The Resistance of Directed Compact Percolation Clusters.

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Abstract

The resistance of the directed compact percolation clusters of the Domany and Kinzel model is analysed numerically. The resistance R of such clusters of length t is a random variable which lies in the range $r_{sq}(t) \leq R \leq t$, where $r_{sq}(t)$ is the corner to corner resistance of a square cluster. Numerical analysis of the sequence of exact rational values of $r_{sq}(2n)$ for $n \leq 29$ gives

$$r_{sq}(2n) = \frac{4}{\pi}(log(n) + 1/n) + 0.0773188939... + O(\frac{1}{n^2}).$$

Analysis of the average $R_{av}(t)$ of the resistance of all compact clusters of length t for $t \leq 11$ is consistent with $R_{av}(t) \sim t^{\frac{1}{2}}$ as $t \to \infty$. Together with $\nu_{\parallel} = 2$ this implies that the critical exponent of the resistance scale is $\zeta = 1$.

1 Introduction

The distribution of current in the elements of a random diode-insulator network has been considered previously [1,2,3]. In this earlier work the network was represented by the standard directed percolation model in which the directed bonds are independently open to the flow of current with probability p. Here we extend the investigation to the compact directed percolation model of Domany and Kinzel [4] which in many respects is exactly soluble.

The model is defined on a directed square lattice the sites of which are the points in the t, y plane with integer co-ordinates such that $t \geq 0$, $y \geq 0$ and $t + y$ even. A random cluster, c, grows from a seed occupying m contiguous sites in the column $t = 0$. The growth rule is that the site (t,y) becomes occupied by an atom (a) with certainty if both the sites $(t-1, y \pm 1)$ are occupied, or (b) with probability p if one these sites is occupied and the other is not. These conditions imply that for any column t the occupied sites (if any) will be contiguous which is why the clusters are said to be compact. Previous calculations [5] have shown that the critical exponents are independent of m so here we consider only $m = 1$.

The cluster c may be finite or infinite and the probability that the cluster is infinite is the percolation probability P(p) which becomes positive above the critical probability $p_c = \frac{1}{2}$ with critical exponent $\beta = 1$ [4]. In the case that the cluster is finite the number of growth stages, t , before termination is a random variable whose mean value $L_b(p)$ diverges at p_c with critical exponent $\tau = 1$ [4]. The expected number of atoms $S_a(p)$ in a finite cluster also diverges at p_c but with critical exponent $\gamma = 2$ [5].

Here we consider the current distribution when a current I is injected at the seed and removed at the terminal site of the cluster. Let c denote a particular cluster and suppose that $i_b(c)$ is the current in nearest neighbour bond $b \in B(c)$. Following de Arcangelis et al [7], the kth moment of the current distribution is defined by

$$L_k(c) = \sum_{b \in B(c)} [i_b(c)/I]^k \tag{1}$$

in terms of which the number of bonds in c is $L_0(c)$ and the number of growth stages t before termination is $L_1(c)$ [3]. In general $L_0(c)$ is the number of bonds which carry a positive current and in the standard directed percolation model these are known as backbone bonds. For compact percolation all bonds belong to the backbone. By consideration of power dissipated it may be seen that $L_2(c)$ is the end-to-end resistance of c. Finally the limit of $L_k(c)$ as $k \to \infty$ gives the number of nodal bonds in c, that is the number of bonds for which $i_b(c) = I$.

The average value, $L_{av}(k,t)$, of $L_k(c)$ over the set $C(t)$ of all possible compact clusters terminating after t growth stages is defined by

$$L_{av}(k,t) = w_t^{-1} \sum_{c \in C(t)} L_k(c). \tag{2}$$

where $w_t = |C(t)|$, the number of different compact clusters of length t. Delest and Viennot (1984) showed that w_t is a Catalan number,

$$w_t = \frac{1}{t+2}\binom{2t+2}{t+1} = C_{t+1}. \tag{3}$$

The probability that a given cluster occurs is given by

$$pr(\text{c occurs}) = p^t(1-p)^{t+2} \qquad \text{for} \qquad c \in C(t) \tag{4}$$

so that the probability that the growth results in a cluster which terminates after t stages is

$$r_t(p) = w_t p^t (1-p)^{t+2}. \tag{5}$$

Hence the probability $Q(p) = 1 - P(p)$ that c is finite is

$$Q(p) = \sum_c pr(\text{c occurs}) = \sum_{t=0}^{\infty} r_t(p). \tag{6}$$

Using (3)

$$Q(p) = \begin{cases} 1 & \text{for} \quad p < p_c \\ \\ (1-p)^2/p^2 & \text{for} \quad p \geq p_c. \end{cases} \tag{7}$$

