Rudin: Numerical Sequences and Series

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December 2023

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1 Convergent Sequences

1.1 Defintion

We say that the sequence a_n in a metric space X **converges** to a point $A \in X$ if for all $\epsilon > 0$, there exists an integer N such that

$$N \leq n \implies d(a_n, A) < \epsilon.$$

If a_n converges to A, we write that $\lim_{n \to \infty} a_n = A$. If a_n fails to converge, we say it **diverges**.

Theorem 1. *The limit is unqiue.*

Proof. Suppose for contradiction that $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} a_n = B$ such that $A \neq B$. Then $d(A, B) > 0$, so there exist N_1, N_2 such that

$$\begin{aligned} N_1 \leq n &\implies d(a_n, A) < \tfrac{d(A,B)}{2} \\ N_2 \leq n &\implies d(a_n, B) < \tfrac{d(A,B)}{2}. \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then $N \leq n$ implies that

$$d(a_n, A) + d(a_n, B) < \tfrac{d(A,B)}{2} + \tfrac{d(A,B)}{2} = d(A, B).$$

This violates the Triangle Inequality, implying that $A = B$. $\square$

Theorem 2. $\lim_{n \to \infty} a_n = A$ *if and only if every neighborhood of* A *contains* a_n *for all but finitely many* n.

Proof. Suppose $\lim_{n \to \infty} a_n = A$. An arbitrary neighborhood $\mathcal{N}$ of A must contain an open ball centered at A; let its radius be r. Then there exists N such that

$$N \leq n \implies d(a_n, A) < r.$$

Thus, a_n for $N \leq n$ lies inside $\mathcal{N}$; only finitely many a_n from $n \in \{1, \ldots, n-1\}$ may lie outside $\mathcal{N}$.

Conversely, suppose every neighborhood of A contains a_n for all but finitely many n. Then define $\mathcal{N}_r$ as the open ball with radius ϵ, and $N_r = \max\{n \mid a_n \notin \mathcal{N}_r\} + 1$. Then

$$N_r \geq n \implies d(a_n, A) < \epsilon,$$

so $\lim_{n \to \infty} a_n = A$. $\square$

Theorem 3. *If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.*

Proof. Suppose $\lim_{n\to\infty} a_{n=A}$. Then there exists N such that

$$N \le n \implies d(a_n, A) < 1.$$

Thus, the maximum distance from a_n to A is less than or equal to

$$M = \max\{d(a_1, A), d(a_2, A), \dots, d(a_{N-1}, A), 1\}.$$

The open ball at A with radius $M + 1$ thus bounds a_n. $\square$

Theorem 4. *If $E \subseteq X$ and if A is a limit point of E, then there exists a sequence a_n such that $\lim_{n\to\infty} a_n = A$.*

Proof. For a positive integer n, let $\mathcal{N}_n$ be the open ball at A with radius $\frac{1}{m}$. Because A is a limit point, there exist $a_n \in X$ inside $\mathcal{N}_n$ for all integers n. Then for all $\epsilon > 0$,

$$\begin{aligned} \lfloor \tfrac{1}{\epsilon} \rfloor + 1 \le n \implies d(a_n, A) &< \frac{1}{\lfloor \frac{1}{\epsilon} \rfloor + 1} \\ &< \frac{1}{\left(\frac{1}{\epsilon}\right)} \\ &= \epsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} a_n = A$. I've condensed this quite a lot. $\square$

1.2 Normed Vector Spaces

A **normed vector space** is a vector A **normed vector space** is a vector space X over $\mathbb{C}$ equipped with a mapping $\|\cdot\| : X \to \mathbb{R}$ that satisfies the following properties:

1. **Positivity**: $\|\mathbf{x}\| \ge 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$.
2. **Homogenity**: $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ for all $\lambda \in \mathbb{C}$.
3. **Triangle Inequality**: $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$

It is clear that such a norm induces a metric on X. Unless otherwise stated, these theorems assume $\mathbf{x}_n$ and $\mathbf{y}_n$ are sequences in a normed vector space X.

Theorem 5. $\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{X} + \mathbf{Y}$.

Proof. For all $\epsilon > 0$, there exist integers N_1 and N_2 such that

$$\begin{aligned} N_1 \le n &\implies \|\mathbf{x}_n - \mathbf{X}\| < \tfrac{\epsilon}{2} \\ N_2 \le n &\implies \|\mathbf{y}_n - \mathbf{Y}\| < \tfrac{\epsilon}{2} \end{aligned}$$

Define $N = \max\{N_1, N_2\}$. Then for all $\epsilon > 0$, $N \le n$ implies that

$$\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{X} + \mathbf{Y})\| \le \|\mathbf{x}_n - \mathbf{X}\| + \|\mathbf{y}_n - \mathbf{Y}\| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon,$$

as desired. □

Theorem 6. *If* $\lambda \in \mathbb{C}$*, then* $\lim_{n\to\infty}\lambda(\mathbf{x}_n) = \lambda(\mathbf{X})$.

