Theory of resistor networks: The two-point resistance

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Abstract

The resistance between arbitrary two nodes in a resistor network is obtained in terms of the eigenvalues and eigenfunctions of the Laplacian matrix associated with the network. Explicit formulas for two-point resistances are deduced for regular lattices in one, two, and three dimensions under various boundary conditions including that of a Möbius strip and a Klein bottle. The emphasis is on lattices of finite sizes. We also deduce summation and product identities which can be used to analyze large-size expansions of two-and-higher dimensional lattices.

Key words: Resistor network, two-point resistance, finite lattices, summation and product identities.

1 Introduction

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (for a list of relevant references up to 2000 see, e.g., [1]). Besides being a central problem in electric circuit theory, the computation of resistances is also relevant to a wide range of problems ranging from random walks (see [2, 3] and discussions below), the theory of harmonic functions [4], first-passage processes [5], to lattice Green's functions [6]. The connection with these problems originates from the fact that electrical potentials on a grid are governed by the same difference equations as those occurring in the other problems. For this reason, the resistance problem is often studied from the point of view of solving the difference equations, which is most conveniently carried out for infinite networks. Very little attention has been paid to finite networks, even though the latter are the ones occurring in real life. In this paper we take up this problem and present a general formulation for finite networks. Particularly, known results for infinite networks are recovered by taking the infinite-size limit. In later papers we plan to study effects of lattice defects and carry out finite-size analyses.

The study of electric networks was formulated by Kirchhoff [7] more than 150 years ago as an instance of a linear analysis. Here, we analyze the problem along the same line of approach. Our starting point is the consideration of the Laplacian matrix associated with a network, which is a matrix whose entries are the conductances connecting pairs of nodes. Just as in graph theory that everything about a graph is described by its adjacency matrix (whose elements is 1 if two vertices are connected and 0 otherwise), we expect everything about an electric network is described by its Laplacian. Indeed, in Section 2 below we shall derive an expression of the two-point resistance between arbitrary two nodes in terms of the eigenvalues and eigenvectors of the Laplacian matrix [8]. In ensuing sections we apply our formulation to networks of one-dimensional and two-dimensional lattices under various boundary conditions including those embedded on a Möbius strip and a Klein bottle, and lattices in higher spatial dimensions. We also deduce summation and product identities which can be used to reduce the computational labor as well as analyze large-size expansions in two-and-higher dimensions. In subsequent papers we shall consider large-size expansions and effects of defects in finite networks, the latter a problem that has been studied in the past only for infinite networks [9].

Let Łbe a resistor network consisting of $\mathcal{N}$ nodes numbered by $i = 1, 2, \cdots, \mathcal{N}$. Let $r_{ij} = r_{ji}$ be the resistance of the resistor connecting nodes i and j. Hence, the conductance is

$$c_{ij} = r_{ij}^{-1} = c_{ji}$$

so that $c_{ij} = 0$ (as in an adjacency matrix) if there is no resistor connecting i and j.

Denote the electric potential at the ith node by V_i and the net current flowing *into* the network at the ith node by I_i (which is zero if the ith node is not connected to the external world). Since there exist no sinks or sources of current including the external world, we have the constraint

$$\sum_{i=1}^{\mathcal{N}} I_i = 0. \tag{1}$$

The Kirchhoff law states

$$\sum_{j=1}^{\mathcal{N}}{}' \; c_{ij}(V_i - V_j) = I_i, \qquad i = 1, 2, \cdots, \mathcal{N}, \tag{2}$$

where the prime denotes the omission of the term $j = i$. Explicitly, (2) reads

$$\mathbf{L}\,\vec{V} = \vec{I} \tag{3}$$

where

$$\mathbf{L} \;=\; \begin{pmatrix} c_1 & -c_{12} & \ldots & -c_{1\mathcal{N}} \\ -c_{12} & c_2 & \ldots & -c_{2\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1\mathcal{N}} & -c_{2\mathcal{N}} & \ldots & c_{\mathcal{N}} \end{pmatrix} \tag{4}$$

and

$$c_i \equiv \sum_{j=1}^{\mathcal{N}}{}' \; c_{ij}. \tag{5}$$

Here $\vec{V}$ and $\vec{I}$ are $\mathcal{N}$-vectors whose components are V_i and I_i respectively. We note in passing that if all nonzero resistances are equal to 1, then the off-diagonal elements of the matrix $-\mathbf{L}$ are precisely those of the adjacency matrix of $\mathcal{L}$.

The Laplacian matrix $\mathbf{L}$ of the network $\mathcal{L}$ is also known as the Kirchhoff matrix, or simply the tree matrix; the latter name derived from the fact that all cofactors of $\mathbf{L}$ are equal, and equal to the spanning tree generating function for $\mathcal{L}$ [10].

To compute the resistance $R_{\alpha\beta}$ between two nodes α and β, we connect α and β to the two terminals of an external battery and measure the current I going through the battery while no other nodes are connected to external sources. Let the potentials at the two nodes be, respectively, V_α and V_β. Then, the desired resistance is the ratio

$$R_{\alpha\beta} = \frac{V_\alpha - V_\beta}{I}. \tag{6}$$

The computation of the two-point resistance $R_{\alpha\beta}$ is now reduced to solving (3) for V_α and V_β with the current given by

$$I_i = I(\delta_{i\alpha} - \delta_{i\beta}). \tag{7}$$

A probabilistic interpretation:

The two-point resistance has a probabilistic interpretation. Consider a random walker walking on the network $\mathcal{L}$ with the probability

$$p_{i\to j} = c_{ij}/c_i \tag{8}$$

of hoping from node i to node j, where $p_{i\to j}$ can be different from $p_{j\to i}$. Let $P(\alpha;\beta)$ be the probability that the walker starting from node α will reach node β before returning to α, which is the probability of first passage. Then one has the relation [2, 5]

$$P(\alpha;\beta) = \frac{1}{c_\alpha R_{\alpha\beta}} \tag{9}$$

where c_α is defined in (5). If all resistances are 1, then (9) becomes

$$P(\alpha;\beta) = \frac{1}{z_\alpha R_{\alpha\beta}}. \tag{10}$$

where z_α is the coordination number of, or the number of nodes connected to, the node α.

2 The two-point resistance: A theorem

Let Ψ_i and l_i be the eigenvectors and eigenvalues of $\mathbf{L}$, namely,

$$\mathbf{L}\ \Psi_i = l_i\ \Psi_i, \qquad i = 1, 2, \cdots, \mathcal{N}.$$

Let $\psi_{i\alpha}, \alpha = 1, 2, \cdots, \mathcal{N}$ be the components of Ψ_i. Since $\mathbf{L}$ is Hermitian, the Ψ_i's can be taken to be orthonormal satisfying

$$(\Psi_i^*, \Psi_j) = \sum_\alpha \psi_{i\alpha}^* \psi_{j\alpha} = \delta_{ij}.$$

Now the sum of all columns (or rows) of $\mathbf{L}$ is identically zero, so one of the eigenvalues of $\mathbf{L}$ is identically zero. It is readily verified that the zero eigenvalue $l_1 = 0$ has the eigenvector

$$\psi_{1\alpha} = 1/\sqrt{\mathcal{N}}, \qquad \alpha = 1, 2, \cdots, \mathcal{N}.$$

We now state our main result as a theorem.

Theorem:

Consider a resistance network whose Laplacian has nonzero eigenvalues l_i *with corresponding eigenvectors* $\Psi_i = (\psi_{11}, \psi_{12}, \cdots, \psi_{1\mathcal{N}})$, $i = 2, 3, \cdots, \mathcal{N}$. *Then the resistance between nodes* α *and* β *is given by*

$$R_{\alpha\beta} = \sum_{i=2}^{\mathcal{N}} \frac{1}{l_i} \Big| \psi_{i\alpha} - \psi_{i\beta} \Big|^2. \tag{11}$$

Proof.

We proceed by solving Eq. (3) by introducing the inverse of the Laplacian $\mathbf{L}$, or the Green's function [6]. However, since one of the eigenvalues of $\mathbf{L}$ is zero, the inverse of $\mathbf{L}$ must be considered with care.

We add a small term $\epsilon\,\mathbf{I}$ to the Laplacian, where $\mathbf{I}$ is the $\mathcal{N} \times \mathcal{N}$ identity matrix, and set $\epsilon = 0$ at the end. The modified Laplacian

$$\mathbf{L}(\epsilon) = \mathbf{L} + \epsilon\,\mathbf{I}$$

is of the same form of $\mathbf{L}$ except that the diagonal elements c_i are replaced by $c_i + \epsilon$. It is clear that $\mathbf{L}(\epsilon)$ has eigenvalues $l_i + \epsilon$ and is diagonalized by the same unitary transformation which diagonalizes $\mathbf{L}$.

