Real Analysis: Boundedness Theorem Attempt

James Pagan

August 2023

Contents

1 INCORECT Boundedness Theorem

Theorem. *If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$, then $f(x)$ is bounded on $[a, b]$.*

Proof. Suppose for contradiction that $f(x)$ is not bounded on $[a, b]$; that is, for all $M \in \mathbb{R}$, there is some $y \in [a, b]$ such that $f(y) > M$. We will prove that this implies the existence of a real number inside $[a, b]$ outside the domain of $f(x)$.

For all $M \in \mathbb{Z}_{\geq 0}$, let $S_M = \{x \mid x \in [a, b], f(x) \geq M\}$ — we supposed that S_M is nonempty.

> **Claim.** *S_M contains a closed proper interval for all $M \in \mathbb{Z}_{\geq 0}$.*
>
> *Proof.* For some $M \in \mathbb{Z}_{\geq 0}$, let c be a real number belonging to S_{M+1}. Observe that $c \in [a, b]$ and $f(c) > M + 1$.
>
> If all reals in $x \in [a, c)$ satisfy $f(x) > M$, then $[a, c] \subseteq S_M$ — and likewise, if all $x \in (c, b]$ satisfy $f(x) > M$, then $[c, b] \subseteq S_M$. Otherwise, there exist real numbers $\alpha \in [a, c)$ and $\beta \in (c, b]$ such that $f(\alpha) \leq M$ and $f(\beta) \leq M$.
>
> Let $p = \sup\{x \mid x \in [\alpha, c), f(x) = M\}$ and $q = \inf\{x \mid x \in (c, \beta], f(x) = M\}$. The Intermediate Value Theorem guarantees that both sets are nonempty, so each set possess a supremum and infimum — furthermore, it trivially guarantees that $f(x) \geq M$ for all $x \in [p, c]$ and $x \in [c, q]$. Then $[p, q] \subseteq S_M$, and S_M contains a closed interval for all $M \in \mathbb{Z}_{\geq 0}$.

=== PLEASE READ ===

There is a flaw in my definition of a characteristic of an interval — it's literally just the floor of p (or floor of q). Every point is contained within at most one maximal interval *for a given S_M, where M is a nonnegative integer.* Oops!

Suppose we have that $f(x)$ is defined on $[0, 1]$ and is continuous on all points that *are not* the reciprocals of powers of three; all these points have infinite limits. Our iterative (2^n slices) idea would converge on 0 — which is actually defined! Oops!

Yes, the limit of 0 is not defined — but that's not how I originally conceived of this proof :)

=== THE FOLLOWING PROOF IS HIGHLY FLAWED ===

We now develop the notion of a *maximal interval of S_n*, which we define to be a closed proper interval in S_n that satisfies three criteria:

1. Its endpoints $p < q$ satisfy $f(p) = f(q) = M$;
2. Either $p = a$ or there exists $\epsilon_1 > 0$ such that $0 < p - x < \epsilon_1$ implies $f(x) < M$.
3. Either $q = b$ or there exists $\epsilon_2 > 0$ such that $0 < x - q < \epsilon_2$ implies $f(x) < M$.

Maximal intervals satisfy key proprties that enable us to construct a real number inside $[a, b]$ that lies outside the domain of $f(x)$:

