MATH-UA 329: Honors Analysis II

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January 2024

Contents

1 Exposition

MATH-UA 329 expands upon Honors Analysis I and will discuss two topics:

1. The theory of differentiation and integration of multiavariable functions.
2. Measure Theory and Lebesgue integration

The instructor is Sinan Gunturk, available at gunturk@cims.nyu.edu. Professor Gunturk's office hours are at WWH 829 in Courant from 2:00-3:30 PM. The TA is Keefer Rowan. The grade distribution is as follows:

- 40%: the final exam.
- 20%: the midterm exam.
- 10-15%: quizzes.
- 15-20%: homework assignments.

The course will not follow one particular textbook; potential textbooks are enumerated on Brightspace, with particular emphasis on Rudin's *Principles of Mathematical Analysis.*

2 Metric Spaces

2.1 Metric Spaces

Definition

A **metric space** is a set X equipped with a binary mapping $d : X \times X \to \mathbb{R}$ called a **metric** such that the following properties are satisfied for all $x, y, z \in X$:

1. **Positivity**: $d(x, y) \geq 0$, with equality if and only if $x = y$.
2. **Symmetry**: $d(x, y) = d(y, x)$.
3. **Triangle Inequality**: $d(x, y) \leq d(x, z) + d(z, y)$.

Metric spaces generalize the notion of distance to arbitrary sets.

Examples

1. **Euclidean Distance**: In $\mathbb{R}$, the Euclidean distance $d(x, y) = |x - y|$ is a metric. The complex absolute value is also a metric of $\mathbb{C}$.

   In general, the Euclidean distance over $\mathbb{R}^n$ is defined as follows:

   $$d_2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

2. **Taxicab Metric**: in $\mathbb{R}^n$, the taxicab metric is defined as follows for $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$:

   $$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

3. **Supremum Distance**: For $\mathbb{R}^n$, the d_∞ metric is as follows:

   $$d_\infty(\mathbf{x}, \mathbf{y}) = \max |x_i - y_i \mid i \in \{1, \ldots, n\}|.$$

   It is denoted by infinity since

   $$\lim_{m \to \infty} d_n(x, y) = \lim_{m \to \infty} \sqrt[m]{\sum_{i=1}^{n} |x_i - y_i|^m} = d_\infty(x, y).$$

4. **Discrete Metric** The discrete metric over any set X is defined as follows:

   $$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}.$$

   It is easy to verify that the discrete metric is a metric; it is primarily used in examples.

Open Balls

For a metric space X, the **open ball** of radius r centered at $x \in X$ is the set

$$B_r(\mathbf{x}) = \{y \in X \mid d(x, y) \leq 1\}.$$

Here are examples of the unit disc $B_1(0)$ in the above metrics in $\mathbb{R}^2$.

- Under the Euclidean metric, the unit disc is the standard unit circle.

- Under d_∞, it is the unit square:

$$B_1(0) = \{\mathbf{y} \in \mathbb{R}^2 \mid \max y_i < 1 \text{ for } i \in \{1, 2\}\}.$$

- Under d_1, the unit disc is a diamond:

$$B_1(0) = \{\mathbf{y} \in \mathbb{R}^2 \mid |y| \leq 1\}.$$

- Open balls under the discrete metric are defined as follows:

$$B_r(x) = \begin{cases} \{x\} & \text{if } r \leq 1, \\ X & \text{if } r > 1. \end{cases}$$

We encourage the reader to graph these examples for further understanding.

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Continuity

Let X and Y be metric spaces. A function $f : X \to Y$ is **continuous** at $x \in X$ if for all $\epsilon > 0$, there exists δ such that

$$0 < d(x, y) < \delta \implies d(f(x) - f(y)) < \epsilon.$$

f itself is continuous on X if it is continuous at every $x \in X$.

2.2 The Metric Space $\mathcal{BC}(X)$

On Metric Sets

The next section will utilize the following definition:

$$\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous on } X\}$$

$\mathcal{C}(X)$ is a vector space over $\mathbb{R}$ under addition of functions and scalar multiplication. The natural question is: is $\mathcal{C}(X)$ a metric space? Since a norm on a vector space V satisfies positivity, the symmetry Triangle Inequality, it induces a metric for $\mathbf{v}, \mathbf{w} \in V$:

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

$\mathcal{C}(X)$ does not possess a clear norm. We must define a subspace B of $C(X)$ as follows:

$$\mathcal{BC}(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous and bounded on } X\}.$$

The natural norm of this space is the **supremum norm**, defined as follows:

$$\|f\|_X = \sup_{x\in X} |f(x)|.$$

This norm fashions $\mathcal{BC}(X)$ into a metric space. The supremum norm encapsulates the concept of uniform convergence quite precisely.