Proof. If $c = 0$, then $\lim_{n\to\infty} 0(\mathbf{x}_n) = \mathbf{0} = 0(\mathbf{X})$. Otherwise, for all $\epsilon > 0$, there exists an integer N such that

$$N \le n \implies \|\mathbf{x}_n - \mathbf{X}\| < \tfrac{\epsilon}{|c|}.$$

Then $N \le n$ implies that

$$\|c(\mathbf{x}_n) - c(\mathbf{X})\| = |c|\|\mathbf{x}_n - \mathbf{X}\| < |c|\tfrac{\epsilon}{|c|} = \epsilon,$$

as required. □

1.3 Inner Product Spaces

In the following theorems, suppose that X is an **inner product space** (see LinearAlgebra/axler6.tex). For the following theorems, $\mathbf{x}_n$ and $\mathbf{y}_n$ be sequences of vectors in X.

Theorem 7. *If* $\mathbf{c} \in X$*, then* $\lim_{n\to\infty} \mathbf{c} \cdot \mathbf{x}_n = \mathbf{c} \cdot \mathbf{X}$ *and* $\lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{c} = \mathbf{X} \cdot \mathbf{c}$ *.*

Proof. If $\mathbf{c} = \mathbf{0}$, then $\lim_{n\to\infty} \mathbf{0} \cdot (\mathbf{x}_n) = 0 = \mathbf{0} \cdot (\mathbf{X})$. Otherwise, for all $\epsilon > 0$, there exists an integer N such that

$$N \le n \implies \|\mathbf{x}_n - \mathbf{X}\| < \tfrac{\epsilon}{\|\mathbf{c}\|}.$$

Then $N \le n$ implies that

$$|(\mathbf{c} \cdot \mathbf{x}_n) - (\mathbf{c} \cdot \mathbf{X})| = |\mathbf{c} \cdot (\mathbf{x}_n - \mathbf{X})| \le \|\mathbf{c}\|\|\mathbf{x}_n - \mathbf{X}\| < \|\mathbf{c}\|\tfrac{\epsilon}{\|\mathbf{c}\|} = \epsilon.$$

Similarly, $|(\mathbf{x}_n \cdot \mathbf{c}) - (\mathbf{X} \cdot \mathbf{c})| \le \|\mathbf{x}_n - \mathbf{X}\|\|\mathbf{c}\|$, and the proof follows like above. □

Theorem 8. $\lim_{n\to\infty}(\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{X}\cdot\mathbf{Y}$.

Proof. For all $\epsilon > 0$, there exist integers N_1, N_2 such that

$$N_1 \le n \implies |\mathbf{x}_n - \mathbf{X}| < \sqrt{\epsilon}$$
$$N_2 \le n \implies |\mathbf{y}_n - \mathbf{Y}| < \sqrt{\epsilon}.$$

Define $N = \max\{N_1, N_2\}$. Then $N \le n$ implies through Cauchy-Schwarz that

$$|(\mathbf{x}_n - \mathbf{X})\cdot(\mathbf{y}_n - \mathbf{Y})| \le \|\mathbf{x}_n - \mathbf{X}\|\|\mathbf{y}_n - \mathbf{Y}\| < \sqrt{\epsilon}^2 < \epsilon,$$

so $\lim_{n\to\infty}(\mathbf{x}_n - \mathbf{X})\cdot(\mathbf{y}_n - \mathbf{Y}) = 0$. Therefore,

$$\begin{aligned}
0 &= \lim_{n\to\infty}(\mathbf{x}_n - \mathbf{X})\cdot(\mathbf{y}_n - \mathbf{Y}) \\
&= \lim_{n\to\infty}(\mathbf{x}_n\cdot\mathbf{y}_n - \mathbf{X}_n\cdot\mathbf{Y} - \mathbf{X}\cdot\mathbf{y}_n + \mathbf{X}\cdot\mathbf{Y}) \\
&= \lim_{n\to\infty}(\mathbf{x}_n\cdot\mathbf{y}_n) - \lim_{n\to\infty}(\mathbf{x}_n\cdot\mathbf{Y}) - \lim_{n\to\infty}(\mathbf{X}\cdot\mathbf{y}_n) + \lim_{n\to\infty}(\mathbf{X}\cdot\mathbf{Y}) \\
&= \lim_{n\to\infty}(\mathbf{x}_n\mathbf{y}_n) - (\mathbf{X}\cdot\mathbf{Y}) - (\mathbf{X}\cdot\mathbf{Y}) + (\mathbf{X}\cdot\mathbf{Y}) \\
&= \lim_{n\to\infty}(\mathbf{x}_n\cdot\mathbf{y}_n) - (\mathbf{X}\cdot\mathbf{Y}).
\end{aligned}$$