> **Claim.** *WRONG! WRONG! WRONG! WRONG! At most one maximal interval contains any real $r \in [a, b]$.*
>
> *Proof.* Let $[p, q]$ and $[s, t]$ be distinct maximal intervals that contain r.
>
> Suppose for contradiction that $q > t$. If $q \notin [s, t]$, then $[p, q]$ and $[s, t]$ are entirely disjoint, which contradicts the definition of r. Otherwise, $q \in [s, t]$. Now as $q \neq b$, $0 < x - q < \min\{\frac{\epsilon_2}{2}, t\}$, implies $f(x) < M$. These x-values are contained within $[s, t]$ — therefore, $[s, t]$ is not maximal, a contradiction. An identical argument shows that $s < t$ leads to contradiction. Thus, we must have that $t = s$.
>
> Suppose for contradiction that $p < s$. If $s \notin [p, q]$, then $[p, q]$ and $[s, t]$ are entirely disjoint, which contradicts the definition of r. Otherwise, $s \in [p, q]$. Now as $s \neq a$, $0 < s - x < \min\{\frac{\epsilon_2}{2}, q\}$, implies $f(x) < M$. These x-values are contained within $[p, q]$ — therefore, $[p, q]$ is not maximal, a contradiction. An identical argument shows that $p > s$ leads to a contradiction. Thus, w must have that $p = s$.
>
> Then $[p, q]$ and $[s, t]$ are the same interval — at most one maximal interval contains r.

We thus have that maximal intervals are non-overlapping; most notably, they have distinct endpoints.

> **Claim.** *Any interval is S_M is contained within a unique maximal interval of S_M.*
>
> *Proof.* Let $[p, q]$ be an interval contained within S_M. Then the interval
>
> $$[\max\{a, \sup\{x \mid x \geq M, x \leq p\}\}, \min\{b, \inf\{x \mid x \geq M, x \geq q\}\}]$$
>
> exists by the Intermediate Value Theorem, contains $[p, q]$ — and if we define s and t such that the interval is $[s, t]$ — we have that $f(s) = f(t) = M$.

We deduce from our claims that S_M contains a maximal interval for all $M \in \mathbb{Z}_{\geq \nvdash}$. Define the *characteristic* of a maximal interval with endpoints p and q as $\lfloor p \rfloor$ — or equivalently, $\lfloor q \rfloor$.

For $n \in \mathbb{Z}_{\geq 0}$, consider the 2^N closed intervals of size $\frac{b-a}{2^N}$ between a and b. Let I_N be unique interval that these that — among those that contain *maximal subintervals* of arbitrarily large characteristic — with the greatest upper bound.

> **Claim.** *I_N exists for all $n \in \mathbb{Z}_{\geq 0}$.*
>
> *Proof.*

We now have two cases — one in which there exists an $M \in \mathbb{Z}_{\geq 0}$ such that S_n contains finitely many maximal intervals for all integers $M < n$, or whether no such n exists. We will refer to these as the *finite case* and *infinite case*. We wish to prove that in both cases, there exists a real number x that lies inside intervals of arbitrarily large characteristic.

> **Lemma 1.** *If $f(x)$ satisfies the finite case on $[a, b]$ for $M \in \mathbb{Z}$, then there exists a real number x that lies inside maximal intervals of arbitrarily large characteristic.*
>
> *Proof.* Suppose for contradiction that all maximal intervals have finite characteristic. Then there does not exist a maximal interval with characteristic $\max\{f(I_1), f(I_2), f(I_3), \ldots f(I_j)\} + 1$. This contradicts our finding that there exists a maximal interval for every $M \in \mathbb{Z}_{\geq 0}$, so some maximal interval must have infinite characteristic.
>
> Furthermore, note that the maximal intervals of

Claim. *There exists a sequence of closed intervals* I_M *such that* $I_M \subseteq S_M$ *and* $I_{M+1} \subseteq I_M$ *for all* $M \in \mathbb{Z}_{>0}$.

Proof. Suppose for contradiction that no such sequence exists. The prior claim establishes the existence of a sequence of closed intervals I_M for all $M \in \mathbb{Z}_{>0}$ such that $I_M \subseteq S_M$ — thus, all such sequences must fail to satisfy the second requirement. For each sequence, there is a positive integer q such that $I_q < I_n$.i

closed interval interval inside S_M for all $M \in \mathbb{Z}_{>0}$, so the second condition must be false — namely, that all closed intervals in S_{M+1} are not contained within S_M for some positive integer M.

However, *all* closed intervals of S_{M+1} lie inside intervals of S_M —

□