The inverse of $\mathbf{L}(\epsilon)$ is now well-defined and we write

$$\mathbf{G}(\epsilon) = \mathbf{L}^{-1}(\epsilon).$$

Rewrite (3) as $\mathbf{L}(\epsilon)\vec{V}(\epsilon) = \vec{I}$ and multiply from the left by $\mathbf{G}(\epsilon)$ to obtain $\vec{V}(\epsilon) = \mathbf{G}(\epsilon)\vec{I}$. Explicitly, this reads

$$V_i(\epsilon) = \sum_{j=1}^{\mathcal{N}} G_{ij}(\epsilon) I_j, \qquad i = 1, 2, \cdots, \mathcal{N}, \tag{12}$$

where $G_{ij}(\epsilon)$ is the ij-th elements of the matrix $\mathbf{G}(\epsilon)$.

We now compute the Green's function $G_{ij}(\epsilon)$.

Let $\mathbf{U}$ be the unitary matrix which diagonalizes $\mathbf{L}(\epsilon)$ and $\mathbf{L}$, namely,

$$\begin{aligned} \mathbf{U}^\dagger \mathbf{L} \mathbf{U} &= \mathbf{\Lambda} \\ \mathbf{U}^\dagger \mathbf{L}(\epsilon) \mathbf{U} &= \mathbf{\Lambda}(\epsilon). \end{aligned} \tag{13}$$

It is readily verified that elements of $\mathbf{U}$ are $U_{ij} = \psi_{ji}$, and $\mathbf{\Lambda}$ and $\mathbf{\Lambda}(\epsilon)$ are, respectively, diagonal matrices with elements $l_i\ \delta_{ij}$ and $(l_i + \epsilon)\ \delta_{ij}$.

The inverse of the second line of (13) is

$$\mathbf{U}^\dagger \mathbf{G}(\epsilon) \mathbf{U} = \mathbf{\Lambda}^{-1}(\epsilon)$$

where $\mathbf{\Lambda}^{-1}(\epsilon)$ has elements $(l_i + \epsilon)^{-1}\delta_{ij}$. It follows that we have

$$\mathbf{G}(\epsilon) = \mathbf{U} \mathbf{\Lambda}^{-1}(\epsilon) \mathbf{U}^\dagger$$

or, explicitly,

$$\begin{aligned} G_{\alpha\beta}(\epsilon) &= \sum_{i=1}^{\mathcal{N}} U_{\alpha\, i} \left(\frac{1}{l_i + \epsilon} \right) U^*_{\beta\, i} \\ &= \frac{1}{\mathcal{N}\epsilon} + g_{\alpha\beta}(\epsilon) \end{aligned} \tag{14}$$

where

$$g_{\alpha\beta}(\epsilon) = \sum_{i=2}^{\mathcal{N}} \frac{\psi_{i\alpha}\psi^*_{i\beta}}{l_i + \epsilon}. \tag{15}$$

The substitution of (14) into (12) now yields, after making use of the constraint (1),

$$V_i(\epsilon) = \sum_{j=1}^{\mathcal{N}} g_{ij}(\epsilon) I_j.$$

It is now safe to take the $\epsilon \to 0$ limit to obtain

$$V_i = \sum_{j=1}^{\mathcal{N}} g_{ij}(0) I_j. \tag{16}$$

Finally, by combining (6), (7) with (16) we obtain

$$R_{\alpha\beta} = g_{\alpha\alpha}(0) + g_{\beta\beta}(0) - g_{\alpha\beta}(0) - g_{\beta\alpha}(0)$$

which becomes (11) after introducing (15). QED •

The usefulness of (11) is illustrated by the following examples.

Example 1: Consider the 4-node network shown in Fig. 1 with the Laplacian

$$\mathbf{L} = \begin{pmatrix} 2c_1 & -c_1 & 0 & -c_1 \\ -c_1 & 2c_1 + c_2 & -c_1 & -c_2 \\ 0 & -c_1 & 2c_1 & -c_1 \\ -c_1 & -c_2 & -c_1 & 2c_1 + c_2 \end{pmatrix},$$

where $c_1 = 1/r_1, c_2 = 1/r_2$. The nonzero eigenvalues of $\mathbf{L}$ and their orthonormal eigenvectors are

$$\begin{aligned} l_2 &= 4c_1, & \Psi_2 &= \frac{1}{2}(1, -1, 1, -1) \\ l_3 &= 2c_1, & \Psi_3 &= \frac{1}{\sqrt{2}}(-1, 0, 1, 0) \\ l_4 &= 2(c_1 + c_2), & \Psi_4 &= \frac{1}{\sqrt{2}}(0, -1, 0, 1). \end{aligned}$$

Using (11), we obtain

$$\begin{aligned} R_{13} &= \frac{1}{l_2}\Big(\psi_{21} - \psi_{23}\Big)^2 + \frac{1}{l_3}\Big(\psi_{31} - \psi_{33}\Big)^2 + \frac{1}{l_4}\Big(\psi_{41} - \psi_{43}\Big)^2 = r_1, \\ R_{12} &= \frac{1}{l_2}\Big(\psi_{21} - \psi_{22}\Big)^2 + \frac{1}{l_3}\Big(\psi_{31} - \psi_{32}\Big)^2 + \frac{1}{l_4}\Big(\psi_{41} - \psi_{42}\Big)^2 = \frac{r_1(3r_1 + 2r_2)}{4(r_1 + r_2)}. \end{aligned}$$

Example 2: We consider $\mathcal{N}(\mathcal{N}-1)/2$ resistors of equal resistance r on a complete graph of $\mathcal{N}$ nodes. A complete graph is a network in which every node is connected to every other node. The Laplacian is therefore

$$\mathbf{L}^{\text{complete graph}} = r^{-1}\begin{pmatrix} \mathcal{N}-1 & -1 & \cdots & -1 & -1 \\ -1 & \mathcal{N}-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & \mathcal{N}-1 & -1 \\ -1 & -1 & \cdots & -1 & \mathcal{N}-1 \end{pmatrix}. \qquad (17)$$

It is readily verified that the Laplacian (17) has eigenvalues $\lambda_0 = 0$ and $\lambda_n = \mathcal{N}r^{-1}, n = 1, 2, ..., \mathcal{N}-1$, with corresponding eigenvectors Ψ_n having components

$$\psi_{n\alpha} = \frac{1}{\sqrt{\mathcal{N}}}\exp\left(i\,\frac{2\pi n\alpha}{\mathcal{N}}\right), \qquad n, \alpha = 0, 1, ..., \mathcal{N}-1.$$

It follows from (11) the resistance between any two nodes α and β is

$$R_{\alpha,\beta} \;=\; r\sum_{n=1}^{\mathcal{N}} \frac{\left|\psi_{n\alpha}-\psi_{n\beta}\right|^2}{\mathcal{N}} = \frac{2}{\mathcal{N}}\,r. \qquad (18)$$

In subsequent sections we consider applications of (11) to regular lattices.

3 One-dimensional lattice

It is instructive to first consider the one-dimensional case of a linear array of resistors. We consider free and periodic boundary conditions separately.

Free boundary condition:

Consider $N-1$ resistors of resistance r each connected in series forming a chain of N nodes numbered from $0, 1, 2, \cdots$, to $N-1$ as shown in Fig. 2, where each of the two end nodes connects to only one interior node. This is the Neumann (or the free) boundary condition. The Laplacian (4) assumes the form

$$\mathbf{L}^{\text{free}}_{\{N\times 1\}} = r^{-1}\,\mathbf{T}^{\text{free}}_N$$

where $\mathbf{T}_N^{\mathrm{free}}$ is the $N \times N$ matrix

$$\mathbf{T}_N^{\mathrm{free}} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \tag{19}$$

The eigenvalues and eigenvectors of $\mathbf{T}_N$ can be readily computed (see, for example, [11]), and are

$$\begin{aligned} \lambda_n &= 2\Big(1-\cos\phi_n\Big), && n = 0,1,\cdots,N-1 \\ \psi_{nx}^{(N)} &= \frac{1}{\sqrt{N}}, && n=0, \ \text{all } x \\ &= \sqrt{\frac{2}{N}}\cos\Big((x+1/2)\phi_n\Big), && n = 1,2,\cdots,N-1, \ \text{all } x, \end{aligned} \tag{20}$$

where $\phi_n = n\pi/N$. Thus, using (11), the resistance between nodes x_1 and x_2 is

$$\begin{aligned} R_{\{N\times 1\}}^{\mathrm{free}}(x_1,x_2) &= \frac{r}{N}\sum_{n=1}^{N-1}\frac{\Big[\cos\big(x_1+\frac12\big)\phi_n - \cos\big(x_2+\frac12\big)\phi_n\Big]^2}{1-\cos\phi_n} \\ &= r\Big[F_N(x_1+x_2+1) + F_N(x_1-x_2) \\ &\qquad -\frac12 F_N(2x_1+1) - \frac12 F_N(2x_2+1)\Big], \end{aligned} \tag{21}$$

where

$$F_N(\ell) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos(\ell\phi_n)}{1-\cos\phi_n}. \tag{22}$$

Note that without loss of generality we can take $0 \geq \ell < 2N$.