We denote the expected value of $L_k(c)$, given that c is finite, by $\Lambda_k(p)$ and in terms of

$$\Lambda_k(p) = \frac{1}{Q(p)} \sum_c pr(\text{c occurs}) L_k(c) = \frac{1}{Q(p)} \sum_{t=0}^{\infty} r_t(p) L_{av}(k,t). \tag{8}$$

It follows from (5) that

$$r_t(p) \cong \pi^{-\frac{1}{2}} t^{-\frac{3}{2}} \exp(-\frac{t}{\xi_{\|}(p)}) \tag{9}$$

where the parallel connectedness length $\xi_{\|}(p)$ diverges at p_c with exponent $\nu_{\|} = 2$. This asymptotic form was obtained by Domany and Kinzel [4] and it will be used here to show that if as $t \to \infty$,

$$L_{av}(k,t) \sim t^{\zeta_k/\nu_{\|}} \tag{10}$$

then as $p \to \frac{1}{2}$

$$\Lambda_k(p) \sim |1-2p|^{-\tau_k}. \tag{11}$$

where

$$\tau_k = \zeta_k - \beta. \tag{12}$$

Now $\Lambda_0(p)$ is the expected number of bonds, $S_b(p)$, in c and it is easy to show that this has the same critical exponent as $S_a(p)$. Hence, using the result quoted above, $\tau_0 = \gamma = 2$ and ζ_0 =3. Also, since $L_1(c)$ is the cluster length, $\zeta_1 = \nu_{\|} = 2$ and hence $\tau_1 = \tau = 1$ which is the result quoted above for $L_b(p) = \Lambda_1(p)$. The second moment is the mean end-to-end resistance, that is $\Lambda_2(p) = R(p)$, and the corresponding exponent $\tau_R = \tau_2$ will be the main subject of this investigation. We shall find that

2 Moments of the current distribution for a square cluster.

When t is even, the cluster of maximum size is a square with $\frac{t}{2}$ bonds along each edge and the moments of the current distribution for this cluster provide bounds for those of an arbitrary cluster in $C(t)$. For $k \geq 1$ and $c \in C(t)$,

$$L_k(square) \leq L_k(c) \leq t \tag{13}$$

and hence

$$L_k(square) \leq L_{av}(k,t) \leq t. \tag{14}$$

It was hoped that this lower bound could be found in closed form but even for $k = 2$ it appears to be a rather complex function of t. In the remainder of this section we investigate $L_k(square)$ firstly by a continuum approximation and then by extrapolation from small values of t using differential approximants.

2.1 Continuum approximation

Labelling the columns $t' = 0, \ldots t$, the column $t' = \frac{t}{2}$ is an equipotential and it follows by symmetry that

$$L_k(square) = 2L_k(triangle) \tag{15}$$

where $L_k(triangle)$ denotes the kth moment for the triangle consisting of the columns $0 \leq t' \leq \frac{t}{2}$ when the last column is maintained at constant potential.

For $t \to \infty$, it is possible to approximate $L_k(triangle)$ by the kth moment of the current distribution in a plate of uniform conductivity bounded by the lines $(r = a, r = \frac{t}{2}, \theta = \pm\frac{\pi}{4})$ in polar co-ordinates where the first two lines are maintained at constant potential. The current for this configuration will be radial and the current per unit length at distance r from the origin is $\frac{2I}{\pi r}$ independently of θ. Replacing the sum in (1) by an integral we obtain

$$\begin{aligned} L_k(triangle) &\approx \int_a^{\frac{t}{2}} (\frac{\pi r}{2})^{1-k} dr \\ &\approx (\frac{\pi}{2})^{1-k}[(\frac{t}{2})^{2-k} - a^{2-k}]/(2-k) \quad \text{for} \quad k \neq 2 \\ &\approx \frac{2}{\pi}[ln(t) - ln(2a)] \quad \text{for} \quad k = 2. \end{aligned} \tag{16}$$

Lower order corrections to this formula may be found using a harmonic expansion and ... The resistance (k=2) is seen to diverge logarithmically.

