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Terms, Definitions and Operations in Quantum Computing Theory

Abstract
In quantum computing theory, one often encounters tensor products and operations on them, and different authors use different notations. To the uninitiated this can seem esoteric and confusing. In the spirit that a few good examples are worth thousands of equations, we clarify the notation and give a few examples.
Key Words: tensor product, qubit, superposition, entangled states, Bell states.

Basic Notation and Definitions

We assume that you are familiar with the basic quantum theory and Dirac notation. A quantum state is denoted by what is called a ket vector (or simply, ket), $|\alpha\rangle$, which can be represented (for example) as a two-component column vector in the form $|\alpha\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The values α_1 and α_2 can be complex.

States $|0\rangle$ and $|1\rangle$ are called basis states, and $|\alpha\rangle = \alpha_1 |0\rangle + \alpha_2 |1\rangle$ is said to be a superposition of $|0\rangle$ and $|1\rangle$. Other bases besides these can be defined. In quantum computing, two-component states, such as $|0\rangle$, $|1\rangle$ and $|\alpha\rangle$ are called qubits. Multicomponent states are also possible.

The dual of $|\alpha\rangle$, denoted by $\langle\alpha|$ is called a bra vector, and can be represented as row vector $\langle\alpha| = [\alpha_1 \quad \alpha_2] = \alpha_1 \langle 0| + \alpha_2 \langle 1|$. The scalar product $\langle\alpha|\alpha\rangle = \begin{bmatrix} \alpha_1^* & \alpha_2^* \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = |\alpha_1|^2 + |\alpha_2|^2$, where * denotes the complex conjugate. The scalar product is sometimes called the inner product or the dot product. If $\langle\alpha|\alpha\rangle = 1$, $|\alpha\rangle$ is said to be normalized. Note that $\langle 0|1\rangle = \langle 1|0\rangle = 0$ and $\langle 0|0\rangle = \langle 1|1\rangle = 1$, and the basis states are said to be orthonormal. Let $|\alpha\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $|\beta\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$, then the scalar product of $|\alpha\rangle$ and $|\beta\rangle$ is $\langle\alpha|\beta\rangle = \begin{bmatrix} \alpha_1^* & \alpha_2^* \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$. Note that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$. $\langle 0|\alpha\rangle = \alpha_1$ is the amplitude that $|\alpha\rangle$ is in state $|0\rangle$, and $|\alpha_1|^2$ is the probability that, when measured $|\alpha\rangle$ is in state $|0\rangle$. Likewise for $\langle 1|\alpha\rangle = \alpha_2$.

Tensor Products

A tensor product of two states is denoted as $|\alpha\rangle \otimes |\beta\rangle$, $|\alpha\rangle|\beta\rangle$ or $|\alpha\beta\rangle$. These are equivalent expressions. Mathematicians prefer to write $|\alpha\rangle \otimes |\beta\rangle$.

Let $|\alpha\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $|\beta\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$. Then by definition $|\alpha\rangle \otimes |\beta\rangle$ is given by

$$|\alpha\rangle \otimes |\beta\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \end{bmatrix}. \tag{1}$$

Similarly for $\langle\alpha| = [\alpha_1 \quad \alpha_2]$ and $\langle\beta| = [\beta_1 \quad \beta_2]$,

$$\langle\alpha| \otimes \langle\beta| = [\alpha_1 \quad \alpha_2] \otimes [\beta_1 \quad \beta_2] = \big[\alpha_1[\beta_1 \quad \beta_2] \quad \alpha_2[\beta_1 \quad \beta_2]\big] = [\alpha_1\beta_1 \quad \alpha_1\beta_2 \quad \alpha_2\beta_1 \quad \alpha_2\beta_2]. \tag{2}$$

Note that $|\beta\rangle \otimes |\alpha\rangle \neq |\alpha\rangle \otimes |\beta\rangle$ because

$$|\beta\rangle \otimes |\alpha\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \otimes \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ \beta_2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \beta_1\alpha_1 \\ \beta_1\alpha_2 \\ \beta_2\alpha_1 \\ \beta_2\alpha_2 \end{bmatrix}. \tag{3}$$

Outer Product

The inner product of a bra and a ket yields a scalar, the tensor product of a ket with a bra vector, $|\alpha\rangle \otimes \langle\beta|$, is often called the outer product and is usually written as $|\alpha\rangle\langle\beta|$. The outer product yields a matrix. Following the definition of tensor product

$$|\alpha\rangle\langle\beta| = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \otimes [\beta_1^* \quad \beta_2^*] = \begin{bmatrix} \alpha_1 [\beta_1^* \quad \beta_2^*] \\ \alpha_2 [\beta_1^* \quad \beta_2^*] \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1^* & \alpha_1\beta_2^* \\ \alpha_2\beta_1^* & \alpha_2\beta_2^* \end{bmatrix}. \tag{4}$$

For example

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes [1 \quad 0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes [0 \quad 1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5}$$

yields the two-dimensional unit matrix.