The function $F_N(\ell)$ can be evaluated by taking the limit of $l \to 0$ of the function $I_1(0) - I_1(\ell)$ evaluated in (61) below. It is however instructive to evaluate $F_N(\ell)$ directly. To do this we consider the real part of the summation

$$T_N(\ell) \equiv \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-e^{i\ell\phi_n}}{1-e^{i\phi_n}}.$$

First, writing out the real part of the summand, we obtain

$$\begin{aligned} \mathcal{R}e\, T_N(\ell) &= \frac{1}{N}\sum_{n=1}^{N-1} \frac{1-\cos\phi_n - \cos\ell\phi_n + \cos(\ell-1)\phi_n}{2(1-\cos\phi_n)} \\ &= \frac{1}{2}\Big[F_N(1) + F_N(\ell) - F_N(\ell-1)\Big]. \end{aligned} \tag{23}$$

Second, we expand the summand to obtain

$$T_N(\ell) = \frac{1}{N}\sum_{n=1}^{N-1}\sum_{\ell'=0}^{\ell-1} e^{i\pi n\ell'/N}$$

and carry out the summation over n. The term $\ell' = 0$ yields $F_N(1)$ and terms $\ell' \geq 1$ can be explicitly summed, leading to

$$T_N(\ell) \;=\; F_N(1) + \frac{1}{N}\sum_{\ell'=1}^{\ell-1}\left[\frac{1-(-1)^{\ell'}}{1-e^{i\pi\ell'/N}} - 1\right], \quad \ell < 2N. \tag{24}$$

We now evaluate the real part of $T_N(\ell)$ giving by (24). Using the identity

$$\mathcal{R}e\left(\frac{1}{1-e^{i\theta}}\right) = \frac{1}{2}, \qquad 0 < \theta < 2\pi, \tag{25}$$

we find

$$\mathcal{R}e\, T_N(\ell) = F_N(1) - \frac{1}{4N}\Big[2\ell - 3 - (-1)^{\ell}\Big]. \tag{26}$$

Equating (23) with (26) and noting $F_N(1) = 1 - 1/N$, we are led to the recursion relation

$$F_N(\ell) - F_N(\ell-1) = 1 - \frac{1}{2N}\Big[2\ell - 1 - (-1)^{\ell}\Big],$$

which can be solved (Cf. section 9 below) to yield

$$F_N(\ell) = |\ell| - \frac{1}{N}\left(\frac{\ell^2 + |\ell|}{2} - \left[\frac{|\ell|}{2}\right]\right) \tag{27}$$

where $[x]$ denotes the integral part of x. The substitution of (27) into (21) now gives the answer

$$R^{\text{free}}_{\{N\times 1\}}(x_1, x_2) = r\,|x_1 - x_2| \tag{28}$$

which is the expected expression.

Periodic boundary conditions:

Consider next periodic boundary conditions for which nodes 0 and $N-1$ are also connected as shown in Fig. 3. The Laplacian (4) of the lattice is therefore $\mathbf{L}^{\rm per}_{\{N\times 1\}} = r^{-1}\,\mathbf{T}^{\rm per}_N$ where

$$\mathbf{T}^{\rm per}_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{29}$$

The eigenvalues and eigenfunctions of $\mathbf{T}^{\rm per}_N$ are well-known and are, respectively,

$$\begin{aligned} \lambda_n &= 2\Big(1-\cos 2\phi_n\Big), \\ \psi^{\rm per}_{nx} &= \frac{1}{\sqrt{N}}\, e^{i2x\phi_n}, \qquad\quad n, x = 0, 1, \cdots, N-1. \end{aligned} \tag{30}$$

Substituting (30) into (11), we obtain the resistance between nodes x_1 and x_2

$$\begin{aligned} R^{\rm per}_{\{N\times 1\}}(x_1,x_2) &= \frac{r}{N}\sum_{n=1}^{N-1} \frac{\Big|e^{i2x_1\phi_n} - e^{i2x_2\phi_n}\Big|^2}{2(1-\cos 2\phi_n)} \\ &= r\,G_N(x_1-x_2) \end{aligned} \tag{31}$$

where

$$G_N(\ell) = \frac{1}{N}\sum_{n=1}^{N-1} \frac{1-\cos(2\ell\phi_n)}{1-\cos 2\phi_n}.$$

The function $G_N(\ell)$ is again evaluated in a special case of the identity (62) below. Following analyses in section 9, one obtains the recursion relation

$$G_N(\ell) - G_N(\ell-1) = 1 - \frac{1}{N}(2\ell-1)$$

which can be solved to yield

$$G_N(\ell) = |\ell| - \ell^2/N. \tag{32}$$

It follows that we have

$$R^{\rm per}_{\{N\times 1\}}(x_1,x_2) = r\,|x_1-x_2|\bigg[1-\frac{|x_1-x_2|}{N}\bigg]. \tag{33}$$

The expression (33) is the expected resistance of two resistors $|x_1-x_2|\,r$ and $(N-|x_1-x_2|)\,r$ connected in parallel as in a ring.

4 Two-dimensional network: Free boundaries

Consider a rectangular network of resistors connected in an array of $M\times N$ nodes forming a network with free boundaries as shown in Fig. 4.

Number the nodes by coordinates $\{m,n\}$, $0\le m\le M-1$, $0\le n\le N-1$ and denote the resistances along the two principal directions by r and s. The Laplacian is therefore

$$\mathbf{L}^{\rm free}_{\{M\times N\}} = r^{-1}\mathbf{T}^{\rm free}_M\otimes\mathbf{I}_N + s^{-1}\mathbf{I}_M\otimes\mathbf{T}^{\rm free}_N \tag{34}$$

where $\otimes$ denotes direct matrix products and $\mathbf{T}^{\rm free}_N$ is given by (19). The Laplacian can be diagonalized in the two subspaces separately, yielding eigenvalues

$$\lambda_{(m,n)} = 2r^{-1}(1-\cos\theta_m)+2s^{-1}(1-\cos\phi_n), \tag{35}$$

and eigenvectors

$$\psi^{\rm free}_{(m,n);(x,y)} = \psi^{(M)}_{mx}\psi^{(N)}_{ny}. \tag{36}$$

It then follows from (11) that the resistance $R_{\rm free}$ between two nodes $\mathbf{r}_1=(x_1,y_1)$ and $\mathbf{r}_2=(x_2,y_2)$ is

$$\begin{aligned}
R^{\rm free}_{\{M\times N\}}(\mathbf{r}_1,\mathbf{r}_2) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}{}_{(m,n)\neq(0,0)}\frac{\Big|\psi^{\rm free}_{(m,n);(x_1,y_1)}-\psi^{\rm free}_{(m,n);(x_2,y_2)}\Big|^2}{\lambda_{(m,n)}}\\
&= \frac{r}{N}\Big|x_1-x_2\Big| + \frac{s}{M}\Big|y_1-y_2\Big| + \frac{2}{MN}\\
&\times\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\frac{\Big[\cos\Big(x_1+\frac12\Big)\theta_m\cos\Big(y_1+\frac12\Big)\phi_n-\cos\Big(x_2+\frac12\Big)\theta_m\cos\Big(y_2+\frac12\Big)\phi_n\Big]^2}{r^{-1}(1-\cos\theta_m)+s^{-1}(1-\cos\phi_n)},
\end{aligned} \tag{37}$$

where

$$\theta_m = \frac{m\pi}{M}, \qquad \phi_n = \frac{n\pi}{N}.$$

Here, use has been made of (28) for summing over the $m = 0$ and $n = 0$ terms. The resulting expression (37) now depends on the coordinates x_1, y_1, x_2 and y_2 individually.

The usefulness of (37) is best illustrated by applications. Several examples are now given.

