MATH UA-129: Homework 6

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1 Section 3.4

Problem 3

Using Lagrange multipliers, we have that for some $\lambda \in \mathbb{R}$,

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \nabla f = \lambda \nabla g = \begin{bmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{bmatrix}$$

We thus have that $1 = 2\lambda x$, so $\frac{1}{2\lambda} = x$; similarly, $-\frac{1}{2\lambda} = y$ and $\frac{1}{2\lambda} = z$. Therefore,

$$2 = x^2 + y^2 + z^2 = \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2}.$$

We deducec that $\lambda^2 = \frac{3}{8}$, so $\lambda = \pm\frac{\sqrt{6}}{4}$. Substituting for x, y, and z, we find two critical points of $f \mid S$:

$$(x, y, z) = \left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}\right) \qquad \text{and} \qquad (x, y, z) = \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}\right)$$

Evaluating $x - y + z$ at these two points, we find that the first is greater than the second. Hence, $\boxed{\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}\right) \text{ is the maximum and } \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}\right) \text{ is minimum of } f \mid S}$.

Problem 8

We have that all $(x, y) \in S$ satisfy $y = 2$, so if $(x, y) \in S$,

$$f(x, y) = x^2 + y^2 = x^2 + 4.$$

As the right-hand side is a quadratic, it has a minimum at $x = 0$ and is unbounded above. Therefore, $\boxed{(0, 2) \text{ is a minimum of } f \mid S}$.

Problem 10

We have that all $(x, y) \in S$ satisfy $y = \cos(x)$, so if $(x, y) \in S$,

$$f(x, y) = x^2 - y^2 = x^2 - \cos^2(x).$$

Dub the right-hand side $g(x)$; maximizing $f \mid S$ is equivalent to maximizing g. See that

$$\begin{aligned} g'(x) &= 2x - 2\sin(x)\cos(x) = 2x - \sin(2x), \\ g''(x) &= 2 + 2\cos(2x). \end{aligned}$$

Then $g'(x) = 0$ implies that $2x = \sin(2x)$, which trivially happens exclusively when $x = 0$ (this may be proven using the tangent line trick or taylor series).

As $g''(0) = 2 + 2\cos(0) = 4$, the second derivative test implies that $x = 0$ is a global minimum of g, so $\boxed{(0, 1) \text{ is a global minimum of } f \mid S}$.

Problem 11

Observe that across all (x, y, z),

$$\nabla f = \mathbf{0} \implies \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \mathbf{0},$$

so the only critical point of f is $(0, 0, 0)$, which does not lie within S. Therefore, any local extrema of f must be attained on the boundary — namely, all elements of $\{(x, y, z) \in \mathbb{R}^3 \mid z = 2 + x^2 + y^2\}$.

Defining $g(x,y,z) = x^2+y^2-z+2$, we use Lagrange Multipliers to deduce that there exists $\lambda \in \mathbb{R}$ such that

$$\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \nabla f = \lambda \nabla g = \begin{bmatrix} 2\lambda x \\ 2\lambda y \\ -\lambda \end{bmatrix}$$

If at least one of x and y is nonzero, then $\lambda = 1$, and $z = -\frac{1}{2}$. We conclude that

$$\frac{1}{2} = x^2 + y^2 + 2 \geq 0 + 0 + 2 > \frac{1}{2},$$

a contradiction. Therefore, $x = y = 0$. As $z = -\frac{\lambda}{2}$,

$$-\frac{\lambda}{2} = x^2 + y^2 + 2 \implies \lambda = -4,$$

and $z = 2$. Then $(0,0,2)$ is the only extrema of f at x. A trivial calculation verifies that this is a minimum of f, so $\boxed{(0,0,2) \text{ is a minimum of } f \mid S}$.