Rearranging this equation yields $\lim_{n\to\infty}(\mathbf{x}_n\cdot\mathbf{y}_n) = \mathbf{X}\cdot\mathbf{Y}$. □

1.4 Complex Vectors and Complex Numbers

We now turn our attention to the inner product spaces $\mathbb{C}^k$. Suppose that $\mathbf{z}_n$ is a sequence in $\mathbb{C}^k$ with coordinates $\mathbf{z}_n = (z_{n1}, \ldots, z_{nk})$ and suppose $\mathbf{Z} = (Z_1, \ldots, Z_k)$

Theorem 9. $\lim_{n\to\infty}\mathbf{z}_n = \mathbf{Z}$ *if and only if* $\lim_{n\to\infty} z_{nj} = Z_j$ *for all* $j \in \{1, \ldots, k\}$.

Proof. Suppose that $\lim_{n\to\infty} z_{nj} = Z_j$ for all $j \in \{1, \ldots, k\}$. Then for all $\epsilon > 0$, there exist integers $N_1, \ldots, N_k$ such that

$$N_1 \le n \implies |z_{n1} - Z_1| < \tfrac{\epsilon}{\sqrt{k}}$$
$$\vdots$$
$$N_k \le n \implies |z_{nk} - Z_k| < \tfrac{\epsilon}{\sqrt{k}}$$

Define $N = \max\{N_1, \ldots, N_k\}$. Then $N \le n$ implies

$$\begin{aligned}
\|\mathbf{z}_n - \mathbf{Z}\| &= \sqrt{|z_{n1} - Z_1|^2 + \cdots + |z_{nk} - Z_k|^2} \\
&< \sqrt{\tfrac{\epsilon^2}{k} + \cdots + \tfrac{\epsilon^2}{k}} \\
&= \sqrt{\epsilon^2} \\
&= \epsilon,
\end{aligned}$$

so $\lim_{n\to\infty} \mathbf{z}_n = \mathbf{Z}$. Now, suppose that $\lim_{n\to\infty} \mathbf{z}_n = \mathbf{Z}$. Then for all $\epsilon > 0$, there exists an integer N such that

$$N \leq n \implies \|\mathbf{z}_n - \mathbf{Z}\| < \epsilon.$$

Then $N \leq n$ implies for each $j \in \{1, \dots, k\}$ that

$$\begin{aligned} |z_{nj} - Z_j| &= \sqrt{|z_{nj} - Z_j|^2} \\ &\leq \sqrt{|z_{n1} - Z_1|^2 + \cdots + |z_{nk} - Z_k|^2} \\ &= \|\mathbf{z}_n - \mathbf{Z}\| \\ &< \epsilon, \end{aligned}$$

so $\lim_{n\to\infty} z_{nj} = Z_j$ for all $j \in \{a, k\}$. This completes the proof. □

Now, we examine $\mathbb{C}$. Suppose that $\{z_n\}$ and $\{w_n\}$ are sequences in $\mathbb{C}$, that $\lim_{n\to\infty} z_n = Z$, and that $\lim_{n\to\infty} w_n = W$. The above results imply that:

- $\lim_{n\to\infty} (z_n + w_n) = Z + W$.
- $\lim_{n\to\infty} c(z_n) = c(Z)$ for all $c \in \mathbb{C}$.
- $\lim_{n\to\infty} z_n w_n = ZW$.