For General Sets

For any set E, we may define a similar function space:

$$\mathcal{B}(E) = \{f : E \to \mathbb{R} \mid f \text{ is bounded on } E\}.$$

This set $\mathcal{B}(E)$ is a normed vector space under the supremum norm:

$$\|f\|_E = \sup_{x\in E} |f(x)|.$$

Theorem 1. *$\mathcal{B}(E)$ is a complete metric space — hence a Banach space.*

Proof. Suppose (f_n) is a Cauchy sequence under the supremum norm: that for all $\epsilon > 0$, there exists N_ϵ such that

$$N_\epsilon \leq i, j \implies \|f_i - f_j\|_E < \epsilon.$$

Then for all $x \in E$,

$$N_\epsilon \leq i, j \implies \|f_i(x) - f_j(x)\|_E < \epsilon.$$

Then the sequence $f_1(x), f_2(x), \ldots$ is a Cauchy sequence in $\mathbb{R}$ under the supremum norm. Then let f be the function that maps x to the limit of $f_1(x), f_2(x), \ldots$. Clearly, $f \in \mathbb{R}^E$. We must demonstrate that this convergence is uniform.

Now, let $N_\epsilon \leq i, j$. Then

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|. \\ &< |f(x) - f_m(x)| + \epsilon. \end{aligned}$$

Observe that $\inf_{N_\epsilon \leq m} |f(x) - f_m(x)| = 0$ by the convergence. Therefore, we may take the infimum of both sides of the above equation:

$$\begin{aligned} |f(x) - f_n(x)| &= \inf_{N_\epsilon \leq m} |f(x) - f_n(x)| \\ &< \inf_{N_\epsilon \leq m} |f(x) - f_m(x)| + \epsilon \\ &= \epsilon. \end{aligned}$$

Thus, $N_\epsilon < i$ implies $\|f - f_n\| = \sup_{x\in E} |f(x) - f_n(x)|_E < \epsilon$. We conclude that (f_n) converges, so $\mathcal{B}(E)$ is complete. □

If we would like to prove that $\mathcal{BC}(X)$ is continuous, we only need demonstrate that the limit of a Cauchy sequence (f_n) is continuous — which is true, since $\mathcal{BC}(X)$ is a closed subspace of the complete metric space $\mathcal{B}(X)$.

Uniform Continuity

Let $f : (X, d_x) \to (Y, d_y)$ map between metric spaces. Then f is **uniformly continuous** if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon.$$

This now leads us to define the following two spaces:

$$\begin{aligned}\mathcal{UC}(X) &= \{f : X \to R \mid f \text{ is uniformly continuous on } X\},\\ \mathcal{BUC}(X) &= \{f : X \to R \mid f \text{ is bounded and uniformly continuous on } X\}.\end{aligned}$$

Both are subspaces of $\mathcal{C}(X)$, but only $\mathcal{BUC}(X)$ is a normed vector space. The exact same proof as Theorem 1 demonstrates that $\mathcal{BUC}(X)$ is a Banach space.

Special case: When $X = K$ is compact, all continuous $f : K \to \mathbb{R}$ are bounded and uniformly continuous. Hence,

$$\mathcal{C}(K) = \mathcal{BC}(K) = \mathcal{BUC}(K)$$

For non-compact X, we can only write

$$\mathcal{C}(X) \supset \mathcal{BC}(X) \supset \mathcal{BUC}(X).$$

2.3 Modulus of Continuity

Definition

Let $f : (X, d_X) \to (Y, d_Y)$ map between metric spaces. Then the **modulus of continuity** $\omega_f : [0, \infty) \to [0, \infty]$ is defined as

$$\omega_f(t) = \sup_{d_X(x_1, x_2) \leq t} d_Y(f(x_1), f(x_2)).$$

The modulus of continuity "measures" the uniform continuity of a function, as observed by the following facts:

Theorem 2. *f is uniformly continuous if and only if* $\lim_{t\to 0^+} \omega_f(t) = 0$.

