MATH-UA 129: Homework 8

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1 Section 5.2

Problem 8

The volume is equivalent to the volume of $f(x, y) = x^2 + y^4$ over the unit square, which is

$$\int_0^1 \int_0^1 (x^2 + y^4)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^1 x^2\,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^1 y^4\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{3} + \frac{1}{5} = \boxed{\frac{8}{15}}.$$

Problem 14

We seek to evaluate the integral defined as f:

$$\begin{aligned}
f(m,n) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos(nx)\sin(ny)\,\mathrm{d}x\,\mathrm{d}y \\
&= \left(\int_{-\pi}^{\pi} \sin(my)\,\mathrm{d}y\right)\left(\int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x\right) \\
&= \left(\frac{-\cos(m\pi)+\cos(-m\pi)}{m}\right)\left(\frac{\sin(n\pi)-\sin(-n\pi)}{n}\right) \\
&= (0)\left(\frac{2\sin(n\pi)}{n}\right) \\
&= 0.
\end{aligned}$$

Therefore, $\lim\limits_{m,n\to\infty} f(m,n) = 0$.

Problem 18

Suppose for contradiction that there exists $\mathbf{a} \in R$ such that $f(\mathbf{a}) > 0$. As f is continuous and R is an open set, there exists δ such that

$$\|\mathbf{x}-\mathbf{a}\| < \delta \implies \mathbf{x} \in R \quad \text{and} \quad |f(\mathbf{x}) - f(\mathbf{a})| < \frac{f(\mathbf{a})}{2}$$

Denote the open ball defined by $\|\mathbf{x}-\mathbf{a}\| < \delta$ as B. We then have that for all $\mathbf{x} \in B$,

$$-\frac{f(\mathbf{a})}{2} < f(\mathbf{x}) - f(\mathbf{a}) < \frac{f(\mathbf{a})}{2} \implies \frac{f(\mathbf{a})}{2} < f(\mathbf{x}) < \frac{3f(\mathbf{a})}{2},$$

so $f(\mathbf{x}) > 0$ on B. We defined that $B \subset R$, so $B \cup (R \setminus B) = R$; hence,

$$0 < \iint_B f\,\mathrm{d}A \le \iint_B f\,\mathrm{d}A + \iint_{B\setminus R} f\,\mathrm{d}A = \iint_R f\,\mathrm{d}A\,,$$

which yields the desired contradiction. We conclude that $f = 0$ on R.

2 Section 5.3

Problem 6

Observe that the ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore, the area is given by evaluating the following integral:

$$\begin{aligned}
\int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} 1 \,\mathrm{d}y \,\mathrm{d}x &= \int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}} \,\mathrm{d}x \\
&= \left[ab \arcsin\left(\tfrac{x}{a}\right) + bx\sqrt{1 - \tfrac{x^2}{a^2}}\right]_{-a}^{a} \\
&= ab \arcsin(1) + 0 - ab \arcsin(-1) - 0 \\
&= \boxed{ab\pi}.
\end{aligned}$$

Problem 9

The area is given by evaluating the following integral:

$$\begin{aligned}
\iint_D x^3 y \,\mathrm{d}x \,\mathrm{d}y &= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{3}} \int_{0}^{-4y^2+3} x^3 y \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\frac{x^4 y}{4}\right]_0^{-4y^2+3} \mathrm{d}y \\
&= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{y(3-4y^2)^4}{4} \,\mathrm{d}y \\
&= -\frac{1}{32} \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} (-8y)(3-4y^2)^4 \,\mathrm{d}y \\
&= -\frac{1}{32} \left[\frac{(3-4y^2)^5}{5}\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \\
&= \boxed{0}.
\end{aligned}$$

Problem 15

Note that for a given z, the 2D region bounded by $x^2 + y^2 = z$ is a circle with radius $\sqrt{z}$, so it has area πz. The volume is thus

$$\int_0^{10} \pi z \,\mathrm{d}z = \left[\frac{\pi z^2}{2}\right]_0^{10} = \boxed{50\pi}.$$

Problem 16

This is equivalent to the solid bounded by $z = 0$, $z = 10$, and $x^2 + y^2 = (10 - z)^2$. The integral for this equation is given by

$$\boxed{\int_0^{10} \pi(10 - z)^2 \,\mathrm{d}z} \qquad \text{or} \qquad \boxed{\int_0^{10} \int_{z-10}^{10-z} 2\sqrt{(10 - z)^2 - x^2} \,\mathrm{d}x \,\mathrm{d}z}$$