Problem 12

Observe that

$$\nabla f = \begin{bmatrix} 2x-1 \\ 2y-1 \end{bmatrix}.$$

Thus, the only critical point of f not on the boundary is $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$, where f evaluates to $\frac{1}{2}$. On the boundary of the unit disc, define $g(x,y) = x^2+y^2-1$; we have that for some $\lambda \in \mathbb{R}$,

$$\begin{bmatrix} 2x-1 \\ 2y-1 \end{bmatrix} = \nabla f = \lambda \nabla g = \begin{bmatrix} 2\lambda x \\ 2\lambda y \end{bmatrix}$$

Thus, $2x - 1 = 2\lambda x$, and $x = \frac{1}{2-2\lambda}$. Similarly, $y = \frac{1}{2-2\lambda}$. Thus,

$$1 = x^2 + y^2 = 2x^2 = 2y^2,$$

so our four points to consider are $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$. We find that

$$\begin{aligned} f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) &= 2-\sqrt{2}, \\ f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) &= 2, \\ f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) &= 2, \\ f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) &= 2+\sqrt{2}. \end{aligned}$$

Comparing all of these, we find that the $\boxed{\left(\frac{1}{2}, \frac{1}{2}\right) \text{ is an absolute minimum}}$ and $\boxed{\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \text{ is an absolute maximum}}$ of f on the unit disc.

Problem 13

Trivially, the only critical point of f outside the boundary is $(0,0)$, in which f evaluates to 0. Defining $g(x) = x^2+y^2-1$, Lagrange mulitpliers yield that there exists $\lambda \in \mathbb{R}$ such that

$$\begin{bmatrix} 2x+y \\ 2y+x \end{bmatrix} = \nabla f = \lambda \nabla g = \begin{bmatrix} 2\lambda x \\ 2\lambda y \end{bmatrix}.$$

Thus, $(2-2\lambda)x + y = 0 = x + (2-2\lambda)y$. Adding these two equations, we find that

$$(3-2\lambda)(x+y) = 0.$$

Either $\lambda = \frac{3}{2}$ — in which case $x = y$ — or $x = -y$. In each case, $x^2 = y^2$, so

$$1 = x^2 + y^2 = 2x^2.$$

Thus, we have four points on the boundary to consider: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{-\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Comparing these, we find that $\boxed{(0,0) \text{ is an absolute minimum}}$ and $\boxed{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \text{ are absolute maxima}}$ of f on the unit disc.

Problem 21

We seek to minimize the surface area $2\pi r^2 + 2\pi rh$ of the cylinder under the constrant that its volume $\pi r^2 h$ is 1000. Letting the surface area be $S(r,h)$ and the volume be $V(r,h)$, Lagrange multipliers ensure that there is $\lambda \in \mathbb{R}$ such that

$$\begin{bmatrix} 4\pi r + 2\pi h \\ 2\pi r \end{bmatrix} = \nabla S = \lambda \nabla V = \begin{bmatrix} 2\lambda \pi r h \\ \lambda \pi r^2 \end{bmatrix}.$$

From $2\pi r = \lambda \pi r^2$, we find that $2 = \lambda r$ (as clearly $r \neq 0$). Thus, the top equation yields that

$$4\pi r + 2\pi h = 4\pi h,$$

so $2r = h$. Therefore,

$$1000 = \pi r^2 h = 2\pi r^3,$$

so $\boxed{r = \sqrt[3]{\frac{1000}{2\pi}} \text{ centimeters}}$ and $\boxed{h = \frac{1}{2}\sqrt[3]{\frac{1000}{2\pi}} \text{ centimeters}}$.

Problem 31

Part (a): We define

$$A = \begin{bmatrix} a & b & c \\ b & a & d \\ c & d & a \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2}(A\mathbf{x}) \cdot \mathbf{x} \\ &= \frac{1}{2} \begin{bmatrix} ax + by + cz \\ bx + ay + dz \\ cx + dy + za \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \frac{1}{2}(ax^2 + bxy + czx) + \frac{1}{2}(bxy + ay^2 + dyz) + \frac{1}{2}(cxz + dyz + z^2 a) \\ &= \tfrac{1}{2}ax^2 + \tfrac{1}{2}by^2 + \tfrac{1}{2}cz^2 + bxy + dyz + czx. \end{aligned}$$

Thus,

$$\nabla f = \begin{bmatrix} ax + by + cz \\ by + bx + dz \\ cz + dy + cx \end{bmatrix} = A\mathbf{x}.$$

Part (b): We have that by Lagrange multipliers, there exists λ such that

$$A\mathbf{x} = \nabla f = \lambda \nabla g = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = 2\lambda \mathbf{x}.$$

Therefore, A must have an eigenvalue $\mathbf{x}$ with eigenvalue 2λ.