Proof. The line of reasoning is not particularly difficult; the two expressions communicate the same idea, buried under different notation. For all $\epsilon > 0$,

$$\begin{aligned} f \text{ is uniformly continuous} &\iff \exists\delta \text{ such that } d_X(x_1, x_2) \le \delta \text{ implies} \\ &\quad\; d_Y(f(x_1), f(x_2)) < \epsilon \text{ for all } x_1, x_2 \in X. \\ &\iff \exists\delta \text{ such that } d_X(x_1, x_2) \le \delta \text{ implies} \\ &\quad\; \sup\big(d_Y(f(x_1), f(x_2))\big) \le \epsilon \\ &\iff \exists\delta \text{ such that } \sup_{d_X(x_1,x_2)\le\delta} d_Y(f(x_1), f(x_2)) \le \epsilon. \\ &\iff \exists\delta \text{ such that } \omega_f(\delta) \le \epsilon \\ &\iff \exists\delta \text{ such that } t < \delta \text{ implies } |\omega_f(t)| \le \epsilon \\ &\iff \lim_{t\to 0^+} \omega_f(t) = 0. \end{aligned}$$

We replaced $<$ by $\le$ wherever necessary; their presence or absence yields an equivalent $\epsilon - \delta$ definition of the limit. ☐

Theorem 3. $d_Y(f(x_1), f(x_2)) \le \omega_f\big(d_X(x_1, x_2)\big)$ *for all* $x_1, x_2 \in X$.

Proof. Set $t = d_X(x_1, x_2)$ when computing the modulus of continuity: we find that

$$d_Y(f(x_1), f(x_2)) \le \sup_{d_X(y_1,y_2)<d_X(x_1,x_2)} d_Y(f(y_1), f(y_2)) = \omega_f\big(d_X(x_1, x_2)\big),$$

as required. ☐

To witness examples of the Modulus of Continuity, we encourage the reader to examine its implications for two types of continuity for a function f:

1. **Hölder Continutiy**: If there exists $\alpha \in (0, 1]$ such that $\omega_f(t) \le Ct^\alpha$. Setting $\alpha \ge 1$ actually implies f is constant, by Problem 2 in Homework 1.

2. **Lipschitz Continuity**: If $\omega_f(t) \le Ct$ for all $t \ge 0$, or if $d_Y(f(x_1), f(x_2)) \le Cd_X(x_1, x_2)$ for all $x \in X$.

It is clear that all Lipschitz continuous functions are Hölder continuous, by setting $\alpha = 1$.

Piecewise Linear Approximation

Let $I = [a, b]$ and $f \in \mathcal{C}(I)$; clearly f is bounded on I. Let L be the affine function interpolating f at the endpoints: $L(a) = f(a)$ and $L(b) = f(b)$.

Theorem 4. *If terms are defined like above, then*

$$\|f - L\|_I \leq \omega_f(b - a)$$

Proof. Recall the definition of the supremum norm:

$$\|f - L\|_I = \sup_{x \in [a,b]} |f(x) - L(x)|.$$

Let $L(x) = y$. Observe that since L is affine, y lies between $L(a)$ and $L(b)$; therefore, between $f(a)$ and $f(b)$. The Intermediate Value Theorem implies the existence of $c \in [a, b]$ such that $f(c) = y$. Then by properties discuseed prior,

$$|f(x) - L(x)| = |f(x) - f(c)| \leq \omega_f|c - x| \leq \omega_f(b - a).$$

□

Corollary 1. *Every $f \in \mathcal{C}(I)$ can be approximated uniformly by piecewise linear continuous functions, with arbitrarily small modulus of continuity.*

Proof. Relatively trivial: divide $[b - a]$ into n segments of length $\frac{b-a}{n}$, and observe how $n \to \infty$ implies $\omega_f\left(\frac{b-a}{n}\right) \to 0$. □

We eventualy conclude that the set of piecewise linear continuous functions on I is *dense* in $\mathcal{C}(I)$. In fact, the set of such functions with rational values for break points is countable.