3 Section 5.4

Problem 4

Part (a): We have that

$$\begin{aligned}
\int_{-1}^{1} \int_{|y|}^{1} (x + y)^2 \,\mathrm{d}x \,\mathrm{d}y &= \int_0^1 \int_{-x}^{x} (x + y)^2 \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^1 \left[\frac{(x + y)^3}{3}\right]_{-x}^{x} \mathrm{d}x \\
&= \int_0^1 \frac{8x^3}{3} \,\mathrm{d}x \\
&= \left[\frac{2x^4}{3}\right]_0^1 \\
&= \boxed{\frac{2}{3}}.
\end{aligned}$$

Part (b): We have that

$$\begin{aligned}
\int_{-3}^{1} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 \,\mathrm{d}x \,\mathrm{d}y &= \int_{-3}^{1} \left[\frac{x^3}{3}\right]_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \mathrm{d}y \\
&= \int_{-3}^{1} \frac{2\sqrt{(9 - y^2)^3}}{3} \,\mathrm{d}y
\end{aligned}$$

which can be simlpified to quite a complex answer involving inverse trigonometric functions.

Part (c): We have that

$$\begin{aligned}
\int_0^4 \int_{\frac{y}{2}}^2 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \int_0^{2x} e^{x^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left[ye^{x^2}\right]_0^{2x}\,\mathrm{d}x \\
&= \int_0^2 2xe^{x^2}\,\mathrm{d}x \qquad = \left[e^{x^2}\right]_0^2 = \boxed{e^4 - 1}.
\end{aligned}$$

Part (d): We have that

$$\begin{aligned}
\int_0^1 \int_{\arctan(y)}^{\pi/4} \sec^5(x)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^{\pi/4} \int_0^{\tan(x)} \sec^5(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^{\pi/4} \sec^5(x)\tan(x)\,\mathrm{d}x \\
&= \left[\frac{\sec^5(x)}{5}\right]_0^{\pi/4} \\
&= \boxed{\frac{\sqrt{2}}{40}}.
\end{aligned}$$

Problem 5

We have that

$$\begin{aligned}
\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^{x^2} e^{x^3}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 x^2 e^{x^3}\,\mathrm{d}x \\
&= \left[\frac{e^{x^3}}{3}\right]_0^1 \\
&= \boxed{\frac{e-1}{3}}
\end{aligned}$$

Problem 9

Observe that the minimmum and maximum values of $\frac{1}{x^2+y^2+1}$ on D are $\frac{1}{6}$ and 1. Thus

$$1 = \iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{6} \leq \iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2+1} \leq \iint_D \mathrm{d}x\,\mathrm{d}y = 6.$$

Problem 10

Observe that the minimum and maximum values of $\frac{1}{y-x+3}$ on D are $\frac{1}{3}$ and $\frac{1}{2}$ respectively. Thus,

$$\frac{1}{6} = \iint_D \frac{\mathrm{d}A}{3} \leq \iint_D \frac{\mathrm{d}A}{y-x+3} \leq \iiint_D \frac{\mathrm{d}A}{2} \leq \frac{1}{4}.$$

Problem 11

Observe that an ellipsoid with axes a, b, and c is a unit sphere under the linear transformation

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

The volume of any figure under a linear transformation (or more generally, the Lebesgue measure of a measurable subset of $\mathbb{R}^n$) is scaled precisely by the absolute value of determininat of the transformation: the volume we seek is thus

$$\frac{4\pi}{3} \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \boxed{\frac{4\pi abc}{3}}$$

(I am indeed familiar with the Lebesgue measure from Baby Rudin.)

Problem 18

If we let an antiderivative of f be F, then

$$\begin{aligned} 2\int_a^b \int_x^b f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x &= 2\int_a^b f(x)(F(b) - F(x))\,\mathrm{d}x \\ &= F(b)2\int_a^b f(x) - 2\int_a^b f(x)F(x) \\ &= 2F(b)(F(b) - F(a)) - \Big[F(x)^2\Big]_a^b \\ &= 2F(b)^2 - 2F(b)F(a) - F(b)^2 + F(a)^2 \\ &= (F(b) - F(a))^2 \\ &= \left(\int_b^a f(x)\,\mathrm{d}x\right)^2, \end{aligned}$$

as desired.