2.2 Extrapolation from small clusters using differential approximants.

The current distribution for square clusters may be rapidly calculated recursively using the following formula for the input conductance matrix $\sigma_{ij}(m)$ of a general cluster having m columns. This is defined such that if the seed atom is earthed and potential $V_j(m)$ is applied to the jth atom in column m then the current which enters the network at the ith atom is given by

$$I_i(m) = \sum_{j=1}^{n_m} \sigma_{ij}(m) V_j(m). \tag{17}$$

Applying Kirchoff's laws it may be shown that

$$\sigma(m+1) = \mathbf{d}^-(m+1) - \mathbf{A}^T(m)[\mathbf{d}^+(m) + \sigma(m)]^{-1}\mathbf{A}(m) \tag{18}$$

where the ith row of $\mathbf{A}(m)$ has a 1 in column j if there is a connection from atom i of column m to atom j of column $m+1$ but is zero if not. $\mathbf{d}^+(m)$ is a diagonal matrix the ith diagonal element of which is the number of atoms in column $m+1$ which are adjacent to the ith atom of column m. $\mathbf{d}^-(m+1)$ is a diagonal matrix the jth diagonal element of which is the number of atoms in column m which are adjacent to the jth atom of column $m+1$. $\sigma(1)$ is just a unit matrix of dimension equal to the number of atoms in column 1. The recurrence relation is used to calculate all of the conductance matrices as far as $m = \frac{t}{2}$ and the voltage vectors are then calculated starting at column $\frac{t}{2}$ where $V_j(\frac{t}{2}) = V$ and using

$$\mathbf{V}(m) = [\mathbf{d}^+(m) + \sigma(m)]^{-1}\mathbf{A}(m)\mathbf{V}(m+1). \tag{19}$$

The current distribution is then calculated by taking differences of the voltage vector elements between adjacent columns. If only the resistance is required when the second electrode feeds all the atoms in column $\frac{t}{2}$ this may be obtained from

$$R = [\sum_{i=1}^{n_t} \sum_{j=1}^{n_t} \sigma_{ij}(t)]^{-1}. \tag{20}$$

The results for the resistance of square clusters, denoted by $r_{sq}(t)$ and obtained from that of a sequence of triangular clusters, is shown in table 1.

Since relatively large numbers of terms in the sequence are available and a logarithmic singularity is expected from the continuum approximation the differential approximant method reviewed by Guttmann [8] is used for extrapolation. The terms are first fitted to a recurrence relation of order n with coefficients which are polynomials in t of degree k. This is then used to obtain a differential equation of order k with polynomial coefficients of degree n for the generating function $R_{sq}(x)$ defined by

$$R_{sq}(x) = \sum_{n=1}^{\infty} r_{sq}(2n) x^n. \tag{21}$$

$t/2$	$r_{sq}(t)$
1	1
2	3/2
3	13/7
4	47/22
5	1171/495
6	6385/2494
7	982871/360161
8	441083/153254
9	427854195/142065451
10	4769986941/1522703822
11	182523584723/56281934513
12	1612811746276087/482203589139798
13	21115942380885185009/6140471000326322381
14	2754705496774325/781147089062918
15	164598359758038442278129/45613817209147281353759
16	268629483927520086956437679041/72886377588656409729470151586
17	227222554503337141416085954361/60459337600769642139861952905
18	37806228153341258635300652841047751795/ 9878743555300458521950915571159588866
19	129548954101732562831760781545158173626645023/ 33283688571680493510612137844679320717594861
20	419042400527118675393410058599143/ 105970629639182547997859101316134
21	11807887565239485583092996488079504567366282667/ 2942039683673607873726698522584668254140514263
22	1191119486680158965128657219586022670627617181987/ 292654382297485257149326291789974519871388673274
23	13898237618359293023471700691938503460418509899519847/ 3369920812539617835952536669540941991969432449062337
24	1204758903953529250749819166882371696191434872488771308477 6625097/ 2884860113866651633138018319822114845194359410308884260356 653122
25	3325978605945240676707300228316244752739833556936198820630 95678730825876965849/ 7870185275820877897241673425471991759976350140256190757497581 4599754803354449
26	1472792064901952271974886838374845549473492585169848897900 92630667740027/ 344587508877407909223015223866763966411096926273509884665 37600022522978
27	3742257276619349254764082805349546048658780516830822516707 5286645172287989/ 866192688792147807435042298663389438234831522047502422232 5114510353569847
28	13486422094945804042763570897289079449310950852700136941141 08650635330418620423/ 3089666699526950708083835760297089707981794235365590298012 41058957134347025366
29	62315604094544214643364424973581177729498057922260511236230 31436688938877/ 1413644350916825786727486081546185662453087677204680232008 658349779821625

Table 1: Resistance between opposite corners of square clusters.

