MATH-UA 140: Assignment 8

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1 Problem 1

Part (a): Suppose for contradiction that two lists of reals $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ — not all equal — satisfy

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n.$$

We may simplfy this equation to get that

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0}.$$

As $\mathbf{v}_1, \dots, \mathbf{v}_n$ are independent, the only solution to this equation is when

$$\begin{aligned} \alpha_1 - \beta_1 &= 0 \\ &\vdots \\ \alpha_n - \beta_n &= 0. \end{aligned}$$

Then $\alpha_j = \beta_j$ for all $j \in \{1, \dots, n\}$, which contradicts our definition of $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ as distinct lists. We conclude that the choice of $\alpha_1, \dots, \alpha_n$ is unique.

Part (b): Such a matrix J maps the canonical basis of $\mathbb{R}^n$ to $\mathbf{v}_1, \dots, \mathbf{v}_n$ — so each $\mathbf{v}_1, \dots, \mathbf{v}_n$ lie in the range of J. This list spans $\mathbb{R}^n$, so range $J = \mathbb{R}^n$. Then as $\dim \operatorname{range} J = n$, the operator J is surjective. Hence,

$$n = \dim \mathbb{R}^n = \dim \operatorname{range} J + \dim \operatorname{null} J = n + \dim \operatorname{null} J$$

so $\dim \operatorname{null} J = 0$. Then J is injective, and is thus invertible.

Part (c): Let v_{ij} be the i-th entry of the j-th vector — so $\mathbf{v}_j = (v_{1j}, v_{2j}, \dots, v_{nj})$. Then the following matrix J maps $\mathbf{e}_j$ to $\mathbf{v}_j$ for all $j \in \{1, \dots, n\}$:

$$J = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix}.$$

By expanding the j-th column in the matrix resulting from the product

$$\begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix},$$

we find that the sum $\gamma_{1j} v_{r1} + \gamma_{2j} v_{r2} + \cdots + \gamma_{nj} v_{rn}$ for row $r \in \{1, \dots, n\}$ is one when $r = j$

and zero when $r \neq j$. Then for all $j \in \{1, \ldots, n\}$,

$$\begin{aligned}
\gamma_{1j}\mathbf{v}_1 + \gamma_{2j}\mathbf{v}_2 + \cdots + \gamma_{nj}\mathbf{v}_n &= \gamma_{1j}\begin{bmatrix} v_{11} \\ \vdots \\ v_{n1} \end{bmatrix} + \cdots + \gamma_{nj}\begin{bmatrix} v_{1n} \\ \vdots \\ v_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \gamma_{1j}v_{11} + \cdots + \gamma_{nj}v_{1n} \\ \vdots \\ \gamma_{1j}v_{n1} + \cdots + \gamma_{n}v_{nn} \end{bmatrix} \\
&= \mathbf{e}_j,
\end{aligned}$$

as desired.

2 Problem 2

Part (a): We have that

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \boxed{\lambda^2 - (a+d)\lambda + ad - bc}.$$

Part (b): The number of real roots depends on the sign of the discriminant $\beta^2 - 4\alpha\gamma$:

- **No Real Zeros**: If $\beta^2 - 4\alpha\gamma < 0$
- **One Real Zero**: If $\beta^2 - 4\alpha\gamma = 0$.
- **Two Real Zeroes**: If $\beta^2 - 4\alpha\gamma > 0$.

Part (c): The discriminant of our polynomial in Part (a) is

$$(-a-d)^2 - 4(ad-bc)(1) = a^2 + 2ad + d^2 - 4ad + 4bc = (a-d)^2 + 4bc.$$

The number of **real** eigenvalues thus depends on the sign of this quantity:

- **No Real Eigenvalues**: If $(a-d)^2 + 4bc < 0$.
- **One Real Eigenvalue**: If $(a-d)^2 + 4bc = 0$.
- **Two Real Eigenvalues**: If $(a-d)^2 + 4bc > 0$

Part (d): The matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has no real eigenvalues; it is a rotation matrix by 90°, so it displaces every nonzero vector off its span. The matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has one real eigenvalue — that being 1, held by every nonzero vector — and the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

has two real eigenvalues — them being 1 and 2, held by the vectors $(0, 1)$ and $(1, 0)$.

3 Problem 3

Part (a): We have that for all eigenvalues λ,

$$0 = \begin{vmatrix} 1-\lambda & \frac{1}{4} \\ \frac{1}{4} & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \frac{1}{16}$$

so $\boxed{\lambda = \frac{3}{4}, \frac{5}{4}}$. Two eigenvectors with these eigenvalues are

$$\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

respectively, as verified by a trivial computation.

Part (b): We have that for all eigenvalues λ,

$$0 = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 0 = (1-\lambda)(2-\lambda)$$

so $\boxed{\lambda = 1, 2}$. Two eigenvectors with these eigenvalues are

$$\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

respectively, as verified by a trivial computation.

Part (c): We have that for all eigenvalues λ,

$$0 = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda - 4),$$

so $\boxed{\lambda = 0, 4}$. Two eigenvectors with these eigenvalues are

$$\boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

respectively, as verified by a trivial calculation.

4 Problem 4

Part (a): By the basic properties of limits,

$$\lim_{\lambda\to\infty} -\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0 = -\infty \qquad \text{and} \qquad \lim_{\lambda\to-\infty} -\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0 = \infty.$$

Therefore, we should *expect* the Intermediate Value Theorem to guarantee that this polynomial always achieves at least one real zero — or for 3-by-3 matricies to have exactly one real eigenvalue.

Part (b): Consider the 3-by-3 identity matrix: for all its eigenvalues,

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3,$$

so $\lambda = 1$ is the only eigenvalue; every nonzero vector is an eigenvector with this eigenvalue. Thus, one such matrix is

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$