is dense in X. $\square$

3.2 Part (b)

Proof. We address each part separately:

Part (i): We must perform rather routine calculations to verify that $d_1 \times d_2$ is a metric:

1. **Positivity**: Since $(d_1 \times d_2)\big((x_1, x_2), (y_1, y_2)\big)$ is a sum of two distances, it is nonnegative. Equality is obtained precisely when $d_1(x_1, y_1) = d_2(x_2, y_2) = 0$ — that is, when $(x_1, x_2) = (y_1, y_2)$.

2. **Symmetry**: We have that

$$\begin{aligned} (d_1 \times d_2)\big((x_1, x_2), (y_1, y_2)\big) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &= d_1(y_1, x_1) + d_2(y_2, x_2) \\ &= (d_1 \times d_2)\big((y_1, y_2), (x_1, x_2)\big). \end{aligned}$$

3. **Triangle Inequality**: For all (x_1, x_2), (y_1, y_2), and (z_1, z_2) in $X_1 \times X_2$, observe that

$$\begin{aligned} (d_1 \times d_2)\big((x_1, x_2), (y_1, y_2)\big) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &\leq d_1(x_1, z_1) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2) \\ &= (d_1 \times d_2)\big((x_1, x_2), (z_1, x_2)\big) \\ &\quad + (d_1 \times d_2)\big((z_1, z_2), (y_1, y_2)\big), \end{aligned}$$

which is the triangle inequality.

We conclude that $d_1 \times d_2$ is a metric of $X_1 \times X_2$.

Part (ii): Select $(z_1, z_2) \in Z_1 \times Z_2$ arbitrarily. For all $\epsilon > 0$, there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that

$$d_1(x_1, z_1) < \frac{\epsilon}{2}$$
$$d_2(x_2, z_2) < \frac{\epsilon}{2}.$$

Considering the pair (x_1, x_2), we deduce that

$$(d_1 \times d_2)\big((z_1, z_2), (x_1, x_2)\big) = d_1(z_1, x_1) + d_2(z_2, x_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $Z_1 \times Z_2$ is dense in $X_1 \times X_2$.

Part (iii): If X_1 and X_2 are separable, then there exist (at most) countable and dense subsets $Z_1 \subset X_1$ and $Z_2 \subset X_2$. Then the product $Z_1 \times Z_2$ is (at most) countable; the prior lemma establishes it is dense in $X_1 \times X_2$. We deduce that $X_1 \times X_2$ is separable. □

3.3 Part (c)

Proof. Let X be a discrete metric space. We utilize the following claim:

Claim 1. *Let $S \subset X$. Then S is dense in X if and only if $S = X$.*

Proof. Suppose S is dense in X. Then for all $x \in X$, there exists $s \in S$ such that

$$d(x, s) < \frac{1}{2}.$$

Since the discrete metric is either 0 or 1, we find $d(x, s) = 0$ and $x = s$. Then $x \in S$, so $S = X$. The proof concludes by noting that $S = X$ implies S is dense in X.

We use our claim in the following chain of equivalecies:

$$\begin{aligned} X \text{ is separable} &\iff \text{there exists dense } S \subseteq X \text{ which is countable} \\ &\iff X \text{ is (at most) countable,} \end{aligned}$$

as desired. □

4 Problem 4

Proof. Suppose that (X,d) is a metric space. Then the following holds for all $x \in X$:

$$\begin{aligned}
(X,d) \text{ is separable} &\iff X \text{ has a countable dense subset} \\
&\iff \text{There is } (x_n)_1^\infty \subseteq X \text{ which is dense in } X \\
&\iff \text{For every } x \in X \text{ and all } \epsilon > 0, \text{ there is } x_m \in (x_n)_1^\infty \text{ such that} \\
&\qquad d(x_m, x) < \epsilon \\
&\iff \text{For every } x \in X, \text{ we have } \liminf_{n\to\infty} d(x_n, x) = 0,
\end{aligned}$$

as required. □

5 Problem 5

5.1 Part (a)

Proof. Suppose that S is a dense subset of $\ell^\infty(\mathbb{N})$. We will prove that D is uncountable.