2.4 Separable Metric Spaces

Definition and Examples

Suppose (X, d) is a metric space and $Z \subseteq X$ is a subset. We say Z is **dense** in X if any of the equivalent definitions are defined:

- For all $x \in X$ and $\epsilon > 0$, there exists $z \in Z$ such that $|x - z| < \epsilon$.
- For all $x \in X$ and $\epsilon > 0$, then $B_\epsilon(x) \cap Z \neq \varnothing$.
- $\bar{Z} = X$, the closure of Z.
- For all $x \in X$, there exists $(z_n) \in Z$ such that $\lim_{n \to \infty} z_n = x$.

Densitiy is transitive: suppose $S \subseteq Z \subseteq X$, where S is dense in Z and Z is dense in X; then S is dense in Z. The metric space (X, d) is **separable** if X has a countable dense subset.

Some examples of dense subsets include:

1. $\mathbb{R}$ with the Euclidean metric, the countable dense subset being $\mathbb{Q}$. We could also conside rthe diatic rationals: $\{\frac{n}{2^m}\}$.
2. $\mathbb{C}^n$ with the Euclidean metric, using the same methods as above.
3. $\mathbb{R}^n$ with the Taxicab metric, using the product metric discussed below.
4. $\mathcal{C}(I)$, discussed prior. The set of all piecewise linear continuous functions with rational values at break points — it is countable yet dense.

For two metric spaces (X, d_X) and (Y, d_Y), the **product metric** is a metric over $X \times Y$ defined as follows:

$$(d_1 \times d_2)\big((x_1, y_1), (x_2, y_2)\big) = d(x_1, x_2) + d(y_1, y_2).$$

We could also consider $\mathbb{R}^n$ to be dense under the product metric, considering $\mathbb{R}^n$ as a direct product of $\mathbb{R}^n$. We would yield the taxicab metric, which is equivalent.

2.5 Polynomial Approximation

Theorem 5 (Weierstrauss Approximation Theorem)**.** *The set of all polynomial functions is dense on $\mathcal{C}(I)$: if $f \in \mathcal{C}(I)$ and for all $\epsilon > 0$, there exists a polynomial P of finite degree such that $\|f - P\|_I < \epsilon$.*

Proof. The proof was discovered by Bernstein in the 1910s, found in the file RealAnalysis/babyrudin7.tex. □

Thus, polynomials are a countable dense subset of I.

2.6 Equivalent Metrics

Two metrics d and ρ on X are **equivalent** if there exists $0 < c \leq C < \infty$ such that for all $x, y \in X$,

$$c\rho(x, y) \leq d(x, y) \leq C\rho(x, y).$$

Density is invariant of equivalent metrics; in fact their topologies are the same. A set $S \subseteq X$ is open under d if and only if S is open under ρ. In particular, metrics in Banach spaces are equivalent if

$$c\|\mathbf{x}\| \leq \|\mathbf{x}\|' \leq C\|\mathbf{x}\|$$

for all $\mathbf{x} \in X$. As an example, the Power Mean Inequality yields in $\mathbb{C}^n$ that

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le \sqrt{d}\,\|\mathbf{x}\|_2 \le d\,\|\mathbf{x}\|_\infty.$$

These relations do *not* extend to infinite dimensional vector spaces, like $\ell_p(\mathbb{N})$ for $1 \le p \le \infty$. A counterexample is given by $(1, \ldots, 1, 0, 0, \ldots)$. As a reminder, this norm is defined as

$$\|\mathbf{x}\|_p \quad \stackrel{\text{def}}{=} \quad \begin{cases} \left(\sum\limits_{n=1}^{\infty} |x_n|^p \right)^{1/p} & \text{if } 1 \le p < \infty \\ \sup\limits_{n \in \mathbb{N}} |x_n| & \text{if } p = \infty. \end{cases}$$

Hence p-norms are not equivalent on spaces of infinite sequences.

Though worth noting, we do have the following:

$$c_{00} \subset \ell_1(\mathbb{N}) \subset \ell^2(\mathbb{N}) \subset c_0 \subset c \subset \ell^\infty(\mathbb{N})$$

All inclusions are clearly proper.

2.7 Normed Vector Space

A **normed vector space** is a complex vector space X equipped with a mapping $\|\cdot\| : X \to \mathbb{R}$ that satisfies the following properties:

1. **Positivity**: $\|\mathbf{x}\| \ge 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$.
2. **Homogenity**: $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$ for all $\lambda \in \mathbb{C}$.
3. **Triangle Inequality**: $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$

It is clear that such a norm induces a metric on X. This metric is **translation invariant** — namely, for all $z \in X$, we have $d(x, y) = d(x+z, y+z)$. In fact, we have $B_r(x)+z = B_r(x+z)$.

