Vakil: Some Category Theory

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1 Categories and Functors

1.1 Definition of a Category

A **cagetory** $\mathscr{C}$ is a collection of **objects** and a collection of **morphisms** between pairs of objects. Morphisms are written $f : A \to B$, where A is the **source** of f and B is the **target** of f. A category must further contain a product $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$; if $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, their composition is denoted $g \circ f \in \mathrm{Mor}(A, C)$. The following properties must also be satisfied:

1. Composition of morphisms is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.
2. Identity morphisms must exist: If $f : A \to B$, then we must have $\mathrm{id}_B \circ f = f$ and $f \circ \mathrm{id}_A = f$.

Morphisms form a monoid under composition; the identity morphisms are thus unique. The usual types of morphisms have definitions in categories, too:

1. A morphism $f : A \to B$ is an **isomorphism** if there exists a morphism $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.
2. A morphism $f : A \to A$ is an **epimorphism**.
3. An isomorphic epimorphism is an **automorphism**.

We do not impose that the classes of objects and morphisms form sets — for instance, consider the cateogry **Set** of all sets under set mappings. Its objects do not form a set, or else both Cantor's and Russel's paradoxes arise. In general, we will avoid foundational issues by neglecting to perform set theory on classes.

1.2 Examples

Let A be an object in a category $\mathscr{C}$. The **automorphism group** $\mathrm{Aut}(A)$ of A is the group of all invertible elements of $f \in \mathrm{Mor}(A, A)$. If two objects A and B are isomorphic, then there exists a group isomorphism $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

Two particularly critical examples of categories are **Ab** and $\mathbf{Mod}_R$ — the categories of Abelian groups and modules over a (commutative) ring R. These two categories constitute the chief examples of an Abelian category. It is clear that $\mathbf{Mod}_k = \mathbf{Vec}_k$ (the category of vector spaces) when k is a field, and $\mathbf{Mod}_{\mathbb{Z}} = \mathbf{Ab}$.

Familiar categories are **Grp** and **Ring**, groups and rings with homomorphisms, and **Top**, topological spaces with continuous maps.

Recall that a partially ordered set $(S, \geq)$ is a set S endowed with a relation $\geq$ such that

1. **Reflexivity**: $x \geq x$ for all $x \in S$.
2. **Antisymmetry**: If $x \geq y$ and $y \geq x$, then $x = y$.
3. **Transitivity**: If $x \geq y$ and $y \geq z$, then $x \geq z$.

We can describe S as a category — one where objects are the elements of S and where a single morphism $f : x \to y$ exists if and only if $y \geq x$. The reflexivity of $\geq$ ensures the existence of identity mappings, while transitivity ensures composition of morphisms is associative. As an example, here is a partially ordered set with four elements:

$$\begin{tikzcd}
x \arrow[loop above] & y \arrow[l] \arrow[loop right] \\
w \arrow[u] \arrow[loop left] & z \arrow[l] \arrow[u] \arrow[ul] \arrow[loop below]
\end{tikzcd}$$

in which $x \geq y \geq z$ and $x \geq w \geq z$. A totally ordered set would look like this:

$$\begin{tikzcd}
\cdots \arrow[r] & z \arrow[r] \arrow[loop above] \arrow[rr, bend right] & y \arrow[r] \arrow[loop above] & x \arrow[r] \arrow[loop above] & \cdots
\end{tikzcd}$$

in which $x \geq y \geq z$. If X is a set, then the subsets of X are partially ordered by inclusion — thus such a category may be constructed. Most notably, one can construct a category modeling the open sets of a topological space by inclusion.

A **subcategory** $\mathscr{A}$ of a category $\mathscr{B}$ is a category whose objects lie in $\mathscr{B}$ and whose morphisms include the identities of its objects and are closed under composition. For instance, the category $A \to B$ (with identity morphisms ommitted) is a subcategory of $A \to B \to C$. There is also the **inclusion functor** defined as the embedding $i : \mathscr{A} \to \mathscr{B}$.

1.3 Functors

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **covariant functor** $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is a map $\mathcal{F} : \mathrm{obj}(A) \to \mathrm{obj}(B)$ that "preserves morphisms" in the following sense: for all morphisms $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_3$,

1. There exists a morphism $\mathcal{F}(f_1) : \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathscr{B}$.
2. $\mathcal{F}$ maps identities in $\mathscr{A}$ to identities: $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$.
3. $\mathcal{F}$ preserves morphism structure: $\mathcal{F}(f_2 \circ f_1) = \mathcal{F}(f_2) \circ \mathcal{F}(f_1)$.