Example 3: For $M = 5, N = 4, r = s$, the resistance between nodes $\{0, 0, \}$ and $\{3, 3\}$ is computed from (37) as

$$\begin{aligned} R^{\text{free}}_{\{5\times 4\}}(\{0,0,\},\{3,3\}) &= \left(\frac{3}{4} + \frac{3}{5} + \frac{9877231}{27600540}\right) r \\ &= (1.70786...)\, r. \end{aligned} \tag{38}$$

Example 4: For $M = N = 4$, we find

$$R^{\text{free}}_{\{4\times 4\}}(\{0,0\},\{3,3\}) = \frac{(r+s)(r^2+5rs+s^2)(3r^2+7rs+3s^2)}{2(2r^2+4rs+s^2)(r^2+4rs+2s^2)}. \tag{39}$$

Example 5: We evaluate the resistance between two nodes in the interior of a large lattice. Consider, for definiteness, both $M, N =$ odd (for other parities the result (40) below is the same) and compute the resistance between two nodes in the center region,

$$\begin{aligned} \mathbf{r}_1 &= (x_1, y_1) = \Big(\frac{M-1}{2} + p_1,\ \frac{N-1}{2} + q_1\Big) \\ \mathbf{r}_2 &= (x_2, y_2) = \Big(\frac{M-1}{2} + p_2,\ \frac{N-1}{2} + q_2\Big), \end{aligned}$$

where $p_i, q_i << M, N$ are integers. The numerator of the summand in (37) becomes

$$\begin{aligned} \Big[\cos\Big(\frac{m\pi}{2} &+ p_1\theta_m\Big)\cos\Big(\frac{n\pi}{2} + q_1\phi_n\Big) - \cos\Big(\frac{m\pi}{2} + p_2\theta_m\Big)\cos\Big(\frac{n\pi}{2} + q_2\phi_n\Big)\Big]^2 \\ &= \Big(\cos p_1\theta_m \cos q_1\phi_n - \cos p_2\theta_m \cos q_2\phi_n\Big)^2, \quad m, n = \text{even} \\ &= \Big(\sin p_1\theta_m\ \sin q_1\phi_n - \sin p_2\theta_m\ \sin q_2\phi_n\Big)^2, \quad m, n = \text{odd} \\ &= \Big(\sin p_1\theta_m\ \cos q_1\phi_n - \sin p_2\theta_m\ \cos q_2\phi_n\Big)^2, \quad m = \text{odd}, n = \text{even} \\ &= \Big(\cos p_1\theta_m\ \sin q_1\phi_n - \cos p_2\theta_m\ \sin q_2\phi_n\Big)^2, \quad m = \text{even}, n = \text{odd} \end{aligned}$$

and the summation in (37) breaks into four parts. In the $M, N \to \infty$ limit the summations can be replaced by integrals. After some reduction we arrive at the expression

$$R_\infty(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \left(\frac{1 - \cos\Big(x_1 - x_2)\theta \, \cos(y_1 - y_2)\phi}{r^{-1}(1 - \cos\theta) + s^{-1}(1 - \cos\phi)} \right). \tag{40}$$

Our expression (40) agrees with the known expression obtained previously [1]. It can be verified that the expression (40) holds between any two nodes in the lattice, provided that the two nodes are far from the boundaries.

5 Two-dimensional network: periodic boundary conditions

We next consider an $M \times N$ network with periodic boundary conditions. The Laplacian in this case is

$$\mathbf{L}^{\mathrm{per}}_{\{M\times N\}} = r^{-1}\mathbf{T}^{\mathrm{per}}_M \otimes \mathbf{I}_N + s^{-1}\mathbf{I}_M \otimes \mathbf{T}^{\mathrm{per}}_N \tag{41}$$

where $\mathbf{T}^{\mathrm{per}}_N$ is given by (29). The Laplacian (41) can again be diagonalized in the two subspace separately, yielding eigenvalues and eigenvectors

$$\begin{aligned} \lambda_{(m,n)} &= 2r^{-1}(1 - \cos 2\theta_m) + 2s^{-s}(1 - \cos 2\phi_n) \\ \psi_{(m,n);(x,y)} &= \frac{1}{\sqrt{MN}} e^{i2x\theta_m} e^{i2y\phi_n}. \end{aligned} \tag{42}$$

This leads to the resistance between nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$

$$\begin{aligned} R^{\mathrm{per}}_{\{M\times N\}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1}{}_{(m,n)\neq(0,0)} \frac{\Big|\psi_{(m,n);(x_1,y_1)} - \psi_{(m,n);(x_2,y_2)}\Big|^2}{\lambda_{(m,n)}} \\ &= \frac{r}{N}\left[\Big|x_1 - x_2\Big| - \frac{(x_1 - x_2)^2}{M} \right] + \frac{s}{M}\left[\Big|y_1 - y_2\Big| - \frac{(y_1 - y_2)^2}{N} \right] \\ &+ \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1 - \cos\Big[2(x_1 - x_2)\theta_m + 2(y_1 - y_2)\phi_n\Big]}{r^{-1}(1 - \cos 2\theta_m) + s^{-1}(1 - \cos 2\phi_n)}, \end{aligned} \tag{43}$$

where the two terms in the second line are given by (33). It is clear that the result depends only on the differences $|x_1 - x_2|$ and $|y_1 - y_2|$, as it should under periodic boundary conditions.

Example 6: Using (43) the resistance between nodes $\{0, 0\}$ and $\{3, 3\}$ on a 5×4 periodic lattice with $r = s$ is

$$\begin{aligned} R^{\rm per}_{\{5\times 4\}}\Big(\{0,0\},\{3,3\}\Big) &= \Big(\frac{3}{10} + \frac{3}{20} + \frac{1799}{7790}\Big) r \\ &= (0.680937...)\, r. \end{aligned} \tag{44}$$

This is to be compared to the value $1.707863...\,r$ for free boundary conditions given in Example 3. It can also be verified that the resistance between nodes $(\{0, 0\})$ and $(\{2, 1\})$ is also given by (44) as it must for a periodic lattice.

In the limit of $M, N \to \infty$ with $|\mathbf{r}_1 - \mathbf{r}_2|$ finite, (43) becomes

$$\begin{aligned} R_\infty(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \frac{1 - \cos\Big[(x_1 - x_2)\theta + (y_1 - y_2)\phi\Big]}{r^{-1}(1-\cos\theta) + s^{-1}(1-\cos\phi)} \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \frac{1 - \cos(x_1 - x_2)\theta\cos(y_1 - y_2)\phi}{r_1^{-1}(1-\cos\theta) + s^{-1}(1-\cos\phi)}, \end{aligned} \tag{45}$$

which agrees with (40).

6 Cylindrical boundary conditions

Consider an $M \times N$ resistor network embedded on a cylinder with periodic boundary in the direction of M and free boundaries in the direction of N.

The Laplacian is

$$\mathbf{L}^{\rm cyl}_{\{M\times N\}} = r^{-1}\,\mathbf{T}^{\rm per}_M \otimes \mathbf{I}_N + s^{-1}\,\mathbf{I}_M \otimes \mathbf{T}^{\rm free}_N$$

which can again be diagonalized in the two subspaces separately. This gives the eigenvalues and eigenvectors

$$\begin{aligned} \lambda_{(m,n)} &= 2r^{-1}(1 - \cos 2\theta_m) + 2s^{-1}(1 - \cos\phi_n), \\ \psi^{\rm cyl}_{(m,n);(x,y)} &= \frac{1}{\sqrt{M}}\, e^{i2x\theta_m}\psi^{(N)}_{ny}. \end{aligned}$$

It follows that the resistance $R_{\rm free}$ between nodes $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ is

$$R^{\rm cyl}_{\{M\times N\}}(\mathbf{r}_1,\mathbf{r}_2) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}{}_{(m,n)\neq(0,0)} \frac{\Big|\psi^{\rm cyl}_{(m,n);(x_1,y_1)} - \psi^{\rm cyl}_{(m,n);(x_2,y_2)}\Big|^2}{\lambda_{(m,n)}}$$
$$= \frac{r}{N}\bigg[\Big|x_1-x_2\Big| - \frac{(x_1-x_2)^2}{M}\bigg] + \frac{s}{M}\Big|y_1-y_2\Big|$$
$$+\frac{2}{MN}\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\bigg(\frac{{C_1}^2+{C_2}^2-2C_1C_2\cos 2(x_1-x_2)\theta_m}{r^{-1}(1-\cos 2\theta_m)+s^{-1}(1-\cos\phi_n)}\bigg),$$

where

$$C_1 = \cos\Big(y_1+\frac{1}{2}\Big)\phi_n, \qquad C_2 = \cos\Big(y_2+\frac{1}{2}\Big)\phi_n. \tag{46}$$

It can be verified that in the $M, N \to \infty$ limit (46) leads to the same expression (40) for two interior nodes in an infinite lattice.

Example 7: The resistance between nodes $\{0,0\}$ and $\{3,3\}$ on a 5×4 cylindrical lattice with $r = s$ is computed to be

$$\begin{aligned} R^{\rm cyl}_{\{5\times4\}}\Big(\{0,0\},\{3,3\}\Big) &= \Big(\frac{3}{10}+\frac{3}{5}+\frac{5023}{8835}\Big)r \\ &= (1.46853...)\,r. \end{aligned} \tag{47}$$

This is compared to the values of $(1.70786...)\,r$ for free boundary conditions and $(0.680937...)\,r$ for periodic boundary conditions.