Scalar Product of Two Tensor Products

From the definition of tensor product it can be shown that the scalar product of $\big(\langle\alpha'| \otimes \langle\beta'|\big)\big(|\alpha\rangle \otimes |\beta\rangle\big)$ is given by

$$\big(\langle\alpha'| \otimes \langle\beta'|\big)\big(|\alpha\rangle \otimes |\beta\rangle\big) = \langle\alpha'|\alpha\rangle\langle\beta'|\beta\rangle \tag{6}$$

Proof:

$$\begin{aligned} \big(\langle\alpha'| \otimes \langle\beta'|\big)\big(|\alpha\rangle \otimes |\beta\rangle\big) &= \big[\alpha_1'[\beta_1' \quad \beta_2'] \quad \alpha_2'[\beta_1' \quad \beta_2']\big] \begin{bmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \end{bmatrix} \\ &= \alpha_1'\alpha_1 \langle\beta'|\beta\rangle + \alpha_2'\alpha_2 \langle\beta'|\beta\rangle \\ &= \langle\alpha'|\alpha\rangle\langle\beta'|\beta\rangle. \end{aligned} \tag{7}$$

Explicit evaluation yields the equivalent result.

$$\left(\langle\alpha'|\otimes\langle\beta'|\right)\left(|\alpha\rangle\otimes|\beta\rangle\right)=\begin{bmatrix}\alpha_1'\beta_1' & \alpha_1'\beta_2' & \alpha_2'\beta_1' & \alpha_2'\beta_2'\end{bmatrix}\begin{bmatrix}\alpha_1\beta_1\\ \alpha_1\beta_2\\ \alpha_2\beta_1\\ \alpha_2\beta_2\end{bmatrix}$$
$$=\alpha_1\alpha_1'\beta_1\beta_1'+\alpha_1\alpha_1'\beta_2\beta_2'+\alpha_2\alpha_2'\beta_1\beta_1'+\alpha_2\alpha_2'\beta_2\beta_2. \qquad (8)$$

Action of Operators on Tensor Product States

Let A and B be two operators (called observables in quantum mechanics, gates in quantum computing). For simplicity let $A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$ and $B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & _{22}\end{bmatrix}$, then by definition

$$(A\otimes B)\left(|\alpha\rangle\otimes|\beta\rangle\right)=A|\alpha\rangle\otimes B|\beta\rangle, \qquad (9)$$

which is equivalent to $(A\otimes B)|\alpha\rangle|\beta\rangle=A|\alpha\rangle\otimes B|\beta\rangle=A|\alpha\rangle B|\beta\rangle$.

Expectation Value of a Tensor Product State

Let $|\psi\rangle=|\alpha\rangle|\beta\rangle$ be a normalized state. The expectation value of $A\otimes B$ with respect to $|\psi\rangle$ is $\langle\psi|A\otimes B|\psi\rangle$. From (9)

$$A\otimes B|\psi\rangle=(A\otimes B)(|\alpha\rangle\otimes|\beta\rangle)=\left(A|\alpha\rangle\otimes B|\beta\rangle\right) \qquad (10)$$

and from (4)

$$\begin{aligned}\langle\psi|A\otimes B|\psi\rangle&=\left(\langle\alpha|\otimes\langle\beta|\right)\left(A|\alpha\rangle\otimes B|\beta\rangle\right)\\ &=\langle\alpha|A|\alpha\rangle\langle\beta|B|\beta\rangle.\end{aligned} \qquad (11)$$

Entangled States

A state (or qubit) is said to be an entanglement of two (or more) states is said to be entangled if it cannot be expressed as a tensor product of simpler ones. For example $|\Phi^+\rangle=\dfrac{|0\rangle|0\rangle+|1\rangle|1\rangle}{\sqrt{2}}$, which is called a Bell state [1] cannot be factored into the tensor product of two other states. Sometimes $|\Phi^+\rangle$ is expressed as $|\Phi^+\rangle=\dfrac{|0\rangle_A\otimes|0\rangle_B+|1\rangle_A\otimes|1\rangle_B}{\sqrt{2}}$. This latter form emphasizes that $|0\rangle_A$ is acted upon by operator A and $|0\rangle_B$ by operator B. This clarifies that $(A\otimes B)\left(|0\rangle_A\otimes|0\rangle_B\right)$ $=A|0\rangle_A\otimes B|0\rangle_B$. Note that both $|0\rangle_A$ and $|0\rangle_B$ are represented by $\begin{bmatrix}1\\0\end{bmatrix}$. Thus (9) is equivalent to $(A\otimes B)\left(|\alpha\rangle_A\otimes|\beta\rangle_B\right)=A|\alpha\rangle_A\otimes B|\beta\rangle_B$. On the other hand $(B\otimes A)\left(|\alpha\rangle_B\otimes|\beta\rangle_A\right)$ $=B|\alpha\rangle_B\otimes A|\beta\rangle_A$.

Exercise 1

Using the rules for tensor products express $|\Phi^+\rangle$ as a four-component column vector.

Exercise 2

Using the definitions given in [2] $A_0 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B_0 = \frac{-(\sigma_z + \sigma_z)}{\sqrt{2}}$ and $B_1 = \frac{\sigma_z - \sigma_z}{\sqrt{2}}$, express $A_0 \otimes B_0$ as a 4X4 matrix.

Exercise 3

Reference [2] discusses measurement of the Bell state qubit $\left|\Psi^-\right\rangle = \frac{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle}{\sqrt{2}}$. Show that

$\left\langle \Psi^- \middle| A_0 \otimes B_0 \middle| \Psi^- \right\rangle = \left\langle \Psi^- \middle| A_0 \otimes B_1 \middle| \Psi^- \right\rangle = \left\langle \Psi^- \middle| A_1 \otimes B_0 \middle| \Psi^- \right\rangle = \frac{1}{\sqrt{2}}$ and that

$\left\langle \Psi^- \middle| A_1 \otimes B_1 \middle| \Psi^- \right\rangle = \frac{-1}{\sqrt{2}}$.

[1] https://en.wikipedia.org/wiki/Bell_state

[2] https://en.wikipedia.org/wiki/Bell's_theorem