An **inner product space** is a complex vector space X equipped with a mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ that satisfies the following properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ and $\lambda \in \mathbb{C}$:

1. **Conjugate Symmetry**: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
2. **Positive-Definiteness**: $\langle \mathbf{x}, \mathbf{x} \rangle \ge 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$.
3. **Additivity in First Argument**: $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.
4. **Homogenity in First Argument**: $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$.

More theorems about these spaces may be found in axler6.tex. It is clear that by setting $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle$, all inner product spaces are normed vector spaces. Hence,

$$\text{inner product spaces} \;\subseteq\; \text{normed vector spaces} \;\subseteq\; \text{metric spaces} \;\subseteq\; \text{topological spaces}.$$

A complete normed vector space is a **Banach space**< while a complete inner product space is a **Hilbert space**. These spaces need not be finite-dimensional.

Theorem 6. *Let X be a finite dimensional vector space over $\mathbb{C}$ (or $\mathbb{R}^n$). Then any two norms on X are equivalent.*

Proof. Let $\dim X = n$. We first prove the theorem for $\mathbb{C}^n$; let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis of $\mathbb{C}^n$, and suppose $\|\cdot\|_1 : \mathbb{C}^n \to [0, \infty)$ is a norm. We prove that $\|\mathbf{z}\|_1$ is equivalent to the canonical norm $\|\mathbf{z}\|$.

Consider the boundary of the unit ball (in the canonica norm) in $\mathbb{C}^n$. Since $\|\cdot\|_1$ is continuous, the Extreme Value Theorem guarantees that there exists $\mathbf{u}, \mathbf{s}$ with norms 1 such that

$$\|\mathbf{u}\|' = \inf_{\|\mathbf{z}\|=1} \{\|\mathbf{z}\|'\} \qquad \text{and} \qquad \|\mathbf{s}\|' = \sup_{\|\mathbf{z}\|=1} \{\|\mathbf{z}\|'\}$$

Then for all $\mathbf{z} \in \mathbb{C}^n$, the constants $\|\mathbf{u}\|'$ and $\|\mathbf{s}\|'$ allow for norm equivalence:

$$\|\mathbf{u}\|'\|\mathbf{z}\| \;\leq\; \left\| \frac{\mathbf{z}}{\|\mathbf{z}\|} \right\|' \|\mathbf{z}\| \;=\; \|\mathbf{z}\|' \;=\; \left\| \frac{\mathbf{z}}{\|\mathbf{z}\|} \right\|' \|\mathbf{z}\| \;\leq\; \|\mathbf{s}\|'\|\mathbf{z}\|.$$

We conclude that all norms on $\mathbb{C}^n$ are equivalent to the canonical norm. $\square$

Since open sets are the same for equivalent metrics, we obtain that there is only one norm-based topology on $\mathbb{R}^n$ — the Euclidean topology. This proof fails on $\ell^p(\mathbb{N})$, since the unit sphere is not compact. Realize that for all $\mathbf{e}_i$ for $i \in \mathbb{Z}_{>0}$,

$$\|\mathbf{e}_i - \mathbf{e}_j\| \geq 1.$$

Thus the set of all $\mathbf{e}_1, \ldots$ contains no convergent subsequence, so it is not compact. Thus the Heine-Borel Theorem fails for $\ell^p(\mathbb{N})$.

2.8 Linear Maps on Normed Vector Spaces

Let (X, d_X) and (Y, d_Y) be normed vector spaces. The set $\mathcal{L}(X, Y)$ denotes the set of all linear maps between normed vector spaces X and Y. With the following operations, $\mathcal{L}(X, Y)$ is a vector space: for $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{L}(X, Y)$,

$$(\mathbf{T}_1 + \mathbf{T}_2)\mathbf{x} = \mathbf{T}_1\mathbf{x} + \mathbf{T}_2\mathbf{x}$$
$$(\lambda \mathbf{T})\mathbf{x} = \lambda(\mathbf{T}\mathbf{x}).$$

Linear maps between normed vector spaces are not necessarily continuous!

