Prove that the family of sets $\boldsymbol{\mathcal{V}}_\gamma = \left\{\boldsymbol{x} \in \boldsymbol{\mathcal{R}}^p : \sum_{i=1}^{N_\phi} \alpha_i \cdot \phi_i(\boldsymbol{x}) \leq \gamma\right\}$ with $\alpha_i > 0$ are forward invariant.

Proof:

According to the spectral property of the Koopman operator in [41], an eigenfunction $\varphi(\boldsymbol{x})$ and its corresponding eigenvalue $\lambda \in \boldsymbol{\mathcal{C}}$ of the Koopman operator $\boldsymbol{K}^t$ satisfies

$$\boldsymbol{K}^t \varphi(\boldsymbol{x}) = e^{\lambda t} \cdot \varphi(\boldsymbol{x})$$

$$\frac{d\varphi(\boldsymbol{x})}{dt} = \lambda \cdot \varphi(\boldsymbol{x})$$

For each $\phi_i(\boldsymbol{x}) \coloneqq \varphi_i(\boldsymbol{x}) \cdot \bar{\varphi}_i(\boldsymbol{x}) = |\varphi_i(\boldsymbol{x})|^2$ $(i = 1, 2, \dots, N_\phi)$,

$$\dot{\phi}_i(\boldsymbol{x}) = \frac{d\phi_i(\boldsymbol{x})}{dt} = \frac{d\varphi_i(\boldsymbol{x})}{dt} \cdot \bar{\varphi}_i(\boldsymbol{x}) + \varphi_i(\boldsymbol{x}) \cdot \frac{d\bar{\varphi}_i(\boldsymbol{x})}{dt}$$

Note $\frac{d\bar{\varphi}_i(x)}{dt} = \overline{\frac{d\varphi_i(x)}{dt}} = \overline{\lambda_i \cdot \varphi_i(\boldsymbol{x})} = \bar{\lambda}_i \cdot \overline{\varphi_i(\boldsymbol{x})}$, thus

$$\dot{\phi}_i(\boldsymbol{x}) = \left(\lambda_i + \bar{\lambda}_i\right) \cdot |\varphi_i(\boldsymbol{x})|^2 = 2 \cdot \mathrm{Re}[\lambda_i] \cdot \phi_i(\boldsymbol{x})$$

Since the Koopman eigenfunctions are approximated using the SoC algorithm, there are approximation errors $\mathfrak{e}_i(\boldsymbol{x})$'s. With a mild assumption that the errors are bounded such that $|\mathfrak{e}_i(\boldsymbol{x})| \leq \zeta_i \cdot \phi_i^2(\boldsymbol{x}) + \eta_i$ for some positive constants ζ_i and η_i,

$$\dot{\phi}_i(\boldsymbol{x}) = 2 \cdot \mathrm{Re}[\lambda_i] \cdot \phi_i(\boldsymbol{x}) + \mathfrak{e}_i(\boldsymbol{x}) \leq 2 \cdot \mathrm{Re}[\lambda_i] \cdot \phi_i(\boldsymbol{x}) + |\mathfrak{e}_i(\boldsymbol{x})|$$

$$= \zeta_i \cdot \phi_i^2(\boldsymbol{x}) + 2 \cdot \mathrm{Re}[\lambda_i] \cdot \phi_i(\boldsymbol{x}) + \eta_i$$

If $\dot{\phi}_i(\boldsymbol{x}) \leq 0$ always holds for certain interval $(\underline{\gamma_i}, \overline{\gamma_i}) \subset \boldsymbol{\mathcal{R}}_{>0}$, the minimum of the above quadratic function of $\phi_i(\boldsymbol{x})$ should be negative, which leads to the condition $(\mathrm{Re}[\lambda_i])^2 > \zeta_i \cdot \eta_i$. It also implies that the $\overline{\gamma_i}$-sublevel set of $\phi_i(\boldsymbol{x})$ is forward invariant.

Further, define $\gamma \coloneqq \min_i(\alpha_i \cdot \overline{\gamma_i})$ for $\alpha_i > 0$ ($i = 1, 2, \dots, N_\phi$). From $\sum_{i=1}^{N_\phi} \alpha_i \cdot \phi_i(\boldsymbol{x}) \leq \gamma$, we have

$$\alpha_i \cdot \phi_i(\boldsymbol{x}) \leq \gamma = \min_i(\alpha_i \cdot \overline{\gamma_i})$$

Thus, $\phi_i(\boldsymbol{x}) \leq \overline{\gamma_i}$ holds for $i = 1, 2, \dots, N_\phi$.

Now, define $\beta \coloneqq 2 \cdot \min_i |\mathrm{Re}[\lambda_i]|$. Then,

$$\sum_{i=1}^{N_\phi} \alpha_i \cdot \dot{\phi}_i(\boldsymbol{x}) \leq \sum_{i=1}^{N_\phi} \alpha_i \cdot (2 \cdot \mathrm{Re}[\lambda_i]) \cdot \phi_i(\boldsymbol{x}) + \sum_{i=1}^{N_\phi} \alpha_i \cdot [\zeta_i \cdot \phi_i^2(\boldsymbol{x}) + \eta_i]$$

$$\leq \sum_{i=1}^{N_\phi} \alpha_i \cdot (-\beta) \cdot \phi_i(\boldsymbol{x}) + \sum_{i=1}^{N_\phi} \alpha_i \cdot (\zeta_i \cdot \overline{\gamma_i}^2 + \eta_i)$$

Therefore, if $\gamma \cdot \beta \geq \sum_{i=1}^{N_\phi} \alpha_i \cdot (\zeta_i \cdot \overline{\gamma_i}^2 + \eta_i)$,

$$\sum_{i=1}^{N_\phi} \alpha_i \cdot \dot{\phi}_i(\boldsymbol{x}) \leq \sum_{i=1}^{N_\phi} \alpha_i \cdot (-\beta) \cdot \phi_i(\boldsymbol{x}) + \gamma \cdot \beta = \beta \cdot [\gamma - \sum_{i=1}^{N_\phi} \alpha_i \cdot \phi_i(\boldsymbol{x})]$$

This suggests that the γ-sublevel set of $\sum_{i=1}^{N_\phi} \alpha_i \cdot \phi_i(\boldsymbol{x})$ is forward invariant. It should also be noted that the condition $\gamma \cdot \beta \geq \sum_{i=1}^{N_\phi} \alpha_i \cdot (\zeta_i \cdot \overline{\gamma_i}^2 + \eta_i)$ can be easily met if β is large enough, which translates to that all λ_i's ($i = 1, 2, \dots, N_\phi$) have a sufficiently large negative real part.

Q.E.D.