An obvious example of a functor is the **identity functor** $\mathrm{id} : \mathscr{A} \to \mathscr{A}$. Some more examples:

1. A **forgetful functor** is a functor that loses additional structure of the source category. One example is the functor $\mathbf{Vec}_k \to \mathbf{Set}$ that maps each vector space to its underlying set; another is the mapping $\mathbf{Mod}_R \to \mathbf{Ab}$ by isolating each module's additive structure.

2. Let X be a topological space, and select $x_0 \in X$ arbitrarily. The fundamental group functor ϕ_1 maps the topological space X to the group $\phi_1(X, x_0)$, and the i-th homology functor $\mathbf{Top} \to \mathbf{Ab}$ sends all topological spaces X to their i-th homology group $H_i(X, \mathbb{Z})$.

3. Suppose A is an object in a category $\mathscr{C}$. There is a functor $\mathrm{h}^A : \mathcal{C} \to \mathbf{Set}$ that sends each object $B \in \mathscr{C}$ to $\mathrm{Mor}(A, B)$ and each morphism $f : B_1 \to B_2$ to $\mathrm{Mor}(A, B_1) \to \mathrm{Mor}(A, B_2)$ described by

$$[g : A \to B_1] \quad \mapsto \quad [f \circ g : A \to B_1 \to B_2].$$

   This little guy ends up becoming surprisingly important.

The **composition** of functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ and $\mathcal{G} : \mathscr{B} \to \mathscr{C}$ is denoted $\mathcal{G} \circ \mathcal{F} : \mathscr{A} \to \mathscr{C}$, and is defined in the obvious way. Composition of functors is associative in an evident sense. The following terms describe the extent to which a covariant functor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ respects the morphisms of $\mathscr{B}$:

1. $\mathcal{F}$ is **faithful** if the mapping $\mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \to \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is injective.

2. $\mathcal{F}$ is **full** if the mapping $\mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \to \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is surjective.

3. $\mathcal{F}$ is **fully faithful** if the mapping $\mathrm{Mor}_{\mathscr{A}}(A_1, A_2) \to \mathrm{Mor}_{\mathscr{B}}(\mathcal{F}(A_1), \mathcal{F}(A_2))$ is bijective.

Let $i : \mathscr{A} \to \mathscr{B}$ be a subcategory. Since inclusions are always faithful, we need not use the phrase "faithful subcategory". $\mathscr{A}$ is a **full subcategory** if i is full. For instance, the category of finitely-generated R modules is a full subcategory of $\mathbf{Mod}_R$; the mapping $\mathbf{Vec}_k \to \mathbf{Set}$ is faithful but not full.

A **contravariant functor** $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ is defined equivalently to covariant functors, except morphisms are reversed; a morphism $f : A_1 \to A_2$ induces a morphism $\mathcal{F}(f) : \mathcal{F}(A_2) \to \mathcal{F}(A_1)$. The composition law is also reversed: $\mathcal{F}(f_2 \circ f_1) = \mathcal{F}(f_1) \circ \mathcal{F}(f_2)$.

For a category $\mathscr{A}$, the **opposite category** $\mathscr{A}^{\mathrm{opp}}$ of A is the category formed by reversing the arrows of $\mathscr{A}$. Hence contravariant functors $\mathcal{F} : \mathscr{A} to \mathscr{B}$ are equivalent to covariant functors $\mathcal{F} : \mathscr{A}^{\mathrm{opp}} \to \mathscr{B}$.

For examples: there is a contravariant functor **Top** $\to$ **Ring** mapping a topological space X to the ring of real-valued functions on X. A morphism of topological spaces $X \to Y$ induces the pullback map from functions on Y to functions on X. In fact the i-th cohomology functor $\mathcal{H}_i(\cdot, \mathbb{Z}) : \mathbf{Top} \to \mathbf{Ab}$ is a contravariant functor.

In the category $\mathbf{Vec}_k$, taking duals gives the contravariant functor $(\cdot)^\vee : \mathbf{Vec}_k \to \mathbf{Vec}_k$. It induces upon each linear transformation $\mathbf{T} : V \to W$ a dual transformation $\mathbf{T}^\vee : W^\vee \to V^\vee$, with $(g \circ f)^\vee = f^\vee \circ g^\vee$.

A critical example is the **functor of points**: suppose A is an object of $\mathscr{C}$. There is a contravariant functor $h_A : \mathscr{C} \to \mathbf{Set}$ sending each $B \in \mathscr{C}$ to $\mathrm{Mor}(B, A)$, and sending the morpshim $f : B_1 \to B_2$ to the morphism $\mathrm{Mor}(B_2, A) \to \mathrm{Mor}(B_1, A)$ via

$$[g : B_2 \to A] \quad \mapsto \quad [g \circ f : B_1 \to B_2 \to A].$$

This looks quite weird, but the examples from linear algebra and the functor **Top** $\to$ **Ring** are merely special cases of the functor of points. A more natural name would be the "functor of maps", but alas — it is too late to change it.