Theorem 7. *Let (X, d_X) and (Y, d_Y) be normed vector spaces. A linear map $\mathbf{T} \in \mathcal{L}(X, Y)$ is continuous if and only if it is continuous at $\mathbf{0}_X$.*

Proof. Suppose that $\mathbf{T}$ is continuous at $\mathbf{0}$. Then

$$\lim_{\mathbf{x}\to\mathbf{y}} \mathbf{Tx} \;=\; \lim_{\mathbf{x}-\mathbf{y}\to\mathbf{0}} \mathbf{Tx} \;=\; \lim_{\mathbf{x}-\mathbf{y}\to\mathbf{0}} (\mathbf{T}(\mathbf{x}-\mathbf{y})) + \mathbf{Ty} \;=\; \mathbf{0} + \mathbf{Ty} \;=\; \mathbf{Ty}.$$

Therefore, $\mathbf{T}$ is continuous at all $\mathbf{x} \in X$. □

Corollary 2. $\mathbf{L} \in \mathcal{L}(X, Y)$ *is continuous if and only if it is uniformly continuous.*

Bounded Linear Operators

Nonzero linear maps are never "bounded"; if $\mathbf{T} \in \mathcal{L}(X, Y)$ is nonzero, let $\mathbf{Tx} \neq 0$; then nonzero $\lambda \in \mathbb{C}$ implies

$$\|\lambda \mathbf{Tx}\| = |\lambda| \|\mathbf{Tx}\| \geq 0$$

can attain any nonzero complex value. Thus we formulate an alternative, relaxed condition of boundedness: $\mathbf{T}$ is **bounded** if it maps bounded sets in X to bounded sets in Y. Equivalently, $\mathbf{T}$ is bounded if for a bounded set $\Omega \subseteq X$, there exists $r > 0$ such that

$$\mathbf{T}(\Omega) \subseteq B_R[0],$$

where $B_r[0]$ is the closed ball of radius r. It is clear that $\mathbf{T}$ is bounded if and only if $\mathbf{T}$ is bounded over the unit ball.

Theorem 8. *Let X be a finite-dimensional normed vector space, and let Y be a normed vector space. Then all $\mathbf{T} \in \mathcal{L}(X, Y)$ are bounded.*

Proof. Let $\dim X = n$ and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis of X. Then for all $\mathbf{z} = z_1\mathbf{e}_1 + \cdots + z_n\mathbf{e}_n$, we have

$$\|\mathbf{Tz}\| \;\leq\; |z_1| \|\mathbf{Te}_1\| + \cdots + |z_n| \|\mathbf{Te}_n\| \;\leq\; C(|z_1| + \cdots + |z_n|),$$

where $C = \max\{\|\mathbf{Te}_i\|\}$. Realize that $\|\mathbf{z}\|_1 = |z_1| + \cdots + |z_n|$ defines a norm on X; since all norms finite-dimensional vector spaces are equivalent, there exists another constant M such that $|z_1| + \cdots + |z_n| = \|\mathbf{z}\|_1 \leq M\|\mathbf{z}\|$. Therefore

$$\|\mathbf{Tz}\| \leq CM\|\mathbf{z}\|,$$

so $\mathbf{T}$ is bounded. This completes the proof. □

2.9 Matrix Norm

If $\mathbf{T}$ is bounded, the **norm** of $\mathbf{T}$ is as follows:

$$\|\mathbf{T}\| = \sup\{\|\mathbf{Tz}\| \mid \mathbf{z} \in \mathbb{C}^n, \|\mathbf{z}\| \leq 1\}.$$

Notice that the matrix norm is nonnegative. The **critical vector** of $\mathbf{T}$ is the vector $\mathbf{z} \in X$ such that $\|\mathbf{z}\| \leq 1$ and $\|\mathbf{Tz}\| = \|\mathbf{T}\|$; the critical vector always has norm 1. Naturally, $\|\mathbf{Tz}\| \leq \|\mathbf{T}\| \cdot \|\mathbf{z}\|$; since equality is attained, $\|\mathbf{Tz}\| \leq \lambda \mathbf{z}$ implies $\|\mathbf{T}\| \leq \lambda$.

