

Vakil: Some Category Theory

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1 Categories and Functors

1.1 Definition of a Category

A **category** \mathcal{C} is a collection of **objects** and a collection of **morphisms** between pairs of objects. Morphisms are written $f : A \rightarrow B$, where A is the **source** of f and B is the **target** of f . A category must further contain a product $\text{Mor}(A, B) \times \text{Mor}(B, C) \rightarrow \text{Mor}(A, C)$; if $f \in \text{Mor}(A, B)$ and $g \in \text{Mor}(B, C)$, their composition is denoted $g \circ f \in \text{Mor}(A, C)$. The following properties must also be satisfied:

1. Composition of morphisms is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.
2. Identity morphisms must exist: If $f : A \rightarrow B$, then we must have $\text{id}_B \circ f = f$ and $f \circ \text{id}_A = f$.

Morphisms form a monoid under composition; the identity morphisms are thus unique. The usual types of morphisms have definitions in categories, too:

1. A morphism $f : A \rightarrow B$ is an **isomorphism** if there exists a morphism $g : B \rightarrow A$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.
2. A morphism $f : A \rightarrow A$ is an **epimorphism**.
3. An isomorphic epimorphism is an **automorphism**.

We do not impose that the classes of objects and morphisms form sets — for instance, consider the category **Set** of all sets under set mappings. Its objects do not form a set, or else both Cantor's and Russell's paradoxes arise. In general, we will avoid foundational issues by neglecting to perform set theory on classes.

1.2 Examples

Let A be an object in a category \mathcal{C} . The **automorphism group** $\text{Aut}(A)$ of A is the group of all invertible elements of $f \in \text{Mor}(A, A)$. If two objects A and B are isomorphic, then there exists a group isomorphism $\text{Aut}(A) \cong \text{Aut}(B)$.

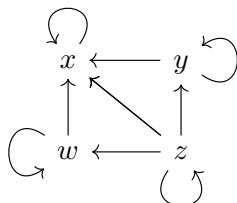
Two particularly critical examples of categories are **Ab** and **Mod_R** — the categories of Abelian groups and modules over a (commutative) ring R . These two categories constitute the chief examples of an Abelian category. It is clear that **Mod_k** = **Vec_k** (the category of vector spaces) when k is a field, and **Mod_ℤ** = **Ab**.

Familiar categories are **Grp** and **Ring**, groups and rings with homomorphisms, and **Top**, topological spaces with continuous maps.

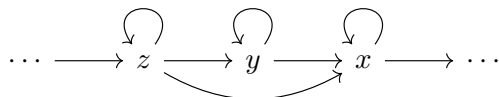
Recall that a partially ordered set (S, \geq) is a set S endowed with a relation \geq such that

1. **Reflexivity:** $x \geq x$ for all $x \in S$.
2. **Antisymmetry:** If $x \geq y$ and $y \geq x$, then $x = y$.
3. **Transitivity:** If $x \geq y$ and $y \geq z$, then $x \geq z$.

We can describe S as a category — one where objects are the elements of S and where a single morphism $f : x \rightarrow y$ exists if and only if $y \geq x$. The reflexivity of \geq ensures the existence of identity mappings, while transitivity ensures composition of morphisms is associative. As an example, here is a partially ordered set with four elements:



in which $x \geq y \geq z$ and $x \geq w \geq z$. A totally ordered set would look like this:



in which $x \geq y \geq z$. If X is a set, then the subsets of X are partially ordered by inclusion — thus such a category may be constructed. Most notably, one can construct a category modeling the open sets of a topological space by inclusion.

A **subcategory** \mathcal{A} of a category \mathcal{B} is a category whose objects lie in \mathcal{B} and whose morphisms include the identities of its objects and are closed under composition. For instance, the category $A \rightarrow B$ (with identity morphisms omitted) is a subcategory of $A \rightarrow B \rightarrow C$. There is also the **inclusion functor** defined as the embedding $i : \mathcal{A} \rightarrow \mathcal{B}$.

1.3 Functors

Let \mathcal{A} and \mathcal{B} be categories. A **covariant functor** $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is a map $\mathcal{F} : \text{obj}(\mathcal{A}) \rightarrow \text{obj}(\mathcal{B})$ that “preserves morphisms” in the following sense: for all morphisms $f_1 : A_1 \rightarrow A_2$ and $f_2 : A_2 \rightarrow A_3$,

1. There exists a morphism $\mathcal{F}(f_1) : \mathcal{F}(A_1) \rightarrow \mathcal{F}(A_2)$ in \mathcal{B} .
2. \mathcal{F} maps identities in \mathcal{A} to identities: $\mathcal{F}(\text{id}_A) = \text{id}_{\mathcal{F}(A)}$.
3. \mathcal{F} preserves morphism structure: $\mathcal{F}(f_2 \circ f_1) = \mathcal{F}(f_2) \circ \mathcal{F}(f_1)$.