Problem 35

Let $x = 2$ and $z = \frac{1}{2}$. Then if we let $y = \frac{M}{2+\frac{1}{2}}$ for any $M \in \mathbb{R}$,

$$xy + yz = y(x + z) = M$$

so the expression $xy + yz$ can assume any real value. We conclude that it has no minimum or maximum.

Problem 37

We wish to maximize the function

$$\sqrt{\cos^2(t) + \sin^2(t) + \sin^2\left(\frac{t}{2}\right)} = \sqrt{1 + \frac{1 - \cos(\theta)}{2}}.$$

This function is clearly maximized whenever $\cos(t) = -1$, in which case the maximum is $\boxed{\sqrt{2}}$.

2 Section 3.5

Problem 12

Part (a): We have that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(x) + r\sin^2(\theta) = r.$$

At $(r, \theta) = (r_0, \theta_0)$, the Jacobian determininant evaluates to r_0.

Part (b): Clearly $f(r, \theta) = (r\cos(\theta), r\sin(\theta))$ is C^1. Then f is invertible if the Jacobian determinant is nonzero — at $\boxed{\text{all points except the origin}}$. Clearly, we cannont establish an inverse at the origin.

Part (c): Using the Rule of Sarrus, we have that

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{vmatrix}, \\
&= \rho^2\sin(\phi)\cos^2(\phi)\cos^2(\theta) + \rho^2\sin^3(\phi)\sin^2(\theta) + \rho^2\sin(\phi)\cos(\phi)^2\sin^2(\theta) \\
&\quad + \rho^2\sin^3(\phi)\cos^2(\theta), \\
&= \rho^2\sin(\phi)\cos^2(\phi) + \rho^2\sin^3(\phi), \\
&= \rho^2\sin(\phi).
\end{aligned}$$

Part (d): We can solve for (ρ,ϕ,θ) whenever the Jacobian determininat is nonzero, which occurs for $\boxed{\text{all } (\rho,\phi,\theta) \text{ such that } \rho \neq 0 \text{ and } \phi \notin \{\pi n \mid n \in \mathbb{Z}\}}$.

Problem 17

Part (a): We define

$$\begin{aligned}
F_1(x,y,u,v) &= x^2 - y^2 - u^3 + v^2 + 4, \\
F_2(x,y,u,v) &= 2xy + y^2 - 2u^2 + 3v^4 + 8.
\end{aligned}$$

We have that

$$\begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_2}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix} = (-3u^2)(12v^3) - (2v)(-4u) = -36u^2v^3 + 8uv.$$

At $(x,y,u,v) = (2,-1,2,1)$, this determiniant evaluates to $-128 \neq 0$. The Inverse Function Theorem thus gurantees that there exists functions $u(x,y)$ and $v(x,y)$ near the point $(2,-1,2,1)$.

Part (b): Using implicit differentiation,

$$\begin{aligned}
0 &= 2x - 3u^2\left(\frac{\partial u}{\partial x}\right) + 2v\left(\frac{\partial v}{\partial x}\right), \\
0 &= 2y - 4u\left(\frac{\partial u}{\partial x}\right) + 12v^3\left(\frac{\partial v}{\partial x}\right).
\end{aligned}$$

Substituting $(x,y,u,v) = (2,-1,2,1)$ yields

$$\begin{aligned}
0 &= 4 - 12\left(\frac{\partial u}{\partial x}\right) + 2\left(\frac{\partial v}{\partial x}\right), \\
0 &= -2 - 8\left(\frac{\partial u}{\partial x}\right) + 12\left(\frac{\partial v}{\partial x}\right).
\end{aligned}$$

Subtracting six times the top equation to the second equation, we find that

$$0 = -26 - 64\frac{\partial u}{\partial x},$$

so $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{13}{32}}$.