Claim 2. $\{0,1\}^{\mathbb{N}}$ *is an uncountable set to which* $\ell^\infty(\mathbb{N})$ *reduces to the discrete metric.*

> *Proof.* Cantor's diagonal argument implies that $\{0,1\}^{\mathbb{N}}$ is an uncountable set. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be sequences in $\{0,1\}^{\mathbb{N}}$. It is clear that $d_\infty(x,y)$ is 0 or 1; we have
>
> $$\begin{aligned}
d_\infty(x,y) = 0 &\iff |x_i - y_i| = 0 \text{ for all } i \in \mathbb{N} \\
&\iff x_i = y_i \text{ for all } i \in \mathbb{N} \\
&\iff x = y.
\end{aligned}$$
>
> Thus d_∞ is the discrete metric on $\{0,1\}^{\mathbb{N}}$, which completes the proof of our claim.

Associate to each $x \in \{0,1\}^{\mathbb{N}}$ the following set:

$$I_x \overset{\text{def}}{=} \left\{ s \in S \;\middle|\; d(x,s) < \frac{1}{2} \right\}.$$

Each I_x is infinite since x is a limit point of D. Observe that I_x and I_y for $x \neq y$ are disjoint. If we suppose otherwise, there would exist $s \in S$ such that $d(x,s) < \frac{1}{2}$ and $d(y,s) < \frac{1}{2}$, which yields the following contradiction:

$$1 \;=\; d(x,y) \;\le\; d(x,s) + d(s,y) \;<\; \frac{1}{2} + \frac{1}{2} \;=\; 1$$

By the Axiom of Choice, we may form a set I consisting of one element of I_x for each $x \in \{0,1\}^{\mathbb{N}}$. Observe that I and $\{0,1\}^{\mathbb{N}}$ are in bijection, so I is uncountable; then $I \subseteq D$, implies that D is uncountable.

We conclude that $\ell^\infty(\mathbb{N})$ is not separable. Once an uncountable subset reduced to the discrete metric is identified, **the argument above applies to any metric space** and will be reinvoked in Problem 6. $\square$

5.2 Part (b)

(Aside: We assume $\mathbb{N}$ does not include 0; this choice is irrelevant to the proof)

Proof. We address each part separately:

Part (i): We define the family of sets $S_1 = (q_1, 0, 0, 0, \ldots)$, $S_2 = (q_1, q_2, 0, 0, \ldots)$, $S_3 = (q_1, q_2, q_3, 0, \ldots)$ so on for all $q_1, q_2, \ldots \in \mathbb{Q}$. Let $S_1 \cup S_2 \cup \cdots = S$; since each S_n is countable, S is countable.

Select $\{x_n\} = (x_1, x_2, \ldots, x_m, 0, 0, \ldots) \in c_{00}$ arbitrarily, where m is the largest integer such that x_m is nonzero. For all $\epsilon > 0$, there exist rationals $q_1, \ldots, q_m$ such that

$$\begin{gathered} |x_1 - q_1| < \epsilon \\ \vdots \\ |x_m - q_m| < \epsilon. \end{gathered}$$

Set $\{q_n\} = (q_1, q_2, \ldots, q_m, 0, 0, \ldots)$. Then

$$d_\infty\big(\{x_n\}, \{q_n\}\big) = \max_{n \in \mathbb{Z}_{>0}} |q_n - s_n| < \epsilon,$$

Since $\{q_n\} \in S$, we deduce that S is dense in c_{00} and countable — so c_{00} is a separable metric space.

Part (ii): Select $\epsilon > 0$ and $\{x_n\} \in c_0$ arbitrarily. Since $\lim\limits_{n\to\infty} x_n = 0$, there exists N such that

$$N \leq n \implies |x_n| < \epsilon.$$

Now, define $\{y_n\} \in c_{00}$ such that $y_n = x_n$ if $N > n$ and $y_n = 0$ if $N \leq n$. Then

$$d_\infty\big(\{x_n\}, \{y_n\}\big) = \sup_{n \in \mathbb{Z}_{>0}} |q_n - s_n| \leq \epsilon.$$

Thus c_{00} is dense in c_0. Since density is transitive, we conclude that S is dense in c_0, so c_0 is separable.

Part (iii): Let T be the set of eventually constant rational sequences, and select $\{x_n\} \in c$ arbitrarily with components $(x_1, x_2, \ldots)$ and limit L. It is clear that T is countable.