4 Secction 5.5

Problem 3

We have that

$$\begin{aligned}\iiint_B x^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z &= \int_0^1 \int_0^1 \int_0^1 x^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \int_0^1 \int_0^1 \frac{1}{3} \, \mathrm{d}y \, \mathrm{d}z \\ &= \int_0^1 \frac{1}{3} \, \mathrm{d}z \\ &= \boxed{\frac{1}{3}}\end{aligned}$$

Problem 11

We must compute the curve where the two regions intersect to find bounds of integration. We have that if

$$x^2 + y^2 = z = 10 - x^2 - 2y^2,$$

then

$$10 = 2x^2 + 3y^2 \implies \frac{5}{3} = \left(\frac{x}{\sqrt{3}}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 .$$

Thus, the boundary we seek is an ellipse, which we will denote E. The volume we seek is thus given by the integral

$$\begin{aligned}\iint_E (10 - 2x^2 - 3y^2) \, \mathrm{d}y \, \mathrm{d}x &= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\frac{\sqrt{30-6x^2}}{3}}^{\frac{\sqrt{30-6x^2}}{3}} (10 - 2x^2 - 3y^2) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\frac{\sqrt{30-6x^2}}{3}}^{\frac{\sqrt{30-6x^2}}{3}} (10 - 2x^2 - 3y^2) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{-\sqrt{5}}^{\sqrt{5}} \Big[10y - 2x^2y - y^3\Big]_{-\frac{\sqrt{30-6x^2}}{3}}^{\frac{\sqrt{30-6x^2}}{3}} \mathrm{d}x \\ &= \int_{-\sqrt{5}}^{\sqrt{5}} \Big[y(10 - 2x^2 - y^2)\Big]_{-\frac{\sqrt{30-6x^2}}{3}}^{\frac{\sqrt{30-6x^2}}{3}} \mathrm{d}x \,.\end{aligned}$$

Observe that on the boundary, $10 - 2x^2 = 3y^2$, so this evaluates to

$$\begin{aligned}
\int_{-\sqrt{5}}^{\sqrt{5}} \left[y(3y^2 - y^2) \right]_{-\frac{\sqrt{30-6x^2}}{3}}^{\frac{\sqrt{30-6x^2}}{3}} \mathrm{d}x &= \int_{-\sqrt{5}}^{\sqrt{5}} \left[2y^3 \right]_{-\frac{\sqrt{30-6x^2}}{3}}^{\frac{\sqrt{30-6x^2}}{3}} \mathrm{d}x \\
&= \int_{-\sqrt{5}}^{\sqrt{5}} 4 \left(\frac{\sqrt{30-6x^2}}{3} \right)^3 \mathrm{d}x \\
&= \boxed{\frac{25\pi\sqrt{6}}{3}}
\end{aligned}$$

Problem 16

We have that

$$\begin{aligned}
\int_0^1 \int_0^x \int_0^y (y + xz) \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x &= \int_0^1 \int_0^x \left[yz + \frac{xz^2}{2} \right]_0^y \mathrm{d}y \,\mathrm{d}x \\
&= \int_0^1 \int_0^x \left(y^2 + \frac{xy^2}{2} \right) \mathrm{d}y \,\mathrm{d}x \\
&= \int_0^1 \left[\frac{y^3}{3} + \frac{xy^3}{6} \right]_0^x \mathrm{d}x \\
&= \int_0^1 \left(\frac{x^3}{3} + \frac{x^4}{6} \right) \mathrm{d}x \\
&= \left[\frac{x^4}{12} + \frac{x^5}{24} \right]_0^1 \\
&= \frac{1}{12} + \frac{1}{24} \\
&= \boxed{\frac{1}{8}}.
\end{aligned}$$

Problem 19

We have that

$$\begin{aligned}
\iiint_W x^2\cos(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^\pi \int_0^1 \int_0^{1-y} x^2\cos(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\pi \int_0^1 \left[\frac{x^3\cos(z)}{3}\right]_0^{1-y} \mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\pi \int_0^1 \left(\frac{(1-y)^3\cos(z)}{3}\right) \mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\pi \left[-\frac{(1-y)^4\cos(z)}{12}\right]_0^1 \mathrm{d}z \\
&= \int_0^\pi \frac{\cos(z)}{12}\,\mathrm{d}z \\
&= \left[\frac{\sin(z)}{12}\right]_0^\pi \\
&= \boxed{0}.
\end{aligned}$$

Problem 21

We have that

$$\begin{aligned}
\iiint_W (1-z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^1 \int_0^{1-z} \int_0^{1-z} (1-z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 (1-z^2)\left(\int_0^{1-z}\int_0^{1-z} 1\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}z \\
&= \int_0^1 (1-z^2)\left(\int_0^{1-z} (1-z)\,\mathrm{d}y\right)\mathrm{d}z \\
&= \int_0^1 (1-z^2)(1-z)^2\,\mathrm{d}z \\
&= \int_0^1 1-2z+2z^3-z^4\,\mathrm{d}z \\
&= \left[z-z^2+\frac{z^4}{2}-\frac{z^5}{5}\right]_0^1 \\
&= \boxed{\frac{3}{10}}.
\end{aligned}$$

Problem 28

The area of W may be represented by a triple integral as follows:

$$\boxed{\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}$$