7 Möbius strip

We next consider an $M\times N$ resistor lattice embedded on a Möbius strip of width N and length M, which is a rectangular strip connected at two ends after a 180^o twist of one of the two ends of the strip. The schematic figure of a Möbius strip is shown in Fig. 5(a). The Laplacian for this lattice assumes the form

$$\mathbf{L}^{\rm Mob}_{\{M\times N\}} = r^{-1}\Big[\mathbf{H_M}\otimes\mathbf{I}_N - \mathbf{K}_M\otimes\mathbf{J}_N\Big] + s^{-1}\mathbf{I}_M\otimes\mathbf{T}^{\rm free}_N \tag{48}$$

where

$$
\begin{aligned}
\mathbf{K}_N &= \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \mathbf{J}_N = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \\
\mathbf{H}_N &= \mathbf{T}_N^{\text{per}} + \mathbf{K}_N = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}
\end{aligned}
$$

are $N \times N$ matrices. Now $\mathbf{T}_N^{\text{free}}$ and $\mathbf{J}_N$ commute so they can be replaced by their respective eigenvalues $2(1 - \cos\phi_n)$ and $(-1)^n$ and we need only to diagonalize an $M \times M$ matrix. This leads to the following eigenvalues and eigenvectors of the Laplacian (48) [11, 12]:

$$
\begin{aligned}
\lambda_{(m,n)} &= 2r^{-1} \cos\left[\Big(4m + 1 - (-1)^n\Big)\frac{\pi}{2M}\right] + 2s^{-1}\Big(1 - \cos\frac{n\pi}{N}\Big), \\
\psi^{\text{Mob}}_{(m,n);(x,y)} &= \frac{1}{\sqrt{M}} \exp\Big[i\Big(4m + 1 - (-1)^n\Big)\frac{x\pi}{2M}\Big] \cdot \psi^{(N)}_{ny}
\end{aligned} \tag{49}
$$

where $\psi^{(N)}_{ny}$ is given in (20). Substituting these expressions into (11) and after a little reduction, we obtain

$$
\begin{aligned}
R^{\text{Mob}}_{\{M\times N\}}(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1}{}_{(m,n)\neq(0,0)} \frac{\Big|\psi^{\text{Mob}}_{(m,n);(x_1,y_1)} - \psi^{\text{Mob}}_{(m,n);(x_2,y_2)}\Big|^2}{\lambda_{(m,n)}} \\
&= \frac{r}{N}\left[\Big|x_1 - x_2\Big| - \frac{(x_1 - x_2)^2}{M}\right] \\
+\frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} & \frac{{C_1}^2 + {C_2}^2 - 2C_1C_2 \cos\left[(x_1 - x_2)(4m + 1 - (-1)^n)\frac{\pi}{2M}\right]}{r^{-1}\Big[1 - \cos\big(4m + 1 - (-1)^n\big)\frac{\pi}{2M}\Big] + s^{-1}(1 - \cos\phi_n)},
\end{aligned} \tag{50}
$$

where C_1 and C_2 have been given in (46).

Example 8: The 2×2 Möbius strip is a complete graph of $\mathcal{N} = 4$ nodes. For $r = s$ the expression (50) gives a resistance $r/2$ between any two nodes which agrees with (18).

Example 9: The resistance between nodes $(0, 0)$ and $(3, 3)$ on a 5×4 Möbius strip with $r = s$ is computed from (50) as

$$\begin{aligned} R^{\rm Mob}_{\{5\times 4\}}\Big(\{0,0\},\{3,3\}\Big) &= \Big(\frac{3}{10}+\frac{1609}{2698}\Big) r \\ &= (0.896367...)\, r. \end{aligned} \tag{51}$$

This is to be compared to the corresponding values for the same network under other boundary conditions in Examples $3, 6$, and 7.

8 Klein bottle

A Klein bottle is a Möbius strip with a periodic boundary condition imposed in the other direction. We consider an $M \times N$ resistor grid embedded on a Klein bottle, a schematic figure of which is shown in Fig. 5(b).

Let the network have a twisted boundary condition in the direction of the length M and a periodic boundary condition in the direction of the width N. Then, in analogous to (48), the Laplacian of the network assumes the form

$$\mathbf{L}^{\rm Klein}_{\{M\times N\}} = r^{-1}\Big[\mathbf{H_M}\otimes \mathbf{I}_N - \mathbf{K}_M \otimes \mathbf{J}_N\Big] + s^{-1}\mathbf{I}_M \otimes \mathbf{T}^{\rm per}_N. \tag{52}$$

Now the matrices $\mathbf{J}_N$ and $\mathbf{T}^{\rm per}_N$ commute so they can be replaced by their respective eigenvalues ± 1 and $2(1 - \cos 2\phi_n)$ in (52) and one needs only to diagonalize an $M \times M$ matrix. This leads to the following eigenvalues and eigenvectors for $\mathbf{L}^{\rm Klein}_{\{M\times N\}}$ [11, 12]:

$$\begin{aligned} \lambda_{(m,n)}(\tau) &= 2r^{-1}\bigg[1-\cos\Big((2m+\tau)\frac{\pi}{M}\Big)\bigg] + 2s^{-1}\Big(1-\cos\frac{2n\pi}{N}\Big), \\ \psi^{\rm Klein}_{(m,n);(x,y)} &= \frac{1}{\sqrt{M}}\exp\Big[i\Big(2m+\tau\Big)\frac{x\pi}{M}\Big]\cdot \psi^{(N)\dagger}_{ny}, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \tau = \tau_n &= 0, \qquad n = 0, 1, \cdots, \Big[\frac{N-1}{2}\Big], \\ &= 1, \qquad n = \Big[\frac{N+1}{2}\Big], \cdots, N-1, \end{aligned}$$

and

$$
\begin{aligned}
\psi_{ny}^{(N)\dagger} &= \frac{1}{\sqrt{N}}, && n=0, \\
&= \sqrt{\frac{2}{N}}\cos\Big[(2y+1)\frac{n\pi}{N}\Big], && n=1,2,\cdots,\Big[\frac{N-1}{2}\Big], \\
&= \frac{1}{\sqrt{N}}(-1)^y && n=\frac{N}{2}, \quad \text{for even } N \text{ only}, \\
&= \sqrt{\frac{2}{N}}\sin\Big[(2y+1)\frac{n\pi}{N}\Big], && n=\Big[\frac{N}{2}\Big]+1,\cdots,N-1.
\end{aligned}
$$

Substituting these expressions in (11), separating out the summation for $n=0$, and making use of the identity

$$
\sin\Big[(2y+1)\Big(\frac{N}{2}+n\Big)\frac{\pi}{N}\Big] = (-1)^y\cos\Big[(2y+1)\frac{n\pi}{N}\Big],
$$

we obtain after some reduction

$$
\begin{aligned}
R^{\text{Klein}}_{\{M\times N\}}(\mathbf{r}_1,\mathbf{r}_2) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}{}_{(m,n)\neq(0,0)} \frac{\Big|\psi^{\text{Klein}}_{(m,n);(x_1,y_1)}-\psi^{\text{Klein}}_{(m,n);(x_2,y_2)}\Big|^2}{\lambda_{(m,n)}(\tau_n)} \\
&= \frac{r}{N}\Bigg[\Big|x_1-x_2\Big|-\frac{(x_1-x_2)^2}{M}\Bigg] \\
&\quad + \sum_{m=0}^{M-1}\sum_{n=1}^{N-1}\frac{1}{l_{(m,n)}(\tau_n)}\Big|\psi^{\text{Klein}}_{(m,n);(x_1,y_1)}-\psi^{\text{Klein}}_{(m,n);(x_2,y_2)}\Big|^2 \\
&= \frac{r}{N}\Bigg[\Big|x_1-x_2\Big|-\frac{(x_1-x_2)^2}{M}\Bigg]+\Delta_N \\
&\quad + \frac{2}{MN}\sum_{\tau=0}^{1}\sum_{m=0}^{M-1}\sum_{n=1}^{[\frac{N-1}{2}]}\frac{C_1^2+C_2^2-2(-1)^{(y_1-y_2)\tau}C_1C_2\cos\big[\,2(x_1-x_2)\Theta_m(\tau)\,\big]}{l_{(m,n)}(\tau)},
\end{aligned}
\tag{54}
$$

where

$$
\begin{aligned}
\Theta_m(\tau) &= \Big(m+\frac{\tau}{2}\Big)\frac{\pi}{M}\ , \\
\Delta_N &= \frac{2}{MN}\sum_{m=0}^{M-1}\frac{1-(-1)^{y_1-y_2}\cos\big[\,2(x_1-x_2)\Theta_m(1)\,\big]}{\lambda_{(m,N/2)}(1)}, \quad N=\text{even} \\
&= 0, \qquad N=\text{odd},
\end{aligned}
\tag{55}
$$

and $C_i = \cos[(y_i + 1/2)n\pi/N], i = 1, 2,$ as defined in (46).