An obvious example of a functor is the **identity functor** $\text{id} : \mathcal{A} \rightarrow \mathcal{A}$. Some more examples:

1. A **forgetful functor** is a functor that loses additional structure of the source category. One example is the functor $\mathbf{Vec}_k \rightarrow \mathbf{Set}$ that maps each vector space to its underlying set; another is the mapping $\mathbf{Mod}_R \rightarrow \mathbf{Ab}$ by isolating each module's additive structure.
2. Let X be a topological space, and select $x_0 \in X$ arbitrarily. The fundamental group functor ϕ_1 maps the topological space X to the group $\phi_1(X, x_0)$, and the i -th homology functor $\mathbf{Top} \rightarrow \mathbf{Ab}$ sends all topological spaces X to their i -th homology group $H_i(X, \mathbb{Z})$.
3. Suppose A is an object in a category \mathcal{C} . There is a functor $h^A : \mathcal{C} \rightarrow \mathbf{Set}$ that sends each object $B \in \mathcal{C}$ to $\text{Mor}(A, B)$ and each morphism $f : B_1 \rightarrow B_2$ to $\text{Mor}(A, B_1) \rightarrow \text{Mor}(A, B_2)$ described by

$$[g : A \rightarrow B_1] \mapsto [f \circ g : A \rightarrow B_1 \rightarrow B_2].$$

This little guy ends up becoming surprisingly important.

The **composition** of functors $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{C}$ is denoted $\mathcal{G} \circ \mathcal{F} : \mathcal{A} \rightarrow \mathcal{C}$, and is defined in the obvious way. Composition of functors is associative in an evident sense. The following terms describe the extent to which a covariant functor $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ respects the morphisms of \mathcal{B} :

1. \mathcal{F} is **faithful** if the mapping $\text{Mor}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{Mor}_{\mathcal{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is injective.
2. \mathcal{F} is **full** if the mapping $\text{Mor}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{Mor}_{\mathcal{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is surjective.
3. \mathcal{F} is **fully faithful** if the mapping $\text{Mor}_{\mathcal{A}}(A_1, A_2) \rightarrow \text{Mor}_{\mathcal{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is bijective.

Let $i : \mathcal{A} \rightarrow \mathcal{B}$ be a subcategory. Since inclusions are always faithful, we need not use the phrase “faithful subcategory”. \mathcal{A} is a **full subcategory** if i is full. For instance, the category of finitely-generated R modules is a full subcategory of \mathbf{Mod}_R ; the mapping $\mathbf{Vec}_k \rightarrow \mathbf{Set}$ is faithful but not full.

A **contravariant functor** $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ is defined equivalently to covariant functors, except morphisms are reversed; a morphism $f : A_1 \rightarrow A_2$ induces a morphism $\mathcal{F}(f) : \mathcal{F}(A_2) \rightarrow \mathcal{F}(A_1)$. The composition law is also reversed: $\mathcal{F}(f_2 \circ f_1) = \mathcal{F}(f_1) \circ \mathcal{F}(f_2)$.

For a category \mathcal{A} , the **opposite category** \mathcal{A}^{opp} of \mathcal{A} is the category formed by reversing the arrows of \mathcal{A} . Hence contravariant functors $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ are equivalent to covariant functors $\mathcal{F} : \mathcal{A}^{\text{opp}} \rightarrow \mathcal{B}$.

For examples: there is a contravariant functor $\mathbf{Top} \rightarrow \mathbf{Ring}$ mapping a topological space X to the ring of real-valued functions on X . A morphism of topological spaces $X \rightarrow Y$ induces the pullback map from functions on Y to functions on X . In fact the i -th cohomology functor $\mathcal{H}_i(\cdot, \mathbb{Z}) : \mathbf{Top} \rightarrow \mathbf{Ab}$ is a contravariant functor.

In the category \mathbf{Vec}_k , taking duals gives the contravariant functor $(\cdot)^\vee : \mathbf{Vec}_k \rightarrow \mathbf{Vec}_k$. It induces upon each linear transformation $\mathbf{T} : V \rightarrow W$ a dual transformation $\mathbf{T}^\vee : W^\vee \rightarrow V^\vee$, with $(g \circ f)^\vee = f^\vee \circ g^\vee$.

A critical example is the **functor of points**: suppose A is an object of \mathcal{C} . There is a contravariant functor $h_A : \mathcal{C} \rightarrow \mathbf{Set}$ sending each $B \in \mathcal{C}$ to $\text{Mor}(B, A)$, and sending the morphism $f : B_1 \rightarrow B_2$ to the morphism $\text{Mor}(B_2, A) \rightarrow \text{Mor}(B_1, A)$ via

$$[g : B_2 \rightarrow A] \mapsto [g \circ f : B_1 \rightarrow B_2 \rightarrow A].$$

This looks quite weird, but the examples from linear algebra and the functor $\mathbf{Top} \rightarrow \mathbf{Ring}$ are merely special cases of the functor of points. A more natural name would be the “functor of maps”, but alas — it is too late to change it.