n	k	x_{21} , x_{22}	x_1	α_1 , α_2		β_1 , β_2	
4	2	0.999640628	0.999810728	-1.000990771	$\pm$	0.069893994	i
		1.000098695		-1.004887210	$\pm$	0.040190678	i
5	2	0.999979800	0.999996950	-1.000146873	$\pm$	0.001631800	i
		1.000015981		-1.000073309	$\pm$	0.009994289	i
6	2	1.000000008	1.000000012	-0.999766592		-1.000231625	
		1.000000008		-0.999766592		-1.000231625	
7	2	0.999999951	1.000000000	-0.999999195	$\pm$	0.000853023	i
		1.000000049		-0.999150292		-1.000851329	
4	3	0.999980649	0.999995784	-1.000160015	$\pm$	0.023275403	i
		1.000012662		-0.993638355		-1.007682755	
5	3	1.000000000	1.000000000	-0.999850480		-1.000149415	
		1.000000000		-0.999850480		-1.000149415	

Table 2: Differential approximant analysis of $R_{sq}(x)$. k is the order of the differential equation and n is the degree of the polynomial coefficients. $x_{2,1}$ and $x_{2,2}$ are the nearly equal roots of Q_k and x_1 is the corresponding root of Q_{k-1}. α_i and β_i are the roots of the indicial equation corresponding to $x_{2,i}$

For a recurrence relation with quadratic coefficients ($k = 2$) the differential equation is of the form

$$Q_2(x)(x\frac{d}{dx})^2 R_{sq}(x) + Q_1(x)(x\frac{d}{dx})R_{sq}(x) + Q_0(x)R_{sq}(x) = P(x) \qquad (22)$$

For general k the roots of $Q_k(x)$ are the possible positions of singular points in the generating function. Confluent singularities are detected by the presence of a double root in $Q_k(x)$ and an equal root in $Q_{k-1}(x)$. The exponents of the singularities are given by the roots (α and β) of a quadratic indicial equation [8].

Approximants were calculated with a range of values of n and k and $k = 2$ was found to give a rapidly converging sequence with increasing n. The results are shown in Table 2. For each value of n, Q_k has two roots ($x_{2,1}$ and $x_{2,2}$) and Q_{k-1} has one root (x_1) all of which are very close to $x_c = 1$. The coefficients of the quadratic indicial equation were evaluated at $x_{2,1}$ and $x_{2,2}$ and corresponding values of the roots α and β were found. These are listed in the last two columns of the table. The top row of each pair corresponds to $x_{2,1}$ and the columns contain either α and β or, when these exponents form a complex pair, the real and imaginary parts are given instead.

This data strongly suggests that $R_{sq}(x)$ is well approximated by a function having confluent singularities at $x_c = 1$ both with exponent -1 corresponding to a simple pole and a pole modified by a logarithmic factor, i.e.

m	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
20	0.0773044951	0.0773176008	0.0773192086	0.0773189708	0.0773189018
25	0.0773114406	0.0773185009	0.0773190077	0.0773189126	0.0773188951
30	0.0773145781	0.0773187483	0.0773189420	0.0773188998	0.0773188942
35	0.0773161837	0.0773188322	0.0773189168	0.0773188962	0.0773188940
40	0.0773170854	0.0773188651	0.0773189059	0.0773188949	0.0773188939
45	0.0773176289	0.0773188795	0.0773189006	0.0773188944	0.0773188939
50	0.0773179754	0.0773188863	0.0773188979	0.0773188942	0.0773188939
55	0.0773182063	0.0773188898	0.0773188964	0.0773188940	0.0773188939
60	0.0773183661	0.0773188916	0.0773188955	0.0773188940	0.0773188939

Table 3: Neville table determination of the constant b.

$$R_{sq}(x) \sim (1-x)^{-1}(A\ \ log(1-x)+B). \tag{23}$$

This implies that as $t \to \infty$,

$$r_{sq}(2n) \sim \ \ a\ \ log(n)+b. \tag{24}$$

Fitting the data in table 1 to this asymptotic form together with inverse power corrections gives an excellent fit with $a = \frac{4}{\pi}$, the continuum value, and strongly suggests the presence of a further term $\frac{4}{\pi t}$. To determine b we have made a Neville table analysis [8] of the sequence

$$c_n = r_{sq}(2n) - (\frac{4}{\pi})(log(n)+\frac{1}{n}) \tag{25}$$

with the results in Table 3. The limit of this sequence should be b and the inverse power corrections should start at order 2. The kth column of the table approximates b by fitting a polynomial in $\frac{1}{t}$ of degree k and the mth row fits to the terms c_n with $n = m, m-1, \ldots, m-k$.

Our final asymptotic approximation is

$$r_{sq}(2n) = \frac{4}{\pi}(log(n)+1/n)+0.0773188939...+O(\frac{1}{n^2}) \tag{26}$$

The leading order asymptotic form of $L_k(square)$ for $k=2$ has now been seen to be the same as for the continuum. This is also clearly the case for $k=0$ and 1 since in these cases it is respectively the dependence of the area and length of the cluster on t which is being estimated by $L_k(square)$. For $k>2$ the moments of the continuum distribution no longer diverge with t and the exponent $2-k$ in equation (16) determines the nature of the approach to the finite limit. We would expect this exponent to be the same for discrete network.

