Some Integrals

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(Dated: October 19, 2020)

Here we will evaluate a couple of integrals which might come up in the tutorial problems. These integrals may seem challenging, but can in fact be solved using only elementary substitutions, as will be demonstrated.

We begin by considering the integral

$$I = \int dx \frac{x}{(a^2 + x^2)^{3/2}}. \tag{1}$$

To evaluate this, we only need the substitution $u = a^2 + x^2$.

$$\begin{aligned}
I &= \int dx \frac{x}{(x^2 + a^2)^{3/2}} \\
u &= a^2 + x^2 \\
dx &= \frac{du}{2x} \\
\Rightarrow I &= \int \frac{du}{2x} \frac{x}{u^{3/2}} \\
I &= \frac{1}{2} \int \frac{du}{u^{3/2}}
\end{aligned}$$

This can now be integrated in the usual way.

$$\begin{aligned}
I &= \frac{1}{2} \int \frac{du}{u^{3/2}} \\
&= -\frac{1}{\sqrt{u}} \\
\Rightarrow \int dx \frac{x}{(x^2 + a^2)^{3/2}} &= -\frac{1}{\sqrt{a^2 + x^2}}.
\end{aligned}$$

Next, we consider

$$I = \int dx \frac{1}{(x^2 + a^2)^{3/2}}. \tag{2}$$

Again, this can be solved by substitution however must now use $x = a \tan u$. Along the way, we will also make use of

$$1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad \frac{d}{dx} \tan x = \frac{1}{\cos^2 x}.$$

So, we make the substitution

$$x = a \tan u \qquad dx = a \frac{1}{\cos^2 u} du,$$

so that the integral becomes

$$\begin{aligned} I &= \int a \frac{1}{\cos^2 u} du \frac{1}{(a^2 \tan^2 u + a^2)^{3/2}} \\ &= a \int \frac{du}{\cos^2 u} \frac{1}{a^3 (\tan^2 u + 1)^{3/2}} \\ &= \frac{1}{a^2} \int \frac{du}{\cos^2 u} \cos^3 u \\ &= \frac{1}{a^2} \int du \cos u. \end{aligned}$$

Now we can integrate this easily!

$$I = \frac{1}{a^2} \sin u.$$

All we have to do now is undo the substitution to get everything in terms of x again. To do this, we need to remember that x and a are the opposite and adjacent sides of a triangle of hypotenuse $h = \sqrt{x^2 + a^2}$. Noticing that u just defines the angle of the triangle, and that $\sin \theta = \sin u = x/h$, we can write

$$\begin{aligned} u &= \operatorname{atan} \frac{x}{a} \\ I &= \frac{1}{a^2} \sin \left(\operatorname{atan} \frac{x}{a} \right) \\ \Rightarrow \int dx \frac{1}{(x^2 + a^2)^{3/2}} &= \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}} \end{aligned}$$

Next, we consider the integral

$$I = \int \frac{1}{\sqrt{a^2 + x^2}} dx. \tag{3}$$

Applying the substitution

$$x = a \sinh u \qquad\qquad dx = a \cosh u du,$$

noting that

$$\cosh^2 \theta - \sinh^2 \theta = 1.$$

This gives

$$\begin{aligned}
I &= \int \frac{1}{\sqrt{a^2 + a^2 \sinh^2 u}} a \cosh u du \\
&= \int \frac{1}{a\sqrt{1 + \sinh^2 u}} a \cosh u du \\
&= \int \frac{1}{a\sqrt{\cosh^2 u}} a \cosh u du \\
&= \int du \\
\Rightarrow I &= u.
\end{aligned}$$

Undoing the substitution yields the result of the integral,

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \operatorname{arcsinh}\left(\frac{x}{a}\right).$$