Theorem 10. *If $Z \neq 0$, then $\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{Z}$.*

Proof. For all $\epsilon > 0$, there exist integers N_1, N_2, and N_3 such that

$$\begin{aligned} N_1 \leq n &\implies |z_n - Z| < \epsilon\left(\tfrac{|Z|^2}{2}\right) \\ N_2 \leq n &\implies |z_n - Z| < \tfrac{|Z|}{2}. \end{aligned}$$

Realize that for $N_2 < N$,

$$0 = |z_n - Z| - |z_n - Z| < \tfrac{|Z|}{2} - |z_n - Z| < |Z| - |z_n - Z| \leq |z_n|$$

so $\frac{1}{z_n}$ is defined. We also find that $N_2 \leq n$ implies that $\frac{|Z|}{2} < |z_n|$. Then defining $N = \max\{N_1, N_2\}$, we have that $N \leq n$ implies that

$$\left|\frac{1}{z_n} - \frac{1}{Z}\right| = \frac{|z_n - Z|}{|z_n||Z|} < \frac{\epsilon\left(\frac{|Z|^2}{2}\right)}{\left(\frac{|Z|}{2}\right)|Z|} = \epsilon,$$

as required. □

Theorem 11. *If $W \neq 0$, then* $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{Z}{W}$.

Proof. Realize that

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \lim_{n\to\infty} = \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} \frac{1}{w_n}\right) = Z\left(\frac{1}{W}\right) = \frac{Z}{W},$$

as desired. $\square$

2 Some Special Sequences

Theorem 12. *If $p > 0$, then* $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Proof. For all $\epsilon > 0$, realize that

$$\begin{aligned}\left(\tfrac{1}{\epsilon}\right)^{\frac{1}{p}} < n \implies \left|\frac{1}{n^p}\right| &< \left|\frac{1}{\left(\left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}\right)^p}\right| \\ &= \epsilon,\end{aligned}$$

as desired. $\square$

Theorem 13. *If $p > 0$, then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

Proof. If $p > 1$: For all $\epsilon > 0$, we have that $\log_{\epsilon+1}(p) < n$ implies $\frac{1}{\log_{\epsilon+1}(p)} > \frac{1}{n}$, so

$$\begin{aligned}\log_{\epsilon+1}(p) < n \implies |\sqrt[n]{p} - 1| &= p^{\frac{1}{n}} - 1 \\ &< p^{\frac{1}{\log_{\epsilon+1}(p)}} - 1 \\ &= p^{\log_p(\epsilon+1)} - 1 \\ &= (\epsilon + 1) - 1 \\ &= \epsilon.\end{aligned}$$

If $p < 1$: then $\frac{1}{p} > 1$, so

$$\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{p}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}}} = \frac{1}{1} = 1.$$

The case $p = 1$ is trivial. $\square$

Theorem 14. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. See that $\sqrt[n]{n} - 1 > 0$ for $n > 1$. Then by the Binomial Theorem,

$$n = (1 + (\sqrt[n]{n} - 1))^n \geq \frac{n(n-1)}{2}(\sqrt[n]{n} - 1)^2.$$

Therefore,

$$\sqrt{\frac{2}{n-1}} \geq \sqrt[n]{n} - 1 \geq 0.$$

We now apply the Squeeze Theorem, which shall be proven at another time:

$$\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} \geq \lim_{n\to\infty} \sqrt[n]{n} - 1 \geq 0.$$

The left-hand side of this equation equals 0 by Theorem 12; then $\lim\limits_{n\to\infty}(\sqrt[n]{n} - 1) = 0$, and we attain the desired $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\square$

Theorem 15. *If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \frac{n^\alpha}{(p+1)^n} = 0$.*

Proof. Let $k = \max\{\lfloor \alpha \rfloor + 1, 1\}$ so that $k > \alpha$. Then $n > 2k$ implies

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence,

$$\frac{2^k k!}{p^k} n^{\alpha - k} > \frac{n^\alpha}{(1+p)^n} > 0.$$

Then by the Squeeze Theorem,

$$\frac{2^k k!}{p^k} \lim_{n\to\infty} n^{\alpha-k} \geq \lim_{n\to\infty} \frac{n^\alpha}{(p+1)^n} \geq 0.$$

As $\alpha - k < 0$, the left-hand side of this equation equals 0 by Theorem 12. We thus attain the desired $\lim\limits_{n\to\infty} \frac{n^\alpha}{(p+1)^n} = 0$. $\square$

Theorem 16. *If $|x| < 1$, then $\lim\limits_{n\to\infty} x^n = 0$.*

Proof. If we let $p = \frac{1}{|x|} - 1$ and $\alpha = 0$ in Theorem 15, we find that

$$0 = \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = \lim_{n\to\infty} \frac{1}{(1 + (\frac{1}{|x|} - 1))^n} = \lim_{n\to\infty} |x|^n.$$

We can remove the absolute value with no issues, as the limit evaluates to 0. $\square$

3 Subsequences

Let n_i be a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$. Then if a_n is a sequence, a_{n_i} is a **subsequence** of a_n. If a_{n_i} converges, its limit is called a **subsequential limit** of $\{a_n\}$.