3 Scaling relation.

The scaling relation between ζ_k and τ_k which was quoted in the introduction will now be derived in a way which applies to a more general class of percolation models. We assume that $\Lambda_k(p)$ may be written in the form (8) where for the present model $L_{av}(k,t)$ is defined in (2) but more generally $L_{av}(k,t)$ would be the expected value of $L_k(c)$ given that c has t growth stages. Now the percolation probability is given by

$$P(p) = 1 - Q(p) = \sum_{t=0}^{\infty}(r_t(p_c) - r_t(p)) \sim (p_c - p)^{\beta} \tag{27}$$

and if as $t \to \infty$

$$r_t(p) \sim t^{-a} exp(-t/\xi_{\|}(p)) \tag{28}$$

then it follows that

$$a = 1 + \beta/\nu_{\|} \tag{29}$$

For the present model $a = 3/2$. Combining (8), (10) and (28)

$$\Lambda_k(p) \sim \int^{\infty} t^{\zeta_k/\nu_{\|}} t^{-a} exp(-t/\xi_{\|}(p))dt \sim \xi_{\|}(p)^{(\zeta_k - \beta)/\nu_{\|}} \tag{30}$$

from which (12) follows by comparison with (11).

4 Geometric lower bound.

In the previous section it was seen that replacing the triangular plate by a quadrant in order to make the current density independent of θ had no effect on the leading order asymptotic form. A similar approximation will now be made for an arbitrary percolation cluster. Suppose that there are $b_{t'}$ bonds connecting columns $t'-1$ and t'. If we suppose that each of these bonds carries a current $I/b_{t'}$ then for $c \in C(t)$, substituting in (1),

$$L_k(c) \geq L_k^-(c) = \sum_{t'=1}^{t} b_{t'}(c)^{1-k}. \tag{31}$$

which would be an equality if all atoms in a given column were at the same potential. This could be achieved by connecting them by a wire having zero resistance and hence the lower bound. Substitution in (8) gives

$$\Lambda_k(p) \geq \Lambda_k^-(p) \equiv \frac{q^2}{Q(p)} \sum_{t=1}^{\infty} \sum_{c\in C(t)} (pq)^t \sum_{t'=1}^{t} b_{t'}(c)^{1-k}. \tag{32}$$

Now rearrange the sum by collecting the terms for which $b_{t'}(c) = 2n$ and those for which $b_{t'}(c) = 2n-1$ for each $n = 1, 2, \ldots$. In the first case there are n and $n+1$ atoms in the adjacent columns and in the second there are n atoms in both columns. Thus

$$\Lambda_k^-(p) = \frac{2pq^3}{Q(p)} \sum_{n=1}^{\infty} \left[(2n)^{1-k} A_n(p) + (2n-1)^{1-k} B_n(p)\right] \tag{33}$$

where

$$
\begin{aligned}
A_n(p) &= \sum_{t=0}^{\infty}(pq)^{t-1}\sum_{t'=1}^{t} r_{t'-1,n}r_{t-t',n+1} = R_n(pq)R_{n+1}(pq) && (34)\\
B_n(p) &= \sum_{t=0}^{\infty}(pq)^{t-1}\sum_{t'=1}^{t} r_{t'-1,n}r_{t-t',n} = R_n(pq)^2 && (35)\\
& && (36)
\end{aligned}
$$

and

$$r_{t,n} = \frac{n}{t+1}\binom{2t+2}{t-n+1}, \tag{37}$$

the number of clusters of length t with 1 atom in column 0 and n atoms in column t, so that

$$\omega R_n(\omega) = \sum_{t=n-1}^{\infty} r_{t,n}\omega^{t+1} = v^n. \tag{38}$$

where

$$v = \frac{1-2\omega-\sqrt{1-4\omega}}{2\omega} \tag{39}$$

Setting $\omega = p(1-p)$ gives

$$v = \frac{1-2p+2p^2-|1-2p|}{2p(1-p)} = \begin{cases} p/q & \text{for} \quad p \le p_c \\ q/p & \text{for} \quad p \ge p_c. \end{cases} \tag{40}$$

$$R_n(pq) = \frac{v^n}{pq} \tag{41}$$

and hence

$$\Lambda_k^-(p) = 2\sum_{b=1}^{\infty} b^{1-k}v^b \tag{42}$$

$L_0(c)$ is manifestly independent of the current distribution and carrying out the sum in (42) gives the expected number of bonds

$$S_b(p) \equiv \Lambda_0(p) = \Lambda_0^-(p) = \frac{2p(1-p)}{(1-2p)^2}. \tag{43}$$