Example 10: The resistance between nodes $(0, 0)$ and $(3, 3)$ on a 5×4 ($N =$ even) Klein bottle with $r = s$ is computed from (54) as

$$R^{\rm Klein}_{\{5\times4\}}\Big(\{0,0\},\{3,3\}\Big) = \Big(\frac{3}{10}+\frac{5}{58}+\frac{56}{209}\Big)r = (0.654149...)\, r, \qquad (56)$$

where the three terms in the first line are from the evaluation of corresponding terms in (54). The result is to be compared to the corresponding value for the same 5×4 network under the Möbius boundary condition considered in Example 9, which is the Klein bottle without periodic boundary connections.

9 Higher-dimensional lattices

The two-point resistance can be computed using (11) for lattices in any spatial dimensionality under various boundary conditions. To illustrate, we give the result for an $M \times N \times L$ cubic lattice with free boundary conditions.

Number the nodes by $\{m, n.\ell\}$, $0 \le m \le M-1$, $0 \le n \le N-1, 0 \le \ell \le L-1$, and let the resistances along the principal axes be, respectively, r, s, and t. The Laplacian then assumes the form

$$\mathbf{L}^{\rm free}_{\{M\times N\times L\}} = r^{-1}\mathbf{T}^{\rm free}_M \otimes \mathbf{I}_N \otimes \mathbf{I}_L + s^{-1}\mathbf{I}_M \otimes \mathbf{T}^{\rm free}_N \otimes \mathbf{I}_L + t^{-1}\mathbf{I}_M \otimes \mathbf{I}_N \otimes \mathbf{T}^{\rm free}_L$$

where $\mathbf{T}^{\rm free}_N$ is given by (19). The Laplacian can be diagonalized in the three subspaces separately, yielding eigenvalues

$$\lambda_{(m,n,\ell)} = 2\, r^{-1}(1-\cos\theta_m) + 2\, s^{-1}(1-\cos\phi_n) + 2\, t^{-1}(1-\cos\alpha_\ell), \qquad (57)$$

and eigenvectors

$$\psi^{\rm free}_{(m,n,\ell);(x,y,z)} = \psi^{(M)}_{mx}\psi^{(N)}_{ny}\psi^{(L)}_{\ell z}$$

where $\psi^{(M)}_{mx}$ is given by (20) and $\alpha_\ell = \ell\pi/L$. It then follows from (11) that the resistance $R_{\rm free}$ between two nodes $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ is

$$R^{\rm free}_{\{M\times N\times L\}}(\mathbf{r}_1,\mathbf{r}_2) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{\substack{\ell=0\\(m,n,\ell)\neq(0,0,0)}}^{L-1} \lambda^{-1}_{(m,n,\ell)} \times \Big|\psi^{\rm free}_{(m,n,\ell);(x_1,y_1,z_1)} - \psi^{\rm free}_{(m,n,\ell);(x_2,y_2,z_2)}\Big|^2 \qquad (58)$$

The summation can be broken down as

$$
\begin{aligned}
R^{\rm free}_{\{M\times N\times L\}}(\mathbf{r}_1,\mathbf{r}_2) &= \sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\sum_{\ell=1}^{L-1}\frac{\Big|\psi^{\rm free}_{(m,n,\ell);(x_1,y_1,z_1)}-\psi^{\rm free}_{(m,n,\ell);(x_2,y_2,z_2)}\Big|^2}{\lambda_{(m,n,\ell)}}\\
&+\frac{1}{L}R^{\rm free}_{\{M\times N\}}\Big(\{x_1,y_1\},\{x_2,y_2\}\Big)+\frac{1}{M}R^{\rm free}_{\{N\times L\}}\Big(\{y_1,z_1\},\{y_2,z_2\}\Big)\\
&+\frac{1}{N}R^{\rm free}_{\{L\times M\}}\Big(\{z_1,x_1\},\{z_2,x_2\}\Big)\\
&-\frac{1}{MN}R^{\rm free}_{\{L\times 1\}}(x_1,x_2)-\frac{1}{NL}R^{\rm free}_{\{M\times 1\}}(y_1,y_2)-\frac{1}{LM}R^{\rm free}_{\{N\times 1\}}(z_1,z_2). \quad (59)
\end{aligned}
$$

All terms in (59) have previously been computed except the summation in the first line.

Example 11: The resistance between the nodes $(0,0,0)$ and $(3,3,3)$ in a $5\times5\times4$ lattice with free boundaries and $r=s=t$ is computed from (59) as

$$
\begin{aligned}
R^{\rm free}_{\{5\times5\times4\}}(\{0,0,0\};\{3,3,3\}) &= \left(\frac{327687658482872}{352468567489225}\right) r\\
&= (0.929693...)\, r. \qquad (60)
\end{aligned}
$$

Example 12: The resistance between two interior nodes $\mathbf{r}_1$ and $\mathbf{r}_2$ can be worked out as in Example 4. The result is

$$
\begin{aligned}
R_\infty(\mathbf{r}_1,\mathbf{r}_2) &= \frac{1}{(2\pi)^3}\int_0^{2\pi}d\phi\int_0^{2\pi}d\theta\int_0^{2\pi}d\alpha\\
&\times\left(\frac{1-\cos\Big(x_1-x_2)\theta\,\cos(y_1-y_2)\phi\cos(z_1-z_2)\alpha}{r^{-1}(1-\cos\theta)+s^{-1}(1-\cos\phi)+t^{-1}(1-\cos\alpha)}\right),
\end{aligned}
$$

which is the known result [1].

10 Summation and product identities

The reduction of the two-point resistances for one-dimensional lattices to the simple and familiar expressions of (31) and (33) is facilitated by the use of the summation identities (27) and(32). In this section we extend the consideration and generalize these identities which can be used to reduce the computational labor for lattice sums as well as analyze large-size expansions in two-and-higher dimensions.

We state two new lattice sum identities as a Proposition.

Proposition: Define

$$I_\alpha(\ell) = \frac{1}{N}\sum_{n=0}^{N-1} \frac{\cos(\alpha\,\ell\,\frac{n\pi}{N})}{\cosh l - \cos(\alpha\,\frac{n\pi}{N})}, \qquad \alpha = 1, 2.$$

Then the following identities hold for $l \geq 0$, $N = 1, 2, \cdots$,

$$I_1(\ell) = \frac{\cosh(N-\ell)l}{(\sinh l)\sinh(Nl)} + \frac{1}{N}\left[\frac{1}{\sinh^2 l} + \frac{1-(-1)^\ell}{4\cosh^2(l/2)}\right], \; 0 \leq \ell < 2N, \tag{61}$$

$$I_2(\ell) = \frac{\cosh(\frac{N}{2}-\ell)l}{(\sinh l)\sinh(Nl/2)}, \qquad 0 \leq \ell < N. \tag{62}$$

Remarks:

1. It is clear that without the loss of generality we can restrict ℓ to the ranges indicated.

2. For $\ell = 0$ and $l \to 0$, $I_1(0)$ leads to (27) and $I_2(0)$ leads to (32).

3. In the $N \to \infty$ limit both (61) and (62) become the integral

$$\frac{1}{\pi}\int_0^\pi \frac{\cos(\ell\theta)}{\cosh l - \cos\theta} d\,\theta = \frac{e^{-\ell l}}{\sinh l} \qquad \ell \geq 0. \tag{63}$$

4. Set $\ell = 0$ in (61), multiply by $\sinh l$ and integrate over l, we obtain the product identity

$$\prod_{n=0}^{N-1}\left(\cosh l - \cos\frac{n\pi}{N}\right) = (\sinh Nl)\tanh(l/2). \tag{64}$$

5. Set $\ell = 0$ in (62), multiply by $\sinh l$ and integrate over l. We obtain the product identity

$$\prod_{n=0}^{N-1}\left(\cosh l - \cos\frac{2n\pi}{N}\right) = \sinh^2(Nl/2). \tag{65}$$

Proof of the proposition:

It is convenient to introduce the notation

$$S_\alpha(\ell) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{\cos(\ell\,\theta_n)}{1+a^2-2a\cos\theta_n}, \qquad a<1, \quad \alpha = 1,2 \tag{66}$$

so that

$$I_\alpha(\ell) = 2\,a\,S_\alpha(\ell), \qquad a = e^{-l}. \tag{67}$$

It is readily seen that we have the identity

$$S_\alpha(1) = \frac{1}{2a}\Big[(1+a^2)S_\alpha(0) - 1\Big]. \tag{68}$$

1. Proof of (61):

First we evaluate $S_1(0)$ by carrying out the following summation, where $\mathcal{R}e$ denotes the real part, in two different ways. First we have

$$\begin{aligned}
\mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{1-a\,e^{i\theta_n}} &= \mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1-a\,e^{-i\theta_n}}{\left|1-a\,e^{i\theta_n}\right|^2} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\frac{1-a\cos\theta_n}{1+a^2-2a\cos\theta_n} \\
&= S_1(0) - aS_1(1) \\
&= \frac{1}{2}\Big[1+(1-a^2)S_1(0)\Big].
\end{aligned} \tag{69}$$