Theorem 17. *a_n converges to A if and only if all subsequences of a_n converge to A.*

Proof. Suppose that a_n converges to A. Then for all $\epsilon > 0$, there exists an integer N such that

$$N \leq n \implies |a_n - A| < \epsilon.$$

Let a_{n_i} be a subsequence of a_n. For each $\epsilon > 0$ and N, define $I = \min\{i \mid N \leq n_i\}$ Then $I \leq i$ implies $N \leq n_i$, so

$$I \leq i \implies |a_{n_i} - A| < \epsilon.$$

We conclude that $\lim_{i\to\infty} a_{n_i} = A$. If all subsequences of a_n converge to A, then $\lim_{n\to\infty} a_n = A$ since a_n is a subsequence of itself. □

Theorem 18 (Bolzano-Weierstrauss)**.** *The following two results hold:*

1. *If $\{a_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point X.*
2. *Every bounded sequence in $\mathbb{C}^n$ contains a convergent subsequence.*

Proof. For (1), there are two cases. If $\{a_n\}$ is finite, then there is an infinite subsequence of $\{a_n\}$ that are all equal: a sequence $n_1 < n_2 < \cdots$ and and a point $A \in E$ such that

$$a_{n_1} = a_{n_2} = \cdots = A.$$

The subsequence $\{a_{n_i}\}$ converges to A. If $\{a_n\}$ is infinite, then the compactness of X implies that there exists a limit point $A \in X$ of $\{a_n\}$.

If we select n_1 such that $d(a_{n_1}, A) < 1$, we can construct a sequence $n_1 < n_2 < \cdots$ such that $d(a_{n_i}, A) < \frac{1}{i}$ for $i \in \mathbb{Z}_{>0}$. The convergence of this sequence to A is a straightforwards calculation. We deduce that (1) holds.

For (2), realize that each bounded sequence in $\mathbb{C}^n$ lies in a compact subset of $\mathbb{C}^n$ (an n-pseudocell!), from which (1) implies the existence of a convergent subsequence. □

Isn't that lovely? The idea of the Bolzano-Weierstrauss Theorem in RealAnalysis/proofs.tex is a special case of the general result about the compactness of k-pseudocells.

3.1 Cauchy Sequences

A **Cauchy sequence** is a sequence $\{a_n\}$ in a metric space X such that for all $\epsilon > 0$, there is an integer N such that

$$N \leq n, m \implies d(a_n - a_m) < \epsilon.$$

Let E be a nonempty subset of a metric space X, and define $S = \{d(x, y) \mid x, y \in E\}$. Then $\sup S$ is called the **diameter** of E. If we define E_N for each $N \in \mathbb{Z}_{>0}$ as $\{a_N, a_{N+1}, a_{N+2}, \cdots\}$; then $\{a_n\}$ is a Cauchy sequence if and only if

$$\lim_{N \to \infty} \operatorname{diam} E_N = 0.$$

Theorem 19. *The following two results hold:*

1. *If $\overline{E}$ is the closure of a subset E in a metric space X, then*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

2. *If K_n is a sequence of compact sets in X such that $K_n \subseteq K_{n+1}$ for each $n \in \mathbb{Z}_{>0}$, and if*

$$\lim_{n \to \infty} \operatorname{diam} K_n = 0,$$

then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Proof. For (1): since $E \subseteq \overline{E}$, it is natural that $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$. For the converse, select $x, y \in \overline{E}$ such that diam; then each x, y is either a point of E or a limit point of E. In either case, there exist points $x', y' \in E$ such that $d(x, x') < \epsilon$ and $d(y, y') < \epsilon$. Hence

$$\begin{aligned} d(x, y) &\leq d(x, x') + d(x', y) \\ &\leq d(x, x') + d(x', y') + d(y', y) \\ &< 2\epsilon + \operatorname{diam} E. \end{aligned}$$

Then $\operatorname{diam} E$ is an upper bound of $d(x, y)$, so $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$. This imples the desired $\operatorname{diam} \overline{E} = \operatorname{diam} E$

For (2): As discussed in RealAnalysis/babyrudin2.tex, $\bigcap_{n=1}^{\infty} K_n$ is nonempty. Suppose that it contains two distinct elements $j \neq k$; then $d(j, k) > 0$ and

$$d(j, k) < \operatorname{diam} K_n$$

for each $n \in \mathbb{Z}_{>0}$; we deduce that

$$0 < d(j, k) \leq \operatorname{diam} K_n.$$

Taking the contrapositive yields the desired result. □