Since $i_b(c)$ when summed over any column of bonds is equal to the total current I then $L_1(c) = t$ for all distributions and hence the expected cluster length measured by bonds (the number of bonds in any directed path connecting the root and terminal point) is

$$L_b(p) \equiv \Lambda_1(p) = \Lambda_1^-(p) = \frac{1}{|1-2p|} - 1. \tag{44}$$

which agrees with the expected length $L_a(p) = 1 + L_b(p)$ measured by atoms given in [5]. The inequality in (32) may also be replaced by equality in the limit $k \to \infty$ which gives the expected number of nodal bonds

$$N_b(p) \equiv \Lambda_\infty(p) = \Lambda^-_\infty(p) = 2v. \tag{45}$$

In [6] an expression (eq.3.15) was given for the probability mass function of the number of nodal bonds. The result of (45) disagrees with the mean calculated using this expression and we take this opportunity to give a corrected expression. For $p \le p_c$

$$Pr(c \text{ has exactly } b \text{ nodal bonds}|c \text{ finite}) = d(1-c-d)^b(1+\frac{c}{1-c})^{b+1} \tag{46}$$

where $c = p^2$ and $d = q^2$, and for $p \ge p_c$, c and d should be interchanged.

The case $k = 2$ gives the expected resistance

$$\Lambda_k(p) \ge \Lambda^-_k(p) = 2\sum_{b=1}^{\infty}\frac{v^b}{b} = 2\log(\frac{1}{1-v}) \tag{47}$$

and hence $\tau_2 \ge 0$. Using scaling gives $\zeta_2 \ge \beta = 2$. It follows that $L_{av}(2,t)$ diverges at least as fast as $t^{\frac{1}{2}}$.

$$\Lambda^-_k(p)/q^2 = \sum_{t=1}^{\infty}\omega^t L^-_{tot}(k,t) = \sum_{t=1}^{\infty}\frac{\omega^t}{t+1}\sum_{b=1}^{u}\frac{b+1}{b^{k-1}}\binom{2t+2}{t-b} \tag{48}$$

For $k > 2$ the singular part of $\Lambda^-_k(p)$ may be obtained from the formula

$$\begin{aligned}
\phi_k(z) &\equiv \sum_{n=k}^{\infty}\frac{z^n}{n(n-1)\dots(n-k+1)} && (49)\\
&= \frac{(-1)^{k+1}}{(k-1)!}\Bigg[(1-z)^{k-1}\log(\frac{1}{1-z})+ && (50)\\
&\qquad \sum_{m=1}^{k-1}(-z)^m\binom{k-1}{m}\sum_{n=0}^{m-1}\frac{1}{k-n-1}\Bigg] && (51)
\end{aligned}$$

5 Estimation of ζ_k using the ratio method applied to $L_{av}(k,t)$ for small values of t.

All clusters with $t \le 11$ were enumerated with a backtracking algorithm. The end-to-end resistance of each cluster was calculated using the recurrence relation (18). In order to improve efficiency the sigma matrices for the partial clusters were kept so that on removing a growth stage the sigma matrix for the previous stage could rapidly be recovered. The third column of Table 4 shows the sum of the resistances over clusters with a given value of t. $R_{av}(t)$

t	$L_{av}(2,t)$	t	$L_{av}(2,t)$
1	1.0000000000000000000000000000	2	1.8000000000000000000000000000
3	2.48571428571428571428571428571	4	3.09234693877551020408163265306
5	3.64020096607159481292008061256	6	4.14228383179819665824922947207
7	4.60751136042521142372048010292	8	5.04229788758704271560925902324
9	5.45143199743052975666625042644	10	5.83859506757113002678283196733
11	6.20668579585467074410949852441	12	6.55803247345942969270979740650
13	6.89453695872051268863281350455	14	7.21777533961243535846263851439
15	7.52907015934567925872233507194	16	7.82954340876486028127457917272

t	$L_{av}(3,t)$	t	$L_{av}(3,t)$
1	1.0000000000000000000000000000	2	1.7000000000000000000000000000
3	2.24228571428571428571428571429	4	2.68242695751770095793419408580
5	3.05137873704335391346875820940	6	3.36800472961235736050657050261
7	3.64463979836518256573623503137	8	3.88977728453815354639873058597
9	4.10950522051166986163780270416	10	4.30833036221944207035319657159
11	4.48967828404733852880596125497	12	4.65621097322583940790324507004
13	4.81003625251827140554557368947	14	4.95285032350022194525064171638
15	5.08603745716070771545198007896	16	5.21074137240384277272413170191

