Axler: Vector Spaces

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Contents

1 Vector Spaces

An **vector space** over a field F is an Abelian group V (with operation written additively) endowed with a mapping $\mu : F \times V \to V$ (written multiplicatively) such that the following axioms are satisfied for all $\mathbf{v}, \mathbf{w} \in V$ and $a, b \in R$:

1. $1\mathbf{v} = \mathbf{v}$;
2. $(ab)\mathbf{v} = a(b\mathbf{v})$;
3. $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$;
4. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Elements of V are called **vectors**. Since $(V, +)$ is an Abelian group, it has a unique additive identity, unique inverses, and satisfies $-(-\mathbf{v}) = \mathbf{v}$ and $-(\mathbf{v}+\mathbf{w}) = -\mathbf{v}-\mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in V$. The additive identity of V is denoted $\mathbf{0}$ and the additive inverse of $\mathbf{v}$ is denoted $-\mathbf{v}$.

Theorem 1. *Let V be a F-vector space. Then the following holds for all $\mathbf{v}, \mathbf{w} \in V$ and $a \in F$:*

1. $0\mathbf{v} = \mathbf{0}$.

2. $a\mathbf{0} = \mathbf{0}$.

3. $(-1)\mathbf{v} = -\mathbf{v}$.

Proof. All three properties follow from the distributive laws. For (1), we have that

$$0\mathbf{v} + 0\mathbf{v} \;=\; (0 + 0)\mathbf{v} = 0\mathbf{v}.$$

Subtracting both sides by $0\mathbf{v}$ yields that $0\mathbf{v} = \mathbf{0}$. For (2), a similar proof holds:

$$a\mathbf{0} + a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0},$$

hence $a\mathbf{0} = \mathbf{0}$. The third property is quite easy as well: we have that

$$-1\mathbf{v} + \mathbf{v} \;=\; (-1 + 1)\mathbf{v} \;=\; 0\mathbf{v} \;=\; \mathbf{0} \;=\; 0\mathbf{v} \;=\; (1 - 1)\mathbf{v} \;=\; \mathbf{v} + (-1\mathbf{v})$$

Hence $-1\mathbf{v}$ is the unique inverse of $\mathbf{v}$, that being $-\mathbf{v}$. □

A vector space over $\mathbb{R}$ is a **real vector space**, while vector spaces over $\mathbb{C}$ are **complex vector spaces**.

2 Subspaces

A subset $U \subseteq V$ is a **subspace** if it is a vector space under the field and operations of V.

Theorem 2. *A subset $U \subseteq V$ is a subspace if and only if $\mathbf{0} \in U$ and U is closed under addition and scalar multiplication.*

Proof. Suppose U satisfies the three desired properties. Then $(U, +) \subseteq (V, +)$ is an Abelian subgroup; once multiplicative closure is ensured, the four other properties are inherited from V. $\square$

Let V be an F-vector space with subspaces $V_1, \ldots, V_m$. We consider two crucial operations on these subspaces:

1. **Sum**: The sum $V_1 + \cdots + V_n$ is the set of all sums $m_1 + \cdots + m_n$, where $m_i \in V_i$ $(i \in \{1, \ldots, n\})$. It is the smallest subspace of V that contains all $V_1, \ldots, V_n$.
2. **Intersection**: The intersection $V_1 \cap \cdots \cap V_n$ is the largest subspace of V that is contained inside each $V_1, \ldots, V_n$.

Let $V_1, \ldots, V_n$ be F-vector spaces. The **direct sum** $V_1 \oplus \cdots \oplus V_n$ is the set of all formal pairs $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, with addition and scalar multiplication defined componentwise.

Theorem 3. *Let $V_1, \ldots, V_n \subseteq V$ be F-subspaces. Then the following holds:*

1. *$V_1 + \cdots + V_n \cong V_1 \oplus \cdots \oplus V_n$ if and only if $\mathbf{v}_1 + \cdots + \mathbf{v}_n = \mathbf{0}$ for $\mathbf{v}_i \in V_i$ implies that $\mathbf{v}_1 = \cdots = \mathbf{v}_n = \mathbf{0}$.*
2. *$V_1 + \cdots + V_n \cong V_1 \oplus \cdots \oplus V_n$ if and only if each vector in the former decomposes as a sum of $\mathbf{v}_i \in V_i$ uniquely.*

Proof. Suppose that $\mathbf{v}_1 + \cdots + \mathbf{v}_n = \mathbf{0}$ for $\mathbf{v}_i \in V_i$ implies that $\mathbf{v}_1 = \cdots = \mathbf{v}_n = \mathbf{0}$. Then define a linear map

$$V_1 \oplus \cdots \oplus V_n \mapsto V_1 + \cdots + V_n \quad \text{by} \quad \mathbf{v}_1, \ldots, \mathbf{v}_n \rightsquigarrow \mathbf{v}_1 + \cdots + \mathbf{v}_n.$$

By definition, this map is surjective; it is injective by our hypothesis. Hence the two vector spaces are isomorphic. The converse is easy to deduce — while result (2) is a mere corollary of the first. $\square$