Theorem 9. *If $\mathbf{T}, \mathbf{S} \in \mathcal{L}(X, Y)$, then $\|\mathbf{T} + \mathbf{S}\| \leq \|\mathbf{T}\| + \|\mathbf{S}\|$. If $X = Y$, then $\|\mathbf{TS}\| \leq \|\mathbf{T}\|\|\mathbf{S}\|$.*

Proof. Let $\mathbf{z}$ be the critical vector of $\mathbf{T} + \mathbf{S}$. Then

$$\|\mathbf{T} + \mathbf{S}\| = \|(\mathbf{T} + \mathbf{S})\mathbf{z}\| \leq \|\mathbf{Tz}\| + \|\mathbf{Sz}\| \leq \|\mathbf{T}\|\|\mathbf{z}\| + \|\mathbf{S}\|\|\mathbf{z}\| = \|\mathbf{T}\| + \|\mathbf{S}\|.$$

Now suppose $X = Y$ and let $\mathbf{w}$ be the critical vector of $\mathbf{TS}$. Then

$$\|\mathbf{TS}\| = \|\mathbf{TSw}\| \leq \|\mathbf{T}\|\|\mathbf{Sw}\| \leq \|\mathbf{T}\|\|\mathbf{S}\|\|\mathbf{w}\| = \|\mathbf{T}\|\|\mathbf{S}\|.$$

This completes the proof. □

Theorem 10. *The matrix norm is a metric of all bounded linear maps in $\mathcal{B}(X, Y)$.*

Proof. Suppose $\mathbf{T}, \mathbf{S} \in \mathcal{B}(X, Y)$ are bounded. We must perform four rather routine calculations:

1. **Positivity**: The matrix norm is nonnegative. If $\|\mathbf{T} - \mathbf{S}\| = 0$, then $\|\mathbf{x}\| = 1$ implies $(\mathbf{T} - \mathbf{S})\mathbf{x} = \mathbf{0}$; hence for *all* $\mathbf{x} \in X$,

$$(\mathbf{T} - \mathbf{S})\mathbf{x} = \|\mathbf{x}\| \left((\mathbf{T} - \mathbf{S}) \left(\frac{\mathbf{x}}{\|\mathbf{x}\|} \right) \right) = \|\mathbf{x}\|(0) = 0;$$

   thus $\mathbf{T} - \mathbf{S} = \mathbf{0}$ and $\mathbf{T} = \mathbf{T}$.

2. **Symmetry**: Notice that $(\mathbf{T} - \mathbf{S})\mathbf{x} = -(\mathbf{S} - \mathbf{T})\mathbf{x}$ for all $\mathbf{x} \in X$. Naturally if $\mathbf{w}$ is the critical vector of $\mathbf{T} - \mathbf{S}$, then $-\mathbf{w}$ is the critical vector of $\mathbf{S} - \mathbf{T}$; thus

$$\|\mathbf{T} - \mathbf{S}\| = \|\mathbf{w}\| = \|-\mathbf{w}\| = \|\mathbf{S} - \mathbf{T}\|.$$

3. **Triangle Inequality**: For all bounded $\mathbf{R} \in \mathcal{L}(X, Y)$,

$$\|\mathbf{T} - \mathbf{S}\| = \|(\mathbf{T} - \mathbf{R}) + (\mathbf{R} - \mathbf{S})\| \leq \|\mathbf{T} - \mathbf{R}\| + \|\mathbf{R} - \mathbf{S}\|.$$

We conclude that the matrix norm is a metric of the bounded matricies of $\mathcal{L}(X, Y)$. □

It is straightforward that $\|\lambda \mathbf{T}\| = |\lambda|\|\mathbf{T}\|$ for all $\lambda \in \mathbb{C}$ as well.

Theorem 11. *$\mathbf{T} \in \mathcal{L}(X, Y)$ is bounded if and only if $\mathbf{T}$ is uniformly continuous.*

Proof. Let $\mathbf{T}$ be bounded. If $\epsilon > 0$, then $0 \le \|\mathbf{z} - \mathbf{w}\| < \frac{\epsilon}{\|\mathbf{T}\|}$ implies

$$\|\mathbf{Tz} - \mathbf{Tw}\| \le \|\mathbf{T}\| \cdot \|\mathbf{z} - \mathbf{w}\| < \|\mathbf{T}\| \left(\frac{\epsilon}{\|\mathbf{T}\|} \right) = \epsilon.$$

Thus, $\mathbf{T}$ is uniformly continuous. If we suppose that $\mathbf{T}$ is uniformly continuous, then it is clear that $\mathbf{T}$ maps compact sets to bounded sets — hence, the image of the unit ball is bounded. $\square$

Hence, all $\mathbf{T}$ from finite dimensional vector spaces are uniformly continuous and admit a matrix norm. For infinite dimensional vector spaces, it depends precisely on whether $\mathbf{T}$ is bounded.

