# Atiyah-MacDonald: Modules Exercises

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## 1 Problem 1

*Proof.* Let the modular inverse of $m \pmod{n}$ be $m^{-1}$. Then for all $a, b \in \mathbb{Z}_n \otimes \mathbb{Z}_m$,

$$a \otimes b \,=\, mm^{-1}a \otimes b \,=\, m_{-1}a \otimes mb \,=\, m^{-1}a \otimes 0 \,=\, 0.$$

We conclude that $\mathbb{Z}_n \otimes \mathbb{Z}_m \,=\, 0$. $\square$

## 2 Problem 2

*Proof.* Consider the exact sequence

$$\mathfrak{a} \xrightarrow{\;i\;} A \xrightarrow{\;\pi\;} A \,/\, \mathfrak{a} \longrightarrow 0,$$

where $i$ is the inclusion map and $\pi$ is the canonical epimorphism. Tensoring with $M$, we find that

$$\mathfrak{a} \otimes_A M \xrightarrow{i \otimes_A 1} A \otimes_A M \xrightarrow{\pi \otimes_A 1} (A \,/\, \mathfrak{a}) \otimes_A M \longrightarrow 0$$

is an exact sequence. Observe that $A \otimes_A M \;\cong\; M$ by the mapping $f(a, x) \,=\, ax$; hence there exists a surjective mapping

$$M \xrightarrow{(\pi \otimes_A 1) \circ g} (A \,/\, \mathfrak{a}) \otimes_A M,$$

where $g(x) \,=\, 1 \otimes_A x$. It is easy to verify that the kernel of this homomorphism is all elements of the form $a \otimes_A x$ for all elements $a \in \mathfrak{a}$ — in other words, $\mathfrak{a}M$. Hence the First Isomorphism Theorem yields

$$M \,/\, \mathfrak{a}M \;\cong\; (A \,/\, \mathfrak{a}) \otimes_A M.$$

This completes the proof. $\square$

## 3 Problem 3

*Proof.* Let $\mathfrak{m}$ be the sole maximal ideal of $A$. Realize that $M \otimes_A N \,=\, 0$ implies that

$$(A \,/\, \mathfrak{m}) \otimes_A (M \otimes_A N) \otimes_A (A \,/\, \mathfrak{m}) \,=\, 0 \;\implies\; M_{(A/\mathfrak{m})} \otimes_A N_{(A/\mathfrak{m})} \,=\, 0.$$

However, $M_{(A/\mathfrak{m})}$ are vector spaces over the field $A \,/\, \mathfrak{m}$. Thus we have (probably)

$$0 \,=\, \dim \left(M_{(A/\mathfrak{m})} \otimes_A N_{(A/\mathfrak{m})}\right) \,=\, \dim M_{(A/\mathfrak{m})} \times \dim N_{(A/\mathfrak{m})}.$$

Thus one of $M_{(A/\mathfrak{m})}$ or $N_{A/\mathfrak{m}}$ must be zero. Without loss of generality, let $M_{(A/\mathfrak{m})}$ be zero; thus by exercise 2,

$$M_{(A/\mathfrak{m})} = 0 \implies (A/\mathfrak{m}) \otimes_A M = 0 \implies M/\mathfrak{m}M = 0.$$

Thus since $M$ is finitely-generated, $M = \mathfrak{m}M$. By Nakayama's Lemma, we conclude $M = 0$. This completes the proof. $\square$

# 4 Problem 4

*Proof.* We utilize the following lemma. The proof is straightforward, omitted for brevity:

**Lemma 1.** *Let* $P_i \xrightarrow{f_i} Q_i$ *be homomorphisms of* $A$*-modules. Then*

$$\bigoplus_i P_i \xrightarrow{\bigoplus_i f_i} \bigoplus_i Q_i$$

*is injective if and only if each* $f_i$ *is injective.*

We are ready to tackle the problem at hand. Let $N_1 \xrightarrow{f} N_2$ be any monomorphism of $A$-modules. Then

$$\begin{aligned}
\bigoplus_i M_i \text{ is flat} &\iff N_1 \otimes \bigoplus_i M_i \xrightarrow{f \otimes \sum_i 1_i} N_2 \otimes \bigoplus_i N_i \text{ is injective} \\
&\iff \bigoplus_i (N_1 \otimes M_i) \xrightarrow{\bigoplus_i (f \otimes 1_i)} \bigoplus_i (N_2 \otimes M_i) \text{ is injective} \\
&\iff N_1 \otimes M_i \xrightarrow{f \otimes 1_i} N_2 \otimes M_i \text{ is injective for each } i \\
&\iff M_i \text{ is flat for each } i.
\end{aligned}$$

This completes the proof. Hence, all free modules are flat. $\square$