Problem 19

Let the roots of $x^3 + ax^2 + bx + c$ for $a, b, c \in \mathbb{R}$ be r, s, and t. Vieta's Formulas return that

$$\begin{aligned} a &= -r - s - t \\ b &= rs + st + tr \\ c &= -rst. \end{aligned}$$

Viewing a, b, and c as functions of r, s, and t, we have that

$$\begin{aligned} \begin{vmatrix} \frac{\partial a}{\partial r} & \frac{\partial a}{\partial s} & \frac{\partial a}{\partial t} \\ \frac{\partial b}{\partial r} & \frac{\partial b}{\partial s} & \frac{\partial b}{\partial t} \\ \frac{\partial c}{\partial r} & \frac{\partial c}{\partial s} & \frac{\partial c}{\partial t} \end{vmatrix} &= \begin{vmatrix} -1 & -1 & -1 \\ s+t & t+r & r+s \\ -st & -tr & -rs \end{vmatrix} \\ &= rs(t+r) + st(r+s) + tr(s+t) - st(t+r) - rs(s+t) - tr(r+s) \\ &= r^2s + s^2t + t^2r - rs^2 - st^2 - tr^2 \\ &= (r-s)(s-t)(t-s). \end{aligned}$$

This determinant is nonzero if and only if r, s, and t are all distinct. In this case, the Inverse Function Theorem guarantees that the vector-valued function $(a(r,s,t), b(r,s,t), c(r,s,t)$ has a smooth inverse — namely, a function that maps a, b, and c to the roots of the polynomial given by $x^3 + ax^2 + cx + d = 0$.

3 Section 4.1

Problem 19

We assume the object lies in 3D space — the argument here may be easily generalized to higher dimensions.

If the velocity vector of the object is $\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t))$ and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t)$, then we are given that

$$\mathbf{v} \cdot \mathbf{a} = v_1(t)v_1'(t) + v_2(t) + v_2'(t) + v_3(t) + v_3'(t) = 0.$$

We thus calculate the derivative of the speed of the object, which is the norm of $\mathbf{v}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{v_1^2(t)+v_2^2(t)+v_3^2(t)} = \frac{v_1(t)v_1'(t)+v_2(t)v_2'(t)+v_3(t)v_3'(t)}{\sqrt{v_1^2(t)+v_2^2(t)+v_3^2(t)}} = 0.$$

As the derivative of the speed is zero, the object's speed must be constant.

Problem 21

We have that if the period is T in miles, the mass of the earth is M , and the gravitational constant is G,

$$T \approx \sqrt{(6.436\times 10^6+500)^3\frac{(2\pi)^2}{GM}}.$$

Problem 23

The general solution of $\mathbf{c}'(t) = (t, e^t, t^2)$ is $\mathbf{c} = (\frac{1}{2}t^2 + c_1, e^t + c_2, \frac{1}{3}t^3 + c_3)$ for $c_1, c_2, c_3 \in \mathbb{R}$. Solving for these constants, we have that at $t = 0$,

$$\begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(0)^2 + c_1 \\ e^0 + c_2 \\ \frac{1}{3}(0)^3 + c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 1 + c_2 \\ c_3 \end{bmatrix},$$

so $c_1 = 0$, $c_2 = -6$, and $c_3 = 1$. The path we desire is thus $\boxed{\mathbf{c}(t) = \left(\frac{1}{2}t^2, e^t - 6, \frac{1}{3}t^3 + 1\right)}$.

Problem 26

Defining $\mathbf{c}(t)$ as $(f_1(t), f_2(t), f_3(t))$, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{c}(t)\times\mathbf{v}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\begin{bmatrix} f_2f_3' - f_2'f_3 \\ f_3f_1' - f_1f_3' \\ f_1f_2' - f_2f_1' \end{bmatrix}\right) \\
&= m\begin{bmatrix} (f_2'f_3' + f_2f_3'') - (f_2''f_3 + f_2'f_3') \\ (f_3'f_1' + f_3f_1'') - (f_3''f_1 + f_3'f_1') \\ (f_1'f_2' + f_1f_2'') - (f_1''f_2 + f_1'f_2') \end{bmatrix} \\
&= m\begin{bmatrix} f_2f_3'' - f_2''f_3 \\ f_3f_1'' - f_1f_3'' \\ f_1f_2'' - f_2f_1'' \end{bmatrix} \\
&= m(\mathbf{c}(t)\times\mathbf{a}(t)) \\
&= \mathbf{c}(t)\times\mathbf{F}(\mathbf{c}(t)),
\end{aligned}$$

as desired.