Secondly by expanding the summand we have

$$\mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{1-a\,e^{i\theta_n}} = \mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\sum_{\ell=0}^{\infty}a^\ell e^{i\ell n\pi/N}$$

and carry out the summation over n for fixed ℓ. It is clear that all $\ell =$ even terms vanish except those with $\ell = 2mN, m = 0,1,2,...$ which yield $\sum_{m=0}^{\infty} a^{2mN} = 1/(1-a^{2N})$. For $\ell = \text{odd} = 2m+1$, $m = 0,1,2,...$ we have

$$\mathcal{R}e\sum_{n=0}^{N-1} e^{i(2m+1)n\pi/N} = \mathcal{R}e\,\frac{1-(-1)^{2m+1}}{1-e^{i(2m+1)\pi/N}} = 1$$

after making use of (25). So the summation over ℓ = odd terms yields $N^{-1}\sum_{m=0}^{\infty} a^{2m+1} = a/N(1-a^2)$, and we have

$$\mathcal{R}e \sum_{n=0}^{N-1} \frac{1}{1-a\,e^{i\theta_n}} = \frac{1}{1-a^{2N}} + \frac{a}{N(1-a^2)} \tag{70}$$

Equating (69) with (70) we obtain

$$S_1(0) = \frac{1}{1-a^2}\left[\left(\frac{1+a^{2N}}{1-a^{2N}}\right) + \frac{2a}{N(1-a^2)}\right]. \tag{71}$$

To evaluate $S_1(\ell)$ for general ℓ, we consider the summation

$$\begin{aligned}\mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1-(a\,e^{i\theta_n})^{\ell}}{1-a\,e^{i\theta_n}} &= \mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{(1-a^{\ell}\,e^{i\ell\theta_n})(1-a\,e^{-i\theta_n})}{|1-a\,e^{i\theta_n}|^2}\\ &= S_1(0) - aS_1(1) - a^{\ell}S_1(\ell) + a^{\ell+1}S_1(\ell-1)\end{aligned} \tag{72}$$

where the second line is obtained by writing out the real part of the summand as in (69). On the other hand, by expanding the summand we have

$$\begin{aligned}\mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1-(a\,e^{i\theta_n})^{\ell}}{1-a\,e^{i\theta_n}} &= \mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{\ell-1} a^m e^{i\pi mn/N}\\ &= 1 + \mathcal{R}e\,\frac{1}{N}\sum_{m=1}^{\ell-1} a^m\left(\frac{1-(-1)^m}{1-e^{i\pi m/N}}\right)\\ &= 1 + \frac{a(1-a^{\ell})}{N(1-a^2)}, \qquad \ell = \text{even} < 2N\\ &= 1 + \frac{a(1-a^{\ell-1})}{N(1-a^2)}, \qquad \ell = \text{odd} < 2N,\end{aligned} \tag{73}$$

where again we have used (25).

Equating (73) with (72) and using (68) and (71), we obtain the recursion relation

$$S_N(\ell) - a\,S_N(\ell-1) = A\,a^{-\ell} + B_{\ell} \tag{74}$$

where

$$A = \frac{a^{2N}}{1-a^{2N}}, \qquad B_{\ell} = \frac{a^{(1+(-1)^{\ell})/2}}{N(1-a^2)}. \tag{75}$$

The recursion relation (74) can be solved by standard means. Define the generating function

$$G_\alpha(t) = \sum_{\ell=0}^{\infty} S_\alpha(\ell)\, t^\ell, \qquad \alpha = 1, 2. \tag{76}$$

Multiply (74) by t^ℓ and sum over ℓ. We obtain

$$(1 - at)G_1(t) - S_1(0) = \frac{A\, a^{-1} t}{1 - a^{-1} t} + \frac{t + at^2}{N(1 - a^2)(1 - t^2)}. \tag{77}$$

This leads to

$$\begin{aligned} G_1(t) &= \frac{1}{1 - at}\left[S_1(0) + \frac{A\, a^{-1} t}{1 - a^{-1} t} + \frac{t + at^2}{N(1 - a^2)(1 - t^2)}\right] \\ &= \frac{1}{(1 - a^2)(1 - a^{2N})}\left[\frac{1}{1 - at} + \frac{a^{2N}}{1 - a^{-1} t}\right] \\ &\quad + \frac{1}{2N(1 - a)^2(1 - t)} - \frac{1}{2N(1 + a)^2(1 + t)}, \end{aligned}$$

from which one obtains

$$\begin{aligned} S_1(\ell) &= \frac{a^\ell + a^{2N-\ell}}{(1 - a^2)(1 - a^{2N})} + \frac{1}{2N(1 - a)^2} - \frac{(-1)^\ell}{2N(1 + a)^2} \\ &= \frac{a^\ell + a^{2N-\ell}}{(1 - a^2)(1 - a^{2N})} + \frac{1}{2N}\left[\frac{4a}{(1 - a^2)^2} + \frac{1 - (-1)^\ell}{(1 + a^2)^2}\right]. \end{aligned} \tag{78}$$

It follows that using $I_1(\ell) = 2\, a\, S_1(\ell)$ we obtain (61) after setting $a = e^{-l}$. QED •

2. Proof of (62):

Again, we first evaluate $S_2(0)$ by carrying out the summation

$$\mathcal{R}e\, \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a\, e^{i2\theta_n}}, \qquad a < 1 \tag{79}$$

in two different ways. First as in (69) we have

$$\mathcal{R}e\, \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{1 - a\, e^{i2\theta_n}} = \frac{1}{2}\Big[1 + (1 - a^2) S_2(0)\Big], \tag{80}$$

where $S_2(\ell)$ is defined in (66). Secondly by expanding the summand we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{1-a\,e^{i2\theta_n}}=\frac{1}{N}\sum_{n=0}^{N-1}\sum_{\ell=0}^{\infty}a^{\ell}e^{i2\ell n\pi/N}=\frac{1}{1-a^N} \tag{81}$$

where by carrying out the summation over n for fixed ℓ all terms in (70) vanish except those with $\ell = mN, m = 0, 1, 2, ...$ Equating (81) with (80) we obtain

$$S_2(0)=\frac{1}{1-a^2}\left(\frac{1+a^N}{1-a^N}\right) \tag{82}$$

and from (68)

$$S_2(1)=\frac{1}{1-a^N}.$$

We consider next the summation

$$\mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1-(a\,e^{i2\theta_n})^{\ell}}{1-a\,e^{i2\theta_n}}\qquad a<1. \tag{83}$$

Evaluating the real part of the summand directly as in (72), we obtain

$$\mathcal{R}e\,\frac{1}{N}\sum_{n=0}^{N-1}\frac{1-(a\,e^{i2\theta_n})^{\ell}}{1-a\,e^{i2\theta_n}}=S_2(0)-aS_2(1)-a^{\ell}S_2(\ell)+a^{\ell+1}S_2(\ell-1). \tag{84}$$

Secondly, expanding the summand in (83) we obtain

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\frac{1-(a\,e^{i2\theta_n})^{\ell}}{1-a\,e^{i2\theta_n}} &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{\ell-1}a^m e^{i2\pi mn/N}\\ &= \frac{1}{N}\left[N+\sum_{m=1}^{\ell-1}\frac{1-e^{i2m\pi}}{1-e^{i2m\pi/N}}\right]\\ &= 1 \qquad\qquad m<\ell\leq N.\end{aligned} \tag{85}$$

Equating (85) and (84) and making use of (82) for $S_2(0)$, we obtain

$$S_2(\ell)-aS_2(\ell-1)=\frac{a^{N-\ell}}{1-a^N} \tag{86}$$

The recursion relation (86) can be solved as in the above. Define the generating function $G_2(t)$ by (76). We find

$$\begin{aligned} G_2(t) &= \frac{1}{1-at}\left[S_2(0) + \frac{a^{N-1}t}{(1-a^N)(1-a^{-1}t)}\right] \\ &= \frac{1}{(1-a^2)(1-a^{2N})}\left[\frac{1}{1-at} + \frac{a^N}{1-a^{-1}t}\right], \end{aligned} \tag{87}$$

from which one reads off

$$S_2(\ell) = \frac{a^\ell + a^{N-\ell}}{(1-a^2)(1-a^{2N})}. \tag{88}$$

Using the relation $I_2(\ell) = 2\,a\,S_2(\ell)$ with $a = e^{-l}$, we obtain (62). QED •

Acknowledgments

I would like to thank D. H. Lee for discussions and the hospitality at Berkeley where this work was initiated. I am grateful to W.-J. Tzeng for a critical reading of the manuscript and helps in clarifying the Möbius strip and Klein bottle analyses, and to W. T. Lu for assistance in preparing the graphs. Work is supported in part by NSF Grant DMR-9980440.