t	$L_{av}(4,t)$	t	$L_{av}(4,t)$
1	1.0000000000000000000000000000	2	1.6500000000000000000000000000
3	2.12331428571428571428571428571	4	2.48704925905128815374546319986
5	2.77755992659021168877410791823	6	3.01629703994418840729667812883
7	3.21685069868057146226126326158	8	3.38830202040068026181478588194
9	3.53697927590622985181765567662	10	3.66744455556952281570608795599
11	3.78308013663679668379962305206	12	3.88645331983753189184024731573
13	3.97955229216106305596177580379	14	4.06394364187020341451994766322
15	4.14088052672874423232708271785	16	4.21137877722983180600458133888

t	w_t	$w_t R_{av}(t)$	$t(\mu_t - 4)/4$	$(t+3)(\mu_t - 4)/4$	$(t - 0.75)(\mu_R(t) - 1)$
0	1	0.00000000	0.0	-1.5	-0.750000
1	2	2.00000000	-0.375	-1.5	0.100000
2	5	9.00000000	-0.6	-1.5	0.306122
3	14	34.80000000	-0.75	-1.5	0.391576
4	42	129.87857144	-0.857143	-1.5	0.435087
5	132	480.50652752	-0.9375	-1.5	0.459862
6	429	1777.03976384	-1.0	-1.5	0.474973
7	1430	6588.74124540	-1.05	-1.5	0.484603
8	4862	24515.65232944	-1.090909	-1.5	0.490910
9	16796	91562.25182884	-1.125	-1.5	0.495099
10	58786	343227.64964228	-1.153846	-1.5	0.497886
11	208012	1291065.12576728			

Table 4: The resistance sums for all clusters with up to 11 growth stages. $\mu_t = w_{t+1}/w_t$.

can be determined from this by dividing by w_t which is given in equation (3) and listed in the second column.

The last two columns of Table 4 illustrate the Ratio method as applied to w_t for which the asymptotic form is known. From (3) it follows that

$$w_t \sim 4^t t^{-3/2} \tag{52}$$

and if the exponent $-3/2$ was not known the Ratio method would estimate it by calculating $\mu_t = w_{t+1}/w_t$ which approaches the limit 4 and then forming the sequence $t(\mu_t - 4)/4$ the limit of which determines the exponent. The limit is normally determined graphically by plotting the estimates against $1/t$ which should give a horizontal line in the limit. Due to corrections of order $1/t^2$ the line is curved and this must be allowed for. One way of doing this is by plotting against $1/(t + \Delta t)$ where the shift Δt is chosen to minimize the curvature. In the case of w_t a shift of $\Delta t = 3$ produces a perfect horizontal line as can by seen from the data of the last column. The index -1.5 is thus confirmed numerically. We now apply the same technique to $R_{av}(t)$. In fact we use $1+R_{av}(t)$ since it is known that the expected value of $1+t$ has a simpler form than that of t. Defining

$$\mu_R(t) = (1 + R_{av}(t+1))/(1 + R_{av}(t)) \tag{53}$$

which by equation (10) should have limit 1 and the sequence $(t+\Delta t)(\mu_R(t)-1)$ should tend towards the index $\zeta_2/\nu_{\|}$. The shift $\Delta t = -0.75$ gives the most slowly varying sequence of estimates which are shown in the last column of Table 4 and are consistent with the prediction $R_{av}(t) \sim t^{\frac{1}{2}}$.

We now examine the generating function for $R_{av}(t)$ which is as follows, the constant 1 has been added so that no terms are lost on taking the *Dlog*.

$$\begin{aligned}1. + x + 1.8x^2 + 2.48571x^3 + 3.09235x^4 + 3.6402x^5 + 4.14228x^6 + \\ 4.60751x^7 + 5.0423x^8 + 5.45143x^9 + 5.8386x^{10} + 6.20669x^{11}\end{aligned} \tag{54}$$

First we look at the Adler Graph which is the gamma exponent (subtract 1 for g) plotted against the correction to scaling exponent delta.