Let $\epsilon > 0$ be arbitrary. Since $\{x_n\}$ converges and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists an integer N and rationals $q_1, q_2, \ldots q_{N-1}$ such that

$$N \leq n \implies |x_n - L| < \frac{\epsilon}{2} \tag{1}$$

$$j \in \{1, \ldots, n-1\} \implies |x_j - q_j| < \epsilon. \tag{2}$$

Let Q be a rational such that $|Q - L| < \frac{\epsilon}{2}$. Then define $\{q_n\}$ as the sequence in T with terms $(q_1, \ldots, q_{N-1}, Q, Q, Q, \ldots)$. We claim that $|x_j - q_j| < \epsilon$ for each $j \in \mathbb{Z}_{>0}$, as verified by examining two cases:

1. If $j \in \{1, \ldots, N-1\}$, then $|x_j - q_j| < \epsilon$ by equation (2).
2. If $j \geq N$, then $|x_j - q_j| = |x_j - Q| \leq |x_j - L| + |L - Q| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

We deduce that $\sup\limits_{j \in \mathbb{Z}_{>0}} \big(\{x_n\}, \{q_n\}\big) \leq \epsilon$. Thus T is dense in c, so c is separable. $\square$

6 Problem 6

Proof. We construct a mapping $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{BUC}(\mathbb{R})$ be defined recursively. For all $s = (s_1, s_2, \ldots) \in \{0,1\}^{\mathbb{N}}$, set $s_0 = 0$. Let $\phi_s(x)$ be zero for $x \leq 0$, and define ϕ_s on each interval $(n-1, n]$ for $n \in \mathbb{Z}$ as follows:

$$\phi_s(x) = \begin{cases} 0 & \text{if } s_{n-1} = s_n = 0 \\ x - (n-1) & \text{if } s_{n-1} = 0 \text{ and } s_n = 1 \\ 1 & \text{if } s_{n-1} = s_n = 1 \\ n - x & \text{if } s_{n-1} = 0 \text{ and } s_n = 1 \end{cases}$$

Two facts about ϕ_s follow: that $\phi_s(n) = s_n$ for each $n \in \mathbb{Z}_{>0}$ and that ϕ_s is continuous. Unless s is exclusively zeros or ones, ϕ_s has a maximum of 1 and a minimum of 0; the derivative of $\phi_s(x)$ for when $x \notin \mathbb{Z}_{\geq 0}$ is either -1, 0, or 1.

Claim 3. *ϕ_s is Lipschitz continuous:* $|\phi_s(x) - \phi_s(y)| \leq |x - y|$.

Proof. If $x < 0$ or $y < 0$, the desired relation is trivial/ If $x, y \geq 0$, we have three cases to consider:

1. If $|x - y| \geq 1$, then the boundedness of ϕ_s implies $|\phi_s(x) - \phi_s(y)| \leq 1 \leq |x - y|$.
2. If $|x - y| < 1$ and $\lfloor x \rfloor = \lfloor y \rfloor$, then ϕ is a linear function between x and y with slope ± 1, so $|\phi_s(x) - \phi_s(y)| = |x - y|$.

3. If $|x-y| < 1$ and $\lfloor x \rfloor \neq \lfloor y \rfloor$, then without loss of generality, let $x \in (n-1, n)$ and $y \in (n, n+1)$. There are precisely eight cases for the function ϕ_s on the interval $[n-1, n+1]$; in each case, it is trivial that $|f(x) - f(y)| \geq |x - y|$.

We conclude that $|\phi_s(x) - \phi_s(y)| \leq |x - y|$, so ϕ_s is Lipschitz continuous.

Thus, ϕ_s is uniformly continuous, so $\phi_s \in \mathcal{BUC}(\mathbb{R})$ for each ϕ_s. From here, it is trivial that the supremum norm reduces $\phi\big(\{0,1\}^{\mathbb{N}}\big)$ to the discrete metric. Then the proof in Problem 5, Part (a) applies here: we may construct the same open balls of radius $\frac{1}{2}$ and utilize the Axiom of Choice to deduce that all dense subsets of $\mathcal{BUC}(\mathbb{R})$ are uncountable. $\square$