References

[1] J. Cserti, Application of the lattice Green's function for calculating the resistance of an infinite network of resistors, *Am. J. Phys.* **68** 896-906 (2002).

[2] P. G. Doyle and J. L. Snell, Random walks and electric networks, The Carus Mathematical Monograph, Series 22 (The Mathematical Association of America, USA, 1984), pp. 83-149; Also in arXiv:math.PR/0001057.

[3] L. Lovász, Random walks on graphs: A survey, in *Combinatorics, Paul Erdös is Eighty*, Vol. 2, Eds. D. Miklós, V. T. Sós, and T. Szónyi (János Bolyai Mathematical Society, Budepest, 1996) pp. 353-398. Also at http://research.microsoft.com∼lovasz/ as a survey paper. I am indebted to S. Redner for calling my attention to this reference.

[4] B. van der Pol, The finite-difference analogy of the periodic wave equation and the potential equation, in *Probability and Related Topics in Physical Sciences*, Lectures in Applied Mathematics, Vol. 1, Ed. M. Kac (Interscience Publ. London, 1959) pp. 237-257.

[5] S. Redner, A Guide to First-passage Processes (Cambridge Press, Cambridge, UK 2001).

[6] S. Katsura, T. Morita, S. Inawashiro, T. Horiguchi, and Y. Abe, Lattice Green's function: Introduction, *J. Math. Phys.* **12** 892-895 (1971).

[7] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, *Ann. Phys. und Chemie,* **72** 497-508 (1847).

[8] A study of random walks using matrices can be found in [3].

[9] J. Cserti, G. Dávid, and Attila Piróth, Perturbation of infinite networks of resistors, *Am. J. Phys.* **70** 153-159 (2002). Also in arXiv:cond-mat/0107362.

[10] See, for example, F. Harary, Graph theory (Addison-Wesley, Reading, MA 1969).

[11] W.-J. Tzeng and F. Y. Wu, Spanning Trees on Hypercubic Lattices and Non-orientable Surfaces, *Appl. Math. Lett.* **13** (7) 19-25 (2000).

[12] I am indebted to W.-J. Tzeng for working out (49) and (53).

Figure Captions

Fig. 1. A network of 4 nodes.

Fig. 2. A one-dimensional network of $\mathcal{N}$ nodes with free ends.

Fig. 3. A one-dimensional network of $\mathcal{N}$ nodes with periodic boundary conditions

Fig. 4. A 5×4 rectangular network.

Fig. 5 (a) The schematic plot of an Möbius strip. (b) The schematic plot of a Klein bottle.

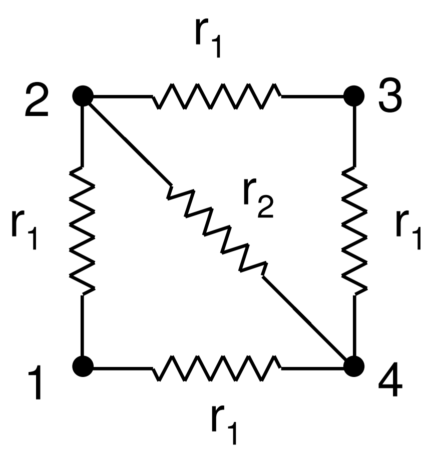


Fig.1

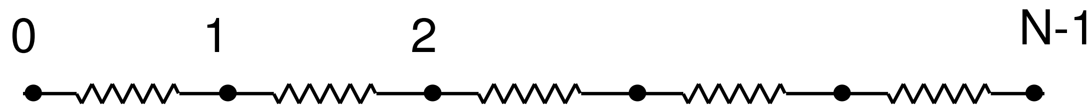


Fig.2

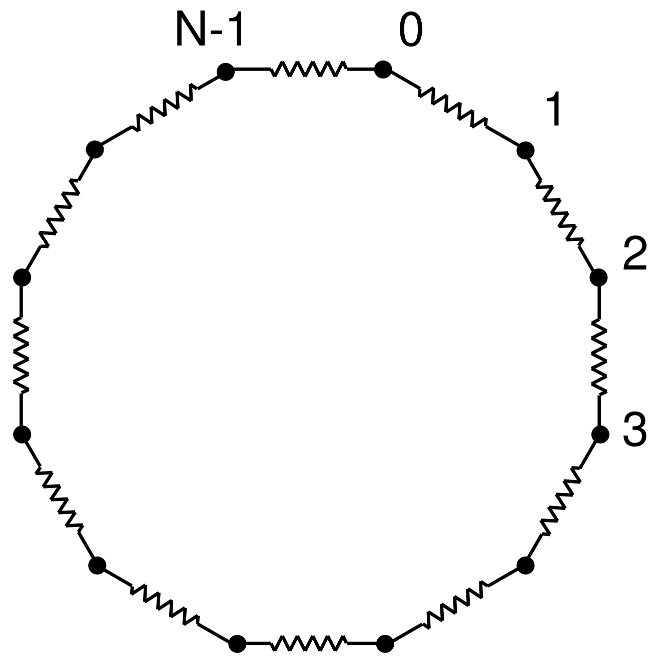


Fig.3

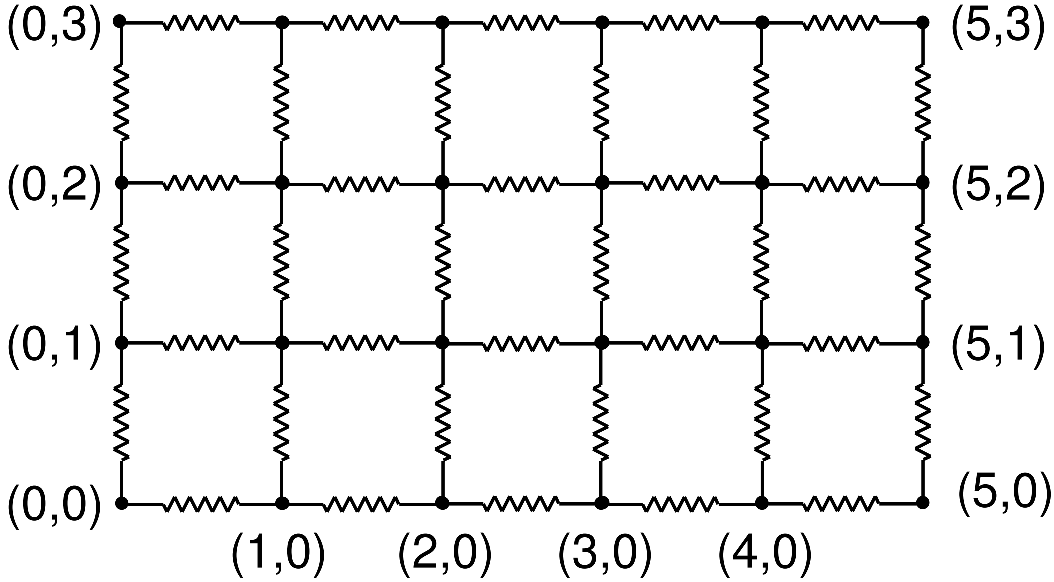
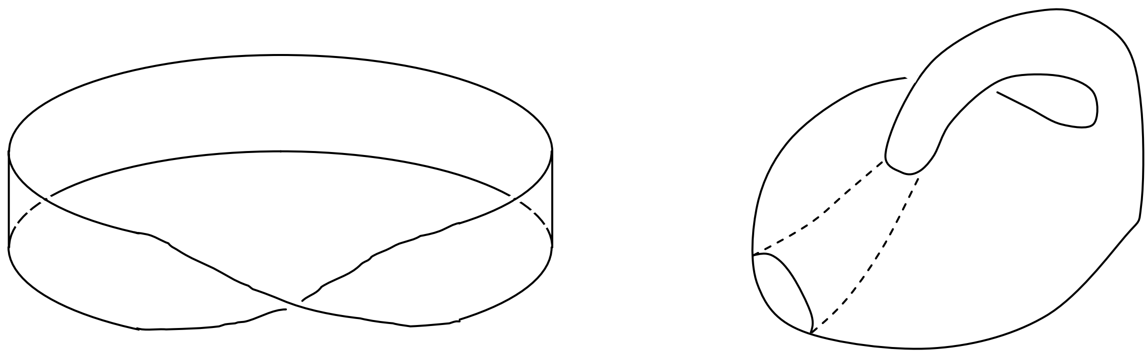


Fig.4



(a) (b)

Fig.5