AdlerGraph[Rbar,x,1.0,0.5,2.0, 3,4,4,3,4,4,4,5,5,4,5,5,"Rbar",0.5,1.4]

There is no strong indication of correction to scaling although the best convergence is with delta = 0.6. With delta =1 (the biased DLog Pade) we get the following results

and with delta = 0.6 the better converged results suggest a limit of 1.447

6 A fractal having similar behaviour to a random compact cluster.

Consider again the square cluster of section 2. The structure of this cluster will now be modified so as to yield an exactly soluble model whose resistance

-1	0	1
0.665399	3.24000	3.05714
-4.55221	1.79868	2.75279
1.72818	1.71937	6.51934
1.68626	1.62298	1.67395
1.59271	1.59190	1.64454
1.57412	1.56090	1.57276
1.54488	1.54467	1.55923
1.53360	1.52833	1.53320
1.52089	1.52085	1.52549

Table 5: Adler method. AdlerTable[RbarGen,x,1.0,1.0]

-1	0	1
0.197492	4.11111	4.04762
-0.149107	1.2793	3.69285
1.51057	1.45782	-47.5083
1.50198	1.50225	1.49284
1.52009	1.49071	1.49057
1.43889	1.44397	1.49347
1.44889	1.4487	1.30048
1.44751	1.44768	1.44736
1.44731	1.44729	1.44729

Table 6: Adler method. AdlerTable[RbarGen,x,1.0,0.6]

0.25						
-0.1	-0.45					
-0.267857	-0.603571	-0.680357				
-0.375425	-0.69813	-0.792688	-0.830131			
-0.452605	-0.761322	-0.856112	-0.898394	-0.91546		
-0.511514	-0.806062	-0.895541	-0.93497	-0.953258	-0.960817	
-0.55832	-0.839154	-0.921883	-0.957006	-0.973534	-0.981644	-0.985115
-0.59659	-0.864484	-0.940477	-0.971466	-0.985925	-0.99336	-0.997265
-0.628571	-0.884417	-0.954181	-0.981591	-0.994247	-1.0009	-1.00468
-0.65576	-0.900462	-0.964641	-0.989045	-1.00023	-1.00621	-1.00974

is a weak upper bound for that of the square . First notice that since there is no net current in the y-direction across any node with $y = 0$ we may split each of these nodes into two nodes of degree 2 so that the $y > 0$ (and $y < 0$) bonds are still connected. This operation has no effect on the current distribution. If we make a similar splitting of all other degree 4 nodes we obtain the fractal structure shown in Fig. 1 whose current distribution approximates that of the square by ignoring transverse currents. Denote the moments for this fractal by $L_k(fractal)$. It is easy to show by symmetry that $L_k(fractal) = L_k(upperhalf)/2^{k-1}$ where $L_k(upperhalf)$ is the kth moment of the distribution in the fractal when the bonds in the $y < 0$ region are removed. Since there is further symmetry of the current distribution about the line $t' = \frac{t}{2} = s$, if we denote by $U_k(s)$ the contribution to $L_k(upperhalf)$ from the bonds connecting nodes with $t' \leq s$ then

$$L_k(fractal) = U_k(s)/2^{k-2}. \tag{55}$$

Note that $R(s) = U_2(s)$ is the resistance of the fractal of length $t = 2s$. The fractal of length $2s + 2$ may be obtained from that of length $2s$ by series/parallel combination which leads to the recurrence relations

$$R(s+1) = 1 + \frac{sR(s)}{(s + R(s))} \tag{56}$$

and

$$U_k(s+1) = 1 + s\left(\frac{R(s)}{s+R(s)}\right)^k + \left(\frac{s}{s+R(s)}\right)^k U_k(s) \tag{57}$$

with initial conditions $R(1) = U_k(1) = 1$. The case $k = 0$ is a first order difference equation with constant coefficients which may be solved to yield

$$U_0(s) = \frac{s(s+1)}{2} \tag{58}$$

which is the number of bonds in one quarter of the original square as expected. The case $k = 1$ is solved by $U_1(s) = s$, half the length of the square. We have been unable to solve for the resistance, $k = 2$, in closed form but for $t \to \infty$ an asymptotic expansion may be obtained in inverse powers of $s^{\frac{1}{2}}$ by iterating equation (56). This may then be used in (57) to obtain a similar expansion of $U_k(s)$ for integer $k \geq 2$.

$$U_k(s) \cong A_k s^{\frac{1}{2}} + B_k + C_k s^{-\frac{1}{2}} + O(s^{-1}) \tag{59}$$

where

$$\begin{aligned} A_k &= \frac{1}{k} + \frac{1}{2}\delta_{k,2} \\ B_k &= \frac{2k^2 + k - 2}{4k^2} - \frac{1}{4}\delta_{k,2} + \frac{1}{3}\delta_{k,3} \\ C_k &= \frac{8k^3 - 11k - 12}{96k^2} - \frac{15}{64}\delta_{k,2} - \frac{1}{6}\delta_{k,3} + \frac{1}{4}\delta_{k,4} \end{aligned} \tag{60}$$

this is the upper bound for the square cluster but the resistance has the same exponent as the average resistance of the compact clusters. It would be interesting to check this for other moments.

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