Theorem 12. *If Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof. Let $(\mathbf{T}_n)$ be a Cauchy sequence in $\mathcal{B}(X, Y)$; for all $\epsilon > 0$, there exists N such that

$$N \le n, m \implies \|\mathbf{T}_n - \mathbf{T}_m\| < \epsilon. \tag{1}$$

We will define the limit of $(\mathbf{T}_n)$. For any $\mathbf{x} \in X$, we have that

$$\|\mathbf{T}_n\mathbf{x} - \mathbf{T}_m\mathbf{x}\| \ \le \ \|\mathbf{T}_n - \mathbf{T}_m\|\|\mathbf{x}\| \le \epsilon\|\mathbf{x}\|.$$

By selecting $\frac{\epsilon}{\|\mathbf{x}\|}$ in equation (1), we find that $(\mathbf{T}_n\mathbf{x})$ is a Cauchy sequence in in Y. Thus, it converges to a unqiue vector in Y. Define a mapping $\mathbf{T} : X \to Y$ as follows:

$$\mathbf{Tz} \quad \overset{\text{def}}{=} \quad \lim_{n\to\infty} \mathbf{T}_n\mathbf{z}$$

It is relatively easy to show that $\mathbf{T}$ is linear: for all $\mathbf{x}, \mathbf{y} \in X$ and $\lambda \in \mathbb{C}$, we have

$$\mathbf{T}(\mathbf{x} + \mathbf{y}) \ = \ \lim_{n\to\infty} \mathbf{T}_n(\mathbf{x} + \mathbf{y}) \ = \ \lim_{n\to\infty} \mathbf{T}_n\mathbf{x} + \ \lim_{n\to\infty} \mathbf{T}_n\mathbf{y} \ = \ \mathbf{TxTy}$$
$$\mathbf{T}(\lambda\mathbf{x}) \ = \ \lim_{n\to\infty} \mathbf{T}_n(\lambda\mathbf{x}) \ = \ \lambda \lim_{n\to\infty} \mathbf{Tx} \ = \ \lambda\mathbf{T}_x.$$

We must show that $\mathbf{T}$ is bounded and is the limit of $(\mathbf{T}_n)$. Observe that

$$\|\mathbf{Tz}\| \ \le \ \|\mathbf{Tz} - \mathbf{T}_n\mathbf{z}\| + \|\mathbf{T}_n\mathbf{z}\|.$$

Observe that the transformation $\phi : \mathbf{y} \to \|\mathbf{T}_n\mathbf{x} - \mathbf{y}\|$ is continuous, since it is a composition of continuous functions. Hence

$$\|\mathbf{T}_n\mathbf{zx} - \mathbf{T}_m\mathbf{z}\| \le \epsilon\|\mathbf{z}\| \implies \|\mathbf{T}_n - \mathbf{Tz}\| = \lim_{m\to\infty} \|\mathbf{T}_n\mathbf{z} - \mathbf{T}_m\mathbf{z}\| \le \epsilon\|\mathbf{x}\|.$$

Then pick $\epsilon = 1$ and $n = N$. We find that

$$\begin{aligned} \|\mathbf{T}\mathbf{x}\| &\leq \|(\mathbf{T} - \mathbf{T}_n)\mathbf{x}\| + \|\mathbf{T}_n\mathbf{x}\| \\ &\leq \|\mathbf{x}\| + \|\mathbf{T}_N\|\|\mathbf{x}\| \\ &\leq (1 + \|\mathbf{T}_\mathbf{N}\|)\mathbf{x}. \end{aligned}$$

Letting $c = 1 + \|\mathbf{T}_N\|$ yields that $\mathbf{T}$ is bounded. As per the limit condition, we have that

$$\|\mathbf{T}_n - \mathbf{T}\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|(\mathbf{T}_n - \mathbf{T})\mathbf{x}\|}{\|\mathbf{x}\|} \leq \epsilon,$$

which completes the proof. $\square$